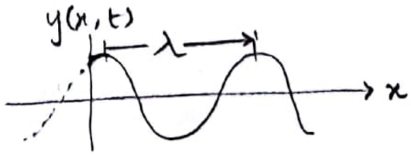
TRAVELLING WAVES OR. SINUSOIDAL WAVES.



Consider a periodic wave in which the displacement $y(x,t)$ has the form

$$y(x,t) = a \cos[K(x \mp vt) + \phi]$$

$(x \mp vt)$ represents a wave traveling in $+x$ & $-x$ direction. $\{(x - vt) \approx +x$ direction.$\}$

ϕ = phase of the wave.
a = amplitude of the wave
assume $\phi = 0$ & at time $t = 0$

$$y(x) = a \cos kx$$

Any two points separated by $\lambda = \frac{2\pi}{K}$ have identical displacements in time

Similarly,

$$y(t) = a \cos wt$$

where $w = kv$

The displacement repeats itself after $T = \frac{2\pi}{w}$

$$WT = 2\pi \quad \text{or} \quad KvT = 2\pi$$

or $T \frac{2\pi}{\lambda} v = 2\pi$ or $\boxed{\lambda = vT}$

$\frac{1}{T} = \frac{v}{\lambda}$ = frequency No. of oscillations of a particle in one second.

Equivalent expressions for (solutions to the wave equation :) expressing wave motion

$$y = a \sin k(x - vt)$$

$$y = a \sin(kx - kvt)$$

$$y = a \sin(kx - \omega t)$$

$$y = a \sin \frac{2\pi}{\lambda}(x - vt)$$

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - \frac{v}{\lambda}t\right)$$

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - ft\right)$$

$$y = a \sin(Kx - \omega t)$$

$$y = a \sin \omega\left(\frac{x}{v} - t\right)$$

$\omega = kv$ $\quad$ $k = \frac{2\pi}{\lambda}$ $\quad$ $f = \frac{v}{\lambda}$

K is the Propagation number.

Particle & wave Velocity

The individual oscillators that make up the medium do not progress through the medium with the waves. Their motion is simple harmonic & is limited to oscillations, transverse or longitudinal, about their equilibrium positions. It is their phase relationships we observe as waves, not their progressive motion through the medium

There are three velocities in wave motion that are connected mathematically

(1) Particle velocity
It is the simple harmonic velocity of the oscillator about its equilibrium position

(2) Wave or Phase velocity
The velocity with which the planes of equal (crests waves or troughs) phase, progress through the medium

(3) Group velocity
The number of waves of different frequency, wavelengths & velocities may be superposed to form a group. It is the velocity with which the energy in the wave group is transmitted

Particle velocity

$$y = a\cos[k(x - vt)]$$

velocity of the particle $= \frac{dy}{dt} = u$

$$u = -a(-kv)\sin[k(x - vt)]$$

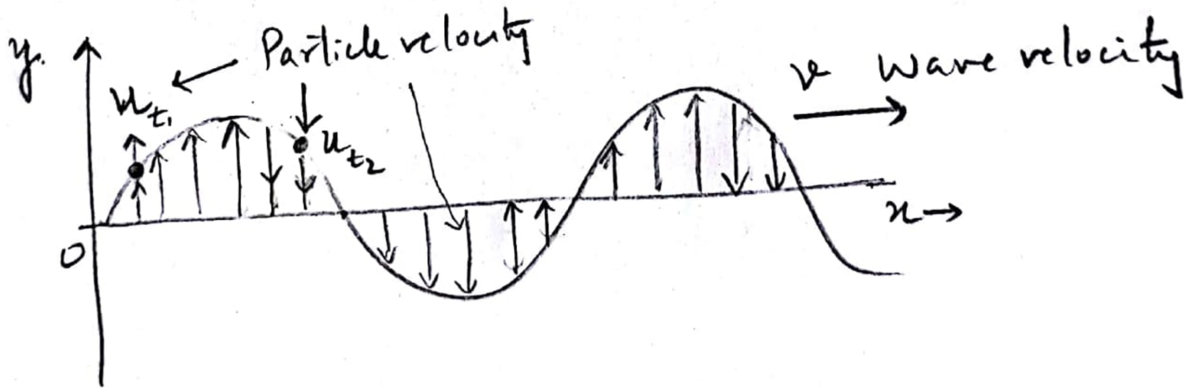
$$u = kav \sin[k(x - vt)]$$

$$u_{max} = \frac{2\pi}{\lambda} a v$$

Particle acceleration $= \frac{du}{dt}$

$$a_p = \frac{d^2y}{dt^2} = -\frac{2\pi}{\lambda} v \cdot \frac{2\pi}{\lambda} v \cdot y$$

$$a_{p_{max}} = -\left(\frac{4\pi^2}{\lambda^2} v^2\right) a$$



Scanned by CamScanner

PHASE

let $Y = a \sin[kx - \omega t]$

then here $\phi = kx - \omega t$ = phase

It describes the state of motion of a particle on the wave.

