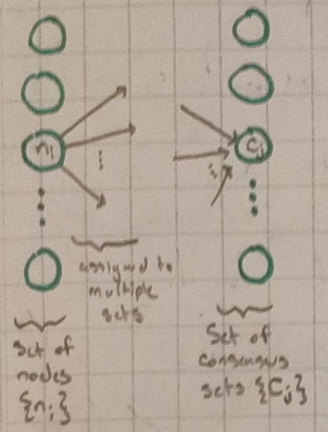
Generalized Fisher Yates

We would like to randomly allocate a set of nodes $\mathcal{N}$ to different consensus sets $\{C\}$ of varying size. We represent this via a directed, bipartite graph:



Here we have a set of nodes $\mathcal{N} = \{n_i\}$ being assigned to different consensus $C = \{C_j\}$.

↳ For a given node $n_i \in \mathcal{N}$, the amount of different sets it is assigned to is given by its degree $\deg(n_i)$ (the number of arrows coming out of n_i). Note that $\dim n_i = 1$ as it is a singular node.

↳ For a given consensus set $C_j \in C$, its size corresponds to its degree: $\dim C_j = \deg(C_j)$

We can also consider the set of arrows connecting elements of $\mathcal{N}$ to elements in C via the following:

$$E = \{(n_i, (C_j, \dots, C_k))\}$$

↳ Node n_i is assigned to the sets $C_j, \dots, C_k$

Formally we can combine $\mathcal{N}, C, E$ to form the structure of a directed, bipartite graph $T = (\mathcal{N}, C, E)$ where $\mathcal{N}, C$ are vertex sets & E is an edge set.

↳ We want an algorithm that permutes the assignment of elements of $\mathcal{N}$ to elements of C. We begin by denoting the information of a vertex as not merely its <u>index</u> but its <u>degree</u>:

$$\left.\begin{array}{l} n_i \longmapsto \deg n_i \\ C_j \longmapsto \deg C_j \end{array} \Rightarrow \begin{array}{l} \mathcal{N} = \{\deg n_i\}_{i \in \mathbb{Z}_+} \\ C = \{\deg C_j\}_{j \in \mathbb{Z}_+} \end{array}\right\}$$

Notice they are now <u>sequences</u> instead of just <u>sets</u> as it is a directed graph

From this we define the edge set E as the set of ordered pairs of vertices:

$$E = \{(\deg n_i, \deg C_j)\}$$

Now, to permute the assignments of different n_i to multiple different C_j, we must permute the degrees of the elements in $\mathcal{N}$

↳ We want: $\{\deg n_1, \dots, \deg n_{|\mathcal{N}|}\} \longmapsto \{\dots, \deg n_{|\mathcal{N}|}, \dots, \deg n_1, \dots\}$

In order to do so, we will need to make use of symmetry groups. In our case of permutations, we need only consider the symmetric group $\mathrm{sym}(\mathcal{N}) \equiv G$ where its elements are all possible bijections of $\mathcal{N}$ to itself.

↳ For $g \in G$, $g: \mathcal{N} \longrightarrow \mathcal{N}$ which is specific permutation:
$$n_i \longmapsto g(n_i)$$

↳ For example: $\mathcal{N} = (n_1, n_2, n_3)$ & $g \in G$ then say $g(n_1) = n_2$, $g(n_2) = n_1$, $g(n_3) = n_3 \Rightarrow \underbrace{g(\mathcal{N}) = (n_2, n_1, n_3)}_{\text{1 possible permutation of } \mathcal{N}}$

The group operation of G is function composition for multiple elements of G. The group action can be denoted as:

$$\phi : G \times \mathcal{N} \longrightarrow \mathcal{N}$$
$$n \longmapsto g(n)$$

