

Distributed consensus networks

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Blockchains rely on distributed consensus algorithms to decide whether a proposed transaction is valid and should be added to the blockchain. The purpose of consensus is to act as an independent arbiter for transactions, robust against adversarial manipulation. This can be achieved by choosing random subsets of nodes to form consensus sets. In an economy where consensus is the commodity, consensus must be secure, computationally efficient, fast and cheap. Most current blockchains operate in the context of open networks, where algorithms such as proof-of-work are highly inefficient and resource-intensive, presenting long-term scalability issues. Inefficient solutions to allocating consensus sets equates to high transaction costs and slow transaction times. Closed networks of known nodes afford more efficient and robust solutions. We describe a secure distributed algorithm for solving the random subset problem in networks of known nodes, bidding to participate in consensus for which they are rewarded, where the randomness of set allocation cannot be compromised unless all nodes collude. Unlike proof-of-work, the algorithm allocates all nodes to consensus sets, ensuring full resource utilisation, and is highly efficient. While the protocol is synchronous, a staking mechanism ensures self-synchronisation with no dependence on an external reference. The protocol follows self-enforcing where adversarial behaviour only results in self-exclusion. Signature-sets produced by the protocol act as timestamped, cryptographic proofs-of-consensus, a commodity with market-determined value. The protocol is highly strategically robust against collusive adversaries, affording subnetworks defence against denial-of-service and majority takeover.

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I. INTRODUCTION

Decentralised finance (DeFi) is the longstanding vision of a truly global and borderless financial system, the promise of universal, low-cost access to banking services and international financial markets, eliminating geographic and socio-political barriers of entry and reliance upon centralised financial institutions, subverting

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the power-base of monopolistic and hegemonic access to capital, facilitating a highly competitive global economic landscape.

Smart-contracts enabling arbitrary, self-executing algorithmic exchange facilitate the synthesis of sophisticated financial instruments in the absence of conventional centralised exchanges. The algorithmic versatility of smart-contracts affords their use as a primitive in the construction of sophisticated distributed protocols such as Decentralised Autonomous Organisations (DAOs).

The key characteristics of DeFi are (??):

- Non-custodial: users assert full control over their own assets.
- Open: protocols are borderless and globally accessible.
- Transparency: protocols and algorithms are open-source.
- Composable: code may be modularised and layered, enabling thriving development and innovation of the ecosystem.
- Decentralised: the absence of top-down control and centralisation of power.

In a decentralised environment transactions are improved via *consensus* (Sec. II) whereby a small number of randomly chosen network nodes (*consensus sets*) collectively act as delegates for the network to authorise transactions. Consensus outcomes should with high probability reflect honest outcomes, a function of consensus set size and the ratio of dishonest nodes.

The randomisation of nodes forming consensus is vital to ensure the integrity of consensus outcomes (Sec. III). If the choice of consensus nodes were known in advance this would provide avenues for malicious parties to compromise security by targeting the known allocation. Randomisation effectively erases all assignment information of consensus node assignment, eliminating all avenues for strategic alignment.

The Bitcoin protocol (Nakamoto, 2008) first introduced proof-of-work (PoW) as mechanism for securely randomly selecting a small number of network nodes at random to form consensus (Sec. III.B). This protocol operates in the context of a network of unknown and unidentifiable nodes in which anyone may freely participate. The algorithm requires nodes to compete to solve inverse-hashing problems, a randomised algorithm that effectively chooses winners by lottery.

While the proof-of-work mining algorithm provides an ingenious solution to the random subset problem it is highly inefficient, resulting in enormous net energy consumption to maintain the network. Currently, annualised energy consumption of the Bitcoin network is $\sim 155\text{TWh}$

(Cambridge Centre for Alternative Finance, 2023)¹, comparable to the total electricity consumption of medium sized countries².

The inefficiency of proof-of-work presents a significant obstacle for scalability, motivating the development of more efficient alternate consensus algorithms. Proof-of-stake (PoS) (Bentov *et al.*, 2017; King and Nadal, 2012) is a leading alternative, recently adopted by the Ethereum network to improve scalability and reduce transaction costs. Ethereum’s transition to proof-of-stake, famously known as *The Merge*, reduced its annualised energy consumption from $\sim 21\text{TWh}$ (PoW) to $\sim 2\text{GWh}$ (PoS) (Cambridge Centre for Alternative Finance, 2023)³, a spontaneous reduction in energy consumption of 99.99%. However, despite its radically improved energy efficiency, proof-of-stake has been criticised for affording reduced security owing to its increased vulnerability to manipulation and reduced level of randomisation. Proof-of-work based on quantum sampling problems has also been investigated as a means for enhanced energy efficiency (Singh *et al.*, 2023).

Networks comprising known nodes afford more efficient distributed algorithms for solving the random subset problem. By utilising secure shared randomness (Sec. III.C) all network nodes may be simultaneously allocated to random subsets (Sec. III.D), ensuring full resource utilisation.

The generalisation of the random subset problem is the *consensus assignment problem* (Sec. IV), enabling the simultaneous allocation of network nodes to arbitrary sets of consensus sets under asymmetric load.

Distributed consensus networks (DCNs) combine the consensus assignment problem with an economic model that self-incentivises the honest participation of nodes (Sec. V). The product of DCNs is proofs-of-consensus (PoC) — timestamped, cryptographic proofs that a sets of nodes form secure random subsets of a network established via distributed execution of the consensus assignment problem (Sec. V.D).

DCNs treat consensus as an abstract market commodity, an enabler of generic digital trade. The internal operation of DCNs is inherently non-monetary, a barter economy in which nodes contribute to consensus load in equal exchange for the consensus load they request. Nodes act as gateways to the DCNs to which they belong, and may independently utilise and monetise consensus load, collectively facilitating an externally-facing competitive bidding market for consensus (Sec. VII).

Quantum consensus networks (QCNs) (Sec. VI) have

¹ Estimate as of January 19, 2024.

² Estimated annual electricity consumption of Sweden (2022): $\sim 164\text{TWh}$.

³ Estimates immediately prior and subsequent to *The Merge* on September 15, 2022.

quantum-enabled nodes, possessing quantum computational or communications resources, acting as secure, certifiable quantum random number generators (QRNGs) upon forming consensus, a commodity with market value. Other (classical) DCNs may observe QCNs as oracles, utilising their random bitstreams to add entropy (Sec. VI.A) to their own global keys, enabling quantum-random consensus assignment.

Existing blockchain design philosophy couples cryptocurrencies with the technological implementation of their mechanism for trade, a vertically integrated paradigm which frustrates inter-ledger exchange (Sec. X). The hard-wiring of currencies with specific consensus algorithms couples their cost of execution with monetary dynamics. The abstract separation between currencies and their mechanism for trade enables horizontal competition between currencies, likewise between competitors offering consensus as a commodity.

Sec. VIII

* Detached from specific

Sec. IX

II. CONSENSUS

The purpose of consensus is to provide a decentralised mechanism for sets of parties to collectively act as an independent arbiter in judging and signing off on the validity of statements. In the context of blockchains, this is employed to determine whether a newly submitted transaction block is legitimate and should be added to the blockchain.

In an environment where a minority subset of parties are dishonest and conspiring to form false consensus, the purpose of consensus protocols is to uphold the will of the majority, independent of the behaviour of collusive, dishonest parties. As dishonest parties are assumed to be in the minority, this can always be achieved by ensuring that all parties are involved in consensus. However, this is highly resource-intensive and consensus needn't involve all parties. A subset of parties suffices to form consensus if their decision reflects the will of the majority, enabling them to act as delegates.

Choosing a subset of parties to form consensus there is some probability of dishonest parties forming a false majority by chance. Consensus sets should therefore be chosen to upper-bound this probability, independent of the strategy employed by dishonest parties.

Consensus is formed by majority vote on the validity of some statement (id) by members of a consensus set (\mathcal{C}) chosen from a pool of consensus nodes in a network (\mathcal{N}),

$$\mathcal{C} \subseteq \mathcal{N}. \quad (2.1)$$

The majority vote of the entire network is a deterministic decision function defining the *source of truth*,

$$\text{MAJORITYVOTE}(\mathcal{N}, \text{id}) \rightarrow \{0, 1\}, \quad (2.2)$$

which consensus sets must uphold. Based on their statistical composition, the majority votes of consensus sets are in general probabilistic,

$$\text{MAJORITYVOTE}(\mathcal{C}, \text{id}) = \begin{cases} \Pr(1 - P_c), & \text{MAJORITYVOTE}(\mathcal{N}, \text{id}) \\ \Pr(P_c), & \neg \text{MAJORITYVOTE}(\mathcal{N}, \text{id}) \end{cases},$$

where P_c is the probability of compromise, whereby \mathcal{C} comprises a majority of dishonest parties attempting to subvert honest outcomes.

For $P_c = 0$ where an honest majority is guaranteed, this reduces to the deterministic case as per Eq. (2.2). The purpose of distributed consensus algorithms is to choose consensus sets \mathcal{C} operating in this regime to a close approximation.

III. RANDOM SUBSET PROBLEM

The *random subset problem* is to choose a random subset A uniformly from B , where $A \subseteq B$, of which there are $\binom{|B|}{|A|}$ combinations. If the proportion of parties acting dishonestly and collusively is r , the probability a random

subset of size N contains at least k dishonest parties is,

$$\begin{aligned} P(N, k, r) &= \sum_{n=k}^N \binom{N}{n} r^n (1-r)^{N-n} \\ &= I_r(k, N - k + 1), \end{aligned} \quad (3.1)$$

where $I_x(a, b)$ is the regularised incomplete beta function. Compromising vote integrity via false-majority requires $k = \lfloor N/2 \rfloor + 1$. Hence, the likelihood of compromise is,

$$P_c(N, r) = I_r(\lfloor N/2 \rfloor + 1, N - \lfloor N/2 \rfloor). \quad (3.2)$$

While in principle all parties are associated with independent likelihood of compromise, r_i , choosing subsets uniformly at random ensures the probabilities of set members being dishonest are independently and identically sampled with the average-case probability r . Hence,

P_c is independent of how dishonest nodes are distributed and what strategies they might employ (Fig. 1).

