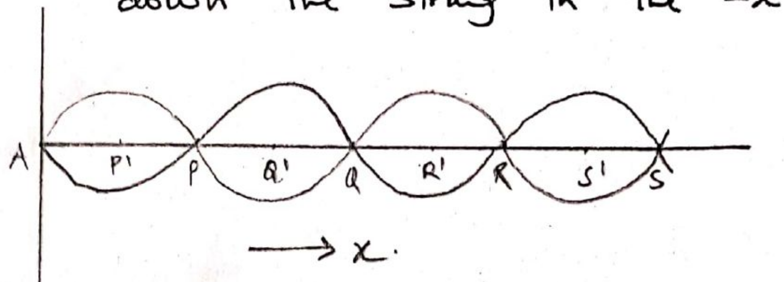
Stationary waves on a string (STANDING WAVES)
(one free end)

Consider a string which is fixed at the point A. A transverse sinusoidal wave is sent down the string in the $-x$ direction (incident wave) i



$x=0$

The displacement of this wave at any point in the string is given by

$$y_i = a \sin\left[\frac{2\pi}{\lambda}(x+vt) + \varphi\right]$$

without loss of generality $\Rightarrow \phi = 0$

$$\Rightarrow y_i = a \sin\left[2\pi\left(\frac{x}{\lambda} + ft\right)\right] \quad \because \left(\text{frequency} = f = v/\lambda\right)$$

displacement at point A, $x=0$ $\quad y_i\big|_{x=0} = a\sin(2\pi f t)$

Since the point A is fixed, there must be a reflected wave such that displacement of the reflected wave at point A is equal & opposite to y_i

$$\therefore y_R\big|_{x=0} = -a\sin(2\pi f t)$$

y_R propagates in $+x$ direction

$$\therefore y_R = a \sin\left[2\pi\left(\frac{x}{\lambda} - ft\right)\right]$$

Resultant displacement $= Y = y_i + y_R = a\left[\sin 2\pi\left(\frac{x}{\lambda}+vt\right) + \sin 2\pi\left(\frac{x}{\lambda}-vt\right)\right]$

$$\boxed{Y = 2a\sin\left(\frac{2\pi x}{\lambda}\right)\cdot\cos 2\pi f t}$$

$$Y = 2a \sin\left(\frac{2\pi x}{\lambda}\right) \cdot \cos(2\pi f t)$$

$Y = 0$ for values of x such that $\sin\left(\frac{2\pi x}{\lambda}\right) = 0$

or $\frac{2\pi x}{\lambda} = n\pi$ or $x = \frac{n\lambda}{2}$

$\therefore$ for point $x = 0, \lambda/2, \lambda, 3\lambda/2 \ldots$ $Y = 0$ &

these points are called nodes

(These correspond to points A, P, Q, R, S in figure)

when $\sin\left(\frac{2\pi x}{\lambda}\right) = 1$ we get Y is maximum or

the amplitude of vibration is maximum.

$\therefore \frac{2\pi x}{\lambda} = \frac{\pi}{2}(2m+1)$ or for $\cancel{x = \frac{(1+m)\lambda}{4}}$ - $m = 1, 2, 3 \ldots$

ie At $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots$

$$\frac{2\pi x}{\lambda} = (2m+1)\frac{\pi}{2} \Rightarrow x_m = \frac{(2m+1)\lambda}{4} \; ; m = 0, 1, 2, 3, 4.$$

these points are called Antinodes

$$y_{x_m} = \pm 2a\cos(2\pi f t)$$

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One end fixed one end open.

$$Y_{(x,t)} = (A \sin kx + B \cos kx)(C \sin \omega t + D \cos \omega t)$$

$x = 0 \quad Y = 0 \quad \Rightarrow \quad B = 0$

$$Y_{(x,t)} = (A \sin kx)(C \sin \omega t + D \cos \omega t)$$

$Y = \max$ at $x = L$

i.e. $A \sin(KL) = A$.

