3.3 DISPLACEMENT VECTOR

One of the characteristics of a moving object is that its position changes with time. The position of a point particle traces out a trajectory in space as it moves from one point to the next. To describe this change, we define a quantity called the **displacement vector** or simply displacement. Let the positions of a particle be $P_1(x_1, y_1, z_1)$ at time t_1 and $P_2(x_2, y_2, z_2)$ at time t_2. Then, the vector $\vec{s}_{12}$ from point P_1 to point P_2 is called the displacement vector as illustrated in Fig. 3.3. The magnitude of $\vec{s}_{12}$ is the direct distance between

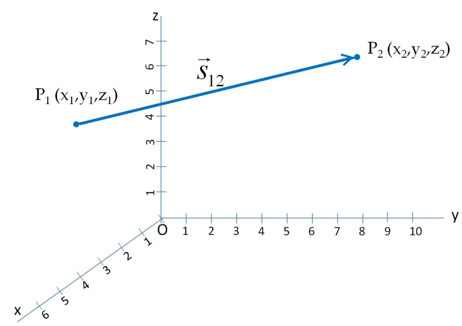


Figure 3.3: The displacement vector $\vec{s}_{12}$ from point P_1 to point P_2 in space.

points P_1 and P_2 and the direction is in the direction of the arrow from P_1 to P_2. We will use the symbol $\vec{s}$ for the displacement vector rather than the symbol $\vec{d}$ since we will soon take the derivative of displacement and other functions and the letter d will be used for that purpose.

In Fig. 3.4 shows four displacement vectors in the motion of a planet around the Sun. The planet moves in an elliptical orbit, but the displacements between points on the trajectory are given by straight segments.

Note that the definition of the displacement vector does not mention the origin or the reference point: **the displacement vector is the vector from the initial position to the final position**, period. We illustrate this fact in Fig. 3.4 by drawing four displacement vectors in the motion of a planet. It is clear that you do not need any coordinate system to define a displacement vector.

Although we do not need any reference point or coordinate system for a definition of the displacement vector, it is often helpful to write the displacement vector in terms of the position vectors of the particle, which does require a reference point, which we will place at the origin of a coordinate system.

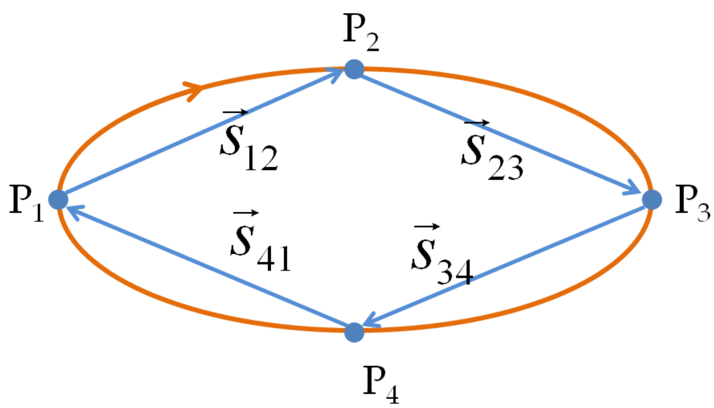


Figure 3.4: Displacement vectors between four successive points in the motion of a planet in an elliptical path around the sun.

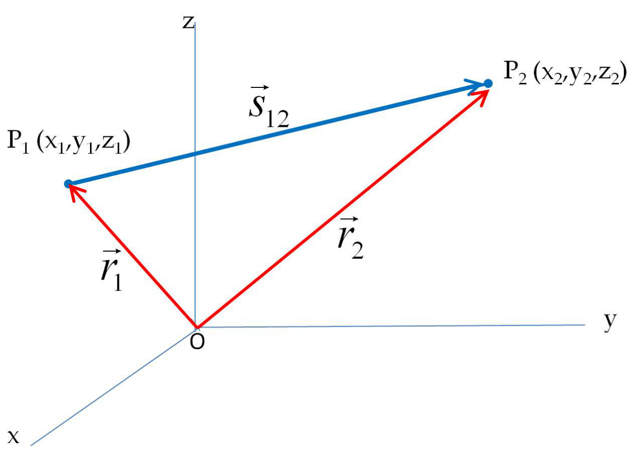


Figure 3.5: The displacement vector $\vec{s}_{12}$ from point P_1 to point P_2 in space can be written in terms of position vectors of the initial and final points as $\vec{s}_{12} = \vec{r}_2 - \vec{r}_1$ as the triangle of vectors shows here.

