# Continuous variable

(Dated: March 1, 2018)

## I. MOTIVATION

Up to this moment we have considered systems with a discrete number of dimensions, we now consider a system with continuous variables, i.e. an infinite number of dimensions, which is based on the quantum harmonic oscillator. The main motivation for using continuous variables, however, comes from the fact that all the main quantum information protocols can be implemented efficiently in an experiment [**?** ].

Here we focus on optical Gaussian states as they are the natural choice for communication protocols. Before delving into quantum communication and computation, in Sec. **??** we first intoduce the basic tools of quantum optics.

## II. CONTINUOUS VARIABLES IN QUANTUM OPTICS

In quantum optics, the quantized electromagnetic modes correspond to to quantum harmonic oscillators. Unlike for classical harmonic oscillators, in quantum mechanics, the position and momentum observables are non-commuting Hermitian operators:

$$[\hat{x}, \hat{p}] = i\hbar \tag{1}$$

For each single mode $k$, the Hamiltonian for a quantum harmonic oscillator is

$$\begin{aligned}\hat{H}_k &= \frac{1}{2}(\hat{p}_k^2 + \omega_k \hat{x}_k^2) && (2)\\ &= \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) && (3)\\ &= \hbar\omega_k(n + \frac{1}{2}) && (4)\end{aligned}$$

where $n$ corresponds to the number of quanta of energy of the oscillator. The creation and annihilation operators can be written in terms of the $\hat{x}_k$ and $p_k$ operators

$$\hat{a}_k^\dagger = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{x}_k - i\hat{p}_k), \tag{5}$$

$$\hat{a}_k = \frac{1}{\sqrt{2\hbar\omega_k}}(\omega_k \hat{x}_k + i\hat{p}_k), \tag{6}$$

or conversely

$$\hat{x}_k = \sqrt{\frac{\hbar}{2\omega_k}}(\hat{a}_k + \hat{a}_k^\dagger), \tag{7}$$

$$\hat{p}_k = -i\sqrt{\frac{\hbar\omega_k}{2}}(\hat{a}_k - \hat{a}_k^\dagger). \tag{8}$$

The position and momentum operators represent the quadratures of a the mode, and correspond to the real and imaginary parts of the oscillator's amplitude, see Fig. **??**.

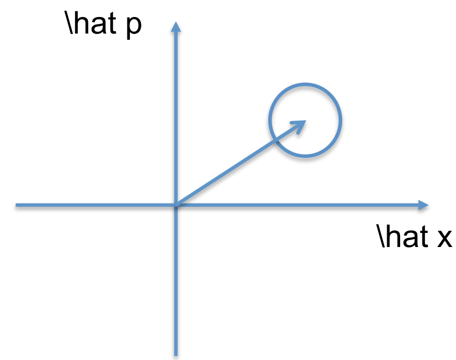


FIG. 1. Representation of a state vector in phase space with position ($\hat{x}$) and momentum ($\hat{p}$) quadratures. The circle indicates the uncertainty associated with the quadrature, where the Heisenberg uncertainty principle imposes that $\Delta x \Delta p \geq \hbar/2$.

The position representation corresponds to expressing a state vector $|\psi\rangle$ in the position basis:

$$|\psi\rangle = \int dx \, \langle x|\psi\rangle \, |x\rangle = \int dx \psi(x) \, |x\rangle \,, \tag{9}$$

where $|x\rangle$ is the eigenstate of the position operator. Here the wave function in the position representation is defined as

$$\psi(x) = \langle x|\psi\rangle \,. \tag{10}$$

Let us now find the wavefunction. Since $\hat{a}\,|0\rangle = 0$, we know that

$$\frac{1}{\sqrt{2}}(\sqrt{m\omega}\hat{x} + \frac{1}{\sqrt{m\omega}} i\hat{p}) \, |0\rangle = 0. \tag{11}$$

Given $\psi_0(x) = \langle x|0\rangle$,

$$\begin{aligned}&(m\omega x + \frac{d}{dx})\psi_0(x) = 0 && (12)\\ &\to \psi_0(x) \propto e^{-m\omega x^2/2}. && (13)\end{aligned}$$

The other position eigenstates can be built using Hermite Polynomials $H_n(x)$, with solutions

$$\psi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{2^n n!}} H_n(x)\psi_0(x). \tag{14}$$

The quadrature eigenstates are mutually related to each other by a Fourier transform:

$$|x\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-2ixp} |p\rangle \, dp, \tag{15}$$

$$|p\rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{+2ixp} |x\rangle \, dx. \tag{16}$$

$$\tag{17}$$

It is worth mentioning that the quadrature eigenstates are unphysical and unormalized, but they are very useful in performing the idealized quantumn information protocols.

