1.6 Some Consequences of Lorentz Transformations

Lorentz transformations can be applied to study the relativity of physical quantities such as length, time, etc. However, it is a little tricky to use them correctly since some of our ordinary notions of time and space have to be carefully revised. There are basically two main departures in thinking compared to Newtonian thinking that we must pay particular attention to.

First, there is no “universal time” since each frame has its own time. Times in different inertial frames with non-zero relative speed will be different and run at different speeds. For instance, simultaneous events in one frame will not be simultaneous in another, and even the time-order of events can be reversed.

Second, the clocks at different locations require synchronization by light signals which gives meaning to the “observed time” in one particular reference frame. To synchronize clocks at different locations you would need one observer at each space point. That is, the notion of an observer in a frame is actually replaced by infinitely many observers, one at each space point of the stationary space. That makes simple questions about observers confusing.

For instance, consider the following problem, “An observer sends a light pulse from the origin towards the positive x-axis. Where will he/she find the pulse at time t?” You may be tempted to answer that he/she will find the pulse at point P whose x-coordinate is equal to ct. But, how can he/she verify that the pulse is actually at P at time t? He/she will need to be at point P ahead of the arrival of the pulse so that he/she can make the measurement about the arrival of the pulse. This is impossible since that would require the observer to travel faster than the speed of light.

Thus, instead of one observer in a frame, we will say that there are infinitely many observers, for instance, one at P also. The clock of the observer at P is synchronized with the observer at the origin, so they have same time readings. The observer at P will observe the arrival of the pulse at time t. The event of the arrival in this frame will be $(t, ct, 0, 0)$. This observer can then relay this information to the observer at the origin by sending another pulse (e.g. by reflecting the pulse) to the observer at the origin. The observer at the origin will find the return pulse at time $2t$ from which he/she can conclude that the pulse must have gone to P located a distance ct for the origin and must have arrived at P at time t. This circuitous logic is necessary if we are keep faith with a definition of time based on observable properties.

1.6.1 Simultaneity of Events

Simultaneous events refer to two or more events happening at the same time, say two lightning occurring at the same time or two trains arriving at the station at the same time, etc. With the implicit assumption of a "universal time" in the Newtonian thinking, two events that were simultaneous in one frame would be automatically simultaneous in all frames. That is, simultaneity was an *absolute* concept. Now, since we do not have the notion of a "universal time" we need to examine whether two events that are simultaneous in one frame are also simultaneous in other frames. Our analysis will show that simultaneity is not absolute but a relative concept.

Let us use the same two frames S and S' as we have used to deduce the Lorentz transformation equations above. Consider two events E_1 and E_2, whose coordinates and times in the the coordinate system for the S' frame will be denoted by (t'_1, x'_1, y'_1, z'_1) and (t'_2, x'_2, y'_2, z'_2) respectively, and without the primes on the symbols in the S frame. Let us take a concrete example of two simultaneous events E_1 and E_2 as observed in the S' frame that occur at $t' = 0$ at $x'_1 = 0$ and $x'_1 = a$ respectively.

$$\left.\begin{array}{ll} E_1: & t'_1 = 0,\ x'_1 = 0,\ y'_1 = 0,\ z'_1 = 0 \\ E_2: & t'_2 = 0,\ x'_2 = a,\ y'_2 = 0,\ z'_2 = 0 \end{array}\right\} \tag{1.48}$$

Since $t'_1 = t'_2$ the two events are simultaneous in S' frame. Now, we ask if $t_1 = t_2$? According to the Lorentz transformations worked out in the last section, the coordinates and times for these events in the S frame would be

$$\left.\begin{array}{ll} E_1: & t_1 = 0,\ x_1 = 0,\ y_1 = 0,\ z_1 = 0 \\ E_2: & t_2 = \gamma \dfrac{V}{c^2}\, a,\ x_2 = \gamma\, a,\ y_2 = 0,\ z_2 = 0 \end{array}\right\} \tag{1.49}$$

Since times $t_1 \neq t_2$, the two events will not be simultaneous in S frame even though they are simultaneous in the $S6\prime$ frame! As a matter of fact, for $a > 0$, the second event will occur later than the first event, and for $a < 0$, i.e. an event occurring on the negative x'-axis, the second event will occur earlier than the first event.

