Midterm in Class March 29. See Webiste for more Information!

Fundamental Algorithms, Assignment 7

Due March 22 in Recitation.

What you need is that your brain is open. – Paul Erdős

1. Determine an LCS of 10010101 and 010110110 using the algorithm studied.

2. Write all the parenthesizations of $ABCDE$. Associate them in a natural way with (setting $n = 5$) the terms $P(i)P(n-i)$, $i = 1, 2, 3, 4$ given in the recursion for $P(n)$.

3. Let $x_1, \ldots, x_m$ be a sequence of distinct real numbers. For $1 \le i \le m$ let $INC[i]$ denote the length of the longest increasing subsequence ending with x_i. Let $DEC[i]$ denote the length of the longest decreasing subsequence ending with x_i. `Caution:` The subsequence must *use* x_i. For example, $20, 30, 4, 50, 10$. Now $INC[5] = 2$ because of $4, 10$ – we do *not* count $20, 30, 50$.

   (a) Find an efficient method for finding the values $INC[i]$, $1 \le i \le n$. (You should find $INC[i]$ based on the previously found $INC[j]$, $1 \le j < i$. Your algorithm should take time $O(i)$ for each particular i and thus $O(n^2)$ overall.)

   (b) Let LIS denote the length of the longest increasing subsequence of $x_1, \ldots, x_m$. Show how to find LIS from the values $INC[i]$. Your algorithm, *starting with* the $INC[i]$, should take time $O(n)$. Similarly, let DIS denote the length of the longest decreasing subsequence of $x_1, \ldots, x_m$. Show how to find DIS from the values $DEC[i]$.

   (c) Suppose $i < j$. *Prove* that it is impossible to have $INC[i] = INC[j]$ *and* $DEC[i] = DEC[j]$. (`Hint:` Show that *if* $x_i < x_j$ then $INC[j] \ge INC[i] + 1$.)

   (d) Deduce (*assume* (3c)) the following celebrated result (called the Monotone Subsequence Theorem) of Paul Erdős and George Szekeres: Let $m = ab + 1$. Then any sequence $x_1, \ldots, x_m$ of distinct real numbers either $LIS > a$ or $DIS > b$. (Idea: Assume not and look at the pairs $(INC[i], DEC[i])$.) Paul Erdős was a great twentieth century mathematician, whose work remains highly influential in many areas.

4. Find an optimal parenthesization of a matrix-chain product whose sequence of dimensions is $5, 10, 3, 12, 5, 50, 6$.

5. Some exercises in logarithms:

   (a) Write $\lg(4^n/\sqrt{n})$ in simplest form. What is its asymptotic value.

   (b) Which is bigger, 5^{313340} or 7^{271251}? Give reason. (You can use a calculator but you can't use any numbers bigger than 10^9.)

   (c) Simplify $n^2 \lg(n^2)$ and $\lg^2(n^3)$.

   (d) Solve (for x) the equation $e^{-x^2/2} = \frac{1}{n}$.

   (e) Write $\log_n 2^n$ and $\log_n n^2$ in simple form.

   (f) What is the relationship between $\lg n$ and $\log_3 n$?

   (g) Assume $i < n$. How many times need i be doubled before it reaches (or exceeds) n?

   (h) Write $\lg[n^n e^{-n}\sqrt{2\pi n}]$ precisely as a sum in simplest form. What is it asymptotic to as $n \to \infty$? What is interesting about the bracketed expression?

There is a theory which states that if ever anybody discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.
Douglas Adams