2.2 THE UNIVERSAL LAW OF GRAVITATION

In *Principia*, Newton presented his three laws of motion, and also stated the law of gravitation. Newton stated that any two particles in the universe exert an attractive force on one another whose magnitude is directly proportional to their masses m_1 and m_2 and decreases with the distance between the two masses as the square of the direct distance r between them.

$$\boxed{\text{Magnitude:} \quad F = G_N \frac{m_1 m_2}{r^2}.} \tag{2.2}$$

The direction of the force on each mass is based on the attractive nature of the force - the force on m_1 is towards m_2 and on m_2 it is towards m_1. Here G_N is a constant, called the universal gravitational constant, which has the same value for the force between any two bodies. The constant G_N is considered to be truly universal, being same between any two objects anywhere in the universe. Do not confuse this G_N with acceleration due to gravity g. The value of G_N has been determined experimentally to be approximately 6.67428×10^{-11} m^3/kg.s^2.

Further Remarks: The great English physicist Paul A. M. Dirac has hypothesized that the value of G_N may have been different in the past. A non-constant G_N will have significant cosmological consequences, which is not fully understood yet.

Law for Spherical Bodies

Although Newton's law of gravitation stated in Eq. 2.2 is for point particles, the same form of the law applies to spherical objects provided we use the center-to-center distance for r. An important

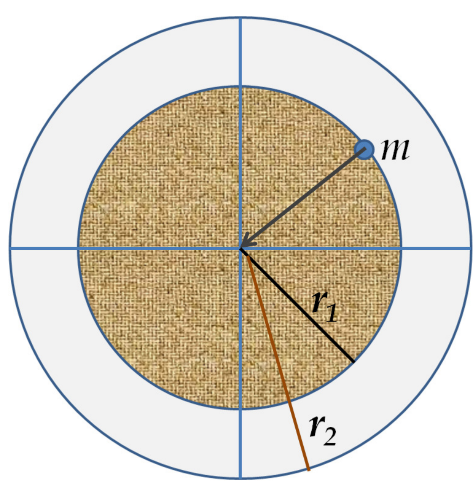

Figure 2.3: The net gravitational force on a mass m placed at a point inside a spherical body is equal to the gravitational force of the particle of the sphere that is within the distance r_1 of the center of the sphere.

consequence of the inverse-square distance nature of the gravitational force is that, if you drill a hole inside a solid sphere of radius r_2 and place a mass m some distance r_1 from the center as shown in Fig. 2.3, then the force on mass will come from only the mass of the sphere that is inside the radius $r < a$, and none of the masses between radius $r = r_1$ and $r = r_2$ will count if the sphere has uniform density. You can show it mathematically that the gravitational force on the test mass m from the particles of the sphere for whom the radial distance r is greater than r_1, i.e. $r_1 < r \leq r_2$, cancels out.

One of the consequences of this result is that if you sit inside a spherical shell, the gravitational force on you will be zero no matter where you sit inside the shell! Can you prove this assertion?

Law for Arbitrary Bodies

If the distance between two bodies is large compared to their sizes, the bodies can be replaced by point masses and then we can use the same form of the law as given in Eq. 2.2 for point particles. The force of attraction between non-spherical objects which are not far apart compared to their sizes is complicated and can only be computed by dividing the bodies into smaller parts and vectorially summing up the forces between every pair of masses.

Example 2.2.1. Newton's law of gravitation from a falling apple and falling Moon.

It is interesting to note that a straightforward analysis of falling of an apple and the continuous falling of the Moon towards the Earth can lead one to the discovery of the universal law of gravitation. Although, Newton did not come upon the universal law of gravitation in this way, the analysis is none-the-less instructive, and will be presented here.

From the observations on freely falling objects near the Earth, you know that the apple falls with an acceleration of 9.81 m/s^2. From observations on the Moon, i.e. from the distance to the Moon and its time period, it is also known that the Moon has a centripetal acceleration of 0.00272 m/s^2 in its circular orbit about Earth. Thus, the acceleration of the Moon is approximately 3600 times less than that of apple.

$$\frac{a_{\text{Moon}}}{a_{\text{apple}}} = \frac{0.00272\ \text{m/s}^2}{9.81\ \text{m/s}^2} \approx \frac{1}{3600}$$

Since the Moon is at a distance of approximately 60 times the radius of the Earth, it can be concluded that the acceleration due to the gravitation of the Earth drops as the inverse of the square of the distance.

$$\frac{a_{\text{Moon}}}{a_{\text{apple}}} = \frac{r^2_{\text{apple}}}{r^2_{\text{Moon}}}$$

Since, force on an object is proportional to the acceleration, the force must also vary as $1/r^2$ if the acceleration varies as such. The genius

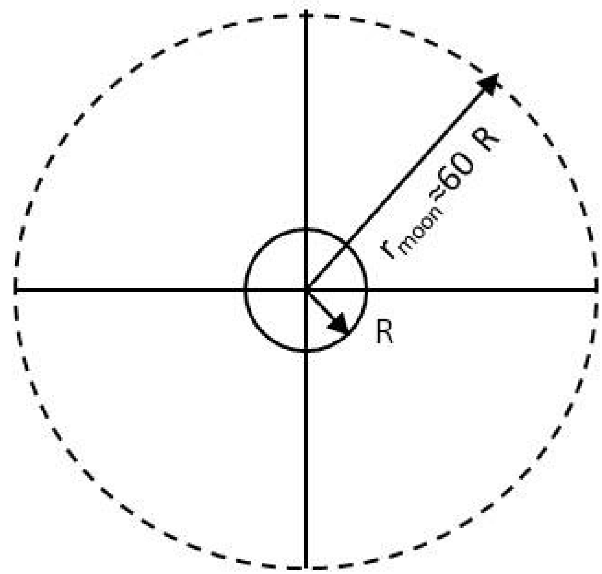

Figure 2.4: The Moon traveling in circular motion about the Earth completes one orbit in approximately 29 days, which gives it's a centripetal acceleration of 0.00272 m/s^2.