## III. UNITARY OPERATORS IN CV

### A. The displacement operator

The one-mode displacement operator generates the coherent states from vacuum.

$$D(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}. \tag{18}$$

An important property (BCH)

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \tag{19}$$

This acts on the vacuum to give the coherent state, here $|0\rangle$ denotes a vacuum state.

$$\begin{aligned} |\alpha\rangle &= D(\alpha) |0\rangle & (20) \\ &= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle & (21) \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle & (22) \end{aligned}$$

### B. Phase rotation

Another important feature in the CV formalism is the phase degree of freedom. The phase of an electromagnetic field has no meaning on its own, and is usually defined with respect to a strong local oscillator. The phase rotation operator is given by

$$R(\theta) = e^{i\theta(\hat{a}^\dagger \hat{a})} \tag{23}$$

In the Heisenberg picture,

$$R^\dagger(\theta) a R(\theta) = a e^{-i\theta}, \tag{24}$$

and the more general quadrature operators

$$\begin{aligned} \hat{x}^\theta &= \hat{a} e^{-i\varphi} + \hat{a}^\dagger e^{-i\varphi} & (25) \\ \hat{p}^\theta &= \hat{a} e^{-i\varphi} - \hat{a}^\dagger e^{-i\varphi} & (26) \end{aligned}$$

can be obtained by applying $R(\theta)$. They transform as

$$\begin{pmatrix} \hat{x}^\theta \\ \hat{p}^\theta \end{pmatrix} = \begin{pmatrix} \hat{\cos}(\theta) & \sin(\theta) \\ \hat{-}\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}. \tag{27}$$

### C. Entanglement generation

The essential ingredient to generate entanglement in CV is squeezed light. A widely used method for generating single photons is via spontaneous parametric down conversion in a nonlinear medium [? ? ]. Here, a "classical" (i.e. strong) electromagnetic field drives the medium, and pairs of correlated photons of the same frequency can be generated. The effective Hamiltonian of one such process can be written as

$$H = \epsilon \hat{a}_1^\dagger \hat{a}_2^\dagger + \epsilon^* \hat{a}_1 \hat{a}_2, \tag{28}$$

$\epsilon$ is the amplitude of the driving field strength.

#### *1. One-mode squeezing*

The one-mode squeeze operator is given by

$$\hat{S}(\zeta) = \exp(\zeta^* \frac{\hat{a}^2}{2} - \zeta \frac{\hat{a}^{\dagger 2}}{2}) \tag{29}$$

where $\zeta = re^{i\phi} = -i\epsilon t/h$, $r$ is known as the squeeze parameter, which will determine the size of the squeezing and $\varphi \in [0, 2\pi]$. The parameter $\zeta$ is a function of the int feraction time (i.e. length of the medium), the amplitude of the driving field strength $\epsilon$.

The creation and annihilation operators transform under the squeezing operator in the following way

$$\begin{aligned} \hat{S}^\dagger(\zeta) \hat{a} S(\zeta) &= \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r & (30) \\ \hat{S}^\dagger(\zeta) \hat{a}^\dagger S(\zeta) &= \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r & (31) \end{aligned}$$

These can be derived from the BakerCampbellHausdorff (BCH) relations, here we derive the first equation in (**??**). Firstly, we need the following BCH identity:

$$\begin{aligned} e^X Y e^{-X} &= Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] \\ &+ \frac{1}{3!}[X, [X, [X, Y]]] + ... \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{S}^\dagger(\zeta) =& \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) \\ =& \hat{a} + \underbrace{[\zeta \frac{\hat{a}^{\dagger 2}}{2} - \zeta^* \frac{\hat{a}^2}{2}, \hat{a}]}_{= -\zeta \hat{a}^\dagger} + \\ & \frac{1}{2!} \underbrace{[\zeta \frac{\hat{a}^{\dagger 2}}{2} - \zeta^* \frac{\hat{a}^2}{2}, -\zeta \hat{a}^\dagger]}_{= \zeta^* \zeta \hat{a}} + \\ & \frac{1}{3!} [\zeta \frac{\hat{a}^{\dagger 2}}{2} - \zeta^* \frac{\hat{a}^2}{2}, +|\zeta|^2 \hat{a}] \\ =& \sum_m \frac{|\zeta|^{2m}}{(2m)!} \hat{a} - \frac{|\zeta|^{2m} \zeta}{(2m+1)!} \hat{a}^\dagger \\ =& \hat{a} \cosh |\zeta| - \hat{a}^\dagger \sinh |\zeta| e^{i\theta}. \end{aligned} \tag{33}$$

