6.1 Survey of Nuclear Properties

6.1.1 Nucleons

All nuclei consist of two types of particles, collectively called nucleons: protons and neutrons. The number of protons Z in a nucleus is called the **atomic number** or charge number and the number of neutrons N is called the **neutron number**. Together, the total number A of nucleons in a nucleus is called the **mass number**.

$$\begin{aligned} Z &\equiv \text{Number of protons} \\ N &\equiv \text{Number of neutrons} \\ \boxed{A = Z + N} &\equiv \text{Number of nucleons} \end{aligned} \tag{6.1}$$

To represent a nucleus we use the symbol of the atom and label the symbol with the Z and A values. For instance, we denote the nucleus of a carbon (C) atom with $Z = 6$, $A = 12$ by ${}^{12}_{6}\mathrm{C}$. In general, the notation is

$$ {}^{A}_{Z}\mathrm{X}, \tag{6.2}$$

for the nucleus of an atom with symbol X. Table 6.1 contains nuclear properties of some example nuclides.

The chemical properties of an element are largely determined by the electrons surrounding the nucleus. In a neutral atom the number of electrons must equal the number of protons. In negative ions of an element, the number of electrons are more than the number of protons and in a positive ion the number of electrons are less than the number of protons. Despite the different number of electrons in a neutral atom and charged atom of an element, they all have the same number of protons. Hence, we identify chemical elements by the number of protons in their nuclei.

Although all atoms of a particular chemical element have the same number of protons Z, they may differ in the number of neutrons N, and hence in A. For instance, atoms of hydrogen come with three different values of N, viz., $N = 0, 1, 2$. They are ordinary hydrogen nucleus ${}^{1}_{1}\mathrm{H}$, the deuterium ${}^{2}_{1}\mathrm{H}$, and tritium ${}^{3}_{1}\mathrm{H}$, which are also denoted as H, D, and T respectively. Nuclei of different A but of same Z are called **isotopes**.

Nuclear properties and nuclear reactions are often studied by utilizing atoms rather than bare nuclei. The atomic species are then referred to as a **nuclides**. Thus, when we study the nulcear properties of the atoms ${}^{12}_{6}\mathrm{C}$, ${}^{14}_{7}\mathrm{N}$, ${}^{16}_{8}\mathrm{O}$ we refer to them as nuclides of carbon, nitrogen and oxygen respectively. Referring ${}^{12}_{6}\mathrm{C}$ nucleus or nuclide is a matter of preference since we would normally not be dealing with a bare nucleus for nuclei beyond the helium nucleus ${}^{4}_{2}\mathrm{He}$.

Table 6.1: Nuclear Properties of Select Nuclides

Nuclide	Z	N	A	Atomic Mass (u)	Radius (fm)	Spin ($\hbar$)	Magnetic Dipole Moment (μ_N)	Binding Energy per Nucl. (MeV)
Hydrogen (${}^1_1$H)	1	0	1	1.007825	1.2	1/2	+2.79	-
Deuterium (${}^2_1$H)	1	1	2	2.014102	1.5	1	+0.857	1.1
Tritium (${}^2_1$H)	1	2	3	3.016049	1.7	1/2	+2.98	2.8
Helium-3 (${}^3_2$He)	2	1	3	3.016029	1.7	1/2	−2.13	2.6
Helium (${}^3_2$He)	2	2	4	4.002603	1.9	0	0	7.1
Lithium (${}^7_3$Li)	3	4	7	7.016004	2.3	3/2	+3.26	5.6
Beryllium (${}^9_4$Li)	4	5	9	9.012182	2.5	3/2	−1.18	6.7
Carbon (${}^{12}_6$C)	6	6	12	12.000000	2.75	0	0	7.7
Nitrogen (${}^{14}_7$N)	7	7	14	14.003074	2.89	1	+0.404	7.5
Oxygen (${}^{16}_8$O)	8	8	16	15.994915	3.02	0	0	8.0
Phosphorous (^{31}P)	15	16	31	30.973762	3.77	1/2	+1.13	8.5
Chlorine (${}^{35}_{17}$Cl)	17	18	35	34.968853	3.93	3/2	+0.822	8.5
Krypton (${}^{84}_{36}$Kr)	36	48	84	83.911507	5.26	0	0	8.7
Barium (${}^{138}_{56}$Kr)	56	82	138	137.905232	6.20	0	0	8.4
Gold (${}^{197}_{56}$Kr)	79	118	197	196.966543	6.98	3/2	+0.148	7.9
Lead (${}^{208}_{82}$Kr)	82	126	208	207.976627	7.11	0	0	7.9
Uranium (${}^{238}_{92}$Kr)	92	146	238	238.0507847	7.44	0	0	7.6

