Linearity & Superposition principle

Equations describing simple harmonic motion are linear, homogeneous, second order differential equations

for example $\ddot{x} + \omega^2 x = 0$ .. (1)

$$\frac{d^2X}{dt^2} + \omega^2 X = 0$$ equation describing SHM.

Combination of two or more simple harmonic oscillations is generally encountered or we can say two or more waves can arrive at the same place in space & overlap.

Consider for example solutions to equation (1).

$x = A\cos\omega t$ is a solution to equation (1)

[A has same dimensions as X]

$\ddot{x} = -A\omega^2\cos(\omega t) = -\omega^2 x.$

Another solution is $x = B\sin\omega t$

[B has same dimensions as X & A]

$\dot{x} = \omega B\cos\omega t$

$\ddot{x} = -\omega^2 B\sin\omega t = -\omega^2 x$

$\therefore$ $x = A\cos\omega t + B\sin\omega t$ should also be a solution & is indeed one.

Say $A = a\sin\phi$ & $B = a\cos\phi$

then $A^2 + B^2 = a^2(\sin^2\phi + \cos^2\phi) = a^2$

& $x = a\sin\phi\cos\omega t + a\cos\phi\sin\omega t$

$X = a\sin(\omega t + \phi)$

In short.

Sum of solutions to the equation of motion is also a solution as long as the equation of motion is linear

Superposition :→

When we superpose the initial conditions corresponding to the velocities and amplitudes, the resultant displacement of the two or more harmonic waves will simply be the algebraic sum of the individual displacements at all subsequent times

Net amplitude caused by two or more waves traversing the same space is the sum of the amplitudes that would have been produced by individual waves separately

Superposition of two collinear harmonic oscillations having same frequencies

Colinear → along same line

Let the displacement of the two colinear SHM oscillations be

$$x_1(t) = a_1 \cos(\omega_0 t + \phi_1) \quad \cdots (1)$$

& $$x_2(t) = a_2 \cos(\omega_0 t + \phi_2) \quad \cdots (2)$$

According to principle of superposition, the resultant displacement

$$x(t) = x_1(t) + x_2(t)$$

$$x(t) = a_1 \cos(\omega_0 t + \varphi_1) + a_2 \cos(\omega_0 t + \varphi_2)$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B.$$

$$x(t) = a_1 \cos \omega_0 t \cdot \cos \varphi_1 - a_1 \sin \omega_0 t \sin \varphi_1 + a_2 \cos \omega_0 t \cdot \cos \varphi_2 - a_2 \sin \omega_0 t \cdot \sin \varphi_2$$

$$x(t) = (a_1 \cos \varphi_1 + a_2 \cos \varphi_2) \cos \omega_0 t - (a_1 \sin \varphi_1 + a_2 \sin \varphi_2) \sin \omega_0 t$$

let $a_1 \cos \varphi_1 + a_2 \cos \varphi_2 = a \cos \varphi \quad \cdots (3)$

$a_1 \sin \varphi_1 + a_2 \sin \varphi_2 = a \sin \varphi \quad \cdots (4)$

Then $x(t) = a \cos \omega_0 t \cos \varphi - a \sin \omega_0 t \sin \varphi$

$$x(t) = a \cos(\omega_0 t + \varphi) \quad \cdot (5)$$

This equation is of the same form as either of our original equations for Harmonic oscillations

$\therefore$

Sum of two collinear harmonic oscillations of the same frequency is also an harmonic oscillation of the same frequency along the same line. But it has a new amplitude & a new phase constant

squaring (3) & (4) & adding them

$$a^2\cos^2\varphi + a^2\sin^2\varphi = a_1^2\cos^2\varphi_1 + a_2^2\cos^2\varphi_2 + 2a_1a_2\cos\varphi_1\cos\varphi_2 + a_1^2\sin^2\varphi_1 + a_2^2\sin^2\varphi_2 + 2a_1a_2\sin\varphi_1\sin\varphi_2$$

