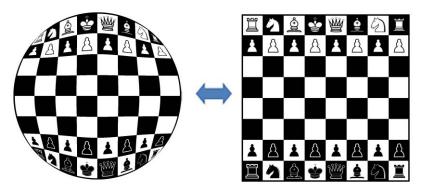
## **Analytical Methods for Squaring the Disc**

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Abstract – We present and discuss several old and new methods for mapping a circular disc to a square. In particular, we present analytical expressions for mapping each point (u,v) inside the circular disc to a point (x,y) inside a square region. Ideally, we want the mapping to be smooth and invertible. In addition, we put emphasis on mappings with desirable properties. These include conformal, equiareal, and radially-constrained mappings. Finally, we present applications to logo design, panoramic photography, and hyperbolic art.

Keywords – Squaring the Disc, Mapping a Circle to a Square, Mapping a Square to a Circle, Squircle, Conformal Mapping, Circle and Square Homeomorphism, Schwarz-Christoffel Mapping, Barrel Distortion, Defishing



#### 1 Introduction

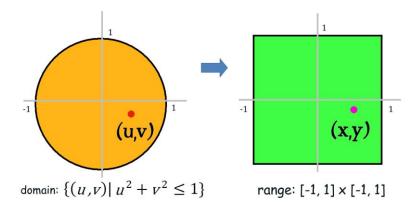
The circle and the square are among the most common shapes used by mankind. It is certainly worthwhile to study the mathematical correspondence between the two. In this paper, we shall discuss ways to map a circular region to a square region and back. There are infinitely many ways of doing this mapping. Of particularly interest to us are mappings with nice closed-form invertible equations. We emphasize the importance of invertible equations because we want to perform the mapping back and forth between the circular disc and the square. We shall present and discuss several such mappings in this paper.

#### 1.1 Organization of this Paper

We anticipate that there will be two types of people who might read this paper. The 1<sup>st</sup> kind would be those who just want to get the equations to map a circular region to a square; and do not really care about proofs or derivations. The 2<sup>nd</sup> kind would be more interested in the mathematical details behind the mappings. Therefore, we shall organize this paper into two parts. The 1<sup>st</sup> part will only contain equations for mapping a circular disc to a square region and back. The 2<sup>nd</sup> part will delve more into mathematical details and discuss some desirable properties of the different mappings. In addition to this, we will also discuss real-world applications of these mappings.

There are accompanying presentation slides and C++ implementation of this paper available on these websites: http://www.slideshare.net/chamb3rlain/analytical-methods http://squircular.blogspot.com

## 1.3 Canonical Mapping Space



The canonical space for the mappings presented here is the unit disc centered at the origin with a square circumscribing it. This unit disc is defined as the set  $\mathcal{D} = \{(u,v) | u^2 + v^2 \le 1\}$ . The square is defined as the set  $\mathcal{S} = [-1,1] \times [-1,1]$ . This square has a side of length 2. We shall denote (u,v) as a point in the interior of the unit disc and (x,y) as the corresponding point in the interior of the square after the mapping. In this paper, we shall present several equations that relate (u,v) to (x,y).

Mathematically speaking, we want to find functions f that maps every point (u,v) in the circular disc to a point (x,y) in the square region and vice versa. In others word, we want to derive equations for f such that (u,v) = f(x,y) and  $(x,y) = f^1(u,v)$ .

#### 1.4 A Classic Problem in a Modern Guise?

The mapping of the circular disc to a square region is similar but not equivalent to the classic mathematical problem of "squaring the circle". For one thing, in the classic mathematical problem, one is restricted to using only a straightedge and a compass. Our problem concerns finding mapping equations that a computer can calculate. In particular, we want explicit equations to mapping each point (u,v) to point (x,y). The two problems are superficially similar but ultimately quite different. One problem has to do with geometric construction; while the other problem has to do with finding a two-dimensional mapping function.

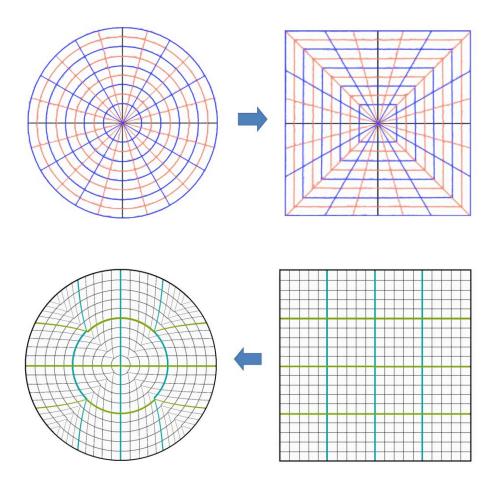
## 1.5 The Mappings

In the next four pages, we shall present four mappings for converting the circular disc to a square and vice versa. We include pictures of a radial grid inside the circle converted to a square; and a square grid converted to a disc. This is followed by equations for the mappings. In these equations, we shall make use a common math function called the signum function, denoted as sgn(x). The signum function is defined as

$$sgn(x) = \frac{|x|}{x} = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Also note that for the sake of brevity, we have not singled out cases when there are divisions by zero in the mapping equations. For these special cases, just equate x=u, y=v and vice versa when there is an unwanted division by zero in the equations. This usually happens when u=0 or v=0 or both.

# Simple Stretching



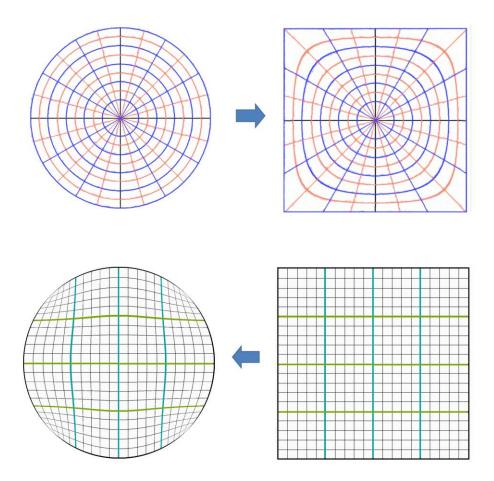
Disc to square mapping:

$$x = \begin{cases} sgn(u)\sqrt{u^2 + v^2} & when \ u^2 \ge v^2 \\ sgn(v) \frac{u}{v}\sqrt{u^2 + v^2} & when \ u^2 < v^2 \end{cases} \qquad y = \begin{cases} sgn(u) \frac{v}{u}\sqrt{u^2 + v^2} & when \ u^2 \ge v^2 \\ sgn(v)\sqrt{u^2 + v^2} & when \ u^2 < v^2 \end{cases}$$

$$u = \begin{cases} sgn(x) \frac{x^2}{\sqrt{x^2 + y^2}} & when \ x^2 \ge y^2 \\ sgn(y) \frac{x \ y}{\sqrt{x^2 + y^2}} & when \ x^2 < y^2 \end{cases}$$

$$v = \begin{cases} sgn(x) \frac{x \ y}{\sqrt{x^2 + y^2}} & when \ x^2 \ge y^2 \\ sgn(y) \frac{y^2}{\sqrt{x^2 + y^2}} & when \ x^2 < y^2 \end{cases}$$

# FG-Squircular Mapping

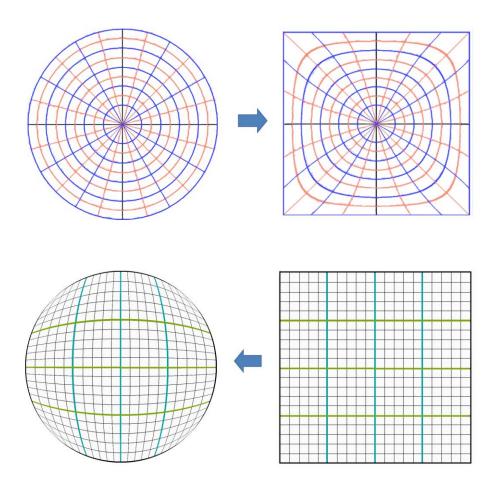


Disc to square mapping:

$$x = \frac{sgn(uv)}{v\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$
$$y = \frac{sgn(uv)}{u\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

$$u = \frac{x\sqrt{x^2 + y^2 - x^2y^2}}{\sqrt{x^2 + y^2}} \qquad v = \frac{y\sqrt{x^2 + y^2 - x^2y^2}}{\sqrt{x^2 + y^2}}$$

# Elliptical Grid Mapping

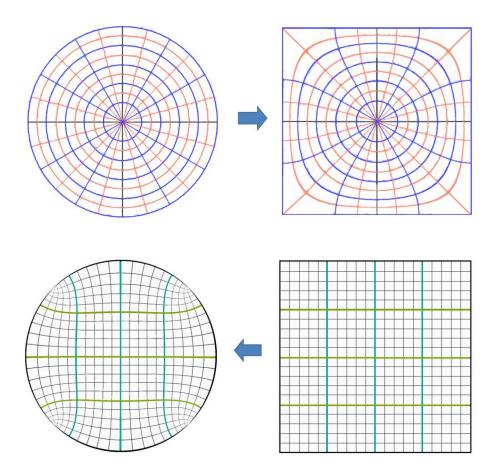


Disc to square mapping:

$$x = \frac{1}{2} \sqrt{2 + u^2 - v^2 + 2\sqrt{2} u} - \frac{1}{2} \sqrt{2 + u^2 - v^2 - 2\sqrt{2} u}$$
$$y = \frac{1}{2} \sqrt{2 - u^2 + v^2 + 2\sqrt{2} v} - \frac{1}{2} \sqrt{2 - u^2 + v^2 - 2\sqrt{2} v}$$

$$u = x\sqrt{1 - \frac{y^2}{2}} \qquad \qquad v = y\sqrt{1 - \frac{x^2}{2}}$$

# Schwarz-Christoffel Mapping



Disc to square mapping:

$$x = Re\left(\frac{1-i}{-K_e}F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}(u+v\,i)\right), \frac{1}{\sqrt{2}}\right)\right) + 1$$
$$y = Im\left(\frac{1-i}{-K_e}F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}(u+v\,i)\right), \frac{1}{\sqrt{2}}\right)\right) - 1$$

Square to disc mapping:

$$u = Re\left(\frac{1-i}{\sqrt{2}} cn\left(K_e \frac{1+i}{2} (x+yi) - K_e, \frac{1}{\sqrt{2}}\right)\right)$$

$$v = Im\left(\frac{1-i}{\sqrt{2}} cn\left(K_e \frac{1+i}{2} (x+yi) - K_e, \frac{1}{\sqrt{2}}\right)\right)$$

where **F** is the incomplete Legendre elliptic integral of the 1<sup>st</sup> kind **cn** is a Jacobi elliptic function

$$K_e = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - \frac{1}{2}\sin^2 t}} \approx 1.854$$

## Part II. Mathematical Details

## 2 Desirable Properties

## 2.1 Conformal and Equiareal Maps

In our mappings, we want to stay faithful to the source and minimize distortion as much as possible. Two standard metrics used in differential geometry [Feeman 2002][Kuhnel 2006][Floater 2005] are measurements of shape distortion and size distortion. Shape distortion is measured in terms of angle variation between the source and target. Size distortion is measured in terms of area variation between the source and target.

