Forced Oscillations Under A General External Force

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EP17BTECH11001[a,b]

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Abstract: We look at solving the general damped oscillation differential equation and then look at various techniques to solve damped oscillations in the presence of an external force. Using this general solution to evaluate a few simple cases, results are verified by numerical analysis.

1. Introduction

The general damped oscillations are described by the differential equation $\frac{d^2x}{dt^2} + 2\gamma\omega_0\frac{dx}{dt} + \omega_0^2 x = 0$ where ω_0 is called the natural frequency of the system and γ is called the damping ratio of the system[1]. This equation has been historically solved and simplified into three cases[2]:

- $\gamma < 1$: The system oscillated with a frequency less than ω_0 and the amplitude decreases to zero over time. The oscillation frequency is given by $\omega_1 = \omega_0\sqrt{1-\gamma^2}$
- $\gamma = 1$: The system returns to rest without any oscillation.
- $\gamma > 1$: The amplitude exponentially decays and overshoots, eventually coming to rest, but for larger values of γ, the system takes more and more time to come to a stop.

2. General Case Solution for Oscillation under an external Force

Starting from a force equation we include another term $F(t)$ to represent the external force. Here m is the mass, k is a spring constant of oscillation, b is damping coefficient.

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -b\frac{dx}{dt} - kx + F(t)\\
\frac{d^2x}{dt^2} &= -(\frac{b}{m})\frac{dx}{dt} - (\frac{k}{m})x + \frac{1}{m}F(t)\\
\frac{d^2x}{dt^2} &= -2\gamma\omega_0\frac{dx}{dt} - \omega_0^2 x + \frac{1}{m}F(t)\\
\frac{d^2x}{dt^2} + 2\gamma\omega_0\frac{dx}{dt} + \omega_0^2 x &= \frac{1}{m}F(t)
\end{aligned}$$

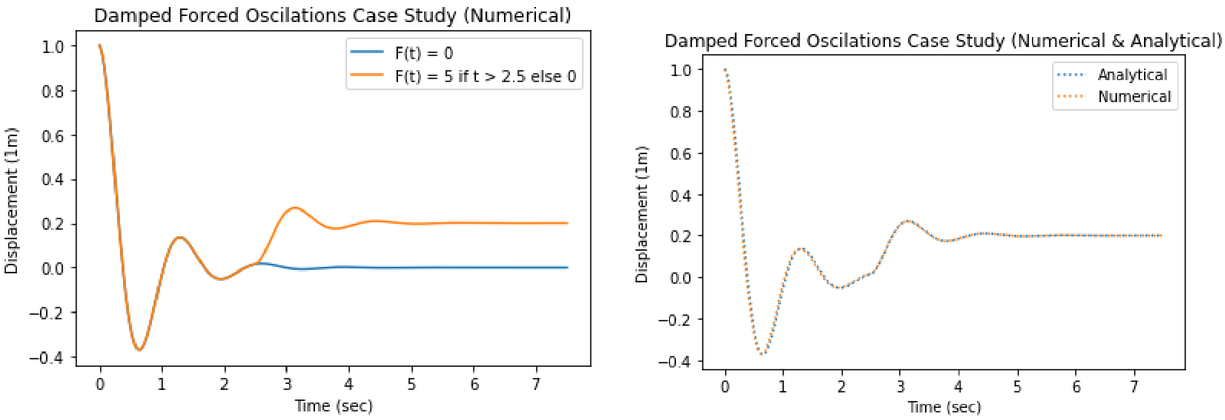
$$(\frac{d^2}{dt^2} + 2\gamma\omega_0\frac{d}{dt} + \omega_0^2)x = \frac{1}{m}F(t) \tag{1}$$

Having put our equation in the desired form, we will now use the fact that $(\frac{d^2}{dt^2} + 2\gamma\omega_0\frac{d}{dt} + \omega_0^2)$ is a linear differential operator which can be factored into $(\frac{d}{dt} - (-\gamma\omega_0 + \iota\omega))(\frac{d}{dt} - (-\gamma\omega_0 - \iota\omega))$ where $\omega = \omega_0\sqrt{1-\gamma^2}$. We also define $(-\gamma\omega_0 + \iota\omega) = \alpha$ and $(-\gamma\omega_0 - \iota\omega) = \beta$. Our equation now becomes:

$$(\frac{d}{dt} - \alpha)(\frac{d}{dt} - \beta)x = \frac{1}{m}F(t) \tag{2}$$

We recall that $x = x(t)$ and that $(\frac{d}{dt} - \delta)f(t) = e^{\delta t}\frac{d}{dt}(e^{-\delta t}f(t))$. We will use this property twice, first $\delta \to \beta$ and then $\delta \to \alpha$, to get the final integrable equation:

$$e^{\alpha t}\frac{d}{dt}(e^{-\alpha t}e^{\beta t}\frac{d}{dt}(e^{-\beta t}x)) = \frac{1}{m}F(t) \tag{3}$$