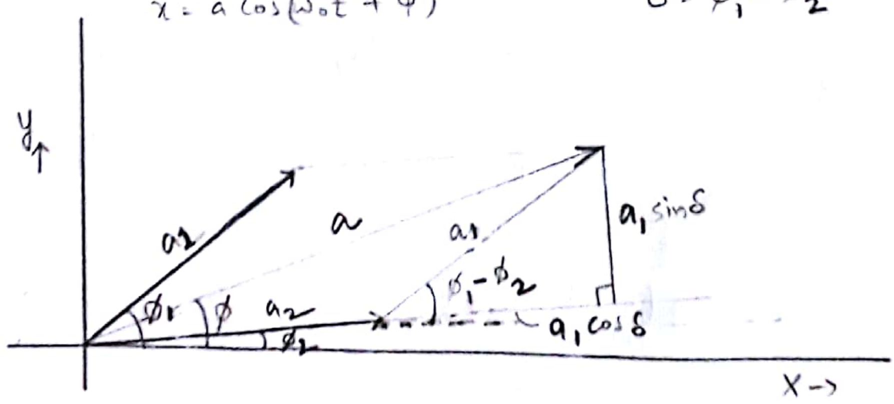
$$a^2 = a_1^2 + a_2^2 + 2a_1a_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2)$$

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\varphi_1 - \varphi_2) \quad \ldots (6)$$

dividing (3) & (4) we get

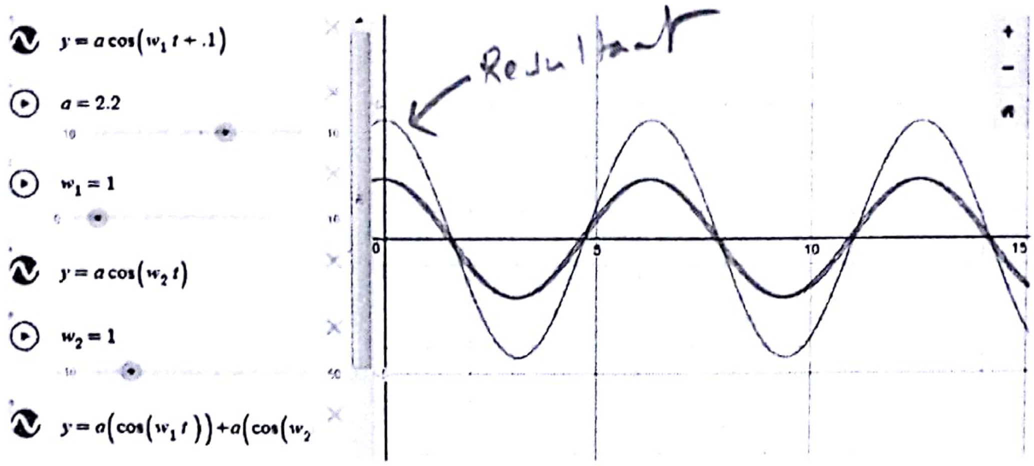
$$\phi = \tan^{-1}\left[\frac{a_1\sin\varphi_1 + a_2\sin\varphi_2}{a_1\cos\varphi_1 + a_2\cos\varphi_2}\right] \quad \ldots (7)$$

$x = a\cos(\omega_0 t + \phi)$ $\delta = \phi_1 - \phi_2$



Superposition of two Collinear waves of having equal frequencies with constructive superposition

https://www.desmos.com/calculator



$$\delta = \phi_1 - \phi_2 = 2n\pi$$

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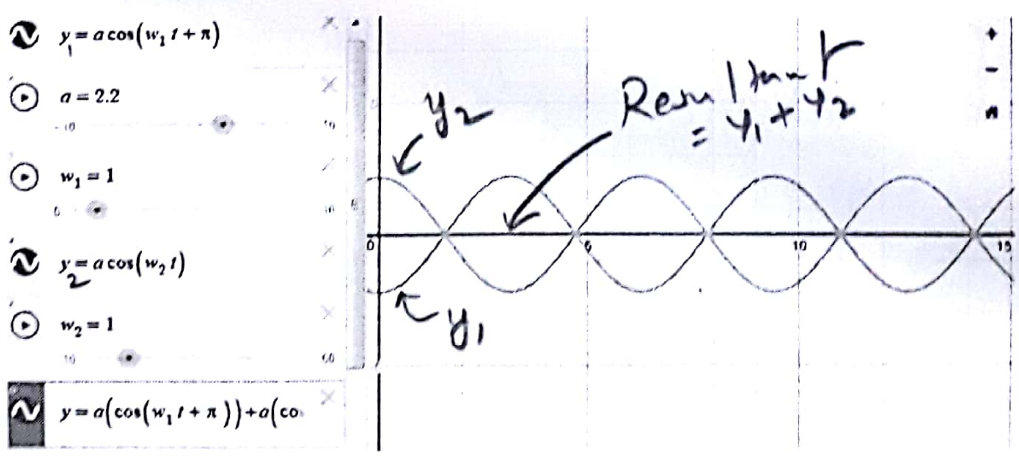


