INDIAN INSTITUTE OF TECHNOLOGY
HYDERABAD

SOLUTIONS TO THE EQUATION $0 \cdot x = const.$

ABHISHEK AGARWAL
EP17BTECH11001

November 2019

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

1 Abstract

The basic motivation is to be able to write formally the solutions to the equation $0 \cdot x = a; a \in \boldsymbol{R}$. Clearly there isn't any solution for x where x is also a real number, Thus we proceed here step by step to define what kind of rules would allow us to write such solutions.

2 Axioms and Structure

We begin by declaring a new set $\boldsymbol{T}$, which is a super set of $\boldsymbol{R}$ along with a new multiplication rule $\times$ which is different from regular real multiplication (denoted as $\cdot$ hereafter). Formally:

Axiom 1 *There exists $\boldsymbol{T}$ such that $\boldsymbol{R} \subset \boldsymbol{T}$ and define $\times : \boldsymbol{R} \times \boldsymbol{T} \to \boldsymbol{R}$, such that*

1. *$a \cdot (b \times x) = (a \cdot b) \times x$ where $a, b \in \boldsymbol{R}$ and $x \in \boldsymbol{T}$*
2. *For every equation $a \times x = b$ where $a, b \in \boldsymbol{R}$, there exists a unique $x \in \boldsymbol{T}$ which satisfies the equality.*

Theorem 1 *If $a \times x = a \times y$ where $a \in \boldsymbol{R}$ and $x, y \in \boldsymbol{T}$ then $x = y$.*

The reason is simply that axiomatically we have assumed such equations to have a unique solution. This also suggests that all elements in $\boldsymbol{T}$ can be labelled by two real numbers.

Definition 1 *For every x such that $a \times x = b$; where $a, b \in \boldsymbol{R}$, x is denoted by $T(b, a)$.*

Since $\boldsymbol{R} \subseteq \boldsymbol{T}$, $\boldsymbol{R} \times \boldsymbol{R} \subseteq \boldsymbol{R} \times \boldsymbol{T}$. Also note that $\cdot$ is a binary relation from $\boldsymbol{R} \times \boldsymbol{R}$ to $\boldsymbol{R}$. Thus if can preserve all the mappings $\cdot$ defines and have $\times$ behave the same way with some additional mappings, we can say that $\cdot \subseteq \times$. This allows us to rewrite the equation $0 \cdot x = a; a \in \boldsymbol{R}$ as $0 \times x = a; a \in \boldsymbol{R}\ \&\ x \in \boldsymbol{T}$.

Corollary 1 *$a \times T(b, c) = (a \cdot b)/c$ if $c \neq 0$.*

Proof:

$$\begin{aligned} a \times T(b,c) &= ((a \cdot c)/c) \times T(b,c) && \text{if c} \neq 0 \\ &= (a/c) \cdot (c \times T(b,c)) && \text{if c} \neq 0 \\ &= (a/c) \cdot (b) && \textit{Def 1}, \text{ if c} \neq 0 \\ &= (a \cdot b)/c && \text{if c} \neq 0 \end{aligned}$$

Theorem 2 *Any Real Number $a = 1 \times T(a, 1)$.*

Proof flows directly from Corollary 1.

Theorem 3 *$T(0, a) = T(0, b)$ where $a, b \in \boldsymbol{R}\ \&\ a, b \neq 0$.*

Proof: Consider $a, b, r \in \boldsymbol{R}$

$$\begin{aligned} r \times T(0,a) &= 0 && \textit{Cor 1}, \text{ if a} \neq 0 \\ &= r \times T(0,b) && \textit{Cor 1}, \text{ if b} \neq 0 \\ \implies T(0,a) &= T(0,b) && \textit{Th 1}, \text{ if a,b} \neq 0 \end{aligned}$$

It is remarkable to know that $T(0, r)$ is the same number for all values for $r \in \boldsymbol{R}$ except 0.

Axiom 2 *Define a new relation,* $\circ : \boldsymbol{T} \times \boldsymbol{T} \to \boldsymbol{T}$ *such that*

$$(a \times T_1) \times T_2 = a \times (T_1 \circ T_2) \text{ where } a \in \boldsymbol{R} \;\&\; T_1, T_2 \in \boldsymbol{T}$$

Theorem 4 *The tuple* $(\boldsymbol{T}, \circ)$ *forms a group.*

Proof:

1. **Closure** This is trivial since $\boldsymbol{T}$ is closed under $\circ$.

