Using non-linear circuits to study chaotic dynamics

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(Dated: March 14th 2023)

Abstract

The study of chaos dynamics using non-linear circuit systems yields agreement within theoretical and experimental models of the system. Theoretical models for the system are built using modern python techniques to provide insight into chaos dynamics of increasing order of chaotic systems. Theoretical models are validated in the experimental results and thus can be trusted to apply to high order chaotic systems. Experimental results yield 5 bifurcation points at $55, 64, 78, 109, 119 (k\Omega)$. Theoretical and experimental models fall in agreement with a discrepancy of 0.04%.

INTRODUCTION

The concepts of nonlinear systems and chaos are a captivating introduction to the world of research. One way we can study chaotic behavior is through the use of nonlinear electronic circuits, which serve as an effective tool. The use of non-linear circuits to study chaotic systems and nonlinear dynamics is prevalent in research [1]. One such specific type of circuit is the Chua circuits [1]. The initial configuration of this circuit incorporates an inductor, which presents challenges in terms of modeling and adapting it to different frequencies. However, alternative variations of Chua's circuit have been proposed that operate without the use of an inductor. These inductorless versions are discussed in several works. These circuits are based on simple third-order differential equations and are made up of only a few basic electronic components such as diodes, op amps, and resistors. They can be easily adjusted to operate at different frequencies and with slight modifications, can be used to compare theory with experimental results very accurately [2]. Using basic op-amp circuits like the integrator circuit, inverting amplifier and summing amplifier circuits to create a chaotic circuit which is described in the theory section. We linked several circuits of such capacity together to study chaos in higher dimensions.

We will be using such a circuit to study chaotic systems and understanding the underlying behaviour which leads to chaotic outputs from the nonlinear circuits. Work is done to construct such a circuit and then gather meaningful data from the system. Additionally, developing theoretical models of the chaotic system using python will aid in providing a comparison tool as well as being able to adapt the software for further research into chaos theory. Developing theoretical models for the circuit is going to be important for future use cases of increasingly complex nonlinear systems. The theoretical and experimental models will be compared in the search for agreement within the system.

During this report, the work of the Kiers. paper [2] will be referenced extensively as it was used as a guiding tool for the experiment. The theory and methods follow a closely similar path of research. The Kiers. paper [2] provides this experiment with the circuit and tools of comparison. Thus for completeness, we will also be comparing the values to the values found in the Kiers. paper [2], ensuring that our work is sound throughout the experiment.

THEORY

We are using circuits to model chaotic behaviours and measure results of fully chaotic systems. The equation we are trying to simulate in the lab is,

$$\dddot{x} = -A\ddot{x} - \dot{x} + D(x) - \alpha, \tag{1}$$

This equation is the general *jerk* equation used to model the third derivative of motion in many systems, and the circuits used to model this equation are called *jerk circuits*. Figure 1 shows a circuit that includes three inverting integrators, with the outputs labeled as V_2, V_1, and x, along with a summing amplifier with its output at V_3. By applying Kirchhoff's rules at nodes a-d, along with the "golden rules" for operational amplifiers, we can derive relationships between the voltages. Using these relationships we obtain the following equation from the Kiers paper [2]

$$\dddot{x} = -\frac{R}{R_v}\ddot{x} - \dot{x} + D(x) - \frac{R}{R_o}V_o, \tag{2}$$

where each dot over the x represents a derivation, the R values represent resistors from fig 1, R_v is the variable resistor which acts as our independent variable. V_o represents the input voltage into the system which is kept constant throughout the system. In this equation $D(x) = |x|$, the block of $D(x)$ represents the non-linear component of the circuit which is shown in fig 2. This equation is similar to eq 1, and is a third order differential equation. The solution to this equation is going to be 3 coupled ODE equations, which are going to be used to analyse the chaos. The three equations are as follows,

$$x_1 = x, \tag{3}$$

$$x_2 = \dot{x}_1, \tag{4}$$

$$x_3 = \dot{x}_2, \tag{5}$$

$$\dot{x}_3 = \dddot{x} = -\frac{R}{R_v}\ddot{x} - \dot{x} + D(x) - \frac{R}{R_o}V_o, \tag{6}$$