If we choose $\theta = 0$, the quadrature operators transform as

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger) && (34)\\
&\to \sqrt{\frac{\hbar}{2\omega}}\hat{S}^\dagger(\zeta)(\hat{a} + \hat{a}^\dagger)\hat{S}(\zeta) && (35)\\
&= \sqrt{\frac{\hbar}{2\omega}}(\hat{a}\cosh r - \hat{a}^\dagger \sinh r + && (36)\\
&\quad \hat{a}^\dagger \cosh r - \hat{a}\sinh r) && (37)\\
&= \hat{x}e^{-r}, && (38)\\
\hat{p} &\to \hat{p}e^{r}. && (39)
\end{aligned}$$

That is, the squeezing operator amplifies (the uncertainty of) one quadrature at the expense of the other.

If we want to write $\hat{S}(\zeta)\left|0\right\rangle$ in the photon number basis, we need to express $\hat{S}(\zeta)$ in a more convenient form. It can be shown [**?** ] that if the operators $\hat{A}$ and $\hat{B}$ obey the commutation relations

$$[\hat{A}, \hat{B}] = \hat{H}, \quad [\hat{H}, \hat{A}] = 2\hat{B}, \qquad [\hat{H}, \hat{B}] = 2\hat{A} \quad (40)$$

then

$$e^{t(\hat{A}+\hat{B})} = e^{\tanh r\hat{B}} e^{\log(\cosh t)\hat{H}} e^{\tanh t\hat{A}} \quad (41)$$

If we choose

$$X = \hat{a}^2 e^{-i\theta}/2, \qquad Y = -\hat{a}^\dagger e^{i\theta}/2, \quad (42)$$

the squeeze operators can now be factored into a product of exponentials

$$\begin{aligned}
&\exp(-(\hat{a}^\dagger)^2 e^{i\theta}\tanh r/2)\times && (43)\\
&\exp[-\frac{1}{4}[\hat{a}^2\hat{(a^\dagger)}^2 - (\hat{a}^\dagger)^2\hat{a}^2]\log(\cosh r)]\times\\
&\exp(\hat{a}^2 e^{i\theta}\tanh r/2).
\end{aligned}$$

Using the commutation relation

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad (44)$$

after some simple algebra, it can be shown that

$$\hat{S}(\zeta)\left|0\right\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}(\tanh r)^n \frac{\sqrt{(2n)!}}{2^n n!}\left|2n\right\rangle. \quad (45)$$

*2. Two-mode squeezing*

The two-mode squeeze operator is then given by

$$\hat{S}_{12}(\zeta) = \exp(\zeta^*\hat{a}_1\hat{a}_2 - \zeta\hat{a}_1^\dagger\hat{a}_2^\dagger) \quad (46)$$

It can be easily verified that the creation/annihilation operators for mode 1 and 2 transform under the operator as

$$\begin{aligned}
S_{12}(\zeta)^\dagger a_1 S_{12}(\zeta) &= \hat{a}_1\cosh r - \hat{a}_2^\dagger e^{i\theta}\sinh r && (47)\\
S_{12}(\zeta)^\dagger a_2 S_{12}(\zeta) &= \hat{a}_2\cosh r - \hat{a}_1^\dagger e^{i\theta}\sinh r && (48)
\end{aligned}$$

One can create a two-mode squeezed state by in mode 1 (2) we choose $\theta = 0(\pi/2)$

$$\begin{aligned}
\hat{a}_1 &= a_1(0)\cosh r - a_1(0)^\dagger \sinh r && (49)\\
\hat{a}_2 &\to a_2(0)\cosh r + a_2(0)^\dagger \sinh r && (50)\\
& && (51)
\end{aligned}$$

After the beam splitter:

$$\begin{aligned}
\hat{b}_1(0) &= 1/\sqrt{2}(a_1 + a_2) && (52)\\
\hat{b}_2(0) &= 1/\sqrt{2}(a_1 - a_2) && (53)\\
b_1 &= b_1(0)\cosh r - b_2^\dagger(0)\sinh r && (54)\\
b_2 &= b_2(0)\cosh r - b_1^\dagger(0)\sinh r && (55)
\end{aligned}$$

The resulting state is a two-mode squeezed state .

