Category Theory

Categories, functors, natural transformations

Def (Categories):

A category C consists of the following data:

- A class $ob(C)$ of objects
- $\forall$ pair $X, Y \in ob(C)$, a set of morphisms $C(X,Y)$
- $\forall$ $X \in ob(C)$, an identity morphism $1_X \in C(X,X)$
- $\forall$ triple $X, Y, Z \in ob(C)$, a composition map $C(X,Y) \times C(Y,Z) \longrightarrow C(X,Z)$, written $(f,g) \mapsto g \circ f$.

These data that define the category are required to satisfy:

- $1_Y \circ f = f$ & $f \circ 1_X = 1_X$ - $(h \circ g) \circ f = h \circ (g \circ f)$

↳ Some convention includes writing $X \in C$ instead of $X \in ob(C)$ & $f: X \longrightarrow Y$ to mean $f \in C(X,Y)$

Def (Isomorphisms):

If $X, Y \in C$, then $f: X \longrightarrow Y$ is an isomorphism if $\exists$ $g: Y \rightarrow X$ w/ $f \circ g = 1_Y$ & $g \circ f = 1_X$.

first

↳ We write $f: X \xrightarrow{\cong} Y$ to indicate its an isomorphism

There are many common mathematical structures which can be arranged into categories

↳ Sets (objects) & functions (morphisms) between them form a category: **Set**

↳ Abelian groups & homomorphisms form a category: **Ab**

↳ Topological spaces & continuous maps for a category:

↳ A monoid is the same as a category w/ one object, where the elements of the monoid are the morphisms of the category (small category)

Def (Functors):

Let C, D be categories. A functor $F: C \longrightarrow D$ consists of the data of:

- An assignment $F: ob(C) \longrightarrow ob(D)$
- $\forall X, Y \in ob(C)$, a function $F: C(X,Y) \longrightarrow D(F(X), F(Y))$.

These data are required to satisfy the following two properties:

- $\forall X \in ob(C)$, $F(1_X) = 1_{F(X)} \in D(F(X), F(X))$
- $\forall$ composable pairs of morphisms f, g in C, $F(g \circ f) = F(g) \circ F(f)$

↳ A few common functors include:

$Sin_n: Top \longrightarrow Set$, $S_n: Top \longrightarrow Ab$, $H_n: Top \longrightarrow Ab$

↳ We can also consider "morphisms between functors"

$\partial: S_n(X) \longrightarrow S_{n-1}(X)$ (boundary map for simplices)

Def (Natural Transformations):

Let $F, G: C \longrightarrow D$ be two functors. A natural transformation or natural map $\theta: F \longrightarrow G$ consists of maps $\theta_X: F(X) \longrightarrow G(X)$ $\forall X \in ob(C)$ st $\forall f: X \longrightarrow Y$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\theta_Y"] & G(Y)
\end{tikzcd}$$

Suppose C, D are categories & that C is small. We may form the category of functors $\mathrm{Fun}(C,D)$

↳ Its objects are the functors from C to D

↳ Its morphisms are natural transformations, such as between functors F, G as $\underbrace{\mathrm{Fun}(C,D)(F,G)}_{C(X,Y) \text{ anal}}$

Categorical Language

Let $Vect_k$ be the category of vs over a field k, & liner transformations between them

- ↳ Given a vs $V \in ob(Vect_k)$, you can consider the dual $V^* = Hom(V, k) \in ob(Vect_k)$ which is the set of morphisms from V to k.

↳ Does this give a functor? Given a liner transformation $f: V \to W$, you get $f^*: W^* \to V^*$ which looks like a functor but the induced map goes the wrong way

↳ This is a contravariant functor : $f: X \to Y$, $F(f): F(Y) \to F(X)$ opposite dir

There is infact a universal definition of a contravariant functor out of a category C, $C \to C^{op}$, where C^{op} has the same objects as C but $C^{op}(X,Y)$ is the set $C(X,Y)$

↳ A contravariant functor from $C \to D$ is the same as a covariant functor from $C^{op} \to D$

- ↳ For $Y \in ob(C)$, the map $C^{op} \to Set$ takes $X \mapsto C(X,Y)$ & takes a map $X \to W$. This is called the functor represented by Y

Def (Split Endomorphism):

A morphism $f: X \to Y$ in a category C is a split endomorphism if $\exists$ $g: Y \to X$ (called a section (splitting)) st the composite $Y \xrightarrow{g} X \xrightarrow{f} Y$ is the identity.