Let $\vec{r}_1$ be the position vector of the particle when it is at P_1 and $\vec{r}_2$ be the position vector when it is at P_2, as shown in Fig. 3.5. Then, it is easy to see that vectors $\vec{r}_1$, $\vec{s}_{12}$ and $(-\vec{r}_2)$ form a closed triangle of vectors. In other words, the sum of vectors $\vec{r}_1$ and $\vec{s}_{12}$ is equal to $\vec{r}_2$. Therefore, the displacement from P_1 to P_2 is equal to the vector obtained by subtracting the initial position vector from the final position vector.

$$\boxed{\vec{s}_{12} = \vec{r}_2 - \vec{r}_1.} \tag{3.2}$$

Since, each position vector can be written in terms of the coordinates of the corresponding points, we can write the displacement vector in terms of the coordinates of the two points also.

$$\begin{aligned} \vec{s}_{12} &= \vec{r}_2 - \vec{r}_1 \\ &= (x_2\hat{u}_x + y_2\hat{u}_y + z_2\hat{u}_z) - (x_1\hat{u}_x + y_1\hat{u}_y + z_1\hat{u}_z) \\ &= (x_2 - x_1)\hat{u}_x + (y_2 - y_1)\hat{u}_y + (z_2 - z_1)\hat{u}_z. \end{aligned} \tag{3.3}$$

An example of the resolution of a displacement vector in the xy-plane is displayed in Fig. 3.6. Note that regardless of the placement of the

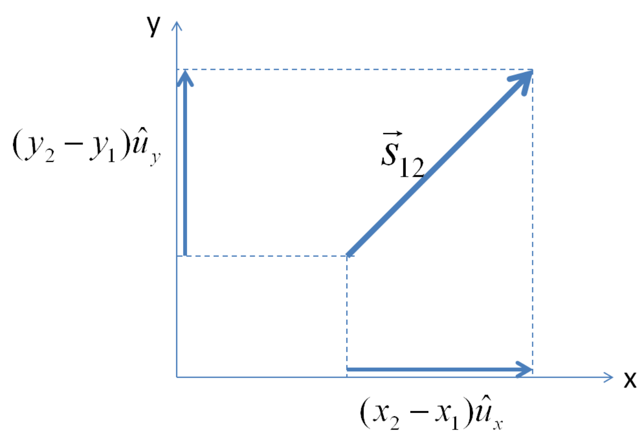
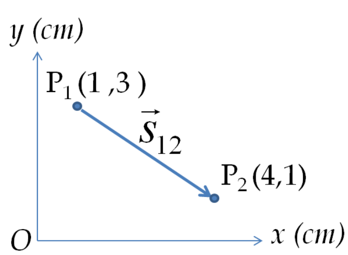


Figure 3.6: Displacement vector in terms of the base vectors shown for a displacement in the xy-plane.

Example:



$\vec{r}_1 = 1\hat{u}_x + 3\hat{u}_y$ [cm]

$\vec{r}_2 = 4\hat{u}_x + 1\hat{u}_y$ [cm]

$\vec{s}_{12} = (4-1)\hat{u}_x + (1-3)\hat{u}_y$ [cm]

displacement vector, the projections of the beginning and ending of the displacement vector on the Cartesian axes give the length of vectors along the Cartesian axes that make up the original displacement vectors. You can say that $(x_2 - x_1)\hat{u}_x$ along the x-axis and $(y_2 - y_1)\hat{u}_y$ along the y-axis add up to the displacement vector shown in the figure.