$$\sin[kx - \omega t + 2\pi] = \sin[kx - \omega t]$$

We know that all points on the wave separated by one wavelength or multiples of λ are in same phase. i.e. $x' = x + \lambda$

$$\therefore \sin(kx' - \omega t) = \sin\{k(x+\lambda) - \omega t\}$$

$$= \sin\{kx + k\lambda - \omega t\} = \sin\{kx + 2\pi - \omega t\}$$

$$= \sin\{kx - \omega t + 2\pi\} = \cancel{\sin\{kx - \omega t\}} \qquad k = \frac{2\pi}{\lambda}$$

$$= \sin(kx - \omega t)$$

or $x' = x + n\lambda$.

$$\& \ \sin(kx - \omega t \pm 2n\pi) = \sin(kx - \omega t)$$

for $n = 0, 1, 2 \ldots$

PHASE VELOCITY.

A travelling wave in an isotropic medium will travel at a constant velocity.

We know,

Phase is given by $\phi(x,t) = kx - \omega t$

$$d\phi = \left(\frac{\partial \phi}{\partial t}\right)_x dt + \left(\frac{\partial \phi}{\partial x}\right)_t dx$$

is the infinitesimal change in $\phi(x,t)$.

$$\left(\frac{\partial \phi}{\partial t}\right)_x = -\omega \quad \& \quad \left(\frac{\partial \phi}{\partial x}\right)_t = K$$

$$\therefore \; d\phi = -\omega dt + K dx$$

Since phase velocity is the velocity of a point of constant phase, $d\phi = 0$

or $\omega dt = k dx$

$$\Rightarrow \left(\frac{dx}{dt}\right)_\phi = \frac{\omega}{K} = v_p. \quad \text{(Phase velocity)}$$

ω is in rad/sec & K is in rad/m.

So v_p is in m/s

In the argument, $\Psi = A \sin[k(x \mp vt)]$

$K(x \mp vt)$ is constant

wave or phase velocity $= \frac{\partial x}{\partial t}$

$$y = a \cos[k(x - vt)]$$

$$\frac{dy}{dt} = \text{particle velocity} = kva \sin[k(x-vt)]$$
$$= \omega a \sin[k(x-vt)]$$

$$\frac{dy}{dx} = -ak \sin[k(x-vt)]$$

$$\frac{\partial y}{dt} = -\frac{\omega}{k}\frac{dy}{dx} = -v\frac{\partial y}{dx}$$

$v = \frac{\partial x}{dt}$

or $$\boxed{\frac{\partial y}{\partial t} = u_p = -v\frac{dy}{\partial x}}$$

particle velocity is given by product of wave velocity times the gradient of the wave profile.

The 1D wave equation $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$ is a homogeneous differential equation. It does not contain a term (such as force) involving only independent variables. (ψ is in each term of the equation)

This equation represents the wave equation for undamped systems. Effects of damping can be considered by adding $\frac{\partial \psi}{\partial t}$ term.

WAVE EQUATION IN THREE DIMENSIONS.

$$\nabla^2 \Psi = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$ where

$$\nabla^2 \Psi = \frac{\partial^2}{\partial x^2} \Psi + \frac{\partial^2}{\partial y^2} \Psi + \frac{\partial^2}{\partial z^2} \Psi$$

$\Psi(x, y, z, t)$ & $\Psi = X(x)\, Y(y)\, Z(z)\, T(t)$

$w^2 = K^2 v^2$ & $K^2 = K_x^2 + K_y^2 + K_z^2$

Solution is of the form.

