3.4 THE DAMPED HARMONIC OSCILLATOR

3.4.1 Under, over and critical damping

Frictional forces damp the motion of a moving object by taking energy away from the object. A particularly simple type of a viscous force whose magnitude is proportional to the velocity of the oscillator has many applications in physics of oscillatory systems.

$$\vec{F}_{visc} = -b\,\vec{v} \quad (b \geq 0), \tag{3.73}$$

where b is the proportionality constant, called the **viscous damping coefficient**, that depends on the viscous medium and the geometry of the oscillating mass. The minus sign makes sure that the viscous force is pointed in the opposite direction to the velocity. Thus, this force will slow the motion of the object. Viscous forces of this type act on objects moving in a fluid if their speeds are not too great. Pictorially, it is customary to represent the damping force by attaching a dash pot to the oscillating block attached to the spring as shown in Fig. 3.15.

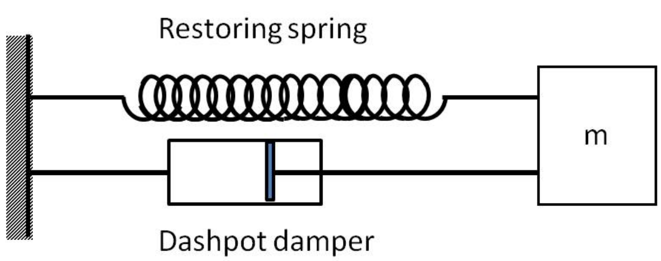

Figure 3.15: A harmonic oscillator with a dashpot. The dashpot has a piston that moves through a liquid providing velocity dependent damping force.

The net force on the oscillator will now be the vector sum of the restoring force by the spring and the viscous force. To cast the problem in analytic terms, we choose Cartesian coordinates. Once again, we will place the origin at the equilibrium position of the block, and point the positive x-axis in the direction that corresponds to the extension of the spring. Then the x-component of the second law of motion would take the following form.

$$m\frac{d^2x}{dt^2} = -kx - b\,\frac{dx}{dt}, \tag{3.74}$$

where I have replaced the x-component of the acceleration by the second derivative of the x-coordinate of the block and the x-component of the velocity by the first derivative. It is helpful to divide both sides of this equation by m, and introduce new constants, β and ω_0 as follows so that the solution of Eq. 3.74 can be written with fewer parameters.

Note that some authors use $\gamma = b/m$ in place of $\beta = b/2m$.

$$2\beta = b/m, \tag{3.75}$$
$$\omega_0 = \sqrt{k/m}, \tag{3.76}$$

where a factor of 2 in the definition of β is included to simplify the formulas that appear below. The quantity ω_0 is called the "natural frequency" of the oscillator. It refers to the innate oscillation frequency at which the block will oscillate if there were no damping or other forces on the block. The quantity β is called **damping constant** or damping parameter.

Note that now we are using a subscript zero for the natural frequency of the undamped oscillator because we will be encountering other frequencies in the same system, and we do not want to get confused about them. We will see below that a damped oscillator does not oscillate at the natural frequency, but with a new frequency depending on both β and ω_0.

With these new parameters, Eq. 3.74 can be written as

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2 x = 0. \tag{3.77}$$

The oscillating characteristics of a damped oscillator depends on whether ω_0 is less than, equal to, or greater than β. If $\omega_0 > \beta$, then the mass oscillates about the equilibrium, successively damping out each cycle, and the system is said to be under damped; if $\omega_0 < \beta$, then the system does not oscillate at all, and we say that the system is overdamped; finally, if $\omega_0 = \beta$, the system is called critically damped, which separated the underdamped from the over-damped cases, and where, again, there is no oscillation. The mathematical expressions of the three solutions are as follows.