or $KL = \frac{(2n-1)\pi}{2}$ $\quad n = 1, 2, 3$

$$\therefore \quad K_n = \frac{(2n-1)\pi}{2L}$$

or $K_n = \frac{2\pi}{\lambda_n}$, $\therefore \lambda_n = \frac{4L}{(2n-1)}$

& $\omega_n = K_n v$ $\quad \omega_n = 2\pi f_n \Rightarrow f_n = v/\lambda_n$

$$\omega_n = \frac{(2n-1)\pi v}{2L}$$

& $f_n = \frac{(2n-1)v}{4L}$

As $\left.\frac{dy}{dt}\right|_{t=0} = 0 \Rightarrow C = 0$

$$\therefore Y_{(x,t)} = C_n \cdot \sin k_n x \cdot \cos(\omega_n t)$$

Both ends open

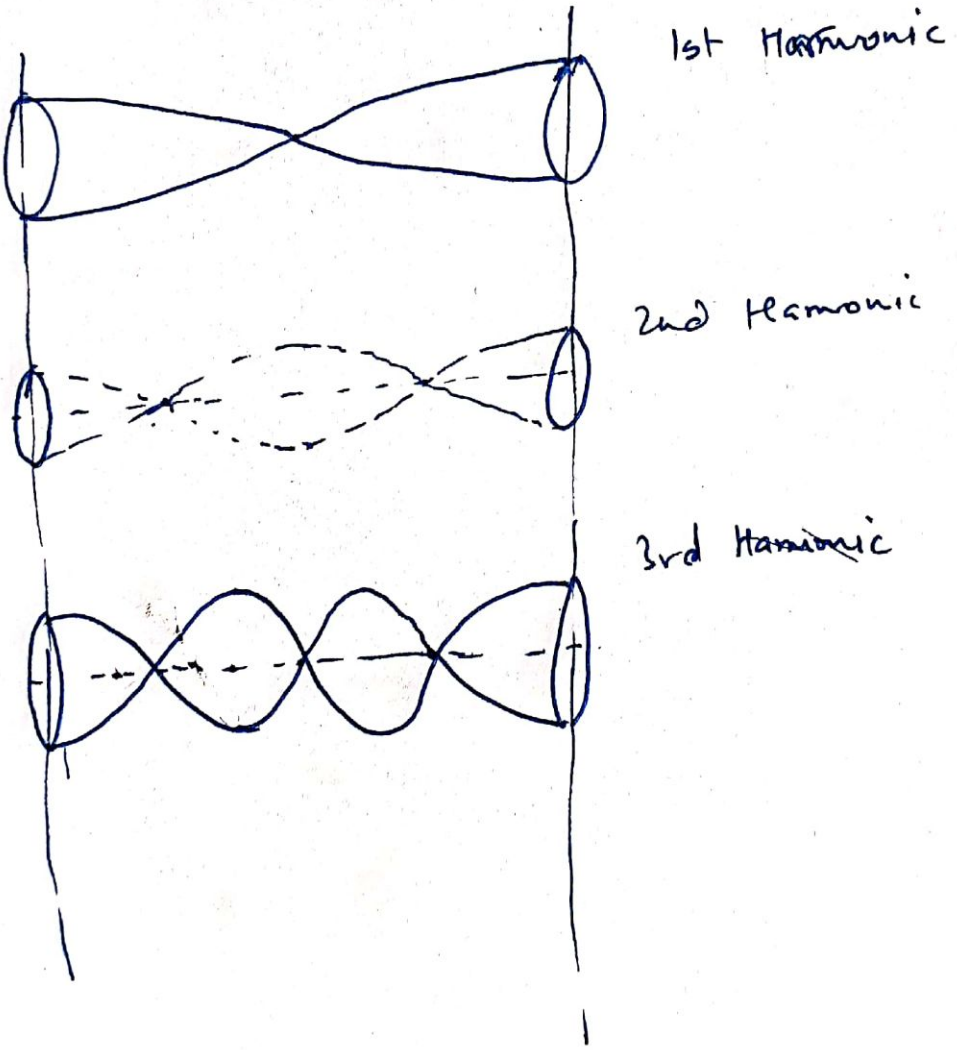
$$Y(x,t) = (A \sin kx + B \cos kx)(C \sin \omega t + D \cos \omega t)$$

at $x = 0$ & $x = L$ Y is maximum.

$\Rightarrow$ $A = 0$.

