## 3.2 EXAMPLES OF SIMPLE HARMONIC MOTION

Simple harmonic motion appears in a wide variety of physical settings. When a system is displaced from a stable equilibrium, the restoring force is usually proportional to the displacement if the displacement is not too great. In this section, we study some commonly encountered systems that exhibit simple harmonic motion.

### Plane Pendulum

A pendulum consists of a bob of mass $m$ suspended from a light inextensible cord of length $l$. If the physical dimension of the bob is much smaller than the length of the cord, we can treat the bob as a point mass. A pendulum has a stable equilibrium position when the bob is hanging vertically down from the suspension point. The angular displacement of the bob from the equilibrium is given in terms of angle $\theta$ that the cord makes with the vertical line as shown in Fig. 3.11. The displacement angle is positive for the counter-clockwise change in angle and negative for a clockwise change in angle when viewed from the axis coming out-of-page in the figure.

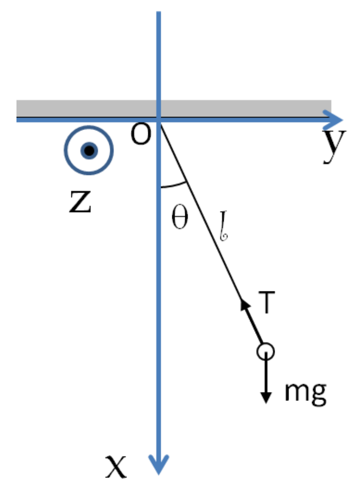


Figure 3.11: A plane pendulum oscillates in a plane, rotating about an axis through the point of suspension O and perpendicular to the plane of the pendulum motion. The angle $\theta$ that the cord makes with the vertical line gives the displacement of the pendulum from a stable equilibrium.

**Equation of Motion Approach:**

Because we wish to study the angular displacement, it is more convenient to treat the pendulum problem as a rotation problem — the rotation of the point mass $m$ about an axis through the point of suspension $O$ and perpendicular to the plane of oscillation. We choose a Cartesian coordinate system as shown in Fig. 3.11 so that angle $\theta$ corresponds to the $z$-component of the angular displacement, $\theta_z$. The $z$-component of the equation of motion for rotation based on the $z$-components of the torque and angular acceleration, and the $I_{zz}$ component of the moment of inertia.

$$\tau_z = I_{zz}\frac{d^2\theta}{dt^2}. \tag{3.39}$$

To find the torque on the bob, we note that there are only two forces on the bob: the weight of the bob and the tension in the cord. Since the force of tension goes through the point of suspension $O$, it does not exert any torque about O. The torque about $O$ comes only from the weight, which gives the following $z$-component:

$$\tau_z = -mgl\sin\theta, \tag{3.40}$$

where the minus sign refers to the fact that torque is pointed towards the negative $z$-axis when $\theta$ is positive and towards the positive $z$-axis when $\theta$ is negative. The moment of inertia component $I_{zz}$ of the bob about the axis is

$$I_{zz} = m(x^2 + y^2) = ml^2. \tag{3.41}$$

Therefore, the $z$-component of the rotational equation of motion of a pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{3.42}$$

Here, the angular acceleration is not proportional to the angular displacement, but to the sine of the angular displacement. Therefore, a plane pendulum does not execute a Simple Harmonic Motion, which requires that the acceleration be proportional to displacement. Another problem with the equation of motion of pendulum, Eq. 3.42, is that it is not easily solvable. But, we note that for small angles (expressed in radians), the sine of the angle can be approximated by the angle itself.

$$\sin(\theta) \approx \theta \text{ (small angles in radians)}. \tag{3.43}$$

With the small angle approximation, the equation of motion of a pendulum, Eq. 3.42, can be simplified to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta, \tag{3.44}$$

where I have replaced the approximate sign ($\approx$) with the equal sign ($=$) since we intend to be within the limit of the applicability of the approximation being utilized. The modified equation of motion, Eq. 3.44 shows clearly that, if the oscillations are kept to small angles, the angular acceleration (two-time derivatives of $\theta$) is proportional to the angular displacement ($\theta$) and is opposed to the angular displacement. Therefore, for small-angle displacements, a pendulum will execute a simple harmonic motion.

