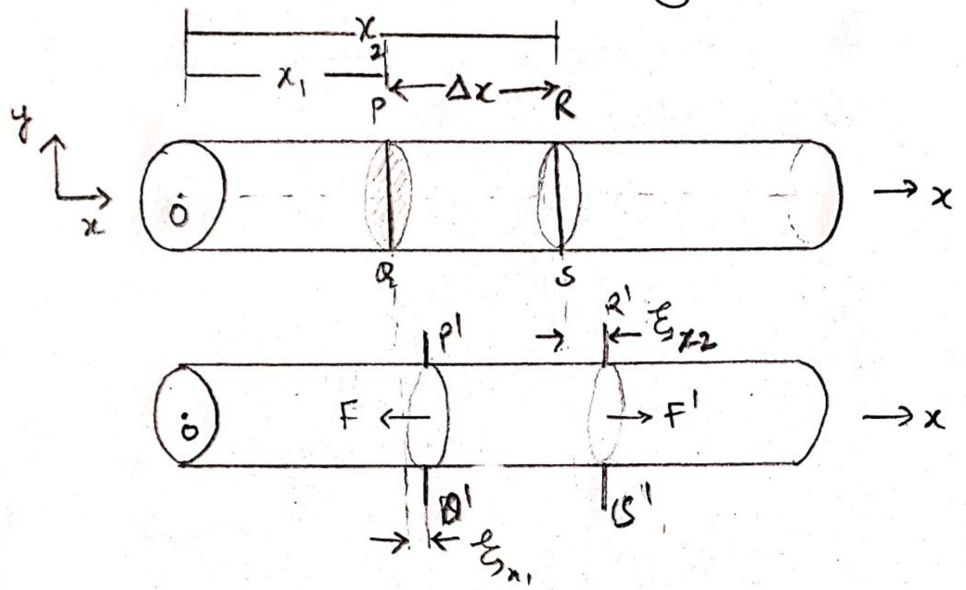
Longitudinal sound waves in a solid.

Consider a solid cylindrical rod (elastic solid) of cross sectional area A. When this rod is struck with a hammer at one end, the disturbance will propagate along it with a speed determined by its physical properties



$$\xi_{x_2} = \xi(x+\Delta x)$$
$$= \Delta x + \frac{\partial \xi}{\partial x}\Delta x$$

Let PQ & RS be two transverse sections of the rod at distances x & $x+\Delta x$ from a fixed point O. Here the rod lies with its length along the x-axis with origin O at the left end.

Let the longitudinal displacement be denoted by $\xi(x)$ since the rod has been struck at origin O lengthwise.

Distance between planes P'Q' & R'S':

$$\xi_{(x+\Delta x)} + \Delta x - \xi_x = \xi_x + \frac{\partial \xi}{\partial x}\Delta x + \Delta x - \xi_x$$

neglect higher order terms in Taylor series expansion

$$= \Delta x + \frac{\partial \xi}{\partial x}\Delta x$$

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longitudinal strain = $\dfrac{\text{change in length}}{\text{original length}} = \dfrac{\frac{\partial \xi}{\partial x} \cdot \Delta x}{\Delta x} = \dfrac{\partial \xi}{\partial x}$

Young's modulus = $\dfrac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = Y$

$\therefore \quad \dfrac{F}{A} = Y \cdot \dfrac{\partial \xi}{\partial x}$ or $F = Y \cdot A \dfrac{\partial \xi}{\partial x}$

here F is the force acting on element $P'Q'$ (-ve X)

A is the cross sectional area.

$$\therefore \quad \frac{\partial F}{\partial x} = YA \frac{\partial^2 \xi}{\partial x^2}$$

Force F' is also acting on the element $R'S'$ along +ve x direction. $\therefore$

Resultant force

$$F(x+\Delta x) - F(x) = F' - F = \frac{\partial F}{\partial x} \cdot \Delta x$$

$$= YA \frac{\partial^2 \xi}{\partial x^2} \cdot \Delta x$$

let ρ be the density, then mass of the element $P'Q'S'R' = \rho A \cdot \Delta x$. Thus the equation of motion becomes

$$\rho A \Delta x \frac{\partial^2 \xi}{\partial t^2} = YA \, \Delta x \frac{\partial^2 \xi}{\partial x^2}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{Y} \frac{\partial^2 \xi}{\partial t^2}$$