$$Y(x,t) = A \cos(K_n x) \cos(\omega_n t)$$

$$K_n = \frac{n\pi}{L} \quad \& \quad f_n = \frac{nv}{2L}$$



String Instruments

String instruments make sound with vibrating strings & the pitch is modified by the thickness, tension & length of the string

for L, μ, T $\quad v = \sqrt{\frac{T}{\mu}}$ & $\lambda = \frac{v}{f}$ $\quad \lambda_n = \frac{2L}{n}$ & $f_n = \frac{nv}{2L}$

for $n=1$ $\quad \lambda_1 = 2L$ & $f_1 = v/2L$

for $n=2$ $\quad \lambda_2 = L$ & $f_2 = \frac{v}{L}$

WIND INSTRUMENTS

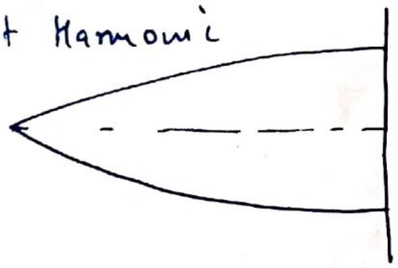
for open end. & closed end together

$\therefore \lambda_n = \frac{4L}{(2n-1)}$

&

$\therefore f_n = \frac{(2n-1)v}{4L}$

1st Harmonic

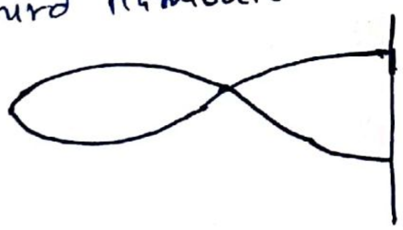


$n=1$

$\lambda_1 = 4L$

$F_1 = \frac{v}{4L}$

Third Harmonic



$n=2$

$\lambda_2 = \frac{4}{3}L$

$f_2 = \frac{3v}{4L} = 3f_1$

fifth harmonic

$n=3$

$\lambda_3 = \frac{4L}{5}$

$f_3 = \frac{5v}{4L} = 5f_1$

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ENERGY OF A VIBRATING STRING.

A vibrating string possesses both kinetic & potential Energy

Kinetic energy $= \frac{1}{2} dm \left(\frac{dy}{dt}\right)^2$

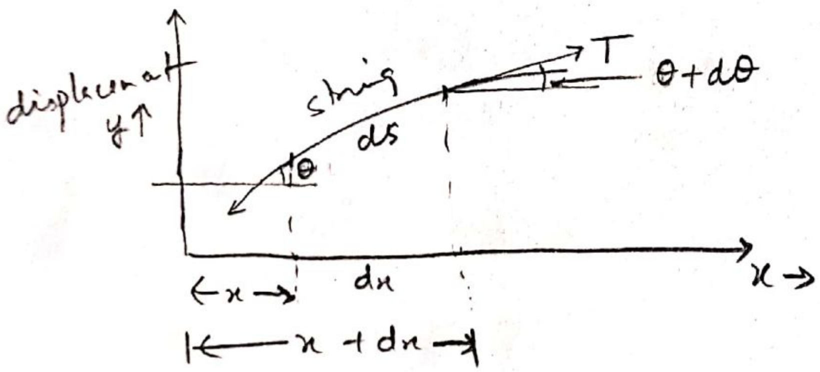
for an element of length dx & linear density ρ.

$$K.E = \frac{1}{2} \rho \, dx \left(\frac{dy}{dt}\right)^2$$

Total K.E. along the length of the string

$$E_{kin} = \frac{1}{2} \int_0^L \rho \left(\frac{dy}{dt}\right)^2 dx$$

Potential Energy



$$ds = \left[1 + \left(\frac{\partial y}{dx}\right)^2\right]^{1/2} dx$$

$$ds \approx dx$$

The potential energy is the work done by the Tension T in extending an element dx to a new length ds when the string is vibrating

$$E_{pot} = \int T(ds - dx) = \int T\left[\left(1 + \frac{1}{2}\left(\frac{\partial y}{dx}\right)^2\right) dx - dx\right]$$

$$E_{pot} = \frac{T}{2} \int \left(\frac{\partial y}{dx}\right)^2 dx$$

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RATE OF ENERGY TRANSFER

$$\int dK = \frac{1}{2}\rho \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx$$

$y = a\cos(kx - wt)$ $\qquad \frac{\partial y}{\partial t} = -aw\sin(kx - wt)$

$$KE = \frac{1}{2}\rho \int_0^L a^2 w^2 \sin^2(kx - wt)\, dx$$

$$= \frac{1}{2}\rho a^2 w^2 \frac{L}{2} = \frac{1}{4}\rho a^2 w^2 L$$

$PE = \frac{1}{4}\rho a^2 w^2 L$. (see below)... (2)

$\therefore$ $T.E. = \frac{1}{2}\rho a^2 w^2 L$.

