9.6 ROLLING MOTION IN A STRAIGHT LINE

Rolling motion in a straight line is an example of motion in which both translation and rotation take place at the same time but the axis of rotation is always pointed in the same direction. The rolling motion is different from a pure rotation since, in a rolling motion, the axis of rotation moves with the body. However, if the rolling motion occurs in a straight line, then the direction of the axis remains unchanged. It is possible to separate the translational motion of the center of mass (CM) from the rotational motion about an axis through CM as we will show now.

Separation of angular momentum

Consider a disk rolling on a plane surface as shown in Fig. 9.41. To be concrete, we choose the z-axis of a coordinate system that is fixed to the plane surface along the axis of rotation. Then, the z-component

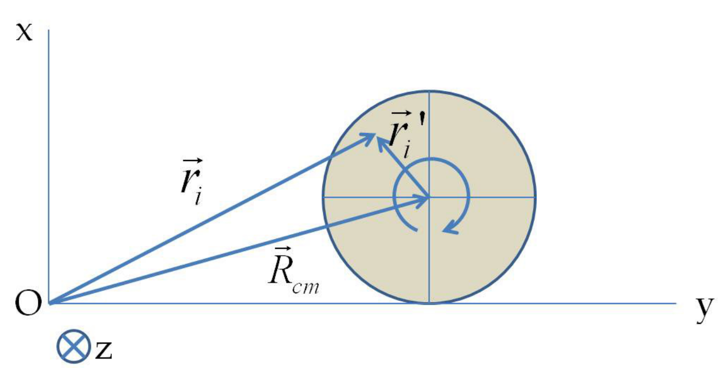


Figure 9.41: Rolling motion. Here z-axis is pointed in-the-page, indicated by x in a circle.

of the angular momentum of a rolling body about the origin O of the fixed coordinate system can be obtained by summing over the angular momenta of the elements of the disk as we have done before. We can prove that the z-component of the angular momentum in the rolling motion separates into two terms.

$$L_z^{\text{about O}} = I_{zz}^{\text{CM}}\omega_z + \left(\vec{R}_{\text{CM}} \times M\vec{V}_{\text{CM}}\right)_z \tag{9.95}$$

where I_{zz}^{CM} is the moment of inertia about an axis through the CM that is parallel to the z-axis through O.

Proof:

Consider the rolling object as collection of masses m_i. Then the angular momentum about O is the sum of angular momentum of each mass.

$$L_z^{\text{about O}} = \sum_i \left(\vec{r}_i \times m_i \vec{v}_i\right)_z = \sum_i \left(\vec{r}_i \times m_i \frac{d\vec{r}_i}{dt}\right)_z \tag{9.96}$$

The position vector of m_i with respect to O, labeled $\vec{r}_i$, differs from its position $\vec{r}_i{}'$ with respect to the CM.

$$\vec{r}_i = \vec{R}_{\text{CM}} + \vec{r}_i{}' \tag{9.97}$$

Substituting this equation into Eq. 9.96, and then expanding the terms in the sum, we find the angular momentum to be

$$\begin{aligned} L_z^{\text{about O}} = \quad & \left[\vec{R}_{\text{CM}} \times \sum_i m_i \frac{d\vec{R}_{\text{CM}}}{dt} + \left(\sum_i m_i \vec{r}_i{}'\right) \times \frac{d\vec{R}_{\text{CM}}}{dt}\right. \\ & \left. + \vec{R}_{\text{CM}} \times \left(\sum_i m_i \frac{d\vec{r}_i{}'}{dt}\right) + \sum_i \left(\vec{r}_i{}' \times m_i \frac{d\vec{r}_i{}'}{dt}\right)\right]_z \end{aligned} \tag{9.98}$$

The middle two terms can be shown to be zero as follows. The second term becomes zero using the definition of center of mass,

$$\sum_i m_i \vec{r}_i{}' = \sum_i m_i \left(\vec{r}_i - \vec{R}_{\text{CM}}\right) = M\vec{R}_{\text{CM}} - M\vec{R}_{\text{CM}} = 0,$$

which makes the third term in Eq. 9.98 zero since it implies

$$\sum_i m_i \frac{d\vec{r}_i{}'}{dt} = 0.$$

The first term in Eq. 9.98 simplifies to $\left(\vec{R}_{\text{CM}} \times M\vec{V}_{\text{CM}}\right)_z$. The fourth term gives motion about the CM and an be shown to be an angular momentum.

$$\left(\sum_i m_i \vec{r}_i{}' \times \frac{d\vec{r}_i{}'}{dt}\right)_z = \left(\vec{r}_i{}' \times \vec{p}_i{}'\right)_z,$$

where $\vec{r}_i{}'$ and $\vec{p}_i{}'$ are the position and momentum of m_i with respect to the CM. Therefore, this term gives the z-component of angular momentum with respect to the CM.

