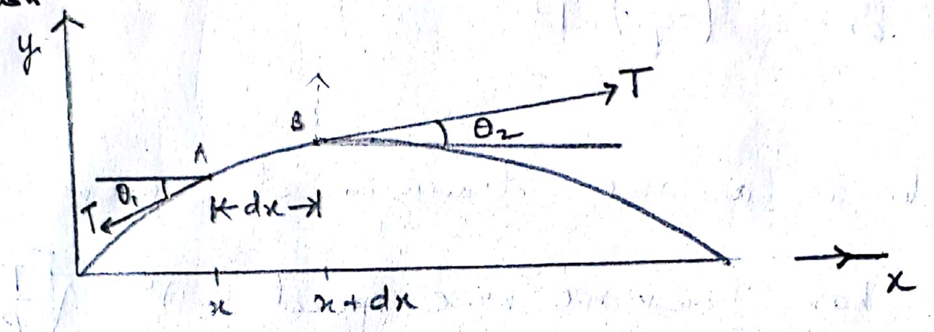
TRANSVERSE VIBRATION - STRETCHED STRING.

Consider a stretched string having tension T.
In the equilibrium position, the string is assumed to lie on the x-axis. If the string is pulled in the y-direction, then forces will act on the string to bring it back to its equilibrium position.



Let us consider a small length AB of the string & calculate the net force acting on it in the y-direction. Due to tension T, the end points A & B experience force in the direction shown.

force at A in upward direction is

$$-T\sin\theta_1 = -T\tan\theta_1 = -T\frac{\partial y}{\partial x}\Big|_x$$

force at B in upward direction is

$$+T\sin\theta_2 = T\tan\theta_2 = T\frac{\partial y}{\partial x}\Big|_{x+dx}$$

we have assumed θ_1 & θ_2 to be small

$$\text{Net force in the } y\text{-direction} = T\left(\frac{\partial y}{\partial x}\Big|_{x+dx} - \frac{\partial y}{\partial x}\Big|_x\right)$$

[Using Taylor Series Expansion

$$\left(\frac{\partial y}{\partial x}\right)\Big|_{x+dx} = \left(\frac{\partial y}{\partial x}\right)\Big|_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}y\Big|_x dx$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots \frac{h^n}{n!}f^n(x)$$

$$\Delta m \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} dx$$

where Δm is the small mass element AB.

$\Delta m = \rho dx$; where ρ is mass per unit length.

$$\therefore \boxed{\frac{\partial^2 y}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 y}{\partial t^2}} \quad \ldots$$

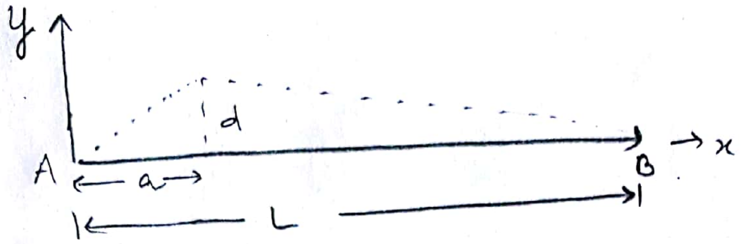
which is a one dimensional wave equation

which has transverse wave speed $v = \sqrt{\frac{T}{\rho}}$

$\therefore$ this wave, phase velocity depends on the density of the string & tension applied to the string.

Any function $f(x \pm vt)$ satisfies the differential equation

Transverse vibrations of Plucked string



equilibrium position is along the x-axis

A point of the string is moved upwards by a distance d: If the displacement occurs at a distance 'a' from the origin the distance

$$y = \frac{d}{a} x \quad \text{for } 0 < x < a \quad \cdots (1)$$

$$y = \frac{d}{L-a}(L-x) \quad \text{for } a < x < L \quad - (2)$$

If the string is released, determine the shape of the string at any subsequent time t.

we know $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \cdots (3)$ where $v = \sqrt{T/\rho}$

Using boundary conditions

(1) $y = 0$ at $x = 0$ & $x = L$.

(2) $t = 0$ $\frac{dy}{dt} = 0$

let $y = X(x) \cdot T(t)$ & $T = \cos\omega t$.

then $y = X_{(x)} \cos\omega t \cdots (4)$ substitute in (3)

$$\frac{d^2X}{dx^2} = -\frac{\omega^2}{v^2} X \quad \text{or} \quad \frac{d^2X}{dx^2} + \frac{\omega^2}{v^2} X = 0$$

if $k = \omega/v$ then solution

$$X = (A \sin kx + B \cos kx)$$

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$y(x,t) = (A \sin kx + B \cos kx)(C \cos \omega t + D \sin \omega t)$

we know.

$y(x,t) = 0$ at $x = 0$

$\Rightarrow$ $B(C \cos \omega t + D \sin \omega t) = 0$ or $B = 0$.

$y(x,t) = A \sin kx \, (C \cos \omega t + D \sin \omega t)$

$y(x,t) = 0$ at $x = L$.

