THE PROBLEM OF APOLLONIUS

H. S. M. COXETER, University of Toronto

1. Introduction. In the third century B.C., Apollonius of Perga wrote two books on Contacts (ἐπαφαι), in which he proposed and solved his famous problem: given three things, each of which may be a point, a line, or a circle, construct a circle which passes through each of the points and touches the given lines and circles. The easy cases are covered in the first book, leaving most of the second for the really interesting case, when all the three "things" are circles. As Sir Thomas Heath remarks [**10**, p. 182], this problem "has exercised the ingenuity of many distinguished geometers, including Vieta and Newton." I will not ask you to look at any of their methods for solving it. These are adequately treated in the standard textbooks [e.g. **11**, p. 118]. Nor will I inflict on you an enumeration of the possible relations of incidence of the three given circles. This was done with great skill in 1896 by Muirhead [**12**].

2. The Descartes Circle Theorem. I will confine the discussion to the two cases that were described by Descartes in letters of November 1643 to his favourite disciple, Princess Elisabeth, daughter of King Frederick of Bohemia [**8**, pp. 37–50]. The first letter deals with three nonintersecting circles, entirely outside one another. Descartes finds some relations between the radii and central distances. The details are clumsy but clear. The second letter deals with the limiting case when the three given circles are mutually tangent at three distinct points. He uses d, e, f to denote the radii of these three circles, and x for the radius of a fourth circle that touches them all externally. Thus d, e, f, x are the radii of four circles in mutual (external) contact. Unfortunately there is a gap in the argument (between pages 48 and 49) which precludes any clear understanding of the crucial steps leading to his conclusion:

$$\begin{aligned} &ddeeff + ddeexx + ddffxx + eeffxx \\ &\quad = 2deffxx + 2deeffx + 2deefxx + 2ddeffx + 2ddefxx + 2ddeefx. \end{aligned}$$

It seems strange today that he did not express this equation more concisely as

$$\frac{1}{dd} + \frac{1}{ee} + \frac{1}{ff} + \frac{1}{xx} = \frac{2}{ef} + \frac{2}{fd} + \frac{2}{de} + \frac{2}{dx} + \frac{2}{ex} + \frac{2}{fx}$$

or

$$(2.1) \qquad 2\left(\frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{x^2}\right) = \left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f} + \frac{1}{x}\right)^2.$$

This beautiful result, which Pedoe [**13**, p. 634] very properly calls *The Descartes Circle Theorem*, was rediscovered almost exactly 200 years later by Mr. Philip Beecroft of Hyde, Cheshire. He published it in a journal not often consulted nowadays: "The Lady's and Gentleman's Diary for the year of our

Lord 1842, being the second after Bissextile, designed principally for the amusement and instruction of Students in Mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit" [1].

Dr. Leon Bankoff of Los Angeles is a dentist who spends his spare time on the same delightful pursuit. When he saw the announced title of my Presidential Address, he kindly sent me a copy of this and two other papers by Beecroft. Although the Descartes circle theorem was rediscovered again in 1936 by Frederick Soddy [14], neither Soddy nor anyone else followed Beecroft in his brilliant idea of regarding the configuration of four circles in mutual contact as part of a configuration of eight circles, each passing through the three points of contact of three others, as in Figure 1.

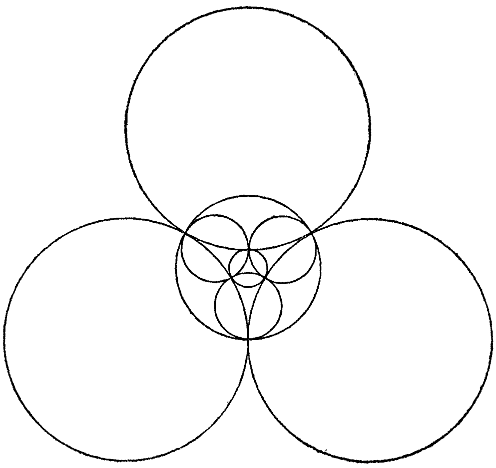

FIG. 1

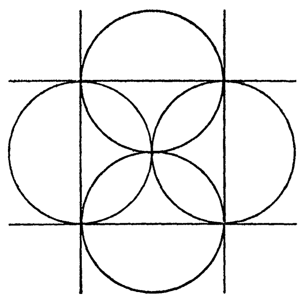

FIG. 2.