In differential geometry parlance, mappings that preserve angles are called conformal. Similarly, mappings that preserve area are called equiareal [Brown 1935]. The figure below shows an example photograph with contrasting results between a conformal map and equiareal map.

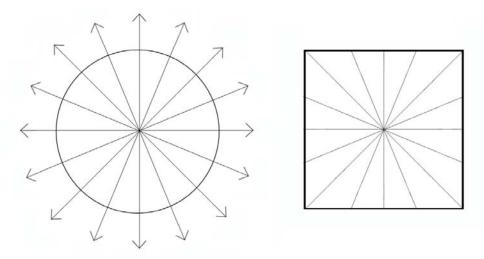


In this paper, we will discuss one conformal mapping -- the Schwarz-Christoffel mapping, which preserves angles throughout the whole mapping. The Schwarz-Christoffel mapping can actually map the unit disc to any simple polygonal region conformally, but we will restrict our discussion to the special case of mapping the circular disc to a square region. The three other mappings mentioned in this paper are not conformal. This can be shown formally by using the Cauchy-Riemann equations in complex analysis.

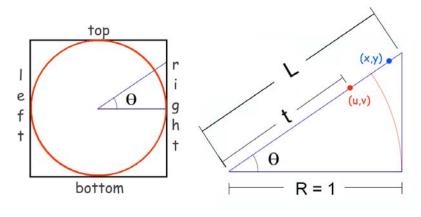
None of the mappings presented in this paper are equiareal. Although it is possible to alter the mappings given in this paper to make them equiareal, we have not found any nice closed-form analytical expressions for these. Shirley and Chiu did present an equiareal mapping between a circular disc and the square in their 1997 paper [Shirley 1997]. Their mapping has a useful application in ray tracing algorithms for computer graphics [Kolb 1995].

Shirley and Chiu's concentric map has closed-form mapping equations. Qualitatively, the Shirley-Chiu concentric map produces results very similar to the Simple Stretching map, which we will discuss further in this paper. However, unlike the Shirley-Chiu concentric map, the Simple Stretching map is not equiareal.

## 2.2 Radial Mappings



Another desirable property that we want for our mappings is being radial. Intuitively, this means that points can only move along radial lines from the center of the disc during the mapping process. Mathematically, this means that the angle that the point (u,v) makes with the x-axis be the same angle as that of point (x,y).



If  $\theta$  is the angle between the point (u,v) and the x-axis, these equations must hold:

$$\cos \theta = \frac{u}{\sqrt{u^2 + v^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin \theta = \frac{v}{\sqrt{u^2 + v^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\tan \theta = \frac{v}{u} = \frac{y}{x}$$

We call these as the *radial constraint equations* for the mapping.

Furthermore, if we write down point (u,v) in polar coordinates (t,  $\theta$ ) inside the unit disc, these equations hold:  $u = t \cos \theta$   $v = t \sin \theta$ 

Using the radial constraint equations above, we can substitute the trigonometric functions out of the equations to get a relationship between (u,v) and (x,y) for radially constrained mappings.

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$$u = t \cos \theta$$
  $\Rightarrow$   $u = t \frac{x}{\sqrt{x^2 + y^2}}$   
 $v = t \sin \theta$   $\Rightarrow$   $v = t \frac{y}{\sqrt{x^2 + y^2}}$ 

We shall denote these equations as the *radial mapping parametric equations*. All radial mappings between the disc and the square have an equation of this form. In general, for all mappings with the radial property, the parameter t can be expressed as an arbitrary function f of x and y.

$$u = f(x, y) \frac{x}{\sqrt{x^2 + y^2}}$$
  $v = f(x, y) \frac{y}{\sqrt{x^2 + y^2}}$ 

#### 3 Simple Stretching Map

One of the simplest ways to map a circular disc to a square region is to linearly stretch the circle to the rim of the inscribing square. The equations for stretching from rim to rim are simple but needs to be broken down to four different cases depending on which side of the square the stretching occurs. We consider the case where the circle extends to the right wall. This occurs for angle  $\theta$  when  $-45^{\circ} \le \theta \le 45^{\circ}$ . If we parameterize t to be linearly proportional to the distance of the destination point (x,y) from the origin, we get:

$$\frac{t}{R} = \frac{\sqrt{x^2 + y^2}}{L}$$

Note that R=1 for our unit circle. Using trigonometry, we have  $\cos\theta = \frac{1}{L}$ , hence  $t = \sqrt{x^2 + y^2} \cos\theta$ . Also from trigonometry, we have  $\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$  so, the equation simplifies to t = x for the right wall. Using the same reasoning, we can get the value of t for the other walls

$$t = \begin{cases} x, & \text{for the right wall } \leftrightarrow x \ge |y| \\ y, & \text{for the top wall } \leftrightarrow |x| \le y \\ -x, & \text{for the left wall } \leftrightarrow x \le -|y| \\ -y, & \text{for the bottom wall } \leftrightarrow |x| \le -y \end{cases}$$

Substituting back into the radial mapping parametric equation, we get an equation that relates the point (u,v) in the circular disc to its corresponding point (x,y) in the square.

$$u = \begin{cases} \frac{x^2}{\sqrt{x^2 + y^2}}, & \text{for the right wall} \\ \frac{xy}{\sqrt{x^2 + y^2}}, & \text{for the top wall} \\ \frac{-x^2}{\sqrt{x^2 + y^2}}, & \text{for the left wall} \\ \frac{-xy}{\sqrt{x^2 + y^2}}, & \text{for the bottom wall} \end{cases} \qquad v = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & \text{for the right wall} \\ \frac{y^2}{\sqrt{x^2 + y^2}}, & \text{for the top wall} \\ \frac{-xy}{\sqrt{x^2 + y^2}}, & \text{for the left wall} \\ \frac{-y^2}{\sqrt{x^2 + y^2}}, & \text{for the bottom wall} \end{cases}$$

Using the signum function and a trick introduced by Dave Cline [Shirley 2011], we can further simplify these equations to:

$$u = \begin{cases} sgn(x) \frac{x^2}{\sqrt{x^2 + y^2}} & when \ x^2 \ge y^2 \\ sgn(y) \frac{x \ y}{\sqrt{x^2 + y^2}} & when \ x^2 < y^2 \end{cases}$$

$$v = \begin{cases} sgn(x) \frac{x \ y}{\sqrt{x^2 + y^2}} & when \ x^2 \ge y^2 \\ sgn(y) \frac{y^2}{\sqrt{x^2 + y^2}} & when \ x^2 < y^2 \end{cases}$$

The inverse equations for this Simple Stretching map can be derived in a similar way.

## 4 Fernandez-Guasti's Squircle

## 4.1 FG-Squircle

In 1992, Manuel Fernandez Guasti introduced an algebraic equation for representing an intermediate shape between the circle and the square [Fernandez Guasti 1992]. His equation included a parameter s that can be used to blend the circle and the square smoothly. In this paper, we shall denote this shape as the *Fernandez Guasti squircle* or *FG-squircle* for short. The figure below illustrates the FG-squircle at varying values of s.

$$s = 0$$
circle
$$s = 0.1$$

$$s = 0.5$$

$$s = 0.8$$

$$s = 0.95$$

$$s = 0.95$$

The equation for this shape is: 
$$x^2 + y^2 - \frac{s^2}{k^2} x^2 y^2 = k^2$$

The squareness parameter s can have any value between 0 and 1. When s = 0, the equation produces a circle with radius k. When s = 1, the equation produces a square with a side length of 2k. In between, the equation produces a smooth curve that interpolates between the two shapes. Unlike the square, the FG-squircle has no tangent discontinuity along its four corners except at s = 1.

In this paper, we shall restrict our discussion and scope of the FG-squircle to  $-k \le x \le k$  and  $-k \le y \le k$ . The FG-squircle equation is valid in the regions |x| > k and |y| > k, but we will ignore those regions.

#### 4.2 The Shrunken FG-Squircle

Consider what happens to the FG-squircle equation when we equate the squareness s with k. The equation reduces to  $x^2 + y^2 - x^2y^2 = s^2$ 

$$s = 0$$
  $s = 0.1$   $s = 0.5$   $s = 0.8$   $s = 0.95$ 

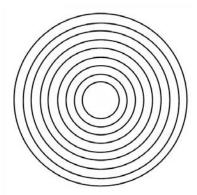
In this equation, we have one parameter s that controls both the squareness and the size of the FG-squircle. Visually, we have a continuum of growing shapes that start as a point at s=0 and end as a square at s=1. At  $s=\frac{1}{2}$ , we have a half-sized FG-squircle that is not quite circular nor square in shape, but in between the two.

This special case of the FG-squircle when s = k is essential to our derivation of several disc-to-square mappings. We shall call this shape as the shrunken FG-squircle.

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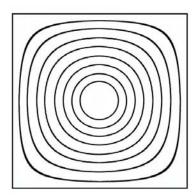
#### 4.3 The Circular Continuum of the Disc

Recall that we defined our input unit disc as the set  $\mathcal{D}=\{(u,v)\in\mathbb{R}^2|\ u^2+v^2\leq 1\}$ . If we think of the unit disc as a continuum of concentric circles with radii growing from zero to one, we can parameterize the unit disc as the set  $\mathcal{D}=\{(u,v)\in\mathbb{R}^2|\ u^2+v^2=t^2\ ,\ 0\leq t\leq 1\}$ . In doing so, we introduced a parameter t that is the distance of point (u,v) to the origin.



## 4.4 The Squircular Continuum of the Square

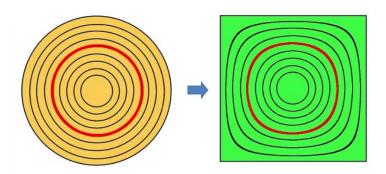
In analogy to the circular continuum of the unit disc, one can write the square region [-1,1] x [-1,1] as the set  $S = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - x^2y^2 = t^2, \ 0 \le t \le 1\}$ 



In other words, the square can be considered as a continuum of concentric shrunken FG-squircles.