The factors $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$ multiplying the unit vectors for the Cartesian axes in Eq. 3.3 are the x, y and z-components of the displacement vector from P_1 to P_2. As explained in the last chapter, components of a vector can be used to determine the magnitude and direction of the vector.

Magnitude of $\vec{s}_{12}$:

$$\boxed{|s_{12}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.}$$

Direction of $\vec{s}_{12}$:
To determine the direction of $\vec{s}_{12}$ either you draw the vector in a properly marked axis system, or give angles with respect to reference directions. Three different cases arise:

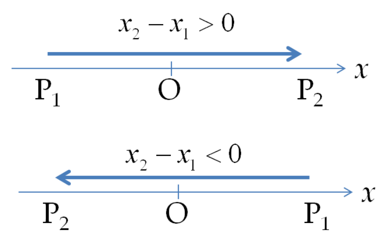


Figure 3.7: Directions along one axis.

1. If a displacement happens along one of the Cartesian axes, the sign of the component determines the direction with respect to the axis. For instance, if the displacement falls on the x-axis, then $\vec{s}_{12} = (x_2 - x_1)\hat{u}_x$. The x-component $(x_2 - x_1)$ will be positive if $x_2 > x_1$, giving a displacement in the direction of the unit vector $\hat{u}_x$, that is, towards the positive x-axis. On the other hand, if $x_2 < x_1$, the component $(x_2 - x_1)$ will be negative, which will give direction opposite to the unit vector $\hat{u}_x$ since multiplication of a vector by a negative number gives a vector that is in the opposite direction.

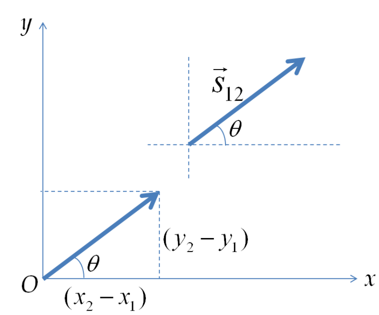


Figure 3.8: To determine the direction in the xy-plane, you can either draw coordinate axes with origin at the tail of the vector or drag the vector to the origin of the coordinate system keeping the physical orientation same.

2. If the displacement vector falls in the xy, yz, or zx plane, or in a plane parallel to these planes, then you need to give only one angle for the direction, usually any of the four angles with respect to the four axis directions in one of these planes would work for this purpose.

   Suppose, the vector falls in xy-plane, then you could give the angle the vector makes with either the positive x-axis, the positive y-axis, the negative x-axis, or the negative y-axis. Let θ be the angle with the positive x-axis, then you can use the following relation to calculate the angle.

   $$\boxed{\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}.}$$

   Beware of the correction needed depending on the quadrant of the point $(x_2 - x_1, y_2 - y_1)$. A more detail discussion can be found in the chapter on vectors.

3. Finally, if the displacement vector is neither along any axis or in any one of the planes mentioned, then we need the direction in the three-dimensional space. Now, you need two angles for

which there are many choices. If you wish to assign the direction by the angles α, β, γ, the vector makes with the x, y and z-axes respectively, then you need only two of them since their cosines obey an identity.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Alternately, you may use the polar and azimuthal angles of a spherical coordinate system to indicate the direction of a vector in three-dimensional space. These have been extensively discussed in the chapter on vectors (see Chapter 2).

Choice of Origin for Displacement Vector

We have already mentioned that the displacement vector does not require any reference point for its definition. We demonstrate here by analytic means that the displacement vector is independent of coordinate system as shown in Fig. 3.9. If you pick another place for the origin, say the point O' [read O-prime], and use a different orientation for the coordinate axes than the one for the origin O, then you will get different numerical values for the coordinates for points P_1 and P_2.