This configuration arises in its most symmetrical form when we consider a regular octahedron and its circumsphere. The eight face-planes of the octahedron cut the sphere in such a set of eight circles, and we can obtain the planar configuration by stereographic projection from an arbitrary point on the sphere. In particular, projection from a vertex of the octahedron yields four lines forming a square and four circles having the sides of this square as diameters, as in Figure 2. Any other case of Beecroft's configuration can be derived from this simple one by an inversion.

In the course of his rediscovery of the Descartes circle theorem, Beecroft noticed that equation 2.1 holds for *any* four circles in mutual contact, provided we make the convention that, when two circles have *internal* contact, we regard the larger circle as having a negative radius. Although he worked with radii, it is obviously more convenient to use curvatures (or, as Soddy would say, "bends"), which are reciprocals of radii. Let

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \eta_1, \eta_2, \eta_3, \eta_4$$

be the curvatures of Beecroft's eight circles. Then the theorem says that

$$2\sum \epsilon^2 = \left(\sum \epsilon\right)^2, \tag{2.2}$$

and of course, we shall have also $2\sum\eta^2 = (\sum\eta)^2$. In the words of Soddy's poem,

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

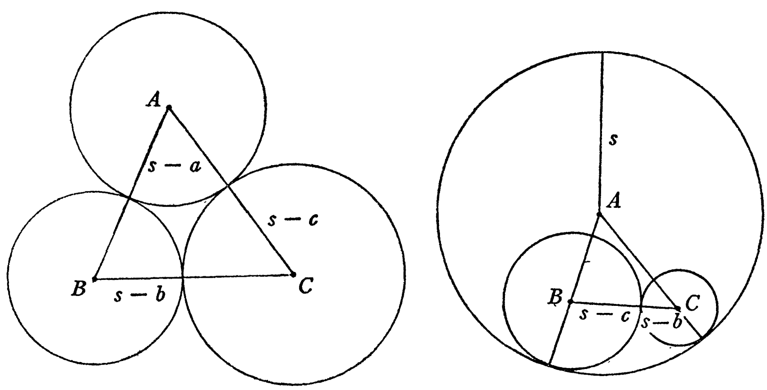

FIG. 3

Here is a simplified version of Beecroft's proof. Let a, b, c, s, r and r_a denote the sides, semiperimeter, inradius and first exradius of a triangle ABC, so that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \quad \text{and similarly} \quad r_a^2 = \frac{s(s-b)(s-c)}{s-a}$$

[7, pp. 60, 164 (Ex. 3)]. Any three mutually tangent circles can be regarded as having centres A, B, C, and radii $s-a$, $s-b$, $s-c$ or else s, $s-c$, $s-b$, as in Figure 3. Accordingly we write in the former case

$$\frac{1}{\eta_1} = r, \quad \frac{1}{\epsilon_2} = s-a, \quad \frac{1}{\epsilon_3} = s-b, \quad \frac{1}{\epsilon_4} = s-c,$$

and in the latter (with the minus sign for internal contact)

$$\frac{1}{\eta_1} = r_a, \quad \frac{1}{\epsilon_2} = -s, \quad \frac{1}{\epsilon_3} = s-c, \quad \frac{1}{\epsilon_4} = s-b.$$

It follows that

$$\epsilon_3\epsilon_4 + \epsilon_4\epsilon_2 + \epsilon_2\epsilon_3 = \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + \frac{1}{\epsilon_4}\right)\epsilon_2\epsilon_3\epsilon_4$$

$$= \left\{\begin{array}{lll} \dfrac{(s-a)+(s-b)+(s-c)}{(s-a)(s-b)(s-c)} = & \dfrac{s}{(s-a)(a-b)(s-c)} & = \dfrac{1}{r^2} \\ \dfrac{s-b-c}{-s(s-c)(s-b)} = & \dfrac{s-a}{s(s-b)(s-c)} & = \dfrac{1}{r_a^2} \end{array}\right\} = \eta_1^2.$$