## 4.5 Squircular Continuum Mapping Equation

Recall from our circular continuum discussion that the unit disc can be represented as a set of concentric circles and parameterized as  $\mathcal{D}=\{(u,v)\in\mathbb{R}^2|\ u^2+v^2=t^2\ ,\ 0\leq t\leq 1\}$ . Likewise, recall from the previous section that our square region can be represented as a set of concentric shrunken FG-squircles and parameterized as  $\mathcal{S}=\{(x,y)\in\mathbb{R}^2|\ x^2+y^2-x^2y^2=t^2\ ,\ 0\leq t\leq 1\}$ .



We can establish a correspondence between the unit disc and the square region by mapping every circular contour in the interior of the disc to a squircular contour in the interior of the square. In other words, we map contour curves in the circular continuum of the disc to those in the squircular continuum of the square. This can be done by equating the parameter t of both sets to get the equation:

$$u^2 + v^2 = x^2 + v^2 - x^2v^2$$

We denote this equation as the squircularity condition for mapping a circular disc to a square region.

## 5 FG-Squircular Mapping

Using the Fernandez Guasti squircle, we can design a way to map a circular disc smoothly to a square region. The main idea is to map each circular contour in the interior of the disc to a squircular contour in the interior of the square. We can combine this with the radial constraint and derive the mapping equations from there.

#### 5.1 Derivation of the Disc to Square Equations

It is easy to derive the FG-Squircular mapping by combining the *squircularity condition* from the previous section:

$$t^2 = u^2 + v^2 = x^2 + y^2 - x^2y^2$$

with the radial mapping parametric equations

$$u = t \frac{x}{\sqrt{x^2 + y^2}}$$

$$v = t \frac{y}{\sqrt{x^2 + y^2}}$$

After substitution of parameter t, we get:

$$u = \frac{x\sqrt{x^2 + y^2 - x^2y^2}}{\sqrt{x^2 + y^2}} \qquad v = \frac{y\sqrt{x^2 + y^2 - x^2y^2}}{\sqrt{x^2 + y^2}}$$

This gives us the FG-Squircular mapping as shown in page 4. This mapping has the nice property of being radial as well as being compliant with the squircularity condition. In other words, this is a radial mapping that converts circular contours on the disc to squircular contours on the square.

Only two of the four mappings presented in this paper are radial -- the other being the Simple Stretching map. However, for imaging applications, the FG-Squircular mapping produces significantly better results than the Simple Stretching map. This is because the latter converts circular contours inside the circular disc to square contours inside the square. The square has a tangent discontinuity on its four corners, so the mapping produces bending discontinuities along the main diagonals

In contrast, the FG-Squircular mapping converts circular contours inside the circular disc to squircular contours inside the square. The FG-squircle does not have tangent discontinuity on its four corners, so there are no diagonal discontinuities in the FG-Squircular mapping.

#### 5.2 Inversion of the FG-Squircular Mapping

We shall now derive the inverse equations for the FG-Squircular mapping. Since the FG-Squircular mapping is radial by design, we can use the *radial constraint equations* to get a relationship between x with y.

$$\tan \theta = \frac{v}{u} = \frac{y}{x}$$
  $\Rightarrow$   $y = \frac{vx}{u}$ 

Substitute this into the equation for the *squircularity condition*:

$$u^{2} + v^{2} = x^{2} + y^{2} - x^{2}y^{2}$$

$$u^{2} + v^{2} = x^{2} + \left(\frac{v x}{u}\right)^{2} - x^{2}\left(\frac{v x}{u}\right)^{2}$$

$$= x^{2} + \frac{v^{2}}{u^{2}}x^{2} - \frac{v^{2}}{u^{2}}x^{4}$$

$$= \left(1 + \frac{v^{2}}{u^{2}}\right)x^{2} - x^{4}\frac{v^{2}}{u^{2}}$$

Rearranging all the terms to one side of the equation, we get

$$\frac{v^2}{u^2}x^4 - \left(1 + \frac{v^2}{u^2}\right)x^2 + u^2 + v^2 = 0$$

$$\Rightarrow \qquad v^2x^4 - (u^2 + v^2)x^2 + u^4 + u^2v^2 = 0$$

This is a special kind of quartic polynomial equation called a biquadratic. Notice that there are no cubic or linear terms in the  $4^{th}$  degree polynomial equation in x. We can solve for  $x^2$  using the quadratic equation with coefficients

$$a = v^2$$
  $b = -(u^2 + v^2)$   $c = u^4 + u^2v^2$ 

This gives us the solution for  $x^2$ 

$$x^{2} = \frac{u^{2} + v^{2} \pm \sqrt{(u^{2} + v^{2})^{2} - 4v^{2}(u^{4} + u^{2}v^{2})}}{2v^{2}}$$

$$= \frac{u^{2} + v^{2} \pm \sqrt{(u^{2} + v^{2})^{2} - 4u^{4}v^{2} - 4u^{2}v^{4}}}{2v^{2}}$$

$$= \frac{u^{2} + v^{2} \pm \sqrt{(u^{2} + v^{2})^{2} - 4u^{2}v^{2}(u^{2} + v^{2})}}{2v^{2}}$$

$$= \frac{u^{2} + v^{2} \pm \sqrt{(u^{2} + v^{2})(u^{2} + v^{2} - 4u^{2}v^{2})}}{2v^{2}}$$

We can then get a quadrant-aware inverse equation for x as

$$x = \frac{sgn(uv)}{v\sqrt{2}}\sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

Using the radial constraint equations and substituting x, we can also get an inverse equation for y

$$y = \frac{vx}{u}$$
  $\Rightarrow$   $y = \frac{sgn(uv)}{u\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$ 

Note that it possible to write these inverse equations in a slightly different form that is more amenable to software implementation in the computer.

We need to use two properties of the signum function. The 1st is a product rule:

$$sgn(uv) = sgn(u) \, sgn(v)$$

Also, we observe that for  $v \neq 0$ ,

$$\frac{|v|}{v} = \frac{v}{|v|}$$

so for non-zero inputs, the signum function is its own reciprocal

$$sgn(v) = \frac{1}{sgn(v)} = \frac{v}{|v|}$$

$$\therefore \frac{sgn(v)}{v} = \frac{1}{|v|}$$

This 2<sup>nd</sup> property follows easily from the definition of the signum function.

So, we can rewrite the equations for x as:

$$x = \frac{sgn(uv)}{v\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

$$= \frac{sgn(u) \, sgn(v)}{v\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

$$= \frac{sgn(u)}{|v|\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

Likewise, we can get the equation for y as:

$$y = \frac{sgn(v)}{|u|\sqrt{2}} \sqrt{u^2 + v^2 - \sqrt{(u^2 + v^2)(u^2 + v^2 - 4u^2v^2)}}$$

#### 6 Elliptical Grid Mapping

In 2005, Philip Nowell introduced a square to disc mapping that converts horizontal and vertical lines in the square to elliptical arcs inside a circular region. This mapping turns a regular rectangular grid into a regular curvilinear grid consisting of elliptical arcs. Nowell provided a mathematical derivation of his mapping in his blog [Nowell 2005] which we will not repeat here. However, he left out the reversal of the process. In this section, we shall provide inverse equations to his mapping.

#### 6.1 Inversion using Trigonometry

We now derive an inverse of Nowell's Elliptical Grid mapping. The derivation involves 3 steps which is summarized as

Step 1: Convert circular coordinates (u,v) to polar coordinates and introduce trigonometric variables  $\alpha$  and  $\beta$ 

Step 2: Find an expression for (x,y) in terms of  $\alpha$  and  $\beta$ 

Step 3: Find an expression for (x,y) in terms of u and v

Start with Nowell's equations:

$$u = x\sqrt{1 - \frac{y^2}{2}} \qquad \qquad v = y\sqrt{1 - \frac{x^2}{2}}$$

Step 1: Convert circle coordinates (u,v) to polar form  $(r, \theta)$ 

$$r = \sqrt{u^2 + v^2}$$
 
$$\tan \theta = \frac{v}{u} \qquad \sin \theta = \frac{v}{\sqrt{u^2 + v^2}} \qquad \cos \theta = \frac{u}{\sqrt{u^2 + v^2}}$$

We introduce intermediate trigonometric angles  $\alpha$  and  $\beta$  such that

$$\cos \alpha = r \cos(\theta + \frac{\pi}{4})$$
$$\cos \beta = r \cos(\theta - \frac{\pi}{4})$$

Note that, since  $r \le 1$  and the cosine value is  $\le 1$ , there exist angles  $\alpha$  and  $\beta$  that satisfies this. We can then expand these expressions using trigonometric expansion formulas.

$$\cos \alpha = r \, \cos(\theta + \frac{\pi}{4}) = r \left( \cos \theta \cos \frac{\pi}{4} - \sin \theta \sin \frac{\pi}{4} \right) = r \frac{\sqrt{2}}{2} (\cos \theta - \sin \theta)$$

Likewise,

$$\cos \beta = r \, \cos(\theta - \frac{\pi}{4}) = r \left( \cos \theta \cos \frac{\pi}{4} + \sin \theta \sin \frac{\pi}{4} \right) = r \frac{\sqrt{2}}{2} (\cos \theta + \sin \theta)$$

Furthermore, we can find expressions for  $\cos\alpha$  and  $\cos\beta$  in terms of u and v.

$$\cos \alpha = r \frac{\sqrt{2}}{2} (\cos \theta - \sin \theta) = r \frac{\sqrt{2}}{2} \left( \frac{u}{\sqrt{u^2 + v^2}} - \frac{v}{\sqrt{u^2 + v^2}} \right) = \sqrt{u^2 + v^2} \frac{\sqrt{2}}{2} \frac{u - v}{\sqrt{u^2 + v^2}} = \frac{\sqrt{2}}{2} (u - v)$$

$$\cos \beta = r \frac{\sqrt{2}}{2} (\cos \theta + \sin \theta) = r \frac{\sqrt{2}}{2} \left( \frac{u}{\sqrt{u^2 + v^2}} + \frac{v}{\sqrt{u^2 + v^2}} \right) = \sqrt{u^2 + v^2} \frac{\sqrt{2}}{2} \frac{u + v}{\sqrt{u^2 + v^2}} = \frac{\sqrt{2}}{2} (u + v)$$

$$\therefore \qquad \cos \alpha = \frac{\sqrt{2}}{2} (u - v) \qquad \cos \beta = \frac{\sqrt{2}}{2} (u + v)$$