$$x(t) = \begin{cases} A_1 \exp\left(-\beta t\right)\cos(\omega_1 t + \phi) \text{ for } \omega_0 > \beta, \text{ under-damped} \\ \exp\left(-\beta t\right)(A_2 t + A_3) \text{ for } \omega_0 = \beta, \text{ critically damped} \\ \exp\left(-\beta t\right)\left[A_4 \exp\left(-\alpha t\right) + A_5 \exp\left(\alpha t\right)\right], \\ \qquad\qquad \text{for } \omega_0 < \beta, \text{ over-damped} \end{cases} \tag{3.78}$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \ \text{ and } \ \alpha = \sqrt{\beta^2 - \omega_0^2}. \tag{3.79}$$

Constants A_1, A_2, A_3, A_4, A_5, and ϕ in Eq. 3.78 are determined by the initial position and initial velocity, as discussed for the undamped oscillator above. It would seem that if you want the oscillations to die out quickly, then may be, you should try overdamping the system. It turns out that overdamping is not the right strategy. Instead, we find that an oscillator is damped more quickly if critically damped as shown in Fig. 3.17. In practical uses of the damping, for example in the shock absorbers for cars, you would want the car not to carry on with bouncing up and down whenever the hits a bump in the road. Therefore, one builds the shock absorbers with the characteristics of critical damping.

Figure 3.16: Rear shock absorber and spring of a BMW R75/5 motorcycle. Photo credits: uploaded by Jeff Dean for Wikicommons.

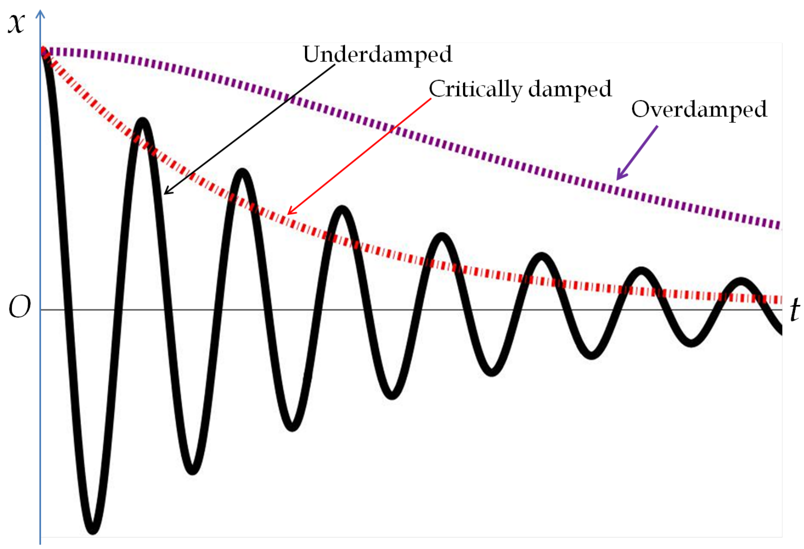

Figure 3.17: The displacement as a function of time for the under-damped, over-damped and critically damped cases. Only the under-damped case is an oscillator. To plot the figures, the following values were used: $x(0) = 1\ cm$, $v(0) = 0$, $\beta = 1\ rad/sec$, $\omega_1 = 10\ rad/sec$, $\alpha = 0.2\ rad/sec$.

The under-damped oscillator provides a physical meaning for the damping parameter β. We see from the graph in Fig. 3.18 that it takes a time of $1/\beta$ for the envelope of the oscillations to decrease by 1/e of its original value. The time to relax to $1/e$ of the original amplitude is called the **time constant** of the oscillator, which is also denoted by the Greek letter τ.

$$\tau = \frac{1}{\beta} \tag{3.80}$$

Therefore, the larger the damping constant, the shorter the time constant, and consequently, faster the damping.