Def (Split Monomorphism):

A map $g: Y \to X$ is a split monomorphism if there is $f: X \to Y$ st $f \circ g = 1_Y$

Lemma (Isomorphism):

- A map is an isomorphism iff is is both a split endo & split mono.

Lemma (Functors):

Any functor sends split endo $\to$ split endo & split mono $\to$ split mono.

Note: We now switch to Goren's notes where some notation changes:

$$X, Y \in ob(C) \longrightarrow A, B \in ob(C)$$
$$C(X,Y) \qquad Mor_C(A,B)$$
$$\theta_X \qquad \varphi_A$$

↳ A functor is faithful if $\forall A, B \in ob(C)$ & $f, g \in Mor_C(A,B)$ we have:

$$F(f) = F(g) \Rightarrow f = g \quad \text{injective}$$

↳ A functor is full if it is surjective on morphisms:

$$F: Mor_C(A,B) \twoheadrightarrow Mor_D(F(A), F(B)) \quad \text{surjective } \twoheadrightarrow$$

↳ Covariant

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"'] & G(B) \end{tikzcd}$$

Contravariant

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_A"] & G(A) \\ F(B) \arrow[r, "\varphi_B"'] \arrow[u, "F(f)"] & G(B) \arrow[u, "G(f)"'] \end{tikzcd}$$

↳ F is isomorphic to G if $\exists \varphi$ st all φ_A are isomorphisms. We say $F \cong G$.

↳ $\mathbb{1}_C$ is the identity functor that takes any object, & any morphism, to itself. It is covariant, full, & faithful.

Equivalence of Categories

Two categories C, D are (anti)-equivalent if $\exists$ (contravariant) covariant functors:

$$F: C \to D \qquad \text{st} \qquad G \circ F \cong \mathbb{1}_C$$
$$G: D \to C \qquad\qquad F \circ G \cong \mathbb{1}_D$$

Furthermore, a functor $F: C \to D$ is essentially surjective if $\forall B \in ob(D) \; \exists A \in ob(C)$ st $F(A) \cong B$.

almost surjectivity of objects

Thm (Equivalence):

A functor $F: C \to D$ is an (anti) equivalence of categories ($\exists G: D \to C$ st $G \circ F \cong \mathbb{1}_C$ & $F \circ G \cong \mathbb{1}_D$) iff F is fully-faithful & essentially surjective.

Adjoint Functors

Let $F: C \to D$ & $G: D \to C$ be functors of the same variance (either both co/contravariant)

↳ We say (F, G) is an <u>adjoint pair</u> (F is the <u>left-adjoint</u> to G, G is the <u>right-adjoint</u> to F) if the following holds:

$\forall A \in ob(C), B \in ob(D)$, we are given isomorphisms of sets:

$$\mathrm{Mor}_D(F(A), B) \xrightarrow[\sim]{\varphi_{A,B}} \mathrm{Mor}_C(A, G(B)),$$

→ same properties as B

st $\forall f \in \mathrm{Mor}_C(A, A')$, $g \in \mathrm{Mor}_D(B, B')$, the following diagrams commute in the covariant case (swap vertical arrow dir for contra):

⇢ changing functors in mor

$$\begin{tikzcd} \mathrm{Mor}_D(F(A), B) \arrow[r, "\varphi_{A,B}"] & \mathrm{Mor}_C(A, G(B)) \\ \mathrm{Mor}_D(F(A'), B) \arrow[r, "\varphi_{A',B}"] \arrow[u, "(-)\circ F(f)"] & \mathrm{Mor}_C(A', G(B)) \arrow[u, "(-)\circ f"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mor}_D(F(A), B) \arrow[r, "\varphi_{A,B}"] \arrow[d, "g\circ(-)"] & \mathrm{Mor}_C(A, G(B)) \arrow[d, "G(g)\circ(-)"] \\ \mathrm{Mor}_D(F(A), B') \arrow[r, "\varphi_{A,B'}"] & \mathrm{Mor}_C(A, G(B')) \end{tikzcd}$$