Similarly $\eta_3\eta_4+\eta_4\eta_2+\eta_2\eta_3=\epsilon_1^2$, and of course we can permute the subscripts 1, 2, 3, 4. Hence

$$\left(\sum \epsilon\right)^2 = \sum \epsilon^2 + 2\epsilon_1\epsilon_2 + \cdots = \sum \epsilon^2 + \sum \eta^2 = \left(\sum \eta\right)^2,$$

and $\sum\epsilon = \sum\eta$. Also

$$\begin{aligned} -\epsilon_1^2 + (\epsilon_2+\epsilon_3+\epsilon_4)^2 &= -\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2+2\eta_1^2 \\ &= 2\eta_1(\eta_2+\eta_3+\eta_4)+2\eta_1^2 \\ &= 2\eta_1\sum\eta = 2\eta_1\sum\epsilon, \end{aligned}$$

whence

$$-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 = 2\eta_1. \tag{2.3}$$

Adding four such equations after squaring each side, we obtain

$$\sum \epsilon^2 = \sum \eta^2,$$

whence $2\sum\epsilon^2 = \sum\epsilon^2+\sum\eta^2 = (\sum\epsilon)^2$. Thus 2.2 is proved.

This version of Beecroft's proof formally resembles one of Pedoe's proofs [13, p. 638], but the meaning is quite different.

One way of expressing the connection between the four ϵ's and the four η's is to remark that they are the roots of quartic equations

$$\begin{aligned} (\epsilon-2u)^2\epsilon^2 + (v-w)\epsilon + 2uw &= 0, \\ (\eta-2u)^2\eta^2 - (v-w)\epsilon + 2uv &= 0. \end{aligned}$$

For instance, in Figure 1, where

$$\begin{aligned} \epsilon_1 = \epsilon_2 = \epsilon_3 &= \sqrt{3}-1, & \epsilon_4 &= \sqrt{3}+3, \\ \eta_1 = \eta_2 = \eta_3 &= \sqrt{3}+1, & \eta_4 &= \sqrt{3}-3, \end{aligned}$$

we have $u=\sqrt{3}$, $v=-2(2+\sqrt{3})$, $w=2(2-\sqrt{3})$. Again, in Figure 2, where

$$\epsilon_1 = \epsilon_2 = \eta_3 = \eta_4 = 0 \quad \text{and} \quad \eta_1 = \eta_2 = \epsilon_3 = \epsilon_4,$$

we have $u=v=0$.

One pretty result which Beecroft seems to have missed is

$$\sum \epsilon\eta = 0. \tag{2.4}$$

This appeared, with a geometric proof, in the "Diary" for 1846 [1]. For an algebraic proof we can use (2.3) in the form

$$\epsilon_1 + \eta_1 = \tfrac{1}{2} \sum \epsilon,$$

whence $\epsilon_1+\eta_1=\epsilon_2+\eta_2=\epsilon_3+\eta_3=\epsilon_4+\eta_4$ and

$$\sum \epsilon\eta = \sum \epsilon(\epsilon+\eta) - \sum \epsilon^2 = \tfrac{1}{2}\left(\sum \epsilon\right)^2 - \sum \epsilon^2 = 0.$$

3. Triads of nonintersecting circles. Although radii and curvatures belong to Euclidean geometry, it should not be forgotten that the problem of Apollonius is still meaningful in the wider field of the *inversive* plane, which may be thought of as the surface of a sphere, or as the Euclidean plane completed by a single point at infinity. In this kind of geometry circles have no "centres," but two intersecting circles still determine an angle, and two nonintersecting or tangent circles have an *inversive distance* δ such that, if an inversion transforms one of the circles into a line and the other into a circle of radius b whose centre is at distance p from the line, $\cosh \delta = p/b$ [7, pp. 130, 176 (Ex. 4)].

In Beecroft's configuration, each of the eight circles is orthogonal to three, tangent to three, and at a certain inversive distance δ from the remaining one. Figure 2 shows that $\cosh \delta = 2$, whence $\delta = \log(2+\sqrt{3})$, the logarithm of the ratio of the radii of the two concentric circles in Figure 1.

In fact, any two nonintersecting circles can be inverted into concentric circles, and their inversive distance is equal to the logarithm of the ratio of the radii (the greater to the smaller) of these two concentric circles [7, pp. 121, 123].