$$\delta = \phi_1 - \phi_2 = (2n+1)\pi$$

Special conditions

(1) $\delta = \phi_1 - \phi_2 = 2n\pi$

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2 = (a_1 + a_2)^2$$

when $a_1 = a_2 = a$, then

$$a_R^2 = (a_1 + a_2)^2$$

$$a_R^2 = (2a)^2$$

$$a_R^2 = 4a^2$$

Constructive Interference

(2) $\delta = \phi_1 - \phi_2 = (2n+1)\pi$

$$a^2 = a_1^2 + a_2^2 - 2a_1a_2$$

$$a^2 = (a_1 - a_2)^2$$

when $a_1 = a_2 = a$

$$a_R^2 = 0$$

Destructive Interference

Superposition of two collinear Harmonic oscillations "but different frequency":

let us consider superposition of two harmonic oscillations having same amplitude 'a' but different (let $\omega_1 > \omega_2$) angular frequencies

$$x_1 = a\cos(\omega_1 t + \phi_1) \quad \text{...(1)}$$

$$x_2 = a\cos(\omega_2 t + \phi_2) \quad \text{...(2)}$$

The phase difference between the two harmonic vibrations is $(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)$

$(\omega_1 - \omega_2)t$ changes continously with time

$\phi_1 - \phi_2$ is constant in time, so assume it to be zero for simplicity

then $x_1(t) = a\cos\omega_1 t$ & ...(3)

$x_2(t) = a\cos\omega_2 t$ ...(4)

∴ superposition of these two waves gives the resultant

$$x(t) = x_1(t) + x_2(t)$$

$$x(t) = a(\cos\omega_1 t + \cos\omega_2 t)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

$$\therefore \quad x(t) = 2a\cos\left(\frac{\omega_1+\omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_1-\omega_2}{2}\right)t$$

$$x(t) = 2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t \cdot \cos\left(\frac{\omega_1+\omega_2}{2}\right)t \quad \text{...(5)}$$

Equation (5) is oscillatory motion with angular frequency $\left(\frac{\omega_1+\omega_2}{2}\right)$ & amplitude $2a\cos\left(\frac{\omega_1-\omega_2}{2}\right)t$

Let w define ;

Average angular frequency , $W_{ave} = \frac{W_1 + W_2}{2}$

& modulated angular frequency, $W_{mod} = \frac{W_1 - W_2}{2}$

∴ modulated amplitude $a_{mod} = 2a \cos(W_{mod} t)$

Varies with frequency $f_{mod} = \frac{W_{mod}}{2\pi} = \frac{W_1 - W_2}{4\pi}$

$\left(\begin{matrix} 2a, & 0, & -2a, & 0 & \& & 2a \\ 0, & \pi/2, & \pi, & \frac{3\pi}{2} & \& & 2\pi \end{matrix} \right)$ { amplitude values for one complete cycle.

Resultant oscillation :→

$x_{(t)} = a_{mod}(t) \cos(W_{ave} t)$

Resultant is periodic but not simple harmonic oscillation.

In general case with S.H.O, different amplitude a_1 & a_2. If their initial phase is zero,