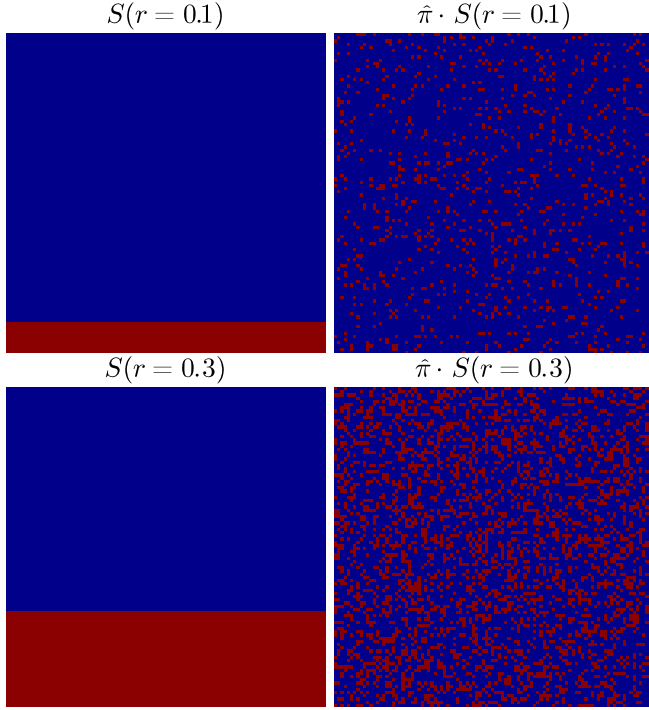


Figure 1: Random sampling as a defence against strategic alignment. (left) A minority of strategically aligned nodes may form false-majority if their alignment overlaps with a known allocation for consensus sets. (right) Random sampling erases strategic alignment and ensures node honesty is sampled with average-case r .

To uphold the integrity of majority votes, we require the likelihood of a minority gaining false-majority by chance to be upper-bounded by an exponentially small threshold $\varepsilon = 2^{-\lambda}$,

$$P_c(N, r) \leq \varepsilon, \quad (3.3)$$

for statistical security parameter $\lambda > 1$, a subjective parameter which may be application- or user-dependent.

We define the required consensus set size, N_{\min} , as the smallest set size satisfying this inequality,

$$N_{\min}(r, \varepsilon) = \arg \min_{N \in \mathbb{Z}^+} (P_c(N, r) \leq \varepsilon). \quad (3.4)$$

For consensus to maintain ε -security, consensus sets should never be smaller than N_{\min} . Simultaneously, choosing consensus sets larger than N_{\min} is unnecessary and wasteful. Tradeoff curves for $N_{\min}(r, \varepsilon)$ are shown in Fig. 2.

A. Centralised algorithm

The random subset problem has a trivial centralised solution (Alg. 1). We assign unique random bit-strings to

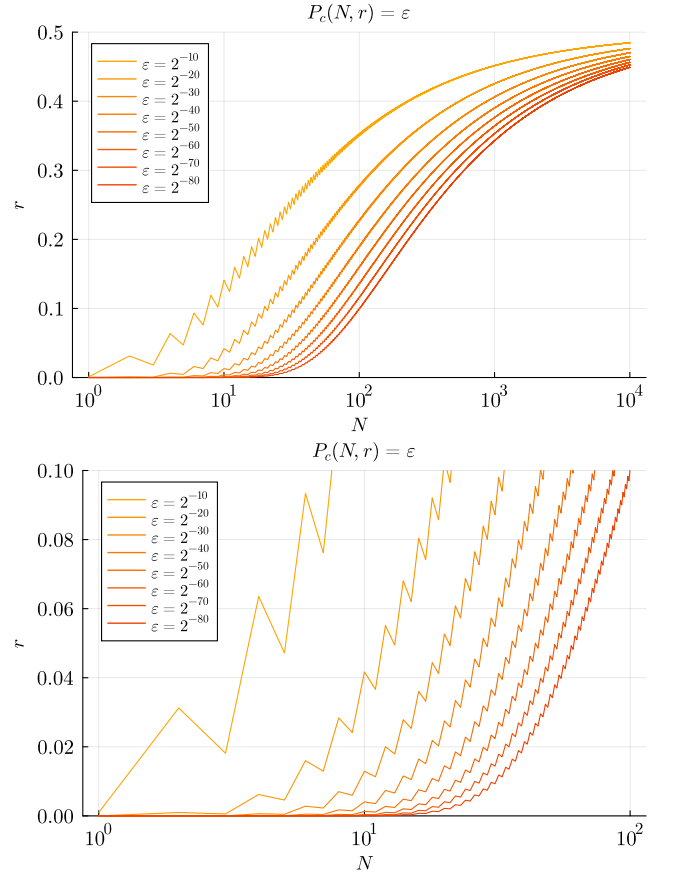


Figure 2: Security tradeoffs with consensus set size. $P_c(N, r) \leq \varepsilon$ is the probability that dishonest nodes within a random subset of size N form a false-majority when the proportion of dishonest nodes is r , where $\varepsilon = 2^{-\lambda}$ is the statistical security parameter. The oscillatory artefacts are associated with the non-linearity of $\lfloor N/2 \rfloor$ for even- and odd-valued N .

each node, order them by their random number, and partition the ordered list piecewise into units of length N_{\min} . These partitions form non-intersecting random subsets, each defining an independent consensus set.

Algorithm 1: Centralised algorithm for the random subset problem, assigning a set of nodes \mathcal{S} to a set of independent random subsets $\{\mathcal{C}\}$ of given sizes $\{|\mathcal{C}|\}$.

```

procedure RANDOMSUBSETS( $\mathcal{S}$ ,  $\{|\mathcal{C}|\}$ )  $\rightarrow \{\mathcal{C}\}$ 
  ▷ Assign a random number to each node
  for  $i \in \mathcal{S}$  do
     $\text{random}_i \leftarrow \text{RANDOM}(\{0, 1\}^n)$ 
  ▷ Sort nodes by their random number
   $\text{ordered} = \text{SORT}(\mathcal{S}, \text{random})$ 
  ▷ Partition the list into consensus sets
   $\{\mathcal{C}\} = \text{PARTITION}(\text{ordered}, \{|\mathcal{C}|\})$ 
  return  $\{\mathcal{C}\}$ 

```

The challenge is to securely implement this algorithm in a decentralised environment, such that nodes are unable to compromise the randomness of the assignments

of consensus sets.

B. Proof-of-work

* Hashcash: (Back, 2002; Dwork and Naor, 1993)

Proof-of-work has been widely employed in blockchains such as Bitcoin (Nakamoto, 2008) as a distributed protocol for choosing random subsets of nodes. Here, nodes compete to find satisfying inputs to hash functions whose outputs satisfy a constraint dictating the likelihood of success. This distributed algorithm effectively asks nodes to find solutions to randomised problems with very low probability of success, $p_{\text{mine}} \ll 1$, such that winners are randomly allocated across nodes in the network.

Since hash functions are pseudo-random and exhibit pre-image resistance⁴, the only viable approach to finding such solutions is via brute-force, repeatedly hashing random bit-strings until a satisfying input is found. Winning nodes are hence allocated at random and cannot be spoofed under a computational hardness assumption. The distributed algorithm for proof-of-work is shown in Alg. 2.

Algorithm 2: Random subsets via proof-of-work. Nodes \mathcal{S} hash random bit-strings, salted by a problem instance specified by the previous block id , where the per-hash success rate is p_{mine} .

```

procedure PROOFOfWork( $\mathcal{S}, \text{id}, \text{size}$ )  $\rightarrow \mathcal{C}$ 
   $\mathcal{C} = \{\}$   $\triangleright$  Consensus set
  while  $|\mathcal{C}| < \text{size}$  do
    for  $i \in \mathcal{N}$  do  $\triangleright$  In parallel
       $\text{random}_i = \text{RANDOM}(\{0, 1\}^n)$ 
       $\text{output}_i = \text{HASH}(\text{id} \parallel \text{random}_i)$ 
      if  $\text{VALID}(\text{output}_i)$  then
         $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{S}_i$ 
  return  $\mathcal{C}$ 

procedure VALID( $x \in \{0, 1\}^n$ )  $\rightarrow \{0, 1\}$ 
  if  $x/2^n \leq p_{\text{mine}}$  then
    return true
  else
    return false

```

Proof-of-work has no requirement that nodes be known or trusted, providing a very general protocol suited to open networks in which anyone can participate. However, while affording a distributed solution to the random subset problem, this approach is inherently wasteful. The expected mining rate is,

$$R_{\text{mine}} = p_{\text{mine}} \cdot R_{\text{hash}}, \quad (3.5)$$

where R_{hash} is the network's net hash-rate⁵. To inhibit the formation of multiple, simultaneous consensus sets (double-mining), potentially manifesting itself as forks in the blockchain, proof-of-work systems may introduce *friction* by algorithmically adjusting the *difficulty parameter* to ensure consistent mining times. To achieve constant mining time, difficulty must scale with cumulative network hash-rate, making the distributed algorithm less efficient as network size grows. For example, maintaining constant mining rate implies efficiency scales inversely with network size,

$$p_{\text{waste}} = 1 - O\left(\frac{1}{n}\right), \quad (3.6)$$

asymptotically perfectly inefficient.

C. Secure shared randomness

An environment comprising a network of known nodes affords a more efficient solution to the random subset problem. The key observation is to utilise *secure shared randomness* to randomly permute and partition sets of nodes as per Alg. 1. The source of shared randomness should be secure, in the sense that its randomness be robust against manipulation by dishonest nodes.

Secure shared randomness can be achieved by first requiring network nodes (\mathcal{N}) to present transaction id s whose hashes act as unique random numbers,

$$\mathcal{X}_i = \text{HASH}(\text{id}_i). \quad (3.7)$$

A global key is then defined as,

$$\mathcal{X}_{\mathcal{N}} = \bigoplus_{i \in \mathcal{N}} \mathcal{X}_i, \quad (3.8)$$

our shared random source, where \oplus denotes the bitwise XOR operation (addition modulo 2).

Under the rules of entropy addition (Sec. VI.A), the randomness of the global key, $\mathcal{X}_{\mathcal{N}}$, cannot be compromised unless all \mathcal{X}_i are compromised.

D. Secure random subsets

From a secure global key, $\mathcal{X}_{\mathcal{N}}$, as per Eq. (3.8) the random subset problem affords a simple solution. Individual keys are assigned to each node,

$$K_i = \text{HASH}(\mathcal{X}_i \parallel \mathcal{X}_{\mathcal{N}}). \quad (3.9)$$

Ordering and partitioning nodes by their associated $\{K_i\}$ assigns them to consensus sets as per Alg. 1.

⁴ For $\text{HASH}(x) \rightarrow y$ it is computationally hard to find a satisfying input $x \in \{0, 1\}^*$ for given output $y \in \{0, 1\}^n$.

⁵ Bitcoin's cumulative network hash-rate was $\sim 500\text{EH/s}$ ($500 \times 10^{18}\text{H/s}$) as of January, 2024 (YCharts, 2024).

Multiple random subset allocations may be derived from a single global key by rehashing individual keys,

$$K_i^{(n)} = \text{HASH}^n(\mathcal{X}_i | \mathcal{X}_{\mathcal{N}}), \quad (3.10)$$

where $K_i^{(n)}$ denote keys for the n th round.

