12.6 AC CIRCUIT PROBLEMS

12.6.1 Series and Parallel Circuits

Complex impedance and admittance are very useful quantities for AC circuit analysis. We have above that impedance of circuit elements in AC circuits is analogous to resistance in DC circuits. Therefore, the complex impedance of circuit elements in series add to yield an equivalent impedance, and inverse of complex impedance add for circuits elements in parallel similar to what happens in resistive circuits.

$$\tilde{Z}_{\text{eq}} = \tilde{Z}_1 + \tilde{Z}_2 + \cdots \quad \text{(Series)} \tag{12.72}$$

$$\frac{1}{\tilde{Z}_{\text{eq}}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \cdots \quad \text{(Parallel)} \tag{12.73}$$

The first step in the analysis of an AC circuit is to replace the circuit elements by their impedances. Each resistor R is replaced by its $Z_R = R$, each inductance L is replaced by its impedance $Z_L = i\omega L$, and each capacitance is replaced by its impedance $Z_C = 1/i\omega C$. Then, we simplify the circuit by replacing the impedances in series by an equivalent resistance and applying the complex Ohm's law to the equivalent impedance.

If the replacement of the impedances in series does not result in one impedance across the EMF source, this step often leaves the resulting circuit with impedances in parallel. These impedances are then replaced by an equivalent impedance.

The process of simplifying the impedances in series and then the impedances in parallel, then the impedances in series in the simplified circuit, then the impedances in parallel, and so on. Often this process yields one final equivalent impedance across the EMF source, called the equivalent impedance of the circuit.

If this process stops at some step because impedances are neither in series nor in parallel, then we use Kirchhoff's rules to solve the circuit as we have done for resistive circuits in an earlier chapter. A circuit where the series-parallel method will fail is the bridge circuit such as the Wheatstone bridge when the bridge is not balanced. We will discuss these circuits below. Here we will work out examples of series/parallel type.

Example 12.6.1. Parallel RL Circuit Consider the circuit given in the Fig. 12.16 where a resistor R and an inductor L are connected in parallel to an AC power generator with voltage $V(t) = V_0 \cos(\omega t)$. Find (a) the current in each branch of the circuit and (b) the power

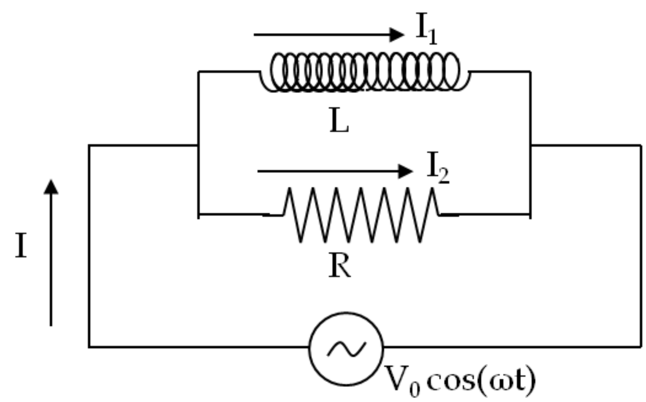

Figure 12.16: A parallel RL circuit with a sinusoidal driving EMF.

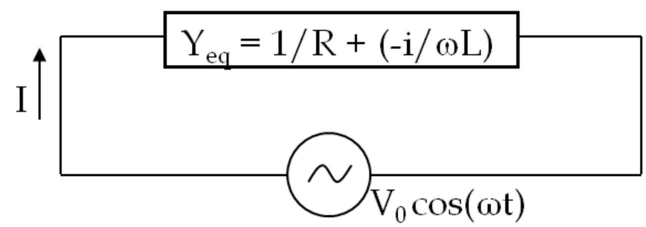

Figure 12.17: The passive elements R and L in Fig. 12.16 are replaced by their impedances for calculations using the complex number methods.

dissipated in the resistor and the power produced by the AC generator.