#### Step 2: Find an expression for x and y in terms of angle $\alpha$ and $\beta$

From step 1, we can write  $\cos \alpha$  and  $\cos \beta$  in terms of x and y

$$\cos \alpha = \frac{\sqrt{2}}{2} \left( x \sqrt{1 - \frac{y^2}{2}} - y \sqrt{1 - \frac{x^2}{2}} \right) = \frac{x}{\sqrt{2}} \sqrt{1 - \frac{y^2}{2}} - \frac{y}{\sqrt{2}} \sqrt{1 - \frac{x^2}{2}}$$

$$\cos \beta = \frac{\sqrt{2}}{2} \left( x \sqrt{1 - \frac{y^2}{2}} + y \sqrt{1 - \frac{x^2}{2}} \right) = \frac{x}{\sqrt{2}} \sqrt{1 - \frac{y^2}{2}} + \frac{y}{\sqrt{2}} \sqrt{1 - \frac{x^2}{2}}$$

We now introduce another set of intermediate trigonometric variables  $\phi$  and  $\lambda$ . Define:

$$\phi = \cos^{-1} \frac{x}{\sqrt{2}}$$

$$\lambda = \sin^{-1} \frac{y}{\sqrt{2}}$$

Note that since  $x \le 1$  and  $y \le 1$ , it is okay to compute their inverse trigonometric values. This implies

$$\cos \phi = \frac{x}{\sqrt{2}}$$

$$\sin \phi = \sqrt{1 - \frac{x^2}{2}}$$

and

$$\cos \lambda = \sqrt{1 - \frac{y^2}{2}} \qquad \qquad \sin \lambda = \frac{y}{\sqrt{2}}$$

So we can write  $\cos \alpha$  and  $\cos \beta$  in terms of  $\phi$  and  $\lambda$ 

$$\cos \alpha = \cos \phi \cos \lambda - \sin \phi \sin \lambda = \cos(\phi + \lambda)$$
$$\cos \beta = \cos \phi \cos \lambda + \sin \phi \sin \lambda = \cos(\phi - \lambda)$$

Hence

$$\alpha = \phi + \lambda = \cos^{-1} \frac{x}{\sqrt{2}} + \sin^{-1} \frac{y}{\sqrt{2}}$$
$$\beta = \phi - \lambda = \cos^{-1} \frac{x}{\sqrt{2}} - \sin^{-1} \frac{y}{\sqrt{2}}$$

Taking the sum and difference of these two equations, we get

$$\alpha + \beta = 2\cos^{-1}\frac{x}{\sqrt{2}}$$
$$\alpha - \beta = 2\sin^{-1}\frac{y}{\sqrt{2}}$$

Rearranging the terms, we get

$$x = \sqrt{2}\cos(\frac{\alpha + \beta}{2})$$
$$y = \sqrt{2}\sin(\frac{\alpha - \beta}{2})$$

Step 3: get an expression for x and y in terms of u and v

We start with the result from step 2 and expand using trigonometric identities for sums and half-angles

$$x = \sqrt{2} \cos(\frac{\alpha}{2} + \frac{\beta}{2}) = \sqrt{2}(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}) = \sqrt{2}(\sqrt{\frac{(1 + \cos\alpha)}{2}} \sqrt{\frac{(1 + \cos\beta)}{2}} - \sqrt{\frac{(1 - \cos\alpha)}{2}} \sqrt{\frac{(1 - \cos\beta)}{2}})$$

$$= \frac{1}{2}\sqrt{2(1 + \cos\alpha)(1 + \cos\beta)} - \frac{1}{2}\sqrt{2(1 - \cos\alpha)(1 - \cos\beta)}$$

$$y = \sqrt{2}\sin(\frac{\alpha}{2} - \frac{\beta}{2}) = \sqrt{2}(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\sin\frac{\beta}{2}) = \sqrt{2}\left(\sqrt{\frac{(1 - \cos\alpha)}{2}}\sqrt{\frac{(1 + \cos\beta)}{2}} - \sqrt{\frac{(1 + \cos\alpha)}{2}}\sqrt{\frac{(1 - \cos\beta)}{2}}\right)$$
$$= \frac{1}{2}\sqrt{2(1 + \cos\alpha)(1 + \cos\beta)} - \frac{1}{2}\sqrt{2(1 - \cos\alpha)(1 - \cos\beta)}$$

Now, recall from step 1 that

$$\cos \alpha = \frac{\sqrt{2}}{2}(u-v) \qquad \qquad \cos \beta = \frac{\sqrt{2}}{2}(u+v)$$

So we can substitute u and v values into  $\cos\alpha$  and  $\cos\beta$ 

$$x = \frac{1}{2}\sqrt{2(1+\cos\alpha)(1+\cos\beta)} - \frac{1}{2}\sqrt{2(1-\cos\alpha)(1-\cos\beta)}$$

$$= \frac{1}{2}\sqrt{2\left(1+\frac{\sqrt{2}}{2}(u-v)\right)\left(1+\frac{\sqrt{2}}{2}(u+v)\right)} - \frac{1}{2}\sqrt{2\left(1-\frac{\sqrt{2}}{2}(u-v)\right)\left(1-\frac{\sqrt{2}}{2}(u+v)\right)}$$

$$= \frac{1}{2}\sqrt{2+u^2-v^2+2\sqrt{2}u} - \frac{1}{2}\sqrt{2+u^2-v^2-2\sqrt{2}u}$$

Likewise,

$$y = \frac{1}{2}\sqrt{2(1-\cos\alpha)(1+\cos\beta)} - \frac{1}{2}\sqrt{2(1+\cos\alpha)(1-\cos\beta)}$$

$$= \frac{1}{2}\sqrt{2\left(1-\frac{\sqrt{2}}{2}(u-v)\right)\left(1+\frac{\sqrt{2}}{2}(u+v)\right)} - \frac{1}{2}\sqrt{2\left(1+\frac{\sqrt{2}}{2}(u-v)\right)\left(1-\frac{\sqrt{2}}{2}(u+v)\right)}$$

$$= \frac{1}{2}\sqrt{2-u^2+v^2+2\sqrt{2}v} - \frac{1}{2}\sqrt{2-u^2+v^2-2\sqrt{2}v}$$

This completes the derivation.

#### 6.2 Inversion using the Biquadratic Equation

The inverse equations for the Elliptical Grid mapping we derived in the previous section are by no means unique in form. We can actually derive another set of inverse equations by using a different method. Of course, these sets of inverse equations are ultimately equivalent to each other. That is, they are just different manifestations of the same inverse equations.

The inverse equations that we will derive in this section are not as mathematically elegant as the ones previously derived, so we prefer to use those. Nevertheless, the inverse equations here are valid and just as useful.

We look at Nowell's square to disc equations and derive another set of the inverse equations for it. Start with

$$u = x\sqrt{1 - \frac{y^2}{2}} \qquad \qquad v = y\sqrt{1 - \frac{x^2}{2}}$$

Isolate y in the 2<sup>nd</sup> equation to get:

$$y = \frac{v}{\sqrt{1 - \frac{x^2}{2}}} = \frac{\sqrt{2}v}{\sqrt{2 - x^2}}$$

$$\Rightarrow \qquad y^2 = \frac{2v^2}{2 - x^2}$$

Substituting back to the 1st equation

$$u = x \sqrt{1 - \frac{2v^2}{2(2 - x^2)}} = x \sqrt{\frac{2 - x^2 - v^2}{2 - x^2}}$$

$$\Rightarrow u^{2} = \frac{x^{2}(2 - x^{2} - v^{2})}{2 - x^{2}}$$

$$\Rightarrow (2 - x^{2})u^{2} = x^{2}(2 - x^{2} - v^{2})$$

$$\Rightarrow x^{4} - x^{2}(2 + u^{2} - v^{2}) + 2u^{2} = 0$$

This is a special kind of quartic polynomial equation called a biquadratic. Notice that there are no cubic or linear terms in the  $4^{th}$  degree polynomial equation in x. We can solve for  $x^2$  using the quadratic equation with coefficients

$$a = 1$$
  $b = -(2 + u^2 - v^2)$   $c = 2u^2$ 

This gives us the solution:

$$x^{2} = \frac{2 + u^{2} - v^{2} \pm \sqrt{(2 + u^{2} - v^{2})^{2} - 8u^{2}}}{2}$$

We can then get a quadrant-aware inverse equation for x as

$$x = \frac{sgn(u)}{\sqrt{2}}\sqrt{2 + u^2 - v^2 - \sqrt{(2 + u^2 - v^2)^2 - 8u^2}}$$

Using a similar approach, one get the equation for y as

$$y = \frac{sgn(v)}{\sqrt{2}}\sqrt{2 - u^2 + v^2 - \sqrt{(2 - u^2 + v^2)^2 - 8v^2}}$$

#### 6.3 Squircularity of the Mapping

It would be appropriate at this point to ask what sort of shape does the Elliptical Grid mapping convert circular contours into. First, recall the circular continuum representation of the unit disc. Now we want to find out what sort of curve each of the concentric circles inside the disc map into inside the square.

Actually, we will show that central circles on the unit disc map to Fernandez Guasti squircles. We do this by looking at an individual circle inside the disc and substitute x and y values into it.

$$u^{2} + v^{2} = \left(x\sqrt{1 - \frac{y^{2}}{2}}\right)^{2} + \left(y\sqrt{1 - \frac{x^{2}}{2}}\right)^{2}$$

$$= x^{2}\left(1 - \frac{y^{2}}{2}\right) + y^{2}\left(1 - \frac{x^{2}}{2}\right)$$

$$= x^{2} - \frac{1}{2}x^{2}y^{2} + y^{2} - \frac{1}{2}x^{2}y^{2}$$

$$= x^{2} + y^{2} - x^{2}y^{2}$$

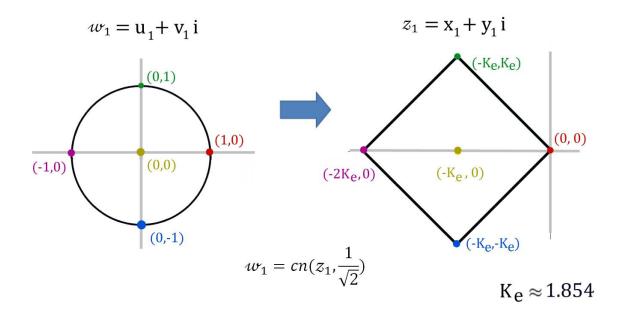
This is just the equation for the squircularity condition of the mapping. From this, we can deduce that the Elliptical Grid mapping also converts central circles in the unit disc to Fernandez Guasti squircles on the square.

#### 7 Schwarz-Christoffel Mapping

In the 1860s, Hermann A. Schwarz and Elwin Christoffel used complex analysis to independently develop a conformal map of the circular disc onto a simple polygonal region. In this paper, we are mainly interested in the special case involving the square. Our goal is to reduce the Schwarz-Christoffel mapping to our specific case and derive an equation relating point (u,v) inside the circular disc to point (x,y) inside the square.

