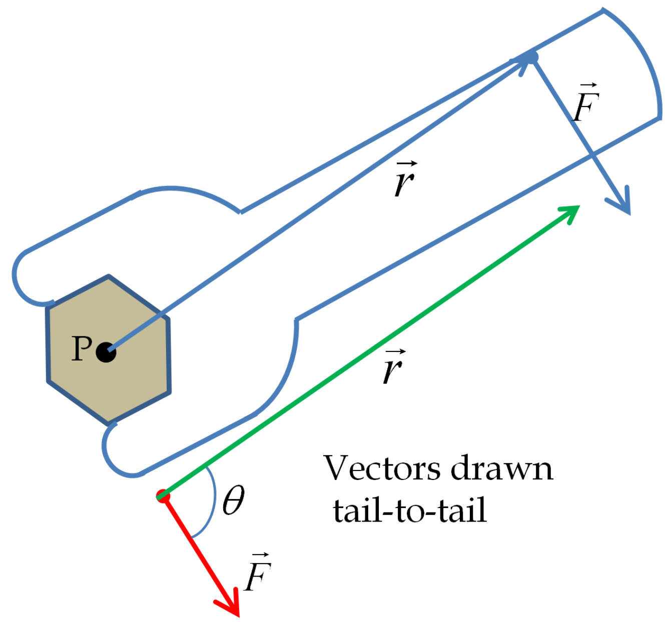


Figure 5.33: Point P about which torque of force $\vec{F}$ requires the displacement vector $\vec{r}$ from P to the point where the force acts. The tail-to-tail diagram helps define the angle between the vectors.

5.5 TORQUE OR MOMENT OF A FORCE

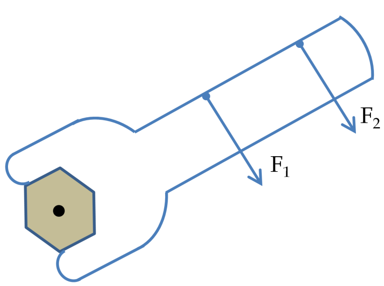


Figure 5.32: Force F_2 is more effective than F_1 in rotating the bolt since it is applied at a greater distance from the axis of rotation.

The moment of a force, also called the torque, is related to the ability of a force to rotate a body about some axis. Suppose you want to rotate a bolt by fitting a wrench on the head of the bolt and applying a force on the handle of the wrench as in Fig. 5.32. The line through the center of the bolt is the axis about which the bolt and the wrench together will rotate. We find that the same force applied on the handle at a larger distance from the axis is more effective than when it is applied nearer to the axis. That is why a longer wrench is more effective than a shorter wrench.

For rotation, we find that the location of the force on the body is as important as the direction and magnitude of the force. This is taken care of by introducing a quantity called **moment or torque of force** which has the following precise definition.

Let $\vec{r}$ be the displacement vector from some point P in space to the point in the body where we apply a force $\vec{F}$ as shown in Fig. 5.33. The moment or torque of the force $\vec{\tau}$ about point P is defined as the cross product of the displacement vector and the force. Note that the names moment and torque mean the same thing and are calculated identically, but the usage usually differs - moments are more commonly used in static situations and torque for both static and dynamic settings.

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}.} \tag{5.14}$$

The unit of torque in the SI system of units is N.m. If the point P, about which the torque is calculated, is fixed during the rotation, then, we say that P is the pivot point of rotation. The definition of the cross product gives us the rules for the magnitude and the direction of torque vector $\vec{\tau}$. We will recall those rules and apply them to the cross product for torque below.

The geometric picture:

To write the magnitude and direction using geometric picture of vectors, we need the angle θ between vectors $\vec{r}$ and $\vec{F}$ when they are drawn tail-to-tail. Then,

Magnitude of torque:

$$\boxed{|\vec{\tau}| = |\vec{r}| \left|\vec{F}\right| \sin\theta.} \tag{5.15}$$

Direction of torque:

Use the right-hand-rule as given in the chapter on vectors. Briefly, hold your right hand such that the thumb points in the direction of $\vec{r}$ and any of the other fingers points in the direction of $\vec{F}$, then the torque will be perpendicular to the two vectors and will be coming out of the palm. There are other ways of doing the right-hand rule also. For instance, if you point $\vec{r}$ in the direction of the positive x-axis, and $\vec{F}$ in the xy-plane, then $\vec{\tau}$ will be in the direction of the positive z-axis of a right-handed Cartesian coordinate system.