Let us denote the two coordinates systems by $Oxyz$ and $O'x'y'z'$ respectively. The coordinates for points P_1 and P_2 in the $O'x'y'z'$ system will be (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2) respectively, which will be different from the coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) for the same points in the $Oxyz$ system. However, you will find that the difference in the coordinates in the two coordinate systems will be the same:

$$(x'_1 - x'_1,\ y'_2 - y'_1,\ z'_2 - z'_1) = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1).$$

You can also see this directly in terms of the triangle of vectors in triangles $\triangle OP_1P_2$ and $\triangle O'P_1P_2$. The vector addition shows that, even though the pairs $\{\ \vec{r}_1, \vec{r}_2\ \}$ and $\{\ \vec{r}_1{}', \vec{r}_2{}'\ \}$ may be different, the displacement vector from P_1 to P_2 is the same in the two coordinate systems:

$$\vec{s}_{12} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2{}' - \vec{r}_1{}'. \tag{3.4}$$

Example 3.3.1. Displacement vector for one-dimensional motion. A box moves a distance of 2 m towards East. What is the displacement vector?

Solution. The statement of the problem already gives both the magnitude and direction of the displacement vector. So, no further work is necessary. The displacement is 2 m towards East as given.

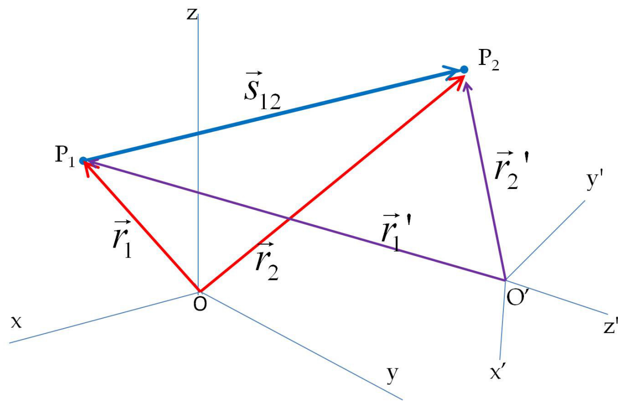


Figure 3.9: The displacement vector $\vec{s}_{12}$ from point P_1 to point P_2 from the perspectives of two different coordinate systems. The triangle of vectors in triangles $\triangle OP_1P_2$ and $\triangle O'P_1P_2$ show that, even though the pairs { $\vec{r}_1$, $\vec{r}_2$ } and { $\vec{r}_1{}'$ $\vec{r}_2{}'$ } may be different, the displacement vector from P_1 to P_2 is the same in the two coordinate systems: $\vec{s}_{12} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2{}' - \vec{r}_1{}'$.

Example 3.3.2. Displacement vector for one-dimensional motion. A box moves a distance of 2 m towards East. A coordinate system is chosen which has the x-axis pointed towards East. What are the components of the displacement vector in the given coordinate system? Write the displacement vector using unit vectors along the Cartesian axes.

Solution. Suppose, the initial point is at the origin, then since the x-axis is in the direction of movement, the final coordinate would be $(2\,\text{m}, 0, 0)$. Therefore, the x, y, and z-components of the displacement vector are 2 m, 0, and 0 respectively. This gives the following representation of the vector with respect to the axes: $\vec{s} = \{2\text{ m},\ East\} = (2\text{ m})\hat{u}_x$.

Example 3.3.3. Displacement vector for one-dimensional motion. A box moves a distance of 2 m towards East. A coordinate system is chosen which has the y-axis pointed towards East. What are the components of the displacement vector in the given coordinate system? Write the displacement vector using unit vectors along the Cartesian axes.

Solution. This example together with Example 3.3.2 illustrates the arbitrariness of choice of coordinates. One may get different numerical values of components, but the choice does not affect the displacement vector. Suppose, the initial point is at the origin, then since the y-axis is now in the direction of movement, the final coordinate is $(0, 2\text{ m}, 0)$. Therefore, the x, y, and z-components of the displacement vector are 0, 2 m, and 0 respectively. Therefore, we now get the following representation of the displacement vector with respect

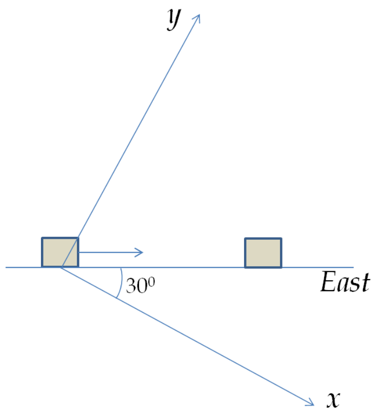


Figure 3.10: Example 3.3.4.

to the new axes: $\vec{s} = \{2\text{ m},\ East\} = (2\text{ m})\hat{u}_y$. Note that the displacement is still 2 m towards East although the representation has changed.

