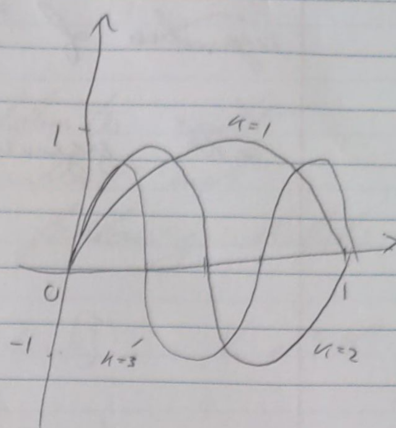
Multi grid Methods

~~Motivation~~

Poisson equation in 1D



$$-u''(x) = f(x) \qquad \forall\, x \in (0,1)$$

$$u(0) = u(1) = 0$$

$u : [0,1] \longrightarrow \mathbb{R}$ solution for some given function $f : [0,1] \to \mathbb{R}$

If solution $\exists$: $\quad u(x) = \sum_k c_k u_k(x) \quad , c_k \in \mathbb{R}$

with the eigenfunctions $u_k(x)$: $\quad -u_k''(x) = \mu_k$

μ_k eigenvalues to eigenfunction u_k $\quad$ | $\quad u(0) = u(1)$

$$\mu_k = k^2 \pi^2$$

$$u_k(x) = (\sin k\pi x) \quad k \in \mathbb{N}$$

Lets discretise (1D)

$$\begin{array}{cccccccc} 0 & & & & & & 1 \\ i = 0 & 1 & 2 & \cdots & N-2 & N-1 & N \end{array}$$

"bulk" interior

Ω_h , $h = \frac{1}{N}$, $x_i = ih$

$0 < i < N$

lin sys $\quad \frac{1}{h^2} T_h v_h = f_h$

$$\mathbb{R}^{n \times n} \ni T_h = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \quad , k = N-1$$

eigenvalues of T_h : $\lambda_k = 4\sin^2\left(\frac{k\pi}{2N}\right) = 2\left(1-\cos\left(\frac{k\pi}{N}\right)\right)$

with eigenvectors $V_h^k = \left(\sqrt{\frac{2}{N}}\right) \begin{pmatrix} \sin\left(\frac{k\pi}{N}\right) \\ \sin\left(2\frac{k\pi}{N}\right) \\ \vdots \\ \vdots \\ \sin\left((N-1)\frac{k\pi}{N}\right) \end{pmatrix} = \begin{pmatrix} \sin(k\pi x_1) \\ \sin(k\pi x_2) \\ \vdots \\ \vdots \\ \sin(k\pi x_{N-1}) \end{pmatrix}$

to normalise

$$= \begin{bmatrix} u_k(x_1) \\ u_k(x_2) \\ \vdots \\ \vdots \\ u_k(x_{N-1}) \end{bmatrix}$$

There is a high frequency part $(k > \frac{N}{2})$
and a low frequency part $(k \le \frac{N}{2}$

($N-1$ = # of nodes in the bulk)

Applying ~~the~~ Richardson Method

$A \in \mathbb{R}^{n\times n}$ $\quad x^{k+1} = (1 - wA)x^k + wb \quad w \in \mathbb{R}$
$b \in \mathbb{R}^n$

$k = 0, 1, \ldots$ initial point $x^0 \in \mathbb{R}^n$ arbitrary

Splitting matrix $M = \frac{1}{w}\cdot\mathbb{1}$ iteration matrix

$$G = (\mathbb{1} - M^{-1}A) = (1 - wA) = G(w)$$

Corollary

Given $A \in \mathbb{R}^{n \times n}$ sym and positive definite with λ_{min} and λ_{max} being the smallest and largest eigenvalue of A the following holds

(a) $\rho(G(w)) = \max\{|1 - w\lambda_{max}|, |1 - w\lambda_{min}|\} \quad \forall w \in \mathbb{R}$

spectral radius

(b) Richardson converges iff $w \in (0, \frac{2}{\lambda_{max}})$

(c) $w_{opt} = w_* = \dfrac{2}{\lambda_{min} + \lambda_{max}}$ minimizes $\rho(G(w)) \; \forall \; w \in \mathbb{R}$

(d) $\rho(G(w_{min})) = \dfrac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}}$

Comment: • typically bad convergence since $\rho \approx 1$

• a quick reduction of the initial $r^0 = b - Ax^0$ while other parts are reduced rather slowly.