While nodes may choose their contributed ids, thereby influencing the established random source, $\mathcal{X}_{\mathcal{N}}$, this does not provide an avenue for compromising the integrity of random subset assignment. Adversarial nodes attempting to compromise ε -security by manipulating $\mathcal{X}_{\mathcal{N}}$ must find the necessary satisfying pre-images id_i . Searching over $\text{poly}(\lambda)$ pre-images, each associated with a unique random instance of $\mathcal{X}_{\mathcal{N}}$, preserves exponentially decreasing likelihood of compromise,

$$\varepsilon' \leq \frac{\text{poly}(\lambda)}{2^\lambda}. \quad (3.11)$$

This approach necessarily assumes the network comprises a known set of nodes, as the global key $\mathcal{X}_{\mathcal{N}}$ cannot be established from an undefined set, nor can an undefined set be ordered.

IV. CONSENSUS ASSIGNMENT PROBLEM

The random subset problem randomly partitions network space into consensus sets, imposing a subset-sum constraint on consensus set sizes. In the context of symmetric load where all nodes request $|\mathcal{C}|$ consensus load the net consensus load is,

$$L = |\mathcal{C}| \cdot |\mathcal{N}|. \quad (4.1)$$

This requires performing $|\mathcal{C}|$ independent re-partitionings within each of which every node will be assigned exactly once. Under this allocation every node both contributes and requests $|\mathcal{C}|$ units of work.

The *consensus assignment problem* generalises the random subset problem to accommodate arbitrary consensus set sizes and asymmetric load, allowing nodes to contribute and request arbitrary consensus workload, a prerequisite for enabling a free consensus market.

Representing consensus assignment via *consensus assignment graphs* (Sec. IV.A) reflecting the contributed and requested load by nodes and consensus sets, the algorithm uniformly samples from the space of satisfying assignments (Sec. IV.C), achieved by defining consensus assignment via its algebraic group structure (Sec. IV.B) and uniformly sampling the group action acting on an initial graph assignment (Sec. IV.C.5), where the pseudo-random algorithm is seeded (Sec. IV.C.4) by a secure random variable established by the network (Sec. III.C) to ensure consistent evaluation in a distributed context. The distributed algorithm for the consensus assignment problem is shown in Alg. 3.

Algorithm 3: Distributed algorithm for the consensus assignment problem. A secure shared random variable (Sec. III.C), $\mathcal{X}_{\mathcal{N}}$, collectively established by the network, \mathcal{N} , seeds pseudo-random sampling (Sec. IV.C) over consensus assignments, E , satisfying the consensus assignment graph constraints, (U, V, d) (Sec. IV.A).

procedure DISTCONSENSUSAMPLING(\mathcal{N}, U, V, d) $\rightarrow E$
 $\mathcal{X}_{\mathcal{N}} \leftarrow \text{SECURESHARED RANDOMNESS}(\mathcal{N})$
 $E \leftarrow \text{INITIALASSIGNMENTGRAPH}(U, V, d)$
 $E \leftarrow \text{CONSENSUSSHUFFLE}(\mathcal{X}_{\mathcal{N}}, E)$
return E

A. Consensus assignment graphs

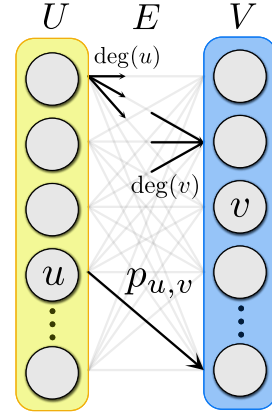


Figure 3: Generalised consensus assignment problem. Consensus assignment may be represented as a directed bipartite graph from a space of network nodes, $U \equiv \mathcal{N}$, to consensus sets, $V \equiv \{\mathcal{C}\}$, where vertex degree represents contributed/consumed consensus load. The set of all satisfying edge-sets $\{E\}$ represents the space of valid assignments. Under sampling, $p_{u,v}$ is the probability of node u being assigned to consensus set v .

Consider a directed bipartite graph (Fig. 3),

$$\mathcal{G} = (U, V, E), \quad (4.2)$$

where vertices $u \in U$ and $v \in V$ represent network nodes (\mathcal{N}) and consensus sets ($\{\mathcal{C}\}$) respectively, and the directed edge set E defines the assignment of nodes to consensus sets where each edge is associated with a unit of consensus work. The constraint that nodes may be assigned at most once to a given consensus set imposes that G is a simple graph.

Consensus load may be arbitrarily partitioned and nodes may make multiple requests for consensus sets, hence in general $|\mathcal{S}| \neq |\{\mathcal{C}\}|$.

Vertex degrees represent workload, where $\deg(u)$ denotes work contributed by node u and $\deg(v)$ the load requested by consensus set v . The degree sum formula for bipartite graphs equates to a conservation of work

constraint in the system,

$$\sum_{u \in U} \deg(u) = \sum_{v \in V} \deg(v) = |E|, \quad (4.3)$$

where $|E|$ represents net consensus work.

The assignment of network nodes to consensus sets is equivalent to assigning edge-sets E subject to the constraints imposed by the degree sequence,

$$d = (\deg(u_1), \dots, \deg(u_{|U|}), \deg(v_1), \dots, \deg(v_{|V|})), \quad (4.4)$$

where,

$$|d| = |U| + |V| = |\mathcal{N}| + |\{\mathcal{C}\}|. \quad (4.5)$$

Expressing \mathcal{G} via its edge-set,

$$\mathcal{G} = \{(u_e \in U, v_e \in V)\}_{e \in E}, \quad (4.6)$$

where the subscript notation u_e (v_e) denotes the node (consensus set) number associated with edge $e \in E$, elements in the two columns have multiplicity given by their degree.

B. Consensus group

* Clarify edge transformations rather than U and V .

Transformations over the space of satisfying graph realisations for given a degree sequence d defines the *consensus group*, $C(d)$. The consensus group is fully characterised by its degree sequence, d . Over the space of satisfying edge-sets, $\{E\}$, the consensus group,

$$\phi : C(d) \times \{E\} \rightarrow \{E\}, \quad (4.7)$$

is identically the symmetric group,

$$\phi : C(d) = \text{Sym}(\{E\}). \quad (4.8)$$

The consensus group may be equivalently defined over the Cartesian product vertex space $U \times V$,

$$\psi : C(d) \times (U \times V) \rightarrow (U \times V), \quad (4.9)$$

the group of non-degenerate vertex permutations which discounts transformations under which edge sets E remain invariant, a subgroup of the symmetric group over vertex space,

$$(\psi, U \times V) \subseteq \text{Sym}(U) \times \text{Sym}(V). \quad (4.10)$$

The group actions of the consensus group defined over vertex- and edge-space are isomorphic,

$$(\phi, \{E\}) \cong (\psi, U \times V). \quad (4.11)$$

While the group action ψ is defined over the joint vertex space of both partitions, all group elements may be realised using permutations over either partition alone,

$$(\psi, U \times V) \cong (\psi, U) \cong (\psi, V). \quad (4.12)$$

In the special case where degree sequences are ones-vectors, $d = (1, \dots, 1)$, the consensus assignment graph, $\mathcal{G}(d)$, defines a bijective map across partitions, $U \xrightarrow{\text{bij}} V$ (where $|U| = |V|$), and the group action over vertex space is identically the symmetric group,

$$\psi : C(d = (1, \dots, 1)) = \text{Sym}(U) = \text{Sym}(V), \quad (4.13)$$

as there are no degeneracies associated with non-unit vertex degrees.

The order of the consensus group, equivalently the number of unique consensus assignments, is given by the multinomial,

$$\begin{aligned} |C(d)| &= \frac{|E|!}{\prod_{u \in U} \deg(u)! \cdot \prod_{v \in V} \deg(v)!} \\ &= \binom{|E|!}{d_1, \dots, d_{|U|+|V|}} \\ &\leq |E|!, \end{aligned} \quad (4.14)$$

reflecting the set of node permutations discounted by their degeneracies, upper-bounded by the order of the symmetric group $|S_n| = n!$ with no degeneracies.

Employing element-wise inequality notation for degree sequences of equal length, $|d| = |d'|$,

$$d' < d := d'_i < d_i, \forall i, \quad (4.15)$$

affords the equivalent relations between degree sequences, assignment graphs and subgroup structures,

$$\begin{aligned} d' < d &\Rightarrow \mathcal{G}(d') \subset \mathcal{G}(d) \\ &\Rightarrow C(d') \triangleleft C(d), \end{aligned} \quad (4.16)$$

where $C(d')$ is a normal subgroup of $C(d)$ and $\mathcal{G}(d')$ is a subgraph of $\mathcal{G}(d)$ under edge-removal.

As edge exchange operations are symmetric across U and V the consensus group may equivalently be associated with the space of edge permutations on either. Despite having support over vertex sets with distinct degree sequences they afford the same space of edge permutations.

The number of bits required to uniquely address the group's orbit scales as (Fig. 4),

$$n = \log_2(|\text{orb}_C(E)|) \leq \log_2(|E|!). \quad (4.17)$$

C. Uniform consensus sampling

Secure consensus assignment requires assigning edge-sets uniformly at random over the space of all satisfying

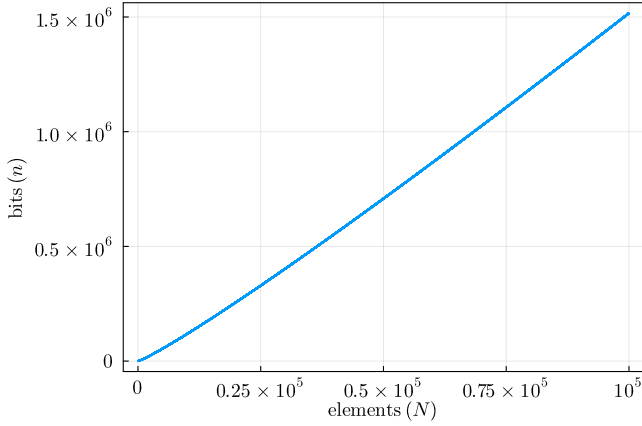


Figure 4: Number of bits (n) required to encode an arbitrary permutation (S_N) over N elements. This upper-bounds the required number of bits to uniquely address elements of the consensus group, $C(d)$.

edge-sets. Uniformity implies maximisation of entropy, given by,

$$H(C(d)) = \log_2(|C(d)|), \quad (4.18)$$

where $|C(d)|$ is the order of the respective consensus group. Degeneracies in permutations result in sampling bias thereby reducing entropy, requiring that algorithmic implementation does not double-count degenerate permutations from the symmetric group.

1. Fisher-Yates shuffle

The Fisher-Yates shuffle (Fisher and Yates, 1953) is an efficient algorithm for applying in-place permutations $\pi \in S_n$ to an array of n elements uniformly at random (Alg. 4).

The algorithm iteratively allows every array index to randomly exchange itself with any element not already assigned. Assuming $O(1)$ random number generation the Fisher-Yates shuffle exhibits $O(n)$ time-complexity. As the operational primitive is the exchange operation the algorithm permutes arrays in-place.