Example 3.3.4. Displacement vector for one-dimensional motion. A box moves a distance of 2 m towards East. A coordinate system is chosen which has the x-axis pointed towards 30° South of East and the y-axis is pointed 60° North of East. What are the components of the displacement vector in the given coordinate system? Write the displacement vector using unit vectors along the Cartesian axes.

Solution. This example together with Examples 3.3.2 and 3.3.3 illustrates the arbitrariness of the choice of coordinates. The choice of coordinates can make a one-dimensional motion problem appear two-dimensional as this example shows, but the physical vector is unaffected by the choice. Supposing the initial point to be at the origin, we can easily work out the coordinates of the final position to be $(2\ m\cos 30°, 2\ m\sin 30°, 0)$. Therefore, the x, y, and z-components of the displacement vector are $\approx$ 1.73 m, 1 m, and 0 respectively. Finally, we have the following representation of the displacement vector with respect to the new axes: $\vec{s} = \{2\text{ m},\ East\} = (1.73\text{ m})\hat{u}_x + (1\text{ m})\hat{u}_y$.

Once again, the displacement is $\{2\text{ m},\ East\}$, but the representation is different here than was in Examples 3.3.2 and 3.3.3. The differences are entirely due to the choice of the Cartesian axes. These examples illustrate that Cartesian axes are only calculational devices. Physics does not depend on your choice of coordinates, but simplicity or tediousness of the calculations may.

Example 3.3.5. Direction of Displacement Vector in a Plane from Components. You walk around in a room and after some time the **change in your coordinates** with respect to a particular Cartesian coordinate system is given as $(3\text{ m}, 4\text{ m}, 0)$. What is your displacement? Find both magnitude and direction.

Solution. Note that the given information allows you to write the displacement vector in the analytic form in terms of the unit vectors along the Cartesian axes.

$$\vec{s} = (3\text{ m})\hat{u}_x + (4\text{ m})\hat{u}_y.$$

The components can be used to determine the magnitude and direction.
Magnitude: $s = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}$.
Direction: Since the motion is in xy-plane, we can find one angle and

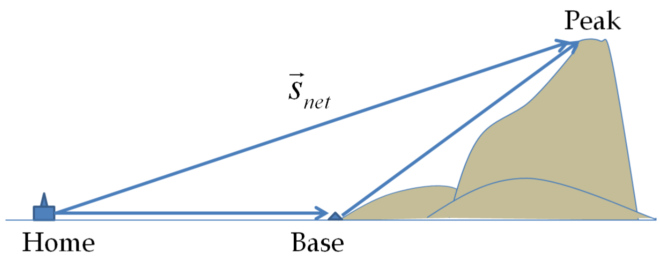


Figure 3.12: Example 3.3.6.

refer the direction using that angle. Since, the vector is in the first quadrant of the xy-plane, we work out the angle the vector makes with respect to the positive x-axis. This is directly given by the arc tangent.

$$\theta = \arctan\left(\frac{4\text{ m}}{3\text{ m}}\right) \approx 53^\circ.$$

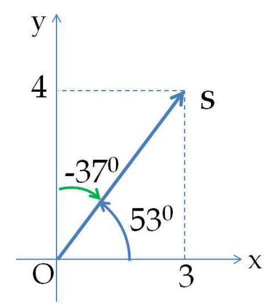


Figure 3.11: Example 3.3.5. The direction of a vector can be specified any angle with reference directions.