When represented in the photon number basis, we see that when the signal and idler modes (labelled with subscripts 1 and 2) are initially in vacuum states, we obtain the interaction [**?** ]

$$\begin{aligned}
\left|\zeta\right\rangle_{12} &= \hat{S}(\zeta)_{12}\left|0\right\rangle_1\left|0\right\rangle_2\\
&= \frac{1}{\cosh r}\sum_{n=0}^{\infty}(e^{i\phi}\tanh r)^n\left|n\right\rangle_1\left|n\right\rangle_2. && (56)
\end{aligned}$$

The two modes of the state are also quantum correlated in photon number and phase. The mean photon number in this state is given by

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{\cosh^2 r}(\tanh r)^{2n} n\\
&= \sinh^2 r. && (57)
\end{aligned}$$

## IV. MEASUREMENT

The most common measurement used for Gaussian states is homodyne detection, which measures the quadrature $\hat{x}$ or $\hat{p}$. In order to detect a quadrature of the mode $b$, it must be combined with a strong local oscillator at a $50:50$ beam splitter. The local oscillator is a coherent state $\left|\alpha_{LO}\right\rangle$, and can be described by a complex classical amplitude. The input and output modes are described by the relations

$$\begin{aligned}
\hat{b}_2^\dagger &= 1/\sqrt{2}(\hat{b}^\dagger + \alpha_{LO}^*) && (58)\\
\hat{a}_2^\dagger &= 1/\sqrt{2}(\hat{b}^\dagger - \alpha_{LO}^*) && (59)\\
\hat{b}_2 &= 1/\sqrt{2}(\hat{b} + \alpha_{LO}) && (60)\\
\hat{a}_2 &= 1/\sqrt{2}(\hat{b} - \alpha_{LO}) && (61)
\end{aligned}$$

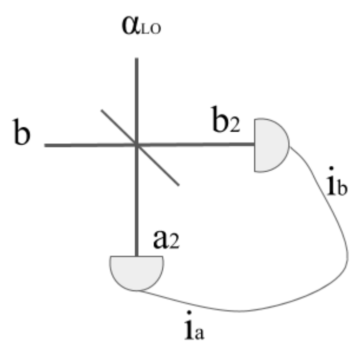


FIG. 2. Schematic for balanced homodyne detection.

If we measure the difference in photocurrent (or photon number)

$$i_b - i_a \propto b_2^\dagger b_2 - a_2^\dagger a_2 \tag{62}$$
$$= b^\dagger \alpha_{LO} + b\alpha_{LO}^*. \tag{63}$$

By introducing the phase $\phi$ of the local oscillator, $\alpha_{LO} = |\alpha|e^{i\theta}$, the expression in Eq. (**??**) simplifies to

$$|\alpha|(b^\dagger e^{i\theta} + be^{-i\theta}). \tag{64}$$

We identify that this is the quadrature observable $\hat{x}^\theta$.

With all the tools in hand, we are now ready to look at quantum communication protocols in CV. [**?** ]

## V. QKD IN CV

A seminar result in CV is the discovery that coherent states are sufficient to perform quantum key distribution. It relies on the fact that coherent states are nonorthogonal. Let us now explicitly describe the coherent beam protocols of this family:

1. Alice draws two random numbers $x_A$ and $p_A$, with zero mean.
2. She sends to Bob the coherent state $|x_A + ip_A\rangle$
3. Bob randomly chooses to measure either $X$ or $P$ using homodyne detection.
4. Using a classical public channel, he informs Alice about the observable that he measured. Analogous to the BB84 protocol, half of the key generated by Alice is discarded.
5. Alie keeps the bit string value $x_A$ or $p_A$ which matched Bob's quadratures.
6. Alice and Bob now share two correlated Gaussian variables. Then they may use classical protocols, e.g. the sliced reconciliation protocol to transform it into a key bit strings.
7. They use privacy amplification to distill the private key.