6.1.2 Charge, Mass, and Radius

The proton has a charge equal to the charge of an electron but of opposite type. The electronic charge has been determined to a very high precision. The following value was posted at the NIST website in April 2014,

$$\boxed{e = 1.602176565 \times 10^{-19}\ \mathrm{C}.}$$

with uncertainty in the last two digits. The neutron is electrically neutral. The rest mass of a neutron, m_n, is a little larger than the rest mass of a proton, m_p. Both are about 1840 times more massive than an electron.

$$m_e = 9.10938291 \times 10^{-31}\ \mathrm{kg}$$
$$m_p = 1.672621777 \times 10^{-27}\ \mathrm{kg} = 1836 \times m_e$$
$$m_n = 1.674927351 \times 10^{-27}\ \mathrm{kg} = 1839 \times m_e$$

The mass is often quoted in unified atomic mass unit (amu) denoted by the letter u. The quantity u is defined by setting the mass of the isotope ${}^{12}_6$C to exactly 12u. This gives the following conversion between u and kg.

$$\boxed{1\ \mathrm{u} = 1.660540 \times 10^{-27}\ \mathrm{kg}.} \tag{6.3}$$

In terms of the unit u the masses of protons and neutrons are

$$m_p = 1.007276466812\,\mathrm{u}$$
$$m_n = 1.00866491600\,\mathrm{u}$$

Recall that the rest mass energy of a particle of rest mass m_0 is given by

$$E_0 = m_0 c^2$$

Therefore, the rest mass energies of proton and neutron are

$$E_{0,p} = 1.672621777 \times 10^{-27}\,\mathrm{kg} \times (3 \times 10^8\,\mathrm{m/s})^2 = 1.50536 \times 10^{-10}\,\mathrm{J}$$
$$E_{0,n} = 1.674927351 \times 10^{-27}\,\mathrm{kg} \times (3 \times 10^8\,\mathrm{m/s})^2 = 1.50743 \times 10^{-10} : \mathrm{J}$$

These energies are often written in units of MeV or GeV, which stand for million electron volts and giga or billion electron volts.

$$E_{0,p} = 938.272046\,\mathrm{MeV} = 0.938272046\,\mathrm{GeV}$$
$$E_{0,n} = 939.565379\,\mathrm{MeV} = 0.939565379\,\mathrm{GeV}$$

If we divide these rest energies by c^2 then we will get the mass back.

$$m_p = \frac{E_{0,p}}{c^2} = 938.272046 \left[\frac{\mathrm{MeV}}{c^2}\right] = 0.938272046 \left[\frac{\mathrm{GeV}}{c^2}\right]$$
$$m_n = \frac{E_{0,p}}{c^2} = 939.565379 \left[\frac{\mathrm{MeV}}{c^2}\right] = 0.939565379 \left[\frac{\mathrm{GeV}}{c^2}\right]$$

The units MeV/c^2 and GeV/c^2 are commonly used units of mass in nuclear physics. The conversion between u and MeV/c^2 is

$$\boxed{1\,\mathrm{u} = 931.494 \left[\frac{\mathrm{MeV}}{c^2}\right].}$$

The estimate of the radius of an atomic nucleus was first obtained by Rutherford and his assistants in their experiments with α-particles on Aluminum targets. He had deduced a formula of scattering of alpha particles assuming purely Coulombic repulsion between the positively charged alpha particles and the positively charge nucleus. The breakdown of the Rutherford scattering formula suggested a closest approach of the alpha particle to the nucleus, which took to be a measure of the radius of the nucleus. Let $d_{\min}$ be the closest approach of the alpha particles (charge $+2e$) to a nucleus (charge Ze), then equating the kinetic energy of the alpha particle to the potential energy at the closest approach give the following equation.

$$\frac{1}{4\pi\epsilon_0}\frac{(2e)(Ze)}{d_{\min}} = \frac{1}{2}m_\alpha v_\alpha^2 = K_\alpha.$$

Now, if we assume that $d_{\min}$ to be a rough measure of the radius R of the nucleus we can set $d_{\min} = R$ and obtain the following for R.

$$R = \frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{K_\alpha}. \tag{6.4}$$

Rutherford found that the energy of alpha particles available to him were not sufficient to penetrate the gold nuclei. Facing this challenge, Rutherford is said to have made the famous remark:

"There is no money for apparatus. We shall have to use our heads!"