Bifurcation plots are widely used in the study of nonlinear dynamical systems and have proved to be a valuable tool in identifying and analyzing chaotic behavior. In a bifurcation plot, the parameter space is scanned, and the outputs from the system are plotted as a function of the control parameter. The resulting diagram shows how the system's dynamics change as the control parameter is varied, highlighting the emergence of stable or unstable periodic orbits, chaos, or other dynamical regimes. By examining the bifurcation plot, we can infer the qualitative nature of the system's behavior and study the transitions between different dynamical regimes. In this lab we used bifurcation plots to study the chaotic circuit's behaviour in reference to R_v. Both theoretical models and experimental data is presented using bifurcation diagrams and phase portrait diagrams. In a phase portrait diagram, the horizontal axis represents the state variable, and the vertical axis represents the derivative of the state variable. The plot consists of a series of curves that correspond to different initial conditions of the system. The curves show the trajectory of the system in phase space. We analyze the projection of 3d phase space through the system.

The set of three coupled ordinary differential equations (ODEs) will be employed to generate bifurcation plots and phase portrait diagrams. Equations 4, 5, and 6 represent the governing dynamics of our chaotic system.

The circuit that we are going to be using to simulate chaos in the lab is fig 1, this circuit is made up of many op-amp integrators and a non-linear portion labelled $D(x)$.

The values of each component in the circuit [2] is discussed in the methods section of the paper. The nodes in the circuit, labelled a, b, c, d are used to derive equations for the output of the circuit. A major component of the circuit which helps in governing the output is the non-linear component marked $D(x)$. The circuit schematic for this non-linear component

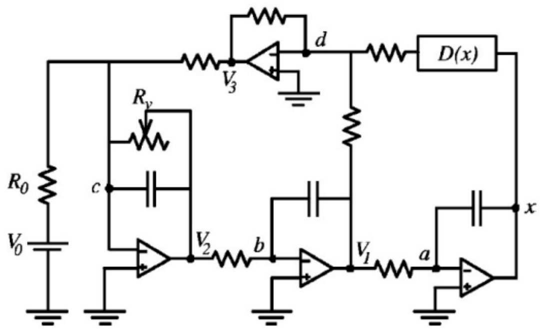


FIG. 1: This is the schematic of the chaotic circuit used in this lab to test chaotic behaviour. The different nodes in the circuit are discussed in detail in the theory section in the paper.

is shown in 2. This circuit [2] functions as an ideal diode and utilizes op-amps to provide stability to the output of the chaotic circuit.

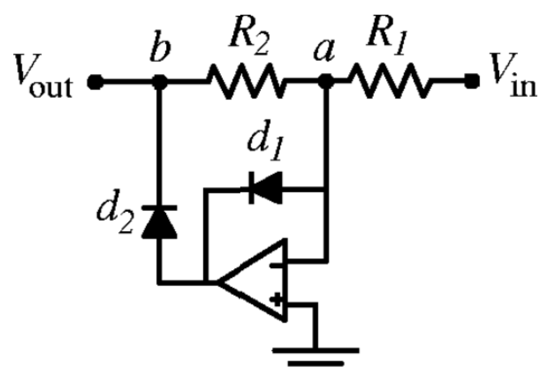


FIG. 2: This is the schematic of the non linear component to the main chaotic circuit. This circuit functions like an ideal diode in it's output and thus does not lead to unbounded solutions