Comparing the approximate equation of motion for a pendulum with the equation for a block attached to a spring, we immediately discover the following correspondences in symbols.

$$x \Longleftrightarrow \theta \tag{3.45}$$

$$(k/m) \Longleftrightarrow (g/l) \tag{3.46}$$

Exploiting this analogy, we can write the solution of the small angle pendulum problem as

$$\theta(t) = A\ \cos(\omega t + \phi) = C_1 \cos\omega t + C_2 \sin\omega t, \tag{3.47}$$

and deduce the formula for the angular frequency of the plane pendulum:

$$\omega = \sqrt{\frac{g}{l}}, \tag{3.48}$$

which gives the period of the pendulum to be:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}. \tag{3.49}$$

The formula for the period shows that the time period of a pendulum does not depend on either the mass $m$ of the pendulum bob or the amplitude $A$ of oscillations, but only on the length $l$ of the pendulum cord and acceleration due to gravity $g$. The dependence of the period on $g$ means that the same pendulum will have different periods on different planets. Also, since the value of $g$ varies over the surface of the Earth, the same pendulum will run at different rate in different locations on the Earth.

Galileo appears to be the first person who noticed this aspect of a pendulum motion when he made the observation that different chandeliers of equal length in a church had the same period regardless of their amplitudes of swing or weights. Apparently Galileo timed the swings of chandeliers using his pulses and found that chandeliers swung at the same period regardless of their masses and swings.

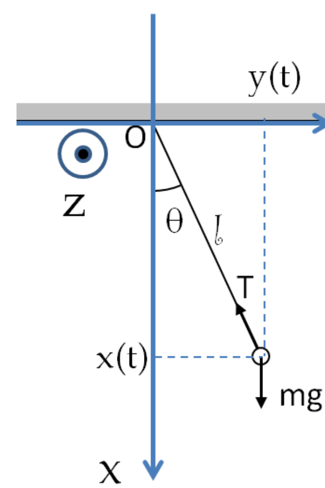


Figure 3.12: The $x$ and $y$ motions of the plane pendulum.

**Energy Approach:**

The formula for the period of the small oscillations of the pendulum can be more easily deduced by examining the energy of the bob as we will show next. The energy of a pendulum can also be written such that it contains two terms, one having the square of velocity for the kinetic energy, and the other the square of displacement for the potential energy, similar to that of a Simple Harmonic Oscillator. Let $x$ and $y$ denote the $x$ and $y$-coordinates of the pendulum bob at an arbitrary time $t$ with respect to the axes shown in Fig. 3.12. Then, in the small angle approximation, we have

$$x = l - l\ \cos\theta \approx \frac{l}{2}\theta^2, \tag{3.50}$$

$$y = l\ \sin\theta \approx l\theta, \tag{3.51}$$

where we have dropped terms which are cubic or higher powers in $\theta$. The velocities are

$$v_x = \frac{dx}{dt} = l\theta\,\frac{d\theta}{dt}, \tag{3.52}$$

$$v_y = \frac{dy}{dt} = l\,\frac{d\theta}{dt}. \tag{3.53}$$

The kinetic energy of the bob is found to be

$$KE = \frac{1}{2}m\left(v_x^2 + v_y^2\right) \approx \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2, \tag{3.54}$$

where we have dropped the term with four powers involving $\theta$ and $d\theta/dt$. The potential energy is

$$PE = mg(l-x) = mg\frac{l}{2}\theta^2. \quad \text{(reference zero at the lowest point)} \tag{3.55}$$

Therefore, the energy of the pendulum in small angle approximation is

$$E = \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mgl\theta^2, \tag{3.56}$$

which has the required form for a simple harmonic motion. Hence, we can also determine the angular frequency also from the ratio of the coefficients of the restoring and inertial terms.

$$\omega = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}, \tag{3.57}$$

which is identical to the expression we obtained from an application of the second law of motion as given in Eq. 3.48.

## Torsion Pendulum

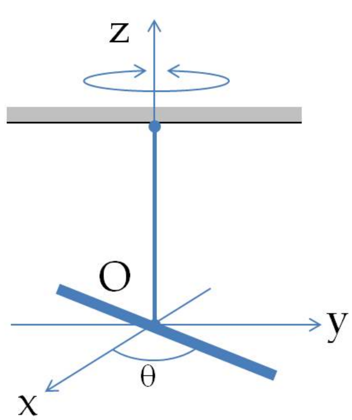

Figure 3.13: The torsion pendulum.