Rutherford correctly noted that lighter nuclei will be penetrated by the alpha particles from radium source. He directed his students to try Aluminum target and sure enough, they found considerable back scatter. Using $Z = 13$ and $K_\alpha = 7.7$ MeV we obtain an estimate of the radius of Aluminum nucleus to be

$$R_{\text{Al}} = [9 \times 10^9] \times \frac{2 \times 13 \times (1.6 \times 10^{-19})^2}{7 \times 10^6 \times 1.6 \times 10^{-19}} = 5.0 \times 10^{-15}\ \text{m}.$$

The nuclear radius is in the range of tens of femtometers. One femtometer is also referred to as 1 **fermi**.

$$1\ \text{fermi} = 1\ \text{fm} = 1.0 \times 10^{-15}\ \text{m}.$$

Electron scattering experiments give a better value for the radius of nuclei. The density ρ of nuclei varies very little among nuclei with the value being

$$\rho = 2.3 \times 10^{17}\ \text{kg/m}^3,$$

which is 14-orders of magnitude more than the density of ordinary matter. Neutron stars are supposed to be this dense. A typical neutron star packs in a 12-13 km ball a mass equal to 1.4-3.2 solar masses with one solar mass being $\sim 2 \times 10^{30}$ kg. For a nucleus, the mass will come from the nucleons. Let m be mass per nucleon, A be the number of nucleon, and R be the radius of a nucleus, then we can write the following equation for the total mass.

$$m \times A = \rho \times \frac{4}{3}\pi R^3.$$

This gives the following useful relation.

$$\boxed{R = R_0 A^{1/3},}$$

with

$$R_0 = \left[\frac{3m}{4\pi\rho}\right]^{1/3}$$

Using $m = 1$ u for the average mass of the nucleon, we fin the value of R_0 to be

$$R_0 = 1.2 \times 10^{-15}\ \text{m}.$$

The relation in Eq. 6.1.2 is direct consequence of the constancy of the nuclear density.

6.1.3 Nuclear Stability and Binding Energy

The electric force between charged protons at nuclear distance can cause enormous acceleration of protons. For instance, if you place two protons at a distance of 1 fm, the force between them will be

$$F = k\frac{e^2}{r^2} = 9\times 10^9\ \frac{(1.6\times 10^{-19})^2}{(10^{-15})^2} = 230\ \text{N}.$$

This is an enormous force on an object which does not have much mass. This force will give rise to a large acceleration of each proton.

$$a = \frac{F}{m_p} = \frac{230\ \text{N}}{1.673\times 10^{-27}\ \text{kg}} = 1.38\times 10^{28}\ \text{m/s}^2.$$

The large electric force would break the nucleus apart. And yet, the nucleus does not fall apart! This is a compelling reason for the existence of some other force that would balance this repulsive force.

Even without the electric force, for instance a neutron, you would expect large speeds of particles when confined to a tiny volume due to the uncertainty principle. For instance, if you confine a neutron to reside within a distance of diameter of a nucleus $D = 2R$, the uncertainty in position will be

$$|\Delta x| = R.$$

Therefore, you would expect the momentum to be quite uncertain as given by Heisenberg's uncertainty principle of position and momentum,

$$\Delta x\, \Delta p \ge \frac{\hbar}{2} \quad\Longrightarrow\quad \Delta p \ge \frac{\hbar}{2\Delta x} = \frac{\hbar}{2R}.$$

With the radius of a nucleus at 10^{-14} m, we will expect

$$\Delta p \ge \frac{1.055\times 10^{-34}\text{J.s}}{2\times 10^{-14}\ \text{m}} = 0.5\times 10^{-20}\ \text{kg.m/s}.$$

The kinetic energy associated with this uncertainty in momentum of the neutron confined in the nucleus will be

$$KE \sim \frac{(\Delta p)^2}{2m_n} = \frac{(0.5\times 10^{-20}\ \text{kg.m/s})^2}{2\times 1.675\times 10^{-27}\ \text{kg}} = 0.8\times 10^{-14}\ \text{J} = 50\ \text{keV}.$$

If there is no energy barrier for the neutron, for instance from an attractive force from inside the nucleus, the neutron would just fly out of the nucleus.

