Q 3.1

Poisson problem on $x = (x_1, x_2) \in \Omega = [0,1]^2$
with function

$f : \Omega \to \mathbb{R}$

$$-\Delta u(x) = f(x) \quad \text{on} \quad \Omega \setminus \partial\Omega$$

$$u(x) = 0 \quad \text{on} \quad \partial\Omega$$

The symmetric finite difference approximation of u is

$$\frac{\partial^2}{\partial x_1^2} u(x) \approx \frac{1}{h^2}\left[u(x+h) - 2u(x) + u(x-h)\right]$$

$$\frac{\partial^2}{\partial x_1^2} u(x_1, x_2) \approx \frac{1}{h^2}\left[u(x_1+h, x_2) - 2u(x_1, x_2) + u(x_1-h, x_2)\right]$$

And similarly for

$$\frac{\partial^2}{\partial x_2^2} u(x_1, x_2) \approx \frac{1}{h^2}\left[u(x_1, x_2+h) - 2u(x_1, x_2) + u(x_1, x_2-h)\right]$$

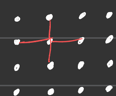
$$\Rightarrow -\Delta u(x) \approx \frac{1}{h^2}\Big[u(x_1+h, x_2) + u(x_1-h, x_2) + u(x_1, x_2+h) + u(x_1, x_2-h) - 4u(x_1, x_2)\Big]$$

using the notation $u_{ij} = u(ih, jh)$ for $0 \le i,j \le N$

$$-\Delta u_{ij}(x) = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right]$$

We can choose to order $\vec{u}$ for $A\vec{u} = \vec{b}$ as

for an $N \times N$ grid



$$\vec{u} = \begin{bmatrix} u_{0,0} \\ u_{0,1} \\ \vdots \\ u_{1,0} \\ u_{1,1} \\ \vdots \\ u_{N,N} \end{bmatrix} \in \mathbb{R}^{N \times N}$$

The matrix A will be constructed by using the structure in the $\vec{u}$ vector and the following

$$-\Delta u_{ij}(x) = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right]$$

which makes use take two neighboring parts of $u_{i,j}$ them being $u_{i,j+1}$, $u_{i,j-1}$ and these three parts sit next to eachother in the vector $\vec{u}$.

Then we are left with $u_{i+1,j}$ and $u_{i-1,j}$ which are located $\pm N$ spaces away from $u_{i,j}$ in the column of the vector $\vec{u}$.

This allows us to build the matrix $A \in \mathbb{R}^{N^2 \times N^2}$

$$A = \frac{1}{h^2}\begin{bmatrix} \ddots & & & & & & & & \\ -1 & \cdots & -1 & 4 & \overbrace{-1 & \cdots & -1}^{n} & & \\ & -1 & \cdots & -1 & 4 & -1 & \cdots & -1 & \\ & \underbrace{-1 & \cdots}_{n} & & -1 & 4 & -1 & \cdots & \cdots & -1 \\ & & & & & & & & \ddots \end{bmatrix}$$

Which allows us to recover the initial finite difference stencil with $A\vec{u} = \vec{b}$ and b holds the function $f(x)$ discrete values similar to how $\vec{u}$ stores values