5.4 ENERGY IN STRAINED MATERIAL

When an external force causes a strain in a body, the external force does work on the body. This work is stored as potential energy in the strained body. The amount of work done by an external force on deforming a body can be calculated similar to the way we have found the energy stored in a spring when the natural length of the spring is changed. In the case of the spring, we can fix one end of the spring to a rigid support and push or pull on the other end by a force $\vec{F}$. It was best to do the calculation analytically by placing the spring along the x axis with origin at the tip of the spring when the spring is not deformed. Then, deformation of the spring will be given by the x-coordinate of the tip of the deformed spring and the work for changing the length from x to $x + dx$ is easily found to be

$$dW = kxdx, \tag{5.39}$$

since the applied force is matched by the internal force of the spring. Here as usual, k is the spring constant. It is best to write this equation in terms of the deformation dl and the internal force F_s of the spring.

$$dW = -F_s dl, \tag{5.40}$$

where the negative sign means that the force is towards the equilibrium while the deformation is pointed away from the equilibrium. Another thing to note in this formula is that l is the deformation and dl is the change in the deformation. That is we have written the work done by an external agent in changing the deformation of the spring from l to $l + dl$. We can integrate this equation to obtain the energy stored in the spring since we know the law for the force in terms of the deformation l.

$$W_{if} = -\int_i^f F_s dl = k\int_i^f l dl = \frac{1}{2}kl_f^2 - \frac{1}{2}kl_i^2, \tag{5.41}$$

where l_i is the deformation of the spring in the initial state and l_f is the deformation in the final state.

Similarly, we can write the work done for various deformations. Thus, the work done by an external agent in a compressive or tensile strain will be obtained as follows. Let the strain be denoted by s and the stress by the Greek letter σ. Suppose an external force $\vec{F}$ is acting on the body that changes the strain of the body from s to $s + ds$. Since the strain for the extension or compression is defined by the change in length divided by the length, the change in the strain is $ds = dl/l$ and the work written in terms of the strain and the force would be

$$dW = Flds. \tag{5.42}$$

Here l is the length of the object being compressed or stretched. We can convert this to an equation in terms of the stress in the body and the strain of the body by replacing the magnitude of the external force F in terms of the stress in the body.

$$dW = -Al\sigma ds. \tag{5.43}$$

Now, the stress of this type is equal to the product of the Young's modulus Y and the strain s. Furthermore, the direction of the stress must be in the opposite direction to the change in the strain since the stress opposes any change in the strain. Therefore,

$$dW = VYsds, \tag{5.44}$$

where I have replaced Al by the volume V of the object. The energy stored in changing the strain from zero strain to some strain s can be obtained by integrating this equation. This would be the potential energy U stored in the state characterized by the strain value s.

$$U = \frac{1}{2}(VY)s^2, \quad \text{(Compressive and Tensile)}. \tag{5.45}$$

We see that the product VY plays the role of "spring constant" in compressive and tensile strains. Similarly, the work done for shear strain can be written as

$$U = \frac{1}{2}(VG)\theta^2, \quad \text{(Shear)}. \tag{5.46}$$

Finally, we will obtain the following expression for the work done on the body by an external force $\vec{F}$ when the volume changes from V to $V + dV$ with the pressure in the body equal to p.

$$dW = -pdV. \tag{5.47}$$

This work will increase the energy of the body if the body is compressed and decrease the energy if the body expands. The energy stored when the volume of the material changes from the equilibrium value to another value due to applied force on the body can be worked similarly.