Note: some authors state this result as "the second event will *appear* to be earlier (or later) than the first event", which is somewhat misleading since it will *indeed* be earlier than the first event, not just appear to be so. Special theory of relativity embodied in the Lorentz transformations clearly leads to an unmistakable conclusion that simultaneity is not an absolute concept. Simultaneous events in one frame will not be simultaneous in another frame.

1.6.2 Order of Events

If simultaneity is a relative concept, then how about the order of events? If one event occurs later than the other in one frame, could the time order of events be opposite in some other frame? Let us see what the Lorentz transformations tell us for two events that are not simultaneous in the S' frame.

1. **Events at the origin of one system**

For simplicity let us consider two events that occur at the origin of S' frame at times $t' = 0$ and $t' = \tau$.

$$\left.\begin{array}{ll} E_1: & t_1' = 0,\ x_1' = 0,\ y_1' = 0,\ z_1' = 0 \\ E_2: & t_2' = \tau,\ x_2' = 0,\ y_2' = 0,\ z_2' = 0 \end{array}\right\} \tag{1.50}$$

According to the Lorentz transformations, the coordinates and times for these events in the frame S would be

$$\left.\begin{array}{ll} E_1: & t_1 = 0,\ x_1 = 0,\ y_1 = 0,\ z_1 = 0 \\ E_2: & t_2 = \gamma\,\tau,\ x_2 = \gamma\,V\tau,\ y_2 = 0,\ z_2 = 0 \end{array}\right\} \tag{1.51}$$

Thus, if $\tau > 0$, then both $t_2' > 0$ and $t_2 > 0$, and if $\tau < 0$, then both $t_2' < 0$ and $t_2 < 0$. The order of events is same in the two frames. You might say, this is particular to the events at the origin.

2. **Events at the same location of one system**

Let us check out two events at a place that is not at the origin of the coordinate system, but let them again occur at the same space point, say $x' = a$.

$$\left.\begin{array}{ll} E_3: & t_3' = 0,\ x_3' = a,\ y_3' = 0,\ z_3' = 0 \\ E_4: & t_4' = \tau,\ x_4' = a,\ y_4' = 0,\ z_4' = 0 \end{array}\right\} \tag{1.52}$$

Now the Lorentz transformations give

$$\left.\begin{array}{ll} E_3: & t_3 = \gamma\,\dfrac{V}{c^2}\,a,\ x_3 = \gamma a,\ y_3 = 0,\ z_3 = 0 \\ E_4: & t_4 = \gamma\left(\tau + \dfrac{V}{c^2}\,a\right),\ x_4 = \gamma\,(a + V\tau)\,,\ y_4 = 0,\ z_4 = 0 \end{array}\right\} \tag{1.53}$$

Again, we find that if $t_4' - t_3' = \tau > 0$ then $t_4 - t_3 = \gamma\tau > 0$ and if $t_4' - t_3' = \tau < 0$ then $t_4 - t_3 = \gamma\tau < 0$. The order of events in not changed. You might now say that, this and the example at the origin are particular to the events at one location.

3. **Events at the two locations of one system**

Let us check out two events at two places, say one event at $t' = 0$ at $x' = 0$ and the later event at $t' = \tau$ at $x' = a$.

$$\left.\begin{array}{ll} E_5: & t_5' = 0,\ x_5' = 0,\ y_5' = 0,\ z_5' = 0 \\ E_6: & t_6' = \tau,\ x_6' = a,\ y_6' = 0,\ z_6' = 0 \end{array}\right\} \tag{1.54}$$

Now the Lorentz transformations give

$$\left.\begin{array}{ll} E_5: & t_5 = 0,\ x_5 = 0,\ y_5 = 0,\ z_5 = 0 \\ E_6: & t_6 = \gamma\left(\tau + \dfrac{V}{c^2}\,a\right),\ x_6 = \gamma\,(a + V\tau)\,,\ y_6 = 0,\ z_6 = 0 \end{array}\right\} \tag{1.55}$$