Solution. (a) First let us convert the given circuit to an equivalent circuit as shown in Fig. 12.17. Since the resistor and the inductor are in parallel, their admittances will add to yield an equivalent admittance $\tilde{Y}_{\text{eq}}$. Therefore current $\tilde{I}$ will be

$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \tilde{Y}\,\tilde{V} = \left(\frac{1}{R} + \frac{1}{i\omega L}\right)\ V_0\ e^{i\omega t}$$

We can use complex algebra to write the equivalent admittance in polar form since that will make it easier to multiply two complex quantities.

$$\tilde{Y} = \left(\frac{1}{R} + \frac{1}{i\omega L}\right) = \left(\frac{1}{R^2} + \frac{1}{\omega^2 L^2}\right)^{1/2}\ \text{Exp}\left[i\tan^{-1}\left(-\frac{R}{\omega L}\right)\right]$$

Put the polar form of $\tilde{Y}$ in the equation for $\tilde{I}$, and then combine the exponents. Now, we compute the real part of $\tilde{I}$ to obtain the physical current and read off the amplitude and phase constant,

$$I = \text{Re}(\tilde{I}) = I_0\ \cos(\omega t + \phi).$$

You will find that the amplitude and phase of the current are given by

$$I_0 = V_0 \left(\frac{1}{R^2} + \frac{1}{\omega^2 L^2} \right)^{1/2},$$
$$\tan\phi = -\frac{R}{\omega L}.$$

To find the current in the two branches, $I_1(t)$ and $I_2(t)$ we can use the Kirchhoff's loop equations. For instance to find $\tilde{I}_1$ we use the loop that contains the power source and the inductor using the impedance of the inductor.

$$\tilde{V} - \tilde{Z}_L \tilde{I}_1 = 0,$$

where $\tilde{Z}_L$ is the complex impedance of the inductor. After solving for the complex $\tilde{I}_1$, we extract physical current I_1 from the real part.

$$I_1 = \mathrm{Re}(\tilde{I}_1) = I_{01}\ \cos(\omega t + \phi_1),$$

with the following amplitude and phase of this current

$$I_{01} = \frac{V_0}{\omega L},$$
$$\phi_1 = -\frac{\pi}{2}.$$

Similarly, you can obtain the following for the current through the resistor.

$$I_{02} = \frac{V_0}{R}, \quad \phi_2 = 0.$$

Note that the peak current does not add up at a junction,

$$I_0 \neq I_{01} + I_{02}.$$

Instead, the condition of the charge conservation at the junction of current will be based on the dynamic currents.

$$I(t) = I_1(t) + I_2(t).$$

(b) For a current $I(t)$ through a circuit element and the voltage $V(t)$ across the element with the phase difference ϕ between the two, the average power is given by

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos(\phi),$$

where I_0 and V_0 are the corresponding amplitudes, i.e. peak values. Because the phase constant ϕ_1 of the current through the inductor is $-\pi/2$, there is no power dissipated there. All the power of the entire circuit is dissipated by the current though the resistor. We can verify

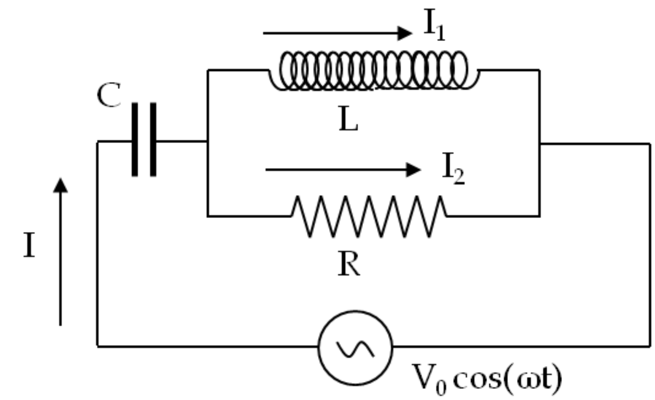

Figure 12.18: An RLC circuit with a sinusoidal driving EMF.

this assertion, by calculating the power from (I, V) and (I_2, V) pairs for the source and the resistor respectively.

