7.8 SYSTEMS WITH TIME VARYING MASS

In many situations of practical interest we find that the mass of the body of interest changes with time. For instance, in the motion of a rocket, the mass of the rocket decreases as burnt fuel is ejected from the rocket. Although the total mass of the rocket and the ejected fuel is constant in this situation, we are normally interested in describing the motion of the rocket only. Similarly, when rain drops fall, they gather more water molecules thus increasing their mass with time. These systems are called syterms with **time-varying mass** or **open systems**.

These problems are best handled by using the complete form of Newton's second law in terms of the change of momentum rather the constant-mas formulation of Newton's second law. In this section, we will show how to adapt Newton's second law for systems with varying mass.

To see the special problem that arises in systems with time-varying mass substitute $\vec{p} = m\vec{v}$ in the full Newton's second law and carry out the derivative with respect to time to obtain the following.

$$\vec{F} = \left(\frac{dm}{dt}\right)\vec{v} + m\vec{a}. \tag{7.67}$$

Let us rearrange the terms in this equation so that we keep $m\vec{a}$ on the one side of the equation and other terms on the other side

$$\boxed{m\vec{a} = \vec{F} - \left(\frac{dm}{dt}\right)\vec{v}.} \tag{7.68}$$

This says that a system whose mass varies with time can have a non-zero acceleration even when there is no external force ($\vec{F} = 0$). The inertial term, $[-\left(\frac{dm}{dt}\right)\vec{v}]$, acts as a force on the system as far as acceleration is concerned. This force is called **thrust** when these equations are applied to the motion of rockets, where it is the force applied on the main body of the rocket by the exiting fuel.

To further elaborate the use of Newton's complete expression for the second law of motion to varying mass systems, we now apply these considerations to the motion of rockets. One complicating and sometimes confusing feature of the calculations for a rocket motion is that the velocity of the ejected fuel is commonly given with respect to the body of the rocket which is not an inertial frame. Since Newton's second law is usually written in an inertial frame, we will need to convert all given data in terms of an inertial frame.

The general strategy for an analysis of a time-varying system involves the following standard steps.

1. Pick an inertial coordinate system.
2. Write down the momenta of all masses at two nearby instants: an arbitrary time t and an infinitesimally instant later $t + dt$. This will be momenta of rocket of mass m at time t and the rocket of mass $m - dm$ and fuel of mass dm at time $t + dt$.
3. Equate rate change in momentum to average force.

7.8.1 Rocket Motion With no External Force

Consider a rocket in outer space with total mass M kg at $t = 0$. The rocket ejects burnt fuel at a constant rate of α kg/s from its rear at a constant speed u m/s with respect to the rocket. We wish to find the speed of the rocket at an arbitrary time t sec before the rocket runs out of fuel.

To deuce the equation of motion of the rocket that has no external force, we compare the momenta of the rocket and the burnt fuel at time t and $t + \Delta t$. To help us do this calculation, we make use of Figure 7.25.

Let m be the mass of the rocket plus remaining fuel at time t, and let v_y be the y-component of its velocity with respect to the inertial frame. The x and z-components of all quantities are zero. Let $|\Delta m|$ be the mass of the ejected burnt fuel between time t and $t+\Delta t$, so that at time $t+\Delta t$, the y-component of the velocity of the rocket is $v_y + \Delta v_y$ and the y-component of the velocity of the burnt fuel is $-u$ with respect to the rocket. Therefore, the y-component of the velocity of the burnt fuel with respect to the inertial observer will be $v_y - u$ by simple vector addition rules. Now, we summarize the y-components of momenta of rockets and fuels at the two nearby instants with respect to the inertial observer are:

$$\text{At time } t: \quad p_y(t) = m v_y \tag{7.69}$$

At time $t + \Delta t$:

$$p_y(t+\Delta t) = (m - |\Delta m|)(v_y + \Delta v_y) + |\Delta m|(v_y - u) \tag{7.70}$$

Here I have put absolute sign around the change in mass so that we do not need to worry about the negative sign at this point. Since there is no external force on the combined system of the rocket and