& $v_l = \sqrt{\dfrac{Y}{\rho}}$ = velocity of longitudinal waves

$\therefore$ the deformation propagates along the rod as a wave

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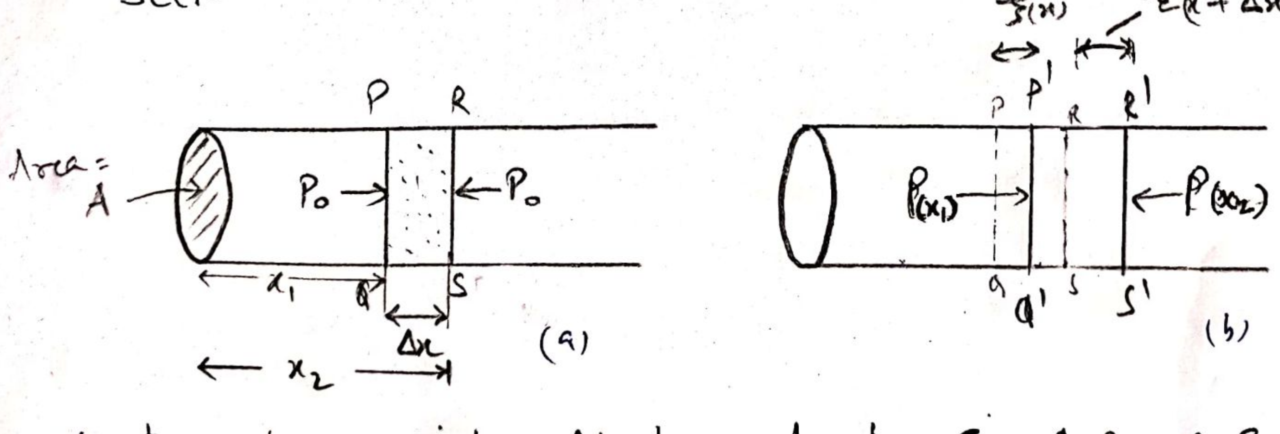
LONGITUDINAL WAVES IN A GAS

Unlike elastic solids, gaseous medium lacks rigidity & so transverse waves cannot propagate through it. However longitudinal waves can propagate through gas, solid & liquid mediums.

Sound waves in air columns.

Gases are compressible & pressure variations in a gas are accompanied with fluctuations in density. In solids (homogeneous), the density essentially remains constant.

∴ Consider an element in gas column 'PQRS' in a long pipe or cylindrical tube of uniform cross-sectional area A & width Δx



Let the longitudinal displacement be $\xi(x)$ at PQ & $\xi_{x+\Delta x}$ at RS

The pressure 'P_0' & density of the gas remains same throughout the volume under equilibrium condition (fig a)

ξ_{x_1} & ξ_{x_2} denote the displacement of particles originally at x_1 & x_2.

Change in thickness $= \Delta l = \xi_{(x_2)} - \xi_{(x_1)}$

$$\Delta l = \Delta x \frac{\partial \xi_x}{\partial x}$$

$$\Delta V = A \Delta l = A \Delta x \frac{\partial \xi_x}{\partial x}$$

volume strain = change in volume per unit volume

$$= \frac{\Delta V}{V} = \frac{A \Delta x \frac{\partial \xi_x}{\partial x}}{A \Delta x} = \frac{\partial \xi_x}{\partial x}$$

The increase in volume is due to decrease in pressure.

as the volume changes due to compressibility, the density will change.

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Let $P(x_1)$ be the pressure at plane $P'Q'$ &
$P(x_2)$ be the pressure at plane $R'S'$. Then
$P(x_1)$ is in positive x-direction &
$P(x_2)$ is in negative x-direction

$$P(x_1) = P_0 + \Delta P(x)$$
$$P(x_2) = P_0 + \Delta P(x+\Delta x)$$

then $\left[\Delta P(x) - \Delta P(x+\Delta x)\right]A$ is the force acting on the column $P'Q'R'S'$.

$$F = \left[-\frac{\partial}{\partial x}(\Delta P)\right]\Delta x \cdot A$$

then the equation of motion will be

$$\rho \cdot A \cdot \Delta x \frac{\partial^2 \xi}{\partial t^2} = \left[-\frac{\partial}{\partial x}(\Delta P)\right]\Delta x \cdot A$$

or $$-\frac{\partial}{\partial x}(\Delta P) = \rho \frac{\partial^2 \xi}{\partial t^2} \quad \cdots (2)$$ ρ = density of gas

Change in pressure causes a change in volume & considering an Adiabatic process.