This circuit's modelling equation is taken from the Kiers paper [2]. It represents the output of the circuit as measured by the components in the circuit. If we employ Kirchhoff's rules to fig. 2 on nodes a and b, along with the Shockley equations to model the $I - V$ curves for the diodes, we get the following equation,

$$V_{out} + \frac{1}{\alpha_2} \ln \left[1 + \frac{V_{out}}{I_{S_2} R_2} \right] = -\frac{1}{\alpha_1} \ln \left[1 + \frac{1}{I_{S_1}} \left(\frac{V_{in}}{R_1} + \frac{V_{out}}{R_2} \right) \right] \tag{7}$$

We take $a_1 \sim a_2 \sim 20\ \mathrm{V}^{-1}$, $I_{S_1} \sim I_{S_2} \sim 3$ nA and resistances of the order $10k\Omega$ and find that the solution of Eq. 7 is very well represented by the approximate expression [2]

$$V_{out} = D(V_{in}) = -\frac{R_2}{R_1} \min(V_{in}, 0). \tag{8}$$

In eq 8 we have chosen values for R_1andR_2 such that $\frac{R_2}{R_1} = 6$. The setup of the circuit shown in Fig. 2is such that the op-amps drive the diodes. As the circuit is designed in such a way that it becomes less affected by the individual characteristics of the diodes, it can be inferred that the solution to Eq. 7 is not significantly dependent on the values of a or I_S to a high degree of accuracy. The circuit's non-linearity is effectively characterized by a straightforward piecewise linear function in Eq. 8, which doesn't require any special treatment and provides highly precise experimental results.

METHODS

The equipment used in the experiment is as follows: Python packages used are SciPy [5], NumPy [4], and Matplotlib [3]. The equipments used in the circuit are oscilloscope [**?**] and the op-amp [**?**]. Rest of the equipment is readily available in any circuitry lab, such as breadboards, wires, resistor, capacitors, etc.

The main circuit used in this experiment is shown in fig 1 and in fig 2, built in the lab with the following values, with the uncertainty of each value coming from a digital multimeter [6],

Component	**Value**	**Unit**
R	470	kΩ
C	1.024	nF
V1	100	mVpp
f	1	kHz
Rv	0-300	kΩ
V_o	0.3	V
R_o	157	kΩ
R_2/R_1	6	kΩ

The experimental procedure involves the construction of a chaotic circuit, followed by a comprehensive testing of each component and node of the circuit to ensure its proper functioning. Any necessary adjustments are made during this process to ensure that the predetermined values are met. The circuit serves as the sole apparatus required for conducting the experiment. The output of the circuit is measured at a specific point denoted as x, using an oscilloscope reader, and is recorded in V_{out} vs *time* format.

A variable resistor (R_v) is utilized, and its value is changed before each data collection step. The data collection process consists of two separate runs, with each run focusing on a different aspect of the output. In each run, the voltage output as a function of time is recorded for a range of R_v values. In the circuit R_v is a connected potentiometer to allow for multiple resistor values to be collected. After each step increase in the potentiometer, data from the oscilloscope is saved onto the computer. The collected data is saved in *.csv* format for ease of analysis using Python. The number of data points collected (N) can be calculated as $N = \frac{range_f - range_i}{step}$. Further details of the two runs are provided in the table below.

Run no.	**Range**	**Step**
Run 1	$45k$ to $70k\Omega$	100Ω
Run 2	$45k$ to $140k\Omega$	$1k\Omega$

To analyze the data collected from the experiment, we compare the theoretical and experimental models. We generate theoretical models using Python code to simulate bifurcation diagrams and phase portrait plots for specific R_v values. To create these models, we first solve Eq. 2, then plot the output x in bifurcation plots and phase diagrams. Throughout the experiment, we compare the accuracy of the data against the theoretical models.

The data collected is formatted into python and then analyzed. We used peak finding functions to detect the peaks in each voltage output and then create bifurcation plots and phase portrait diagrams for each run. The results then are compared to the theoretical model of the same range and checked for variance. The output images are superimposed on top of each other to provide a qualitative representation of the discrepancy between the theoretical and experimental values. The residuals for the discrepancy are also included to quantitatively measure the agreement between theoretical and experimental values. Specific R_v values are chosen for the reader to visualize discrepancy within theoretical and experimental values.