$$\text{Fourth term in Eq. 9.98} = \quad I_{zz}^{\text{CM}} \omega_z.$$

Hence,

$$\boxed{L_z^{\text{about O}} = I_{zz}^{\text{CM}} \omega_z + \left(\vec{R}_{\text{CM}} \times M\vec{V}_{\text{CM}}\right)_z.}$$

Separation of torque

There is a similar separation of torque on the body calculated about the z-axis through O. For each force, the torque separates into a torque about the CM and a torque of the force if the force were acting at the CM rather than where it is actually acting. Let $\vec{F}_i$ be an external force on the body acting at a position vector $\vec{r}_i$. Then, the torque from this force about O is

$$\begin{aligned}
\tau_{i,z}^{\text{about O}} &= \left(\vec{r}_i \times \vec{F}_i\right)_z \\
&= \left(\vec{r}_i{}' \times \vec{F}_i\right)_z + \left(\vec{R}_{\text{CM}} \times \vec{F}_i\right)_z \\
&= \text{Torque about CM} + \text{Torque about O as if } \vec{F}_i \text{ was acting at CM.}
\end{aligned}$$

Now, adding torques of all forces on the body we find

$$\begin{aligned}
\tau_{\text{net,z}}^{\text{about O}} &= \sum_i \left(\vec{r}_i' \times \vec{F}_i\right)_z + \left(\vec{R}_{\text{CM}} \times \vec{F}_{\text{net}}\right)_z \\
&= \tau_{\text{net,z}}^{\text{about CM}} + \left(\text{Torque about O as if } \vec{F}_{\text{net}} \text{ was acting at CM}\right)_z .
\end{aligned}$$

Separation of the equations of motion

Taking the time derivative of the angular momentum, and then equating the result to the torque, we obtain the equation for rotational dynamics for a fixed axis rotation.

$$\tau_z^{\text{about O}} = \frac{dL_z^{\text{about O}}}{dt}.$$

This relation can be expanded in motion of the CM and motion about the CM as follows. For a rigid body we obtain

$$\left(\vec{r}_i{}' \times \vec{F}_i\right)_z + \left(\vec{R}_{\text{CM}} \times \vec{F}_{\text{net}}\right)_z = I_{zz}^{\text{CM}}\alpha_z + \left(\vec{R}_{\text{CM}} \times M\vec{A}_{\text{CM}}\right)_z, \quad (9.99)$$

where α_z is the z-component of the angular acceleration and $\vec{A}_{\text{CM}}$ is the CM acceleration. We know from an earlier chapter that the net external force on a body is equal to the total mass times acceleration of the CM.

$$\vec{F}_{\text{net}}^{\text{ext}} = M\vec{A}_{\text{CM}}$$

Putting this in Eq. 9.99, we can separate out the dynamics of the rotation about the CM from the translation of the CM.

$$\boxed{\text{Rotation about CM:} \quad \tau_z^{\text{about CM}} = I_{zz}^{\text{CM}}\alpha_z} \quad (9.100)$$

$$\boxed{\text{Translation of CM:} \quad \vec{F}_{\text{net}}^{\text{ext}} = M\vec{A}_{\text{CM}}} \quad (9.101)$$

Separation of kinetic energy

Kinetic energy for a rolling motion also separates into the kinetic energy of translation of CM and the kinetic energy of rotation about CM.

$$\boxed{\text{Kinetic Energy} = \frac{1}{2} I_{zz}^{\text{CM}} \omega_z^2 + \frac{1}{2} M V_{\text{CM}}^2.} \tag{9.102}$$

In Table 9.2 the dynamical equations for pure rotations and rolling motion are summarized for a quick reference.

Table 9.2: Pure rotation and rolling motion of a rigid body

Quantity	Pure Rotation	Rolling Motion
Angular momentum	$L_z = I_{zz}\omega_z$	$L_z^{\text{about O}} = I_{zz}\omega_z + \left(\vec{R}_{\text{CM}} \times M\vec{V}_{\text{CM}}\right)_z$
Torque	$\tau_z = \sum \left(\vec{r} \times \vec{F}\right)_z$	$\tau_z = \sum \left(\vec{r} \times \vec{F}\right)_z^{\text{about O}}$ $\tau_z = \left(\vec{R}_{cm} \times \vec{F}_{\text{ext}}\right)_z + \sum \left(\vec{r}' \times \vec{F}\right)_z^{\text{about CM}}$
Equations of motion	$\sum \left(\vec{r} \times \vec{F}\right)_z = \frac{d}{dt}\left(I_{zz}\omega_z\right)$	Rotation about the CM: $\tau_{z,net}^{\text{about CM}} = \frac{d}{dt}\left(I_{zz}\omega_z\right)$ Translation of the CM: $\vec{F}_{\text{ext}} = M\vec{A}_{\text{CM}}$
Kinetic energy	$K = \frac{1}{2} I_{zz}\omega_z^2$	$K = \frac{1}{2} I_{zz}\omega_z^2 + \frac{1}{2} M V_{\text{CM}}^2$

Example 9.6.1. Rolling a drum.

