MAP55672 (2024-25) — Case studies 4

The Multigrid method

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Instructions

- Complete all parts of the assignment by **midnight on Monday 5th May 2025**. Create a git repository with your solutions and submit the link to me by e-mail.
- The git repository should include all source files and a summary pdf containing a description of solutions (incl. tables and plots, if necessary). The summary report should reflect your understanding of the material and contain a short description of your submitted code. As such it should contain all necessary information to (compile and) run your code for validation purposes.
- *It is strictly forbidden to use code generating tools such as LLM's. Their use will result in 0 marks for this case study.*

4 The MG method

The goal of this case study is to implement the Multigrid (MG) algorithm for the solution of a square linear system $Ax = b$ for positive definite, symmetric regular matrices A and right hand sides $b \neq 0$.

4.1 Basics: The Poisson problem, again.

Consider the Poisson problem on a unit square $x = (x_1, x_2) \in \Omega = (0,1)^2$ with function $f : \Omega \to \mathbb{R}$,

$$-\Delta u(x) = f(x)\ , \quad \text{on interior of } \Omega \qquad\qquad \Delta u(x) \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u(x)$$

$$u(x) = 0\ , \quad \text{on the boundary } \partial\Omega\ ,$$

and unknown solution function $u(x) : \Omega \to \mathbb{R}$.

Use *symmetric finite difference* approximations to the partial derivatives appearing in the Laplacian Δ, to express above's partial differential equation with specified boundary conditions as a linear system $Ay = b$ where y is a vector containing the solution function $u(x)$ at discrete values of $x = (ih, jh)$, $0 \leq i, j \leq N$ on a regular 2D grid.

You can take over your notation (introduction) from case study 3 and adopt details wherever necessary.

4.2 Serial implementation of a recursive V-cycle multigrid.

Implement a serial variant of the *recursive V-cycle MG algorithm* using the prescription:

Algorithm 1 `x = Vcycle( Al, xl, bl, omega, nu, lmax)`

`xl = smooth( Al, xl, bl, omega, nu)`
$r_l = b_l - A_l x_l$
$b_{l+1} = R^l_{l+1} r_l$
if `(l+1)==lmax` **then**
    Solve: $A_{l_{\max}} x_{l_{\max}} = b_{l_{\max}}$
else
    x_{l+1} `= Vcycle( Al, xl, bl, omega, nu, lmax)`
$x_l = x_l + P^{l+1}_l x_{l+1}$
`xl = smooth( Al, xl, bl, omega, nu)`

where

- each step reduces the multigrid size by factor 2 in each dimension,
- $l \in \{1, \dots, \texttt{lmax}\}$ are consecutive multigrid levels with `lmax` being the coarsest,
- `smooth()` implements ν iterations of weighted Jacobi (with parameter ω),
- A_l is the problem matrix discretisation at level l,
- R^l_{l+1} is the restriction matrix/mapping from level l to $l+1$, and
- P^{l+1}_l is the prolongation matrix/mapping from level $l+1$ to l.

Your implementation should

1. be able to run more than a single complete V-cycle,
2. contain necessary safety measures to avoid excessive values for algorithmic param.s,
3. have a known solver for the coarsest level solve,
4. have break point(s) where considered necessary to stop MG if converged,
5. be able to stop if divergent or too slow to reach target precision,
6. be efficient.

It is allowed to modify/adapt above's algorithm for your convenience and efficiency purposes.
Explain your implementation and the reasoning behind all your choices.

4.3 Convergence of MG.

Solve the linear system defined in section 4.1 using the function $f(x) = 2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$.

1. Try to solve the system in double precision arithmetic for fixed number of grid points $N = N_x = N_y = 128$ and varying numbers of `lmax` ≥ 2 up to a fixed residual $r = 10^{-7}$. Monitor the total runtime, total no. of coarse level solves, and consecutive residuals r_l of your program.
2. Compare the performance of your program (iteration count, runtime, smallest residual) using $N = 16, 32, 64, 128, 256$ with a 2-level MG setup in contrast to a max-level MG setup for which the coarsest level has $N = 8$.

Present, interpret and discuss all your findings in an appropriate format.
What do you consider as best practice for this particular problem?