4.1 COUPLED VIBRATIONS OF TWO MASSES

In the last chapter we have studied oscillations of one mass about an equilibrium. Wave phenomena arises as a result of the coupling of vibratory motions of different parts of a system. To develop a feel for the consequences of coupling of vibrations, we will study a simple system consisting of two blocks of masses m_1 and m_2 that are connected by a spring and supported on the two sides by two other springs as shown in Fig. 4.2. We will study the one-dimensional motion of the coupled masses analytically by choosing the x-axis to coincide with the line of the springs and the blocks as shown in Fig. 4.2. Let x_1 and x_2 be the displacements of the two blocks from their equilibrium positions. Note that x_1 and x_2 are not the x-coordinates of the blocks but rather the x-components of their displacements from the corresponding equilibrium positions. Let k_{s1} be the spring constant of the spring connecting block m_1 to the left support, k_{s2} the spring constant of the spring connecting block m_2 to the right support, and k_c the spring constant of the spring coupling the two blocks as indicated in the figure.

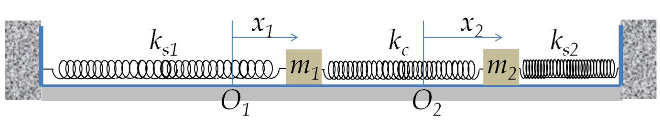

Figure 4.2: Two blocks coupled with each other by a spring and attached to fixed supports with additional springs. At equilibrium, the blocks are at points marked O_1 and O_2. The x-component of the displacements of the two blocks as measured from O_1 and O_2 are denoted by x_1 and x_2. Any change in x_1 affects the forces on block # 2 and any change on x_2 affects the forces on block # 1. We say that the motions of the blocks are coupled.

The x-components of the equations of motion of the two blocks are

$$m_1 \frac{d^2x_1}{dt^2} = -k_{s1}x_1 - k_c(x_1 - x_2), \tag{4.1}$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_{s2}x_2 + k_c(x_1 - x_2). \tag{4.2}$$

Rather than solve these two equations for arbitrary values of the masses and constants, let us simplify the problem by looking at the case where masses are equal and two of the springs connecting the blocks to the support have equal spring constants.

$$m_1 = m_2 \equiv m$$
$$k_{s1} = k_{s2} \equiv k_s$$

Then, the equations of motion for x_1 and x_2 are

$$m\frac{d^2x_1}{dt^2} = -k_s x_1 - k_c(x_1 - x_2), \tag{4.3}$$
$$m\frac{d^2x_2}{dt^2} = -k_s x_2 + k_c(x_1 - x_2). \tag{4.4}$$

To solve these equations, we introduce new composite variables X and x in place of x_1 and x_2 by

$$X = \frac{x_1 + x_2}{2}, \tag{4.5}$$
$$x = x_1 - x_2. \tag{4.6}$$

Equations 4.3 and 4.4 then gives rise to the following equations of motion for $X(t)$ and $x(t)$.

$$m\frac{d^2X}{dt^2} = -k_s X, \tag{4.7}$$
$$m\frac{d^2x}{dt^2} = -\left(k_s + 2k_c\right)x. \tag{4.8}$$

Equation 4.7 is an equation of motion of an oscillator of mass m and spring constant k_s, and Eq. 4.8 is an equation of motion of an oscillator of mass m and spring constant $k_s + 2k_c$. Therefore, we can think of X and x as displacements of two fictitious simple harmonic oscillators of mass m attached to springs of spring constants k_s and $(k_s + 2k_c)$ respectively. The solution of these equations show that the variables X and x will oscillate with angular frequencies given by

$$\text{X oscillates at:} \qquad \omega_X = \sqrt{\frac{k_s}{m}} \tag{4.9}$$
$$\text{x oscillates at:} \qquad \omega_x = \sqrt{\frac{k_s + 2k_c}{m}} \tag{4.10}$$

The variables X and x are called **normal modes** and the angular frequencies ω_X and ω_x the **normal frequencies** of this system of two masses. The normal coordinates X and x execute simple harmonic motions of their corresponding frequencies and are given by

$$X(t) = A\cos(\omega_X t - \delta) \tag{4.11}$$
$$x(t) = B\cos(\omega_x t - \phi) \tag{4.12}$$

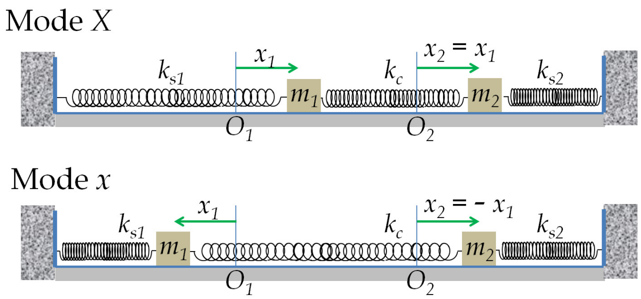

Figure 4.3: Illustrations of the two normal modes X and x. In mode X, the blocks move in tandem, both towards right or both towards left, such that the middle spring in unaffected. In mode x, the blocks move in the opposite directions such that the center of mass of the two blocks remains fixed.

The coefficients of x_1 and x_2 in the definitions of X and x in Eqs. 4.5 and 4.6 show how the two masses move when the system oscillates in mode X or x.

Thus, Eq. 4.5 shows that when the masses are moving in the mode X, the two masses have equal displacements and are in-phase as reflected by the same sign for x_1 and x_2 in the expression for X given in Eq.4.5. When the two masses are moving in this mode only the springs on the outside to the supports contract or expand while the coupling spring remains un-affected as shown in Fig. 4.3. That is why the frequency of this mode does not have dependence on k_c .

On the other hand, Eq. 4.6 shows that when the motion is in the mode x, the displacements of the two masses have equal magnitude but are in opposite directions as reflected in opposite signs for x_1 and x_2 in x given in Eq.4.6. In this mode, the center of mass remains fixed, which can happen if the masses are moving in opposite directions as shown in Fig. 4.3. When the two masses move according to this mode, all three springs participate, which is reflected in the dependence of the frequency of this mode on both k_s and k_c.