11.2 DAMPING IN CIRCUITS WITH RESISTANCE

There is no resistance in the ideal LC circuit studied in the last section. In reality, however, there is always some resistance in the circuit, for instance in the connecting wires and the materials of the inductor and the capacitor. In that case, the energy is dissipated in every cycle through heat. This results in the decrease in the current over time. Oscillatory circuits also radiate electromagnetic waves and lose energy through radiation. We will not discuss the radiation of energy now, and assume that the loss by radiation is negligible compared to the loss due to resistance.

To understand the phenomenon of resistive damping better consider a circuit with a resistor R in series with a capacitor C and an inductor L as shown in Fig. 11.4. Let the capacitor be charged

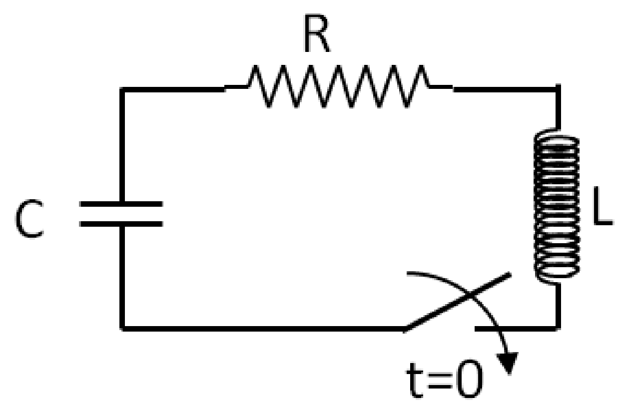


Figure 11.4: A series RLC circuit. Initially the capacitor is charged when the switch is closed at $t = 0$. The current in the circuit flows from one plate to the other till the charges on the plate are neutralized and then the charges continue to flow resulting in reversal of the polarity of the capacitor. The plates are charged again and then the current flows in the opposite direction leading to the plated becoming charged in the same way as before. AT the end of each cycle the net charge on the capacitor is less than in the previous cycle.

initially with charges $\pm Q_0$ on the plates so that there is a voltage $V_0 = Q_0/C$ at time $t = 0$ between the capacitor plates. When the switch is closed at $t = 0$, the current in the circuit will rise from zero to a maximum value. Once again, we can deduce the equation of motion relating the charge $q(t)$ on one of the plates and current $I(t)$ flowing into that plate as done above. We follow the same convention as used above. Performing the Faraday loop integral in the direction of the electric field yields

$$\frac{q}{C} + RI = -L\frac{dI}{dt}, \tag{11.19}$$

with

$$I = \frac{dq}{dt}.$$

Replacing I by the derivative of charge, we arrive at the following differential equation for $q(t)$.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0. \tag{11.20}$$

Experimentally it is far easier to measure voltages than charges on plates. We can deduce the equation of motion for the voltage $V_C(t)$ across the capacitor by using the capacitor relation, $q = CV_C$.

$$L\frac{d^2V_C}{dt^2} + R\frac{dV_C}{dt} + \frac{V_C}{C} = 0. \tag{11.21}$$

In order to determine the time evolution of V_C we need to solve this equation subject to some initial conditions, which are usually $V_C(0)$ and $I(0)$. Dividing both sides by LC leads to the reduction of parameters from three, namely, R, L and C to two, R/L and $1/LC$.

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\,\frac{dV_C}{dt} + \frac{V_C}{LC} = 0. \tag{11.22}$$

Once again, we find the analogy with a mechanical system helpful, this time with the viscously damped harmonic oscillator. In analogy with the mechanical system, we introduce two composite parameters β and ω_0. The quantity ω_0 is called the **natural frequency** of oscillation and β the **damping parameter**.

$$\beta \equiv \frac{R}{2L}, \tag{11.23}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{11.24}$$

I have introduced a factor of $\frac{1}{2}$ in the definition of β to get rid of some factors of 2 that would appear in the future equations. Let us rewrite Eq. 11.22 using β and ω_0.

$$\frac{d^2V_C}{dt^2} + 2\,\beta\,\frac{dV_C}{dt} + \omega_0^2\,V_C = 0. \tag{11.25}$$

This is an equation of motion for a damped harmonic oscillator if V_C is thought of as a displacement variable. You may recall that the solution of this equation depends on the relative values of ω_0 and β, which in turn depend on the values of R, L and C. We find three different behaviors according to the three possibilities of the the relative values of β and ω_0.

1. If $\omega_0 > \beta$, then the voltage V_C across the capacitor plates oscillates about the equilibrium value of zero voltage, successively damping out each cycle. The system is then said to be **under-damped**.