We utilized Python packages to simulate theoretical values of chaotic circuits. Specifically, we solved Equation 2 into a system of coupled ordinary differential equations, which were then plotted over the same run range as previously described. We generated bifurcation plots by plotting the peaks of a function for a specific R_v value and constructed phase plot diagrams using the output of the coupled ODEs at specific R_v values.

To minimize statistical error in the system, we adopted a strategy of collecting a high number of data points and reducing the step value in data collection. The primary source of statistical error in this experiment is the possibility of missing bifurcation points in the results. We addressed this concern by consulting Kiers et al.'s [2] study, which offered valuable insight into the expected timing of bifurcation.

In addition to statistical error, there is a possibility of systemic error in the circuit itself, including discrepancies in circuit values and equipment performance. To minimize these errors, we carefully tested each point while building the circuit. The data analysis portion of this experiment required particular attention due to the potential for inconsistencies within peaks found for solutions of Eq. 2. To avoid errors in data analysis, we meticulously examined each component of the code.

RESULTS

Data collection runs as described in the table within the methods section are stored as a folder of *.csv* files for the range specified. Each file within the folder contains data as shown in Fig. 3. The use of the *find peaks* function to find each peak of the V_{out} is showcased.

The same method is used for the experimental and theoretical data models. Find peaks is applied to find peaks of each output in a run and then plotted in bifurcation diagrams as shown in Fig. 4. Here the plot is made for both runs with the experimental data imposed on the theoretical data. In addition to the bifurcation diagrams, phase plot diagrams are used to asses the agreement of experimental results with theoretical models. The phase plot diagrams for different values of R_v are presented in Fig. 5. The values shown are taken such that chaos and non-chaotic outputs in the system are clearly visible from the phase plot diagrams.

As mentioned in the introduction section of the paper, the Kiers paper [2] was used as a guiding tool for the experiment. It is important that the point of bifurcation within the system be in agreement with the theoretical model and the Kiers paper [2]. The following table shows the R_v values at which the bifurcation of the system occurs. The table adds a

