7.3 TRANSLATIONAL MOTION OF MULTIPARTICLE SYSTEMS

We call a group of particles a multiparticle system when we are interested in describing the motion of the group as a whole. A baton, for instance, with two masses connected by a light rod, as a whole, moves as a two-particle system. When you throw a baton in the air, the two masses tumble about each other and move under the influence of gravity of the Earth. Although the motion of each mass is quite complicated, we are often interested in an overall motion of the baton.

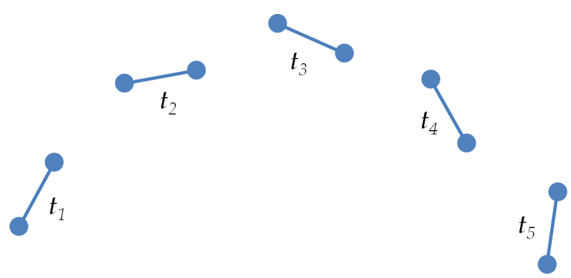

Figure 7.6: A baton with two masses moves as a two-particle system. In this sectio we will find that the total momentum of the two masses is affected only by the external force on the two masses.

The physics of multipoarticle systems are based on the physics of individual particles of the system. When we apply Newton's laws of motion to each particle we obtain the rate of change of momentum of each particle. These rules for individual particles can be combined to deduce the overall motions, such as the overall translational and the rotational motion of the multiparticle system. In this section, we will see how the total momentum of a multiparticle system can be studied. The rotational motion of multiparticle systems will be studied in a later chapter.

7.3.1 Two-Particle System

As a an example of a multiparticle system, let us examine a two-particle system first. We take a familiar example from Astronomy and study the motion of the Earth and the Moon together as one multiparticle system. To keep the physical situation simple we will ignore other planets and consider only the Sun as the sole external object with which the Earth and the Moon can interact as shown in Fig. 7.7 in addition to their interaction with each other.

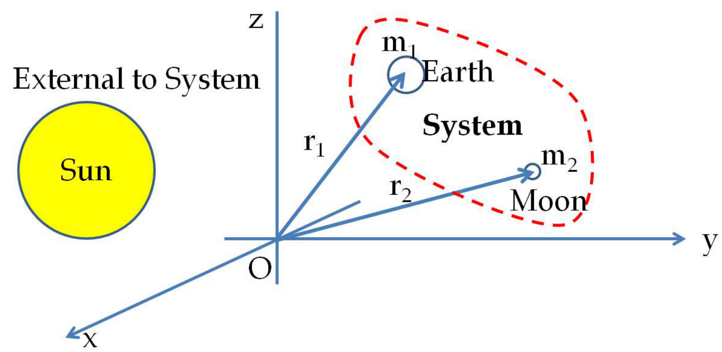

Figure 7.7: In the three-body world of the Earth, the Moon and the Sun, when we wish to study the motions of the Earth and the Moon, then they form a two-particle system and the Sun becomes an external body to the system. Had we decided to study the motion of the Earth only, then the Earth would be a one-body system and the Moon and the Sun would have been the external bodies to the system. Had we decided to study the motions of all three bodies shown, then the system would be a three-body system, and since there is nothign external to this system, the system would have been an isolated system

To simplify the notation we will use subscript 1 for the Earth, 2 for the Moon, and *ext* for the Sun in our calculations below. Let us start with the separate equations of motion of the two objects in the system as given by Newton's second law of motion.

$$\text{Earth:}\quad \frac{d\vec{p}_1}{dt} = \vec{F}_1^{\text{ext}} + \vec{F}_{12} \tag{7.17}$$

$$\text{Moon:}\quad \frac{d\vec{p}_2}{dt} = \vec{F}_2^{\text{ext}} + \vec{F}_{21} \tag{7.18}$$

where $\vec{F}_{12}$ is the force on the Earth by the Moon, $\vec{F}_1^{\text{ext}}$ is the force of the Sun on the Earth, $\vec{F}_{21}$ is the force on the Moon by the Earth, and $\vec{F}_2^{\text{ext}}$ is the force of the Sun on the Moon. Note that forces of the Moon on the Earth and that of the Earth on the Moon are internal forces for the system since the Earth and Moon belong to the multiparticle system. Only the forces by Sun on the Earth and Moon are external to the Earth-Moon system.

