SIMPLE HARMONIC MOTION.

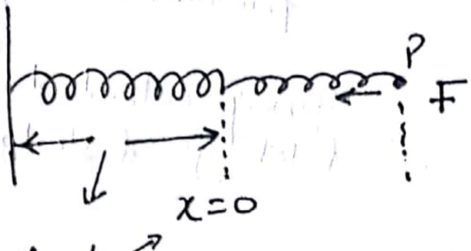
A motion that repeats itself over and over again after regularly recurring intervals of time (is called time period) is referred to as a periodic motion

If a particle covers the same path back and forth about a mean position, is said to undergo an oscillation & such a motion within a well defined limit is termed as Simple Harmonic motion

Since sines and cosines are periodic as well as bounded, the displacement of the particle undergoing S.H.M. is expressed in terms of these functions

SPRINGS & SIMPLE HARMONIC MOTION

A spring has a relaxed length at $x = 0$ and we extend the spring to point P.



then there is a force F that wants to drive the spring back to equilibrium

In case of ideal springs

$$|F| \propto |x| \quad \ldots (1)$$

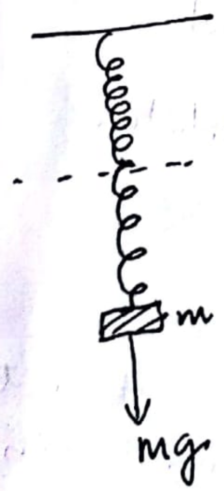
F is called a Restoring force. & in case of ideal springs, is proportional to the displacement x.

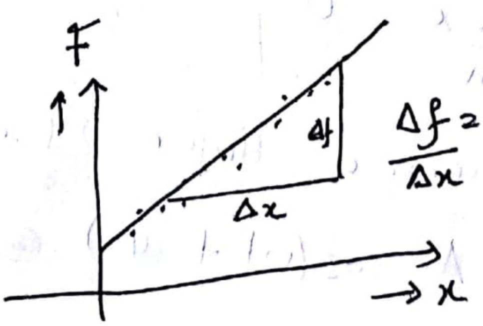
$$F = -kx \quad \ldots (2)$$

K = Spring constant. $\left[\frac{N}{m}\right]$

This relationship (2) is Hooke's law.

TO FIND K :→

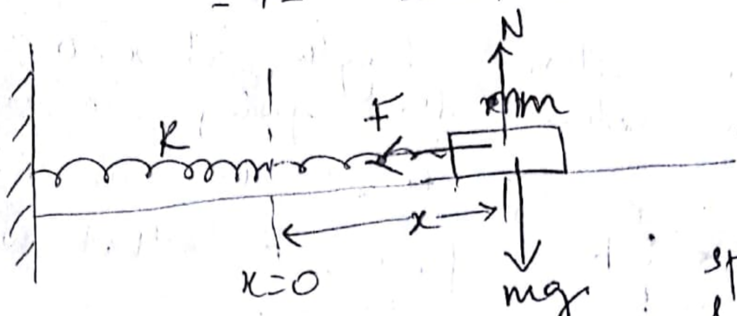




$\frac{\Delta f}{\Delta x} = k$

Since it is an ideal spring, when the force is removed, the spring goes back to its relaxed length. So Hooke's law will hold only in certain limitations.

Dynamic Response



frictionless surface

spring extended to x & let go & starts oscillating

No acceleration in y direction

The oscillation

The period of oscillation, $T = 2\pi\sqrt{\frac{m}{k}}$

No dependence on x (displacement or amplitude)

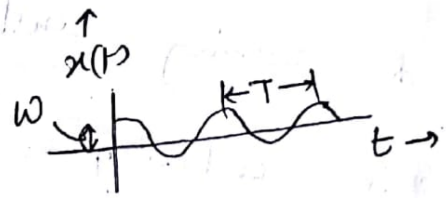
for ideal case (Hooke's law holds + no friction + negligible spring mass)

from Newton's second law:

$$m a_x = -kx$$

$$m\ddot{x} = -kx$$

$$\boxed{\ddot{x} + \frac{k}{m}x = 0} \quad \cdots (1)$$



This is second order Differential Equation

$x(t) = A\cos(\omega t + \phi)$ ... (2) is the solution

A = amplitude

ω = angular frequency = $\frac{2\pi}{T}$ [rad/sec]

if t goes to $\frac{2\pi}{\omega}$, then angle $\omega t \to 2\pi$ so it is the time for one complete revolution/oscillation

frequency of oscillation $f = \frac{1}{T}$ = [Hz]

ϕ = phase angle

So lets check if (2) is indeed the solution to differential equation (1)

$\dot{x} = A - \sin(\omega t + \phi) \cdot \omega$

$\ddot{x} = -A \cos(\omega t + \phi) \omega^2$

$\ddot{x} = -\cancel{A} \omega^2 x(t)$ [from eqn (2)]

(x is a function of t)

$\therefore \ddot{x} + \frac{k}{m} x = 0$ when

$-\cancel{A}\omega^2 x + \frac{K}{m} x = 0 \Rightarrow -\omega^2 x = -\frac{k}{m} x$

or $\omega = \sqrt{K/m}$ . . (3)

So, $x(t) = A \cos(\omega t + \phi)$ is a solution to differential equation (1) when $\omega = \sqrt{K/m}$

$\therefore$ the time period of this oscillation

$T = \frac{2\pi}{\omega}$

or $\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$. (4)

Instead of cos function, we can also choose sine function.

