2.3 The Black-Body Radiation

Every object at a non-zero absolute temperature emits electromagnetic radiation. If you put your hand next to a cup of hot water you will feel warmth due to the infrared radiation from the heated water. If you heat an object at even higher temperatures they will emit radiation at even shorter wavelengths, e.g., the wavelengths in the visible range. This is what happens in a fire or glow. Objects at even higher temperatures emit radiation at even shorter wavelengths, such as ultraviolet and x-rays.

The radiation emitted by a body at constant temperature is called the **black-body radiation**. It is also called **thermal radiation**. After a suggestion by Gustav Kirkchhof in 1859, the characteristics of the radiation emitted by a body at constant temperature is studied experimentally by examining the radiation coming out of a small hole in a cavity [such as an oven] kept at a constant temperature T. By careful experimentation the following facts were established.

Result 1: The radiation from the cavity is independent of the material or shape of the cavity. The intensity of radiation depends only on the temperature of the body and the wavelength of the radiation.

Result 2: The wavelength of the radiation where the radiation intensity is maximum is inversely proportional to the temperature - the higher the temperature the shorter the wavelength of the maximum intensity. Let λ_{max} denote the wavelength where the intensity is maximum when the body is at the absolute temperature T, i.e., the temperature in the Kelvin scale. Then, experiment shows

$$\boxed{\lambda_{\text{max}} T = \text{constant},} \tag{2.9}$$

with constant equal to approximately 2.90×10^{-3} m.K. The equation 2.9 is called **Wien's displacement law**.

Result 3: The total radiation intensity I over all wavelengths increases as the temperature is raised with a fourth power dependence on the absolute temperature T:

$$\boxed{I(T) = \sigma T^4,} \tag{2.10}$$

where σ is a universal constant, called the **Stefan-Boltzmann constant**, with the value

$$\sigma = 5.670 \times 10^{-8} \, \frac{\text{W}}{\text{m}^2\text{K}^4}.$$

Equation 2.10 is called the **Stefan-Boltzmann law**.

Result 4: The Wien's law and Stefan-Boltzmann law are particular aspects of the complete radiation spectrum. Suppose the energy emitted in the wavelength range λ to $\lambda + d\lambda$ in a time interval Δt from a surface of area ΔA be ΔU.

Then, the intensity ΔI of the radiation in this wavelength range would be given by

$$\Delta I[\text{in range } \lambda \text{ to } \lambda + d\lambda] = \frac{\Delta U}{\Delta A \Delta t}.$$

This quantity will be proportional to the the wavelength range $\Delta\lambda$. Therefore, we define another quantity called **spectral radiancy** $R_T(\lambda)$ by writing ΔI as

$$\Delta I = R_T(\lambda)\Delta\lambda.$$

The spectral radiancy $R_T(\lambda)$ is the intensity per unit wavelength at the wavelength λ. If we integrate over all wavelengths we will get the total radiation intensity at temperature T, whose dependence on T is emphasized by writing I as a function of T as $I(T)$.

$$I(T) = \int_0^\infty R_T(\lambda)d\lambda. \tag{2.11}$$

The variation of the total intensity of light at any temperature is given by the Stefan-Boltzmann law given in Eq. 2.10 above.

Fig. 2.3 shows typical radiancy R_T as a function of λ at different temperatures. The plots show that as temperature increases the maximum of intensity occurs at smaller wavelength. From Eq. 2.11 we see that the area under the curves would be equal to the total intensity at temperature T. Clearly, the areas under the curves, $I(T)$, increase with temperature as expected from Stefan-Boltzmann law.