The two arguments of potential instabilities of a nucleus, viz., the electrostatic repulsion and the quantum momentum uncertainty, point to the fact that there must be a strong nuclear force to hold the nucleons together. In 1934 Hideki Yukawa of Japan proposed that the strong interaction among the nucleons is mediated by particles of mass 200 times the mass of an electron. This particle is now known as the pi meson. He proposed an attractive potential V(r) for the stron interaction that has the following form, called the Yukawa potential.

$$V(r) = -g\frac{e^{-\lambda r}}{r}, \tag{6.5}$$

where λ is a constant which is proportional to the inverse of the mass of the particle mediating the force. Cecil Powel discovered three particles in 1948 that matched the Yukawa's predictions. The **strong nuclear force** acts between all three types of nucleon pairs, viz., proton-proton, proton-neutron, and neutron-neutron pairs.

N/Z ratio as a measure of stability

There are 254 known stable nuclides and hundreds of unstable nuclides, some occurring naturally while others made in the laboratory. The unstable nuclides are also called **radionuclides**. To get a better understanding of the type of nucleus that would be stable, in Fig. 6.1 we plot all the stable nuclides found in nature in a diagram with the neutron number on one axis and the proton number on the other axis. The figure shows that small stable nuclides with $A \leq 21$ have a balance between neutrons and protons with $N = Z$. As Z of the nuclei gets larger, the repulsive electric force between the protons increases and it takes additional neutrons to keep them sufficiently separated so as to form stable nuclei. Therefore, stable larger Z nuclides have $N > Z$.

There is no stable nuclei for neuclides that have $Z > 83$. Atoms of all the elements beyond Bismuth, $Z = 83$ are unstable and decay through one or other mechanisms of radioactivity to be discussed later in the chapter. Even the Bi isotope $^{209}_{83}$Bi, which was long thought to be stable, was found to decay with a half life of 1.9×10^{19} yr, which is much longer than the age of the universe.

Evenness of Z, N, and A as measures of stability

Among the 254 or so stable nuclides observed in nature so far, 153 have even A and 101 have odd A. Of the 153 even A stable nuclides 148 have both Z and N even and only 5 have both both Z and N odd. Among the 110 odd A stable nuclides, 53 are even Z and odd N and 48 are odd Z and even N. These abundances suggest that evenness of Z, N, and A contributes to the stability of a nucleus.

Binding energy as a measure of stability

It turns out that the mass of nuclei (except that of the hydrogen atom) is always less than the sum of the masses of the protons and neutrons contained in the nucleus. The difference in mass is called the **missing mass**. From the Einstein's relation,