The sequence of $O(n)$ random numbers chosen during execution defines a decision tree whose i th level assigns the i th array index. Individual execution paths correspond to individual permutations with a one-to-one correspondence. Hence, the algorithm's execution space \mathcal{L}_n is isomorphic to the symmetric group,

$$\mathcal{L}_n \cong S_n, \quad (4.19)$$

under mapping between execution paths $l \in \mathcal{L}_n$ and permutations $\pi \in S_n$. As all execution paths occur with uniform probability $1/n!$ the algorithm uniformly samples from the symmetric group.

Algorithm 4: The Fisher-Yates shuffle algorithm for applying a random permutation $\pi \in S_{|\vec{v}|}$ to the elements a vector \vec{v} . The algorithm permutes vectors in-place with $O(n)$ runtime assuming an $O(1)$ RANDOM(\cdot) function.

```

procedure FISHERYATESSHUFFLE( $\vec{v}$ )  $\rightarrow \vec{v}$   $\triangleright O(n)$ 
  for  $i \leftarrow |\vec{v}| - 1$  to 1 do  $\triangleright O(n)$ 
     $j \leftarrow \text{RANDOM}(0 \leq j \leq i)$   $\triangleright O(1)$ 
     $v_i \leftrightarrow v_j$   $\triangleright \text{Exchange}$ 
  return  $\vec{v}$ 

```

The uniqueness of execution paths may be seen by noting that an element at index $j \leq i$ has exactly one path to be reassigned to index i , via the direct exchange $v_i \leftrightarrow v_j$ when the loop reaches index i .

The execution path $l \in \mathcal{L}_n$ of the algorithm directly relates to the cycle decomposition of the respective group element. As the algorithm iterates through i the series of exchange operations defines a path. When a trivial $v_i \leftrightarrow v_i$ exchange occurs it terminates that path, closing it as a cycle, after which the next i opens a new one.

Closely related to the Fisher-Yates shuffle is Sattolo's algorithm (Sattolo, 1986) for uniformly sampling the cyclic group (C_n) differing only from the Fisher-Yates shuffle in the set of allowed exchanges,

$$\begin{aligned} \text{Fisher-Yates } (S_n) : 0 \leq j \leq i, \\ \text{Sattolo } (C_n) : 0 \leq j < i. \end{aligned} \quad (4.20)$$

Note that this distinction serves to prohibit trivial exchanges ($v_i \leftrightarrow v_i$) thereby forcing all execution paths to define single cycles.

2. Generalised Fisher-Yates shuffle for finite groups

The Fisher-Yates shuffle may be generalised to uniformly sample over other finite groups (Alg. 5).

Alg. 4 may be interpreted as follows: for every index i the associated value v_i is chosen uniformly from the set of unassigned values v_j (where $0 \leq j \leq i$). More generally, we would choose v_i uniformly from the set of elements it is allowed to transition to under the action of the respective group. For the symmetric group, S_n , this includes all elements, whereas other groups in general constrain the set of allowed transitions.

Consider a group G defined over set X ,

$$\phi : G \times X \rightarrow X. \quad (4.21)$$

Elements of the set $x \in X$ individually transform under the group action of G to their orbit,

$$\begin{aligned} \text{orb}_G(x) &= G \times x \\ &= \{y \in X \mid \exists g \in G \mid y = g \cdot x\} \end{aligned} \quad (4.22)$$

defining sets for the allowed transitions of x under the action of G . For the symmetric group we have,

$$\text{orb}_{S_n}(x) = X, \quad (4.23)$$

as all elements may map to all others, while for other groups the orbit of x may be a subset of elements.

We define the *transition set*, t , to be the set of allowed transitions of x under the group action of G , given by the intersection unassigned elements (s) and the orbit of x ,

$$t = \text{orb}_G(x) \cap s. \quad (4.24)$$

Algorithm 5: Generalised Fisher-Yates shuffle for finite groups G .

```

procedure GROUPSHUFFLE( $\vec{v}$ ,  $G$ )  $\rightarrow \vec{v}$   $\triangleright O(n^2)$ 
  for  $i \leftarrow |\vec{v}| - 1$  to 1 do  $\triangleright O(n)$ 
     $s \leftarrow \{\vec{v}_j \mid 0 \leq j \leq i\}$   $\triangleright \text{Unassigned elements}$ 
     $t \leftarrow \text{TRANSITIONSET}(\vec{v}_i, s, G)$   $\triangleright O(n)$ 
     $j \leftarrow \text{RANDOM}(t)$   $\triangleright O(1)$ 
     $v_i \leftrightarrow v_j$   $\triangleright \text{Exchange}$ 
  return  $\vec{v}$ 

```

```

procedure TRANSITIONSET( $x, s, G$ )  $\rightarrow t$   $\triangleright O(n)$ 
  return  $(G \times x) \cap s$   $\triangleright \text{Group action on reduced set}$ 

```

The uniqueness of execution paths in the generalised algorithm follows the same argument as for the original scheme. The probability associated with an execution path is,

$$P(\vec{d}) = \prod_{i=1}^n p(\vec{d}_i) = \prod_{i=1}^n \frac{1}{|t_i|} = \prod_{i=1}^n \frac{1}{|\text{orb}_{G^{(i)}}(x_i)|}, \quad (4.25)$$

where $\vec{d}_i = 1/|t_i|$ is the probability of making choice $j = \vec{d}_i$ at level i under uniform sampling, and $G^{(i)}$ denotes the i th level subgroup of $G = G^{(n)}$ acting on the reduced set predicated on the removal of already assigned elements of X ,

$$\begin{aligned}
 G^{(i)} \times X^{(i)} &\rightarrow X^{(i)}, \\
 X^{(i)} &= X \setminus \{x_k\}_{k>i}, \\
 G^{(i-1)} &\triangleleft G^{(i)},
 \end{aligned} \quad (4.26)$$

defining a composition series followed by the algorithm to iteratively assign set elements,

$$1 = G^{(0)} \triangleleft \dots \triangleleft G^{(n-1)} \triangleleft G^{(n)} = G, \quad (4.27)$$

where each subgroup acts on the set predicated on the removal of an element from the set upon which the supergroup acts.

As the consensus group is both free⁶ and transitive⁷ it

⁶ Free groups: all elements are invariant under the action of all group elements except the identity,

$$g \cdot x = x \Rightarrow g = e.$$

⁷ Transitive groups: all elements $x \in X$ map to all elements $y \in X$ under the action of some group element $g \in G$,

$$x, y \in X \mid \exists g \in G \mid g \cdot x = y.$$

has regular⁸ group action and all group elements have the same orbit. Hence, $|t_i|$ is level-dependent on i but independent of group element $x \in X$ and execution pathway $l \in \mathcal{L}$. $P(\vec{d})$ is therefore a function of the group but uniform across all group elements.

3. Uniformly sampling the consensus group

Uniformly sampling the consensus group is achieved by defining transition sets as per Alg. 6, whereby allowed transitions under edge exchanges are defined by the Boolean non-degeneracy constraint,

$$(u_i \neq u_j) \odot (v_i \neq v_j) = 1, \quad (4.28)$$

where \odot denotes logical XNOR. This constraint is equivalent to,

$$[(u_i \neq u_j) \wedge (v_i \neq v_j)] \vee [(u_i = u_j) \wedge (v_i = v_j)] = 1, \quad (4.29)$$

where the first clause allows exchange operations between nodes u_i and u_j , equivalently between consensus sets v_i and v_j , only when distinct (Fig. 5), and the second clause is equivalent to $i = j$, ensuring inclusion of u_i itself.

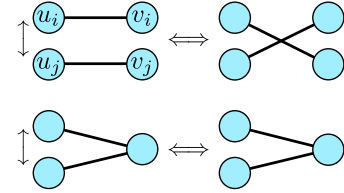


Figure 5: Degeneracy rule for vertex exchanges in the consensus group. Allowed vertex exchanges $u_i \leftrightarrow u_j$ (equivalently $v_i \leftrightarrow v_j$) require vertices in both the U and V partitions of the bipartite graph to be distinct, as per Eq. (4.29).

Algorithm 6: The transition set for the consensus group may be characterised by the constraints imposed by the group relations characterising non-degenerate transitions. The TRANSITION function is a binary operator specifying whether index j is an allowed transition for index i , defining the respective transition set.

```

procedure CONSENSUSHUFFLE( $\mathcal{X}$ ,  $\vec{u}$ ,  $\vec{v}$ )  $\rightarrow \vec{v}$   $\triangleright O(n^2)$ 
  for  $i \leftarrow |\vec{v}| - 1$  to 1 do  $\triangleright O(n)$ 
     $t \leftarrow \{0 \leq k \leq i \mid \text{TRANSITION}(i, k) = 1\}$   $\triangleright O(n)$ 
     $j \leftarrow \text{RANDOM}(\mathcal{X}, t)$   $\triangleright O(1)$ 
     $v_i \leftrightarrow v_j$   $\triangleright \text{Exchange}$ 
  return  $\vec{v}$ 

```

```

procedure TRANSITION( $i, j$ )  $\rightarrow \{0, 1\}$   $\triangleright O(1)$ 
  return  $(u_i \neq u_j) \odot (v_i \neq v_j)$   $\triangleright \text{Non-degeneracy rule}$ 

```

⁸ Regular group action: $\forall x, y \in X \mid g \cdot x = y, g \in G$ is unique.

The regular action of the consensus group implies that under symmetrisation via uniform sampling all satisfying edge assignments $\{E\}$ occur with equal probability, from which it follows,

$$\begin{aligned} p_{u,v} &= \frac{\deg(u)}{|V|}, \\ \sum_{v \in V} p_{u,v} &= \deg(u), \\ \sum_{u \in U} p_{u,v} &= 1, \\ \sum_{u \in U, v \in V} p_{u,v} &= |E|, \end{aligned} \quad (4.30)$$

where $p_{u,v}$ is the probability of node u being assigned to consensus set v . * Regular action.

* Use uniform bidding to preserve average case r . If non-uniform the effective r in consensus sets is biased towards the r of higher bidders.

* ??? * For a given edge $e = (u, v)$ in edge-set $e \in E$, the order of the orbit of vertex v under the group action ψ is,

$$\psi : |\text{orb}_C(v)| = |V/v| = |V| - 1. \quad (4.31)$$

That is, vertex v may permute to any vertex other than itself.

* ??? TODO. Fig. 3

4. Sampling via random bitstreams

As nodes must be in agreement on consensus assignment the RANDOM functions relied upon during execution of Alg. 6 must evaluate consistently across network nodes, for which the established secure random bitstream \mathcal{X}_N (Sec. III.C) is employed as per Alg. 7.