of Newton was in the realization that the same law of force must apply between the Earth and the Moon as applies between the Earth and the apple.

$$\frac{F}{m} \propto \frac{1}{r^2}$$

Here m stands for the mass of the apple or the Moon.

Example 2.2.2. Gravitational force between objects

Find the gravitational forces between objects in the following situations. (a) Two spherical lead balls of masses 10 kg each separated by center-to-center distance of 10 cm. (b) Two people of height approximately 1.5 m and masses 100 kg and 60 kg separated by a distance of 2 m. (c) Same two people 20 m apart. (d) A satellite of mass 1000 kg in orbit 200 km above the Earth; use mass of the Earth = 5.96×10^{24} kg, and mean radius of the Earth = 6.37×10^6 m.

Solution. (a) We have stated above that a spherical object "acts" same as a point particle as far as gravitation outside the sphere is concerned. Therefore, the force between the spherical lead balls can be easily computed giving the following magnitude.

$$F = G_N \frac{m_1 m_2}{r^2} = \left(6.67 \times 10^{-11} \frac{\text{N.m}^2}{\text{kg}^2}\right) \times \frac{10 \text{ kg} \times 10 \text{ kg}}{(0.1 \text{ m})^2} = 6.67 \times 10^{-7} \text{ N}.$$

(b) Since people are not spherical objects, and since their separation is not too great compared to their sizes, we cannot use the same form of the law of gravitation as given for the point particles. If we use the formula given, we will only get some rough estimate, not an exact answer for the magnitude of the force.

$$F \approx G_N \frac{m_1 m_2}{r^2} = \left(6.67 \times 10^{-11} \frac{\text{N.m}^2}{\text{kg}^2}\right) \times \frac{100 \text{ kg} \times 100 \text{ kg}}{(2 \text{ m})^2} = 1.0 \times 10^{-8} \text{ N}.$$

(c) In this part, the people are separated by more than 10-times their sizes. Therefore approximating them as spheres will not give us too different a result than an exact result. Although the result of spherical approximation for a person will not give an exact result, but we expect the result to be more precise here than it was in part (b). The magnitude of the force is

$$F \approx G_N \frac{m_1 m_2}{r^2} = \left(6.67 \times 10^{-11} \frac{\text{N.m}^2}{\text{kg}^2}\right) \times \frac{100 \text{ kg} \times 100 \text{ kg}}{(20 \text{ m})^2} = 1.0 \times 10^{-10} \text{ N}.$$

(d) Although, the satellite is not a spherical object, it is far away from the Earth compared to the size of the satellite. Therefore, we can treat the satellite as a point mass. This gives the following for the magnitude of the force on the satellite.

$$F = \left(6.67 \times 10^{-11} \frac{\text{N.m}^2}{\text{kg}^2}\right) \times \frac{5.97 \times 10^{24} \text{ kg} \times 1000 \text{ kg}}{(6.37 \times 10^6 \text{ m} + 0.200 \times 10^6 \text{ m})^2} = 9.2 \times 10^3 \text{ N}.$$

Example 2.2.3. Gravitational force and circular orbits

A satellite is to be placed in a geocentric circular orbit about Earth. Find the altitude of the circular orbit above Earth.

Solution. A **geocentric orbit** means that the satellite has the same time period as the time period of the rotation of Earth. When a satellite is in a geocentric orbit, the satellite appears at the same spot above the Earth. These satellites are used for telecommunications. Let h be the altitude above Earth, and R_E the radius of the Earth. Then, the radius of the circular motion of the satellite is $R_E + h$. Since the satellite covers a distance of one circumference $2\pi(R_E + h)$ in time T=1 day, the average speed of the satellite must be

$$v = \frac{2\pi(R_E + h)}{T}$$

We will assume that the speed of the satellite is constant. This means that the satellite would be in a uniform circular motion and the magnitude of the centripetal acceleration of the satellite would be

$$a_c = \frac{v^2}{R} = \left[\frac{2\pi(R_E + h)}{T}\right]^2 \frac{1}{R_E + h} = \frac{4\pi^2(R_E + h)}{T^2}$$

The centripetal acceleration is the net acceleration of the satellite in this case. The net force on the satellite is from the force from the Earth. Therefore, we can write the following using the magnitudes of the force on the satellite and the acceleration of the satellite.

$$ma_c = F \quad \Longrightarrow \quad m\frac{4\pi^2(R_E+h)}{T^2} \approx G_N\frac{mM_E}{(R_E+h)^2}$$

We can solve this equation for the altitude of the satellite in a geocentric orbit with the following result.

$$h = \left(\frac{G_N M_E T^2}{4\pi^2}\right)^{1/3} - R_E = 3.6 \times 10^7 \text{ m}.$$