$$E = mc^2,$$

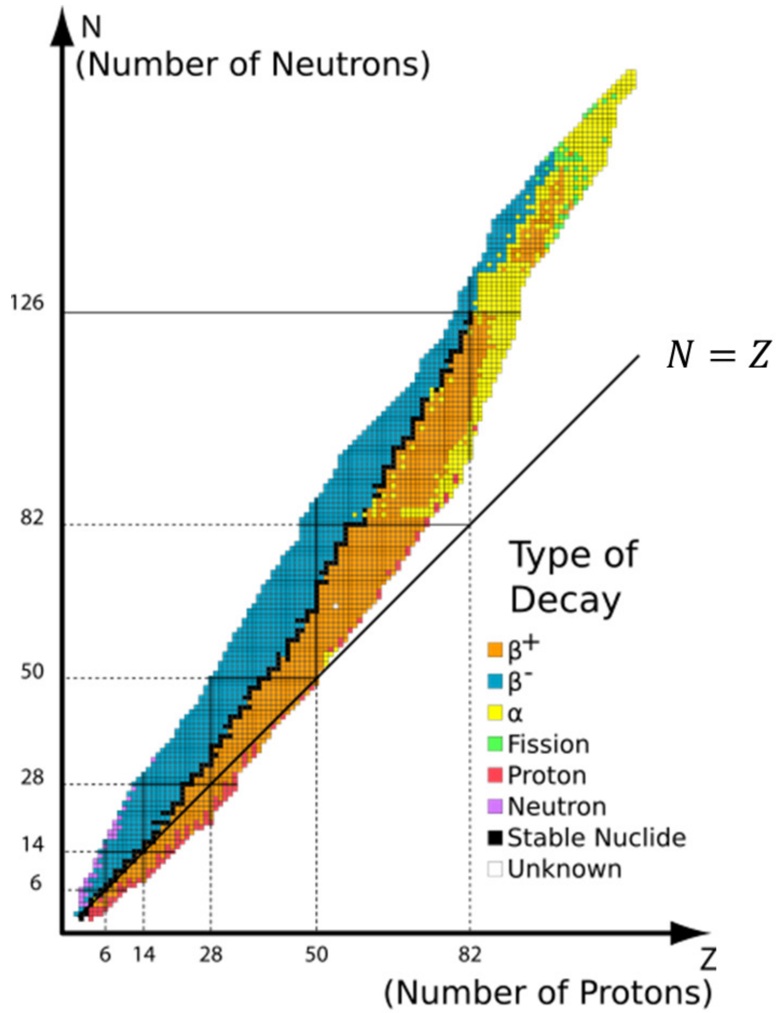

Figure 6.1: Isotopes with number of neutrons in the ordinate axis and the number pf protons in the abscissa. The line is the $N = Z$ line. Stable isotopes are shown as the zig-zag line in the middle of the isotope distributions. Stable nuclei with $Z > 20$ have more neutrons than protons. Credits: Wikimedia Commons.

we interpret the difference in mass to be the energy that must have been taken out of the system when putting them together to form the nucleus. In other words, this energy must be supplied if we were to separate the nucleons of the nucleus far apart from each other. This energy is called the **binding energy** E_b of the nucleus. For a nucleus of mass M_{nuc} with Z protons and N neutrons the binding energy will be

$$E_b = \left(Z m_p c^2 + N m_n c^2 - M_{\text{nuc}} c^2 \right).$$

We can include the electrons with protons and same number of electrons with the atomic nucleus and write this in terms of mass m_H of H atom and the nuclide, M_{atom}.

$$E_b = \left(Z m_H + N m_n - M_{\text{atom}} \right) c^2.$$

From this binding energy we define binding energy per unit nucleon $\mathcal{E}_b$ by dividing it by the number of nucleons.

$$\mathcal{E}_b = \frac{E_b}{A}.$$

Fig. 6.2 shows a plot of binding energy per nucleon versus the mass number A. The plot shows that the highest binding energy per nucleon is near $A = 60$. In general, a nucleus that has higher binding energy per nucleon will be more stable than the nucleus with less.

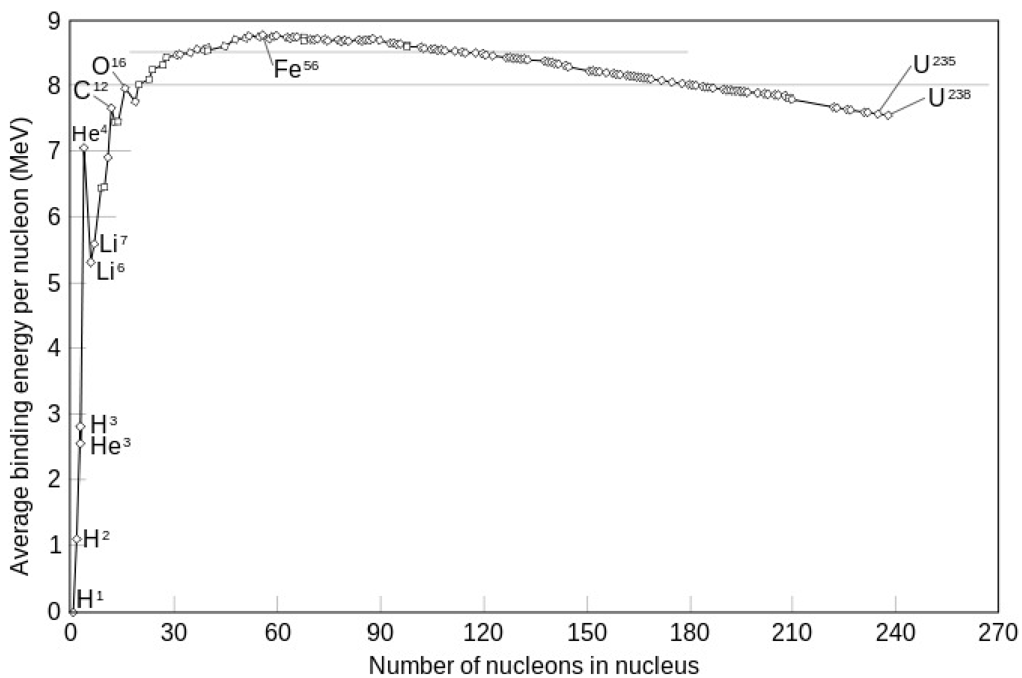

Figure 6.2: Binding energy per nucleon versus number of nucleons (A). The graph shows that the most stable nuclei are in the mid-range of the periodic table with iron isotope, ${}^{56}_{26}$Fe being one of the most stable. Credits: Wikimedia Commons