The lever arm picture:

The magnitude of a torque given in Eq. 5.15 gives rise to popularly used term called the **lever arm**. Suppose you draw the displacement and force vectors in their real settings and not move them for tail-to-tail drawing as illustrated in Fig. 5.34. Now, we drop a perpendicular from point P about which we want torque to the line of action of the force. The perpendicular distance $r_\perp$ is equal to $|\vec{r}|\sin\theta$ and is called the lever arm of the force about P. The larger the lever arm the larger the torque for the same force.

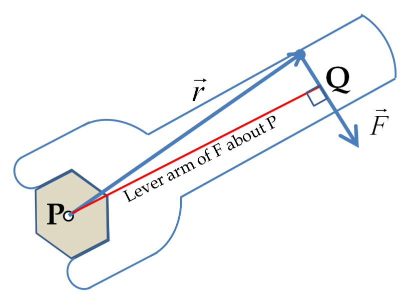


Figure 5.34: The lever arm of force $\vec{F}$ about point P is the distance PQ of the perpendicular from P to the line of the force.

$$\boxed{\text{Lever arm,}\ \ r_\perp = |\vec{r}|\sin\theta.} \tag{5.16}$$

Magnitude of torque in terms of lever arm:

$$\boxed{|\vec{\tau}| = r_\perp \left|\vec{F}\right|.} \tag{5.17}$$

The analytic picture:

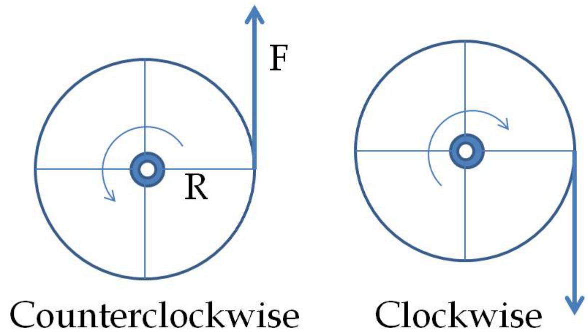


Figure 5.35: The "sense of rotation" for a torque. The torque on the left figure will tend to rotate the body counter-clockwise about an axis through the center if observed from above as in this figure. The body on the right has a clock-wise sense of rotation. Torque vector for counter-clockwise would be in the direction coming-out-of-page and that of the clockwise sense will be in-the-page.

The cross product in Eq. 5.14 can also be calculated analytically. First, we choose a Cartesian coordinate system and express the two vectors $\vec{r}$ and $\vec{F}$ into components. The cross product is usually done by organizing the components in a determinant. We summarize these steps now.

1. Choose a Cartesian coordinate system.
2. Write out the vectors in component forms

$$\vec{r} = x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$$
$$\vec{F} = F_x\hat{u}_x + F_y\hat{u}_y + F_z\hat{u}_z$$

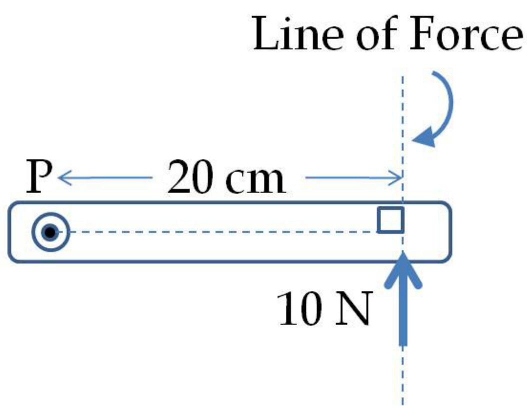
3. Expand the determinant form of the cross product.

$$\boxed{\vec{r} \times \vec{F} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}}$$

