4.3 OSCILLATIONS OF A VERY LONG STRING

In the last section we studied the normal modes of a string of length L that was fixed at both ends. We found that the string supports the normal modes in which the string can oscillate at various frequencies that are integral multiples of a fundamental frequency.

Now, we ask: what happens when the string in very long? Clearly, if the string is "infinitely" long, we will be looking at only a small finite part of the string that would be far away from the two ends. Suppose, again, the string to be along the x-axis. If the string is infinitely long, we can assume that the oscillations occur over the entire x-axis. Sinusoidal oscillations that continue over the entire axis are periodic functions, and we write the modes as

$$y(x) = A\sin(kx), \quad -\infty < x < \infty, \tag{4.17}$$

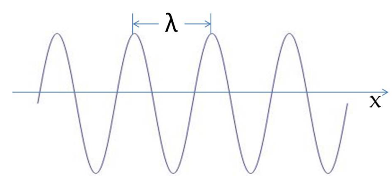

Figure 4.8: Wavelength corresponding to a mode of oscillation of a taut string.

where k is called the **wavenumber**. For an infinitely long string, the wavenumber can assume any real value up to a maximum cutoff value, which depends on the smallest distance between atoms of the string. The period of the sinusoidal mode function given in Eq. 4.17 is called the **wavelength** corresponding to the mode. Wavelength is usually denoted by the Greek letter λ (read: lambda). From the periodicity of Eq. 4.17 along the x-axis,

$$y(x+\lambda) = y(x), \quad -\infty < x < \infty,$$

we see that the wavelength is related to the wave number by

$$\lambda = \frac{2\pi}{k}. \tag{4.18}$$

4.4 TRAVELING WAVES

A traveling or progressive wave represents the movement of a disturbance from one place to another without the actual movement of the underlying medium itself. The property characterizing a disturbance depends on the nature of the wave. For instance, in the case of a mechanical wave or sound wave, the disturbance refers to the displacement of the particles of the medium from their equilibrium positions as we have seen above. In the case of the electromagnetic wave, the disturbance refers to is the change in the electric and magnetic fields in space. The disturbance varies both in space and time. The function that describes the disturbance is called the **wave function**. We have used $y(x, t$ for the transverse vibration modes on a string. We will use symbol ψ (read: psi) for a wave equation which may stand for a traveling transverse vibrations on a string or some other system.

A particularly simple wave function for the mathematical analysis results if the continuous disturbance can be cast as a sine or a cosine function of the space and time variables. These waves are called **sinusoidal waves** and are characterized by periodicities of the wave in space and time. These waves are also called **plane waves** if they travel in a three-dimensional space since the wave function for these waves take the same value in a plane perpendicular to their propagation direction.

The wave function ψ for a sinusoidal wave of amplitude A, wavenumber k, and angular frequency ω traveling towards the positive x-axis is given by

$$\psi(x, t) = A\ cos(kx - \omega t), \quad -\infty < x < \infty,\ -\infty < t < \infty \tag{4.19}$$

assuming $\psi(0, 0) = A$, and k and ω would be taken to be positive. To visualize the motion of a wave we usually draw multiple images of the **wave profile** at successive times.

In Fig. 4.9 I have plotted snapshots of a sinusoidal wave at different times of a plane wave traveling towards the positive x-axis. Note that the wave arrives at a screen at different times with different values of the wave function. The picture of the wave function at different times shows that the wave at the screen or at any other point in space acts as a simple harmonic oscillator, oscillating between $\psi = A$ and $\psi = -A$ with frequency ω as the wave passes through the point.

The wave function in Eq. 4.19 moves towards positive x-axis. If the two terms in the argument of cosine in the wave function in Eq.

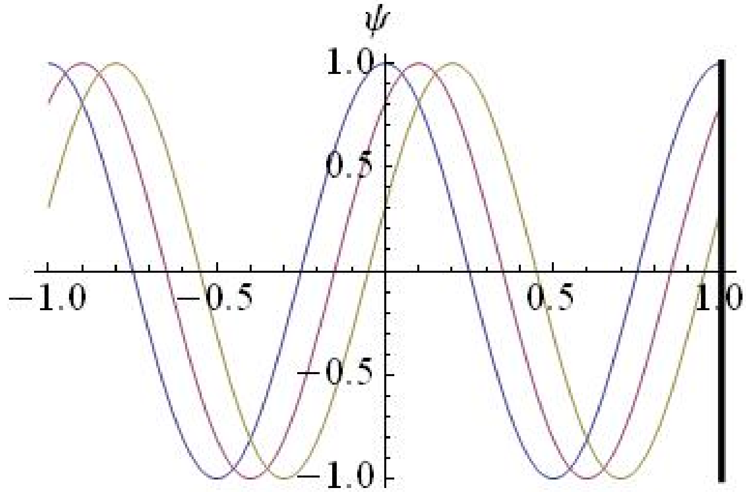

Figure 4.9: A sinusoidal traveling wave along the x-axis for three different instants ($t_3 > t_2 > t_1$). The wave function $\psi = cos[2\pi(x - t)]$ is plotted with $t = t_1 = 0$, $t = t_2 = 0.1$ sec, and $t = t_3 = 0.2$ sec. Note that the wave arrives at the screen with different values of wave function at different times.

4.19 have the same sign, the wave would move towards negative x-axis. Therefore, the following wave function will be for a wave moving towards the negative x-axis.

$$\psi'(x, t) = A \ \cos(kx + \omega t). \tag{4.20}$$

At any time, say $t = 0$, the successive crests or successive troughs of the wave are separated by a wavelength, λ, which is $2\pi/k$. You can see this by setting $t = 0$ in either Eq. 4.19 or 4.20 and finding the period of the resulting function, $A \cos(kx)$.

As seen in Fig. 4.9 the wave $A \ cos(kx - \omega t)$ moves towards the positive x-axis. If we focus on one of the crests, we note that the crest moves over a distance equal to a wavelength in time $2\pi/\omega$. Therefore, the speed of the wave v is given by dividing $|\Delta x| = \lambda = 2\pi/k$ by $\Delta t = 2\pi/\omega$.

$$v = \frac{|\Delta x|}{\Delta t} = \frac{\omega}{k} \tag{4.21}$$

We can write the speed of the wave in terms of the wavelength λ and the regular frequency f rather than in terms of the wavenumber and the angular frequency.

$$v = \frac{\omega}{k} = \lambda f. \tag{4.22}$$