Let $(-)$ be $h \in \mathrm{Mor}_C(A', G(B))$ st $h: A' \to G(B)$

↳ If we want to swap A' to A in h we must compose it w/ $f: A \to A'$

↳ Go from $A' \to G(B)$ to $A \to G(B)$:

$$A' \xrightarrow{h} G(B) \quad \Rightarrow \quad A \xrightarrow{f} A' \xrightarrow{h} G(B)$$

$$\underbrace{A \xrightarrow{h \circ f} G(B)}$$

Unintuitive! I expected to precompose w/ f^{-1} to go from A' to A

↳ Precomposition is contravariant

We say that the isomorphisms $\varphi_{A,B}$ are <u>natural</u>.

Tensor Product Definitions

Before getting into the tensor product definitions between objects, we must briefly review rings & modules

↳ <u>Rings</u> are abelian groups whose operation is called addition, w/ a second binary operation called multiplication that is associative, is distributive over the addition operation, and has a multiplicative identity element

↳ Rings are <u>more general</u> than fields as they don't necessarily have to be commutative, don't necessarily need multiplicative inverses, and don't necessarily need a multiplicative identity.

↳ A <u>module</u> is a generalization of a vecS in which a field of scalars is replaced by a ring. It also generalizes abelian groups (abelian groups are modules over the ring of integers).

So a module in essence is a generalized vcS our a ring

↳ Left modules how the elements of a ring act on the module elements from the left side. If given right modules, it acts from the right side instead.

Now, consider the categories ${}_R\text{Mod}$ or Mod_R.

↳ Let R be a ring (associative w/ 1 but not necessarily commutative). The category ${}_R\text{Mod}$ is the category of left R-modules M.

↳ Namely, M is an abelian group & ∃ a function:

$$R \times M \longrightarrow M, \quad (r, m) \longmapsto rm,$$

st i) $(r_1 + r_2)m = r_1 m + r_2 m$ ii) $(r_1 r_2)m = r_1(r_2 m)$
iii) $1m = m$ iv) $r(m_1 + m_2) = rm_1 + rm_2$

A morphism between two left R-modules M, N is a function:

$$f : M \longrightarrow N,$$

which is a homomorphism that satisfies $f(rm) = r f(m)$
(for right R-modules it satisfies $f(mr) = f(m)r$)

Let R be a ring, let $A \in \text{Mod}_R$ (remember $A \in \text{ob}(\text{Mod}_R)$), $B \in {}_R\text{Mod}$ be modules. An R-biadditive* map is a function:

$$f : A \times B \longrightarrow H,$$

where H is an abelian group. The map satisfies the properties:

i) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$
ii) $f(a_1, b_1 + b_2) = f(a_1, b_1) + f(a_1, b_2)$
iii) $f(ar, b) = f(a, rb)$

We want construct an abelian group, called the tensor product of A and B over R:

$$A \otimes_R B,$$

$$* \; f(x+y, z+q) = \overbrace{f(x, z+q) + f(y, z+q)}^{\text{additive in both arguments}}$$
$$= f(x+y, z) + f(x+y, q)$$

together w/ an R-biadditive function $\varphi : A \times B \longrightarrow A \otimes_R B$ st $\forall$ R-biadditive map $f : A \times B \longrightarrow H$ $\exists$ a unique group homomorphism $g : A \otimes_R B \longrightarrow H$ st the diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[dr, "f"'] & A \otimes_R B \arrow[d, "g"] \\ & H \end{tikzcd}$$

You can define scalar product: $s(a \otimes b) = (sa) \otimes b$

bimodule $s(mr) = (sm)r$

Furthermore, if S is another ring & $A \in {}_S\mathrm{Mod}_R$, then $A \otimes_R B \in {}_S\mathrm{Mod}$ & we get the functors:

