11.3 DRIVEN OSCILLATIONS AND RESONANCE

11.3.1 Driven Oscillations

The damped oscillator was discussed in the last section. The charged capacitor becomes uncharged over time in a purely damped oscillator since energy is dissipated in the resistor and there is no power source to replenish the lost energy. In this section we modify the damped circuit by including an alternating current (AC) generator in the circuit as shown in Fig. 11.7.

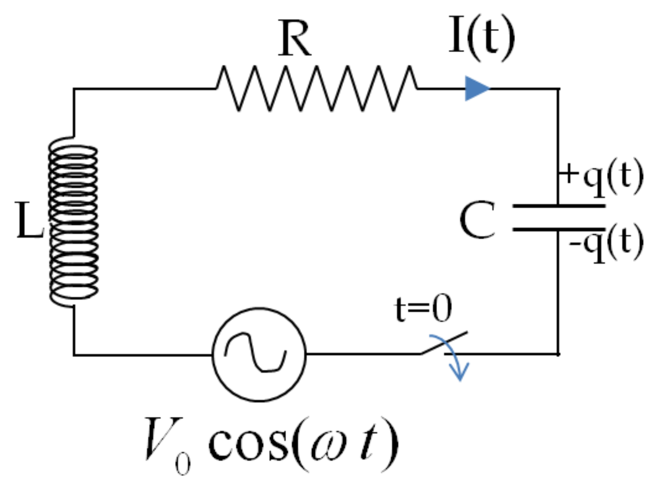


Figure 11.7: Sinusoidally driven electromagnetic oscillator circuit: the resistor R, the inductor L and the capacitor C are in series with a sinusoidal driving EMF that varies in time as $V_0 \cos(\omega t)$.

The concept of voltage for an AC source will be discussed in detail in the next chapter. Briefly, if there is no magnetic field outside of the AC generator, then the electric field there becomes a conservative field, which allows us to define a voltage for the source from the potential difference between the terminals of the generator. Let the voltage of the source be

$$V(t) = V_0 \cos(\omega\, t) \tag{11.34}$$

The angular frequency ω of the source EMF is called the **driving frequency**. As above the sense of current is chosen so that the positive flows towards the positive plate when the current is increasing. The Faraday loop equation gives the equation relating the charge on the capacitor and the current in the circuit as follows by integrating the electric field in the direction of the current.

$$\frac{q}{C} + RI - V_0 \cos(\omega\, t) = -L\frac{dI}{dt}. \tag{11.35}$$

Writing the current in terms of the charge Q, replacing the charge q on the capacitor in terms of voltage V_C across the capacitor and dividing out by LC we find the equation for V_C.

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\,V_C + \frac{1}{LC}\,V_C = \frac{V_0}{LC}\cos(\omega\, t). \tag{11.36}$$

Once again, we introduce composite parameters the natural frequency and the damping parameter

$$\omega_0 = \frac{1}{\sqrt{LC}}, \tag{11.37}$$

$$\beta = \frac{R}{2L}. \tag{11.38}$$

Substituting these parameters in Eq. 11.36 yields

$$\frac{d^2V_C}{dt^2} + 2\beta\, V_C + \omega_0^2\, V_C = \omega_0^2\, V_0\, \cos(\omega\, t). \tag{11.39}$$

Underdamped Driven Oscillator

We wish to study the effect of the driving voltage on the oscillation of the circuit. Therefore, we will choose R, L, and C corresponding to the under-damped electromagnetic oscillation in the absence of the driving EMF. That is, we will require that

$$\omega_0 > \beta \quad \Leftrightarrow \quad \frac{RC}{2} < \sqrt{LC}. \tag{11.40}$$

The solution of the equation of motion (Eq. 11.39) for the driven oscillator is a difficult exercise in mathematics. Here I will state the solution without any proof since we would like to discuss the physics of the system. The solution of Eq. 11.39 consists of two parts,

$$V_C(t) = V_T(t) + V_S(t). \tag{11.41}$$

The part $V_T(t)$ is called the **transient** or the complimentary solution. This part is the solution of Eq. 11.39 without the driving force present. Without the driving term, Eq. 11.39 is the same equation as the damped oscillator, and since we have chosen the parameters for the under-damped oscillator, $V_T(t)$ will have the following form.

$$V_T(t) = e^{-\beta\, t}\, \left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right], \tag{11.42}$$

where A and B are constants of motion containing the initial voltage and initial current information, and