#### 7.1 Fundamental Mapping in the Complex Plane

Without getting much into the mathematical underpinnings of the Schwarz-Christoffel mapping, we show in the figure below a fundamental conformal mapping between the circular disc and the square in the complex plane. Using the complex-valued Jacobi elliptic function  $cn(z,\frac{1}{\sqrt{z}})$ , one can map every point inside the unit disc to a square region conformally.



The main drawback of this diagram on the complex plane is that x and y coordinates are not in our canonical mapping space. As the figure above shows, the square has corner coordinates in terms of a constant  $K_e$  instead of the  $\pm 1$  that we desire. Moreover, the square is tilted by 45° and off-center from the origin.

The constant  $K_e$  has a numerical value of approximately 1.854. Its exact value is the complete Legendre elliptic integral of the 1<sup>st</sup> kind with modulus  $\frac{1}{\sqrt{2}}$ . This number arises from Schwarz and Christoffel's equations for the specific case when the desired polygonal shape is a square.

$$K_e = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - \frac{1}{2}\sin^2 t}} \approx 1.854$$

Note that the complete Legendre elliptic integral of the 1<sup>st</sup> kind can be calculated from the incomplete Legendre elliptic integral of the 1<sup>st</sup> kind; i.e.

$$K_e = K\left(\frac{1}{\sqrt{2}}\right) = F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)$$

#### 7.2 Aligning to the Canonical Mapping Space

In order to get the mapping into our canonical mapping space, we have to do a sequence of affine transformations on the square. For example, we want the square to be centered on the origin with side length value of 2. This is exactly what the square to disc mapping equations below do.

$$u = Re\left(\frac{1-i}{\sqrt{2}} cn\left(K_e \frac{1+i}{2}(x+yi) - K_e, \frac{1}{\sqrt{2}}\right)\right)$$
$$v = Im\left(\frac{1-i}{\sqrt{2}} cn\left(K_e \frac{1+i}{2}(x+yi) - K_e, \frac{1}{\sqrt{2}}\right)\right)$$

In the complex plane, rotation can be done simply by multiplication with the complex number  $e^{i\theta}$ . For our case, we are interested in rotation by  $\pm 45^{\circ}$ , so we have these multipliers

for 45° rotation:

$$e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$$

for -45° rotation:

$$e^{-i\frac{\pi}{4}} = \cos\frac{\pi}{4} - i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1-i)$$

One can see these rotational multipliers along with translational offsets and scale factors in the mapping equations. These affine transformations are there to place the square into our canonical mapping space in the complex plane.

In order to get the disc to square mapping equations, we use the fact that inverse of the Jacobi elliptic function cn is just the Legendre elliptic integral of the 1<sup>st</sup> kind for an arccosine. In other words,

$$cn^{-1}\left(z, \frac{1}{\sqrt{2}}\right) = F(\cos^{-1}z, \frac{1}{\sqrt{2}})$$

Consequently, by using this identity and doing some algebra, the disc to square equations are:

$$x = Re\left(\frac{1-i}{-K_e}F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}(u+v\,i)\right),\frac{1}{\sqrt{2}}\right)\right) + 1$$

$$y = Im\left(\frac{1-i}{-K_e}F\left(\cos^{-1}\left(\frac{1+i}{\sqrt{2}}(u+v\,i)\right),\frac{1}{\sqrt{2}}\right)\right) - 1$$

We would like to point out that the complex-valued Jacobi elliptic function cn is an even function, so  $cn(z, \frac{1}{\sqrt{z}}) = cn\left(-z, \frac{1}{\sqrt{z}}\right)$ . Therefore, the equations below are also valid formulas for the mapping:

$$u = Re\left(\frac{1-i}{\sqrt{2}} cn\left(-K_e \frac{1+i}{2}(x+yi) + K_e, \frac{1}{\sqrt{2}}\right)\right)$$

$$v = Im\left(\frac{1-i}{\sqrt{2}} cn\left(-K_e \frac{1+i}{2}(x+yi) + K_e, \frac{1}{\sqrt{2}}\right)\right)$$

#### 7.3 A Compact Complex Equation

It is possible to get a more compact complex equation for the mapping on the complex plane. First, observe that the  $45^{\circ}$  rotational factor needed for canonical alignment can expressed as the square root of i. In other words,

$$\frac{1}{\sqrt{2}}(1+i) = \sqrt{i}$$

Likewise, the -45° rotational factor can be expressed as the square root of -i.

$$\frac{1}{\sqrt{2}}(1-i) = \sqrt{-i}$$

Note that these two rotational factors are complex conjugates of each other.

If we define complex variables w and z as

$$\mathbf{w} = \mathbf{u} + \mathbf{v} i$$
  $\mathbf{z} = \mathbf{x} + \mathbf{y} i$ 

i.e. w represents complex numbers inside the circular disc and z represents complex numbers inside the square.

We can then simplify the canonically-aligned mapping equation into

$$\mathbf{w} = \sqrt{-i} \ cn\left(K_e \mathbf{z} \sqrt{\frac{i}{2}} - K_e, \frac{1}{\sqrt{2}}\right)$$

along with its inverse

$$\mathbf{z} = -\frac{\sqrt{-2i}}{K_e} F\left(\cos^{-1}\left(\mathbf{w}\sqrt{i}\right), \frac{1}{\sqrt{2}}\right) + 1 - i$$

These are more compact equations that the ones provided in page 6.

#### 7.4 Relation to the Lemniscate Constant

The constant  $K_e$  is actually related to another well-known mathematical constant known as the *lemniscate* constant. The definition of the lemniscate constant is

$$L_e = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{2\sqrt{2\pi}}$$

where  $\Gamma$  is the gamma function.

The numerical value of  $L_e$  is approximately 2.62205755

The constant  $K_e$  is related to  $L_e$  by the following equation:

$$K_e = \frac{L_e}{\sqrt{2}}$$

#### 7.5 Non-standard Notation involving Matrices

It is possible to express the Schwarz-Christoffel mapping in a compact equation involving matrices and vectors.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{K_e} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} F \left( \cos^{-1} \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right), \frac{1}{\sqrt{2}} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} cn \left( \frac{K_e}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} K_e \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \right)$$

In these equations, we substituted the complex rotational multipliers with 2x2 matrices.

$$\frac{1}{\sqrt{2}}(1+i) \quad corresponds \ to \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$\frac{1}{\sqrt{2}}(1-i) \quad corresponds \ to \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Although these mapping equations are nice and compact, they use non-standard notation by mixing complex numbers and 2x1 vectors interchangeably in the equations. This form of the equations is arguably more accessible to programmers who are familiar with linear algebra but unfamiliar with using complex numbers for rotation. In computer science, it is common practice to mix the interpretation of variables for convenience. This practice is called function overloading.

In the matrix form of our mapping equations the following functions are overloaded:

- $F\left(w, \frac{1}{\sqrt{2}}\right)$  Legendre elliptic function of the 1<sup>st</sup> kind
- $\cos^{-1} w$  arccosine function
- $cn(z, \frac{1}{\sqrt{2}})$  Jacobi elliptic function

Whenever one encounters vector-valued inputs and outputs to these functions, the 2x1 vector should be interpreted as a complex number. All three functions listed above are well-defined for complex number inputs.

#### 7.6 Verifying Conformality

One way to verify the conformality of the mapping is by looking at the mapping diagrams in page 6. The top figure shows a radial grid inside a circular disc mapped to a square. The radial lines and circular contours meet at 90° inside the circular disc. Since this mapping is conformal, the corresponding curves and squircle-like contours should also meet at 90° inside the square region. This can be verified visually by inspection.

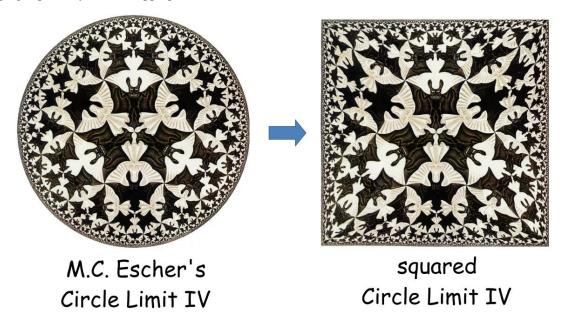
Likewise, we can observe something similar for the bottom figure showing a rectangular grid inside a square region. This rectangular grid is composed of horizontal and vertical lines that meet at 90° inside the square. After mapping to the circular disc, the corresponding curves also meet at 90° inside the circle.

The conformal map between the circular disc and the square has many applications in science and engineering. For example, the Peirce quincuncial map projection [Fong 2011] used in geography and panoramic photography relies on this conformal map as an intermediate step.

## 8 Applications

#### 8.1 The Poincare square and Hyperbolic Art

The Poincare disk is one of the most interesting models to arise from non-Euclidean hyperbolic geometry. In fact, this model of hyperbolic geometry has inspired artwork such as M.C. Escher's circle limit woodcuts. Using the different mappings discussed in this paper, one can convert the Poincare disc to a square. The Schwarz-Christoffel mapping is probably the most appropriate for this task because of its conformal nature.



Herbert Müller [Müller 2013] has used Schwarz-Christoffel transformations on the complex plane to convert M.C. Escher's Circle Limit IV to polygons with different number (n) of sides. His paper shows results for n=3, 6 but not for n=4 (the square).

#### 8.2 Logo Design and Artwork

The circle and the square are very common shapes used in logos. It is certainly useful to have methods to convert from one shape to the other as part of the designer's toolbox. To illustrate this, we convert the Fields medal, which is circular in shape, to a square one below. Of course, the Fields medal awards ceremony is a highlight of the quadrennial International Congress of Mathematicians (ICM) for which this work was shown in 2014.





Simple Stretching



FG-Squircular



Elliptical Grid



Schwarz-Christoffel

The four different mappings presented in this paper can be used to convert square-shaped designs into circular artwork. For example, the figure above show results from mapping a square chessboard into a circular disc. Each of the four mappings produces quite different results and distortion characteristics. These distortions are most apparent at the four corners of the board where the rooks are located.

It is quite evident that the Simple Stretching map has unsightly artifacts along its diagonals. In fact, we only included it in this paper because of its simplicity. It is largely unsuitable for our intended applications.

The Schwarz-Christoffel mapping is conformal. This is quite noticeable on its corresponding circular map. By and large, each of the chess pieces on the board have no shape distortion and very much resemble the shapes on the square chessboard. However, there is significant size distortion in the mapping. This distortion is most prominent with the corner rooks. The rooks are considerably smaller than the other chess pieces.