A torsion pendulum consists of a solid, a dumbbell, a disk, a bar, or an object of any other shape, suspended by a torsion wire from a fixed support as shown in Fig. 3.13. A torsion wire is essentially a wire that could be twisted easily about its length. The twisting of the wire applies a restoring torque on the supported body whose tendency is to bring the body back to the configuration when the wire was not twisted, i.e., to the equilibrium. According to the Hooke's law for elasticity of materials, the restoring torque $\tau$ on the bar should be proportional to the angle of twist, at least for the small angles of twist we would work with. Using a coordinate system in which the $z$-axis is pointed in the direction of the axis of rotation, the $z$-component of the torque would be

$$\tau_z = -\kappa\theta, \tag{3.58}$$

where $\kappa$ (read: kappa) is the torsional constant of the wire, and $\theta$ is the $z$-component of the angular displacement from equilibrium as measured from $x$-axis in the $xy$-plane. The torsional constant for twisting wire is analogous to the spring constant of a spring. The rotational motion of the bar is then given by the $z$-component of the rotational equation of motion.

$$I_{zz}\frac{d^2\theta}{dt^2} = -\kappa\theta, \tag{3.59}$$

where $I_{zz}$ is the moment of inertia of the bar about $z$-axis, which is the axis of rotation. This equation is analogous to the equation for a plane pendulum in small angle approximation. By exploring the mathematical correspondence of symbols in the equations here to the equations of the simple pendulum we can read off the expression for the angular frequency $\omega$ of oscillating motion of the torsion pendulum as

$$\omega = \sqrt{\frac{\kappa}{I_{zz}}}. \tag{3.60}$$

Torsion pendulums are often used for time-keeping purposes, e.g., in the balance wheel of a mechanical watch. A torsion pendulum is also used in Cavendish experiment for determining the value of the Newton's gravitational constant $G_N$.

## Physical pendulum

A rigid body hung from a post swings just like a pendulum. Such oscillating bodies are called physical pendulums. Almost anything can be a physical pendulum. An illustration in shown in Fig. 3.14. The torque responsible for the oscillations comes from the force of gravity on the body, which is calculated by placing the weight vector at the center of mass of the body. What is the frequency of small oscillations in this case?

Let $M$ be the mass and $D$ the distance between the suspension point and the CM of the physical pendulum. From the $z$-component of the torque due to gravity we obtain the following equation of motion for the rotation about the $z$-axis in terms of the angle $\theta$ with respect to the positive $x$-axis.

$$I_{zz}\frac{d^2\theta}{dt^2} = -MgD\sin\theta, \tag{3.61}$$

where $I_{zz}$ is the moment of inertia about $z$-axis passing through the point of suspension. The mathematical situation here is similar to that of the plane pendulum. Once again, we resort to the small angle approximation. For small-angle oscillations, we set $\sin\theta \approx \theta$ in Eq. 3.61, which yields the following approximate equation of motion of a physical pendulum.

$$\frac{d^2\theta}{dt^2} \approx -\frac{MgD}{I_{zz}}\theta. \tag{3.62}$$

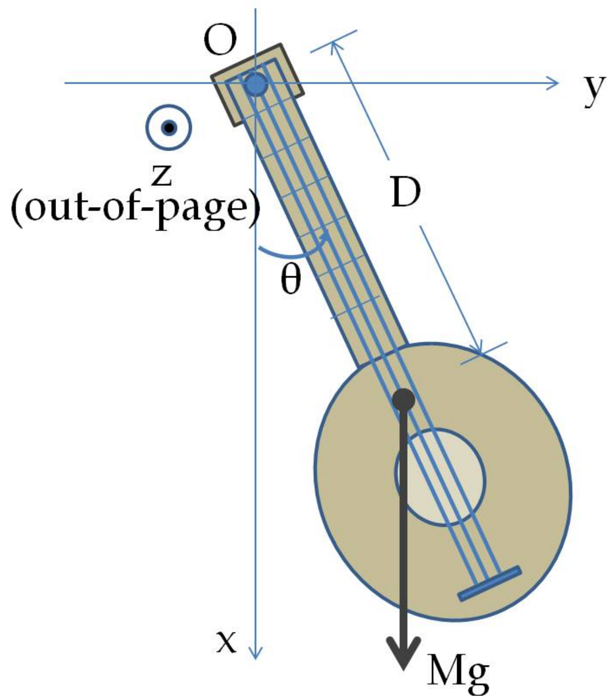


Figure 3.14: A physical pendulum can be anything that can swing about an axis. Here a musical instrument oscillated about a fixed point at one end illustrates a physical pendulum.

Now, by analogy with the plane pendulum, we find that the angular frequency of oscillation of a physical pendulum is given by

$$\omega = \sqrt{\frac{MgD}{I_{zz}}}. \tag{3.63}$$