PV^γ = constant

& ~~Young's modulus~~ $E = \gamma P$
Bulk of elasticity.

differentiating we get
$$\Delta P V^\gamma + \gamma P V^{\gamma-1} \Delta V = 0$$
$$\Delta P = -\frac{\gamma P \Delta V}{V}$$

$\left(\gamma = \frac{C_p}{C_v}\right)$ Ratio of specific heats

$$\Delta p = -E\left(\frac{\Delta V}{V}\right)$$ (negative sign indicates that as pressure increases volume decreases)

change in volume $= \Delta V =$ Area $\times$ change in length

change in length $= \left[\xi(x+\Delta x) - \xi(x) + \Delta x\right] - \Delta x$

change in volume $= \frac{\partial \xi}{\partial x} A \cdot \Delta x$

original volume $= A\,\Delta x$

$$\therefore \quad \Delta P = -E\frac{\partial \xi}{\partial x}$$ $(E = \gamma P)$

$\therefore$ Substituting in equation (2) we get

$$E\frac{\partial^2 \xi}{\partial x^2} = \rho\frac{\partial^2 \xi}{\partial t^2}$$

$$\text{or} \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 \xi}{\partial t^2} \quad \ldots (3)$$

hence $v_g = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$ $\left[10 \text{ dyne}/cm^2 = 1\ Pa = N/m^2\right]$

for air, $\gamma = 1.40$ & $P = 1.01 \times 10^6$ dynes$/cm^2$ & $\rho = 1.3 \times 10^{-3}$ g$/cm^3$ then $v_g = 330$ m/sec

$V_g =$ velocity of propagation of longitudinal sound waves in a gas, depends on Bulk modulus & density of the material.

At a given temperature $P/\rho =$ constant

$[\ \rho = 1.29$ kg$/m^3$ & $P = 1.01 \times 10^5$ N$/m^2$

SUPERPOSITION OF n harmonic waves.

We have already seen that superposition of two waves having the same frequency :

$$x_1 = a_1 \cos(\omega t + \phi_1)$$
$$x_2 = a_2 \cos(\omega t + \phi_2)$$

$$\left.\begin{aligned} a\cos\phi &= a_1\cos\phi_1 + a_2\cos\phi_2 \\ a\sin\phi &= a_1\sin\phi_1 + a_2\sin\phi_2 \end{aligned}\right\}$$

then $x = x_1 + x_2 = a(\cos \omega t + \phi)$

where $a = \left[a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2)\right]^{1/2}$

$$\tan\phi = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1\cos\phi_1 + a_2\cos\phi_2}$$

If we have n displacements, then when $x_1 . x_2 \cdots x_n$.

$x_n = a_n(\cos(\omega t + \phi_n))$

then $x = x_1 + x_2 + \cdots x_n = a\cos(\omega t + \phi)$

where $a\cos\phi = a_1\cos\phi_1 + a_2\cos\phi_2 + \cdots a_n\cos\phi_n$

$a\sin\phi = a_1\sin\phi_1 + a_2\sin\phi_2 + \cdots a_n\sin\phi_n$.

Now if we have n simple harmonic vibrations of equal amplitude & equal successive phase difference δ then;

$R\cos\phi = a + a\cos\delta + a\cos 2\delta + \ldots a\cos(n-1)\delta \quad \cdots (1)$

& $R\sin\phi = a + a\sin\delta + a\sin 2\delta + \cdots a\sin(n-1)\delta \quad \cdots (2)$

$$R\cos(\omega t+\phi) = a\cos\omega t + a\cos(\omega t+\delta) + a\cos(\omega t+2\delta) + \dots$$
$$\dots + a\cos(\omega t+(n-1)\delta) \quad \dots (3)$$

Multiplying equation (1) by $2\sin\delta/2$

$$2R\cos\phi\sin(\delta/2) = a\left[2\sin\delta/2 + 2\sin\delta/2\cos\delta + \dots 2\sin\delta/2\cdot\cos(n-1)\delta\right]$$

