Q3.1

Poisson problem on $x = (x_1, x_2) \in \Omega = [0,1]^2$
with function

$$f : \Omega \to \mathbb{R}$$

$$-\Delta u(x) = f(x) \quad \text{on} \quad \Omega \setminus \partial\Omega$$

$$u(x) = 0 \quad \text{on} \quad \partial\Omega$$

The symmetric finite difference approximation of u is

$$\frac{\partial^2}{\partial x_1^2} u(x) \approx \frac{1}{h^2}\left[u(x+h) - 2u(x) + u(x-h)\right]$$

$$\frac{\partial^2}{\partial x_1^2} u(x_1, x_2) \approx \frac{1}{h^2}\left[u(x_1+h, x_2) - 2u(x_1, x_2) + u(x_1-h, x_2)\right]$$

And similarly for

$$\frac{\partial^2}{\partial x_2^2} u(x_1, x_2) \approx \frac{1}{h^2}\left[u(x_1, x_2+h) - 2u(x_1, x_2) + u(x_1, x_2-h)\right]$$

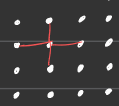
$$\Rightarrow -\Delta u(x) \approx \frac{1}{h^2}\Big[u(x_1+h, x_2) + u(x_1-h, x_2) + u(x_1, x_2+h) + u(x_1, x_2-h) - 4u(x_1, x_2)\Big]$$

using the notation $u_{ij} = u(ih, jh)$ for $0 \le i,j \le N$

$$-\Delta u_{ij}(x) = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right]$$

We can choose to order $\vec{u}$ for $A\vec{u} = \vec{b}$ as

for an $N \times N$ grid



$$\vec{u} = \begin{bmatrix} u_{0,0} \\ u_{0,1} \\ \vdots \\ u_{1,0} \\ u_{1,1} \\ \vdots \\ u_{N,N} \end{bmatrix}$$

This gives us the $NN \times NN$ matrix A to solve with block matrices

$$C = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \qquad -I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Then $$A = \begin{bmatrix} C & -I & & \\ -I & C & -I & \\ & -I & \ddots & -I \\ & & -I & C \end{bmatrix}$$

Q3.2

$$-\Delta u = \nabla^2 u = f(x) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$$

assume $u = \sin(\pi x_1)\sin(\pi x_2)$ this is the analytical solution to the equation

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 \sin(\pi x_1)\sin(\pi x_2)$$

$$\Rightarrow \nabla^2 u = -2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$$