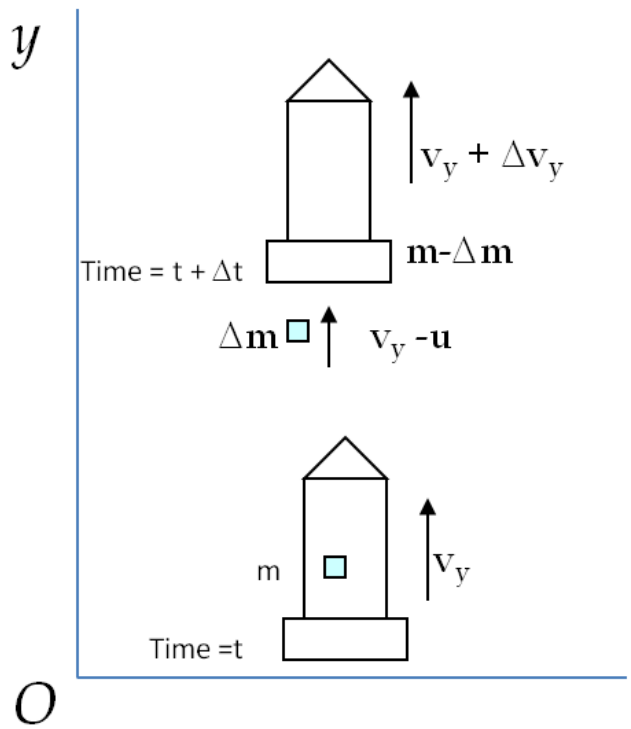

Figure 7.25: Rocket at two instants in time. The burnt fuel of mass Δm which is inside the rocket at time t is ejected from the back of the rocket at speed u with respect to the rocket. The positive y-axis of an inertial frame is pointed towards the forward direction of the rocket. In this frame, the y-component of the velocity of the ejected fuel is $v_y - u$ where v_y is the y-component of the velocity of the rocket at tine t.

the ejected fuel, the total momentum is conserved. Therefore, we equate the momentum of the system at the two instants.

$$mv_y = (m - |\Delta m|)(v_y + \Delta v_y) + |\Delta m|\ (v_y - u), \tag{7.71}$$

which can be simplified to

$$m\Delta v_y - |\Delta m|\Delta v_y - |\Delta m|\ u = 0. \tag{7.72}$$

We need to take the limit of this equation as the duration Δt becomes small. In this limit, we keep only the dominant terms. To see the dominant term, it is best to compare dimensionless quantities. Therefore, let us rearrange the first two terms and combine them to get

$$m\Delta v_y \left(1 - \frac{|\Delta m|}{m}\right) - |\Delta m|\ u = 0. \tag{7.73}$$

For most of the operation of a rocket, the ratio $|\Delta m|/m << 1$, therefore, we can ignore the second term within the parenthesis. Now, taking the infinitesimal limit, the change in y-component of velocity Δv_y is replaced by dv_y and the change in mass $|\Delta m|$ is replaced by $-dm$ since dm is negative.

Using the infinitesimal notation we write this equation as follows.

$$\boxed{m\ dv_y + u\ dm = 0.} \tag{7.74}$$

This equation is called the **rocket equation**. Here, the mass m and the y-components of velocity v_y are functions of time.

To predict the velocity of the rocket at an arbitrary time we divide Eq. 7.74 by m, which separates the equation into two terms, one has only v_y and the other has the constant u and the variable m. This type of equation can be integrated. The v_y integration has limits in v_y and m integration has limits in m. Let M be the mass and v_{0y} be the y-component of the velocity at time $t = 0$, then we obtain

$$\int_{v_{0y}}^{v_y(t)} dv_y + u \int_M^{m(t)} \frac{1}{m} dm = 0, \tag{7.75}$$

which gives

$$v_y(t) - v_{0y} = u \ln\left(\frac{M}{m(t)}\right). \tag{7.76}$$

The ratio the mass at some time t and the initial mass of the rocket is usually written in terms of a quantity, called the **burn rate**, which is defined as

$$\boxed{\text{Burn rate:}\quad \alpha = -\frac{dm}{dt}.} \tag{7.77}$$

There are two types of burn rates of particular practical and theoretical interest.