According to Newton's third law, the internal forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are equal in magnitude but act in opposite directions. Hence their vector sum must be zero.

$$\vec{F}_{12} + \vec{F}_{21} = 0. \tag{7.19}$$

This suggests that we should add equations 7.17 and 7.18, which will result in an equation without any reference to the internal forces.

$$\frac{d\left(\vec{p}_1 + \vec{p}_2\right)}{dt} = \vec{F}_1^{ext} + \vec{F}_2^{ext}, \tag{7.20}$$

which can be re-written more compactly as

$$\frac{d\vec{P}_{\text{system}}}{dt} = \vec{F}^{ext}_{net}. \tag{7.21}$$

where I have written $\vec{P}_{\text{system}}$ for the total momentum $\vec{p}_1 + \vec{p}_2$ and $\vec{F}^{\text{ext}}_{\text{net}}$ for the sum of all the external forces on the Earth-Moon system. Thus, we find that the rate of change of total momentum of a composite system depends only on the external forces, and completely independent of the internal forces. Therefore, even if you do not know anything about the internal forces of a composite system, you can still predict the translational motion of the system as a whole by looking at the total momentum vector.

7.3.2 Generalization

From the procedure outlined above for a two-particle system it is clear what to expect for the total momentum of an arbitrary number N of particles instead of just two. The following is a summary of results one can easily obtain.

Let $\vec{P}_{\text{system}}$ be the total momentum of the system of N particles, i.e.,

$$\vec{P}_{system} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_N\vec{v}_N.$$

The rate of change of the total momentum will be equal to the net external force on the system.

$$\boxed{\frac{d\vec{P}_{\text{system}}}{dt} = \vec{F}^{\text{ext}}_{\text{net}},} \tag{7.22}$$

where $\vec{F}^{\text{ext}}_{\text{net}}$ is the vector sum of external forces on all particles of the multiparticle system.

$$\boxed{\vec{F}^{\text{ext}}_{\text{net}} = \vec{F}^{\text{ext}}_1 + \vec{F}^{\text{ext}}_2 + \cdots + \vec{F}^{\text{ext}}_N.}$$

7.3.3 Motion of Center of Mass

Another useful interpretation of the rate of change of total momentum equation, Eq. 7.22, is that it represents the motion of a mathematical particle of mass equal to the total mass of the system located at a special point in space called the **center of mass** or **CM** of the system as we will see below. The center of mass point does not have to be anywhere in the body or bodies making up the multiplarticle system. For instance, the center of mass of a ring, whose mass is spread out uniformly around the ring, is at the center of the ring where no particle of the ring is located.

Consider a system consisting of N particles of masses m_1, m_2, $\cdots$, m_N, whose position vectors with respect to some origin are given by $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_N$ respectively as shown in Fig. 7.8. Let M be the total

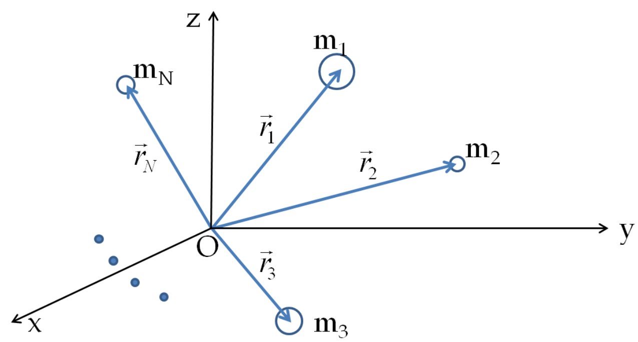

Figure 7.8: The location of the center of mass (CM) of an N-particle system is obtained mass-weigting the positions of each vector as shown in Eq. 7.24.

mass of the system,

$$M = m_1 + m_2 + \cdots + m_N.$$

The position vector of center of mass (CM) of the system, to be denoted by $\vec{R}_{\text{cm}}$, is defined by mass-weighting the position vectors of all the particles.

$$M\vec{R}_{\text{cm}} = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N, \tag{7.23}$$

or,

$$\boxed{\vec{R}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{M}.} \tag{7.24}$$

Taking derivative with respect to time on both sides of Eq. 7.23 we see that the momentum of a particle of mass M moving with R_{cm} is equal to the total momentum of the system.