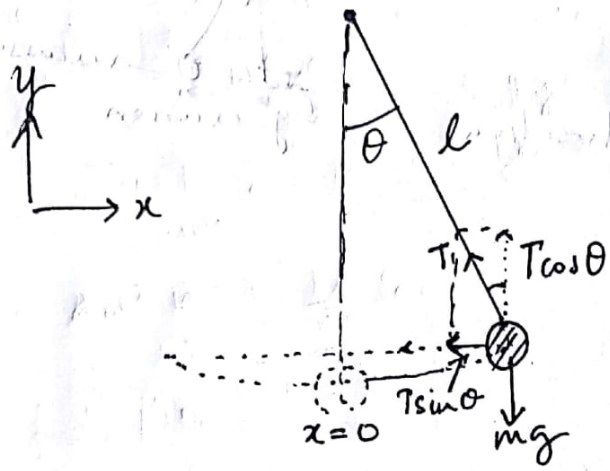
Try this for an Exercise

T is independent of amplitude & phase angle

frequency = Number of oscillations made by particle per second.

$= 1/T = \frac{1}{2\pi}\sqrt{K/m}$

Pendulum.



Consider a simple pendulum that has length 'l'
It is offset by an angle θ
Gravity acts on the mass (mg) & there is
tension T in the string.
When we let go, the mass is going to follow
the arc as shown & oscillate back & forth.

Using Newton's second law.

$\sin\theta = \frac{x}{l}$

for x →

$$ma_x = -T\sin\theta = -\frac{Tx}{l}$$

$$m\ddot{x} = -\frac{Tx}{l} \quad \text{.. (1)}$$

for y: →

$$ma_y = T\cos\theta - mg$$

$$m\ddot{y} = T\cos\theta - mg \quad \text{.. (2)}$$

We need to make some approximations
very common for oscillations ⇒ SMALL ANGLE APPROXIMATION

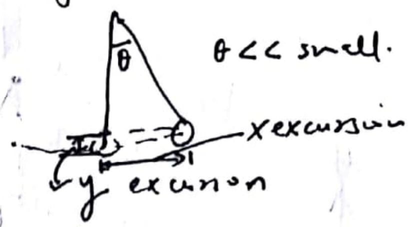
(1) $\theta << 1$ $\quad \cos\theta = 1$

use calculator to check. $\cos(5^\circ) = 0.9962$

Also,

(2) when θ is small.

the excursion in y-direction is very small compared to x direction.



$\therefore$ we can say $\ddot{y} = 0$

no acceleration in y-direction

So from eqn (2) ...

$$m\ddot{y} = T\cos\theta - mg$$

$$0 = T(1) - mg$$

$$\text{or } T = mg \quad \ldots (3)$$

$$m\ddot{x} = -\frac{Tx}{l} \quad \ldots \text{ using eqn (1)}$$

$$\text{So } m\ddot{x} = -\frac{mgx}{l} \quad \text{substitute (3) in (1)}$$

$$\text{or } \boxed{\ddot{x} + \frac{g}{l}x = 0}$$

This again is a simple harmonic motion

[Similar to $\ddot{x} + \frac{k}{m}x = 0$ (of spring)]

So solution is

$$x_{(t)} = A\cos(\omega t + \phi)$$

where $\omega = \sqrt{g/l}$ (solve for this for practice)

& $T = 2\pi\sqrt{\frac{l}{g}}$ is the period.

$$M \frac{d^2x}{dt^2} = -kx$$

$$\text{or } \frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$$\omega = \sqrt{\frac{k}{M}} \text{ , so } \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

Harmonic Oscillator.

The displacement & acceleration always have opposite signs

(a) $x > 0$ negative acceleration
mass goes to rest & comes back to equilibrium position

(b) $x < 0$ positive acceleration
& mass is pulled back

Motion that repeats regularly is called periodic motion.

Simple Harmonic motion is periodic

Sin & cosine functions are periodic

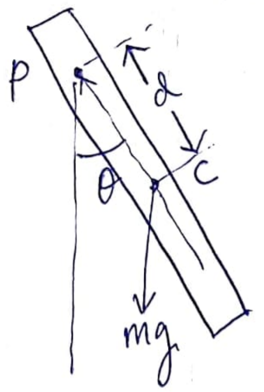
$T_s = 2\pi\sqrt{\frac{m}{K}}$ & $T_p = 2\pi\sqrt{l/g}$

In the case of spring. when extended beyond equlibrium, there is a restoring force that acts & this force is independent of the mass attached to the spring. When mass is changed, the acceleration changes but force remains same. so as acceleration changes T ~~remains~~ has to ~~same~~ change. ∴ T is dependent on m in case of spring.