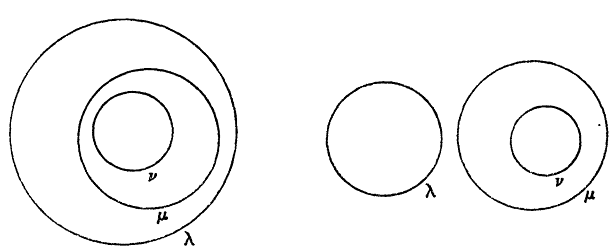

FIG. 4

By thinking of circles on a sphere (without any distinction between "great" and "small" circles), we see that, when the inversive problem of Apollonius is considered for three non-intersecting circles, the number of solutions can only have two possible values: zero or eight. The number is 0 if the circles are *nested*, as in Figure 4 (where every circle tangent to λ and ν intersects μ in two distinct points). It is 8 in the remaining case (Figure 5), where we naturally speak of the three nonintersecting circles as an *Apollonian triad.* In particular, any three circles that belong to a nonintersecting pencil of coaxal circles are nested.

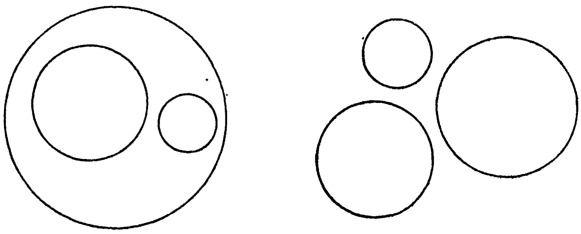

FIG. 5

4. A "Nontriangle Inequality" for nested circles. It is clear from considerations of continuity that, for any three positive numbers α, β, γ, there exists an Apollonian triad of circles whose inversive distances have these values. Accordingly, it is remarkable that the mutual inversive distances of three nested circles satisfy a "nontriangle inequality" (which thus serves as a necessary, but not sufficient, condition for three circles to be nested):

(4.1) *Among the mutual inversive distances between three nested circles, one is greater than or equal to the sum of the other two. Equality holds only when the three circles are coaxal.*

Although this is a theorem of inversive geometry, the simplest proof employs Euclidean ideas. Let λ, μ, ν be the nested circles, as in Figure 4, and let α, β, γ be their inversive distances: λ to μ, μ to ν, ν to λ. Since the circles are nonintersecting, there is at least one circle ρ orthogonal to all of them [11, p. 34]. Since either of the intersections of μ and ρ is the centre of a circle inverting these two circles into perpendicular lines, we lose no generality by taking μ to be a line parallel to the radical axis of λ and ν, as in Figure 6.

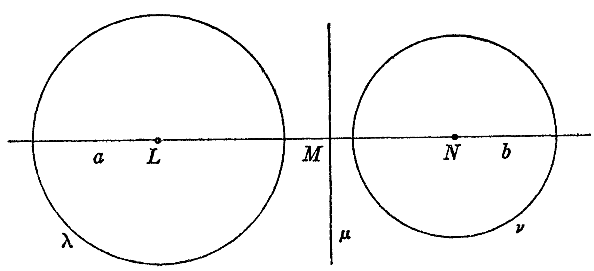

FIG. 6

Let λ and ν have centres L and N, radii a and b, and let μ meet LN in M, so that $LM=a\cosh\alpha$ and $MN=b\cosh\beta$. Since

$$(a\cosh\alpha+b\cosh\beta)^2 = LN^2 = a^2+b^2+2ab\cosh\gamma$$

[6, p. 77], it follows that

$$(a \sinh \alpha - b \sinh \beta)^2 = 2ab \left\{ \cosh \gamma - \cosh (\alpha + \beta) \right\},$$

whence $\gamma \geqq \alpha + \beta$, with equality only when $a \sinh \alpha = b \sinh \beta$. Since $(a \sinh \alpha)^2$ and $(b \sinh \beta)^2$ are the powers of M with respect to λ and ν, this exceptional case is when μ coincides with the radical axis of λ and ν. From the standpoint of inversive geometry, this means that the nested circles λ, μ, ν are coaxal. Thus (4.1) is proved.

