1.5 The Lorentz Transformations

We will now work out the relations between the coordinates and times assigned to the same event in two inertial frames that are in relative motion with respect to each other. As above, we will call one frame S with Cartesian coordinates x, y and z, and the other S' with Cartesian coordinates x', y' and z'. The Cartesian coordinates of the two frames are parallel with each other. Furthermore, let there be identical synchronized clocks at all space points in each frame with symbols t and t' denoting the time readings in the two frames.

Let frame S' be moving relative to frame S towards the positive x-axis direction of the S frame with speed V. You could equivalently say that the frame S is moving relative to the frame S' towards the negative x'-axis direction of the S' with speed V. We will use the first choice and sometimes refer to S as the "stationary system" and S' the "moving system".

Since the two system of coordinates are parallel to each other and in relative motion along the common x-axis direction, at some time the origin of S' will be at the same place as the origin of S. We will set $t = 0$ and $t' = 0$ for the event when the two origins are at top of each other. That is, the event with time and coordinates $(t, x, y, z) = (0, 0, 0, 0)$ in the S frame will correspond to the the time coordinates $(t', x', y', z') = (0, 0, 0, 0)$ in the S' frame.

Now, for any arbitrary event E at some point P in space, the two frames will give different measurements of coordinates and times. Let the values of coordinates and time for this event in the S frame be (t, x, y, z, t) and in the S' frame be (t', x', y', z'). Our task is to find the relation between these values. First, I will tell you what the answer is and then we will work out the derivation.

$$\left.\begin{aligned} t' &= \frac{t - Vx/c^2}{\sqrt{1 - V^2/c^2}} \\ x' &= \frac{x - Vt}{\sqrt{1 - V^2/c^2}} \\ y' &= y \\ z' &= z \end{aligned}\right\} \tag{1.27}$$

Derivation of Lorentz transformations. We want one-to-one relation between events in the two frames. First we note that the relation between the two sets of coordinates and times must be linear. If not, then two or more events in one frame will correspond to one event in the other. Next, we note that the relative motion of the coordinates is along the common x-axis direction. Therefore, the y and z coordinates in the two systems will not change. Since, $y' = y$ and $z' = z$ at $t = 0$, they will remain same at later times.

$$y' = y \tag{1.28}$$

$$z' = z \tag{1.29}$$

Now, to work on the relation between x, t and x', t' we will start with linear relation with unknown coefficients, α, β, γ, and δ.

$$x' = \alpha\, t + \beta\, x \tag{1.30}$$
$$t' = \gamma\, t + \delta\, x \tag{1.31}$$

To find the four coefficients we will apply these relations to following four events.

Event 1: The origin of the S' at time t'. This event has $x = Vt$ and $x' = 0$. Therefore, we obtain the following from Eq. 1.30.

$$0 = \alpha\, t + \beta\, Vt, \implies \boxed{\alpha = -\beta\, V} \tag{1.32}$$

Event 2: The origin of the S frame at time t. This event has $x' = -Vt'$ and $x = 0$. Therefore, Eqs. 1.30 and 1.31 give

$$-Vt' = \alpha\, t \tag{1.33}$$
$$t' = \gamma\, t \tag{1.34}$$

Dividing the two equations we obtain

$$\boxed{\alpha = -\gamma V} \tag{1.35}$$

From Eqs. 1.32 and 1.35 we find

$$\boxed{\beta = \gamma} \tag{1.36}$$

Event 3: Send a light pulse from the origin of the two coordinates when they were at the same location at time $t = 0 = t'$ towards the positive x-axis. The event we will look at is the position of the pulse at time t in the S frame and t' in the S' frame. The location of the pulse in the two systems will be $x = ct$ and $x' = ct'$ since light has same speed in the two systems and the pulse will be traveling towards positive x-axis in both systems. Putting these values for x and x' in Eqs. 1.31 and 1.30 and using $\beta = \gamma$ and $\alpha = -\gamma V$ we get

$$ct' = -\gamma V\, t + \gamma\, ct \tag{1.37}$$
$$t' = \gamma\, t + \delta\, ct \tag{1.38}$$

From these equations it is easy to deduce the following

$$\boxed{\delta = -\gamma\, \frac{V}{c^2}} \tag{1.39}$$

Let us summarize the results so far by rewriting all coefficients in Eqs. 1.30 and 1.31 in terms of γ.

$$x' = -\gamma V\, t + \gamma\, x \tag{1.40}$$
$$t' = \gamma\, t - \gamma \frac{V^2}{c^2}\, x \tag{1.41}$$

Now, we need to find the expression for γ only.

Event 4: Send a light pulse at $t = 0$ from the origin travels towards the positive y axis in the S frame. In the S' frame, the ray will appear diagonal since pulse on a point on the y-axis will have both x' and y' nonzero. The event we will look at is the position of the pulse at time t in the S frame, which is time t' in the S' frame. At time t in S frame and time t' in S' frame, the coordinates of the pulse will be $x = 0$, $y = ct$, $x' = -Vt$, $y' = y = ct$. Since the speed of light is same in the two frames, in frame S' the path length of light in time t' along the diagonal of the triangle will equal ct' giving the following relation.

$$x'^2 + y'^2 = (ct')^2, \implies (-\gamma V t)^2 + (ct)^2 = (c\gamma t)^2 \tag{1.42}$$

Therefore,

$$\gamma^2 = \frac{1}{1 - V^2/c^2}. \tag{1.43}$$

Taking square-root of both sides gives both plus and minus roots for γ. We keep only the positive root since with negative root we would get $x' = -x$ in the limit of $V = 0$ when the two frames become identical.

$$\boxed{\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.} \tag{1.44}$$

Our derivation for the Lorentz transformation is complete with the desired result. It is customary to write the Lorentz transformations in terms of γ rather than the radical.

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{V}{c^2}\,x\right) \\ x' &= \gamma\,(x - Vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \tag{1.45}$$

These equations can be solved for x, y, z, t in terms of x', y, z', t' to yield the following result.

$$\left.\begin{aligned} t &= \gamma\left(t' + \frac{V}{c^2}\,x'\right) \\ x &= \gamma\,(x' + Vt') \\ y &= y' \\ z &= z' \end{aligned}\right\} \tag{1.46}$$

This result is expected from the fact that while frame S' is moving towards the positive x-axis of the S frame, frame S is moving towards the negative x'-axis of the S' frame, which means that V in the earlier transformations will be replaced by $-V$ in the later equations. We note that if $V > c$ then γ becomes imaginary. Therefore, Lorentz transformation is applicable only when the relative speed is less than the speed of light in vacuum.

$$\boxed{\text{Condition on Lorentz transformations:} \quad V < c.} \tag{1.47}$$