2. If $\omega_0 < \beta$, then the system does not oscillate at all, and we say that the system is **over-damped**.

3. Finally, if $\omega_0 = \beta$, the system is called **critically damped**, which separated the under-damped from the over-damped cases, and where, again, there is no oscillation.

These three types of solutions for the voltage across the capacitor plates are plotted in Fig. 11.5. You can see that only the under-damped case corresponds to an oscillator; the voltage decays to zero in the other two cases without making oscillations.

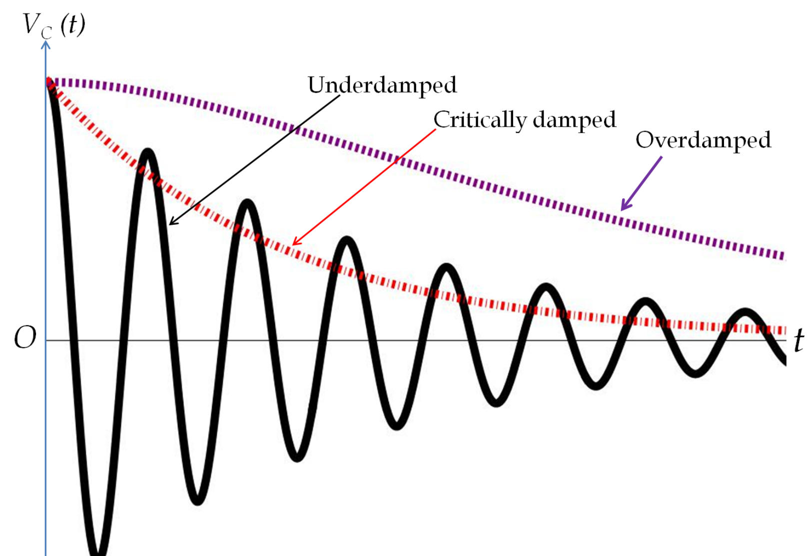


Figure 11.5: The voltage across the capacitor as a function of time for the under-damped, over-damped and critically damped cases. Only the under-damped case is an oscillator. To plot the figures, the following values were used: $V(0) = 1$ V, $I(0) = 0$, $\beta = 1$ rad/sec, $\omega_1 = 10$ rad/sec, $\alpha = 0.2$ rad/sec.

The mathematical expressions of the three solutions are as follows.

$$\boxed{V_C(t) = \begin{cases} A_1 \exp(-\beta t) \cos(\omega_1 t + \phi), \text{ under-damped} \\ \exp(-\beta t)(A_2 t + A_3), \text{ critically damped} \\ \exp(-\beta t) \left[A_4 \exp(-\alpha t) + A_5 \exp(\alpha t)\right], \text{ over-damped} \end{cases}} \quad (11.26)$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \ \text{ and } \ \alpha = \sqrt{\beta^2 - \omega_0^2}. \tag{11.27}$$

The constants A_1, A_2, A_3, A_4, A_5, and ϕ in Eq. 11.26 are determined by the initial V_C and the initial current I, as discussed for the undamped LC-circuit above.

The under-damped oscillator provides a physical meaning for the damping parameter β. We see from the graph in Fig. 11.6 that it takes a time of $1/\beta$ for the envelope of the oscillations to decrease by 1/e of its original value. The time to relax to $1/e$ of the original amplitude is called the **time constant** of the oscillator, which is also denoted by the Greek letter τ.

$$\boxed{\tau = \frac{1}{\beta}.} \tag{11.28}$$

Therefore, the larger the damping constant, the shorter the time constant, and consequently, faster the damping.

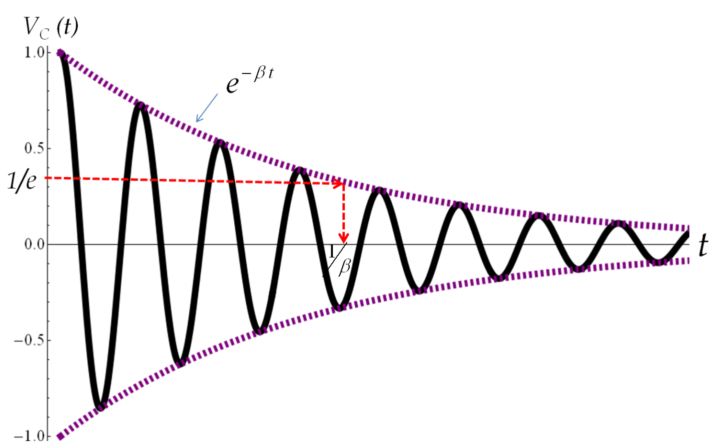


Figure 11.6: The dynamics of under-damped case shows the oscillations of the voltage across the capacitor plates with a decreasing amplitude. The envelop of the decreasing amplitude is used to define the time constant for the under-damped oscillator. The figure shows that in time $t = 1/\beta$, the envelop decreased to by a factor of $1/e$, where e is the Euler's number with value $e = 2.71828....$

