3.5 Reflection from a Finite Potential Step

Consider sending a particle towards a finite barrier as shown in Fig. 3.8.

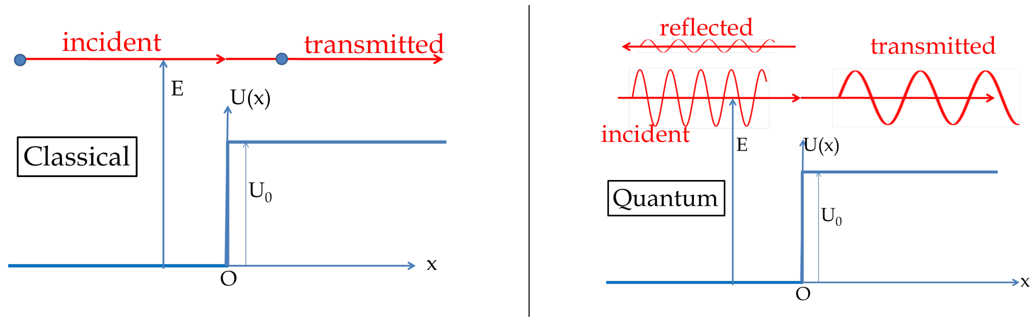

Figure 3.8: The reflection of a particle from a finite potential step when the energy of the particle is greater than the energy of the potential step. Classically, the particle should go over the step and continue on its original path. In quantum mechanics, there is a probability that the particle will be reflected back!

If energy E of the particle is more than the potential energy of the barrier U_0, then in the classical world of everyday life such as plane flying over a mountain, the particle will just continue onward. But in quantum mechanics, the wave function in the $x < 0$ region will be a superposition of a wave moving towards the positive x-axis, ψ_{in}, and another wave moving towards the negative x-axis, ψ_{re}.

$$\psi_I = \psi_{in} + \psi_{re}. \tag{3.50}$$

In the region $x > 0$ we have one wave - the transimtted wave moving towards the positive x-axis, ψ_{tr}.

$$\psi_{II} = \psi_{tr}. \tag{3.51}$$

To write the solutions it is better to introduce the following constants.

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad k' = \frac{\sqrt{2m(E-U_0)}}{\hbar}, \quad \omega = \frac{E}{\hbar}. \tag{3.52}$$

In terms of these constants the incoming, reflected and transmitted waves are:

$$\psi_{in} = Ae^{ikx-i\omega t}, \quad \psi_{re} = Be^{-ikx-i\omega t}, \quad \psi_{tr} = Ce^{ikx-i\omega t}, \tag{3.53}$$

where A, B, and C are the amplitudes of the incoming wave, the reflected wave, and the transmitted wave. The wave function in the two regions are:

$$\psi_I(x,t) = Ae^{ikx-i\omega t} + Be^{-ikx-i\omega t} \tag{3.54}$$
$$\psi_{II}(x,t) = Ce^{ikx-i\omega t} \tag{3.55}$$

The wave functions in the two regions join at $x = 0$ such that they have same amplitudes and same slopes with respect to x at $x = 0$. These are called **boundary**

conditions on the wave function.

$$\psi_I(0,t) = \psi_{II}(0,t) \qquad\qquad A + B = C \tag{3.56}$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0} \qquad\qquad A - B = \frac{k'}{k}C. \tag{3.57}$$

Equations 3.56 and 3.57 can be solved for B/A and C/A as

$$\frac{B}{A} = \frac{k-k'}{k+k'}, \qquad \frac{C}{A} = \frac{2k}{k+k'}. \tag{3.58}$$

The probability current of incoming wave tells us the flux of the incoming wave. Suppose there is a source of particles at $x = -\infty$, then the number of particles arriving at $x = 0$ per unit time will be proportional to j_{in}. In that experiment, the particle leaving $x = 0$ towards $x = -\infty$ will be j_{re} and towards $x = \infty$ will be j_{tr}. Therefore, the flux of reflected and transmitted waves should add up to the flux of the incoming wave. Let us verify this by computing the probability currents for the three waves.

$$j_{in} = \frac{\hbar A^2 k}{m}, \quad j_{re} = -\frac{\hbar B^2 k}{m}, \quad j_{tr} = \frac{\hbar C^2 k'}{m}. \tag{3.59}$$

Therefore,

$$\left|\frac{j_{tr}}{j_{in}}\right| = \frac{k'C^2}{kA^2}, \quad \left|\frac{j_{re}}{j_{in}}\right| = \frac{B^2}{A^2}. \tag{3.60}$$

This gives the desired conservation of probabilities arrivig at $x = 0$ and leaving $x = 0$ as expected.

$$\left|\frac{j_{tr}}{j_{in}}\right| + \left|\frac{j_{re}}{j_{in}}\right| = \frac{k'C^2}{kA^2} + \frac{B^2}{A^2} = 1. \tag{3.61}$$