Due to the higher binding energy per nucleon for intermediate size nuclei, if a larger nucleus, say $A = 120$, splits into two nuclei of $A = 60$, energy will be released. Similarly, if two smaller nuclei, say two nuclei of $A = 30$ were to combine to form one nucleus with $A = 60$, energy will also be released. The process of breaking of nuclei is called **fission** while the process of combination of nuclei is called **fusion**.

Example 6.1. (a) Compute the binding energy of ^{12}C. (b) Compute the binding energy per nucleon of ^{12}C. Data needed: mass of ^{12}C atom = 12.00000 u, $m_p =$ 1.00727647 u, $m_n = 1.0086649$ u, $m_e = 0.000548613$ u.

Solution.

(a) First we compute the missing mass when a carbon atom is constructed from 6 p, 6 n, and 6 e from

$$\Delta m = Zm_p + Zm_e + (A - Z)m_n - M.$$

Here $Z = 6$ and $A = 12$. Therefore,

$$\Delta m = 6\times(1.00727647\ \text{u}+0.000548613\ \text{u})+(12-6)\times1.0086649\ \text{u}-12.00000\ \text{u} = 0.0989399\text{u}.$$

Converting this missing mass into rest energy will give us the total binding energy of the nucleus.

$$e_b = \Delta m\ c^2 = 0.0989399\,\text{u}\ c^2 = 0.0989399\,\text{u} \times 931.494\frac{\text{MeV}}{c^2}\ c^2 = 92.16\,\text{MeV}.$$

(b) By dividing the total binding energy by A we obtain the binding energy per nucleon.

$$\mathcal{E}_b = \frac{E_b}{A} = \frac{92.16\ \text{MeV}}{12} = 7.68\,\text{MeV}.$$

6.1.4 Nuclear Spin and Magnetism

Spin

The protons and neutrons also have spin angular momentum. The nuclear spin is denoted by I rather than S. The magnitude of the spin angular momentum to the spin quantum number is similar here as was the case for electronic spin. Both proton and neutron are spin $\frac{1}{2}$ particles, meaning that the spin quantum number is $\frac{1}{2}$ and the magnitude of their spin angular momentum is

$$I = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\ \hbar = \frac{\sqrt{3}}{2}\ \hbar.$$

Nuclear spin quantum number is also space-quantized. That is, the projection of the spin vector along any axis will take only distinct values. For spin 1/2, the quantum numbers for z-projections are $+1/2$ and $-1/2$ and corresponding angular momenta components are

$$I_z = +\frac{1}{2}\hbar,\ -\frac{1}{2}\hbar.$$

The spins of the protons and neutrons in a nuclei add vectorially to give the total nuclear spin I of a nucleus. A state of total nuclear spin I has the spin angular momentum equal to

$$\text{Spin angular momentum } = \sqrt{I(I+1)}\,\hbar.$$

The space-quantization of the nuclear spin is similar to the space quantization of electron spin. Therefore, the projection of the nuclear spin vector along any axis has values from $-I\hbar$ to $+I\hbar$ in steps of $\hbar$. For instance, the projection on the z-axis of the nuclear spin I state will be

$$I_z = -I\hbar, (-I+1)\hbar, (-I+2)\hbar, \cdots, (I-2)\hbar, (I-1)\hbar, I\hbar.$$

The value I is called the spin quantum number and the values $-I, -I+1, \cdots, I-1, +I$ are called the spin quantum numbers of the z-projection of the spin.

For instance, the spin quantum number of the nucleus of deuterium (D) is 1 since the spins of the proton and neutron in a deuterium nucleus are pointed in the same direction. The nuclear spin angular momentum of the deuterium nucleus is

$$\text{Spin angular momentum of D } = \sqrt{1(1+1)}\,\hbar = \sqrt{2}\,\hbar,$$

which can have three projections along any axis, say on z-axis, giving three possible values of the components of the nuclear spin angular momentum.

$$I_z = -\hbar,\ 0,\ \hbar.$$

Some of the larger nuclei, e.g. ${}^{13}_{6}\mathrm{C}$, has 12 of the spins paired up leaving only one nucleon with unpaired spin. The spin of ${}^{14}_{7}\mathrm{N}$ is 1 and that of ${}^{23}_{11}\mathrm{Na}$ is actually $\frac{3}{2}$.