$$M\frac{d\vec{R}_{\text{cm}}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \cdots + m_N\frac{d\vec{r}_N}{dt}, \tag{7.25}$$

or,

$$M\vec{V}_{\text{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_N\vec{v}_N, \tag{7.26}$$

where we have defined the center of mass velocity by the rate of change of the center of mass position vector.

$$\vec{V}_{\text{cm}} = \frac{d\vec{R}_{\text{cm}}}{dt}. \tag{7.27}$$

Define the center of mass momentum $\vec{P}_{cm}$ as the momentum of a fictitious particle of mass equal to the total mass of the system and moving with center of mass velocity.

$$\vec{P}_{\text{cm}} = M\vec{V}_{\text{cm}}. \tag{7.28}$$

Then, we have the result that the center of mass momentum is equal to the sum of the momenta of all particles in the system.

$$\vec{P}_{\text{cm}} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N = \vec{P}_{\text{system}} \tag{7.29}$$

The acceleration of the center of mass is defined by the derivative of the velocity of the CM.

$$\vec{A}_{\text{cm}} = \frac{d\vec{V}_{\text{cm}}}{dt}. \tag{7.30}$$

Using the result given in Eq. 7.22, we can now write the equation of motion of the center of mass as follows.

$$\boxed{\vec{F}^{\text{ext}}_{\text{net}} = \frac{d\vec{P}_{\text{cm}}}{dt},} \tag{7.31}$$

which for a constant mass system becomes

$$\boxed{\vec{F}^{\text{ext}}_{\text{net}} = M\vec{A}_{\text{cm}}.} \tag{7.32}$$

Equations 7.31 and 7.32 give the translational motion of the system as a whole since the system has been replaced by one point particle of total mass moving with the center of mass. The detailed information about the motions of individual particles has been lost since the motion of any particle depends on both the external forces and the internal forces on that particle.

For instance, if a system consists of two particles of equal mass moving in opposite directions, then we will find that center of mass is at rest. In this example, just because the center of mass is at rest does not imply no motion is taking place in the system. The detailed motion of each particle requires the study of the equations of motion of each particle separately.

Example 7.3.1. CM of an exploding shell

As an example consider the motion of pieces of an exploded shell in free-fall. Although the pieces fly away in various directions, the CM falls as if no explosion had taken place since during the explosion all forces are internal to the system (see Fig. 7.9).

The motion of CM is that of a single particle with total mass of the system. The fictitious particle at the CM falls freely with acceleration

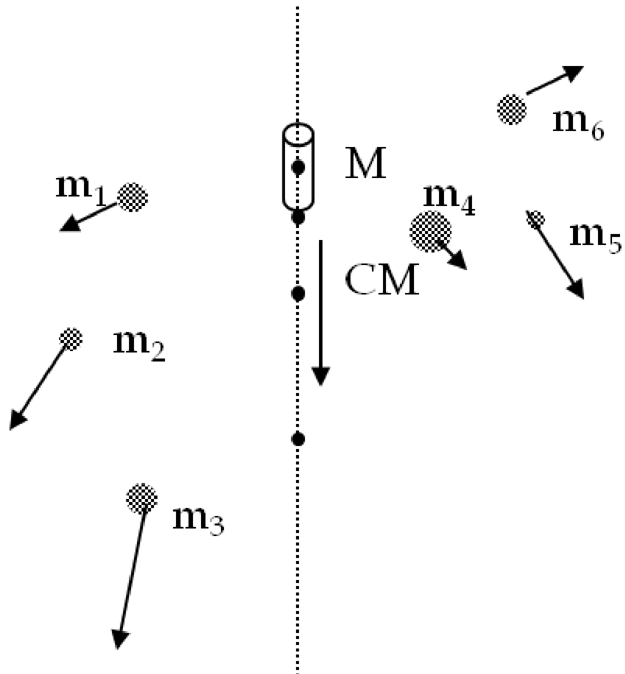

Figure 7.9: A falling explosive explodes but its CM falls as if nothing happened. This continues till the first piece hits the ground.

equal to g until one of the pieces hits the ground. At that instant there would be an additional external force on the CM due to the normal force on the fallen particle from the ground. During the time when the fallen part is coming to rest there is an upward force on the system. As a result, there would be additional force in the equation for the CM which would decelerate the CM.