The sense of rotation and torque

We can assign a **sense of rotation** with a torque whether of not the object rotates. Suppose we place the displacement vector of where the force acts from the reference point, $\vec{r}$, and the force, $\vec{F}$, in the xy-plane. Then, the torque will be pointed towards either the positive z-axis or the negative z-axis, which are also referred to as "out-of-page" and "into-the-page" as shown in Fig. 5.35. If the torque is pointed out-of-page, i.e., towards the positive z-axis, then the torque will tend to rotate the object in the counter-clockwise fashion as observed from the positive z-axis. We say that the torque has a counter-clockwise "sense of rotation". On the other hand, if torque is pointed towards the negative z-axis, the sense of rotation will be clockwise.

Example 5.5.1. A peg goes through a hole in a metal bar at one end so that the bar can rotate about the peg. A force of 10 N is applied on the bar perpendicular to the bar a distance 20 cm from the peg as shown in the figure. What is the torque about the peg?

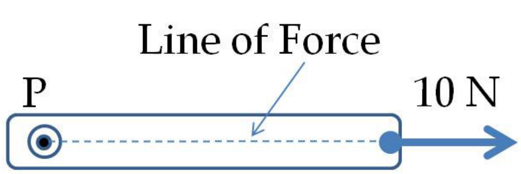


Solution. The lever arm is easy to work out here since the force is perpendicular to the bar. The projection from peg to the line of the force gives the lever arm to be 20 cm, which we convert to meter before using in the calculations. Therefore, the torque has the following magnitude

$$\text{Magnitude:} \quad \tau = (0.2)(10) = 2 \text{ N.m.}$$

The direction of the torque is obtained by the right-had rule. In the given figure, the direction is coming out of the page.

Example 5.5.2. A peg goes through a hole in a metal bar at one end so that the bar can rotate about the peg. A force of 10 N pulls the bar away from the peg as shown in the figure. What is the torque about the peg?



Solution. In this configuration, the line of force goes through point P located at the peg. Therefore, the lever arm is zero and so is the torque.

Example 5.5.3. The two examples presented above can be written more generally. From the definition of the torque, it is clear that torque will be zero if the angle between $\vec{r}$ and $\vec{F}$ is zero or 180°. This says that if we think of the force as a sum of two forces, one parallel to $\vec{r}$ and another perpendicular to this direction, then only the perpendicular part of the force will give a non-zero contribution to the torque. That is, if we write the force vector as

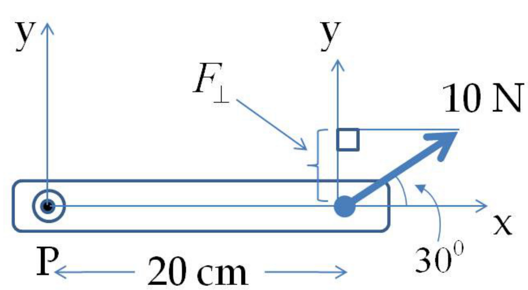
$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp},$$

then

$$\vec{r} \times \vec{F} = r\ F_{\perp},$$

where $F_{\perp}$ and r are magnitudes of $\vec{F}_{\perp}$ and $\vec{r}$ respectively.

Example 5.5.4. A peg goes through a hole in a metal bar at one end so that the bar can rotate about the peg. A force of 10 N is applied on the bar at an angle 30° to the bar a distance 20 cm from the peg as shown in the figure. What is the torque about the peg?



Solution. As illustrated in Example 5.5.3 the magnitude of the torque will be simply product of the perpendicular force and the distance from P. Let us place a coordinate system so that $\vec{r}$ and $\vec{F}$ are in xy-plane. Let the x-axis be along the bar, which is the direction of $vecr$. Then, to find the magnitude of the torque, we just multiply the y-component of the force and the distance from P to the point where force acts.

$$F_{\perp} = F_y = 10\ \mathrm{N} \sin(30°) = 5\ \mathrm{N} \quad \text{and} \quad r = 20\ \mathrm{cm} = 0.2\ \mathrm{m}.$$

Therefore, the magnitude of the torque is

$$\text{Magnitude:} \quad \tau = (0.2)(5) = 1\ \mathrm{N.m}.$$

The direction of the torque is obtained by the right-had rule. In the given figure, the direction is coming out of the page.