Algorithm 7: Deterministically choose an element x from n choices where \mathcal{X} is a secure shared random bitstream. At most $|\mathcal{X}| \leq \lceil \log_2(n) \rceil \cdot \log_2(1/\delta)$ bits are required to ensure success with probability $p \geq 1 - \delta$.

```

procedure RANDOM( $\mathcal{X}, n$ )  $\rightarrow x$ 
  repeat
     $b \leftarrow \lceil \log_2(n) \rceil$   $\triangleright$  Minimum required number of bits
     $x \leftarrow \mathcal{X}.\text{pop}(b)$   $\triangleright$  Pop bits from bitstream
  until  $x < n$   $\triangleright$  Until  $x$  is within bounds
  return  $x$ 

```

While this function is deterministic it is not guaranteed to halt if $n \neq 2^b$ where $b \in \mathbb{Z}^+$. The success probability for a single iteration of the repeat block in Alg. 7 is,

$$p_1(n) = \frac{n}{2^{\lceil \log_2(n) \rceil}} \geq \frac{1}{2}, \quad (4.32)$$

where the worst-case lower-bound of $p = 1/2$ arises when $n = 2^b + 1$ for $b \gg 1$,

$$\begin{aligned} p_1(2^b + 1) &= \frac{1}{2} + \frac{1}{2^{b+1}}, \\ \lim_{b \rightarrow \infty} p_1(2^b + 1) &= \frac{1}{2}. \end{aligned} \quad (4.33)$$

With m rounds the success probability is,

$$p_m(n) = 1 - (1 - p_1)^m > 1 - \frac{1}{2^m}. \quad (4.34)$$

Limiting the probability of failure to $\delta = 1/2^m$ requires at most,

$$m \geq \log_2(1/\delta), \quad (4.35)$$

rounds. Hence, the worst-case consumption of random bits is,

$$|\mathcal{X}^{(n)}| \leq \lceil \log_2(n) \rceil \cdot \log_2(1/\delta). \quad (4.36)$$

The respective number of bits required to address all elements of the symmetric group, S_n , with δ -likelihood of failure for all calls to RANDOM(\mathcal{X}, \cdot) is,

$$\begin{aligned} |\mathcal{X}|^{(S_n)} &= \sum_{i=1}^n |\mathcal{X}^{(i)}| \\ &= \log_2(1/\delta) \cdot \sum_{i=1}^n \lceil \log_2(i) \rceil \\ &\approx \log_2(1/\delta) \cdot \log_2(n!), \end{aligned} \quad (4.37)$$

an effective $\log_2(1/\delta)$ multiplicative overhead on the scaling relationship shown in Fig. 4.

5. Initial assignment

Uniformly sampling consensus allocation requires applying sampled consensus group transformations (Alg. 6) to an initial satisfying graph assignment (Fig. 3) — a solution to the *bipartite realisation problem*. Via the Gale-Ryser theorem (Gale, 1957; Ryser, 1957), realisable bipartite graphs (those with *bigraphic* degree sequences) demands,

$$\begin{aligned} \sum_{i=1}^{|U|} \deg(u_i) &= \sum_{j=1}^{|V|} \deg(v_j), \\ \sum_{i=1}^k \deg(u_i) &\leq \sum_{j=1}^{|V|} \min(\deg(v_j), k), \quad \forall k, \end{aligned} \quad (4.38)$$

where the second constraint assumes U and V are ordered decreasingly by degree. If realisable, Alg. 8 allocates an initial graph realisation for given degree sequence.

The simplest bidding format for ensuring realisable consensus assignment is *uniform bidding* where all nodes

request a single consensus set of size $|\mathcal{C}|$. Now all vertices have constant degree,

$$\deg(u) = \deg(v) = |\mathcal{C}|, \forall u \in U, v \in V, \quad (4.39)$$

which is realisable as per Eq. (4.38) if and only if,

$$|\mathcal{C}| \leq |\mathcal{N}'|, \quad (4.40)$$

where,

$$|U| = |V| = |\mathcal{N}'| = |\mathcal{C}|, \quad (4.41)$$

is the number of participating network nodes, equivalently the number of consensus sets. The same condition holds if all nodes request a constant number of consensus sets of uniform size.

* Asymmetric load effective r.

$$r = \frac{1}{|E|} \sum_{i \in \mathcal{N}} \deg(i) \cdot r_i. \quad (4.42)$$

* Policy approaches to ensure bipartite realisation:

- Uniform bidding: All nodes request consensus sets of equal size as per the random subset problem.
- Multiple uniform bidding: Nodes request multiple consensus sets of varying size, where the arrays of consensus set sizes are uniform across users.
- Hierarchical bidding: ...

Algorithm 8: Deterministic consensus assignment via bipartite graph realisation of a given degree sequence, with $O(|U| \cdot |V|)$ time-complexity.

```

procedure INITIALASSIGNMENTGRAPH( $U, V, d$ )  $\rightarrow E$ 
   $E \leftarrow \{\}$   $\triangleright$  Edge set
   $U \leftarrow \text{SORT}(U \mid \deg(u_i) \geq \deg(u_{i+1}))$   $\triangleright O(|U| \log |U|)$ 
   $V \leftarrow \text{SORT}(V \mid \deg(v_i) \geq \deg(v_{i+1}))$   $\triangleright O(|V| \log |V|)$ 
  for  $u \in U$  do  $\triangleright O(|U| \cdot |V|)$ 
    if  $\deg(u) > 0$  then
      for  $v \in V$  do
        if  $\deg(v) > 0$  then
           $E \leftarrow E \cup (u, v)$   $\triangleright$  Assign edge
           $\deg(u) \leftarrow \deg(u) - 1$   $\triangleright$  Decrement valence
           $\deg(v) \leftarrow \deg(v) - 1$ 
          if  $\deg(u) = 0$  then
            break
   $x \leftarrow \sum_{u \in U} \deg(u) + \sum_{v \in V} \deg(v)$   $\triangleright$  Unassigned edges
  if  $x \neq 0$  then
     $E \leftarrow \emptyset$   $\triangleright$  Failure: unrealisable graph
  return  $E$ 

```

V. DISTRIBUTED CONSENSUS NETWORKS

A. Model

We consider a network of known nodes (\mathcal{N}) with access to a shared, public broadcast channel (\mathcal{B}), which nodes

may **broadcast()** into and **listen()** to. The communications primitives our model relies upon are shown in Alg. 9.

Network nodes BID to participate in consensus by announcing transaction ids with their required consensus set size. The network forms consensus on the set of nodes and bids to accept, thereby establishing the global key $X_{\mathcal{N}}$ which defines the assignment of consensus sets via distributed implementation of the random subset problem.

Algorithm 9: Communications primitives, where \mathcal{B} denotes the shared broadcast channel and numbers denote distinct synchronous steps.

```

procedure ANNOUNCE( $\text{node}, \text{statement}$ )
   $\text{message} = \text{SIGN}(\text{node}, \text{statement})$ 
   $\mathcal{B} \leftarrow \mathcal{B} \cup \text{message}$   $\triangleright$  Broadcast

procedure COMMITREVEAL( $\text{node}, \text{statement}$ )
   $\text{salt} = \text{RANDOM}(\{0, 1\}^n)$ 
   $\triangleright$  1. Commit hash  $\triangleleft$ 
  ANNOUNCE( $\text{node}, \text{HASH}(\text{statement}|\text{salt})$ )
   $\triangleright$  2. Reveal pre-image  $\triangleleft$ 
  ANNOUNCE( $\text{node}, \text{message}|\text{salt}$ )

```

B. Protocol

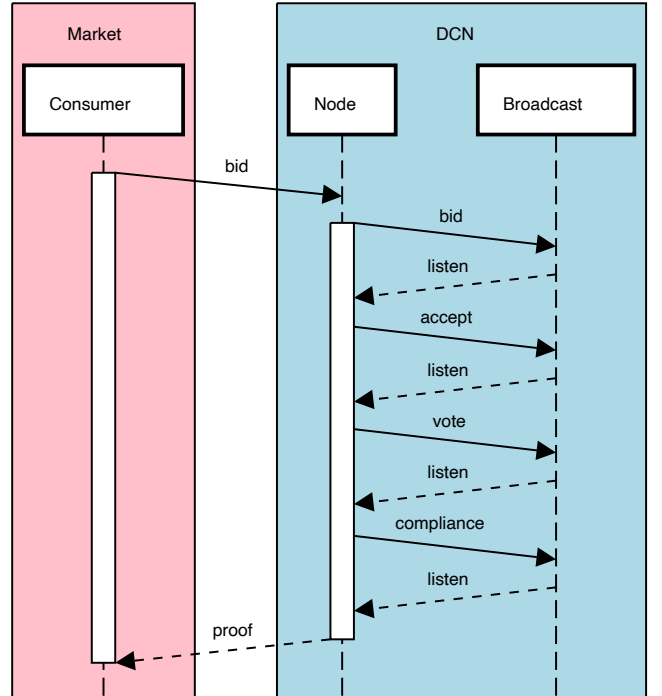


Figure 6: DCN protocol sequence diagram. (blue) DCN internal network. (red) External consensus market.

The DCN protocol is synchronous with the following steps for each network node $i \in \mathcal{N}$:

1. BID: Nodes (i) bid to participate by presenting a set of requests (j) for consensus sets of given sizes ($|\mathcal{C}_{i,j}|$),

$$\begin{aligned}\mathcal{R}_{i,j} &= (\text{id}_{i,j}, |\mathcal{C}_{i,j}|), \\ \mathcal{R}_i &= \bigcup_j \mathcal{R}_{i,j},\end{aligned}\quad (5.1)$$

and stating recognised network nodes via the set of their public keys (PubKey_j),

$$\mathcal{N}^{(i)} = \{\text{PubKey}_j\}. \quad (5.2)$$

All nodes broadcast (commit-reveal):

$$\mathcal{B} \leftarrow (\mathcal{R}_i, \mathcal{N}^{(i)}). \quad (5.3)$$

2. ACCEPT: Observing the broadcast channel \mathcal{B} , all nodes infer the accepted network state, bids and participating nodes (\mathcal{N}'),

$$\begin{aligned}\mathcal{B} &\rightarrow (\tilde{\mathcal{R}}, \tilde{\mathcal{N}}', \tilde{\mathcal{N}}), \\ \tilde{\mathcal{N}} &= \text{MAJORITYVOTE}(\{\mathcal{N}^{(j)}\}), \\ (\tilde{\mathcal{R}}, \tilde{\mathcal{N}}') &= \text{ACCEPT}(\{\mathcal{R}_j\}),\end{aligned}\quad (5.4)$$

where $\text{ACCEPT}(\cdot)$ is a network-agreed deterministic function, and \sim denotes values inferred from the broadcast channel.

In this step consensus is performed at the network level where all network nodes form a single consensus set, requiring that participants form a network majority,

$$|\mathcal{N}'| > \frac{|\mathcal{N}|}{2}. \quad (5.5)$$

All other votes in the protocol are conducted at the level of assigned consensus sets.