As the wave moves along the string, this amount of energy passes by a given point on a string during the time interval of one period of the oscillation

$P = \frac{T.E.}{\Delta t} = \frac{\frac{1}{2}\rho a^2 w^2 L}{T}$ $\qquad (L \sim \lambda.$ for $t = \tau.)$

$$\boxed{P = \frac{1}{2}\rho a^2 w^2 v}$$

solution (2) →

$$E_{pot} = \int \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$= \frac{T}{2}\int_0^L a^2 K^2 \sin^2(kx - wt)\, dx$$

$$= \frac{T}{4} a^2 K^2 L$$

for a stretched string $v^2 = T/\rho$

or $E_p = \frac{1}{4}\rho v^2 a^2 K^2 L = \frac{1}{4}\rho w^2 a^2 L$ $\qquad \boxed{\text{as } w = kv}$

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NORMAL MODES OF STRETCHED STRINGS.

The total displacement y in the string is the superposition of the displacements y_n of the individual harmonics

Kinetic energy of n_{th} harmonic ⇒

$$E_{KE} = \frac{1}{2} \int_0^L \rho \dot{y}_n^2 \, dx \qquad \dot{y}_n = \frac{\partial y_n}{\partial t}$$

Solution y_n for n_{th} harmonic.

$$y_n = (A_n \cos\omega_n t + B_n \sin\omega_n t) \sin\left(\frac{\omega_n x}{v}\right)$$

$$\therefore \dot{y}_n = (-A_n \sin\omega_n t + B_n \cos\omega_n t)\, \omega_n \sin\left(\frac{\omega_n x}{v}\right)$$

$$\therefore E_{KE} = \frac{1}{2} \rho \omega_n^2 \left[-A_n \sin\omega_n t + B_n \cos\omega_n t\right]^2 \int_0^L \sin^2\left(\frac{\omega_n x}{v}\right) dx$$

Potential Energy of the n^{th} harmonic

$$E_{PE} = \frac{1}{2} T \int_0^L \left(\frac{\partial y_n}{\partial x}\right)^2 dx \qquad (v^2 = T/\rho)$$

$$\frac{\partial}{\partial x}(y_n) = (A_n \cos\omega_n t + B_n \sin\omega_n t) \cdot \left[\cos\left(\frac{\omega_n x}{v}\right)\right]\left(\frac{\omega_n}{v}\right)$$

$$\therefore E_{PE} = \frac{1}{2} T \frac{\omega_n^2}{v^2} (A_n \cos\omega_n t + B_n \sin\omega_n t)^2 \int_0^L \cos^2\left(\frac{\omega_n x}{v}\right) dx$$

integral = $L/2$. So total energy

$$E_n = \frac{1}{4} \rho \omega_n^2 l \, (A_n^2 + B_n^2) = \frac{1}{4} m \omega_n^2 (A_n^2 + B_n^2)$$

(m = mass of string, can get A_n & B_n from fourier method)

Total Energy = sum of all E_n's of normal mode

Dispersive Medium

A medium in which the phase velocity is frequency dependent (ω/k is not constant) is known as dispersive medium. If a group contains a number of components of frequencies which are nearly equal then

$$\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

group velocity is the velocity of the maximum amplitude of the group, so that it is the velocity with which the energy is transmitted in the group

$$\omega = k v_p$$

$$\therefore v_g = \frac{d}{dk}(k v_p) = v_p + k\frac{dv_p}{dk} = v_p - \lambda\frac{dv_p}{d\lambda}$$

$k = 2\pi/\lambda$

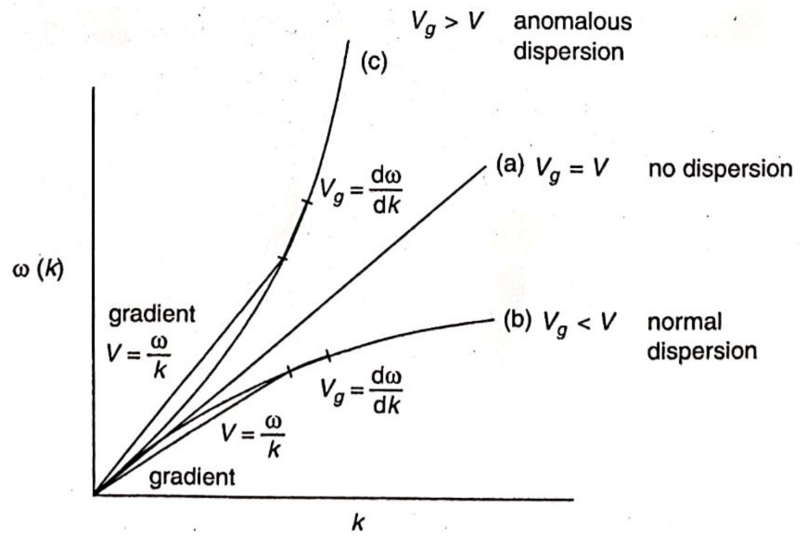


Figure 5.12 Curves illustrating dispersion relations: (a) a straight line representing a non-dispersive medium, $v = v_g$; (b) a normal dispersion relation where the gradient $v = \omega/k > v_g = d\omega/dk$; (c) an anomalous dispersion relation where $v < v_g$

GROUP VELOCITY.