$$\Psi = A \exp [i (\vec{k} \cdot \vec{r} \pm wt + \varphi)]$$

$$\vec{k} \cdot \vec{r} = K_x x + K_y y + K_z z$$

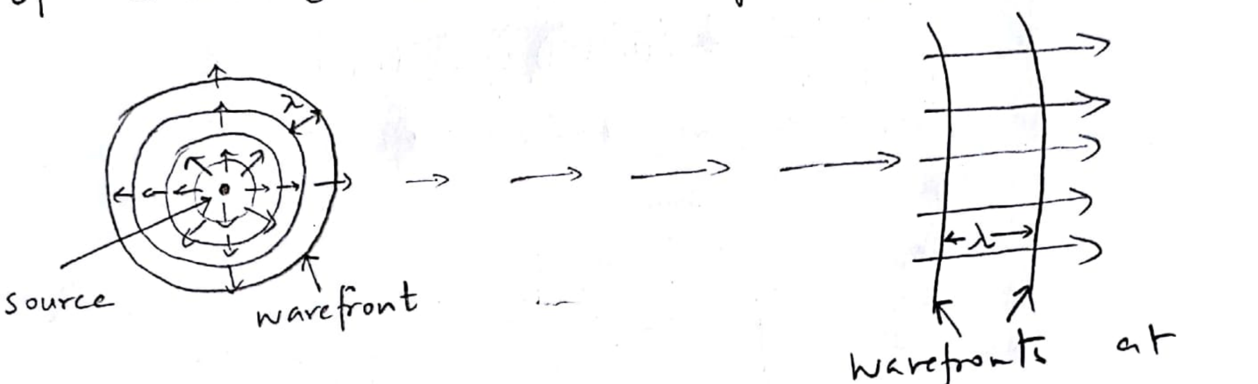
$$\vec{k} = K_x \hat{i} + K_y \hat{j} + K_z \hat{z}$$ &

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

Spherical Waves & Plane Waves

Waves start from a source & spread out into new regions of space.

ex:→ Ripples on a pond surface start from a point of disturbance & expand in form of circles (circles are crests of waves



(wave surfaces separated by one wavelength)

The circles are crests of waves & all particles located at the crest are in the same state of oscillation & hence in same phase.

WAVEFRONT :→
A Continous Locus of all particles which are in the same phase is called a wavefront.
A surface which passes through these points & completely surrounds the source is called the wave surface.

ex:→ Point light source has a spherical wavefront

At large distances from source spherical waves form become large & small portion of it may be considered as plane wave front.

PLANE WAVES.

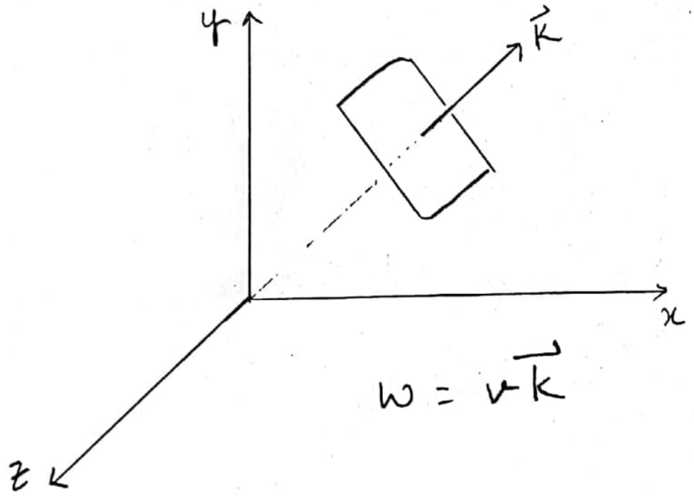
Consider a vector $\vec{r}$ which is normal to $\vec{k}$; thus $\vec{k} \cdot \vec{r} = 0$. $\vec{k}$ is the wave vector.

Consequently at a given time the phase of the disturbance is constant on a plane normal to K

$$\Psi = A \cos(\underbrace{\vec{k} \cdot \vec{r} - \omega t + \varphi}_{\text{constant}})$$

[Plane wave solution]

The direction of propagation of the disturbance is along K & the phase fronts are normal to K; Such waves are plane waves.



$\omega^2 = k^2 v^2$

$k_x = k_y = \frac{k}{\sqrt{2}}$

$k_z = 0$

$\omega = v\vec{k}$

for a given value of frequency, value of k^2 is fixed.

Considering planar waves is easier as all the velocity vectors are pointing in the same direction (parallel to one another) & reduces the problem to 1 Dimension.

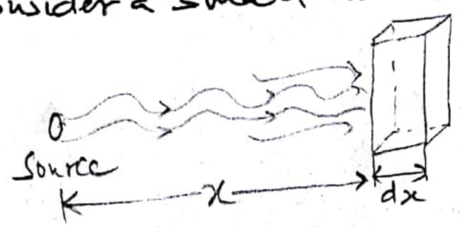
ENERGY TRANSPORTED IN PROGRESSIVE WAVES.

$$Y_{(x,t)} = A \sin(kx - \omega t)$$

Consider 1-D mechanical wave moving in a medium & transporting energy characterized by wave velocity

Consider a small mass element dm.