3. CONSENSUS: The global key $\mathcal{X}_{\mathcal{N}}$ is implied by the accepted parameters (Sec. III.C) as is the network's allocation to consensus sets via the random subset algorithm (Sec. 1).

$$\mathcal{B} \rightarrow \mathcal{X}_{\mathcal{N}} \rightarrow \{\mathcal{C}\}. \quad (5.6)$$

Participating nodes vote (commit-reveal) on their assigned consensus requests,

$$\mathcal{B} \leftarrow \text{VOTE}_i(\tilde{\mathcal{R}}). \quad (5.7)$$

4. COMPLIANCE: Nodes vote (announce) on the compliance of participating nodes and reveal the time-of-receipt of all broadcast messages associated with the respective consensus'. Compliant nodes follow all required protocol steps, are in agreement with the majority on all votes, where timestamps (time-of-receipt) of all announcements are within the network's δ threshold.

$$\mathcal{B} \leftarrow \text{COMPLIANTSET}_i(\mathcal{N}). \quad (5.8)$$

C. Consensus time

* ??? TODO

To enforce synchronisation of the protocol compliance requires nodes' broadcast messages to satisfy timing constraints under majority vote. To establish majority vote outcomes on the timing of messages we introduce *consensus time*, given by the median of the reported times of receipt of messages (Alg. 10).

Algorithm 10: Consensus time of a broadcast message is given by the median of the times of receipt reported by nodes.

```

procedure CONSENSUSTIME( $\mathcal{C}$ , message)  $\rightarrow \mathbb{R}$ 
    times  $\leftarrow \{\text{TIMEOFRECEIPT}(i, \text{message})\}_{i \in \mathcal{C}}$ 
    return MEDIAN(times)

```

Consensus time exhibits the property that if for any majority of consensus nodes,

$$|t_i - \text{CONSENSUSTIME}(\mathcal{C}, \text{message})| \leq \delta, \quad (5.9)$$

no reported t_i for the remaining minority can shift consensus time by more than δ , making consensus time robust against minority manipulation. Consensus time may therefore be utilised as an implied majority vote on nodes' timing compliance,

$$\text{MAJORITYVOTE}(\mathcal{C}, \text{COMPLIANT}(\text{message})). \quad (5.10)$$

Let the worst-case network latencies be,

$$\tau_{\max} = \max_{i,j \in \mathcal{N}}(\tau_{i,j}), \quad (5.11)$$

where $\tau_{i,j}$ is the matrix of point-to-point latencies between nodes i and j . A message broadcast at time t_B will be received by all network nodes by at latest $t_B + \tau_{\max}$. By majority vote, the latest time at which a message could have been broadcast is,

$$t_B \leq \text{CONSENSUSTIME}(\mathcal{C}, \text{message}) - \tau_{\max}. \quad (5.12)$$

Ensuring majority votes on consensus time are well-defined requires,

$$\delta \geq \frac{\tau_{\max}}{2}. \quad (5.13)$$

D. Proof-of-consensus

The sequence of signed, broadcast announcements made by compliant network nodes (i) is:

- \mathcal{R}_i : Request for consensus sets.
- $\mathcal{N}^{(i)}$: Public keys of all network nodes.
- $\text{VOTE}_i(\tilde{\mathcal{R}}_j)$: Votes on assigned consensus'.

- $\text{TIMESTAMP}_i(\mathcal{B})$: Time-of-receipt of broadcast messages from other nodes in the consensus set.
- $\text{COMPLIANTSET}_i(\mathcal{C})$: Recognised compliant nodes in the consensus set.

We denote a complete set of such announcements from a given node ($i \in \mathcal{N}$) participating in consensus set \mathcal{C} as,

$$\mathcal{P}_i(\mathcal{C}). \quad (5.14)$$

A proof-of-consensus for consensus set \mathcal{C} comprises any self-consistent majority set of partial proofs,

$$\mathcal{P}(\mathcal{C}) = \bigcup_{i \in \text{MAJORITY}(\mathcal{C})} \mathcal{P}_i(\mathcal{C}). \quad (5.15)$$

Note that while the proofs for a given consensus are not unique they are equivalent proofs of the same statement.

A proof-of-consensus in a self-contained proof system, comprising all necessary information to verify its validity.

* Variation in consensus time preserves compliance outcomes.

E. Network policy

Let the network maintain its own ledger for tallying the contributed consensus workload of all network nodes, comprising an array of zero-initialised integer registers, one for each node,

$$\text{tally} \in \mathbb{Z}^{|\mathcal{N}|}. \quad (5.16)$$

When node i faithfully participates in consensus its tally register is incremented. Conversely, when requesting consensus load its tally must have sufficient balance to make the request. Under steady-state operation where nodes request what they contribute registers do not change.

When the network accepts new nodes they are unable to immediately make consensus requests and instead make null-bids, offering to contribute load without return. This increases their tally, enabling them to make subsequent requests for consensus load. This effectively forces new nodes to buy into the network by pre-contributing load they will subsequently request.

During the ACCEPT stage of the protocol, the network must also agree on the network's updated **tally** state.

VI. QUANTUM CONSENSUS NETWORKS

Quantum consensus networks (QCNs) have quantum-enabled nodes, whose goal it is to form consensus on the generation of certifiable quantum randomness, an important resource in cryptography and numerous other applications.

As quantum hardware is costly compared to classical hardware it is expected that few networks will be

quantum-enabled. However, they may exploit the quantum randomness provided by dedicated QCNs acting as *quantum random oracles* (Sec. VI.B) to inject entropy into their own global keys using *entropy addition* (Sec. VI.A), effectively enabling classical DCNs to achieve quantum random consensus assignment.

We consider two approaches for quantum random number generation (QRNG):

- Quantum key distribution (QKD): requires quantum communication but no quantum computation (Sec. VI.C).
- Interactive proofs of quantumness: require quantum computation but no quantum communication (Sec. VI.D).

As these are two-party protocols, every instance may be associated with a graph edge between the respective nodes. Random numbers associated with edges may be spoofed if both nodes collude, bypassing the need for expensive quantum resources. However, so long as at least one QRN is genuine combining them under bit-wise XOR yields a collective QRN source.

A. Entropy addition

The bit-wise XOR operator is a strictly non-entropy-decreasing function. For binary random variables,

$$\mathcal{X}_i \rightarrow \{0, 1\}, \quad (6.1)$$

with probability distributions,

$$\mathcal{X}_i : p_i = p_i(0) = 1 - p_i(1), \quad (6.2)$$

the individual binary Shannon entropies are given by,

$$H_2(\mathcal{X}_i) = -p_i \log_2 p_i - (1 - p_i) \log_2 (1 - p_i), \quad (6.3)$$

where,

$$0 \leq H_2(\cdot) \leq 1. \quad (6.4)$$

Their combined entropy under the bit-wise XOR operation is lower-bounded by their maximum,

$$\max_i \{H_2(\mathcal{X}_i)\} \leq H_2\left(\bigoplus_i \mathcal{X}_i\right) \leq 1. \quad (6.5)$$

Hence, a random source derived from multiple sources via entropy addition is at least as random as any of them. Consequently, if any single contributing source is a genuine QRNG, so too will be their the combined source.

B. Quantum random oracles

Let $\mathcal{O}(t)$ denote a dynamic set of contributing random oracles at time t . We define a collective random bit-stream,

$$\mathcal{X}_{\mathcal{O}}(t) = \bigoplus_{i \in \mathcal{O}(t)} \mathcal{X}_i(t), \quad (6.6)$$

whose combined entropy is bounded by,

$$\max_{i \in \mathcal{O}} (\{H_2(\mathcal{X}_i)\}) \leq H_2(\mathcal{X}_{\mathcal{O}}) \leq 1. \quad (6.7)$$

A classical DCN may observe a QRNG oracle and add its entropy $\mathcal{X}_{\mathcal{O}}$ to its own global key. For this to be secure it is required that the DCN's own global key, $\mathcal{X}_{\mathcal{N}}$, be committed prior to the availability of the external entropy source,

$$\tilde{\mathcal{X}}_{\mathcal{N}}(t + \Delta) = \mathcal{X}_{\mathcal{N}}(t) \oplus \mathcal{X}_{\mathcal{O}}(t + \Delta), \quad (6.8)$$

where $\tilde{\mathcal{X}}_{\mathcal{N}}$ is the network's oracle-modulated global key, and $\Delta > 0$ is a pre-agreed future point in time, subsequent to commitment of the networks initially established global key, $\mathcal{X}_{\mathcal{N}}$.

C. Quantum key distribution (QKD)

Quantum key distribution (QKD) (Bennett and Brassard, 1984; Ekert, 1991) enables the secure establishment of shared randomness between two parties with information theoretic security. While ordinarily utilised for secret key exchange, here we exploit not the secrecy of shared randomness but its inability to be spoofed under honest execution of the protocol.

Assuming the existence of a quantum internet (Rohde, 2021) capable of arbitrary point-to-point entanglement routing, all node-pairs (i, j) have access to an indefinite supply of maximally-entangled Bell pairs,

$$|\Psi\rangle_{i,j} = \frac{1}{\sqrt{2}}(|0\rangle_i |0\rangle_j + |1\rangle_i |1\rangle_j), \quad (6.9)$$

requiring full $O(n^2)$ quantum communications connectivity.

For the n th copy of $|\Psi\rangle_{i,j}$ both nodes independently and privately choose measurement bases,

$$b_i(n), b_j(n) \in \{0, 1\}, \quad (6.10)$$

where $b = 0$ denotes the Pauli-Z basis and $b = 1$ the Pauli-X basis, and record their associated measurement outcomes,

$$m_i(n), m_j(n) \in \{0, 1\}. \quad (6.11)$$

The subset of measurement outcomes where both parties operate in the same basis defines a shared random bit-string,

$$s_{i,j} = \{m(n) \mid b_i(n) = b_j(n)\}_n. \quad (6.12)$$

These post-selected bit-strings correspond identically to those provided by the E91 (Ekert, 1991) QKD protocol. The BB84 protocol (Bennett and Brassard, 1984) can be similarly employed with the interpretational difference that for one party b and m denote encoding, for the other measurement.

Physical and implementation errors reduce the otherwise perfect measurement correlations between nodes measuring in the same basis resulting in inconsistent shared strings. However, privacy amplification (Impagliazzo *et al.*, 1989) may be used to reduce an imperfect random bit-string to a shorter one with higher entropy using classical post-processing.