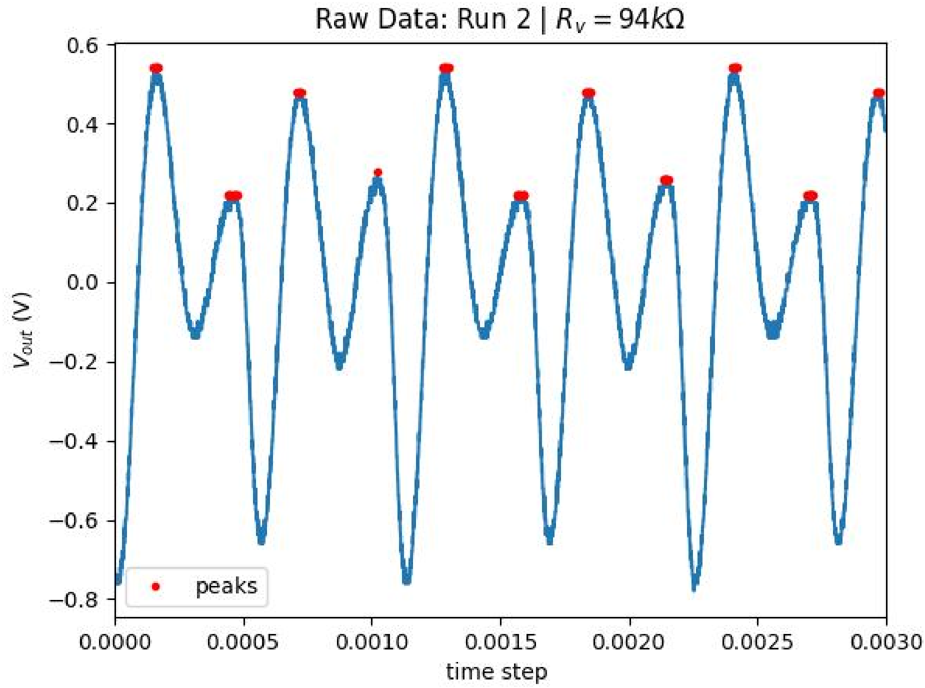


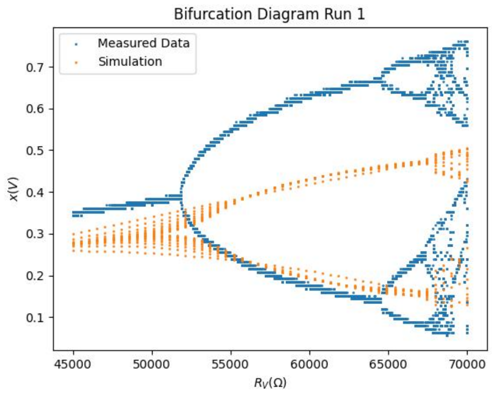
FIG. 3: This figure shows the output data from the oscilloscope that is captured and put into python for plotting. The use of the find peaks function to find each peak of the V_{out} is showcased.

quantitative result to the experiment along with a qualitative result.

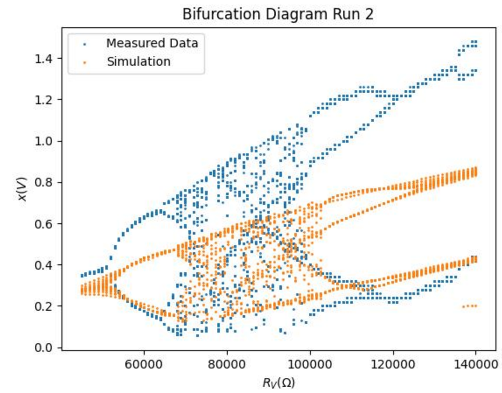
point of bifurcation	**Expt.** ($k\Omega \pm 0.04\%$)	**Theory**($k\Omega \pm 0.04\%$)	**Kiers. Expt.** ($k\Omega \pm 0.04\%$)
1	55	54	53
2	64	62	65
3	78	79	78
4	109	105	101
5	119	122	125

TABLE I: Comparison of experimental and theoretical values for the points of bifurcation.

Figure 5 showcases 4 phase portrait plots for $70k\Omega$, $103k\Omega$, $113k\Omega$, and $134k\Omega$ from top left to bottom right in order. These values were chosen to represent two chaotic resistor values and two non-chaotic values. Values $70k\Omega$, and $103k\Omega$ showcases the chaos in the system while $113k\Omega$, and $134k\Omega$ present non chaos moments in the system.



(a) Figure a



(b) Figure b

FIG. 4: Figure a in the image represents the bifurcation diagram for run 1, as described in the methods section. Figure b in the image represents the bifurcation diagram for run 2, as described in the methods section. In each figure blue dots represent the experimental values while yellow represents the theoretical values.

We had some difficulties in the theoretical simulation of the model, specifically in the findings peak function in python. We were not able to successfully code the program so as to be able to detect every single peak in the system. Additionally there was transient behaviour in the system which led to the bifurcation diagrams in fig 4 to qualitatively be distant.

ANALYSIS

During the analysis we used python to simulate the theoretical model of the chaotic circuit using eq 2. As visible in fig 4 the residuals of the diagram do not tell the full story of the experiment and the results. The important aspect of the analysis is the bifurcation points in the system. That is at what resistor values did the theoretical and experimental values of the system bifurcate. This is a more accurate measurement of "goodness of results" compared to qualitative residuals in our scenario. We see a lot of error in the theoretical model of code used to build it. There is a transient in the parent code of the model leading to a large discrepancy between the theoretical and experimental model.