FURTHER REMARKS:

> When using the average power formula, you need to pay attention to the phase constant that goes into the power factor. The phase constant refers to the phase difference between the current and voltage of the pair that is for the circuit element under question. Thus, when calculating the power dissipated in the resistor, we need the phase difference between the voltage V across the resistor and current I_2 through the resistor. And, when you want the power produced by the source you will use the phase difference between I and V for the source.

The average power dissipated in the resistor is

$$P_{\text{ave, R}} = \frac{1}{2} I_0 V_0 \cos(\phi_{(I_2,V)}) = \frac{1}{2}\frac{V_0^2}{R}.$$

The average power of the source would be

$$\begin{aligned} P_{\text{ave, source}} &= \frac{1}{2} I_0 V_0 \cos(\phi_{(I,V)}) \\ &= \frac{1}{2} V_0^2 \left(\frac{1}{R^2} + \frac{1}{\omega^2 L^2}\right)^{1/2} \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}} = \frac{1}{2}\frac{V_0^2}{R}. \end{aligned}$$

This power is equal to the average power delivered to the resistor, verifying that in an RL-circuit the energy is used only by the resistor.

Example 12.6.2. Circuit Elements in Series and in Parallel Find current through the capacitor in the circuit presented in Fig. 12.18.

Solution. We again start by first replacing the passive elements in the circuit by their impedances. The equivalent impedance $\tilde{Z}_{\text{eq}}$ of the

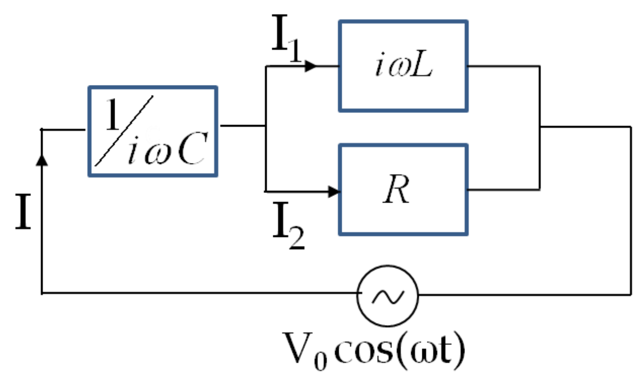

Figure 12.19: The passive elements R, C and L in Fig. 12.18 are replaced by their impedances for calculations using the complex number methods.

given circuit is obtained by adding the impedance of capacitor to the equivalent impedance of the parallel elements R and L.

$$\tilde{Z}_{\text{eq}} = \frac{1}{i\omega C} + \left(\frac{1}{R} + \frac{1}{i\omega L}\right)^{-1}.$$

We can separate the real and imaginary parts to calculate the amplitude and phase of the impedance.

$$\text{Re}\left[\tilde{Z}_{\text{eq}}\right] = \frac{R\omega^2 L^2}{R^2 + \omega^2 L^2}.$$

$$\text{Im}\left[\tilde{Z}_{\text{eq}}\right] = \frac{R^2\omega L}{R^2 + \omega^2 L^2} - \frac{1}{\omega C}.$$

Write this as $|\tilde{Z}|\ \exp(i\phi)$. The phase ϕ is equal to the negative of the the phase constant of the overall current $\tilde{I}$ compared to the phase of the voltage of the source.

$$|\tilde{Z}|^2 = \left(\frac{R\omega^2 L^2}{R^2 + \omega^2 L^2}\right)^2 + \left(\frac{R^2\omega L}{R^2 + \omega^2 L^2} - \frac{1}{\omega C}\right)^2 \tag{12.74}$$

$$\tan(\phi) = \frac{R^2\omega^2 LC - R^2 - \omega^2 L^2}{R\omega^3 L^2 C} \tag{12.75}$$

Since the capacitor is in series with the source, the current $I(t)$ through the equivalent impedance will also go through the capacitor branch, and hence the current through the capacitor.

$$I(t) = \text{Re}[\tilde{I}] = \text{Re}[I_0\ e^{i(\omega t+\phi)}] = I_0\ \cos(\omega t + \phi),$$

where

$$I_0 = \frac{V_0}{|\tilde{Z}|}.$$

Clearly, the formulas obtained for overall current are very complicated. Let us apply them to a choice of numerical values for the