By regarding the inversive plane as a sphere, we see that each circle determines an enveloping cone which can be regarded as the null cone at a point in an *exterior-hyperbolic* space [5, pp. 83–84]. The inversive distance between two nonintersecting circles now appears as the non-Euclidean distance between two points lying on a secant of the sphere, and the three nested circles are represented by the vertices of the kind of triangle for which the nontriangle inequality was observed as long ago as 1907 by E. Study [16, p. 108; see also 3, p. 225].

Du Val [9] proved in 1924 that the events in de Sitter's space-time can be represented by the points of an exterior-hyperbolic 4-space: the part of real projective 4-space that lies outside a nonruled quadric 3-fold. Thus the exterior-hyperbolic 3-space that we have been discussing may be regarded as a 3-dimensional section of de Sitter's 4-dimensional world, and the terminology of space-time is appropriate. For instance, the light-cone at a given event is the enveloping cone from a given point to the absolute quadric surface Ω (the 3-dimensional section of the quadric 3-fold).

The "dictionary" relating the inversive plane to exterior-hyperbolic 3-space begins as follows:

Circle	Point (or "event")
Coaxal circles	Collinear points
Intersecting pencil	Spacelike line
Tangent pencil	Null line (tangent to Ω)
Non-intersecting pencil	Timelike line (secant to Ω)
Limiting points	The beginning and end of eternity
Orthogonal pencils	Polar lines
Angle of intersection of circles	Space interval
Inversive distance	Time interval
Homography	Lorentz transformation

To establish the connection, we regarded Ω as a sphere. Pedoe [13, p. 635] prefers a paraboloid of revolution.

5. Two special solutions of the problem. After that wild excursion, let us return to the Euclidean plane and consider two nonintersecting (or possibly tangent) circles of *equal* radius b. Since their inversive distance α is twice the inversive distance between either circle and their radical axis, the Euclidean distance between their centres in $2b \cosh \frac{1}{2}\alpha$. Hence three nonintersecting (or possibly tangent) circles can be inverted into *congruent* circles if and only if their three inversive distances α, β, γ are such that $\cosh \frac{1}{2}\alpha$, $\cosh \frac{1}{2}\beta$, $\cosh \frac{1}{2}\gamma$ are the mutual (ordinary) distances of three points in the Euclidean plane. Since three congruent circles are tangent to two parallel lines, or to two concentric circles, according as their centres are or are not collinear, we can deduce the following theorem of inversive geometry:

(5.1) *Among the eight circles that touch an Apollonian triad with inversive distances α, β, γ, two are nonintersecting (or tangent) if and only if each of the three numbers* $\cosh \frac{1}{2}\alpha$, $\cosh \frac{1}{2}\beta$, $\cosh \frac{1}{2}\gamma$ *is less than the sum of the other two (or one of them is equal to the sum of the other two).*

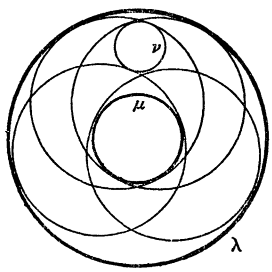

FIG. 7

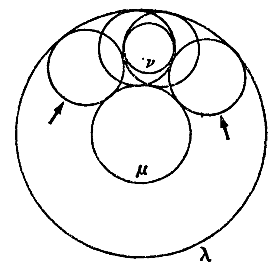

FIG. 8

After noticing that three circles cannot always be inverted into congruent circles, Roger Johnson [11, p. 97] says "This negative result is highly regrettable." In order to refute his pessimistic attitude, let us work again in the Euclidean plane and consider an Apollonian triad consisting of two concentric circles λ, μ, whose radii satisfy $a > b$, and a third circle ν. Since λ and μ are concentric, the circles that touch both consist of two one-parameter families of congruent circles in the closed annulus bounded by λ and μ: one family having radius $\frac{1}{2}(a+b)$ (Figure 7) and one having radius $\frac{1}{2}(a-b)$ (Figure 8). Since the triad $\lambda\mu\nu$ is Apollonian, ν must lie strictly within this annulus and have radius less than $\frac{1}{2}(a-b)$.

