7.2 ENERGY CONSERVATION AND BERNOULLI'S EQUATION

The application of the principle of the conservation of energy to the non-viscous steady flow leads to a very useful relation between the pressure and the flow speed in the fluid. This equation is called Bernoulli's equation after Daniel Bernoulli (1700-1782) who published his studies on fluid motion in the book, Hydrodynamica (1738).

Consider a fluid flowing through a pipe that may have different cross-sections and be located at different heights at along the flow of a fluid as shown in Fig. 7.5. Let us focus on two such places marked 1

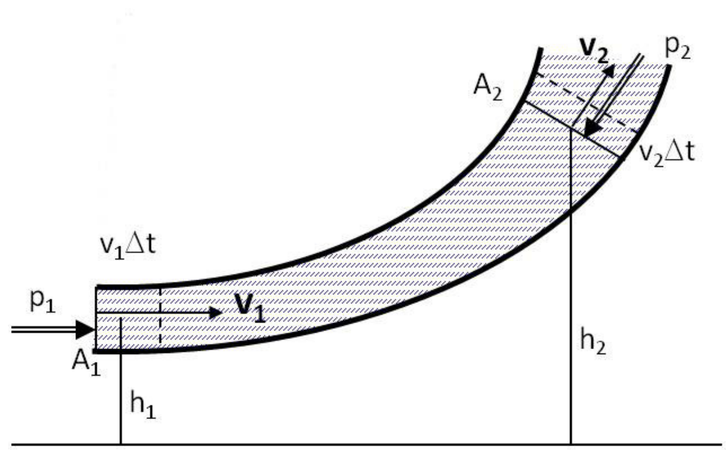

Figure 7.5: Geometry used for derivation of the Bernoulli's equation.

and 2, and denote the area of cross-section A, the speed of flow v, the height from ground h, and the pressure p at these sites by subscripts 1 and 2 respectively. The density at the two sites will be taken to be the same and denoted by ρ without any subscripts.

We will assume that there are no viscous forces in the fluid so that the energy of any part of the fluid would be conserved. To deduce the Bernoulli's equation, we will equate the change in the energy of the fluid between the points 1 and 2 to the work by the fluid outside of 1 and 2 during an interval Δt. That is, we select the fluid between 1 and 2 as our system and the rest of the fluid and gravity become external to the system.

At point 1, the fluid external to the system is pushing on the system with a force $p_1 A_1$ and in the time interval Δt the boundary moves a distance $v_1 \Delta t$. Therefore, the work done on the system at point 1 is

$$W_1 = (p_1 A_1)(v_1 \Delta t). \tag{7.9}$$

On the other end, the system pushes against the external fluid with a force of magnitude $p_2 A_2$, which must push the system back with a force of the same magnitude but in the opposite direction. Since the displacement of the layer at point 2 during the time interval Δt is $v_2 \Delta t$, the work done on the system at point 2, that is W_2 will be

$$W_2 = -\left(p_2 A_2\right)\left(v_2 \Delta t\right). \tag{7.10}$$

Therefore the net work done on the system by the external fluid is the sum of W_1 and W_2.

$$W = W_1 + W_2 = p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t. \tag{7.11}$$

During the interval Δt, some fluid enters the system at height h_1 with speed v_1 while the same amount leaves the system at another height h_2 with speed v_2. Therefore there will be the following changes in the kinetic energy ΔK and the potential energy ΔU.

$$\Delta K = \frac{1}{2} m_2 v_2^2 - \frac{1}{2} m_1 v_1^2 = \frac{1}{2}\left(\rho A_2 v_2 \Delta t\right) v_2^2 - \frac{1}{2}\left(\rho A_1 v_1 \Delta t\right) v_1^2 \tag{7.12}$$

$$\Delta U = m_2 g h_2 - m_1 g h_1 = \left(\rho A_2 v_2 \Delta t\right) g h_2 - \left(\rho A_1 v_1 \Delta t\right) g h_1 \tag{7.13}$$

The work-energy theorem relates the change in the mechanical energy of a system to the work done on the system by external forces. Therefore, we obtain the following relation between the net external work and the change in mechanical energy.

$$W = \Delta K + \Delta U. \tag{7.14}$$

Substituting for W, ΔK and ΔU, rearranging terms and using the equation of continuity $A_1 v_1 = A_2 v_2$ applicable here, we find the following equation, called the Bernoulli's equation.

$$\boxed{p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2.} \tag{7.15}$$

This relation states that the mechanical energy of any part of the fluid changes as a result of the work done by the fluid external to that part due to varying pressure along the way. Since the two points were chosen arbitrarily, we can write the Bernoulli's equation more generally as a conservation principle along the flow.

$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant.}} \tag{7.16}$$

The constancy of the combination of pressure and the sum of kinetic and potential energy densities is not only constant over time but is also constant over space. A special note must be made here of the fact the in a dynamic situation, the pressure at the same height in different parts of the fluid may be different if they have different speeds of flow.

