1.4 FERMAT'S PRINCIPLE OF LEAST TIME

The three laws of geometric optics can be understood from the perspectives of a deeper principle concerning the propagation of light due to Pierre Fermat (1601-1665), the French lawyer and mathematician. In its original form, **Fermat's principle** states that,

"*The actual path between two points taken by a beam of light is the one which is traversed in the least time.*"

This principle can be stated in terms of a quantity called the **optical path length** (OPL) defined as the product of geometric length and the refractive index of the medium. Suppose a ray of light travels a distance l in a medium of refractive index n. Then, we say that the optical path length (OPL) of the ray is

$$\text{OPL} = nl. \tag{1.21}$$

Let $v = c/n$ be the speed of the ray in this medium, then, the time taken will be

$$t = \frac{l}{v} = \frac{nl}{c} = \frac{1}{c} \times \text{OPL}.$$

Since c is a universal constant, a minimum t will correspond to a minimum OPL. Thus, we arrive at an alternate statement of Fermat's principle:

"*The optical path length (OPL) of light must be minimum.*"

The laws of rectilinear motion, reflection off a plane surface and refraction through transparent media can be easily derived from Fermat's principle as we illustrate below.

1.4.1 Deducing the law of reflection from Fermat's principle

Consider two points A and B in the same medium (Fig. 1.21). A ray of light traveling to a mirror reflects in the direction of B. Where on the mirror the light has to hit so that total time between A and B will be minimum? That is, we wish to find the location of point P such that the path AP+PB corresponds to the least time path between A and B that contains a reflection from the mirror.

Let the fixed distances in the figure be AC = BD = L and CD = h. Since we need to find P, let CP = x, unknown. Then, according to Fermat's least time principle, for fixed A and B, point P will be at a

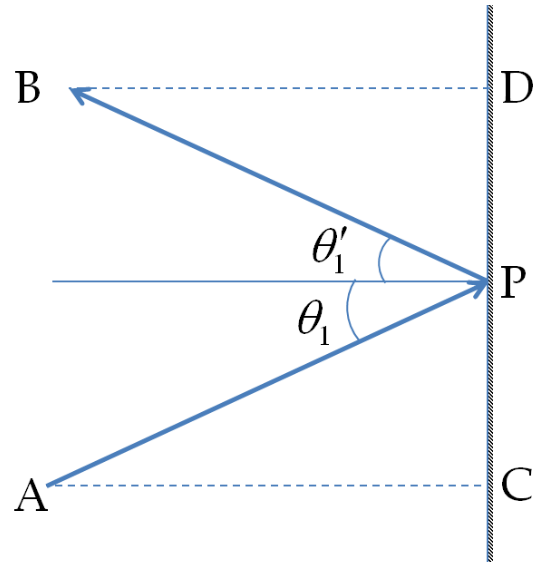

Figure 1.21: Deriving law of reflection using Fermat's principle.

spot such that the time for travel for light of speed v will be smallest. Let time for AP be t_{AP} and time for PB be t_{PB}. Since the rays are in the same medium we will use the same speed for both rays. We can write time t as a function of x.

$$t = t_{AP} + t_{PB} = \frac{AP}{v} + \frac{PB}{v} = \frac{\sqrt{L^2 + x^2}}{v} + \frac{\sqrt{L^2 + (h-x)^2}}{v}. \tag{1.22}$$

To minimize t, we take a derivative of t with respect to the independent variable x and set it to zero.

$$\frac{2x}{v\sqrt{L^2 + x^2}} - \frac{2(h-x)}{v\sqrt{L^2 + (h-x)^2}} = 0, \tag{1.23}$$

which gives

$$\frac{x}{\sqrt{L^2 + x^2}} = \frac{h-x}{\sqrt{L^2 + (h-x)^2}}. \tag{1.24}$$

This relation can be written in terms of the angles θ_1 and θ_1'.

$$\sin\theta_1 = \sin\theta_1' \tag{1.25}$$

Since both angles are less than 90°, we can immediately write down their equality.

$$\theta_1 = \theta_1' \tag{1.26}$$

Therefore, point P has to be such that the angle of incidence will be equal to the angle of reflection.

1.4.2 Law of refraction from Fermat's principle

To deduce the law of refraction based on Fermat's principle, we fix points A and B in the two media, and find a point P at the interface

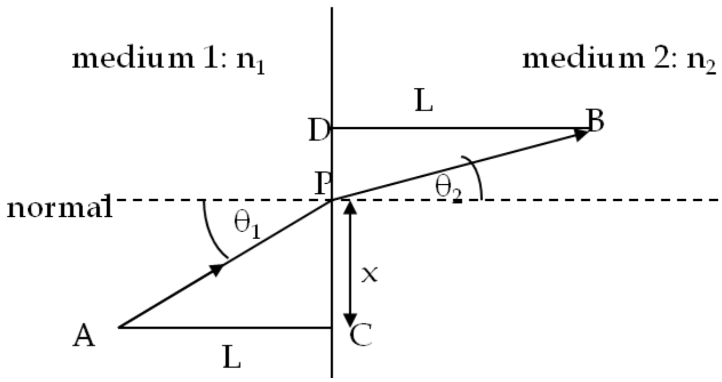

Figure 1.22: Deriving law of refraction using Fermat's principle.

where a ray from point A in medium 1 will refract in the direction of B in medium 2 as illustrated in Fig. 1.22.

Let points A and B be such that AC = BD = L and CD = h. AC and BD are chosen equal for the convenience in calculation. Let point P be at a distance x from C. We need to find point P such that time from A to B is least. Note that light travels with different speeds in the two media.

$$v_1 = \frac{c}{n_1} \tag{1.27}$$

$$v_2 = \frac{c}{n_2} \tag{1.28}$$

where c is the speed of light in vacuum. On path APB we can write the time as a function of x and then minimize this function.

$$t = t_{AP} + t_{PB} = \frac{AP}{v_1} + \frac{PB}{v_2} = \frac{\sqrt{L^2+x^2}}{v_1} + \frac{\sqrt{L^2+(h-x)^2}}{v_2}. \tag{1.29}$$

To minimize t, we take the derivative of t with respect to the independent variable x and set it to zero. This gives the following relation.

$$\frac{1}{v_1}\frac{x}{\sqrt{L^2+x^2}} = \frac{1}{v_2}\frac{h-x}{\sqrt{L^2+(h-x)^2}}. \tag{1.30}$$

Writing the speeds in terms of c and the refractive indices, and replacing the ratios of the distances in the right angled triangles by trigonometric functions we immediately arrive at the Snell's law.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{1.31}$$

The derivations of the laws of reflection and refraction from Fermat's principle illustrates the fundamental importance of Fermat's principle. We can say that Fermat principle "predicts" $\theta_1 = \theta_1'$ and $n_1 \sin\theta_1 = n_2 \sin\theta_2$. A general lesson of Fermat's principle is that the optimization of a physical quantity is at work by nature. The search of optimization of other physical quantities has led to other discoveries in physics.