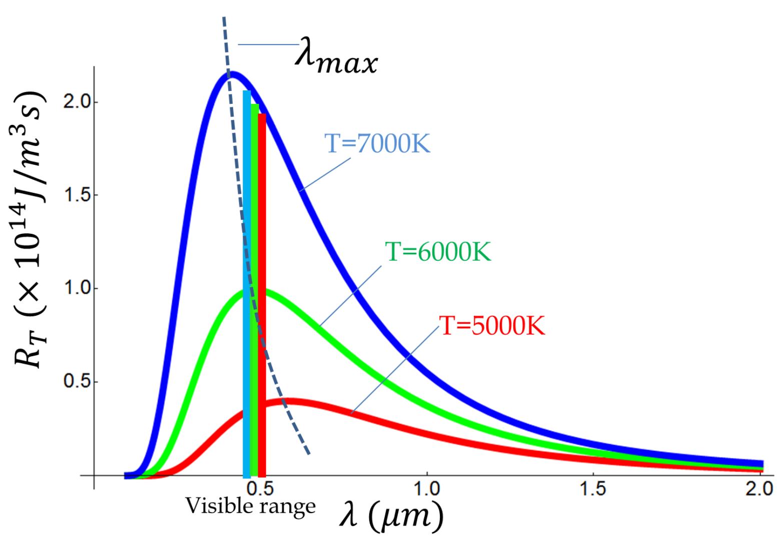

Figure 2.3: The black-body spectrum at three temperatures, $T = 5000\text{K}, 6000\text{K}, 7000\text{K}$. The wavelength at the peak moves towards smaller wavelength as temperature rises in accordance with Wien's law, $\lambda_{\max} = \text{const}/T$. The total intensity at a particular temperature is the area under each curve, which is clearly seen to rise as the temperature rises in accordance with the Stefan-Boltzmann law, $I(T) = \sigma T^4$.

A calculation based on classical physics predicted that

$$R_T(\lambda) = 2\pi c k_B \frac{T}{\lambda^4}. \tag{2.12}$$

This result is called **Raleigh-Jeans Law**. This result agrees with the experimental result given in Fig 2.3 at long wavelengths but as the wavelength becomes smaller it deviates considerably from the experiment. And in the $\lambda \to 0$, $R_T \to \infty$, i.e. classical physics predicts that radiation intensity will become infinite as λ approaches zero. This result is called the **ultraviolet catastrophe**. The calculation of Raleigh and Jeans was done to illustrate that there was something missing in the classical physics.

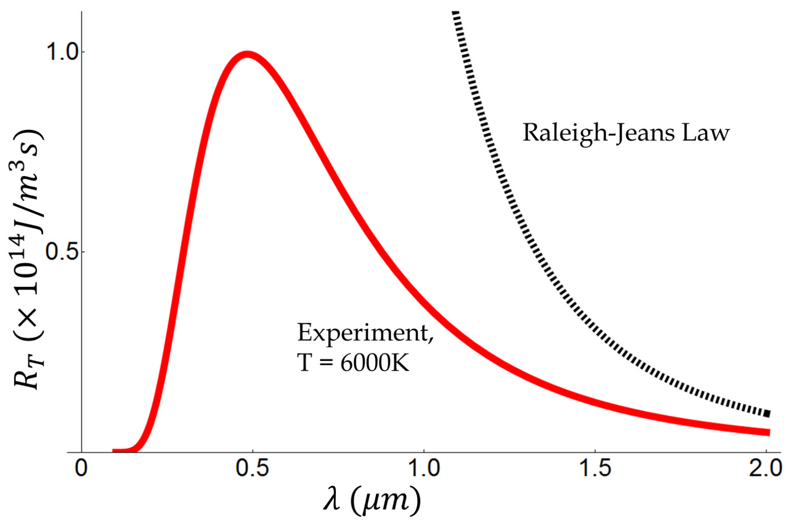

Figure 2.4: The black-body spectrum at T = 6000K and prediction of Raleigh-Jeans Law, Eq. 2.12. Notice that at large wavelengths the Raleigh-Jeans law gives more or less correct prediction that radiacny would go down with increasing wavelength but fails to account for decrease in radiancy for smaller wavelengths.

In 1900 Max Planck derived a formula for the radiancy based on a quantum assumption that fit the data at all wavelengths. Planck modeled the atoms in the cavity wall act as tiny springs which exchange energy with the electromagnetic radiation. The electric field of the electromagnetic radiation would accelerate electrons of the cavity wall which will then oscillate at the frequency of the electromagnetic radiation. Before Planck it was assumed that any amount of energy can be exchanged between the radiation field and the atoms of the cavity wall. The Raleigh-Jeans formula is the result of such an approach. Planck introduced a quantum assumption - he assumed that the exchange would take place only in integral multiples of a quantum of energy proportional to the frequency f of the radiation.

$$E_{\text{quantum}} \propto f, \tag{2.13}$$

and introduced a constant of proportionality h, which we now call Planck's constant.

$$E_{\text{quantum}} = hf. \tag{2.14}$$

Use then used the field of statistical mechanics to derive the following radiancy formula of the radiation in the cavity at thermal equilibrium at temperature T.