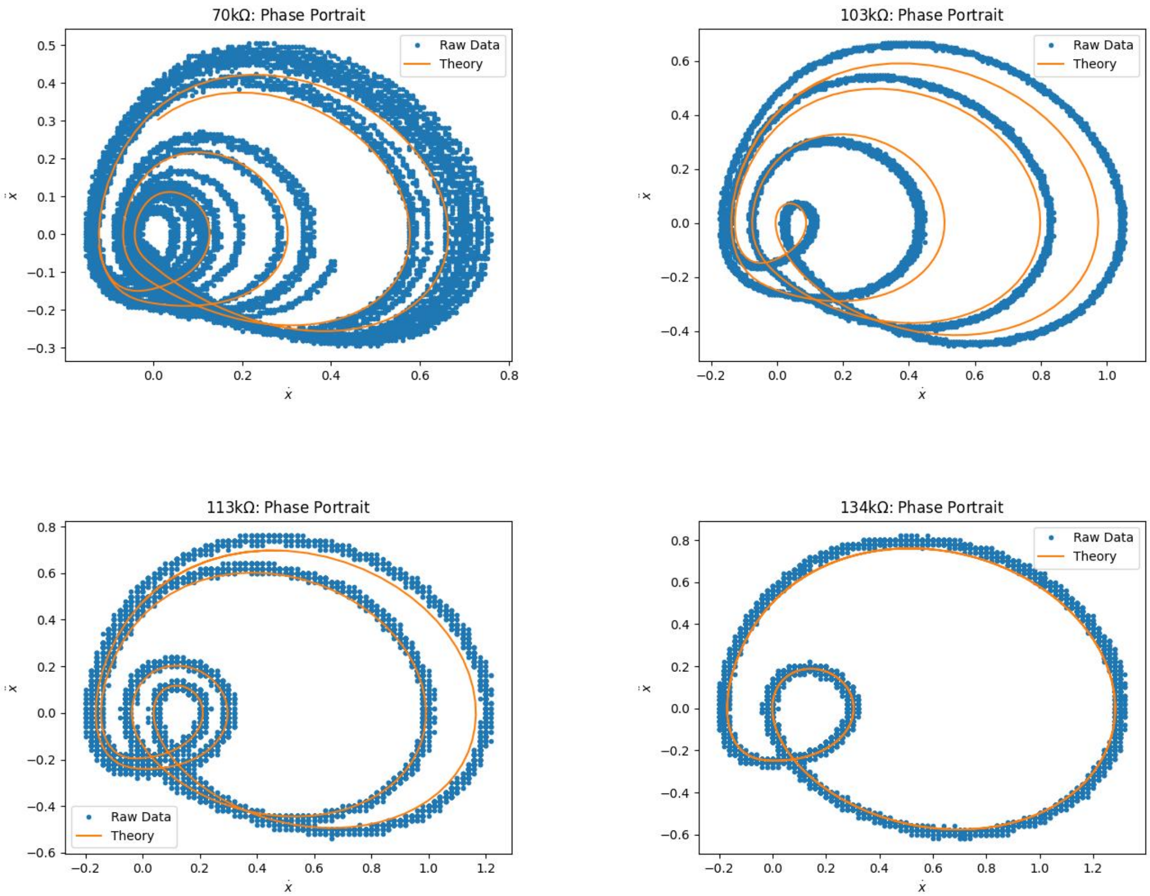


FIG. 5: Several phase portraits were created experimentally by varying the value of the variable resistance R_v. Each portrait consists of a graph where the values of x and $\dot{x}$ are plotted on the horizontal and vertical axes, respectively, measured in volts. The portraits here are $70k\Omega$, $103k\Omega$, $113k\Omega$, $134k\Omega$, from top left to bottom right in order. The theoretical values are in yellow, while the experimental values are in blue.