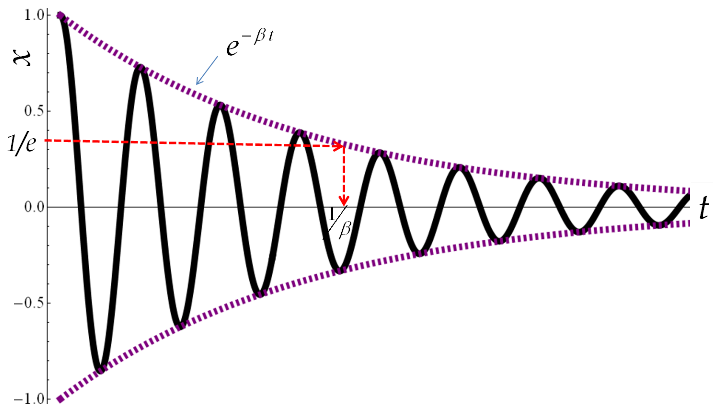

Figure 3.18: The dynamics of under-damped oscillator shows oscillations with decreasing amplitude. The envelop of the decreasing amplitude is used to define the time constant for the under-damped oscillator. The figure shows that in time $t = 1/\beta$, the envelop decreased to by a factor of $1/e$, where e is the Euler's number with value $e = 2.71828....$

3.4.2 Quality factor Q of an oscillator

Under-damped oscillators oscillate with decreasing amplitude and eventually come to rest. A better oscillator oscillates many more cycles than a poorer oscillator before losing its energy. This important aspect of an oscillator is characterized by a quantity called the **Quality** or Q factor. The Q factor measures the persistence of oscillations of an oscillating system. A simple definition would be to divide the average energy in a particular cycle by the energy lost in the cycle, so that the less the energy is lost in a cycle, the more persistent the oscillations, and therefore higher the Quality of the oscillator. For technical reasons, we define the Q factor by dividing the energy at the beginning of a cycle to the energy lost in a fraction of the cycle.

$$Q = \frac{\text{Energy of the oscillator at the beginning of a cycle}}{\text{Energy dissipated per } (1/2\pi) \text{ of the next cycle}}. \tag{3.81}$$

The factor $1/2\pi$ of the cycle refers to per radian of the oscillation when we represent the motion of the oscillator by the motion of a fictitious particle in a circle so that covering 2π radians corresponds to one full cycle. One includes the factor $1/2\pi$ of the cycle in the definition to make the formulas later come out simpler. Although an exact calculation of the Q factor in terms of β and ω_0 of the oscillator is possible, it is not very illuminating. Instead, we will derive an approximate formula for Q of a lightly-damped oscillator defined by the following criterion.

$$\text{Lightly damped oscillator: } \beta << \omega_0. \tag{3.82}$$

First, note that the energy of a lightly-damped oscillator obeys the following equation (see the derivation below).

$$\frac{dE}{dt} = -2\beta E. \tag{3.83}$$

Therefore, the energy dissipated ΔE in a short time Δt is

$$\Delta E = \int_0^{\Delta t} \left|\frac{dE}{dt}\right| dt \approx \left|\frac{dE}{dt}\right| \Delta t = 2\beta E \Delta t. \tag{3.84}$$

For Q, we need the energy loss in $1/2\pi$ of one cycle. Therefore, we choose Δt to be $1/2\pi$ times one time period T , which is

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \approx \frac{2\pi}{\omega_0}, \text{ since } \beta << \omega_0. \tag{3.85}$$

Therefore, Q of a lightly damped oscillator is found to be

$$Q = \frac{E}{\Delta E \text{ for } 1/2\pi \text{ of one cycle}} \approx \frac{\omega_0}{2\beta}. \tag{3.86}$$

Good oscillators such as tuning forks and guitar strings have Q values in the thousands. Laser cavities have much higher Q values, exceeding 10^7. Of course, the undamped oscillator has zero β, and hence infinite Q. There is no Q for the critically damped and over-damped cases since these systems do not oscillate.

