Part I
Introduction to quantum mechanics

I. OVERVIEW

II. QUANTUM STATES

A. State vectors

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \\ &= \sum_n \alpha_n |n\rangle , \end{aligned} \tag{1}$$

where $\alpha_n \in \mathbb{C}$, and for normalisation,

$$\sum_n |\alpha_n|^2 = 1, \tag{2}$$

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \alpha |0\rangle + \beta |1\rangle . \end{aligned} \tag{3}$$

$$\begin{aligned} \langle\psi| &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^\dagger \\ &= (\alpha^*, \beta^*) \\ &= \alpha^* \langle 0| + \beta^* \langle 1| . \end{aligned} \tag{4}$$

$$\begin{aligned} |\psi\rangle &= \sum_n \alpha_n |n\rangle , \\ |\phi\rangle &= \sum_n \beta_n |n\rangle , \end{aligned} \tag{5}$$

The *overlap* between two states is defined as,

$$\langle\psi|\phi\rangle = \sum_n \alpha_n \beta_n^*, \tag{6}$$

The normalisation condition from Eq. (2) implies,

$$\langle\psi|\psi\rangle = 1. \tag{7}$$

Because basis states are orthonormal, this implies that for basis states $|m\rangle$ and $|n\rangle$,

$$\langle m|n\rangle = \delta_{m,n}. \tag{8}$$

B. Composite systems

$$\begin{aligned} |\psi\rangle_{A,B} &= |\psi\rangle_A \otimes |\phi\rangle_B \\ &= \sum_{m,n} \alpha_m \beta_n |m\rangle \otimes |n\rangle . \end{aligned} \tag{9}$$

$$|\psi\rangle_{A,B} = \sum_{m,n} \lambda_{m,n} |m\rangle \otimes |n\rangle . \tag{10}$$

In general, $\lambda_{m,n}$ may not be separable as $\lambda_{m,n} = \alpha_m \beta_n$.

$$\begin{aligned} |\psi\rangle_{A,B} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B). \end{aligned} \tag{11}$$

Cannot be written in separable form as $|\psi\rangle_{A,B} = |\psi\rangle_A \otimes |\phi\rangle_B$. This is a so-called *entangled state*, whereby the two subsystems exhibit a type of quantum correlation with no classical analogue.

C. Density operators

For an n-dimensional Hilbert space, the density operator, $\hat{\rho}$, is an $n \times n$ complex Hermitian matrix, satisfying,

$$\hat{\rho} = \hat{\rho}^\dagger. \tag{12}$$

$$\hat{\rho} = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} .$$

$$\mathrm{tr}(\hat{\rho}) = \sum_i \hat{\rho}_{i,i} = 1, \tag{13}$$

for normalisation.

$$\begin{aligned} \hat{\rho} &= |\psi\rangle \langle\psi| \\ &= \begin{pmatrix} |\alpha|^2 & \alpha\beta \\ \alpha^*\beta^* & |\beta|^2 \end{pmatrix} . \end{aligned}$$

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i, \tag{14}$$

where the probabilities are normalised such that,

$$\sum_i p_i = 1. \tag{15}$$

Purity,

$$\mathcal{P} = \mathrm{tr}(\hat{\rho}^2), \tag{16}$$

where $\mathcal{P} = 1$ only for pure states.

D. Reduced states

$$\hat{\rho}_A = \mathrm{tr}_B(\hat{\rho}_{A,B}). \tag{17}$$

$$\mathrm{tr}_B(\hat{\rho}_A \otimes \hat{\rho}_B) = \mathrm{tr}(\hat{\rho}_B) \cdot \hat{\rho}_A. \tag{18}$$

From the cyclic property of the trace, it follows that,

$$\mathrm{tr}(|\psi\rangle\langle\phi|) = \langle\psi|\phi\rangle . \tag{19}$$

III. EVOLUTION

$$\hat{U} = e^{-i\hat{H}t}. \tag{20}$$

IV. MEASUREMENT

Measurement operator $\hat{M}$, let the eigenvectors be the so-called *measurement projectors*,

$$\hat{M}_i = |m_i\rangle\langle m_i| , \tag{21}$$

where,

$$\sum_i \hat{M}_i = \hat{\mathbb{I}}. \tag{22}$$

$$p_i = \mathrm{tr}(\hat{M}_i \hat{\rho}). \tag{23}$$

$$\hat{\rho}_i = \frac{\hat{M}_i \hat{\rho} \hat{M}_i^\dagger}{\mathrm{tr}(\hat{M}_i \hat{\rho} \hat{M}_i^\dagger)}. \tag{24}$$