$$\boxed{R_T(\lambda) = \left(2\pi hc^2\right) \frac{1}{\lambda^5} \left[\frac{1}{e^{hc/\lambda k_B T} - 1}\right],} \tag{2.15}$$

where k_B is the Boltmann constant and c the speed of light. This formula is known as **Planck's radiation law**. This formula completely agreed with the experimental results at all wavelengths and at all temperatures for a universal value of h which he estimated to be 6.55×10^{-34} J.s. The best experimental values now put the value of Planck constant at

$$h = 6.63 \times 10^{-34} \text{ J.s.}$$

Although it is tempting to say that the fundamental quanta of energy hf correspond to the energy of elementary particles that make up the radiation of frequency f in the cavity, Planck, however, thought of the idea of the fundamental quantum of energy to be only a mathematical device and not representing some reality of the system in the cavity. It was difficult for scientists at the time to believe that the atoms of the cavity could not absorb arbitrary amounts of energy in a continuous spectrum.

In 1916 Einstein gave another derivation of Planck's radiation law based on the quantized energy levels of atoms of the cavity. The quantization of energy of atoms give rise to the radiation with spectral radiancy given by the formula in Eq. 2.15. Although, the quanta of energy in the cavity equal to hf suggests strongly that electromagnetic radiation consists of particles, called photons, we will see that a more compelling case for the particle nature of light comes from Einstein's explanation of the Photoelectric effect and and Compton's explanation of the Compton effect. In modern time we are able to produce, manipulate, and detect single photons as if they were particles.

Example 2.1. Number of photons. A laser beam of wavelength 633 nm and power 3 mW is incident on a plate. How many photons strike the plate per second?

Solution.

Let P be the power of the laser and E be the energy of one photon, then we see that the number N of photons striking the plate per unit time will be

$$N = \frac{P}{E}.$$

The energy of a photon is related to the wavelength λ as

$$E = hf = \frac{hc}{\lambda}$$

Numerically,

$$E = \frac{6.63 \times 10^{-34} \text{ J.s} \times 3 \times 10^3 \text{ m/s}}{633 \times 10^{-9} \text{ m}} = 3.14 \times 10^{-19} \text{ J.}$$

Therefore,

$$N = \frac{3 \times 10^{-3} J/s}{3.14 \times 10^{-19}\ \mathrm{J}} = 9.55 \times 10^{15}.$$

Example 2.2. Sun as a black-body. The solar spectrum above atmosphere is very close to the black-body spectrum at $T = 5500\mathrm{K}$. What is the wavelength of the light emitted by the Sun at the maximum intensity?

Solution.

The Wien's law can be used to find the λ_{max}.

$$\lambda_{\mathrm{max}} T = 2.90 \times 10^{-3}\ \mathrm{m.K}.$$

Putting in the temperature this gives

$$\lambda_{\mathrm{max}} = \frac{2.90 \times 10^{-3}\ \mathrm{m.K}}{5500\mathrm{K}} = 527\,\mathrm{nm},$$

which is in the middle of the visible spectrum.

Example 2.3. Stefan-Boltzman law from Planck's radiation law. Deduce the Stefan-Boltzmann law from Planck's radiation law.

Solution.

Integrating the radiancy $R_T(\lambda)$ over all wavelengths should give the total intensity at temperature T.

$$I(T) = \int_0^\infty R_T(\lambda) d\lambda = \int_0^\infty \left(2\pi hc^2\right) \frac{1}{\lambda^5} \left[\frac{1}{e^{hc/\lambda k_B T} - 1}\right] d\lambda.$$

A change of variable helps isolate a definite integral that will have a purely numerical value. Let

$$x = \frac{hc}{\lambda k_B T}.$$

Then,

$$dx = -\frac{hc}{k_B T} \frac{d\lambda}{\lambda^2}.$$

In terms of integral over x, the intensity becomes

$$I(T) = -\left(2\pi hc^2\right) \left(\frac{k_B T}{hc}\right)^4 \int_\infty^0 \frac{x^3}{e^x - 1}\, dx$$

The integral in this expression is difficult to do. If you look up a standard table of integrals or work it in Mathematica you will find that

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}$$

Therefore, we find the following expression for the total intensity,

$$I(T) = \left[\frac{2\pi^5 k_B^4}{15 h^3 c^2}\right] T^4.$$

The quantity [ ] should equal the Stefan-Boltzmann constant σ.

$$\sigma = \frac{2\pi^5 k_B^4}{15h^3c^2}.$$

Let us check the numerical value.

$$\frac{2\pi^5 k_B^4}{15h^3c^2} = 5.65 \times 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2K^4}},$$

which is close to the standard value $5.670 \times 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2K^4}}$ given above.