Example 5.5.5. How to do the previous example by lever arm method?

Solution. To find the lever arm of the force about point P, we extend the line of force so that a perpendicular line from P on the line of force can be drawn. The extended line and the perpendicular are shown in Fig. 5.36.

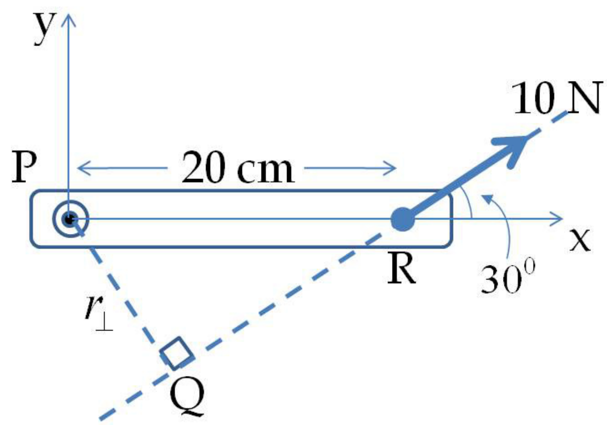


Figure 5.36: Example 5.5.5

The lever arm is equal to the length $r_\perp$, the side PQ in the right-angle triangle $\triangle PRQ$. We immediately find the lever arm to be 10 cm. The magnitude of the torque is equal to the product of the lever arm and the magnitude of the force, which gives the result 1 N.m. The direction of the torque is obtained as before by using the right-hand rule - the direction is coming out of page in the given figure.

Example 5.5.6. How to do the previous example by analytic method?

Solution. We use the coordinate system given in Fig. **??** to write out the vectors in their component form so that we can compute the cross product. We wish to leave out units in the expressions, so we convert all units in meter-kilogram-second system of units before we use in equations.

$$\vec{r} = 0.2\hat{u}_x,$$
$$\vec{F} = 10\ \cos(30^\circ)\hat{u}_x + 10\ \sin(30^\circ)\hat{u}_y.$$

The torque is computed by directly working out the cross product.

$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} \\ &= 0.2\hat{u}_x \times \left(5\sqrt{3}\hat{u}_x + 5\hat{u}_y\right) \\ &= (1.0\ \text{N.m})\ \hat{u}_z.\end{aligned}$$

Putting the units back, the magnitude of the torque is 1 N.m and the direction is towards the positive z-axis, which is pointed out of paper in the figure.

Further Remarks

Do not confuse torque with force. They are entirely different quantities. Some major difference between torque and force to keep in mind are

1. Torque depends on the choice of the point P about which torque is to be evaluated. Force has no such dependence.
2. Torque and force are perpendicular to each other since torque is defined as cross product of a displacement vector and force.
3. There can torque on a system even when net force is zero. For instance, if you apply equal magnitude forces in the opposite directions on opposite ends of a wheel, the net force will be zero, but the net torque about the center will not be zero (see Fig. 5.37).
4. Net force an a system may not be zero even when torque is zero. For instance, if you apply equal forces on opposite ends of a wheel, the net torque about the center will be zero, but the net force will not be zero (see Fig. 5.37).

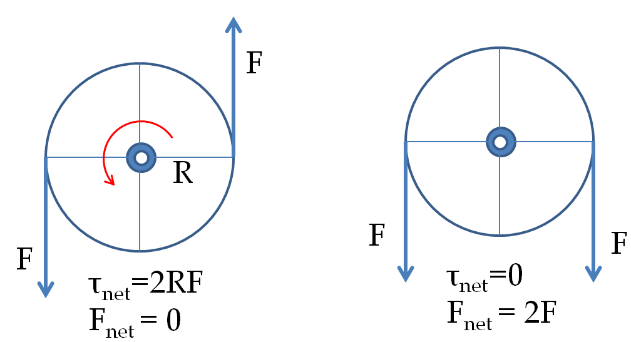


Figure 5.37: Torque versus force.