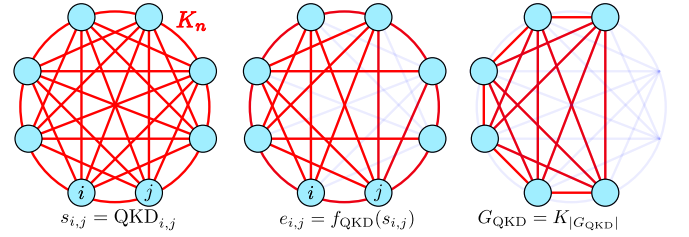


Figure 7: QKD proof-chain for shared quantum randomness. Amongst n nodes with full $O(n^2)$ quantum communications connectivity given by the complete graph K_n , every node executes a QKD protocol with every other node, associating a shared random bit-string $s_{i,j}$ with every edge. Bit-strings passing a QRN certification function $f_{\text{QKD}}(s_{i,j})$ define the edges of a subgraph. Maintaining only vertices connected to a majority of other nodes followed by eliminating those not connected to every other, we obtain a complete subgraph \mathcal{G}_{QKD} reflecting the unanimous majority.

1. Consensus protocol

Associating QKD bit-strings $s_{i,j}$ with graph edges we assume a certification function,

$$f_{\text{QKD}}(s_{i,j}) \rightarrow \{0, 1\}, \quad (6.13)$$

which evaluates **true** if $s_{i,j}$ passes a certification test for randomness. We define a QKD compliance graph with edge-inclusion based on the validity of the respective QKD bit-strings,

$$\mathcal{G}_{\text{QKD}}^{(\text{comp})} : e_{i,j} = f_{\text{QKD}}(s_{i,j}). \quad (6.14)$$

Letting the QKD compliance of nodes be,

$$\mathcal{G}_{\text{QKD}}^{(\text{comp})} : v_i = \text{MAJORITY} \left(\bigcup_{j \neq i} e_{i,j} \right), \quad (6.15)$$

the reduced graph now only contains nodes whose associated QKD bit-strings $s_{i,j}$ are majority valid.

Additionally requiring unanimity demands finding a fully-connected subgraph, or clique⁹, achieved by eliminating all vertices in $\mathcal{G}_{\text{QKD}}^{(\text{comp})}$ not connected by an edge to every other node,

$$\mathcal{G}_{\text{QKD}} : v_i = \begin{cases} 1 & \text{if } |u_i \in \mathcal{G}_{\text{QKD}}^{(\text{comp})}| = |\mathcal{G}_{\text{QKD}}^{(\text{comp})}| - 1 \\ 0 & \text{otherwise} \end{cases}. \quad (6.16)$$

The fully connected $\mathcal{G}_{\text{QKD}} = K_{|\mathcal{G}_{\text{QKD}}|}$ subgraph now represents the accepted subset of QKD-compliant nodes under consensus. The associated collectively established shared random bit-string is defined as,

$$s(\mathcal{G}_{\text{QKD}}) = \bigoplus_{v_i, v_j \in \mathcal{G}_{\text{QKD}}} s_{i,j}. \quad (6.17)$$

Although nodes could commit post-processed QKD strings obtained following post-selection and privacy amplification, this requires interaction between respective nodes. In the interests of maintaining a broadcast-only communications interface nodes may simply commit their raw unprocessed strings (b and m) from which the associated QRNs s are implied under a network-agreed post-processing function $f_{\text{PP}}(\vec{b}, \vec{m}) \rightarrow \vec{s}$.

D. Interactive proofs of quantumness

An interactive proof of quantumness (Liu and Gheorghiu, 2022; Zhu *et al.*, 2023) comprises two parties, a *prover* and a *verifier*, where the goal is for the prover to prove to the verifier that they have honestly executed a quantum implementation of some function $f(\cdot)$ that cannot be spoofed by classical simulation. The verifier has only classical resources and both parties may classically communicate.

While such protocols are not known in general for arbitrary $f(\cdot)$, they have been described in the context of a restricted class of functions known as trapdoor claw-free functions.

1. Trapdoor claw-free (TCF) functions

Trapdoor claw-free functions (TCF) are a class of cryptographic, 2-to-1, one-way functions,

$$f_{\mathcal{I}}(x) \rightarrow w, \quad (6.18)$$

which are classically efficient to evaluate in the forward direction, but for which it is hard to find simultaneously satisfying inputs $\{x_0, x_1\}$ (the ‘claw’) mapping to the same output,

$$f(x_0) = f(x_1) = w, \quad (6.19)$$

where $x \in \{0, 1\}^n$ and $w \in \{0, 1\}^{n-1}$ are bit-strings.

Here, \mathcal{I} denotes a problem instance derived from a secret (the trapdoor). If the secret is known, finding claws $\{x_0, x_1\}$ is classically efficient for any w . Since $f(\cdot)$ is easy to evaluate in the forward direction, verifying solutions is classically efficient, and the problem of claw-finding by definition resides in the complexity class **NP**.

2. The LWE problem

A candidate TCF is the lattice-based learning with errors (LWE) problem (Goldwasser *et al.*, 1985; Regev, 2009, 2010). This problem is believed to be post-quantum, where the associated claw-finding problem lies outside of **BQP**, the class of problems efficiently solvable by quantum computers¹⁰.

For matrix,

$$A \in \mathbb{Z}_q^{m \times n}, \quad (6.20)$$

and vectors,

$$x, y, s, e \in \{0, 1\}^n, \quad (6.21)$$

related by,

$$y = A \cdot s + e, \quad (6.22)$$

under modulo q arithmetic where q is prime, a TCF may be constructed as,

$$f_{\mathcal{I}}(b, x_b) = \lfloor A \cdot x + b \cdot y \rfloor, \quad (6.23)$$

where $b = \{0, 1\}$ is a single bit and claws are related by,

$$x_0 = x_1 + s. \quad (6.24)$$

Here, $\mathcal{I} = \{A, y\}$ specifies the problem instance derived from the secret trapdoor $\mathcal{T} = \{s, e\}$ secretly held by the verifier, enabling efficient classical claw-finding and verification if known.

Since $f(x) \rightarrow w$ is classically efficient to evaluate in the forward direction, it is easy to find a w for which a single satisfying input x is known. The challenge lies in finding simultaneously satisfying pairs of inputs, believed to be hard for both classical and quantum computers.

⁹ While the MAXCLIQUE problem of finding the largest cliques in a graph is known to be **NP**-complete in general, here we are not finding maximal cliques, affording an efficient solution.

¹⁰ An alternate number-theoretic TCF based on Rabin’s function has been described (Goldwasser *et al.*, 1988; Rabin, 1979). Since here the complexity of inverting the trapdoor reduces to integer factorisation this candidate TCF is vulnerable to quantum attack via Shor’s algorithm (Shor, 1997), making it less applicable in the assumed context of universal quantum computation.

3. Interactive proof protocol

Taking a cryptographic TCF function, $f_{\mathcal{I}}(x) \rightarrow w$, an interactive proof of quantumness may be implemented as follows:

1. The verifier specifies a problem instance \mathcal{I} , without revealing the associated secret \mathcal{T} from which it was derived.
2. The prover prepares a uniform superposition of all length- n bit-strings x via a Hadamard transform,

$$|\psi_H\rangle = \hat{H}^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle. \quad (6.25)$$

3. Evaluating $f_{\mathcal{I}}(x)$ into an output register yields¹¹,

$$|\psi_{\mathcal{I}}\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f_{\mathcal{I}}(x)\rangle,$$

which may be efficiently prepared using a quantum circuit with,

$$O(n^2 \log^2 n), \quad (6.26)$$

gate count (Kahanamoku-Meyer *et al.*, 2022).

4. The prover measures the output register, obtaining measurement outcome w which is communicated to the verifier. Measuring w collapses the x -register

¹¹ The unitarity of quantum circuits prohibits direct evaluation of classical functions on quantum registers in general,

$$\hat{U}_f |x\rangle \not\rightarrow |f(x)\rangle,$$

where \hat{U}_f denotes a quantum circuit evaluating classical function $f(\cdot)$. Introducing ancillary quantum register $|y\rangle$, evaluation of,

$$\hat{U}_f |x\rangle |y\rangle \rightarrow |x\rangle |y \oplus f(x)\rangle,$$

may be implemented unitarily in general. Considering a single output bit of $f(\cdot)$, \hat{U}_f admits the decomposition,

$$\hat{U}_f = \hat{\Pi}_0 \otimes \hat{I} + \hat{\Pi}_1 \otimes \hat{X},$$

where,

$$\hat{\Pi}_i = \sum_{x | f(x)=i} |x\rangle \langle x|,$$

are projectors onto the subspaces of x satisfying $f(x) = i$ (Nb: $\hat{\Pi}_0 + \hat{\Pi}_1 = \hat{I}$, $\hat{\Pi}_0 \cdot \hat{\Pi}_1 = 0$). The unitarity of \hat{U}_f follows, independent of $f(\cdot)$,

$$\hat{U}_f^\dagger \cdot \hat{U}_f = (\hat{\Pi}_0 \otimes \hat{I} + \hat{\Pi}_1 \otimes \hat{X})^2 = \hat{I}.$$

Repeating for all output bits and letting $y = 0$ yields,

$$\hat{U}_f |x\rangle |0\rangle \rightarrow |x\rangle |f(x)\rangle.$$

onto the equal superposition of associated satisfying pre-images,

$$|\psi_w\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle) |w\rangle. \quad (6.27)$$

As the x -register was initialised into a uniform superposition over all length- n bit-strings and the TCF is a 2-to-1 function, w is sampled uniformly at random.

5. The verifier specifies a random measurement basis in which the prover should measure qubits in the x register, where $b = \{0, 1\}$ correspond to the Pauli- \hat{Z} and \hat{X} bases respectively.
6. When measuring in the \hat{Z} basis, the prover randomly measures either $m = x_0$ or $m = x_1$, easily verified by directly evaluating $f(\cdot)$ and comparing with the prover's previously reported w . When measuring in the \hat{X} basis verification succeeds if,

$$m \cdot x_0 = m \cdot x_1. \quad (6.28)$$

7. The above is repeated some constant number of rounds, independently randomising the measurement basis b at every round.

The key observation is that since \hat{X} and \hat{Z} measurements do not commute, it is not possible for the prover to know both measurement outcomes simultaneously and therefore must measure in accordance with the verifier's stated measurement basis to pass verification of a single round. While a single round can be classically spoofed if the measurement basis b is known in advance of announcing w , if unknown, b can only be guessed with a probability of $1/2$. Upon repetition, the probability of correctly guessing all measurement bases scales as $p = 1/2^n$ for n rounds, ensuring asymptotic confidence in the honesty of the prover.

4. Consensus protocol

To incorporate IPQs into the QCN framework we require all nodes to act as both prover and verifier for all other nodes.

In the verifier capacity every node prepares a single random TCF instance for all other nodes to prove. Despite solving the same problem instance their proofs will be distinct.

Following the same approach as with QKD we represent the proofs-of-quantumness via a complete graph with the distinction that as this is an asymmetric protocol the graph is now directed (from prover to verifier) with edges in both directions for every node-pair.