This will give

$$t_6 - t_5 = \gamma\left(\tau + \frac{V}{c^2}\,a\right). \tag{1.56}$$

We see that for $t'_6 - t'_5 = \tau > 0$, the order of events will depend on the relative velocity V of the frames.

$$\left.\begin{array}{llll}\text{If } V > -\dfrac{c^2\tau}{a}\,a, & \text{then } t_6 - t_5 > 0, & \text{Order of events same} \\ \text{If } V < -\dfrac{c^2\tau}{a}\,a, & \text{then } t_6 - t_5 < 0, & \text{Order of events opposite} \\ \text{If } V = -\dfrac{c^2\tau}{a}\,a, & \text{then } t_6 - t_5 = 0, & \text{Events simultaneous in S.}\end{array}\right\} \tag{1.57}$$

Three examples in this subsection illustrate that the order of events in two frames may or may not be same which depends on the particulars of the two events in question.

1.6.3 Relativity of Uniformly Moving Clocks

The clock at the origin (or for that matter any other location) of the S' system is moving with respect to the S system. Let $\Delta t'$ be a time interval observed by an observer at rest with the clock in the S' system. That is, we are looking at the time intervals between two events which are given by $(t'_1, 0, 0, 0)$ and $(t'_2, 0, 0, 0)$ with $\Delta t' = t'_2 - t'_1$.

We would like to know what will be the time interval recorded by an observer in the S system. The S' frame in this context is referred to as the **rest frame** of the clock since the system we are examining, here, the clock fixed to the origin of S' frame, is at rest in the S' frame. The time in the rest frame is also called the **proper time**. The other frame (the S frame) is then called the **lab frame**. The system (the clock fixed to the origin of S' frame) is moving with respect to the lab frame. Let Δt be the time interval in the lab frame.

To examine what S frame will record when S' frame records $\Delta t'$ in the clock, we look at the events $E_1 : (t' = 0, x' = 0, y' = 0, z' = 0)$ and $E_2 : (t' = \Delta t', x' = 0, y' = 0, z' = 0)$ and ask the times and coordinates of these events in the S frame. We have already worked out the events in the S frame corresponding to the two events in the last subsection. From Eq. 1.51 the duration Δt in the lab frame (the S frame) will be

$$\Delta t = \gamma \Delta t' = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}}. \tag{1.58}$$

Since for $V \neq 0$, $\gamma > 1$, the time duration in the lab frame will always be greater than the time duration in the rest frame, $\Delta t > \Delta t'$. This means that a moving clock runs slower with respect to the clock of the frame in which the clock is moving.

If you examine a clock fixed to the S frame, say the clock at the origin of the S frame. That clock will be moving with respect to the S' frame. Now, we compare the time intervals for two intervals which occur at the origin of the S frame, $(t_1, 0, 0, 0)$ and $((t_2, 0, 0, 0)$. The corresponding events in the S' frame will be $(\gamma t_1, -\gamma\, V t_1, 0, 0)$ and $(\gamma t_2, -\gamma\, V t_2, 0, 0)$ respectively. Let us denote the time interval between the two events in the S frame by $\tilde{\Delta t}$ and in the S' frame by $\tilde{\Delta t}'$. Here the S frame is the rest frame and the S' frame is the lab frame. We get the same relation as in Eq. 1.58.

$$\tilde{\Delta t}' = \gamma \tilde{\Delta t} = \frac{\tilde{\Delta t}}{\sqrt{1 - V^2/c^2}}. \tag{1.59}$$

It is perhaps better to write the time-dilation relations using the rest frame and lab frame language rather than S and S'.

$$\boxed{\Delta t_{\text{lab}} = \gamma \Delta t_{\text{rest}} = \frac{\Delta t_{\text{rest}}}{\sqrt{1 - V^2/c^2}}.} \tag{1.60}$$

The confusion arises here from the fact that the “rest frame” is the frame which is moving with respect to the lab frame. The reason for calling the “moving frame” the rest frame has to do with the fact that *the clock under study is at rest in this frame.*

Example 1.1. Neutron lifetime. A free neutron at rest in the laboratory decays into a proton, an electron, and an anti-neutrino in approximately 15 min. What will be the lifetime of a neutron that moves at speed that is 99% of the speed of light?