Using the known trignometric identity we get
$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$2R\cos\phi\sin(\delta/2) = a\left[2\sin\delta/2 + \left(\sin\frac{3\delta}{2} - \sin\frac{\delta}{2}\right) + \left(\sin\frac{5\delta}{2} - \sin\frac{3\delta}{2}\right) + \dots\right.$$
$$\left.\dots + \left(\sin(n-\tfrac{1}{2})\delta - \sin(n-3/2)\delta\right)\right]$$

$$\therefore 2R\cos\phi\sin\delta/2 = a\left[\sin\delta/2 + \sin(n-\tfrac{1}{2})\delta\right].$$

$$2R\cos\phi\sin\delta/2 = 2a\left(\sin\frac{n\delta}{2}\right)\cdot\left(\cos\frac{(n-1)\delta}{2}\right)$$

$$\therefore R\cos\phi = \frac{a\sin(n\delta/2)}{\sin(\delta/2)}\cdot\cos\left[\frac{(n-1)\delta}{2}\right] \quad \dots (4)$$

Similarily multiply $2\sin(\delta/2)$ to equation 2 & simplifying gives us

$$R\sin\phi = \frac{a\cdot\sin\left(\frac{n\delta}{2}\right)}{\sin(\delta/2)}\cdot\sin\frac{(n-1)\delta}{2} \quad \dots (5)$$

Squaring 4 & 5 & adding them

$$R^2 = \frac{a^2\sin^2(n\delta/2)}{\sin^2(\delta/2)} \quad \text{or} \quad \boxed{R = \frac{a\sin(n\delta/2)}{\sin(\delta/2)}}$$

$$\text{\& } \tan\phi = \tan\frac{(n-1)\delta}{2} \quad \text{or} \quad \phi = \frac{(n-1)\delta}{2}$$

When n is very large, δ is very small

& $\phi = \frac{(n-1)\delta}{2} \simeq \frac{n\delta}{2}$

& $\sin\left(\frac{\delta}{2}\right) \sim \frac{\delta}{2} \approx \frac{\phi}{n}$

$\therefore \quad R = \frac{a \sin(n\delta/2)}{\sin(\delta/2)} = a\,\frac{\sin\alpha}{\alpha/n} = na\,\frac{\sin\alpha}{\alpha}$

$$\boxed{R = A\,\frac{\sin\alpha}{\alpha}}$$

The figure displays the mathematical expression

$$R\cos(\omega t + \alpha) = a\cos\omega t + a\cos(\omega t + \delta) + a\cos(\omega t + 2\delta) + \cdots + a\cos(\omega t + [n-1]\delta)$$

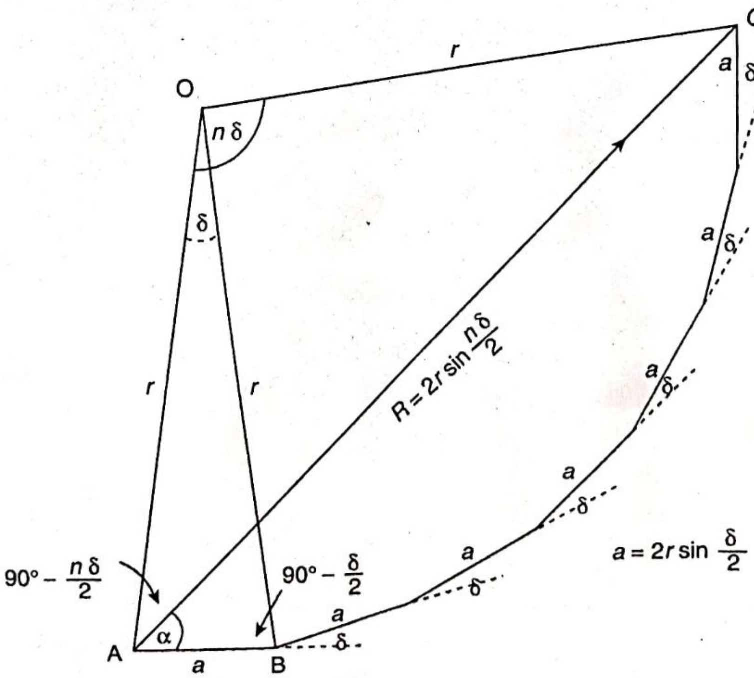

Figure 1.11 Vector superposition of a large number n of simple harmonic vibrations of equal amplitude a and equal successive phase difference δ. The amplitude of the resultant