Special cases of interest:

1. Static situations

Bernoulli's equation includes static behavior as we can see immediately by setting $v_1 = v_2 = 0$ in the equation.

$$p_1 + \rho g h_1 = p_2 + \rho g h_2. \quad \text{(Static)} \tag{7.17}$$

This says that pressure difference between two points in a fluid at rest is directly proportional to the height difference as we have studied in the last chapter.

$$p_1 - p_1 = \rho g \left(h_2 - h_1 \right). \quad \text{(Static)} \tag{7.18}$$

2. Horizontally flowing fluids

For horizontally flowing case $h_1 = h_2$ Bernoulli's equation simplifies.

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2. \tag{7.19}$$

An important interpretation of this equation is that in a fluid, pressure is lower where the speed is higher and vice versa. This conclusion is quite contrary to the "common sense". It would seem that pressure should be greater where the fluid is flowing at a higher speed than where it is flowing at a lower speed, but Bernoulli's equation tells us that just the opposite is true.

To make use of this aspect of the flow of fluids, airplane wings are constructed so that air will flow with a greater speed above the wings than below the wings. This flow pattern gives rise to a lower pressure at the top of the wing than at the bottom. Therefore, the force pushing on the wing from below would be larger than the force pushing down on the wing from the above. This difference in the force is called the **lift force** and is responsible for flying. (Fig. 7.7).

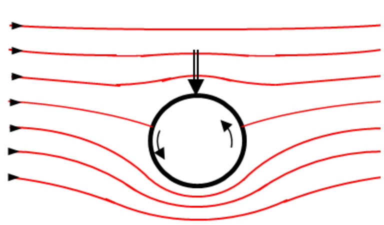

Figure 7.6: Downward force on spinning curve baseball arises due to increased speed of the air flow at the bottom compared to the speed above the ball when the ball is spinning backwards.

Similarly, in atmospheric physics, we find that the wind speed is higher where the pressure is lower, and in circulatory system, blood flows fastest where the pressure is the lowest.

To understand these and other fluid flow problems, it is usually very helpful to apply Bernoulli's equation to standard problems. To illustrate the applications of the Bernoulli's equation, we will start

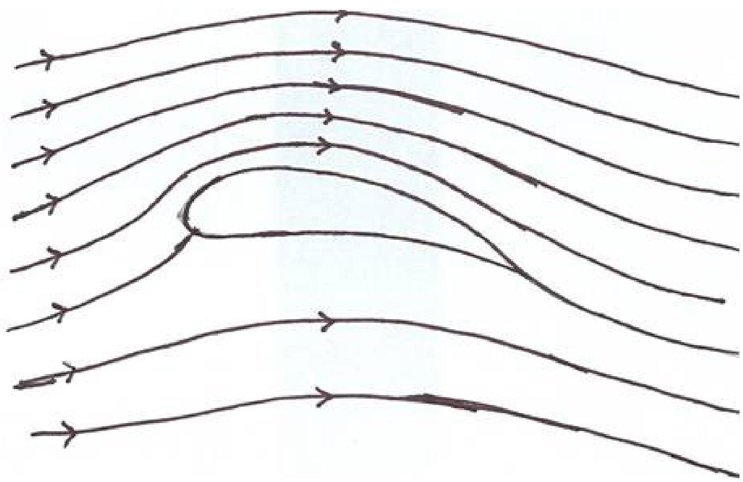

Figure 7.7: Air flows faster above the wing of an airplane than below the wing aided by the design of the foil. Therefore, pressure above the wing is less than below the wing. The product of the area of the wing and the difference in pressure is the lift force on the wing.

with the problem of draining a tank that Bernoulli presented in his book Hydrodynamica.

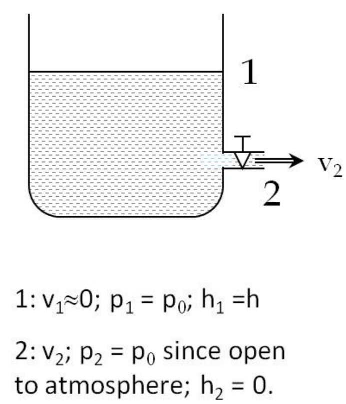

Figure 7.8: Example 7.2.1

Example 7.2.1. Draining a tank.

Consider a tank of water with a tap at depth h from the surface in a container. As the tap is opened what is the speed with which water will come out? Assume steady flow. Also assume that the speed of flow of water at the surface in the tank to be very nearly zero compared to the speed at the tap.

Solution. To apply Bernoulli's equation, we consider two points in the fluid. Point 1 is at the surface and point 2 is just outside the valve at the spigot as shown in Fig. 7.8. Since the area of cross-section at the surface is much larger than the opening, the speed of flow at the surface is much smaller than the speed at the opening. Therefore, I will assume that the speed at a point at the top of the tank may be set to zero without incurring significant error. The pressure at the surface is the atmospheric pressure p_0 since it is freely exposed to the atmosphere.