Find the acceleration of a drum rolling downhill without slipping on a slope inclined at angle ϕ.

Solution. First we draw a figure and identify the relevant dynamical variables and forces as shown in Fig. 9.42. For the translation of the

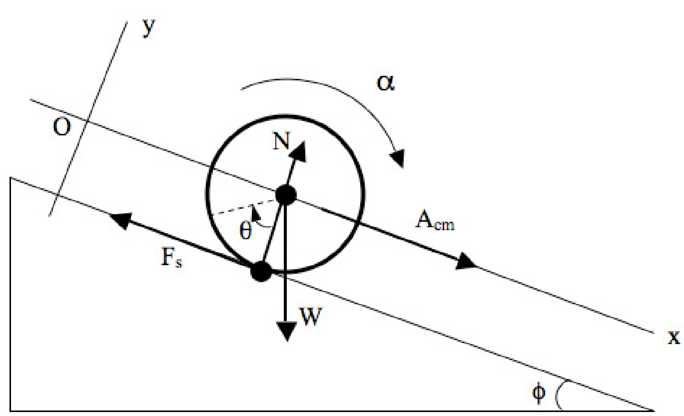


Figure 9.42: Rolling drum down an incline.

CM, it is best to use a Cartesian coordinates with x-axis along the incline as shown in the figure. For the rotation, we can use the angle between a marked line, shown dashed line on the cross-section of the drum, and the vertical to the incline, not the vertical to the ground. From the figure, we find that the angle rotated by the marked line and the distance traveled by the CM have the following relation if the drum rolls without slipping.

$$X_{\text{CM}} = R\theta_z$$

Taking derivative of both sides with respect to time t we find

$$V_x^{\text{CM}} = R\omega_z$$

The x-component of acceleration of the CM is, therefore, related to the z-component of the angular acceleration α_z.

$$A_x^{\text{CM}} = R\alpha_z.$$

The forces acting on the drum are its weight with magnitude $W(= Mg)$, normal force N (unknown) and the static friction F_s (unknown). Now, we will set up the dynamical equations for the translation of the CM and the rotation about the CM. These equations together with the relation between the x-component of the acceleration of the CM and the z-component of the angular acceleration will be sufficient to solve for the acceleration of the CM.

Translation of the CM:

$$\begin{aligned} &\text{x-component: } W\sin\phi - F_s = MA_x^{\text{CM}} \\ &\text{y-component: } N - W\cos\phi = 0 \end{aligned}$$

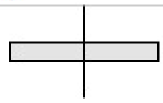
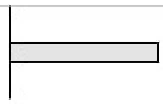
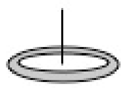
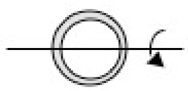
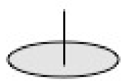
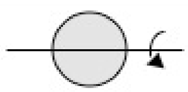
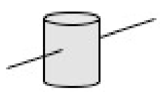
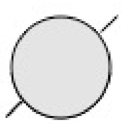
Rotation about the CM:

$$F_s R = \frac{1}{2}MR^2\alpha_z$$

Replace α_z by A_x^{cm}/R, and solving the three equations simultaneously, we obtain the desired result.

$$A_x^{\text{CM}} = \frac{2}{3}g\sin\phi.$$

Table 9.3: Formulas for moment of inertia components

Shape	Axis	Moment of Inertia	Radius of Gyration
Rod		$\frac{1}{12}ML^2$	$\frac{1}{\sqrt{12}}L$
Rod		$\frac{1}{3}ML^2$	$\frac{1}{\sqrt{3}}L$
Ring		MR^2	R
Ring		$\frac{1}{2}MR^2$	$\frac{1}{\sqrt{2}}R$
Disk		$\frac{1}{2}MR^2$	$\frac{1}{\sqrt{2}}R$
Disk		$\frac{1}{4}MR^2$	$\frac{1}{2}R$
Cylinder		$\frac{1}{2}MR^2$	$\frac{1}{\sqrt{2}}R$
Cylinder		$\frac{1}{4}MR^2+\frac{1}{12}ML^2$	$\sqrt{\frac{1}{4}R^2+\frac{1}{12}L^2}$
Sphere		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}\ R$