The eight solutions of the problem of Apollonius for $\lambda\mu\nu$ consist of four members of each family. We see at once from the figures that each of λ, μ, ν is separated from the other two by two of the eight. (Of the four circles in Figure 7, two

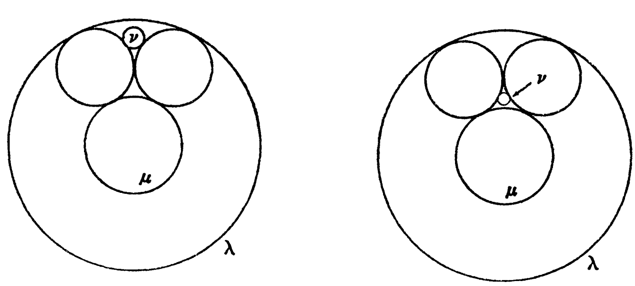

FIG. 9

surround μ and ν, separating them from λ, and two surround μ, separating it from λ and ν. Of the four in Figure 8, two surround ν, separating it from λ and μ.) The two circles emphasized in Figure 8 are special in that they do not separate λ, μ, ν at all. In this figure they happen to be nonintersecting (as in Theorem 5.1), but they could just as easily be tangent, as in Figure 9, or intersecting, as in Figure 10.

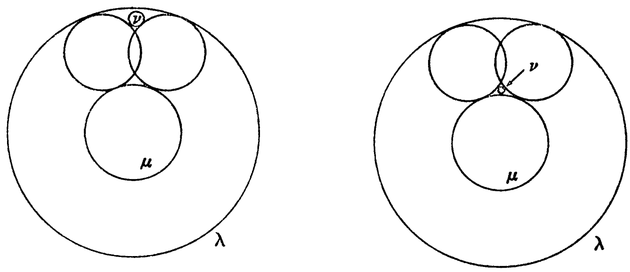

FIG. 10

It is this last possibility that Johnson considered "regrettable." Its redeeming feature is that, of the four "crescents" or *lunes* into which these two intersecting circles decompose the inversive plane, just one contains all three of the original circles λ, μ, ν. Consequently, when the intersecting circles are inverted into intersecting lines, which decompose the Euclidean plane into four angular regions, the new versions of λ, μ, ν are all inscribed in the same one of the four angles, that is, they are homothetic in pairs from the same centre of dilatation. Our conclusion may be summed up as follows:

(5.2) *Every Apollonian triad can be inverted into three circles which are either congruent or homothetic.*

6. Mid-circles. Any two nonintersecting circles have a unique *mid-circle* which inverts them into each other [7, pp. 121–122]. For instance, the mid-circle of two concentric circles is concentric with them, and its radius is the geometric mean of the two radii. Since mid-circles invert into mid-circles, we are now ready to prove the following nice theorem:

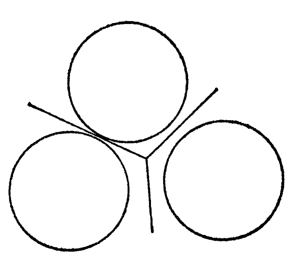

FIG. 11

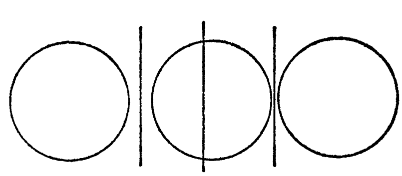

FIG. 12

(6.1) *The three mid-circles of an Apollonian triad are coaxal.*

If the Apollonian triad can be inverted into three congruent circles, their mid-circles become their radical axes [7, p. 35], which are either concurrent (Figure 11) or parallel (Figure 12), and of course three concurrent or parallel lines are a special case of three coaxal circles. If, on the other hand, the Apollonian triad can be inverted into three homothetic circles (inscribed in an angle), their mid-circles become concentric circles (Figure 13), which are another special case of coaxal circles.