(a) Damped Oscillator With and Without External Force

(b) Numerical vs Analytical Solution

Figure 1: my caption

Equation (3) has been solved [3] to give a final solution:

$$x = \frac{e^{-\gamma\omega_0 t}}{\omega}(x_0\omega cos(\omega t) + x_0\omega_0\gamma sin(\omega t) + v_0 sin(\omega t)) + \frac{1}{\omega}\int_0^t dt' e^{-\gamma\omega_0(t-t')} sin(\omega t - \omega t')F(t') \tag{4}$$

where x_0 is $x(0)$ and v_0 is $|\frac{dx}{dt}|_{t=0}$. Very interestingly, $x(t_1)$ is independent of the nature of $F(t)$ $s.t.$ $t > t_1$. This is the beauty of math, since the integral only sums the external force from 0 to t_1, any force that lies in the future, i.e. $F(t)\ \forall\ t > t_1$ will not affect the result now which is $x(t_1)$, whereas all the past forces do affect it.

3. Case Study

Let us study the result for an external force that is zero until a certain time, but is a constant after it.

$$F(t) = \begin{cases} 0 & \text{if } t < t_1 \\ F_0 & \text{if } t > t_1 \end{cases}$$

3.1. Analytical Solution

The analytical solution is given by:

$$x = \frac{e^{-\gamma\omega_0 t}}{\omega}(x_0\omega cos(\omega t) + x_0\omega_0\gamma sin(\omega t) + v_0 sin(\omega t)) + (\frac{F_0}{\omega\omega_0^2})(\omega - e^{\gamma\omega_0(t-t_1)}(\gamma\omega_0 sin(\omega t - \omega t_1) + \omega cos(\omega t - \omega t_1))) \tag{5}$$

To compare answers from a numerical solution, we solve equation (1) with a numerical solver and choose certain values for the variables and initial conditions. $(x_0 = 1, v_0 = 0, \gamma = 0.5, \omega_0 = 5.0, F_0 = 5, t_1 = 2.5)$.

3.2. Analysis

In the numerical solution, we have taken underdamped case $(\gamma < 1)$ in order to see some oscillatory motion. We can clearly see how before 2.5 seconds, the oscillator with non-zero force imitates the oscillator with zero external force. However as soon as the external force starts acting, the resting position of the oscillator changes. It is quite useful to see that under a constant force $(t >> 2.5)$ the only difference between the two oscillators is in the resting positions. This resting position turns out to be $\frac{F_0}{\omega_0^2}$ as can be observed by substituting $t \to \infty$. For our example, this is at exactly $x = 0.2$ for the forced oscillator, as predicted by the analytical solution, while the oscillator for whom external force is zero rests at the default $x = 0$. In both cases, the numerical plots agree with analytical solutions to within 1%.

4. Application

History is riddled with the extraordinary cost of covering up various design issues that have caused structural failures due to resonance. The Tacoma Bridge is a very popular example of this phenomenon[4]. However it is not just the construction designs that can be flawed. It was reported that in 2011, an aerobics class in Seoul, South Korea managed to synchronise their workout tot he natural frequency of the 39 storey building causing violent shakes and emergency evacuation. Coupled systems and damping is used to control vibrations in structures.

There are two ways this is done. One is to change the shape of the structure in order to change the natural frequency so as to be at least three times the frequency of disturbances. The other way is to attach a second heavy body to the primary structure, effectively turning them into a coupled harmonic system[5]. The second heavy body is attached to a damping system which drains energy from the system and saves the structure from failure. This mechanism was employed to secure the James Webb Telescope lenses while testing at NASA's Johnson Space Center.[6].

5. Conclusion

Damped Oscillator problem is an analytically solvable. Any disturbance can be interpreted as a non periodic external force and the damped oscillator can be considered to be under forced oscillations of this external force. While numerical solutions are easy to compute with the resources available today, having written down analytical solutions gave can give us great insight into how the system will evolve over time and how the initial conditions will affect its evolution.

I would like to thank Professor Dr Vandana Sharma (Dept. of Physics, IIT Hyderabad) for her intriguing classes.

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