$\therefore$ $A \sin(kL) = 0$

or $KL = n\pi$ or $k_n = \frac{n\pi}{L}$ $n = 1, 2, 3 ..$ &

$f_n = \frac{nv}{2L}$; $\omega_n = \frac{n\pi v}{L}$ & $\lambda = \frac{2L}{n}$

$\therefore$ $y(x,t) = A \sin\left(\frac{n\pi x}{L}\right)(C \cos \omega t + D \sin \omega t)$

or $y(x,t) = \sum_n \sin(k_n x)(C_n \cos \omega_n t + D_n \sin \omega_n t)$

$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ $\therefore$

$\sin(k_n x)(-C_n \sin \omega_n t + D_n \cos \omega_n t) = 0$

or $D_n = 0$

$\therefore$ $y(x,t) = \sum_{n=1,2,3} C_n \sin k_n x \cdot \cos \omega_n t$

& $y(x,0) = \sum C_n \sin k_n x = \sum_n C_n \sin\left(\frac{n\pi}{L} x\right)$

$y(x,0) = \sum_n C_n \sin\left(\frac{n\pi x}{L}\right)$ is essentially a fourier series. So to get constant C_n multiply both sides by $\sin\left(\frac{m\pi x}{L}\right) dx$ & integrate from 0 to L.

$$C_m = \frac{2}{L} \int_0^L y(x,0) \sin \frac{m\pi x}{L} \, dx$$

where $\int_0^L \sin \frac{n\pi x}{L} \cdot \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$

$$\therefore C_n = \frac{2}{L} \int_0^a \frac{d}{a} x \sin \frac{n\pi x}{L} \, dx + \frac{2}{L} \int_a^L \frac{d}{L-a} (L-x) \sin \frac{n\pi x}{\cancel{dx}} \, dx$$

$$= \frac{2dL^2}{a(L-a)\pi^2 n^2} \sin\left(\frac{n\pi a}{L}\right)$$

$$\text{\&} \quad y(x,t) = \frac{2dL^2}{a(L-a)\pi^2} \sum \frac{1}{n^2} \sin \frac{n\pi}{L} a \quad \sin \frac{n\pi x}{L} \cdot \cos\left(\frac{n\pi f}{L} t\right)$$

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When a string is set into vibrations, the vibrations consist of the fundamental frequency accompanied by certain higher frequencies called overtones.

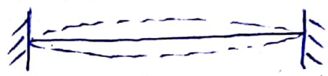
Harmonics are simply integral multiples of the fundamental frequency. If fundamental frequency is n, then harmonics are $2n, 3n, 4n \ldots$

$2n$ is the second harmonics
$3n$ is the third harmonic

$K_n = \frac{n\pi}{L}$ & $\omega_n = kv$. & $\omega_n = 2\pi f_n$

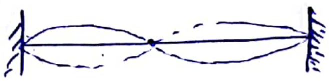
$\therefore f_n = \frac{nv}{2L}$ & $v = \sqrt{\frac{T}{\rho}}$

$\therefore$ fundamental frequency $= f = \frac{v}{2L}$



we know $f = \frac{v}{\lambda}$ $\therefore$ $\lambda_n = 2L$.

for second harmonic, $n = 2$.

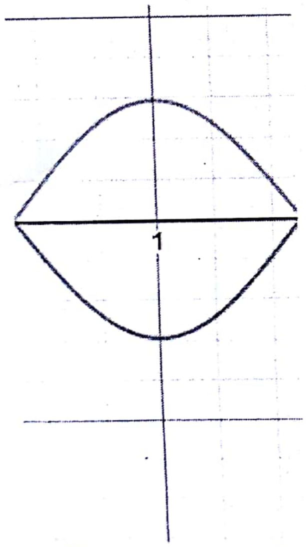


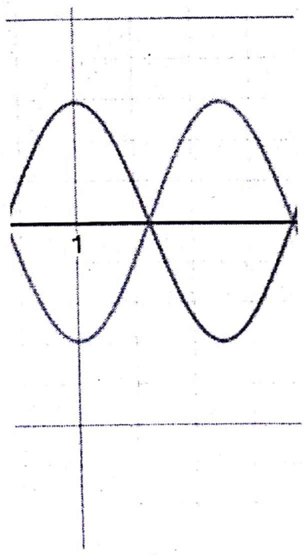
$f_2 = \frac{v}{L}$ & $\lambda_2 = L$.

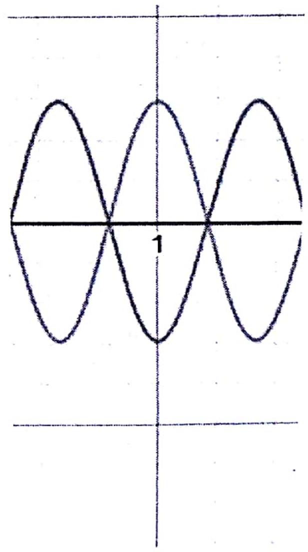
Standing wave pattern created using Desmos graphing calculator

$$\Psi = A \sin(k_n x) \cdot \cos(\omega_n t)$$

$k = \frac{n\pi}{l}$. Choose l, t, ω & A. & vary n.







1st mode $n=1$ fundamental

2nd mode $n=2$ 2nd harmonic

3rd mode $n=3$ 3rd hamonic.