The pressure against which the water comes out at the opening is also the atmospheric pressure p_0 and not $p_0 + \rho g h$ as would be the case if water was not flowing! This aspect is somewhat disturbing to people first learning to think in terms of dynamics rather than static situation, especially after memorizing formulas of static fluid in the last chapter. The reason for the pressure at the exit to be equal to the atmospheric pressure has to do with the fact that the only the force due to the atmospheric pressure is opposing the flow coming

One way to figure out the dynamic pressure at the exit is to pretend that you are the fluid at the exit, and ask, "what force will you be pushing against?" When you think in this language, you would not be confused about the dynamic pressure at any point.

out of the nozzle.

We now put the properties at the two points 1 and 2 in the Bernoulli's equation.

$$p_0 + 0 + \rho g h = p_0 + \frac{1}{2}\rho v_2^2 + 0. \tag{7.20}$$

Therefore, the speed of water at the spigot is

$$v_2 = \sqrt{2gh} \tag{7.21}$$

This formula is also called the **Torricelli's theorem**. It is interesting to note that the speed of a freely falling particle dropped from a height h is given by the same formula.

Example 7.2.2. The Venturi meter

The Venturi meter is a device for measuring the flow speed of gases and liquids of a known density ρ. The fluid at some unknown speed enters at the inlet, passes through a constriction called the throat where a manometer is attached which contains a denser liquid of density ρ_0. The areas of cross-section at the inlet and constriction are A_1 and A_2 respectively.

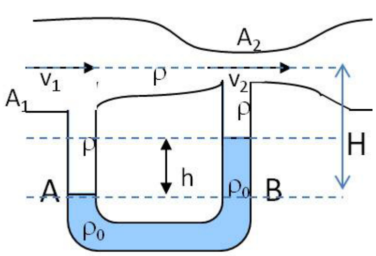

Figure 7.9: Example 7.2.2

Fluid ρ: Applying Bernoulli's equation to the gas at the inlet and outlet at the same height we obtain the following equation.

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \quad \text{(dynamic fluid } \rho) \tag{7.22}$$

Fluid ρ_0: Another relation between the pressures at the inlet and outlet is obtained by applying the static pressure condition on the fluid in the tube. As shown in Fig. 7.9, the pressure at points A and B will be equal.

$$p_A = p_B \quad \text{(static fluid } \rho_0)$$

Now, pressure p_A is equal to $p_1 + \rho g H$ and p_B is equal to $p_2 + \rho g(H - h) + \rho_0 g h$. Therefore,

$$p_1 + \rho g H = p_2 + \rho g(H - h) + \rho_0 g h. \tag{7.23}$$

Subtracting Eq. 7.23 from Eq. 7.22 we find

$$(\rho_0 - \rho)\, g h = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2. \tag{7.24}$$

The equation of continuity for flowing gas between the inlet and outlet gives us the other relation we need to find v_1.

$$A_1 v_1 = A_2 v_2 \tag{7.25}$$

Eliminating v_2 from Eqs. 7.24 and 7.25, we find the following result for the speed of gas at the inlet.

$$v_1 = A_1 \sqrt{\frac{2(\rho_0 - \rho) g h}{\rho \left(A_1^2 - A_2^2\right)}}. \tag{7.26}$$

Therefore, $v_1 = \alpha\sqrt{h}$, where α is a constant. Recording the difference in the two arms of the U-tube for the static fluid of density ρ_0 gives us h which lets us determine the flow speed of the flowing fluid of density ρ at the inlet by Eq. 7.26.

The formula in Eq 7.26 can be simplified by introducing the following ratios.

$$x \equiv \frac{\rho_0}{\rho}$$
$$y \equiv \frac{A_2}{A_1}$$

With these changes Eq 7.26 becomes

$$v_1 = \sqrt{\frac{2(x-1) g h}{(1-y^2)}}, \tag{7.27}$$

which has the added advantage that we do not need to worry about the particular system of units for the densities and the areas because they cancel out in the ratios.

Example 7.2.3. The lift of an airplane

An air plane of mass 1000 kg has air of density 1.2 kg/m^3 flowing above the wing of area 15 m^2 at a speed of 60 m/s over the top surface and 50 m/s past the bottom surface. (a) Find the lift force on the airplane. (b) How does the lift force compare with the weight of the airplane?

Solution. Since the height between the top and bottom of the wing is negligible for the present purpose, we can find the pressure difference between the top and bottom of the wing by using simplified Bernoulli's equation.

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$
$$p_{\text{bottom}} - p_{\text{top}} = \frac{1}{2}\rho_{\text{air}}\left(v_{\text{top}}^2 - v_{\text{bottom}}^2\right) = 660 \text{ Pa}.$$

The magnitude of the lift force F_L is the pressure difference times area of the wing.

$$F_L = 660 \text{ Pa} \times 15 \text{ m}^2 = 9,900 \text{ N}.$$

(b) The weight of the plane is $W = 1000 \text{ kg} \times 9.81 \text{ m/s}^2 = 9800\text{N}$. Hence, in the present case, the lift force is 100 N more than the weight.