- $A \otimes_R (-) : {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod}$
- $\mathrm{Hom}_S(A, (-)) : {}_S\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}$

Moreover, it could be proven that $(A \otimes_R (-), \mathrm{Hom}_S(A, -))$ is an adjoint pair. This means $\forall\, B \in {}_R\mathrm{Mod}$, $C \in {}_S\mathrm{Mod}$ we have a natural isomorphism:

$$\varphi_{B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$

Construction of the Tensor Product

Let R be a ring & $A \in \mathrm{Mod}_R$, $B \in {}_R\mathrm{Mod}$ be modules. First we start w/ a free abelian group G (an abelian group w/ a basis) on the symbols $(a,b) \in A \times B$:

$$G = \bigoplus_{(a,b)} \mathbb{Z} \cdot (a,b)$$

This might seem strange at first, so consider the set $S = \mathbb{Z} \cdot X = \{aX : a \in \mathbb{Z}\}$ & X is a symbol (could be a variable).

↳ To select an element of S, we must select an element of $\mathbb{Z}$ which is labelled by X

↳ To select an element of G, we must select an element of $\mathbb{Z}$, which is labelled by symbols (a,b)

↳ Each element of G is a finite sum as we want only finitely many non-zero coefficients ($\in \mathbb{Z}$), given by a basis $(a,b) \in A \times B$

We can consider a subgroup $N \subset G$ which is generated (you can perform group operations on the generators & recover all the elements of the group which it generates) by:

$\cdot\ (a_1 + a_2, b) - (a_1, b) - (a_2, b)$

$\cdot\ (a, b_1 + b_2) - (a, b_1) - (a, b_2)$

$\cdot\ (ar, b) - (a, rb)$

$a, a_1, a_2 \in A$
$b, b_1, b_2 \in B$
$r \in R$

We thus define the tensor product of modules A, B over a ring R as:

$$A \underset{R}{\otimes} B = G/N$$

This is a quotient group is a set of equivalence classes of elements of G, where elements of the classes differ by elements of N. (alternate is projecting out elements of G which are not invariant under N).

↳ The image of the element (a,b) of G is denoted by $a \otimes b$ & is given by the R-biadditive map:

$$\varphi : A \times B \longrightarrow A \underset{R}{\otimes} B, \quad (a,b) \longrightarrow a \otimes b$$

$\Rightarrow$ The group $A \otimes_R B$ is called the tensor product of A & B over R & its elements are called tensors

Properties of the Tensor Product

Prop (Functoriality of Tensor Products):

Let $f \in Hom_R(A, A')$, $g \in Hom_R(B, B')$. $\exists$ a group homorphism:

$$f \otimes g : A \otimes_R B \longrightarrow A' \otimes_R B' \text{ st } (f \otimes g)(a \otimes b) = \underbrace{f(a)}_{a'} \otimes \underbrace{g(b)}_{b'}.$$

Prop (Properties of Tensor Products):

i) $A \otimes_R R = A$, $R \otimes_R B \cong B$

ii) $(\bigoplus_{i \in I} A_i) \otimes_R B \cong \bigoplus_{i \in I} (A_i \otimes_R B)$, $A \otimes (\bigoplus_{i \in I} B_i) = \bigoplus_{i \in I} A \otimes_R B_i$

iii) If R is commutative then $A \otimes_R B \cong B \otimes_R A$.

ex:

Let k be a field, V, W be finite vecs over k.

Then canonically:

$$Hom_k(V, W) \cong V^* \otimes_k W$$

Tensor Product of Algebras

Let R be a commutative ring. A ring A is called an R-algebra if we are given a homomorphism of rings:

$$i_A : R \longrightarrow A$$

whose image is contained in the centre $Z(A)$ of A (a subring consisting of the elements x st $xy = yx \ \forall y \in A$).