DISCUSSION

I learned that the circuit that is being discussed in this study can be modeled using a simple, third-order differential equation. When the solutions of this equation are observed, they display a range of chaotic and periodic behaviors. has investigated the circuit and found that the theoretical predictions matched very well with experimental results, with high agreement for some quantities such as bifurcation points, phase portraits, and bifurcation points. This high level of agreement and the circuit's stability make it a valuable tool for further research into topics such as synchronization, chaos control, and higher-dimensional chaos within the field of nonlinear dynamics.

Initial conditions play a crucial role in studying chaotic circuits because these systems are highly sensitive to their initial states. Chaotic circuits are nonlinear and dynamic systems that can exhibit unpredictable and complex behavior, making it difficult to predict their behavior without understanding their initial conditions. Initial conditions are possibly the error missing in the theoretical models of the system we have developed leading to a nonsensical initial value for voltage but similar bifurcation points.

During the course of the experiment, the Kiers paper [2], was used as a point of reference for all activities conducted in the lab. The theoretical model developed during this process does not provide an accurate representation of the chaotic circuit we are studying. In comparing our values to the values in the Kiers. paper, we see that our values are in agreement, this is shown in table I. This shows that the experimental data gathered and processed within the lab are accurate for the chaotic circuit. The problem with the theoretical model is not with the value of R_v at which the system bifurcates but rather the voltage output of the system that it is predicting. The percentage error in the system is at a maximum of 0.4% from the values seen in table I, this shows that the experimental values are in agreement with the theoretical model quantitatively. The values shown in table I shows that the bifurcation points are in agreement with the Kiers paper [2].

The phase plot diagrams show that the theoretical and experimental models in the system are in agreement. We see that there is chaos in the system as seen in sub-figures a and b in fig 5. The phase plot diagrams with chaos show the amount of times there is a peak in the output voltages in the system. As a qualitative measurement the bifurcation models offer support of the theory with the experimental results in the system. Equation 2 is well represented by the theoretical model and thus is a good model for the experimental results obtained in the system.

In the future, given more time in this project, I would work extensively on improving the theoretical model for the circuit. It is already a great representative quantitatively, however the transient behaviour needs to be accounted for. Additionally I would like to take the system further into chaos theory and study the system at its limits. We now know that the this system provides a good representation of chaos and thus the circuit can be used to study chaos theories and understand complexities of the systems.

CONCLUSION

In this experiment we studied chaotic systems through circuitry. The aim was to build theoretical models which are able to support experiment results and provide agreement. In this case we were successful in obtaining quantitative agreement between theoretical models and experimental results. Table I showcases the points of bifurcation with agreements of 0.7% in the system between theory and experimentally obtained data

ACKNOWLEDGMENTS

I thank Michael Paulgaard for his support as my lab partner through this experiment, and Dr. Paul Haljan for providing the opportunity and aid throughout the experimental work to understanding the theory and experiment.

[1] Chua, L. O., Roska, T. (1993). Nonlinear electronic circuits: A brief review and some current trends. *Proceedings of the IEEE*, 81(2), 268-291. doi: 10.1109/5.204583

[2] Kiers, H. A. L., Klein, T., Kolb, J., Prive, S., Sprott, J. C. (1991). Chaos in a nonlinear analog computer: bifurcation analysis. International Journal of Bifurcation and Chaos, 1(3), 657-682.

[3] J. D. Hunter, "Matplotlib: A 2D Graphics Environment", Computing in Science & Engineering, vol. 9, no. 3, pp. 90-95, May/June 2007.

[4] T. E. Oliphant, "A Guide to NumPy", Trelgol Publishing, 2006.

[5] E. Jones, T. Oliphant, and P. Peterson, "SciPy: Open Source Scientific Tools for Python", 2001.

[6] Fluke Corporation. (2018). Fluke 117 Electricians True RMS Multimeter. Retrieved April 11, 2023, from https://www.fluke.com/en-us/product/electrical-testing/digital-multimeters/fluke-117-electrician-s-multimeter