The Elliptical Grid mapping produces a nice and uniform checkered pattern on the circular disc. Horizontal and vertical lines on the square chessboard map into a curvilinear grid consisting of elliptical arcs on the circular disc. This characteristic definitely makes the Elliptical Grid mapping quite desirable. However, there is considerable shape distortion at the four corners of the chessboard. In fact, it is quite noticeable that the rooks are horribly stretched and bent out of shape.

The FG-Squircular mapping offers a good compromise between shape and size distortion. The four corner rooks are still noticeably smaller than the other chess pieces, but their shapes are not as deformed as in the Elliptical Grid mapping. The rooks also appear much larger than their counterparts in the Schwarz-Christoffel mapping.

#### 8.3 Azimuthal Panoramas

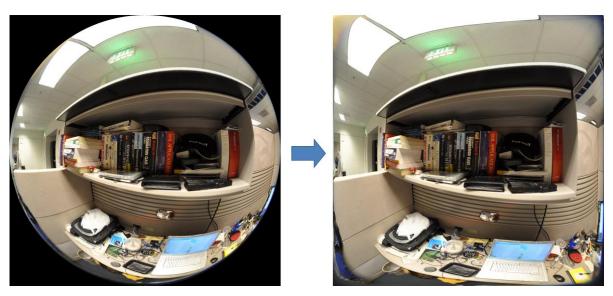
Azimuthal projections of spherical panoramas produce naturally circular images. However, since most of the world's photographs are rectangular in shape, it is desirable to convert these circular images to square ones. An example azimuthal panorama is provided below along with its mapping to a square [Fong 2014]. The FG-Squircular mapping was used for the mapping because of its radial nature. The radial property makes it suitable for extending the radial grid of the azimuthal projection to the square.



#### 8.4 Quick and Dirty Defishing

There are two types of fisheye lens – frame-filling rectangular fisheye lens and circular fisheye lens. Photographs taken using circular fisheye lens are circular in shape. Photographers usually want to convert these circular photographs to square ones in order to straighten out curved lines distorted by the fisheye lens. This process is known as defishing.

Fisheye lens produce distortions known in the optics literature as barrel distortion [Falk 1986]. Visually speaking, this distortion magnifies the center of the image and falls off as you move away from the center. Qualitatively, this distortion looks quite similar to what the Elliptical Grid mapping does to a rectangular grid as shown in page 5. Consequently, it seems viable to use the Elliptical Grid equations to reverse this distortion.



Using the disc to square mappings discussed in this paper, one can do quick and dirty defishing of photographs. We call it "quick and dirty" because the real defishing process is dependent on the optics of the fisheye lens in order to be done correctly. There are actual physical models and equations for barrel distortion in lens [Bailey 2002]. The mappings provided in this paper have no knowledge of the inner workings of fisheye lens optics, and hence give only a poor man's approximation of the defishing process.

Empirically, we have found that the Elliptical Grid mapping works well for defishing when compared to the other mappings discussed with this paper. This is probably because of the qualitative similarity between its distortion and the barrel distortion produced by fisheye lens. The sample image shown above has before and after photographs of the defishing process using the Elliptical Grid mapping.

#### 9 Summary

In this paper, we presented and discussed four different mappings for converting a circular disc to a square region and vice versa. Each of the mappings has different properties and characteristics. The table below summarizes these properties.

mapping	key property	contours inside square	notes/comments
Simple Stretching	radial	squares	Circular disc is just linearly stretched to a square
FG-Squircular	radial	FG-squircles	Good compromise between equiareal and conformal
Elliptical Grid	grid of elliptical arcs	FG-squircles	Rectangular grid mapped to elliptical grid inside circle
Schwarz-Christoffel	conformal	squircle-like curve	Considerable size distortions at the four corners

In the course of our mapping discussions and derivations, we also introduced the concept of the squircular continuum of the square, where the square region in the interval [-1,1] x [-1,1] can be represented as the set  $S = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - x^2y^2 = t^2 \}$  which further simplifies to  $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - x^2y^2 \le 1\}$ .

We also discussed different applications of the mappings between the disc and the square. We thereupon showed pictures of the mappings used for logo design, panoramic photography, and art. In addition, we showed that the elliptical grid mapping can be used for defishing photographs with barrel distortion from fisheye lens.

#### 10 Acknowledgements

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## Appendix A: A Simpler Variation of the FG-Squircular Mapping

The FG-Squircular mapping is by no means the only radial mapping between the circular disc and the square that uses the FG-squircle. In this section, we shall introduce another squircle-based radial mapping with much simpler equations.

1<sup>st</sup> recall the *radial mapping parametric equations* for mappings between the circular disc and the square. All radial mappings between the disc and the square have an equation of this form, where the parameter t can be expressed as an arbitrary function f of x and y.

$$u = t \frac{x}{\sqrt{x^2 + y^2}} \qquad \qquad v = t \frac{y}{\sqrt{x^2 + y^2}}$$

i.e.,

$$u = f(x, y) \frac{x}{\sqrt{x^2 + y^2}}$$
  $v = f(x, y) \frac{y}{\sqrt{x^2 + y^2}}$ 

Next, recall our condition for the squircular continuum of the square. We start from the squircle equation  $x^2 + y^2 - \frac{s^2}{t^2}x^2y^2 = t^2$  and set s=t. This gives the square as  $\mathcal{S} = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - x^2y^2 = t^2 , 0 \le t \le 1\}$  In other words, we have this basic equation for the FG-squircular mapping:

$$x^2 + y^2 - x^2 y^2 = t^2$$

We can actually use any monotonic function h(t) for the squircular continuum of the square where s=h(t) as long as these two conditions hold: h(0)=0 and h(1)=1, i.e. the contour at the center of the square is circular and the contour at the perimeter is a square.

Consider what happens if we set  $s = h(t) = t^2$  in the squircle equation instead, we get this condition

$$x^2 + y^2 - t^2 x^2 y^2 = t^2$$

We then show that this case actually simplifies to produce mapping equations that are simpler than the case s=t. We do this by collecting the  $t^2$  terms into one side of the equation

$$t^{2}(1+x^{2}y^{2}) = x^{2} + y^{2} \implies t = \sqrt{\frac{x^{2}+y^{2}}{1+x^{2}y^{2}}}$$

and plugging back t to the radial mapping parametric equations.

$$u = \sqrt{\frac{x^2 + y^2}{1 + x^2 y^2}} \frac{x}{\sqrt{x^2 + y^2}} \qquad v = \sqrt{\frac{x^2 + y^2}{1 + x^2 y^2}} \frac{y}{\sqrt{x^2 + y^2}}$$

This simplifies to

$$u = \frac{x}{\sqrt{1 + x^2 y^2}}$$
  $v = \frac{y}{\sqrt{1 + x^2 y^2}}$ 

We now solve for the inverse of these mapping equations.

$$u = \frac{x}{\sqrt{1 + x^2 y^2}} \implies u\sqrt{1 + x^2 y^2} = x \implies u^2 (1 + x^2 y^2) = x^2 \implies 1 + x^2 y^2 = \frac{x^2}{u^2}$$

$$\implies x^2 y^2 = \frac{x^2}{u^2} - 1 \implies y^2 = \frac{1}{u^2} - \frac{1}{x^2} \implies y = \sqrt{\frac{1}{u^2} - \frac{1}{x^2}}$$

Substitute y into the other equation

$$v = \frac{y}{\sqrt{1 + x^2 y^2}} = \frac{\sqrt{\frac{1}{u^2} - \frac{1}{x^2}}}{\sqrt{1 + x^2 (\frac{1}{u^2} - \frac{1}{x^2})}} = \frac{\sqrt{\frac{1}{u^2} - \frac{1}{x^2}}}{\sqrt{\frac{x^2}{u^2}}} = \frac{u}{x} \sqrt{\frac{1}{u^2} - \frac{1}{x^2}} = \frac{u}{x} \sqrt{\frac{x^2 - u^2}{u^2 x^2}} = \frac{\sqrt{x^2 - u^2}}{x^2}$$

We now find a polynomial equation for x in terms of u and v.

$$v^2 = \frac{x^2 - u^2}{x^4}$$
  $\Rightarrow$   $x^4 = \frac{x^2 - u^2}{v^2}$   $\Rightarrow$   $x^4v^2 - x^2 + u^2 = 0$ 

This is a biquadratic equation in which we could solve for  $x^2$  using the standard quadratic equation with coefficients:  $a = v^2$  b = -1 and  $c = u^2$ 

$$x^2 = \frac{1 \pm \sqrt{1 - 4u^2v^2}}{2v^2}$$

We can then get a quadrant-aware expression for x by taking the square root.

$$x = \frac{sgn(uv)}{v\sqrt{2}}\sqrt{1-\sqrt{1-4u^2v^2}} = \frac{sgn(uv)}{v}\sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-u^2v^2}}$$

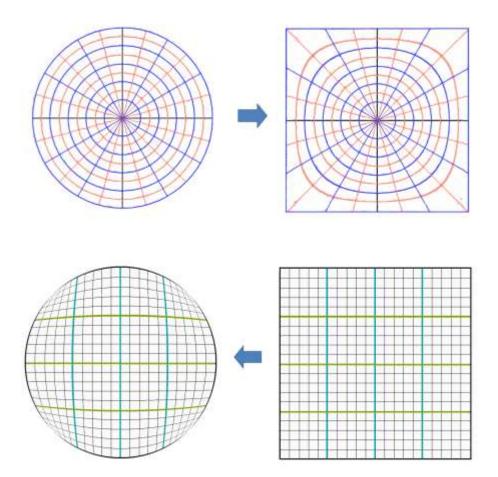
Similarly, we get an expression for y as

$$y = \frac{sgn(uv)}{u\sqrt{2}}\sqrt{1-\sqrt{1-4u^2v^2}} = \frac{sgn(uv)}{u}\sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-u^2v^2}}$$

In summary, we were able to derive another radial mapping based on the FG-squircle by using simple algebraic manipulations on the squircular continuum of the square. We were also able to find simple inverse equations for the mapping. We shall name this mapping as the 2-Squircular mapping.

Note that this case produces mapping equations that are much simpler than the case s=t. The qualitative results of both mappings are very similar. In fact, it is quite difficult to tell the two mappings apart visually. See the figure on the next page and compare with the figure for the FG-Squircular mapping in Section 1 of this paper.