Resultant oscillation,

$$x_{(t)} = a_{mod}(t) \cos(W_{ave} t + \theta_{mod})$$

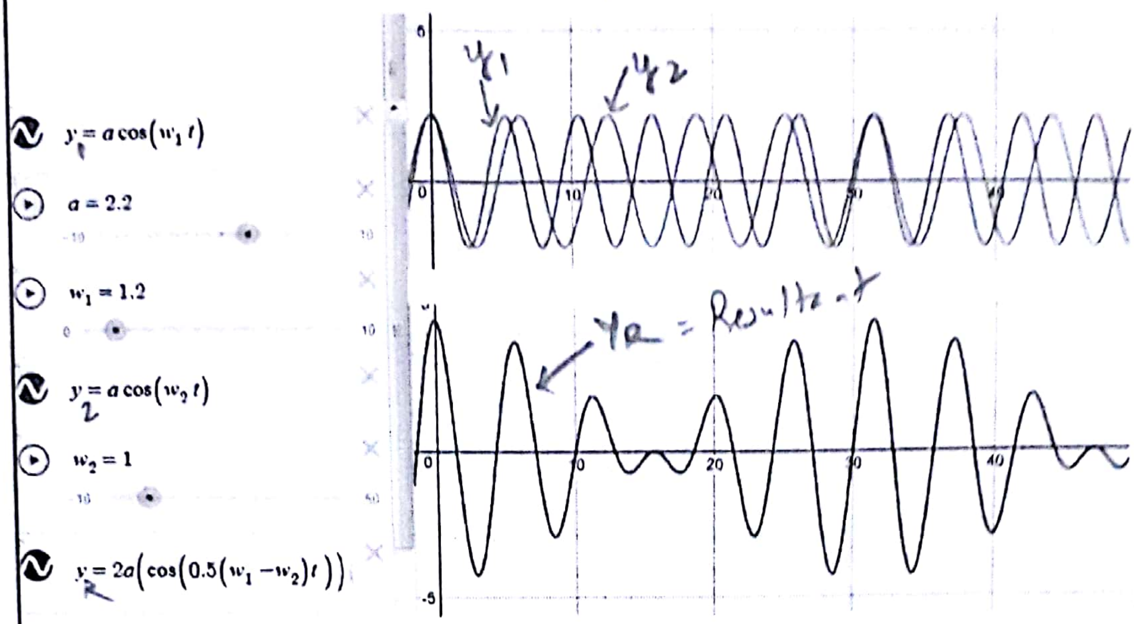
$$a_{mod}(t) = \left[a_1^2 + a_2^2 + 2a_1 a_2 \cos(2W_{mod} t)\right]^{1/2}$$

$$\& \quad \theta_{mod} = \left[\frac{(a_1 - a_2)\sin(W_{mod} t)}{a_1 + a_2 \cos W_{mod} t}\right]$$

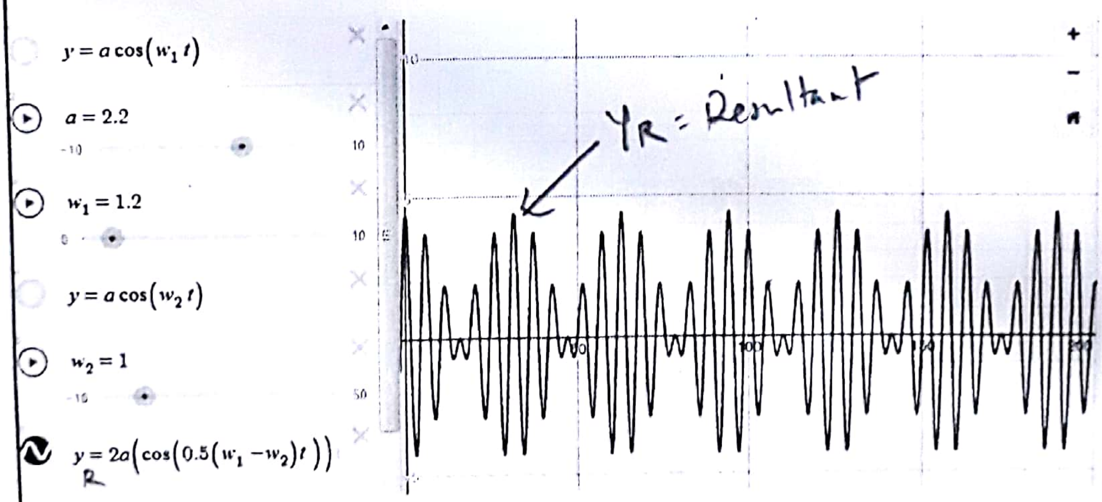
When $a_1 = a_2$; $\theta_{mod} = 0$.

Superposition of two Collinear waves of having different frequencies- BEATS

Two waves of slightly different frequencies



Superposition of two Collinear waves of having different frequencies- BEATS



$f_b = |f_2 - f_1|$

Beat frequency is the absolute value of difference in frequency.

The number of beats per second is equal to the difference in frequency