Example 6.2. The nucleus ${}^{7}_{3}\mathrm{Li}$ has spin quantum number $\frac{3}{2}$. (a) What is its spin angular momentum? (b) What are the spin projection quantum numbers? (c) What are the projections of the spin angular momentum along the z-axis?

Solution.

(a) The spin angular momentum I is obtained from the spin quantum number as usual.

$$I = \sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)}\,\hbar = \frac{\sqrt{15}}{2}\,\hbar.$$

(b) The spin projection quantum numbers just go from −spin to +spin in steps of 1.

$$I_z[\text{quantua}] = -\frac{3}{2},\ -\frac{1}{2},\ +\frac{1}{2},\ +\frac{3}{2}.$$

(c) The allowed values of the component of the spin angular momentum are the corresponding quantum times $\hbar$.

$$I_z = -\frac{3}{2}\hbar,\ -\frac{1}{2}\hbar,\ +\frac{1}{2}\hbar,\ +\frac{3}{2}\hbar.$$

Nuclear Magnetic Moment

If a charged particle has a spin angular momentum $\vec{I}$, then it will have a magnetic dipole moment $\vec{\mu}$ also.

$$\vec{\mu} = \gamma\vec{I}, \tag{6.6}$$

where γ is the gyromagnetic ratio of the particle. For an electron, the gyromagnetic ratio is given by

$$\gamma_e = \frac{e}{2m_e}\,g, \tag{6.7}$$

where g is called the Landé g factor, which has a value close to 2. Similarly, the gyromagnetic ratios of a nucleus is defined similarly,

$$\gamma_n = \frac{e}{2m_p}\, g_n, \tag{6.8}$$

where m_p is the mass of a proton and g_n is the g-factor of the particular nucleus. The constants in this equation are combined to include $\hbar$ and a new quantity called **nuclear magneton** is introduced for convenience.

$$\mu_N = \frac{e\hbar}{2m_p} \equiv 5.05 \times 10^{-27}\ \mathrm{J/T}. \tag{6.9}$$

In terms of μ_N, the gyromagnetic ratios of a nucleus becomes

$$\gamma_n = \frac{g_n}{\hbar}\, \mu_N. \tag{6.10}$$

Putting this in Eq. 6.6 we get the relation between the magnetic dipole moment of a nucleus and its spin angular momentum vector .

$$\vec{\mu} = g_n \frac{\vec{I}}{\hbar}\, \mu_N. \tag{6.11}$$

The components of the magnetic dipole moment are related to the components of the spin angular momentum quantum number as

$$\mu_z = g_n \frac{I_z}{\hbar}\, \mu_N, \tag{6.12}$$

where $I_z/\hbar$ are equal to the quantum numbers since we have divided out $\hbar$. The gyromagnetic ratio and g-factors of proton and neutron are

Particle	$\gamma_n\left(\mathrm{s}^{-1}\mathrm{T}^{-1}\right)$	g_n
Proton	$+2.675222 \times 10^8$	$+5.585694$
Neutron	-1.832471×10^8	-3.82608

Due to the spin angular momentum and charge of the proton, proton has a magnetic dipole moment. The neutron also has a magnetic dipole moment; even though, neutron is overall neutral in charge, it is made up of particles, called quarks, which we will discuss later in the book, that are charged and have spin angular momentum as well. The magnetic dipole moments of the the proton and neutron are

$$\begin{aligned} \text{Magnetic moment of proton } &= +2.7928\mu_N, \\ \text{Magnetic moment of neutron} &= -1.9135\mu_N. \end{aligned}$$

The negative value for the magnetic dipole moment of neutron is indicative of the fact that its spin and magnetic dipole moment are in opposite direction.

Energy in a magnetic field

The magnetic dipole moments of the proton and neutron give rise to a net magnetic dipole moment of a nucleus. As is generally the case, a magnetic dipole $\vec{\mu}$ experiences a torques when placed in a magnetic field $\vec{B}$.

$$\text{Torque on dipole , } \vec{\tau} = \vec{\mu} \times \vec{B}. \tag{6.13}$$

You can also show that the energy of a magnetic dipole in an external magnetic field is given by

$$\text{Energy of dipole , } U = -\vec{\mu} \cdot \vec{B}. \tag{6.14}$$

For concreteness, let the magnetic field be pointed towards the positive z-axis. Then, there will be no torque if the magnetic dipole $\vec{\mu}$ points either towards the magnetic field or opposite to it. The energy of the dipole of proton is lowest when it is aligned with the field and that of the neutron is lowest when it is in the opposite direction to the field.