Example 3.4.1. An under-damped harmonic oscillator. A copper block of mass 1.5 kg is attached to a spring of stiffness 450 N/m and hung from a platform above a beaker that contains a thick liquid so that the block oscillates entirely in the liquid with a damping constant of 3.0 kg/s. (a) How many oscillations will the block make before its amplitude drops by 90%? (b) What is the Quality of this oscillator?

Solution. a) We will first calculate the oscillation frequency ω_1 and the damping constant β for the damped oscillator. Since, the peak of successive cycles drops as $e^{-\beta t}$ we can find the time for the 90% drop in the amplitude using the value of β. Then, we will obtain the required number of oscillations by dividing the time required for the 90% drop by the time period of the oscillator.

$$\beta = \frac{b}{2m} = \frac{3.0 \text{ kg/s}}{2 \times 1.5 \text{ kg}} = 1.0 \text{ rad/sec.}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{450 \text{ N/s}}{1.5 \text{ kg}}} = 17.32 \text{ rad/sec.}$$

Therefore, the angular frequency of oscillation is

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \approx 17.3 \text{ rad/sec},$$

which gives the time period for oscillations to be

$$T = \frac{2\pi}{\omega_1} \approx 0.363 \text{ sec}.$$

Now, we use the decay of the amplitude envelop to find the time for the amplitude to drop by 90%.

$$\frac{\text{Amplitude left}}{\text{Original Amplitude}} = 0.1.$$

Therefore,

$$e^{-\beta t} = e^{(-1)t} = 0.1 \Longrightarrow \ t = 10 \text{ sec}.$$

Hence, the number of cycles in which the amplitude drops by 90% will be

$$\text{Number of cycles } = \frac{t}{T} = \frac{10 \text{ sec}}{0.362 \text{ sec}} = 27.5 \text{ cycles}.$$

(b) The Q factor of the oscillator is

$$Q \approx \frac{\omega_0}{2\beta} = \frac{17.32}{2(1)} = 8.66.$$

This is a poor oscillator, losing almost $2/3^{rd}$ of the amplitude in only 28 cycles.

3.4.3 Rate of change of energy for a lightly damped oscillator

A lightly-damped oscillator has much smaller damping constant (β) compared to the natural frequency (ω_0) of the oscillator.

$$\beta << \omega_0 \ \text{ (Lightly damped oscillator)}. \tag{3.87}$$

We will make use of this approximation to simplify formulas below. To find the rate at which energy is dissipated by a lightly damped oscillator, we can first calculate the energy from the sum of the kinetic and potential energies and then take the time derivative. Since a lightly damped oscillator is an under damped case we use the appropriate solution for the displacement $x(t)$.

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi). \tag{3.88}$$

Note that we could have alternately used $x(t) = e^{-\beta t}\left[C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\right]$, but, for the present calculation, the form given in Eq. 3.88 is better. The velocity is obtained by taking a time derivative of x:

$$v(t) = Ae^{-\beta t}\left[-\beta\cos(\omega_1 t + \phi) - \omega_1 \sin(\omega_1 t + \phi)\right]. \tag{3.89}$$

Putting these in the expression for energy we find the following for the energy at time t.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \approx \frac{1}{2}m\omega_0^2 A^2 e^{-2\beta t}, \tag{3.90}$$

where we have made use of the lightly damped approximation. Therefore, the energy of a lightly damped oscillator varies at the following rate.

$$\frac{dE}{dt} = -2\beta E. \tag{3.91}$$

Note that the energy of the oscillator does not decrease at the same rate as the envelop of the amplitude: it takes a time equal to $1/2\beta$ for the energy to decrease by a factor $1/e$ while it takes twice as much time $1/\beta$ for the amplitude to decrease by a factor $1/e$. The reason for the energy to decrease twice as fast as the amplitude is that energy is proportional to the square of the amplitude. The same effect shows up in the rate at which sound fades compared to the rate at which a tuning fork loses vibrations. Since the intensity of sound generated by the vibrating tuning fork is related to the energy, the intensity of the sound will decrease faster than the amplitude of oscillations of the tuning fork itself.