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}. \tag{11.43}$$

The part $V_S(t)$ is called the **steady-state** or particular solution. This part contains the information about the driving EMF and oscillates with the same frequency as the driving frequency so that we can write V_S as

$$V_S(t) = V_{0C} \cos(\omega\, t + \phi), \tag{11.44}$$

where V_{0C} is called the peak voltage across the capacitor and ϕ is called the phase lag. The expressions for the peak voltage and the phase lag can be deduced by putting Eq. 11.44 in the equation of motion, Eq. 11.39.

$$V_{0C} = \frac{V_0/\omega C}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}, \tag{11.45}$$

and

$$\tan(\phi) = -\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right). \tag{11.46}$$

We find that, while the transient part V_T drops off with time as $e^{-\beta t}$, the steady part V_S oscillates for ever with an undiminished amplitude V_{0C}. Hence, the importance of the transient part V_T diminishes with the passage of time. That is, for times t such that

$$\text{For } \ t >> (1/\beta), \quad V_C(t) \approx V_S(t). \tag{11.47}$$

This is the reason for calling V_S the steady state solution, and V_T the transient solution. The steady state solution tells us about the long-time behavior of the circuit.

Another important aspect of the solution is that the function $V_S(t)$ is also sinusoidal with the same frequency as the driving EMF. Note that the phase of $V_S(t)$ is different from that of the driving EMF, the difference being the angle ϕ. If the angle ϕ is positive, we say that V_S leads the driving EMF V, and if the angle ϕ is negative, we say that V_S lags the driving EMF V.

Let us work out a numerical estimate to get a feel for the time it would take to reach the steady state for some common situation in a laboratory. A typical circuit having $R = 1$ kΩ, $L = 1$ mH and $C = 1$ μF can be considered to have reached the steady state when $t >> 2 \times 10^{-6}$ sec.

The steady current I_S in the circuit is obtained by taking the derivative of the steady state voltage V_S.

$$I_S = C\frac{dV_S}{dt} = -\omega\, C\, V_{0C}\ \sin(\omega t + \phi) \tag{11.48}$$

We rewrite this relation by introducing the symbol I_0 for the amplitude of current, and introducing additional phase $\pi/2$ radians to write cosine in place of sine and absorb the minus sign.

$$I_S = I_0\ \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \tag{11.49}$$

where ϕ is given in Eq. 11.46 as before, and I_0 is

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \tag{11.50}$$

Note that we do not need to label the peak current with more than one subscript 0 since we have only one current in the circuit, unlike the voltage drops across various elements in the circuit, e.g. the peak voltage across the capacitor was labeled by two subscripts 0 and C. In the following sections we analyze the consequences of these formulas for the amplitude and phase of the steady state solution.

11.3.2 Resonance of Voltage Across the Capacitor

An important point about the driven circuit in the steady state is that the voltage across the capacitor depends on the driving frequency ω in relation to the natural frequency ω_0. The situation here is analogous to the situation of a swing in children's playground - the amplitude of swing depends on the frequency with which one pushes on it. By keeping R, L and C fixed, i.e. keeping the circuit same, we can vary the frequency of the driving EMF, and look at the voltage across the capacitor after the steady state has reached. For each driving frequency ω, we find that the peak voltage across the capacitor is different as shown in Fig. 11.8.

An interesting question then arises: how does the peak voltage V_{0C} vary with the driving frequency ω. We have already found the answer to this question. The analytic expression for V_{0C} is given in Eq. 11.45 above. In Fig. 11.9 I have plotted V_{0C} versus ω to show you what the dependence looks like graphically.

We find that there is a driving frequency at which the peak voltage across the capacitor has the largest value. We say that the peak

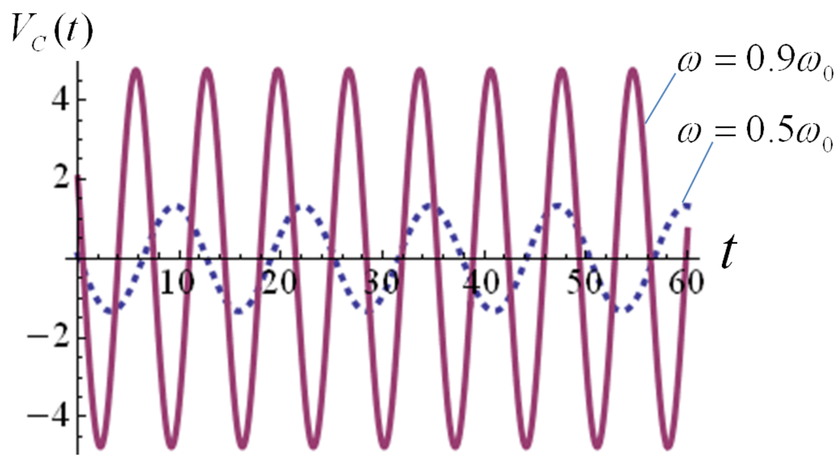


Figure 11.8: Voltage across the capacitor versus time. Note the value of the peak voltage depends on the driving frequency ω. The two curves are for two values of driving frequency: the solid line corresponds to $\omega = 0.9\omega_0$ and the dashed line for $\omega = 0.5\omega_0$. The other parameters are same for the two plots: $R = 0.1\ \Omega$, $L = 1$ H, $C = 1$ F, $V_0 = 1$ V.