This says that the direction of the vector is in the xy-plane of the gives coordinate system at an angle counter-clockwise from the positive x-axis. Note again that the angle itself is not the direction - if you choose to give the angle with respect to the positive y-axis, the value will be -37° for the same direction in the plane. So, the value of angle, in itself, is not the full information about the direction - you need to state what the angle means in the real space. Minus in -37° refers to the clockwise direction as you look down from the positive z-axis.

Example 3.3.6. Adding Displacement Vectors. You travel from your home to the base of a mountain, and then, climb the mountain. The coordinates of your home, base of the mountain and the top of the mountain in a particular coordinate system are: home (0,0,0), base (4000 m, 3000 m, 0), peak (4500 m, 2500 m, 800 m). Find the magnitude and direction of the net displacement from home to the peak (a) directly, and (b) by adding the two displacements, from home to the base and from base to the peak.

Solution. (a) To find the displacement directly, we use the coordinates of the home and the peak. This gives us the representation of the net displacement in the given Cartesian coordinate system, which can be used to determine the magnitude and direction of the net displacement vector. We will suppress writing units in calculations and put the units back at the end. The net displacement written in terms of the unit vectors along the Cartesian axes are

$$\vec{s}_{net} = (4500 - 0)\text{ m}\hat{u}_x + (2500 - 0)\text{ m}\hat{u}_y + (800 - 0)\text{ m}\hat{u}_z.$$

The magnitude of the net displacement is

$$s_{net} = \sqrt{(4500 \text{ m})^2 + (2500 \text{ m})^2 + (800 \text{ m})^2} \approx 5210 \ m. \tag{3.5}$$

The direction can be given by polar and azimuthal angles of a spherical coordinate system with the following values. For clarity in the calculations, we will not writs units in the intermediate steps.

$$\text{Polar angle: } \theta = \arctan\left(\frac{\sqrt{4500^2 + 2500^2}}{800}\right) = \arctan\left(\frac{5}{1}48800\right) \approx 81^\circ.$$

$$\text{Azimuthal angle: } \phi = \arctan\left(\frac{2500}{4500}\right) \approx 29^\circ.$$

Since, the polar angle and azimuthal angles are in the first octant, we do not need to make any adjustments. The angles can be read off to say that the direction is $\approx 29^\circ$ counterclockwise from the positive x-axis and and $\approx 81^\circ$ towards the xy-plane from the z-axis.

As mentioned above, one can also find the direction of a vector by working out the angles, α, β and γ, the vector makes with the x, y and z-axes respectively by using the direction cosines. We will not pause here to elaborate on this. An interested student should try to see how that can be done.

(b) The two displacements from home to base and from base to the peak are

$$\vec{s}_{12} = (4000 - 0) \text{ m}\hat{u}_x + (3000 - 0) \text{ m}\hat{u}_y + (0 - 0)\hat{u}_z$$
$$\vec{s}_{23} = (4500 - 4000) \ \hat{u}_x + (2500 - 3000) \text{ m}\hat{u}_y + (800 - 0) \text{ m}\hat{u}_z$$

Vector equation for the net displacement $\vec{s}_{net}$ is

$$\vec{s}_{net} = \vec{s}_{12} + \vec{s}_{23},$$

which gives the same expression for the $\vec{s}_{net}$ as in Eq. 3.3 since the components will add separately.

$$\begin{aligned}
\vec{s}_{net} &= \vec{s}_{12} + \vec{s}_{23} \\
&= [(4000 - 0) \text{ m}\hat{u}_x + (3000 - 0) \text{ m}\hat{u}_y + (0 - 0)\hat{u}_z] \\
&\quad + [(4500 - 4000) \text{ m}\hat{u}_x + (2500 - 3000) \text{ m}\hat{u}_y + (800 - 0) \text{ m}\hat{u}_z] \\
&= 4500 \text{ m}\hat{u}_x + 2500 \text{ m}\hat{u}_y + 800 \text{ m}\hat{u}_z.
\end{aligned}$$

Putting the units back the answer is

$$\vec{s}_{net} = (4500 \text{ m}) \ \hat{u}_x + (2500 \text{ m}) \ \hat{u}_y + (800 \text{ m}) \ \hat{u}_z.$$