In case of pendulum, when mass changes the tension on the string changes & hence the restoring force changes but acceleration remains the same.

When a spring is stiff. K is high so acceleration is very high & T will be small.

BAR PENDULUM.



Ruler forced to oscillate about point P.

$\tau_P = Mgd \cdot \sin\theta$ $(r \times F)$

$\tau_P = -I_P \alpha$

(Is a restoring torque)

$\alpha = \frac{d\omega}{dt} = \ddot{\theta}$

ω = angular velocity

for small angle θ $\sin\theta \approx \theta$

$\therefore$ $Mgd\theta + I_P \ddot{\theta} = 0$

$$\boxed{\ddot{\theta} + \frac{Mgd}{I_P}\theta = 0}$$

Is a simple harmonic oscillation

$\theta_t = \theta_{max} \cos(\omega' t + \phi)$

ω' = angular frequency. (related to period of oscillation,

where $\omega = \sqrt{\frac{Mgd}{I_P}}$

& $T = 2\pi \sqrt{\frac{I_P}{Mgd}}$

$I_P = \frac{ML^2}{12} + Md^2$.

$\therefore T = 2\pi \sqrt{\frac{\left(\frac{l^2}{12} + d^2\right)}{gd}}$

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Displacement of Simple Harmonic Oscillator (S.H.O.)

$x = A \sin(\omega t + \varphi)$

Velocity of S.H.O.

$v = \frac{dx}{dt} = \dot{x} = a\omega \cos(\omega t + \varphi)$

$v = a\omega \sin(\omega t + \varphi + \pi/2)$

$\sin(\omega t + \varphi) = \frac{x}{a} \Rightarrow \sqrt{1 - \sin^2(\omega t + \varphi)} = \cos(\omega t + \varphi)$

$\cos(\omega t + \varphi) = \sqrt{1 - x^2/a^2}$

$\therefore v = \omega\sqrt{a^2 - x^2}$

$v_{max} = a\omega$ = maximum velocity

Acceleration of S.H.O.

$= \frac{dv}{dt} = -a\omega^2 \sin(\omega t + \varphi) = -a\omega^2 x$

$\frac{dv}{dt} = +a\omega^2 \sin(\omega t + \varphi + \pi) = a_{max} = a\omega^2$

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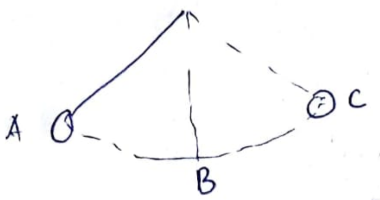
Energy of a Harmonic oscillator



at C only Potential Energy, $KE = 0$

at B only kinetic energy, $PE = 0$

Between B & C both $PE + KE = ME =$ conserved.

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\therefore \text{force} = m\frac{d^2x}{dt^2} = +m\omega^2 x.$$

$\therefore$ work done $= F.dx = +m\omega^2 x\, dx$

$$U = \int_0^x m\omega^2 x\, dx = \frac{1}{2} m\omega^2 x^2$$

$$U = \frac{1}{2} m \frac{K}{m} x^2 = \frac{1}{2} K x^2$$

$$U_{max} = (x = a) = \frac{1}{2} K a^2 = \frac{1}{2} m\omega^2 a^2$$

$$v = \frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$$

$$K.E = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 (a^2 - x^2)$$

$$K.E = \frac{1}{2} m \frac{K}{m} (a^2 - x^2) = \frac{1}{2} K (a^2 - x^2)$$

$$K.E_{max} = \frac{1}{2} K a^2$$

$$\text{Total Energy} = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 a^2 - \frac{1}{2} m\omega^2 x^2$$

$$TE = \frac{1}{2} m\omega^2 a^2 = \frac{1}{2} K a^2$$

$\omega = \frac{2\pi}{T}$ & $\frac{1}{T} = n = $ frequency

$\therefore T.E = \frac{1}{2} m\omega^2 a^2$

$= \frac{1}{2} m \frac{(2\pi)^2}{T^2} a^2 = \frac{2\pi^2 m a^2}{T^2}$

$T.E = 2\pi^2 n^2 m a^2$

$KE_{max} = PE_{max} = TE = \frac{1}{2} m\omega^2 a^2 = \frac{1}{2} k a^2$

Average value $= PE_{avg} = KE_{avg} = \frac{1}{4} m\omega^2 a^2 = \frac{1}{4} k a^2$

$P.E = (x) = \frac{1}{2} m\omega^2 x^2$

$PE_x = \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t + \varphi)$

Average P.E over one complete cycle $= \frac{1}{T} \int_0^T \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t + \varphi)\, dt$