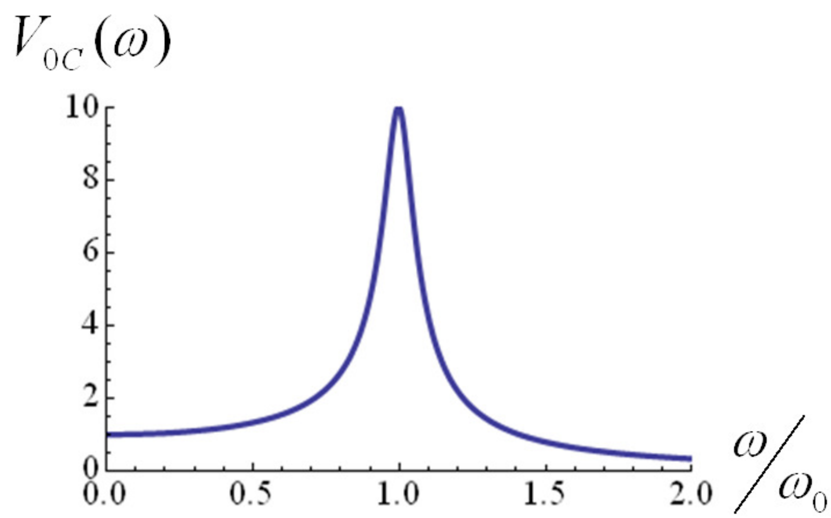


Figure 11.9: **Resonance curve** of the capacitor voltage. The amplitude of oscillations of the voltage across the capacitor plates is plotted versus frequency of the EMF source. We find the amplitude of is maximum for a particular value of the frequency of the source EMF. The following values were kept fixed in the plot: $R = 0.1\ \Omega$, $L = 1$ H, $C = 1$ F, $V_0 = 1$ V.

voltage of capacitor resonates at this frequency, and call the frequency ω_C the resonance frequency of the peak voltage across the capacitor. We will find below that the current in the circuit also depends in a similar manner on the driving frequency and exhibits a resonance phenomena, but the resonance frequency ω_I for the current is different than the resonance frequency ω_C for the voltage across the capacitor.

By using calculus we can predict where the resonance frequency will occur by setting the derivative of $V_{0C}(\omega)$ with ω to zero and solving this equation for ω. We find the the independent variable ω appears only in the denominator. Therefore, instead of taking the derivative of V_{0C} we can get the same result by actually taking the derivative of the quantity inside the radical in the denominator.

$$\frac{dV_{0C}}{d\omega} = 0 \quad \Longrightarrow \quad \frac{d}{d\omega}\left[\omega^2 R^2 C^2 + \left(\omega^2 LC - 1\right)^2\right] = 0 \tag{11.51}$$

This equation yields the following for the resonance frequency ω_C.

$$\omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}, \tag{11.52}$$

which can also be written as

$$\omega_C = \omega_0 \sqrt{1 - \frac{1}{2}\omega_0^2 \tau_C^2}, \tag{11.53}$$

where $\tau_C = RC$ and $\omega_0 = 1/\sqrt{LC}$. The resonance frequency of V_C can also be expressed in terms of the Q-factor of the oscillator by noting that

$$Q \approx \frac{\omega_0}{2\beta} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{\omega_0 \tau_C}. \tag{11.54}$$

Therefore, the resonance frequency ω_C of V_C is given as

$$\boxed{\omega_C = \omega_0 \sqrt{1 - \frac{1}{2}\omega_0^2 \tau_C^2} = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}.} \tag{11.55}$$

The sharpness of the resonance curve depends on how good the oscillator is in maintaining its oscillations as given by the Q-factor of the oscillator. Fig. 11.10 shows plots of the resonance curves for various values of the Q-factor. As the Q-factor increases, the oscillator becomes a better oscillator, the plot of the peak voltage versus the driving frequency becomes sharper, and the resonance frequency of V_C moves closer to ω_0 as shown in the figure.