Majority votes as per Eq. (6.15) are now made from the verifier perspective.

The additional synchronous steps required to accommodate IPQs are:

1. Nodes commit a single random problem instance \mathcal{I}_i .
2. Nodes execute the quantum problem instance specified by every other node j and commit the obtained $w_{i,j}$.
3. Nodes commit the random measurement bases b_i other nodes will be required to measure in.
4. Nodes complete their quantum computations and commit the obtained measurements $m_{i,j}$.
5. Nodes reveal their secrets \mathcal{T}_i .

Assuming a verification function analogous to Eq. (6.13),

$$f_{\text{IPQ}}(\mathcal{I}_i, \mathcal{T}_i, w_{i,j}, b_i, m_{i,j}) \rightarrow \{0, 1\}, \quad (6.29)$$

similarly defines a directed IPQ compliance graph,

$$G_{\text{IPQ}}^{\text{comp}} : e_{i,j} = f_{\text{IPQ}}(\mathcal{I}_i, \mathcal{T}_i, w_{i,j}, b_i, m_{i,j}) \rightarrow \{0, 1\}. \quad (6.30)$$

VII. ECONOMICS

Internally, the DCN operates as a cooperative, profit-neutral, barter economy, whose nodes facilitate an externally-facing competitive bidding market for consensus. Nodes' subjective cost of consensus is identically their own computational execution cost, individually incentivising computational and communications implementation efficiency.

Proof-of-work associates the creation of money with exchange, creating a monetary dependence on algorithmic inefficiency. DCNs are monetarily neutral, providing consensus as a generic and abstract commodity in a competitive market environment.

The DCN is a floating market instrument with instantaneous value. Consensus is necessarily consumed upon production and cannot be saved.

Consensus nodes individually utilise and monetise their gateway to the network facilitating highly competitive and high volume consensus markets.

VIII. APPLICATIONS

A. Protocols

Protocols are user-level applications for consensus, defined as arbitrary time-dependent functions acting on the current state of the broadcast channel,

$$\text{PROTOCOL}(\mathcal{B}(t=0), \star) \rightarrow \star, \quad (8.1)$$

where time is relative to the present. The state of the broadcast channel a time t contains all previous messages broadcasts, where,

$$\mathcal{B}(t=0) = \bigcup_{x \in \mathcal{B}(t \leq 0)} x. \quad (8.2)$$

The state of the broadcast channel is in general subjective as individual users may have imperfect knowledge of \mathcal{B} as a result of information loss. The subjective state of the broadcast channel for user i is,

$$\mathcal{B}_i \subseteq \mathcal{B}. \quad (8.3)$$

Consequently, PROTOCOL outputs are also subjective and may differ in general,

$$\text{PROTOCOL}_i(\mathcal{B}_i, \star) \neq \text{PROTOCOL}(\mathcal{B}, \star) \quad (8.4)$$

B. Ledgers

Consensus is formed on state register transformations.

Ledgers are defined by policies stipulating the consensus networks they recognise. Inter-ledger operations require only mutual recognition of consensus networks.

Ledgers as oracles.

Finite state machine oracles.

The combined public broadcasts across all networks acts as a global oracle for ledger states, facilitating a high arbitrage environment in an algorithmic context, an equilibrating force across the ledgers of parallel markets or interconnected markets.

Inter-ledger transactions require only mutual recognition of consensus, defined by the networks from which they are drawn.

* Sec: ledger transaction queues.

* Hierarchical bidding to maximise resource utilisation.

C. Blockchains

Blockchains are protocol-level applications for consensus, following their own rules on what consensus is formed on. A blockchain's transaction history is immutable and may be retrospectively evaluated. Blockchain implementations typically consider asynchronous operating environments. In this setting simultaneous block additions manifest themselves as forks, requiring error correction mechanisms to maintain the integrity of the chain. Formally, a pool of valid block additions defines a directed tree graph which the blockchain implementation must correct to a directed linear graph.

In a synchronous setting where a ledger is associated with consensus derived from a given network these considerations change. Rather than performing consensus assignment on the basis of unique transaction identifiers we

assign on the basis of transaction queue identifiers associated with individual ledgers. From the pool of accepted bids nodes assigned to a given transaction queue consensus set process all bids associated with that queue, batch processed in accordance with the ledger's transaction amalgamation rules. Employing queue assignment rather than transaction assignment mitigates the possibility of fork formation. In a proof-of-work setting this issue may be addressed by introducing friction. However, while this hinders double-mining it also undermines transaction processing rates, an undesirable tradeoff.

If n consensus sets of size N independently form honest majority their union of size nN necessarily forms honest majority. Hence,

$$P(nN, r) \leq \varepsilon \Rightarrow P(N, r)^n \leq \varepsilon. \quad (8.5)$$

For $n = 1$ this reduces to consensus by majority vote amongst N parties, while the opposing limiting case of $n = N$ reduces to consensus by unanimity amongst N parties. A blockchain with unanimous n -level retrospective verification affords ε^n -security, and the associated tradeoff in required consensus set size scales as,

$$N_{\min}(r, \varepsilon) = N_{\min}^{(n)}(r, \varepsilon^n), \quad (8.6)$$

which implies,

$$P_c(N, r) = P_c(N^{(n)}, r)^n. \quad (8.7)$$

Now ε represents the effective error rate in new block additions while $\varepsilon' = \varepsilon^n$ is the effective security parameter which applies only to blocks at least n steps back, which have been subject to n independent verifications. These blocks are considered *complete* whereas more recent blocks are considered *pending*, potentially still subject to being invalidated (Fig. 8). Only the most recent *complete* block must persist to maintain the blockchain, the point to which the blockchain is reverted if *pending* blocks are invalidated.

D. Distributed computing

The consensus assignment problem may be interpreted more generically as randomised dynamic load allocation.

Consider the case of $|\mathcal{C}| = 1$ consensus sets, the delegation of computational workloads to single randomly allocated nodes. In this context the consensus assignment algorithm facilitates dynamic load balancing across the network. Similarly, $|\mathcal{C}| > 1$ equates to dynamic allocation with $|\mathcal{C}|$ -fold redundancy where consensus is formed on the outcome.

More generally, MAPREDUCE-type (Dean and Ghemawat, 2008) computations may be delegated to consensus sets of arbitrary size, where the MAP routine corresponds to the assignment of consensus nodes and the REDUCE routine is evaluated by consensus.

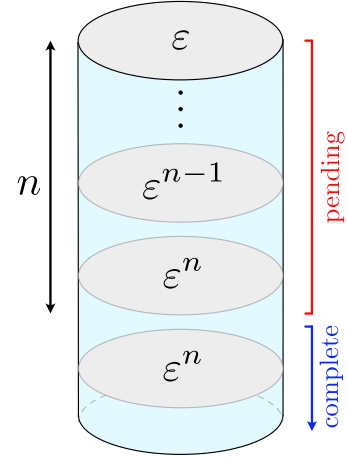


Figure 8: Blockchain with n -level reverse integrity checking. Upon addition of the top-level block consensus requires verifying integrity going back n blocks. If consensus has ε -security, for $i \leq n$ the i th past block will have been verified i times, exhibiting cumulative ε^i -security. Blocks $i \geq n$ all exhibit $\varepsilon' = \varepsilon^n$ integrity, the security parameter of the blockchain. The ε -security of the top block may be interpreted as the effective error rate in block addition, where error correction is implemented at the blockchain's protocol level.

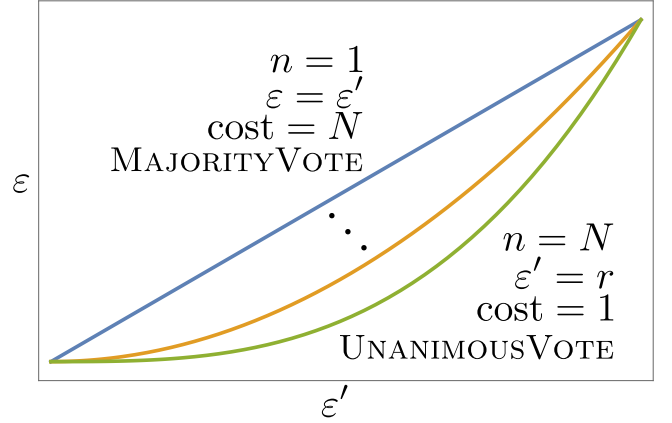


Figure 9: Tradeoffs in blockchain integrity with retrospective consensus verification.

* Distributed queries.

* Consider asymmetric bidding if preserving r is not a consideration.

E. Distributed signature authorities

* Consensus on state of knowledge. Trust in knowledge. Oracle.

IX. STRATEGIC CONSIDERATIONS

Fig. 11

‘Rebasing’ network upon strategic retreat.

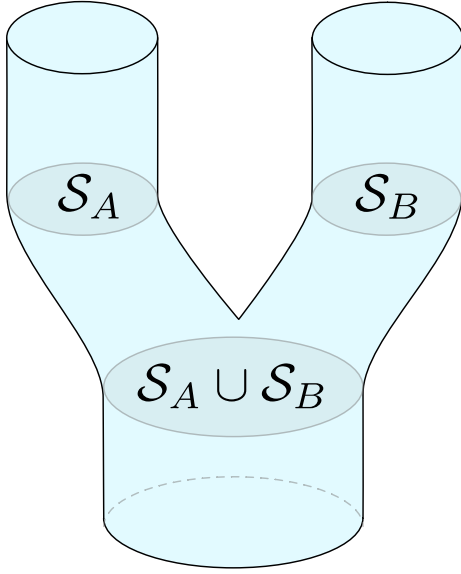


Figure 10: Bifurcation in network trust. When the network $S_A \cup S_B$ supporting a blockchain segregates into two non-interacting networks, S_A and S_B , a fork is created, forming two unique, legitimate blockchains, one associated with each network.

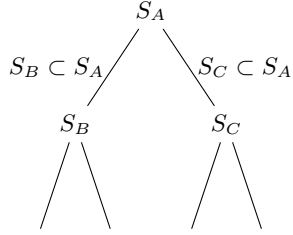


Figure 11: Trust hierarchies represented as subset-trees define retreat strategies.

Retreat strategy in trust hierarchy where root is level $i = 0$. Levels represent set containment, $s_{i+1} \subset s_i$. Retreat to smallest i (i.e largest trusted subset). Assume that $r_{i+1} < r_i$. Inequality could work in either direction??

X. CONCLUSION

Contrary to existing blockchain design philosophy, the abstract separation between digital assets and their technological mechanism for trade provides a framework under which both may compete in a competitive market environment, enabling the emergence of a free banking system.

DCNs are generically applicable, automated digital signature systems and neither inherently transactional nor monetary in nature, reducing exposure to escalating regulatory efforts to restrict the trade of digital assets and

inhibit technological innovation (Warren, 2023).

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