## VI. TELEPORTATION

Fig. **??**

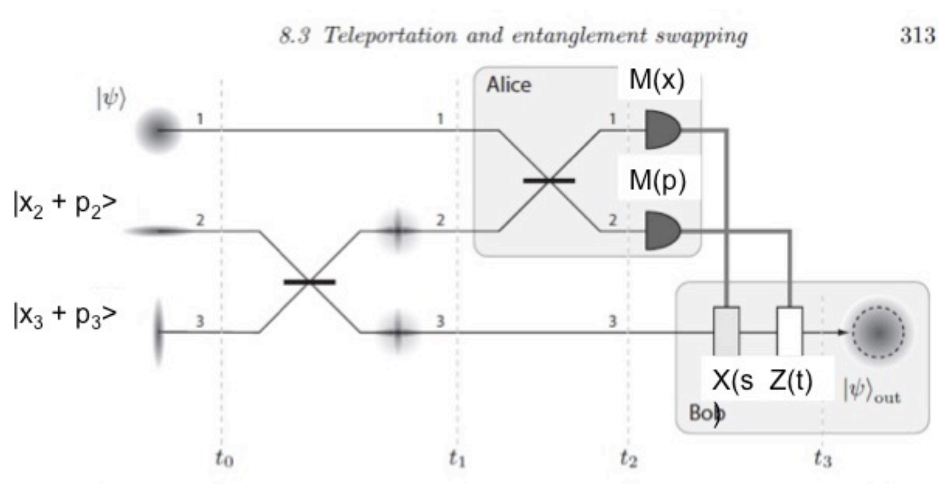


FIG. 3.

Protocol extended to CV by [**? ?** ]

[**?** ]

$$|p\rangle = \int dx e^{i/\hbar px}\,|x\rangle\,, \qquad |x\rangle = \int dx e^{i/\hbar xp}\,|p\rangle \tag{65}$$

Displacement operator

$$X(s) = e^{-i/\hbar x\hat{p}} \tag{66}$$
$$X(s)\left|x\right\rangle = \int dp e^{-i/\hbar px} e^{-i/\hbar x\hat{p}}\left|p\right\rangle \tag{67}$$
$$= dp e^{-i/hbar p(x+s)}\left|p\right\rangle \tag{68}$$
$$= \left|x+s\right\rangle \tag{69}$$

This is the continuous-variable generalization of the Pauli X operator, in that it displaces the value of the state by amount $s$.

The analogue of the Pauli Z operator adds a state-dependent phase

$$Z(t) = e^{i/\hbar t\hat{x}} \tag{70}$$
$$e^{i/\hbar t\hat{x}}\left|x\right\rangle = e^{i/\hbar px}\left|x\right\rangle \tag{71}$$

and acts on the momentum basis state $\left|p\right\rangle$ as

$$Z(t)\left|p\right\rangle = \left|p+t\right\rangle. \tag{72}$$

Before the squeezing operator is applied, we set $t = 0$.

1. At $t = t_0$, the vacuum states in mode 2 and 3 are squeezed. The quadrature operators become

$$x_2(t_0) \to x_2(0)e^r \qquad p_2 \to p_2(0)e^{-r} \tag{73}$$
$$x_3(t_0) \to x_3(0)e^r \qquad p_3 \to p_3(0)e^{-r} \tag{74}$$

2. The 50:50 beam splitter in modes 2 and 3 creates the approximate Bell state necessary for teleportation, and this induces the quadrature transformations

$$x_2(t_1) = \frac{x_2(t_0)+x_3(t_0)}{\sqrt{2}} \qquad x_3(t_1) = \frac{x_2(t_0)-x_3(t_0)}{\sqrt{2}}$$
$$p_2(t_1) = \frac{p_2(t_0)+p_3(t_0)}{\sqrt{2}} \qquad p_3(t_1) = \frac{p_2(t_0)-p_3(t_0)}{\sqrt{2}} \tag{75}$$

3. The second beam splitter induces the transformation

$$x_1(t_2) = \frac{x_1(t_1)-x_2(t_1)}{\sqrt{2}} \tag{76}$$
$$= \frac{x_1(t_1)}{\sqrt{2}} - \frac{x_2(t_0)+x_3(t_0)}{2} \tag{77}$$
$$p_2(t_2) = \frac{p_1(t_1)+p_2(t_1)}{\sqrt{2}} \tag{78}$$
$$= \frac{p_1(t_0)}{\sqrt{2}} + \frac{p_2(t_0)+p_3(t_0)}{2} \tag{79}$$