Now if we look at a node $n \in \mathcal{N}$ (we drop the indices for now), we can look at all possible different outputs it can be sent to via looking at all possible $g \in G$ acting on it, given by its orbit:

$$\underbrace{\mathrm{orb}(n) = \{g(n) : g \in G\}}_{\text{all possible outputs of } G \text{ acting on } n}$$

This is also denoted as $G \cdot n$ but I prefer this less & will stick to $\mathrm{orb}(n)$

↳ The set of all orbits of $\mathcal{N}$ under the action of G is written as the quotient space:

$$\mathcal{N}/G = \{\mathrm{orb}(n) : n \in \mathcal{N}\}$$

How do we interpret this? First consider $a \in \mathcal{N}$ & its equivalence class:

$$[a] = \{n \in \mathcal{N} : n \sim a\}$$

Here $\sim$ is an equivalence relation which tell you how set elements are equivalent.

Here $\{[a]\}$ form a partition of $\mathcal{N}$ (grouping it into subsets). The partition of $\mathcal{N}$ is given by the set:

$$\mathcal{N}/\sim \; = \{[n] : n \in \mathcal{N}\}$$

Now, back to $\mathcal{N}/G$. Here $\mathcal{N}/G$ is the set of equivalence classes of $\mathcal{N}$ under the group action of G. These are precisely the orbits.

↳ Two elements $[n], [m]$ are equivalent if their orbits are the same: $\text{orb}(n) = \text{orb}(m) \Rightarrow [n] = [m] \Rightarrow n \sim m$

——— x ——— Fisher Yates

Now for a set $\mathcal{N}$, the Fisher Yates algorithm makes use of $G = \text{Sym}(\mathcal{N}) = \{g : \mathcal{N} \to \mathcal{N}\}$ of permutative bijections.

↳ Formally the modern Fisher Yates algorithm is done as:

- Consider for $|\mathcal{N}| = k$, $\mathcal{N} = \{n_1, \dots, n_k\}$ & $\text{Sym}(\mathcal{N}) = G$. We randomly select $n_i \in \mathcal{N}$ & fix $g \in G$ st:

$$g(n_i) = n_k, \quad g(n_k) = n_i, \quad g(n_j) = n_j \quad \text{for } j \neq i$$

This gives us $\phi(g, \mathcal{N}) = \{g(n_1), \dots, g(n_i), \dots, g(n_k)\}$

$$= \{n_1, \dots, n_k, \dots, n_i\}$$

Next, we exclude n_i via $\phi(g, \mathcal{N}) \setminus \{n_i\} \equiv \mathcal{N}'$. Now we select $n_j \in \mathcal{N}'$ & fix $f \in G$ st:

$$f(n_j) = n_{k-1}, \quad f(n_{k-1}) = f(n_j), \quad f(n_l) = n_l \quad \text{for } l \neq j$$

- This gives us $\phi(f, \mathcal{N}') = \phi(f, \phi(g, \mathcal{N}) \setminus \{n_i\}) = \phi_f \circ \phi_g(\mathcal{N}) \setminus \{n_i\}$. We play the same game & remove the new endpoint n_j:

$$\mathcal{N}'' = (\phi_f \circ \phi_g(\mathcal{N}) \setminus \{n_i\}) \setminus \{n_a\}$$

Consider this iteratively $k-1$ times w/ $g, f, \dots \longrightarrow f_1, f_2, \dots$ & $\mathcal{N}', \mathcal{N}'', \dots \longrightarrow \mathcal{N}^{(1)}, \mathcal{N}^{(2)}$ & the ordered removed points $\{n_i, n_a, \dots n_{k-1}\} \to \{n_1, n_2, \dots n_{k-1}\}$

$$\mathcal{N}^{(k-1)} = \phi_{f_{k-1}} \circ \Big(\dots \circ (\phi_1(\mathcal{N}) \setminus \{n_1\}) \setminus \dots \Big) \setminus \{n_{k-1}\}$$

Now that we have the selection set after $k-1$ iterations. To get the final permuted set, you reinclude all ordered removed points back:

$$\mathcal{N} = \mathcal{N}^{(k-1)} \bigcup_{a=1}^{k-1} \{n_a\} \quad \Big\} \text{ } k-1 \text{ iterations using group action } \phi_{k-1}$$

— x — Generalized Fisher Yates

Now we consider using a general group G w/ an action ϕ, meaning there is no previously determined output.