1. Constant burn rate

If the rate of fuel burn is constant, i.e. if α is constant, Eq. 7.77 can be integrated to yield the following result for the mass of the rocket at time t if it was M at time $t = 0$.

$$m(t) = M - \alpha t. \tag{7.78}$$

Note that $m(t)$ cannot be negative. If the body of the rocket has mass M_R, then the rocket will run out of fuel at time $t = t_f$ given by

$$M_R = M - \alpha t_f \quad \Longrightarrow \quad t_f = \frac{M - M_R}{\alpha} \tag{7.79}$$

When we substitute Eq. 7.78 in Eq. 7.76 we find the y-component of velocity at an arbitrary time to be

$$v_y(t) - v_{y0} = u \ln\left(\frac{M}{M - \alpha t}\right), \quad (0 \le t \le t_f), \tag{7.80}$$

which says that the motion of the rocket is not a constant acceleration motion since velocity does not change linearly in time. Figure 7.26

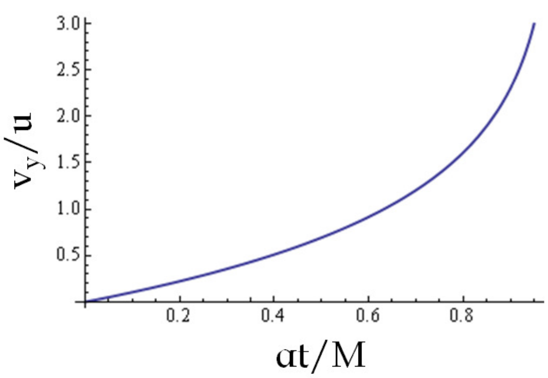

Figure 7.26: Velocity of the rocket as a function of time for constant burn rate case.

shows a plot of the rocket velocity as a function of time. Initially the velocity of the rocket increases quite linearly but when the rocket has less fuel on board the velocity of rocket rises sharply.

2. Variable burn rate

If the burn rate is proportional to the mass of the rocket, say $\alpha = \beta m$, then

$$\frac{dm}{dt} = -\beta m. \tag{7.81}$$

This equation leads to

$$m(t) = M \exp\left(-\beta t\right). \tag{7.82}$$

The final time when the rocket runs out of fuel will be $t = t'_f$. At that time, the mass of the rocket will be only the mass M_R of the rocket body.

$$M_R = M \exp\left(-\beta t'_f\right). \tag{7.83}$$

For this type of burn rate, the mass of the rocket drops exponentially. By substituting Eq. 7.82 into Eq. 7.76 you can easily show that the rocket moves at a constant acceleration .

Figure 7.27: The launch of the Zvezda service module of the International Space Station on a Russian Proton-K rocket. Photcredit: NASA/Wikicommons.

$$v_y(t) - v_{y0} = (u\beta)\ t, \quad 0 \leq \leq t'_f \tag{7.84}$$

which shows the y-component of the acceleration is a constant, $a_y = u\beta$.

7.8.2 Rocket Motion With Constant External Force

Rocket motion near the surface of the Earth surface is subject to approximately constant gravitational force of the Earth. In this subsection we examine the vertical component of the motion of the rocket near the Earth, for instance, when a rocket is launched as shown in Fig. 7.27. Let the fuel be ejected at a constant speed u with respect to the rocket. We set up the problem in the same way as we did for the rocket in space, and add the force of gravity to the equation to obtain the following equation of motion of the y-coordinate of the rocket.

$$\boxed{m\ dv_y + u\ dm = -mgdt.} \tag{7.85}$$

Dividing this equation by m and combining dv_y and dt terms we obtain a simpler form.

$$d\left(v_y + gt\right) + \frac{u}{m}\ dm = 0. \tag{7.86}$$

Let us write

$$w \equiv v_y + gt, \tag{7.87}$$

so that Eq. 7.86 looks simpler.

$$dw + \frac{u}{m}\ dm = 0. \tag{7.88}$$

We integrate this equation from $t = 0$ to $t = t$ to obtain

$$w(t) - w(0) = u \ln\left(\frac{M}{m(t)}\right), \tag{7.89}$$

where M is the mass of the rocket at $t = 0$. Putting the velocity back in we get

$$v_y(t) - v_{y0} = -gt + u \ln\left(\frac{M}{m(t)}\right). \tag{7.90}$$

For a constant burn rate α, we replace $m(t)$ by $M - \alpha t$ to obtain the following for the y-component of the velocity.

$$v_y(t) - v_{y0} = -gt + u \ln\left(\frac{M}{M - \alpha t}\right), \quad 0 \le t \le t_f, \tag{7.91}$$

where $t_f = (M - M_R)/\alpha$ is the time when the rocket runs out of fuel and left with the rocket body only with mass M_R.