Significance of phase angle

The driving EMF V(t) and the steady state capacitor voltage V_S have a difference of phase angle ϕ. What does the angle ϕ mean?

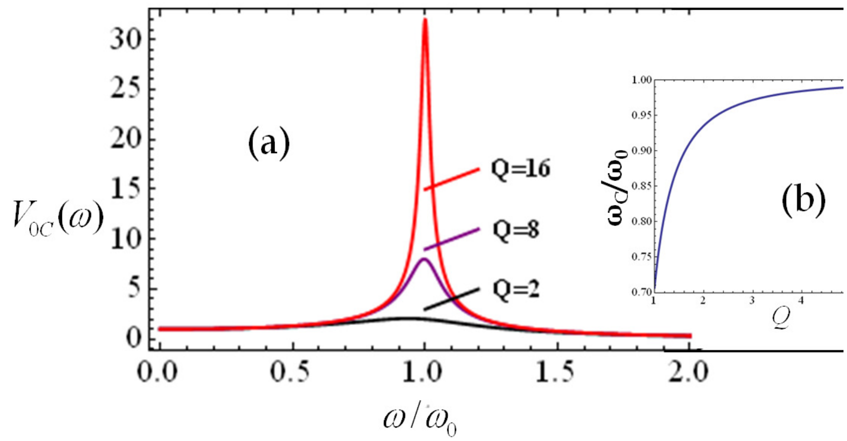


Figure 11.10: (a) Plot of the amplitude of the voltage across the capacitor as a function of the driving frequency for $Q = 2$, 8, and 16. (b) The resonance frequency ω_C, here the ratio ω_C/ω_0 as a function of Q shows that resonance frequency tends to the natural frequency ω_0 as Quality factor Q of the oscillator rises.

One way to look at ϕ is that it tells us about the lag or lead of the phase of the oscillation of the voltage across the capacitor with respect to the oscillation of the driving EMF. To get a better feel of what the difference does, we plot the driving EMF (V) and the steady state capacitor voltage (V_S) in Fig. 11.11 for $\phi = +\pi/2$ rad for amplitude and angular frequency, $\omega = 1$ rad/sec. The plot shows that the maxima of the $V_S(t)$ in this example occur before that of $V(t)$. We say that $V_S(t)$ leads $V(t)$. The lead here is a quarter of the cycle since $\phi = \pi/2$ and the full cycle corresponds to an angle 2π rad.

An easier way to observe the phase difference is through the use of a polar plot, called the **phasor diagram**. Note that a sinusoidal function can be represented by components of a vector, the x-component for cosine and the y-component for the sine. To compare two sinusoidal functions in their vector representations, first you must convert all the sines and cosines to either all sines or all cosines. In our present study we will convert them all into cosines. Then, the x-component of these vectors will contain the physical information. In Fig. 11.12, the vector diagram of $A\cos(\omega t)$ is shown by an arrow pointed in the counterclockwise direction at an angle ωt from the positive x-axis.

The vector rotates counterclockwise in time in keeping with the argument ωt of cosine. The vector is called a **phasor**. When phasors corresponding to the driving EMF and the steady state voltage across the capacitor are plotted in the same diagram, the two phasors point

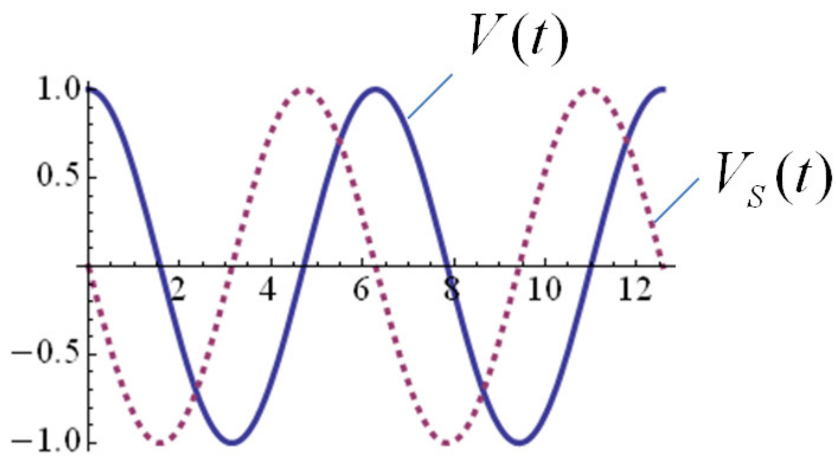


Figure 11.11: Illustration of EMF $V = \cos(t)$ and $V_s = \cos(t + \phi)$ for $\phi = +\pi/2$ rad.