↳ That is: $i_A(r) \cdot a = a \cdot i_A(r) \quad \forall r \in R, a \in A$

Thm (Association):

If A, B are R-algebras, so is $A \otimes_R B$ & we have the following:

$$a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$$

The Adjoint Property for $\otimes$ & Hom

Thm ($\otimes$ Functors):

Let R, S be rings & $A \in {}_S\mathrm{Mod}_R$. The following functors:

$$A \underset{R}{\otimes} (-) : {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod} \ , \ \mathrm{Hom}_S(A, -) : {}_S\mathrm{Mod} \longrightarrow {}_R\mathrm{Mod}$$

are an adjoint pair of functors $(A \otimes_R (-), \mathrm{Hom}_S(A, -))$.

↳ Namely, there are natural isomorphisms $\varphi_{B,C}$ for $B \in {}_R\mathrm{Mod}$, $C \in {}_S\mathrm{Mod}$:

$$\mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow[\cong]{\varphi_{B,C}} \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$

↳ Similarly, if $B \in {}_R\mathrm{Mod}_S$ then $((-) \otimes_R B, \mathrm{Hom}_S(B, -))$ is an adjoint pair.

> We can apply this to something which is known as Frobenius reciprocity:

Let $H \subset G$ be finite groups & k be a field. Any k-representation (ρ, W) of G over a vs W:

$$W = \text{vs over } k \ , \quad \rho : G \longrightarrow GL_k(W),$$

induces a k-representation of H, $(g|_H, W)$ as:

$$g|_H : H \longrightarrow GL_k(W),\ g|_H(h) = g(h)$$

↳ Another way to view this is to think of W as a $k[G]$-module. Here $k[G]$ is a group ring[△] & so the $k[G]$-module which is a k-rep of G is also the group algebra of G over k.

↳ Since $k[H] \subset k[G]$, W becomes a $k[H]$-module.

This gives us a restriction functor:

$$\mathrm{Res}^G_H : {}_{k[G]}\mathrm{Mod} \longrightarrow {}_{k[H]}\mathrm{Mod}$$

On the other hand, $k[G]$ is infact a bimodule in ${}_{k[G]}\mathrm{Mod}_{k[H]}$ & so we get the induction functor:

$$\mathrm{Ind}^G_H : {}_{k[H]}\mathrm{Mod} \rightarrow {}_{k[G]}\mathrm{Mod},\quad \mathrm{Ind}^G_H(V) = k[G] \otimes_{k[H]} V$$

Using the tensor-Hom adjoint property we find that:

$$\mathrm{Hom}_{k[G]}(\mathrm{Ind}^G_H(V), M) = \mathrm{Hom}_{k[H]}(V, \mathrm{Hom}_{k[G]}(k[G], W))$$
$$= \mathrm{Hom}_{k[H]}(V, W) = \mathrm{Hom}_{k[H]}(V, \mathrm{Res}^G_H(W))$$

Thm (Frobenius Reciprocity):

Let k be a field & $H \subset G$ be finite groups. We have:

$$\mathrm{Hom}_{k[G]}(\mathrm{Ind}^G_H(V), W) = \mathrm{Hom}_{k[H]}(V, \mathrm{Res}^G_H W)$$

Tensor Products Over a Commutative Ring

Let R be a commutative ring & A, B be R-modules

↳ We can think of A, B as R-bimodules $(ra = ar,\ rb = br)$ & so $A \otimes_R B$ is an R-module as well w/ $r \cdot a \otimes b = ra \otimes b = a \otimes rb$

△ A ring formed by taking elements of G as a basis & coeffs from k. Otherwise, all formal LC of G w/ coeffs k.

Let H be an R-module & let $f: A \times B \longrightarrow H$ be an R-bilinear map of R-modules from $A \times B$ to H

↳ Thinking of f as R-biadditive, we get a group homomorphism:

$$g: A \otimes_R B \longrightarrow H, \quad g\left(\sum_i a_i \otimes b_i\right) = \sum_i f(a_i, b_i)$$ } is an R-module homomor

It turns out that $A \otimes_R B$ has a universal property in the category of R-modules: relative to R-bilinear maps $A \times B \longrightarrow H$ into R-modules H.