2.2 IMAGE FORMATION BY REFLECTION - ALGEBRAIC METHOD

Image formation can also be studied algebraically. For a plane mirror you can show that the image has the same height and orientation as the object, and it is located at the same distance behind the mirror as the object is in front of the mirror. The situation is much more complicated for curved mirrors. It is however possible to find simple formulas for concave and convex mirrors by using plane geometry. Here, I will work out the derivation for a concave mirror and leave the derivation for a convex mirror as an exercise for you to complete.

2.2.1 Concave Mirror

Consider an object OP located at a distance from a concave mirror so that a real image forms on a screen in front of the mirror as shown in Fig. 2.11.

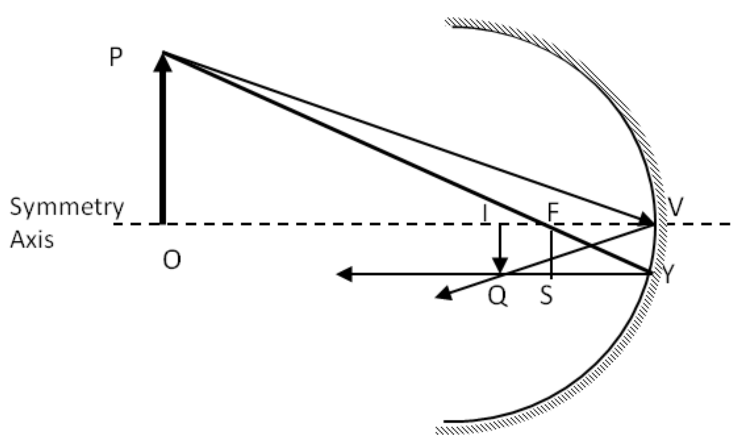

Figure 2.11: Image by a concave mirror.

All distances are measured from the point V called the **vertex**. Thus,

VO = object distance denoted by p.
VI = image distance denoted by q.
VF = focal length denoted by f.

The focal length is equal to $\frac{1}{2}R$ where R is the radius of curvature of the interface.

$$f = \frac{1}{2}R.$$

Now we derive the relation among p, q and R.

$$\text{Since } \triangle OPV \text{ similar to } \triangle IQV,\ \frac{VO}{VI} = \frac{OP}{IQ} \tag{2.1}$$

$$\text{Since } \triangle OPF \text{ similar to } \triangle SFY,\ \frac{FO}{SY} = \frac{OP}{FS} \tag{2.2}$$

By construction, we also have

$$IQ = FS. \tag{2.3}$$

We will do our calculations in the **paraxial approximation**, i.e., for rays that make small angles with the symmetry axis. Paraxial rays make small angle $\theta = \angle PVQ$ at the vertex, therefore we make the approximations

$$\sin\theta \approx \tan\theta \approx \theta.$$
$$\cos\theta \approx 1.$$

Using these approximations we will also have the following relation.

$$SY = FV. \tag{2.4}$$

From Eqs. 2.1 - 2.4, we can deduce the following

$$\frac{VO}{VI} = \frac{FO}{FV}. \tag{2.5}$$

Writing this equation in p, q, and f defined above we obtain

$$\frac{p}{q} = \frac{p-f}{f},$$

which can be rearranges to the following familiar form.

$$\boxed{\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{2}{R}.} \tag{2.6}$$

A similar equation also describes the relation in the case of a convex mirror except that R would be negative there.

Magnification of the Image

To study the size and orientation of the image compared to the object we can study the heights of the object and image points with respect to the symmetry axis. We will take the heights above the axis as positive and below the axis will be negative. In the present example, suppose the point P is at the top point of an object whose bottom point is at the optical axis level. Then the location of Q with respect to the optical axis will give us the size and orientation of the image.

$$h_o = \text{Object height} = OP.$$
$$h_i = \text{Image height} = -IQ.$$

The size and orientation of the image relative to the object is given by the **magnification** m defined by

$$m = \frac{h_i}{h_o} \tag{2.7}$$

Thus if m is positive then image has the same orientation as the object and if it is negative the image is inverted. If $|m| > 1$ then image is larger than the object, and if $|m| < 1$ then image is smaller than the object.

From $\triangle$OPV and $\triangle$IQV we find the following relation

$$\frac{\text{IQ}}{\text{OP}} = \frac{\text{IV}}{\text{OV}}.$$

Therefore, we have the following relation between the vertical and horizontal distances:

$$\boxed{m = \frac{h_i}{h_o} = -\frac{q}{p}.} \tag{2.8}$$

This is a very useful relation since it lets you obtain the magnification of the image from horizontal distances, the object and image distances, which can be obtained from the use of Eq. 2.6.

2.2.2 Convex Mirror

Consider an object OP in front of a convex mirror. We draw two rays from the off-axis point P to the mirror and trace the resulting reflected rays as shown in Fig. 2.12.

A similar calculation to the one shown above for the concave mirror yields the following relation between the object distance p, image distance q and the focal length f which is defined as half of the radius of curvature here.

$$\frac{1}{p} - \frac{1}{q} = -\frac{2}{R}. \tag{2.9}$$

This equation for the convex mirror and Eq. 2.6 for the concave mirror can be written in a unified manner by introducing a sign convention.

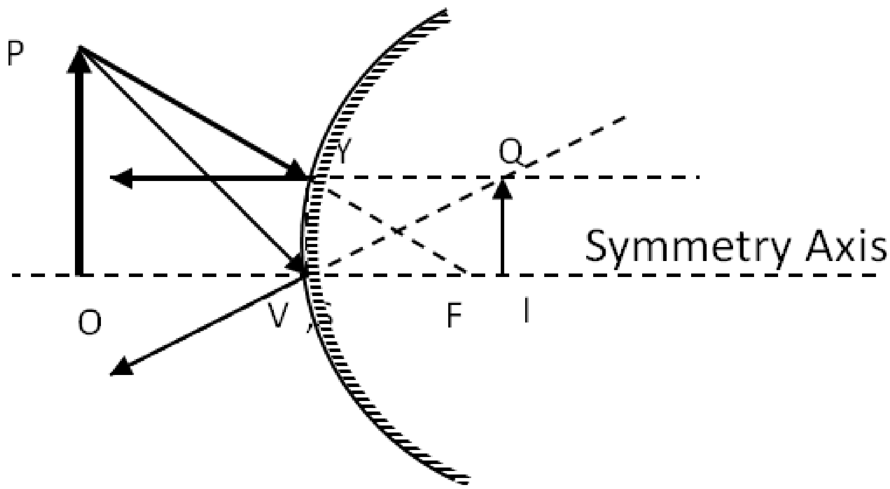

Figure 2.12: Image in a convex mirror.

Sign Convention:

Distances in front of the mirror will be positive, while distances behind the mirror will be negative.

With this sign convention, we obtain the following results.

1. Focal length f is negative for the convex mirror and positive for the concave mirror.

2. If the image forms behind the mirror, the image distance q will be negative, and if it forms in front of the mirror, the image distance q will be positive.

Unified mirror equation with sign convention is then

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \quad \text{(must use sign convention.)} \tag{2.10}$$

The focal length f of concave and convex mirrors are:

$$f = \begin{cases} +R/2 & \text{concave mirror} \\ -R/2 & \text{convex mirror} \end{cases} \tag{2.11}$$

The magnification m is given by the same relation in both cases.

$$m = \frac{h_i}{h_o} = -\frac{q}{p}. \tag{2.12}$$