Let (λ, v) be eigenpair of A

$$G(w)v = (I - wA)v = \underbrace{(1 - w\lambda)}_{\sigma(w)} v$$

$\Rightarrow (\sigma(w), v)$ is eigenpair of $G(w)$

$$\Rightarrow \sigma_k = 1 - \omega\lambda_k = 1 - \omega 2\left(1 - \cos\left(\tfrac{k\pi}{N}\right)\right) \qquad k = 1, \dots, N-1$$

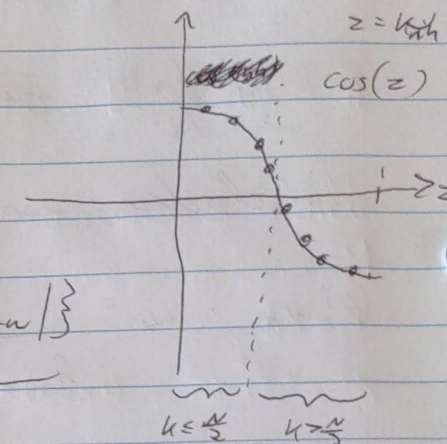
$$= 1 - 2\omega + 2\omega\cos(k\pi h)$$

<u>Low frequency</u> $\left(k \le \frac{N}{2}\right)$

$$-1 \le \cos(k\pi h) \le 1$$

$$\Rightarrow 1 - 4\omega \le \sigma_k \le 1 - 2\omega$$

$$\Rightarrow |\sigma_k| \le \underbrace{\max\{|1-4\omega|, |1-2\omega|\}}_{\text{this is minimal for } \omega = \frac{1}{3} \neq \omega_{opt}}$$



<u>Error?</u>

$$e^k = x^k - x^*$$

$$e^0 = \sum_{j=1}^{N-1} \alpha_j v_j \quad , \alpha_j \in \mathbb{R}$$

$$\Rightarrow \|e^0\|^2 = \sum_{j=1}^{N-1} |\alpha_j|^2$$

$$e^{k+1} = G(\omega)e^k \Rightarrow e^1 = G\left(\tfrac{1}{3}\right) = \sum_{j=1}^{N-1} \overbrace{\left(\tfrac{1}{3} + \tfrac{2}{3}\cos(j\pi h)\right)}^{\sigma_j\left(\frac{1}{3}\right)} \alpha_j v_j$$

$$\|e^1\|^2 = \sum_j \left(\tfrac{1}{3} + \tfrac{2}{3}\cos(\ )\right)^2 |\alpha_j|^2$$

$$\le \left(\tfrac{1}{3} + \tfrac{2}{3}\cos(\pi h)\right)^2 \underbrace{\sum_j |\alpha_j|^2}_{\|e^0\|^2}$$

$$\|e^k\| \le \left(\tfrac{1}{3} + \tfrac{2}{3}\cos(\pi h)\right)\|e^{k-1}\| \le \left(\tfrac{1}{3} + \tfrac{2}{3}\cos(\pi h)\right)^k \|e^0\|$$

In each iteration step, the error reduces at least by a factor

$$\frac{1}{3} + \frac{2}{3}\cos(\pi h) \overset{h\to 0}{\approx} \frac{1}{3} + \frac{2}{3}\left(1 - \frac{1}{2}(\pi h)^2\right)$$

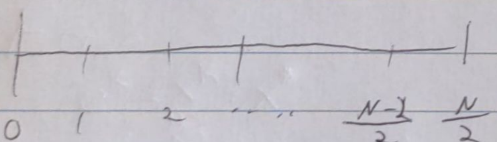
$$= 1 - \frac{\pi^2}{3} h^2$$

$$= 1 - O(h^2)$$

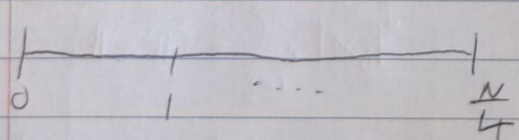
In general: low frequency modes (long range) are harder ~~to treat~~ or update in each iteration ~~that~~ than high frequency modes

Key Insights $\longrightarrow$ Multigrid Methods

"have to increase h for better convergence!!"