The energy of a proton U_p when it is aligned with the external field will be

$$U_p(\text{aligned}) = -\mu_p B,$$

when it is in the opposite direction to the field, it will be

$$U_p(\text{opposite}) = +\mu_p B.$$

Therefore, there will be two energy states for the proton when it is placed in an external magnetic field. The difference in the energy of the two states is ΔE,

$$\Delta E = U_p(\text{opposite}) - U_p(\text{aligned}) = 2\mu_p B.$$

A proton can absorb a photon to make quantum transition between these states. The frequency f of the photon will be given by

$$hf = \Delta E = 2\mu_p B,$$

where h is the Planck constant. Often we write this using this using angular frequency ω rather than f.

$$\hbar\omega = 2\mu_p B,$$

where $\hbar = h/2\pi$. Classically, if the magnetic dipole vector is not pointed towards or opposite to $\vec{B}$, then it will experience a torque which will make the dipole moment vector rotate about the magnetic field. The frequency of precession is called the **Larmor frequency**, and often denoted as ω_L.

$$\omega_L = \frac{2\mu B}{\hbar}. \tag{6.15}$$

Example 6.3. Find the Larmor frequency of a proton in a 1 T magnetic field.

Solution.

We use Eq. 6.15 to compute the Larmor frequency.

$$\omega_L = \frac{2\mu_p B}{\hbar} = \frac{2 \times 2.7928 \times 5.05 \times 10^{-27}\ \text{J/T} \times 1\ \text{T}}{1.055 \times 10^{-34}\ \text{J.s}} = 2.67 \times 10^8\ \text{s}^{-1}.$$

Dividing this by 2π give the frequency of precession of proton to be 42.5 MHz.

The precession of the magnetic dipole of the proton can be used to put energy to the proton nucleus and cause the spin to flip by exciting the precessing proton with a photon of the same frequency as the precession frequency. When the frequencies match, the proton absorbs energy from the photon field in its transition from the lower energy state to the higher energy state. This is the basis of the technique called **Nuclear Magnetic Resonance (NMR)**.

The NMR techniques can be applied to any nucleus that has a net nuclear magnetic dipole moment. For instance, NMR of ${}^{13}_{6}$C and ${}^{31}_{15}$P are commonly used in chemical analysis. Their Larmor frequencies are different from that of the proton given above. The Larmor frequencies of nuclear dipole precession also depend upon the chemical environment of the atom which has made NMR an invaluable tool for chemists.

Example 6.4. A ^{7}Li nucleus has spin quantum number $\frac{3}{2}$. It is placed in a magnetic field 2 T pointed towards the positive z-axis. (a) How many energy levels correspond to spin quantum number $\frac{3}{2}$ due to space quantization? (b) What is the energy of a photon released when the nucleus makes a transition between adjacent states? Data: $g_n(\,{}^7\text{Li}) = +2.17$.

Solution.

(a) In the magnetic field, the levels will spit up into four levels, corresponding to the spin projection quantum numbers $+3/2$, $+1/2$, $-1/2$, $-3/2$ with the negative quantum number higher energy since $U = -\mu B$ and gyromagnetic ratio is positive.

(b) The adjacent levels will be separated equally, with separation of energy

$$\Delta E = |\mu_2 - \mu_1| B.$$

where we can take μ_1 and μ_2 for $+3/2$, $+1/2$ states.

$$\mu_1 = g_n \times \frac{3}{2}\mu_N = 3.26\mu_N, \quad \mu_2 = g_n \times \frac{1}{2}\mu_N = 1.09\mu_N$$

Therefore,

$$\Delta E = 2.17\,\mu_N\,B = 2.17 \times 5.05 \times 10^{-27}\ \text{J/T} \times 2\ \text{T} = 2.19 \times 10^{-26}\ \text{J}.$$

The wavelength of the light will be

$$\frac{hc}{\lambda} = \Delta E.$$

Therefore,

$$\lambda = \frac{hc}{\Delta E} = \frac{1.986 \times 10^{25}\text{J.m}}{2.19 \times 10^{-26}\ \text{J}} = 9.1\ \text{m}.$$