# 2-Squircular Mapping



Disc to square mapping:

$$x = \frac{sgn(uv)}{v\sqrt{2}} \sqrt{1 - \sqrt{1 - 4u^2v^2}}$$
$$y = \frac{sgn(uv)}{u\sqrt{2}} \sqrt{1 - \sqrt{1 - 4u^2v^2}}$$

$$u = \frac{x}{\sqrt{1 + x^2 y^2}}$$
  $v = \frac{y}{\sqrt{1 + x^2 y^2}}$ 

## Appendix B: A Quasi-Symmetric Variation of the FG-Squircular Mapping

In the previous section, we derived a mapping by using the assumption  $s=t^2$ . We can take this step further by considering the case when  $s=t^3$ . The squircle equation then reduces to  $x^2 + y^2 - t^4x^2y^2 = t^2$  or equivalently, we have this biquadratic polynomial equation in t:

$$x^2y^2\,t^4+t^2-x^2-y^2=0$$

We can solve for  $t^2$  using the quadratic equation to get

$$t^2 = \frac{\pm \sqrt{4x^4y^2 + 4x^2y^4 + 1} - 1}{2x^2y^2}$$

or

$$t = \pm \sqrt{\frac{\pm \sqrt{4x^4y^2 + 4x^2y^4 + 1} - 1}{2x^2y^2}} = \frac{\pm \sqrt{\pm \sqrt{1 + 4x^2y^2(x^2 + y^2)} - 1}}{xy\sqrt{2}}$$

Plugging back into the *radial mapping parametric equations* and accounting for the 4 different quadrants, we get these square-to-disc mapping equations

$$u = \frac{sgn(xy)}{y} \sqrt{\frac{-1 + \sqrt{1 + 4x^2y^2(x^2 + y^2)}}{2(x^2 + y^2)}}$$
$$v = \frac{sgn(xy)}{x} \sqrt{\frac{-1 + \sqrt{1 + 4x^2y^2(x^2 + y^2)}}{2(x^2 + y^2)}}$$

For the inverse of these mapping equations, we can start by manipulating the  $\mathbf{u}$  equation

$$uy\sqrt{2(x^{2}+y^{2})} = \sqrt{-1+\sqrt{1+4x^{2}y^{2}(x^{2}+y^{2})}} \Rightarrow 2u^{2}y^{2}(x^{2}+y^{2}) = -1+\sqrt{1+4x^{2}y^{2}(x^{2}+y^{2})} \Rightarrow (2u^{2}y^{2}(x^{2}+y^{2})+1)^{2} = 1+4x^{2}y^{2}(x^{2}+y^{2}) \Rightarrow 4u^{4}y^{4}(x^{2}+y^{2})^{2}+4u^{2}y^{2}(x^{2}+y^{2})+1=4x^{4}y^{2}+4x^{2}y^{4}+1 \Rightarrow 4u^{4}y^{4}(x^{4}+2x^{2}y^{2}+y^{4})+4u^{4}x^{2}y^{2}+4u^{4}y^{4}=4x^{4}y^{2}+4x^{2}y^{4}$$

We can divide the equation by  $4y^2$  to get

$$u^4x^4y^2 + 2u^4x^2y^4 + u^4y^6 + u^4x^2 + u^4y^2 = x^4 + x^2y^2$$

Substitute  $y = \frac{v}{u}x$ , to get

$$u^{2}v^{2}x^{6} + 2v^{4}x^{6} + \frac{v^{6}}{u^{2}}x^{6} + u^{2}x^{2} + v^{2}x^{2} = x^{4} + \frac{v^{2}}{u^{2}}x^{4}$$

Multiply the equation by  $\frac{u^2}{x^2}$  to get

$$u^4v^2x^4 + 2u^2v^4x^4 + v^6x^4 + u^4 + u^2v^2 = u^2x^2 + v^2x^2$$

Collect the terms to get this biquadratic equation in x

$$(u^{4}v^{2} + 2u^{2}v^{4} + v^{6})x^{4} - (u^{2} + v^{2})x^{2} + u^{4} + u^{2}v^{2} = 0$$

$$v^{2}(u^{2} + v^{2})^{2}x^{4} - (u^{2} + v^{2})x^{2} + u^{2}(u^{2} + v^{2}) = 0$$

We can then divide the equation by  $(u^2 + v^2)$  to get this simple biquadratic equation

$$v^2(u^2 + v^2)x^4 - x^2 + u^2 = 0$$

Thus, we can solve for  $x^2$  using the quadratic equation.

$$x^{2} = \frac{1 \pm \sqrt{1 - 4u^{2}v^{2}(u^{2} + v^{2})}}{2v^{2}(u^{2} + v^{2})}$$

Taking the square root of this and accounting for the 4 quadrants, we get this equation in x.

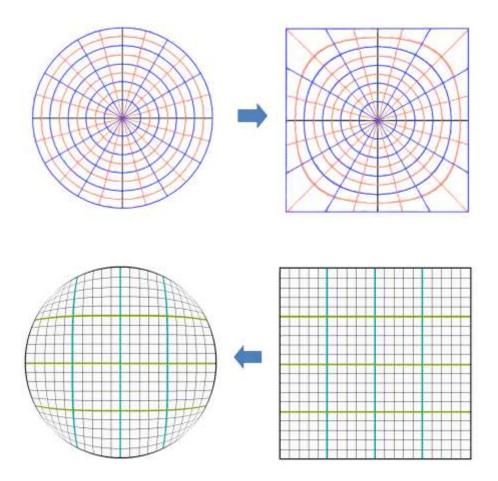
$$x = \frac{sgn(uv)}{v} \sqrt{\frac{1 - \sqrt{1 - 4u^2v^2(u^2 + v^2)}}{2(u^2 + v^2)}}$$

Substitute  $y = \frac{v}{u}x$ , we get the equation for y as

$$y = \frac{sgn(uv)}{u} \sqrt{\frac{1 - \sqrt{1 - 4u^2v^2(u^2 + v^2)}}{2(u^2 + v^2)}}$$

If we compare these inverse equations with the forward equations derived earlier, we can see that the two sets of equations look very similar. In fact, they are almost symmetrical if not for a few changes in sign involved in the addition/subtraction of terms in the numerator. This quasi-symmetrical property of the forward and inverse equations is noteworthy and surprising.

# 3-Squircular Mapping



Disc to square mapping:

$$x = \frac{sgn(uv)}{v} \sqrt{\frac{1 - \sqrt{1 - 4u^2v^2(u^2 + v^2)}}{2(u^2 + v^2)}}$$
$$y = \frac{sgn(uv)}{u} \sqrt{\frac{1 - \sqrt{1 - 4u^2v^2(u^2 + v^2)}}{2(u^2 + v^2)}}$$

$$u = \frac{sgn(xy)}{y} \sqrt{\frac{-1 + \sqrt{1 + 4x^2y^2(x^2 + y^2)}}{2(x^2 + y^2)}}$$
$$v = \frac{sgn(xy)}{x} \sqrt{\frac{-1 + \sqrt{1 + 4x^2y^2(x^2 + y^2)}}{2(x^2 + y^2)}}$$

## Appendix C: More Variations of the FG-Squircular Mapping

We can further generalize the result in Appendix A by setting  $h(t)=t^n$  for different exponents n. Not all exponents are amenable to simplification so we have to carefully select those that produce soluble polynomials. Also, we have to restrict ourselves to n>0 for the exponents. Otherwise, these required conditions will not hold: h(0)=0 and h(1)=1.

Note that we will only solve for square-to-disc mapping equations in this appendix. We have deliberately left out the inverse equations because these are particularly gnarly. For now, we shall relegate the disc-to-square mapping equations as future work.

## **B1. The** $\frac{3}{2}$ **-Squircular Mapping** (case s= $t^{3/2}$ )

In the case  $s=t^{3/2}$ , the squircle equation reduces to  $x^2 + y^2 - tx^2y^2 = t^2$  or equivalently, we have this quadratic polynomial equation in t:

$$t^2 + x^2 y^2 t - x^2 - y^2 = 0$$

We can solve for t using the quadratic equation to get

$$t = \pm \frac{1}{2} \sqrt{x^4 y^4 + 4x^2 + 4y^2} - \frac{1}{2} x^2 y^2$$

Thus, the square-to-disc mapping equations are

$$u = \frac{x}{2\sqrt{x^2 + y^2}} \left( \sqrt{x^4 y^4 + 4x^2 + 4y^2} - x^2 y^2 \right) \qquad v = \frac{y}{2\sqrt{x^2 + y^2}} \left( \sqrt{x^4 y^4 + 4x^2 + 4y^2} - x^2 y^2 \right)$$

## **B2. The ½-Squircular Mapping** (case s=t<sup>1/2</sup>)

In the case  $s=t^{1/2}$ , the squircle equation reduces to  $x^2+y^2-\frac{x^2y^2}{t}=t^2$  or equivalently, we have this depressed cubic polynomial equation in t:

$$t^3 - (x^2 + y^2) t + x^2 y^2 = 0$$

We can solve for t using the cubic formula to get

$$t = \frac{1}{3} \left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}} + \frac{(x^2 + y^2)}{\left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}}}$$

Thus, the square-to-disc mapping equations are

$$u = \frac{x}{\sqrt{x^2 + y^2}} \left( \frac{1}{3} \left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}} + \frac{(x^2 + y^2)}{\left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}}} \right)$$

$$v = \frac{y}{\sqrt{x^2 + y^2}} \left( \frac{1}{3} \left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}} + \frac{(x^2 + y^2)}{\left( \frac{1}{2} \sqrt{729x^4y^4 - 4(3x^2 + 3y^2)^3} - \frac{27}{2} x^2 y^2 \right)^{\frac{1}{3}}} \right)$$

#### **B3.** The 4-Squircular Mapping (case $s=t^4$ )

In the case  $s=t^4$ , the squircle equation reduces to  $x^2 + y^2 - t^6x^2y^2 = t^2$  or equivalently, we have this soluble sextic polynomial equation in t:

$$x^2y^2\,t^6+t^2-x^2-y^2=0$$

We can solve for  $t^2$  using the cubic formula to get

$$t^2 = \frac{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}{3x^2y^2} - \frac{1}{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}$$

Thus, the square-to-disc mapping equations are

$$u = \frac{x}{\sqrt{x^2 + y^2}} \sqrt{\frac{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}{3x^2y^2} - \frac{1}{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}}$$

$$v = \frac{y}{\sqrt{x^2 + y^2}} \sqrt{\frac{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12}x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}{3x^2y^2}} - \frac{1}{\left(\frac{27}{2}x^6y^4 + \frac{27}{2}x^4y^6 + \frac{3}{2}\sqrt{12}x^6y^6 + 81(x^6y^4 + x^4y^6)^2}\right)^{\frac{1}{3}}}$$

#### **B4.** Other Soluble Exponents

One can easily from section B3 that the mapping equations get very complicated quickly when we solve polynomial equations with higher exponents. Moreover, the Abel-Ruffini theorem states that we cannot get explicit solutions for general polynomials with degree greater than or equal to 5 (i.e. the quintic). In other words, we can only choose exponents for  $s=t^n$  where the effective polynomial equation derived from the squircular continuum is linear, quadratic, cubic, or quartic.