Now, if we rearrange Eq. (??) to make $x_2(t_0)$ the subject, and identify that $x_1(t_1) = x_1(t_0)$

$$x_2(t_0) = -2x_1(t_2) + \sqrt{2}x_1(t_1) - x_3(t_0) \tag{80}$$
$$p_2(t_0) = -2p_2(t_2) + \sqrt{2}p_1(t_0) - p_3(t_0) \tag{81}$$

Now, for Bob, $x_3(t_2) = x_3(t_1), p_3(t_2) = p_3(t_1)$, whose state is now

$$x_3(t_2) = \frac{-2x_1(t_2)+\sqrt{2}x_1(t_1)-x_3(t_0)-x_3(t_0)}{\sqrt{2}}$$
$$= -\sqrt{2}x_1(t_2) + x_1(t_0) - \sqrt{2}x_3(0)e^{-r} \tag{82}$$
$$p_3(t_2) = \frac{p_2(t_0)-p_3(t_0)}{\sqrt{2}} \tag{83}$$
$$= \frac{-2p_2(t_2)+\sqrt{2}p_1(t_0)-p_3(t_0)-p_3(t_0)}{\sqrt{2}} \tag{84}$$
$$= p_1(t_0) - \sqrt{2}p_2(t_2) - \sqrt{2}p_3(t_0) \tag{85}$$

4. Alice measures the position quadrature $x_1(t_2)$ in mode 1, yielding the outcome $s/\sqrt{2}$, and she measures the momentum quadrature $p_2(t_2)$ in mode 2, yielding the outcome $t/\sqrt{2}$. She sends these measurement outcomes to Bob. applies the displacements

$$X(u) = \exp(-2is\hat{p}_3) \qquad Z(t) = \exp(2it\hat{x}_3) \tag{86}$$

The output quadratures at Bob's mode become

$$x_3 = x_1 - \frac{\sqrt{2}x_3}{e^r} \qquad p_3 = p_1 - \frac{\sqrt{2}p_3}{e^r} \tag{87}$$

Here the input quadratures are teleported to the output quadratures up to a factor $\frac{\sqrt{2}p_3}{e^r}$. In the limit of infinite squeezing $\to \infty$, the teleportated state is perfect.

## VII. ENTANGLEMENT SWAPPING

## VIII. ENTANGLEMENT DISTILLATION

Entanglement distribution between distant parties is an essential component to most quantum communication protocols [? ]. As it has been discussed, entanglement is a resource in a quantum network, and it is of interest for distant parties to share maximally entangled states.

In practice the transmission channel is never perfect, and noise due to interaction with the environment or imperfect gate operations will reduce the entanglement of a state. If one has several copies of some less than maximally entangled state available, it is possible for two parties to concentrate or distill the entanglement.

In contrast to the qubit version, however, it has been proven that for Gaussian states it is impossible to distill more entanglement by using Gaussian operations [? ? ? ]. In order to to distill from Gaussian input states, non-Gaussian operations are necessary.

Proposal [? ], one variant experimentally realized by photon-subtraction [? ].

| Circuit model | CV cluster state |
|---|---|
| Pauli $X$ | $X(s) = \exp[-is\hat{p}]$ |
| Pauli $Z$ | $Z(t) = \exp[it\hat{q}]$ |
| Phase gate | $\hat{P}(\eta) = \exp[i\eta\hat{x}^2]$ |
| Hadamard | $F = \exp[i\frac{\pi}{8}(\hat{p}^2 + \hat{q}^2)]$ |
| $C_Z$ | $C_Z = \exp[\frac{i}{2}q_1 \otimes q_2]$ |
| CNOT | $C_X = \exp[-2i\hat{x}_1 \otimes \hat{p}_2]$ |

TABLE I.

## IX. QUANTUM COMPUTING

### A. Quantum gates

Gates for the qubit circuit model and the analogous operations for CV are summarised in Table **??**. [**?** ]

The operators $X, Z$ can be interpreted as displacement operators

$$D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{88}$$

which correspond to a translation in phase space. $X$ is a translation in position, and Z is a displacement in momentum.