Ω_{2h}



Recap

$u''(x) = f(x) \quad \forall\ x \in (0,1)$

Dirichlet bc $\quad u(0) = u(1) = 0$

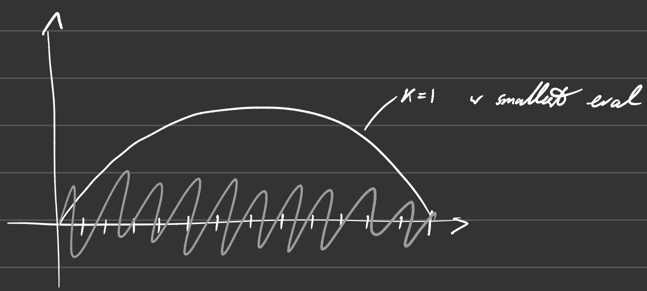
$M = \frac{1}{\omega} D \quad \Rightarrow$ Iteration $\quad G = (\mathbb{I} - M^{-1}A)$

error step size $\propto \boxed{1 - \mathcal{O}(h^2)} \qquad h = \frac{1}{N} \ll 1$

mode decomposition $\qquad$ low freq $\quad k < \frac{N}{2}$

$\qquad\qquad\qquad\qquad$ high freq $\quad k \geq \frac{N}{2}$

typ. method resolve high frequency modes fast
but low frequency modes slow



GRID HIERARCHY

fine grid (original) $A_h u_h = f_h$ $A_h \in \mathbb{R}^{n \times n}$

coarse grid $A_{2h} u_{2h} = f_{2h}$ $A_{2h} \in \mathbb{R}^{\frac{n}{2} \times \frac{n}{2}}$

5-level multigrid

$$\Omega_h = \left\{ \frac{i}{N} \;\middle|\; i = 0, \dots, N \right\}$$

Assume N even

Prolongation

Restriction

$$\Omega_{2h} = \left\{ \frac{2i}{N} \;\middle|\; i = 0, \dots, \frac{N}{2} \right\}$$

$\vdots$

$\Omega_{2^5 h}$

Advantages

- coarse grid faster to solve
- error reduction per iteration $1 - O(h^2)$
- low-freq part on fine grid
  $\Rightarrow$ high freq part on coarse grid
- easy to implement recursively

Example

$N = 8 \Rightarrow h = \frac{1}{8} \Rightarrow A \in \mathbb{R}^{7\times 7}$

2nd eigenvects

$$\Omega_h: \qquad v_h^2 = \begin{bmatrix} \sin(2\pi \frac{1}{8}) \\ \sin(2\pi \frac{2}{8}) \\ \vdots \\ \sin(2\pi \frac{7}{8}) \end{bmatrix} \in \mathbb{R}^7, \quad h = \frac{1}{8}$$

associated to low frequency part of spectrum

$$\downarrow$$

$$\Omega_{2h}: \qquad v_{2h}^2 = \begin{bmatrix} \sin(2\pi \frac{1}{4}) \\ \sin(2\pi \frac{2}{4}) \\ \sin(2\pi \frac{3}{4}) \end{bmatrix} \in \mathbb{R}^3, \quad 2h = \frac{1}{4}$$

associated to the high freq part on Ω_{2h}

Note we have to transfer information between Ω_h and Ω_{2h}

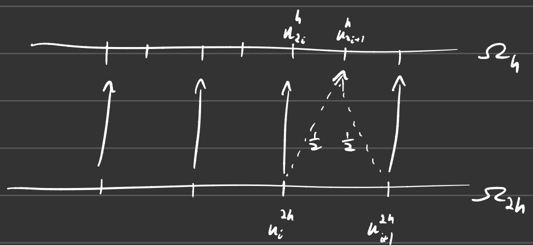
PROLONGATION mapping

$$u_h := P_h^{2h}(u_{2h})$$

$$P_h^{2h} : \mathbb{R}^{\frac{N}{2}+1} \longrightarrow \mathbb{R}^{N+1}$$

(linear and injective)