Furthermore, all these complicated mappings probably produce qualitative results not much different from the FG-Squircular mapping or the 2-Squircular mapping. This means that we get diminishing returns and hardly any improvement for more complicated formulas.

Therefore, we shall not continue examining other exponents and will only list down different possible exponents that produce soluble polynomials equations in t. The table on the next page shows a list of these soluble exponents along with their polynomial equations. All of these exponents except the last entry can be used to produce explicit square-to-disc mapping equations.

function	squircular continuum	effective	equivalent polynomial equation
s=h(t)	equation	type	
	equation $x^2 + y^2 - x^2y^2 = t^2$	simple	$t^2 = x^2 + y^2 - x^2 y^2$
s=t		quadratic	
2	$x^2 + y^2 - t^2 x^2 y^2 = t^2$	simple	$t^2(1+x^2y^2) = x^2 + y^2$
$s=t^2$		quadratic	
2.12	$x^2 + y^2 - tx^2y^2 = t^2$	general	$t^2 + x^2 y^2 t - x^2 - y^2 = 0$
$s=t^{3/2}$		quadratic	
1/2	$x^2 + y^2 - \frac{x^2 y^2}{t} = t^2$	depressed	$t^3 - (x^2 + y^2) t + x^2 y^2 = 0$
$s=t^{1/2}$	t t	cubic	
s=t <sup>5/2</sup>	$x^2 + y^2 - t^3 x^2 y^2 = t^2$	cubic	$x^2y^2 t^3 + t^2 - x^2 - y^2 = 0$
5—•	$x^2 + y^2 - t^4 x^2 y^2 = t^2$	biquadratic	$x^2y^2 \tau^2 + \tau - x^2 - y^2 = 0$
$s=t^3$		1	$x^{2}y^{2} \tau^{2} + \tau - x^{2} - y^{2} = 0$ where $\tau = t^{2}$
	$x^2 + y^2 - t^6 x^2 y^2 = t^2$	depressed	$x^{2}y^{2} \tau^{3} + \tau - x^{2} - y^{2} = 0$ where $\tau = t^{2}$
$s=t^4$	,	bi-cubic	where $\tau = t^2$
	$x^2 + y^2 - t^8 x^2 y^2 = t^2$	depressed	$x^2y^2  \tau^4 + \tau - x^2 - y^2 = 0$
$s=t^5$		bi-quartic	where $\tau = t^2$
	$x^2 + y^2 - x^2 y^2 \sqrt{t} = t^2$	depressed	$\tau^4 + x^2 y^2 \tau - x^2 - y^2 = 0$
$s=t^{5/4}$		quartic	where $\tau = \sqrt{t}$
	$x^2 + y^2 - t^{\frac{2}{3}}x^2y^2 = t^2$	depressed	$\tau^3 + x^2 y^2 \tau - x^2 - y^2 = 0$
$s=t^{4/3}$		cubic	where $\tau^3 = t^2$
	$x^2 + y^2 - t^{2n-2} x^2 y^2 = t^2$	polynomial	$x^2y^2 t^{2n-2} + t^2 - x^2 - y^2 = 0$
s=t <sup>n</sup>			

## Appendix D: Parameterized Squelching of the Elliptical Grid Mapping

In this section, we shall generalize the Elliptical Grid mapping by introducing a squelching parameter q to produce a family of similar but distinct mappings. The parameter q is valid for  $0 < q \le 1$ 

We shall derive the mapping equations in a similar manner as Nowell derived the equations for the elliptical grid mapping. The basic idea is to map horizontal line segments in the square to horizontal elliptical arcs on the circle; and to map vertical line segments in the square to vertical elliptical arcs on the circle.

In other words, we want line segments of constant x in the interior of the square to map to arcs in an ellipse with fixed semi-minor length in the interior of the circular disc. This can be represented as an equation which we shall call the *vertical constraint* of the mapping.

$$1 = \frac{u^2}{x^2} + \frac{v^2}{b^2}$$

Likewise, we want line segments of constant y in the interior of the square to map to arcs in an ellipse with fixed semi-major length in the interior of the circular disc. This can be represented as an equation which we shall call the *horizontal constraint* of the mapping.

$$1 = \frac{u^2}{a^2} + \frac{v^2}{y^2}$$

Start with the vertical constraint equation and isolate  $u^2$ ,

$$1 = \frac{u^2}{x^2} + \frac{v^2}{b^2} \implies 1 - \frac{v^2}{b^2} = \frac{u^2}{x^2} \implies u^2 = x^2 \left(1 - \frac{v^2}{b^2}\right)$$

We can then plug the  $u^2$  value into the horizontal constraint equation

$$1 = \frac{u^2}{a^2} + \frac{v^2}{v^2} \quad \Rightarrow \qquad 1 = \frac{x^2 \left(1 - \frac{v^2}{b^2}\right)}{a^2} + \frac{v^2}{v^2}$$

Multiply both sides of the equation by  $a^2b^2y^2$  to remove the fractions and get

$$a^2b^2v^2 = x^2v^2(b^2 - v^2) + v^2a^2b^2$$

After which, we can isolate v<sup>2</sup> into one side of the equation

$$a^{2}b^{2}y^{2} = x^{2}y^{2}(b^{2} - v^{2}) + v^{2}a^{2}b^{2} = x^{2}y^{2}b^{2} - x^{2}y^{2}v^{2} + v^{2}a^{2}b^{2} \implies$$

$$v^{2}a^{2}b^{2} - x^{2}y^{2}v^{2} = a^{2}b^{2}y^{2} - x^{2}y^{2}b^{2} \implies v^{2} = y^{2}\frac{a^{2}b^{2} - x^{2}b^{2}}{a^{2}b^{2} - x^{2}y^{2}} = y^{2}\frac{x^{2}b^{2} - a^{2}b^{2}}{x^{2}y^{2} - a^{2}b^{2}} \implies$$

$$v = y\sqrt{\frac{x^{2}b^{2} - a^{2}b^{2}}{x^{2}y^{2} - a^{2}b^{2}}}$$

Similarly, we can derive an equation for u in terms x, a, and b

$$u = x \sqrt{\frac{y^2b^2 - a^2b^2}{x^2y^2 - a^2b^2}}$$

For Nowell's Elliptical Grid derivation, he defined the semi-major and semi-minor lengths of the ellipse as

$$a = \sqrt{2 - y^2} \qquad \qquad b = \sqrt{2 - x^2}$$

We can generalize this with a parameter q

$$a = \sqrt{q + 1 - qy^2} \qquad \qquad b = \sqrt{q + 1 - qx^2}$$

Note that when q=1, these equations revert to the ordinary Elliptical Grid mapping. On the other hand, when one approaches the limiting case q=0, we get a=b=1

We can then substitute the values for a and b into the equations for u and v to get these square-to-disc for our parameterized Elliptical Grid mapping.

$$u = x \sqrt{\frac{y^2(q+1-qy^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$

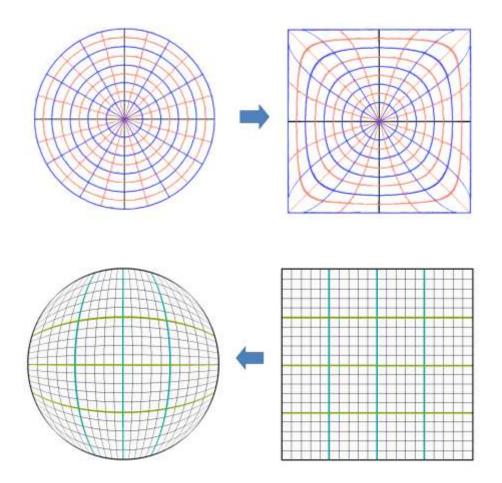
$$v = y \sqrt{\frac{x^2(q+1-qx^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$

The inverse of this squelched Elliptical Grid mapping can be derived using a biquadratic equation similar to section 6.2 to get

$$x = \frac{sgn(u)}{\sqrt{2q}} \sqrt{q + 1 + qu^2 - v^2 - \sqrt{(q + 1 + qu^2 - v^2)^2 - 4q(q + 1)u^2}}$$

$$y = \frac{sgn(v)}{\sqrt{2q}} \sqrt{q + 1 - u^2 + qv^2 - \sqrt{(q + 1 - u^2 + qv^2)^2 - 4q(q + 1)v^2}}$$

## Squelched Elliptical Grid Mapping (q=0.5)



Disc to square mapping:

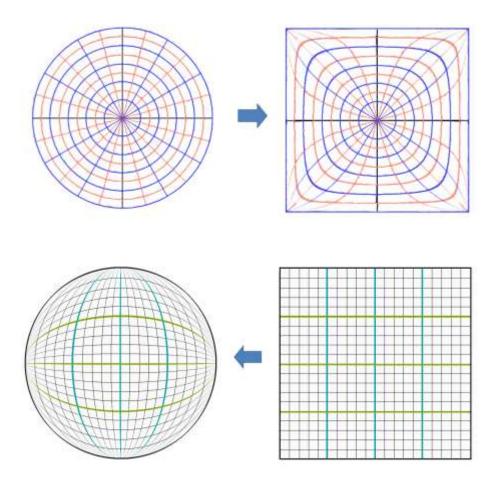
$$x = \frac{sgn(u)}{\sqrt{2q}} \sqrt{q + 1 + qu^2 - v^2 - \sqrt{(q + 1 + qu^2 - v^2)^2 - 4q(q + 1)u^2}}$$

$$y = \frac{sgn(v)}{\sqrt{2q}} \sqrt{q + 1 - u^2 + qv^2 - \sqrt{(q + 1 - u^2 + qv^2)^2 - 4q(q + 1)v^2}}$$

$$u = x \sqrt{\frac{y^2(q+1-qy^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$

$$v = y \sqrt{\frac{x^2(q+1-qx^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$

## Squelched Elliptical Grid Mapping (q=0.00001)



Disc to square mapping:

$$x = \frac{sgn(u)}{\sqrt{2q}} \sqrt{q + 1 + qu^2 - v^2 - \sqrt{(q + 1 + qu^2 - v^2)^2 - 4q(q + 1)u^2}}$$

$$y = \frac{sgn(v)}{\sqrt{2q}} \sqrt{q + 1 - u^2 + qv^2 - \sqrt{(q + 1 - u^2 + qv^2)^2 - 4q(q + 1)v^2}}$$

$$u = x \sqrt{\frac{y^2(q+1-qy^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$

$$v = y \sqrt{\frac{x^2(q+1-qx^2) - (q+1-qx^2)(q+1-qy^2)}{x^2y^2 - (q+1-qx^2)(q+1-qy^2)}}$$