↳ Furthermore, we want to only consider from some allowed transition set. This is the set of allowed possible group actions $\{g\} \in G$ on points $\{n\} \in \mathcal{N}$, which are given by their orbits. For a node $n \in \mathcal{N}$, its orbit is:

$$\mathrm{orb}(n) = \{g(n) : g \in G\}$$

The collection of all node orbits is given by the quotient set:

$$\mathcal{N}/G = \{\mathrm{orb}(n) : n \in \mathcal{N}\}$$

Thus for our considerations we must select elements that belong to the orbits when applying the group action.

↳ This must also overlap w/ $\mathcal{N}$ of current iteration

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For some $\mathcal{N} = \{n_1, \dots, n_N\}$ i.e. for all $|\mathcal{N}| = N$ i.e. under the action of a group G, we consider the endpoint n_N & what it can possibly transition to:

$$\text{orb}(n_N) \cap \mathcal{N}$$

- Notice we select $n_i \in \text{orb}(n_N) \cap \mathcal{N}$ instead of just $\mathcal{N}$ for cases where $\text{orb}(n) \neq \mathcal{N}$ & so can have some cases outside the range of $\mathcal{N}$.
  - ↳ Then we define the swap: $f(n_N) = n_i$, $f(n_i) = n_N$, $f(n_k) = n_k \ \forall k \neq$
  - ↳ Then we remove the last point $f(n_N)$ via $\phi(f, \mathcal{N}) \setminus \{f(n_N)\}$

We play the same game to get a similar result:

$$\mathcal{N} = \mathcal{N}^{(N-1)} \bigcup_{a=1}^{N-1} \{n_a\}$$

—— x —— Removing Degeneracies

- One could consider an isotropy subgroup of G denoted as $\mathcal{G}$ which leave an element of the edge set E invariant
  - ↳ This would mean that for $n_i \longleftrightarrow n_j$, if this means that we require:

$$\Big\{ \underbrace{n_i \neq n_j}_{\substack{\text{distinct} \\ \text{nodes selected}}} \ \& \ \underbrace{g(n_i) \neq g(n_j)}_{\substack{\text{to be swapped} \\ \text{\& allocated into} \\ \text{distinct consensus} \\ \text{sets}}} \Big\}$$

Recall that $\mathcal{N} = \{\deg n_i\}_{i \in \mathbb{Z}_+}$ & $C = \{\deg c_j\}_{j \in \mathbb{Z}_+}$

↳ Thus we want an isotropy subgroup $\mathcal{G}$ that leaves $E(\mathcal{N}, C)$ invariant & we project out degeneracy $n_i = n_j$ & $g(n_i) \neq g(n_j)$

- ↳ We use a reduced group $G/\mathcal{G}$ to thus describe allowed transitions which are now:

$$\text{orb}(n) \cap \mathcal{N} = \{\xi(n) : \xi \in G/\mathcal{G}\} \cap \mathcal{N}$$

— x — Uniform Sampling

To see if the outcomes of selecting n based on $orb(n) \cap \mathcal{N}$ all have equal probability (or not) we can compute the probability of some outcome $\mathcal{N} = \sigma$ as:

$$P(\sigma) = \prod_{a=1}^{k-1} |orb(n_a)|^{-1}$$

Furthermore, we can compute all possible states σ & randomly execute many shuffles & tally up the multiplicity of the states.

↳ We expect $P(\sigma) = 1/N!$ for $G = Sym(\mathcal{N})$ which would be uniformly sampling the bijections $g \in Sym(\mathcal{N})$