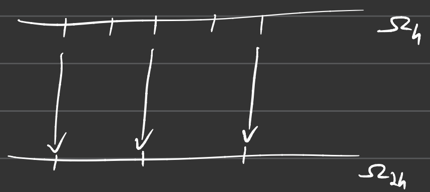
$$\boxed{\begin{aligned} u_{2i}^h &:= u_i^{2h} & \forall\ i \in \{1,2,\dots,\tfrac{N}{2}-1\} \\ u_{2i+1}^h &:= \tfrac{1}{2}\left(u_i^{2h} + u_{i+1}^{2h}\right) & \forall\ i \in \{0,1,\dots,\tfrac{N}{2}-1\} \end{aligned}}$$



$$\begin{bmatrix} u_1^h \\ u_2^h \\ \vdots \\ \vdots \\ u_{N-1}^h \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 1 & 2 & \ddots & \\ & 1 & \ddots & 1 \\ & & & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} u_1^{2h} \\ u_2^{2h} \\ \vdots \\ u_{\frac{N}{2}-1}^{2h} \end{bmatrix}$$

RESTRICTION

$$R_{2h}^{h} : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{\frac{N}{2}+1}$$



$$u_{2h} := R_{2h}^{h}(u_h)$$

(linear and surjective)

(a) a simple restriction

$$u_i^{2h} := u_{2i}^{h} \qquad \forall\ i \in \{0, \dots, \tfrac{N}{2}\}$$

(b) linear restriction

$$u_i^{2h} := \frac{1}{4}\left(u_{2i-1}^{h} + 2\,u_{2i}^{h} + u_{2i+1}^{h}\right) \qquad \forall\ i \in \{1, \dots, \tfrac{N}{2}-1\}$$

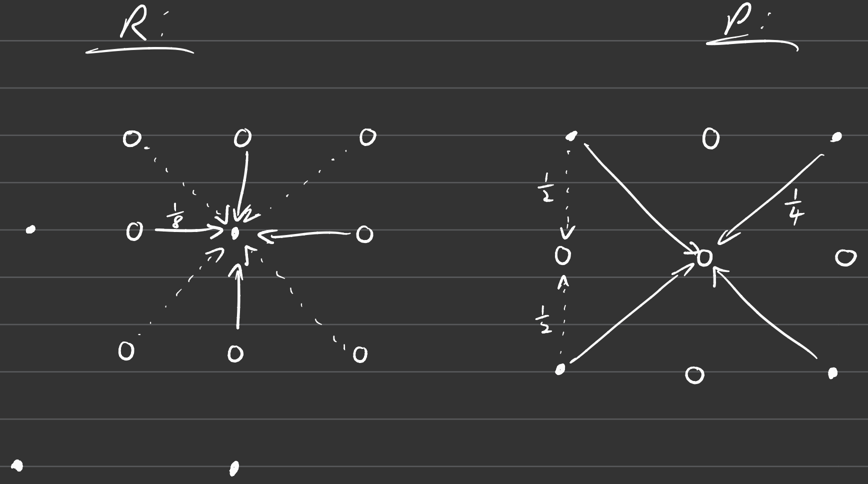
u_0^{2h}, $u_{\frac{N}{2}}^{2h}$ fixed by bc

$$\begin{bmatrix} u_1^{2h} \\ u_2^{2h} \\ \vdots \\ \vdots \\ u_{\frac{N}{2}-1}^{2h} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & \\ & & & \ddots & & \\ & & & & & \ddots \\ & & & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} u_1^{h} \\ \vdots \\ \vdots \\ u_{N-1}^{h} \end{bmatrix}$$