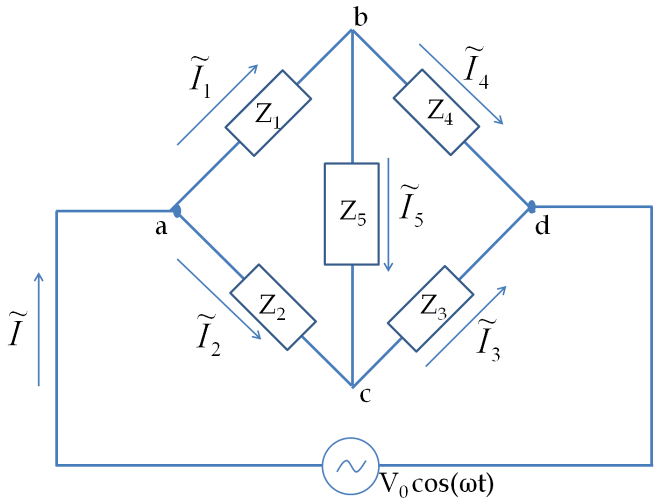

Figure 12.20: A bridge circuit.

circuit elements. Suppose, we have 110 V-rms, 60 Hz supply line for the source. This will mean that the peak voltage $V_0 = 170$ V and the angular frequency will be $\omega = 2\pi \times 60 = 377$ rad/sec. Let the circuit elements have the following values.

$$R = 1000\ \Omega;\quad L = 1\ \text{mH};\quad C = 1\ \mu\text{F}.$$

The value of impedances of the elements are:

$$Z_R = 1000\ \Omega;\quad Z_L = i0.377\ \Omega;\quad Z_C = -i2652\ \Omega.$$

The equivalent impedance becomes

$$\tilde{Z}_{\text{eq}} \approx 1.6 \times 10^{-4}\Omega - i2652\Omega$$

The amplitudes and phase of the current would be,

$$I_0 = 170\ \text{V}\sqrt{(1.6 \times 10^{-4})^2 + 2652^2} \approx 64\ \text{mA}.$$

$$\phi = \tan^{-1}\left(\frac{2652}{1.6 \times 10^{-4}}\right) \approx \frac{\pi}{2}\ \text{rad}.$$

12.6.2 Bridge Circuits

Figure 12.20 shows a bridge circuit. This circuit cannot be simplified by the method presented above. There are six branches in this circuit and therefore we will have six currents in the circuit. Let the complex currents through the impedances be labelled with the same subscript, i.e., $\tilde{I}_1$ through Z_1, $\tilde{I}_2$ through Z_2, etc. and $\tilde{I}$ be the current through the EMF source.

We pretend that the circuit is like a DC circuit and use Kirchhoff's rules to generate six independent equations which must be simultaneously solved. Three of these equations come from the application of Kirchhoff's node rule and the other three from Kirchhoff's loop rule to three non-overlapping loops.

$$\begin{aligned}
&\text{Node a:} \quad \tilde{I} - \tilde{I}_1 - \tilde{I}_2 = 0. \\
&\text{Node b:} \quad \tilde{I}_1 - \tilde{I}_3 - \tilde{I}_5 = 0. \\
&\text{Node c:} \quad \tilde{I}_2 + \tilde{I}_5 - \tilde{I}_3 = 0. \\
&\text{Loop abda:} \quad -\tilde{I}_1\tilde{Z}_1 - \tilde{I}_4\tilde{Z}_4 = \tilde{V}. \\
&\text{Loop acda:} \quad -\tilde{I}_2\tilde{Z}_2 - \tilde{I}_3\tilde{Z}_3 = \tilde{V}. \\
&\text{Loop abca:} \quad -\tilde{I}_1\tilde{Z}_1 - \tilde{I}_5\tilde{Z}_5 + \tilde{I}_2\tilde{Z}_2 = 0.
\end{aligned}$$

Each equation in the list is a collection of two equations, one for the real part and the other for the imaginary part. The algebra for solving for the amplitudes and phases of the currents is tedious but straightforward and is left as an exercise for the student.