4.7 THE PROOF OF CLAUSIUS'S THEOREM

Clausius's theorem states that the following inequality holds for all cyclic processes.

$$\oint \frac{dQ}{T} \leq 0. \tag{4.28}$$

Proof: Consider a system S that undergoes a cyclic process. In each cycle the system receives or gives up heat to a set of thermal baths at temperatures T_1, T_2, $\cdots$, T_n. A Carnot engine is an example of this kind of system with only two heat baths where every process is reversible. Here we assume that the processes of system S may be irreversible. Let Q_1, Q_2, $\cdots$, Q_n be the heat exchanged by the system S and the thermal baths T_1, T_2, $\cdots$, T_n respectively with the sign notation that Q_i will be positive if system S takes in heat and negative if it gives up heat to the bath. We will prove that

$$\sum_{i=1}^{n} \frac{Q_i}{T_i} \leq 0. \tag{4.29}$$

Once we have proved this, we can immediately obtain the theorem by considering the case where system S exchanges heat with infinitely many baths. In that case the sum in Eq. 4.29 will turn into integral in Eq. 4.28.

To proceed with the proof, we introduce reversible engines such as Carnot engines C_1, C_2, $\cdots$, C_n between the baths and a common heat bath at temperature T_0 as shown in Fig. 4.7.

We will pick Carnot engines such that the i^{th} Carnot cycle replenishes the heat exchanged by the i^{th} heat bath and the system S. Each Carnot cycle has the following relation between its temperatures of operation and the corresponding heat exchanged.

$$C_1 : \qquad Q_{01} = \frac{Q_1}{T_1} T_0 \tag{4.30}$$

$$C_2 : \qquad Q_{02} = \frac{Q_2}{T_2} T_0 \tag{4.31}$$

$$\vdots$$

$$C_n : \qquad Q_{0n} = \frac{Q_n}{T_n} T_0 \tag{4.32}$$

Now if we look at one cycle of the complex system consisting of S and the carnot cycles C_1, C_2, $\cdots$, C_n, we find that it absorbs a net heat $Q_0 = Q_{01} + Q_{02} + \cdots + Q_{0n}$, and does the same amount of work.

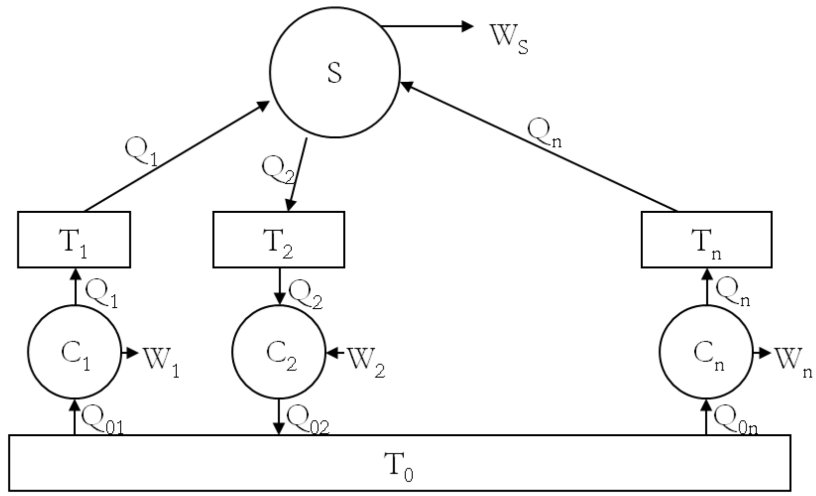

Figure 4.7: System S undergoing a cyclic process interacting with heat baths T_1, T_2, $\cdots$, T_n. Carnot cycles operate between these heat baths and a common heat bath T_0.

Therefore the complex system $\{S,\ C_1,\ C_2,\ \cdots,\ C_n\}$ will violate the Kelvin-Planck statement of the second law of thermodynamics if heat Q_0 was positive. That would mean a net intake of heat to the system in one cycle all of which converted to work without any other change. Therefore, second law forbids Q_0 from being positive. Hence

$$Q_0 = \sum_{i=1}^{n} \frac{Q_i}{T_i} T_0 \leq 0. \tag{4.33}$$

Since $T_0 > 0$, we obtain the desired result.

$$\sum_{i=1}^{n} \frac{Q_i}{T_i} \leq 0.$$

Considering the case where system S exchanges infinitesimal amount of heat with infinitely many baths turns the sum into integral over the cycle of the system S: $\oint \frac{dQ}{T} \leq 0$.