The X and Z operators can be written as X = D(s/2), Z = D(it/2)

The Fourier gate F is the Gaussian analogue of the qubit Hadamard gate, which corresponds to a $\pi/2$ rotation in phase space, and transforms the quadrature eigenstates from one to another:

$$F = \exp(i\pi/4)\exp(\frac{i\pi}{4}\hat{a}^\dagger\hat{a}) = (\hat{p}^2 + \hat{x}^2) \tag{89}$$

$$F\left|s\right\rangle_x = \left|s\right\rangle_p \tag{90}$$

The controlled-phase gate, $C_Z$ is a two-mode Gaussian gate. It is defined as

$$E_z = \exp\left(\frac{i}{2}\hat{x}_1 \otimes \hat{x}_2\right). \tag{91}$$

It transforms on the quadrature eigenstates as

$$C_Z\left|s\right\rangle_1\left|t\right\rangle_2 = \exp(is_1t_2)/2\left|s\right\rangle_1\left|t\right\rangle_2 \tag{92}$$

It transforms the momentum quadratures according to

$$\hat{p}_1 \to \hat{p}_1 + \hat{x}_2, \qquad \hat{p}_2 \to \hat{p}_2 + \hat{x}_1 \tag{93}$$

### B. Cluster state/ measurement based

[**?** ] The advantage of this approach is that both the cluster state preparation and commutation can be performed deterministically.

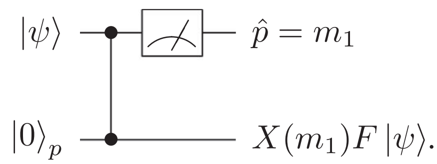


FIG. 4.

Once the state has been created, a sequence of one-mode (separable) measurement

The one-qubit teleportation give insight into how the cluster-state evolves in this model

Consider perfect squeezing, where the second mode is a momentum eigenstate $\left|0\right\rangle_p$. Initially, the state is

$$\left|\psi\right\rangle\left|0\right\rangle_p = \frac{1}{\sqrt{2\pi}}\int dx_1dx_2\psi(x_1)\left|x_1\right\rangle\left|x_2\right\rangle \tag{94}$$

Applying the $C_Z$ gate results in

$$\frac{1}{2\pi}\int dx_1dx_2\psi(x_1)\exp(ix_1x_2/2)\left|x_1\right\rangle\left|x_2\right\rangle \tag{95}$$

After measuring $\hat{p}_1$, the state is projected onto $\left|m_1\right\rangle\left\langle m_1\right|$, we have
$\langle m_1|q_1\rangle = 1/2\pi\exp(-iq_1m_1/2)$

$$\frac{1}{4\pi}\int dx_1dx_2\psi(x_1)\exp(ix_1(q_2 - m_1))q_2 \tag{96}$$

Applying the correction $X(m_1)$ gives back the initial state $\left|\psi\right\rangle$

We can now consider the teleportation of a quantum gate, which is the key of the measurement-based quantum computing. Consider a variation of the above circuit, where the only difference is the addition of a unitary that is diagonal in the computational basis, and therefore commutes with the CPHASE gate, for example $U = \exp(if(\hat{x}))$.

We have just shown that by performing a measurement in the basis $U\hat{x}U^\dagger$, we can absorb the gate into the measurement.

- addition of any non-gaussian projective measurement allows universal QC using CV cluster states
- multimode Gaussian operations can be made in any order

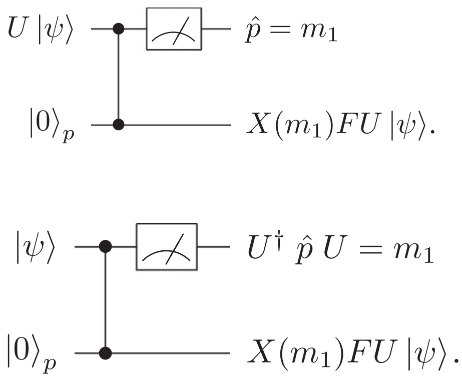
$U\left|\psi\right\rangle$
$\hat{p} = m_1$
$\left|0\right\rangle_p$
$X(m_1)FU\left|\psi\right\rangle.$
$\left|\psi\right\rangle$
$U^\dagger\ \hat{p}\ U = m_1$
$\left|0\right\rangle_p$
$X(m_1)FU\left|\psi\right\rangle.$

FIG. 5.