2. **Associativity** Consider $a \in \boldsymbol{R}$ and $T_1, T_2, T_3 \in \boldsymbol{T}$,

$$\begin{aligned} ((a \times T_1) \times T_2) \times T_3 &= ((a \times T_1) \times T_2) \times T_3 \\ (a \times (T_1 \circ T_2)) \times T_3 &= (a \times T_1) \times (T_2 \circ T_3) \\ a \times ((T_1 \circ T_2) \circ T_3) &= a \times (T_1 \circ (T_2 \circ T_3)) \\ (T_1 \circ T_2) \circ T_3 &= T_1 \circ (T_2 \circ T_3) && \textit{Th 1} \end{aligned}$$

3. **Existence of Identity** Consider $b, a \in \boldsymbol{R}$ where $a \neq 0$ and $T_1, T(a,a) \in \boldsymbol{T}$

$$\begin{aligned} b \times (T_1 \circ T(a,a)) &= (b \times T_1) \times T(a,a) && \textit{Ax 2} \\ &= ((b \times T_1) \cdot a)/a && \textit{Cor 1} \\ &= b \times T_1 \\ \implies T_1 \circ T(a,a) &= T_1 && \textit{Th 1}\text{, if a} \neq 0 \end{aligned}$$

$$\begin{aligned} b \times (T(a,a) \circ T_1) &= (b \times T(a,a)) \times T_1 && \textit{Ax 2} \\ &= ((b \cdot a)/a) \times T_1 && \textit{Cor 1} \\ &= b \times T_1 \\ \implies T(a,a) \circ T_1 &= T_1 && \textit{Th 1}\text{, if a} \neq 0 \end{aligned}$$

Since $T(a,a)$ acts as both left and right identity to T_1 and T_1 is any element in $\boldsymbol{T}$, $T(a,a)$ where $a \neq 0$ is the unique identity element in $\boldsymbol{T}$. This also means that all equations of the form $a \times x = a$ where $a \in \boldsymbol{R}, a \neq 0$ have the same solution chosen arbitrarily as $T(1,1) = I(Notation)$. It is interesting to note that if $a = 0$, then x can be any real number hence the solution becomes $\boldsymbol{R}$.

4. **Existence of Inverse** Consider $a, b \in \boldsymbol{R}$ and $T_1, T_2 \in \boldsymbol{T}$ such that

$$\begin{aligned} a \times T_1 = b \;&\&\; b \times T_2 = a \\ \implies (a \times T_1) \times T_2 = b \times T_2 \;&\&\; (b \times T_2) \times T_1 = a \times T_1 \\ \implies (a \times T_1) \times T_2 = a \;&\&\; (b \times T_2) \times T_1 = b \\ \implies a \times (T_1 \circ T_2) = a \;&\&\; b \times (T_2 \circ T_1) = b && \textit{Ax 2} \\ \implies a \times (T_1 \circ T_2) = a \times I \;&\&\; b \times (T_2 \circ T_1) = b \times I \\ \implies T_1 \circ T_2 = I \;&\&\; T_2 \circ T_1 = I && \textit{Th 1} \end{aligned}$$

Thus T_1 and T_2 are inverses of each other. In formal notation, $T_1 = T(b,a)$ and $T_2 = T(a,b)$. Hence for every $T(a,b) \in \boldsymbol{T}$ we have it's inverse $T(b,a)$ such that $T(a,b) \circ T(b,a) = T(b,a) \circ T(a,b) = i$.

This completes our proof and thus Theorem 4 is proven.

3 Structure of $T(a, b)$

So far we already know three very important things about T:

1. Any real number r can be represented by as $T(r, 1)$ since $1 \times T(r, 1) = r$
2. All $T(0, a)$ are the same number (call it N for *Null*) (except when $a = 0$) which has the property that multiplying it with all real numbers gives us 0.
3. All $T(a, a)$ are the same number which we call I (except when $a = 0$) which has the property that multiplying it with any real number leaves it unchanged.

These rules leave out the very mysterious T(0, 0). So far if we let $T(a, b) = a/b$ and let $T(0, 0)$ be undefined, we land back at the regular real multiplication. In essence the rules for T so far are the rules any multiplication system should to follow to imitate real multiplication anyway. We must in a way identify and re-establish all rules that do not break if we were to define a division by zero. This is exactly what we have done so far. The question of $T(0, 0)$ is a question of dividing by zero, which our rules are not strong enough to include so far and will be possibly for the near future.