Solution. Consider a frame moving with the neutron. Then, this frame will be the rest frame. In this frame neutron will take 15 min to decay.

$$\Delta t_{\text{rest}} = 15\,\text{min}.$$

We wish to determine the corresponding time elapsed in the laboratory frame. Here γ has the following value.

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - 0.99^2}} = 7.1.$$

Therefore, the lifetime of the neutron in the lab frame will be

$$\Delta t_{\text{lab}} = \gamma \Delta t_{\text{rest}} = 7.1 \times 15\,\text{min} = 107\,\text{min}.$$

Example 1.2. Travel time. An airplane takes 5 hours to go from Boston to San Francisco, a distance of 3,000 miles. What will be the total flight time in the cockpit if the plane’s velocity was constant?

Solution. The clock in the cockpit is the clock in the rest frame of the plane. The ground-based frame is thelab frame and we are given $\Delta t_{\rm lab} = 5$ h. We also have the relative speed of the two frames.

$$V = \frac{3,000\ \text{mi}}{5\ \text{h}} = 600\ \text{mi/h} = 268\ \text{m/s}.$$

This is a small fraction of the speed of light.

$$\frac{V}{c} = \frac{268\ \text{m/s}}{3\times 10^8\ \text{m/s}} = 8.93\times 10^{-7}.$$

Therefore, γ will be very close to 1.

$$\gamma = \frac{1}{\sqrt{1-V^2/c^2}} \approx 1 + \frac{1}{2}V^2/c^2 = 1 + 3.99\times 10^{-13}.$$

Let us represent the small number by the suymbol ϵ.

$$\gamma = 1+\epsilon, \quad \epsilon = 3.99\times 10^{-13}.$$

Therefore, the flight time in the rest frame will be

$$\Delta t_{\rm rest} = \frac{\Delta t_{\rm lab}}{\gamma} = \frac{\Delta t_{\rm lab}}{1+\epsilon} \approx \Delta t_{\rm lab}(1-\epsilon) = 5\ \text{hr} - 5\times 3.99\times 10^{-13}\ \text{hr}.$$

The difference is

$$\Delta t_{\rm lab} - \Delta t_{\rm rest} = 5\times 3.99\times 10^{-13}\ \text{hr} = 7.2\ \text{ns}.$$

The difference is too small a fraction of the total time to be observable.

1.6.4 The Twin Paradox

The time dilation formula Eq. 1.60 makes a distinction between the duration in one frame and the duration in the other frame. We have computed the formula by setting the S' frame as the rest frame and S frame the lab frame. Suppose we looked at the clock fixed at the origin of the S frame, then S frame will be the rest frame and S' frame the lab frame. That is if we look at the passage of time in the clock at rest in S frame, it will go slower than the clocks in the S' frame. So, clock in which frame will go slower S or S'?

The problem can be stated more dramatically by the following hypothetical situation about two identical twins Alice and Barbara. Suppose Alice is in a space ship and Barbara be Earth-bound. Suppose Alice takes off from Earth is moving in a straight line with a constant speed V with respect to Barbara. Let us call the time in Alice's clock be the A-time, t_A and that in Barbara's clock to be the B-time, t_B. As far as Barbara is concerned the time elapsed in Alice's clock will be less than the time elapsed in her clock, since her clock is in the lab frame and Alice's clock in the rest frame.

$$\Delta t_B = \gamma t_A. \tag{1.61}$$

However, the situation is different from Alice's frame. When she examines the clock carried by Barbara, she is in the lab frame and Barbara is in the rest frame. Thus, she will record that the time in her clock is more than the time elapsed in Barbara's clock.

$$\Delta t_A = \gamma t_B. \tag{1.62}$$

Suppose after some time they get together. According to Barbara, Alice will be younger, and according to Alice, Barbara should be younger. Both possibilities cannot be right. So, which is the correct possibility? This the paradox. This paradox is called the Twin Paradox.