The persistence of oscillations of an underdamped oscillator is often expressed by a property of the oscillator, called the **Q-factor** or the quality factor, which should not be confused with the same symbol for charge. A simple definition of the Q-factor would be to divide the average energy in a particular cycle by the energy lost in the cycle. For technical reasons, the Q factor is defined by dividing the energy at the beginning of a cycle to the energy lost in a fraction of the cycle.

$$\boxed{Q = \frac{\text{Energy of the oscillator at the beginning of a cycle}}{\text{Energy dissipated per } (1/2\pi) \text{ of the next cycle}}.} \tag{11.29}$$

For low damping the energy loss in each cycle will be small and hence oscillations will persist for many cycles. Therefore, a better oscillator will have a higher Q-factor.

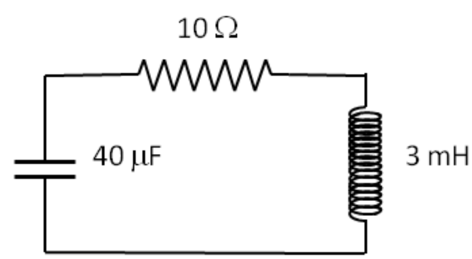
For a lightly-damped circuit, defined as a circuit where $\beta << \omega_0$, the quality factor calculation gives the following formula.

$$\boxed{Q = \frac{\omega_1}{2\beta} \approx \frac{\omega_0}{2\beta} = \sqrt{\frac{L}{RC}} \quad \text{(Lightly-damped RLC circuit)}} \tag{11.30}$$

Clearly, as the resistance R in the circuit becomes smaller, its Q-factor becomes larger. This makes sense because energy is dissipated only in the resistor, and the less the dissipation, the longer the circuit would remain oscillatory.

Example 11.2.1. A damped circuit.

Consider the following damped circuit. (a) Find the natural frequency and the damping parameter, and determine if the circuit is under-damped, over-damped or critically damped.



(b) At $t = 0$, the charge on the capacitor was 15 μC and the current in the circuit was zero, find the current in the circuit at an arbitrary time. (c) How much energy is dissipated in first 2 seconds?

Solution. (a) The natural frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{3\times 10^{-3}\ \text{H} \times 40 \times 10^{-6}\ \text{F}}} = 2887\ \text{rad/sec}$$

The damping parameter,

$$\beta = \frac{R}{2L} = \frac{10\ \Omega}{2 \times 3 \times 10^{-3}\ \text{H}} = 1667\ \text{rad/sec}.$$

Since $\omega_0 > \beta$, the circuit is under-damped.

(b) For an under-damped oscillator, the charge on the capacitor $q(t)$ is given by a similar equation as was given for the voltage V_C across the capacitor. We will write the solution in the following form.

$$q(t) = e^{-\beta t}\left[A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right], \tag{11.31}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = 2357\,\text{rad/sec}$. We find the constants A and B based on the initial conditions on the given charge and current at $t = 0$. The current is derivative of charge.

$$\begin{aligned} I(t) = \frac{dq}{dt} &= -\beta\, e^{-\beta t}\ \left[A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right] \\ &\quad + \omega_1\, e^{-\beta t}\ \left[-A_1 \sin(\omega_1 t) + A_2 \cos(\omega_1 t)\right] \end{aligned} \tag{11.32}$$

Now setting $t = 0$ in Eqs. 11.31 and 11.32 we find respectively

$$\begin{aligned} &A_1 = 15\ \mu\text{C} \\ &2708\, A_2 - 1000\, A_1 = 0. \end{aligned} \tag{11.33}$$

Hence, $A_1 = 15\ \mu$C and $A_2 = 10.6\ \mu$C. Having found A_1 and A_2, we can put them in the expression for the current. But, before we put in any numbers, it pays to simplify the expression for the current. Since the current at $t = 0$ is zero, there will be no cosine terms left in Eq. 11.32. After some algebra you should be able to show that

$$I(t) = -\frac{\omega_0^2}{\omega_1} A_1 e^{-\beta t} \sin(\omega_1 t).$$

Putting in the numbers now we obtain the following for the current in the circuit with t in sec.

$$I(t) = (-53\ \text{mA})\, e^{-1667\, t}\ \sin(2357\ t),$$

(c) Integrate power deposited in duration (0, 2 sec).

$$\text{Energy} = \int_0^2 I^2\ R dt = 2.8\ \mu\text{J}.$$

Note that I have left more digits in the earlier numbers than are significant. The final number is given to two digits of significance present in the data. It is a common practice to carry additional digits in the intermediate steps of calculations and round off the final answer to the significant digits.