Ex Our example $R = \frac{1}{2} P^T$

Note

Typical R and P operation in 2D Poisson case



Multigrid ingredients

(1) Define matrix rep of original problem $Ax = b$

(2) Coarse discretisation

(3) Transfer matrices between diff levels (prolong and restrict)

(4) matrix A_{2h} associated to course grid $A_{2h} = RA_hP$

(5) smoother (Jacobi, Richardson) to resolve
high frequency modes (large eigenvalues)

$$A_{2h} = \frac{1}{(2h)^2} \text{tridiag}(-1, 2, 1)$$

Two level V-cycle multigrid

restrict to course grid $v_{2h} = Rv_h$

perform solve on course grid

$$x_{2h} = (A_{2h})^{-1} R v_h$$

$$= (RA_hP)^{-1} R v_h$$

prolongation to fine grid:

$$v_h' = Px_{2h} = \overbrace{P(RA_hP)^{-1}R}^{S} v_h$$

$$= (A_h)^{-1} v_h$$

multigrid iteration matrix

$$\Rightarrow \quad e_{k+1} = (\mathbb{1} - SA)\, e_k$$

Again low freq components of e_k become high freq modes on the coarse grid

MODE ANALYSIS: eigendecomposition $A = V \Lambda V^T$

$$\Rightarrow A_{2h} = R A_h P = R V \Lambda_h V^T P$$

$$= \tfrac{1}{2} P^T V \Lambda V^T P$$

$$=: W^T W$$

$W^T = P^T V \sqrt{\tfrac{1}{2}\Lambda_h}$

$$= V_{2h} \Lambda_{2h} V_{2h}^T$$

For $N = 8 \Rightarrow$ 1st column W^T

$$(W^T)_1 \propto \begin{bmatrix} \sin\left(\frac{\pi}{N}\right) + 2\sin\left(\frac{2\pi}{N}\right) + \sin\left(\frac{3\pi}{N}\right) \\ \sin\left(\frac{3\pi}{N}\right) + 2\sin\left(\frac{4\pi}{N}\right) + \sin\left(\frac{5\pi}{N}\right) \\ \sin(\quad) + \text{"} \quad \text{"} + \text{"} \quad \text{"} \end{bmatrix} \in \mathbb{R}^3$$

Restriction matrix mixes frequencies of eigenmodes of the fine matrix of the coarse grid

Transfer residual error to coarse grid

$r_{2h} = R r_h$

$\Rightarrow A_{2h} x_{2h} = r_{2h} = R A_h e$

$\Rightarrow x_{2h} = (A_{2h})^{-1} R A_h e$

$= (R A_h P)^{-1} R A_h e$

coarse grid correction

transfer to fine grid

$x_0 \leftarrow x_0 + P (RAP)^{-1} RAe$

We also add a smoothing step (post smoothing)

Two level V-cycle multigrid Algorithm

Until convergence do

(1) Pre-smooth on $Ax = b$ using ν iterations of a given smoothing (Jacobi, Richardson)

(2) Compute the residual $r = b - Ax$

(3) restrict the residual to coarse grid
$r_{2h} = R r_h$

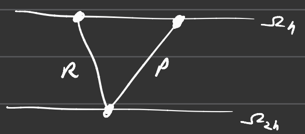
(4) direct solve $A_{2h} x_{2h} = r_{2h}$

(5) Prolong and add that to current solution

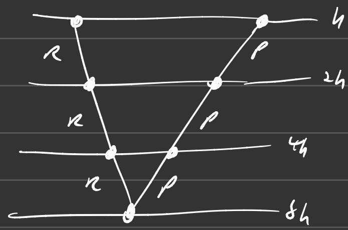
$$x_0 \leftarrow x_0 + P x_{2h}$$

(6) Post smoothing using ν iterations

(7) If converge, break



Iteration



W cycle