The source of the problem is the fact that we have neglected to notice that if Alice has a constant velocity with respect to Barbara, then once Alice has left the Earth she cannot come back to Earth and there is no possibility of an event where Alice's clock and Barbara's clock are at the same position so that their times could be compared. For Alice to come back to Earth she must have deceleration and acceleration. Unlike uniform motion, the decelerating and accelerating frames can detect their own deceleration and acceleration. The period of acceleration and deceleration of Alice's ship breaks the symmetry between the frames, since there is no acceleration or the deceleration of Barbara's frame. Thus, both would agree that Alice's clock is in the moving frame and therefore should run slowly giving us a unique answer of the riddle: Alice would have aged less than Barbara!

1.6.5 Relativity of Length

How do we measure length of a rod? You might say that the length is the distance between the two ends of the rod. Simple enough! However, to deduce the distance between the two ends you need to *locate the positions of the two ends in space at the same time.*

When the rod is stationary in a frame, this process is very simple: You just lay the rod along a well-marked axis and read off the coordinates at the two ends; the absolute value of the difference of the coordinates is the length. What would happen if the rod is moving? *We still need to locate the positions of the two ends in space at the same time.* We will need to look at the two events at the ends of the the rod that occur at the same time in that frame and use Lorentz transformation to deduce their relation.

We have used the frame S' to be the rest frame and frame S as the lab frame. Let us continue to use the same notation for rest and lab frames here. For simplicity, we place the rod in the S' frame with one end at the origin of the S' frame and the other end at a distance l_0 on the x'-axis of that frame since the rod is at rest in this frame. The two ends of the rod at rest in the S' frame correspond to two events in this frame given by the following times and space coordinates.

$$\text{Frame } S': \quad \begin{cases} E_1: & t_1' = 0,\ x_1' = 0,\ y_1' = 0,\ z_1' = 0 \\ E_2: & t_2' = 0,\ x_2' = l_0,\ y_2' = 0,\ z_2' = 0 \end{cases} \tag{1.63}$$

These two events are simultaneous in the S' frame and will not be simultaneous in frame S. To find the length of this rod in the S frame, which is the lab frame here, we need to find where the two ends of the rod will be at the same value of t. That is we seek two events E_3 and E_4 which happen at the same value of t in the S frame. Let us find the two events corresponding to $t = 0$. The left end of the rod is at origin. So, E_3 is same as E_1.

$$E_3: \quad t_3 = 0,\ x_3 = 0,\ y_3 = 0,\ z_3 = 0. \tag{1.64}$$

For the other event we have $t_2 = 0$ and $x_2' = l_0$ and we seek x_2. We will describe this event by giving you the information from two different frames.

$$E_4: \quad t_4 = 0,\ x_4' = l_0,\ y_4 = 0,\ z_4 = 0. \tag{1.65}$$

We do not yet know t_4' or x_4. Let us write down the relevant Lorentz transformation to deduce them.

$$\left.\begin{aligned} t_4' &= -\gamma \frac{V}{c^2} x_4 \\ x_4' &= l_0 = \gamma x_4 \end{aligned}\right\} \tag{1.66}$$

From $x_3 = 0$ and $x_4 = l_0/\gamma$ we find that the length $l = |x_4 - x_3|$ in the lab frame will be

$$l = \frac{l_0}{\gamma} = l_0\sqrt{1 - \frac{V^2}{c^2}}. \tag{1.67}$$

This says that the length l of the bar moving in the direction of the length will be less than the length l_0 when it is not moving. that is length is contracted. This effect is called **Lorentz contraction**. Let us again rewrite this equation in terms of the rest frame and lab frame language, where the rest frame refers to the frame in which the bar is at rest and the lab frame refers to the frame in which the bar is moving in the direction of the length of the bar with speed V.

$$\boxed{l_{\text{lab}} = \frac{l_{\text{rest}}}{\gamma} = l_{\text{rest}}\sqrt{1 - \frac{V^2}{c^2}}.} \tag{1.68}$$

Since y and z coordinates do not change, the transverse dimensions, i.e., the dimensions of the rod perpendicular to the direction of the motion of the rod are unaffected. Let us denote these dimensions with a subscript $\perp$.

$$\boxed{l_\perp = l_{0\perp}.} \tag{1.69}$$

Example 1.3. Size of a spaceship. A spaceship is seen to move past an observer on Earth at a speed of $0.95c$, i.e. at 95% of the speed of light in the direction parallel to the length of the ship. An astronaut in the spaceship has measured the length of the spaceship to be 100 m. What will be the length of the spaceship as observed by the observer on Earth?