thickness = dx
Cross-sectional area = A_r
Density = ρ



mass = dm = $\rho \cdot A_r \cdot dx$

$$dK = \frac{1}{2} v^2 dm$$

$$v = \left(\frac{\partial y}{\partial t}\right)_x$$

$$\therefore v(x,t) = A\omega \cdot \cos(kx - \omega t)$$

$$\therefore dK = \frac{1}{2} A^2 \omega^2 dm \cos^2(kx - \omega t)$$

$\langle KE \rangle$ = Average Kinetic Energy

Kinetic Energy transported by the wave in one time period or one wavelength.

$$\langle KE \rangle = \frac{1}{2} A^2 \omega^2 dm \left[\frac{\int_0^\lambda \cos^2(kx - \omega t)\, dx}{\int_0^\lambda dx} \right]$$

$$\langle KE \rangle = \frac{1}{4} dm \cdot A^2 \omega^2$$

Potential energy stored in mass dm. } displaced from equilibrium by distance of y.

$$PE = -\int_0^y F \cdot dy$$

$$F = \text{mass} \cdot \text{acceleration} = dm \cdot \frac{\partial}{\partial t} v(x,t)$$

$$v(x,t) = A\omega \cdot \cos(kx - \omega t) \qquad \& \; y(x,t) = A \cdot \sin(kx - \omega t)$$

$$\therefore \frac{\partial v(x,t)}{\partial t} = -A\omega^2 \cdot \sin(kx - \omega t)$$

$$F = dm \cdot \frac{\partial}{\partial t} v(x,t) = -\omega^2 dm \, y(x,t)$$

$$\therefore PE = \omega^2 dm \int_0^y y \, dy$$

$$PE = \frac{1}{2} dm \, \omega^2 y^2$$

$$PE = \frac{1}{2} dm \, \omega^2 \cdot A^2 \sin^2(kx - \omega t)$$

Average P.E. over one period $\langle PE \rangle$

$$\langle PE \rangle = \frac{1}{2} dm \, \omega^2 A^2 \left[\frac{\int_0^\lambda \sin^2(kx - \omega t)}{\int_0^\lambda dx} \right]$$

$$\langle PE \rangle = \frac{1}{4} dm \, A^2 \omega^2$$

$$E = \langle KE \rangle + \langle PE \rangle = \frac{1}{2} dm \, A^2 \omega^2$$

Power of a wave.

Average Rate of Energy flow in the medium per cycle. is the Power transmitted by the wave.

$$P = \frac{E}{\Delta t} = \frac{1}{2} dm\, A^2 w^2$$

$$P = \frac{\frac{1}{2} \rho A_r dx\, A^2 w^2}{(\Delta t)} = \frac{\frac{1}{2} \rho A_r dx\, 2^2 \pi^2 f^2 A^2}{\left(\frac{dx}{v}\right)}$$

$$\boxed{P = 2\pi^2 A^2 f^2 \rho \cdot A_r v}$$

To find the variation of strength of a progressive wave with distance from the source we need to know about the Intensity of the wave. $I = \frac{P}{A_r}$

$$\therefore \quad I = 2\pi^2 A^2 f^2 \rho \cdot v$$

Intensity of a wave (Sound wave)

Energy flow per unit time across a unit area perpendicular to the direction of propagation.

Consider a sound wave propagating through a gas, then the energy per unit volume.

$\frac{\varepsilon}{V}\ (v=1) = \quad \varepsilon = \frac{1}{2} dm\, a^2 w^2 \cdot n$ (here n is the number of molecules per unit volume)

$\varepsilon = \frac{1}{2} \rho a^2 w^2$ (here ρ = density of gas = (dm)n) & a = amplitude

$\varepsilon = 2\rho a^2 \pi^2 f^2$ ($w = 2\pi f$; f = frequency).

$$\boxed{I = 2\pi^2 \rho a^2 f^2 v}$$

$I \propto a^2$ & $I \propto f^2$

N v 

(Since N particles are moving with same wave velocity v, then number of particles passing an unit area in unit time will be N v)

Intensity of a spherical wave.

Consider a wave emanating from a point source in a uniform isotropic medium i.e. velocity of the wave is uniform in all directions.

Let source emit W joules / sec. Then, $P = W$.

W joules/sec will cross the spherical surface area $4\pi r^2$ in one sec

Thus Intensity, $I_m = \frac{W}{4\pi r^2}$ --(1)

Inverse square law

$$\frac{W}{4\pi r^2} = 2\pi^2 \rho a^2 f^2 v$$

here a = amplitude

$$a = \left[\frac{W}{8\pi^3 \rho f^2 v}\right]^{1/2} \frac{1}{r}$$

amplitude $\propto \frac{1}{r}$

Displacement for spherical wave. $f_{(x,t)} = \frac{a_0}{r} \sin(kr - \omega t)$

[a_0 = amplitude at unit distance from source]