Waves rarely occur as a single monochromatic component. A white light for example consists of infinite fine spectrum of frequencies & the motion of such a pulse would be described by its group velocity

Such a group of white light would disperse with time in a medium because the wave velocity of each component would be different in all media except free space.

for a monochromatic wave the group velocity & the wave velocity are identical

Consider a group of two plane waves (having the same amplitude A but) with slightly different frequencies $\omega_1 \cancel{+\Delta\omega}$ & $\omega_2 \cancel{-\Delta\omega}$ propagating along the +z direction

then $\Psi_1(z,t) = A\cos[(\omega_1+\Delta\omega)t - (k_1+\Delta k)z]$

$$\Psi_2(z,t) = A\cos[(\omega_2-\Delta\omega)t - (k_2 - \Delta k)z]$$

$k+\Delta k$ & $k-\Delta k$ are the wavenumbers of the corresponding frequencies in the expressions

The superposition of the two waves will be given by

$$\Psi_{(z,t)} = 2A\cos\left[\frac{(\omega_1-\omega_2)t}{2} - \frac{(k_1-k_2)x}{2}\right]\cos\left[\frac{(\omega_1+\omega_2)t}{2} - \cancel{x}\frac{(k_1+k_2)}{2}\right]$$

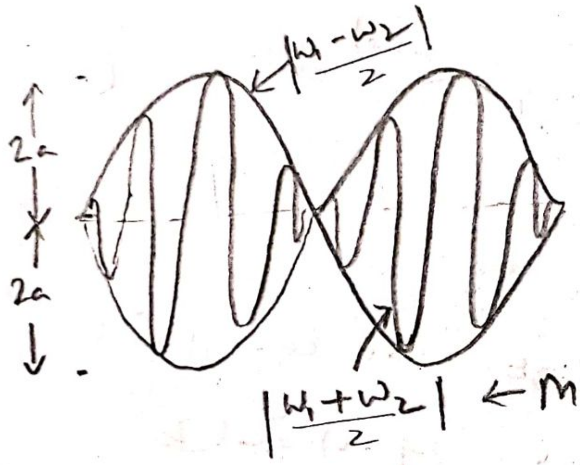
$$\Psi = 2a \cos\left[\frac{(\omega_1+\omega_2)t}{2} - \frac{(k_1+k_2)x}{2}\right] \cos\left[\frac{(\omega_1-\omega_2)t}{2} - \frac{(k_1-k_2)x}{2}\right]$$

wave system similar to 2 waves with slightly different frequency but amplitude 2a

modulation which is varying slowly.

Velocity of the new wave $= v_g = \dfrac{\omega_1 - \omega_2}{k_1 - k_2} = \dfrac{\Delta\omega}{\Delta K}$.

phase velocity $= v_p = \dfrac{\omega}{K}$



The displacement of a 1D - plane wave propagating in +z direction can be written as

$$E(z,t) = A e^{i(\omega t - kz)} \quad \ldots (1)$$

$k(\omega) = \dfrac{\omega}{c} n(\omega)$ (n being the refractive index)

Equation (1) represents a monochromatic wave.

Real part of $E(z,t) = A\cos(\omega t - kz + \varphi)$

The displacement is finite only over a certain domain of time which gives rise to wave packet

A wave packet can always be expressed as superposition of plane waves of different frequencies

$$E_{(z,t)} = \int_{-\infty}^{\infty} A(\omega)\, e^{i\left[\boxed{\omega} t - kz\right]}\, d\omega \quad \ldots (2)$$

$$\text{or } E_{(z=0,t)} = \int_{-\infty}^{\infty} A(\omega)\, e^{i\omega t} \quad \ldots (3)$$

$$\therefore\ A(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{(z=0,t)}\, e^{-i\omega t}\, dt \quad \ldots (4)$$

using fourier transform.

$\therefore$ if we know, $E_{(z=0,t)}$ we can find $A(\omega)$ & then integrate equation (2)