Solution. The length 100 m for the spaceship is the length in the rest frame of the ship. The length observed by the Earth-based observer will be in the lab frame. From our discussion above the lab frame value will be contracted compared to the rest frame value by a factor of γ. Here γ has the following value.

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - 0.95^2}} = 3.2.$$

Therefore, the length observed by the observer on Earth will be

$$L = \frac{L_0}{\gamma} = \frac{100 \text{ m}}{3.2} = 31.3 \text{ m}.$$

Example 1.4. Mystery of muons. Muons are created when cosmic rays made up of protons and other highly energetic particles strike the upper atmosphere approximately 15 km from the surface of Earth. In laboratory experiments it is found that a muon at rest decays into two photons in $\tau = 2.2\,\mu$s. Suppose a muon created at the upper atmosphere has a speed of $0.9999c$. If we use Newtonian mechanics, we will conclude that within a lifetime, the muon will go a distance of $0.9999c\tau$, which would be 660 m. Therefore, we would not expect any muon to make it to the surface of Earth. However, we do observe a considerable fraction of muons make it to the surface of Earth. What is the correct explanation?

Solution. This mystery can be understood if we examine the phenomenon of muon decay by special relativity. We consider two frames: the rest frame in which muon is at rest and the laboratory frame in which Earth is at rest. In the lab frame muon is moving towards Earth and in the rest frame Earth is moving towards the muon. The mystery of muon traveling all the way to Earth can be understood from the perspectives of the two frames, which are

Perspective of Lab frame: Muon is traveling towards Earth at speed $0.9999c$ and lives long enough to travel a distance of 15 km. In this perspective, the muon lifetime is dilated and lives longer than the lifetime of 2.2 μs in the rest frame.

Perspective of muon rest frame: Earth is traveling towards muon at speed $0.9999\,c$ and in 2.2 μs the surface of the Earth to travels through the distance of thickness of the atmosphere. In this perspective, the thickness of the atmosphere is contracted and is less than 15 km, making it possible for the surface of Earth to travel a distance of the thickness of the atmosphere.

Now, let us work out the details of the two perspectives.

Lab Frame:

The lifetime in the lab frame will be dilated with respect to the time in the rest frame by a factor of γ, which is

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - 0.9999^2}} = 70.7.$$

Therefore, the lifetime in the lab frame will be

$$\Delta t_{\text{lab}} = \gamma \Delta t_{\text{rest}} = 70.7 \times 2.2\ \mu\text{s} = 156\ \mu\text{s}.$$

Now, in this frame the speed of muon is $0.9999c$. Therefore, the muon will travel a distance,

$$\Delta x = 0.9999\, c \times \Delta t_{\text{lab}} = 46,795\ \text{m} > 15\ \text{km}.$$

Since $\Delta x > 15$ km this muon will make it to the surface of Earth.

Muon Rest Frame:

Let L_0 = thickness of atmosphere in the Earth frame. We have been given $L_0 = 15$ km. Now, from the perspective of the muon rest frame the length L_0 will be contracted by a factor of γ. Therefore, the thickness of the atmosphere will be less in the muon rest frame.

$$L = \frac{L_0}{\gamma} = \frac{15,000\ \text{m}}{70.7} = 212\ \text{m}.$$

Now, during the interval of 2.2 μs, the surface of Earth will move 660 m, which is more than the thickness of 220 m. Therefore, from the perspective of the muon rest frame, the muon will make it to the surface of Earth. Note that the two frames differ in the reason behind the fact that muon will arrive at the surface of Earth, but they agree on the fundamental physical observation that a muon created at 15 km above the surface of Earth and moving towards Earth at speed $0.9999c$ will make it to the surface of Earth.