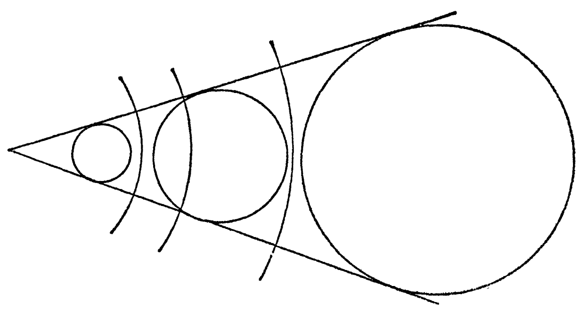

FIG. 13

The most familiar Euclidean example of an Apollonian triad consists of three circles all outside one another. In this case the centres of the three mid-circles are the external centres of similitude of the pairs of circles (Figure 14). Since coaxal circles have collinear centres, the inversive Theorem (6.1) has the Euclidean corollary

(6.2) *If three non-intersecting circles of different sizes are mutually external, so that every two of them have four common tangents, then the three points of intersection of the pairs of external common tangents are collinear.*

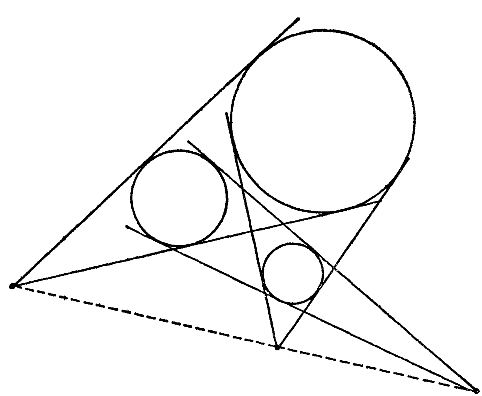

FIG. 14

This result is of special interest in the present context, because it was used as a lemma in Apollonius' own solution of his problem [**10**, p. 182]. Although it follows easily from Menelaus [2, p. 188], it pleased Herbert Spencer so much that he wrote of it as "a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe" [**15**, pp. 187–188; see also pp. 606–608].

A Presidential Address to the Canadian Mathematical Congress, August 28, 1967; reprinted from the Canadian Mathematical Bulletin.

References

1. Philip Beecroft, Properties of circles in mutual contact, Lady's and Gentleman's Diary, 1842, pp. 91–96; 1846, p. 51.

2. N. A. Court, College Geometry, 2nd ed., Barnes and Noble, New York, 1952.

3. H. S. M. Coxeter, A geometrical background for de Sitter's World, this MONTHLY, 50 (1943) 217–228.

4. ———, Introduction to Geometry, Wiley, New York, 1961.

5. ———, Geometry, In vol. 3 of T. L. Saaty's *Lectures on Modern Mathematics*, Wiley, New York, 1965.

6. ———, Inversive distance, Ann. Mat. Pura Appl., (4), 71 (1966) 73–83.

7. ——— and S. L. Greitzer, Geometry Revisited, New Math. Library, Vol. 19, New York and Syracuse, 1967.

8. René Descartes, Œuvres, Publiées par C. Adam et P. Tannery, vol. 4, Paris, 1901.

9. Patrick Du Val, Geometrical note on de Sitter's World, Phil. Mag. (6), 47 (1924) 930–938.

10. Thomas Heath, A History of Greek Mathematics, Vol. 2, Oxford, 1921.

11. R. A. Johnson, Advanced Euclidean Geometry, Dover, New York, 1960.

12. R. F. Muirhead, On the number and nature of the solutions of the Apollonian contact problem, Proc. Edinburgh Math. Soc., 14 (1896) 135–147.

13. Daniel Pedoe, On a theorem in geometry, this MONTHLY, 74 (1967) 627–640.

14. Frederick Soddy, The kiss precise, Nature, 137 (1936) 1021.

15. Herbert Spencer, Autobiography, Vol. 1, New York, 1904.

16. Eduard Study, Beiträge zur nicht-euklidische Geometrie, Amer. J. Math., 29 (1907) 101–167.

ON UNIQUENESS OF OPTIMAL BASIC SOLUTIONS TO LINEAR PROGRAMS

C. B. MILLHAM, Iowa State University

We consider a linear programming problem of the form:

$$\text{maximize } cx \text{ subject to } Ax \leqq b,\ x \geqq 0,$$

where A is an $m\times n$ matrix, x, c are n-dimensional column and row vectors, respectively, and b is an m-dimensional column vector, together with the dual,

$$\text{minimize } ub \text{ subject to } uA \geqq c,\ u \geqq 0,$$

where u is an m-dimensional row vector.