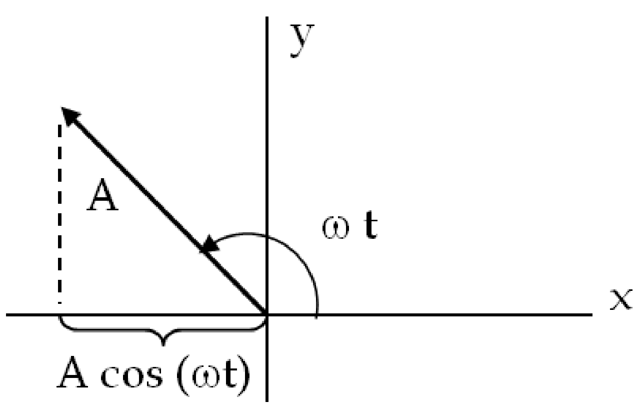


Figure 11.12: The function $Acos(\omega t)$ represented as the x-component of a vector in a phasor diagram. The fictitious vector for representing a cosine or a sine function in this way is called a phasor.

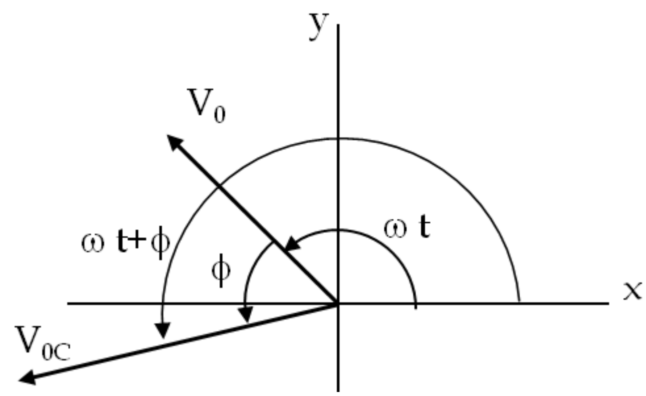


Figure 11.13: Phasor diagram of two phasors drawn on the same diagram for comparing the phases of two cosine or sine functions. Here the phasor of the driving EMF V represents the function $V_0 \cos(\omega t)$ and the phasor for V_S represents the function $V_0 \cos(\omega t + \phi)$ for a positive ϕ.

in different directions, although both rotate at the same rate. If angle $\phi > 0$, then the phasor for V_S is said to be ahead of the phasor for V, while the opposite is the case when $\phi < 0$. In Fig. 11.13, you can examine a situation in which phasor V_S is ahead of phasor V.

11.3.3 Dependence of Phase on the Driving Frequency

We have discussed above how the peak capacitor voltage V_{0C} in a series RLC-circuit driven by a sinusoidal EMF depends on the driving frequency. We found that there is a particular frequency when the resonance of $V_C(t)$ takes place in the circuit. The phase difference of $V_C(t)$ from the driving EMF $V(t)$ also depends on the driving frequency given by Eq. 11.46 which we rewrite here.

$$\tan(\phi) = \frac{1}{\omega\tau_C} - \omega\tau_L, \tag{11.56}$$

where $\tau_C = RC$ and $\tau_L = L/R$. This dependence is shown in Fig. 11.14 for $L = 1$ H, $C = 1$ F, and $R = \frac{1}{2}\Omega$, $\frac{1}{5}\Omega$, and $\frac{1}{20}\Omega$ corresponding to Q-factors, $Q = 2$, 5, and 20 respectively. The plot of ϕ shows a transition from $+\pi/2$ at low frequencies to $-\phi/2$ for high frequencies as the frequency is varied across the resonance frequency. The transition is also sharper for a circuit with a higher Q-factor. The phase ϕ goes through zero at the natural frequency, i.e. when $\omega = \omega_0$ as shown in the figure.

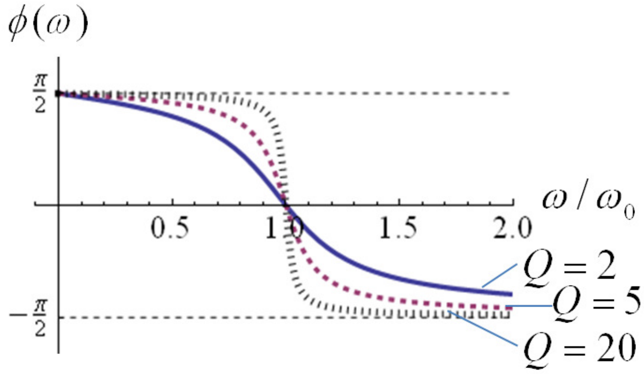


Figure 11.14: Phase angle changes sign at the resonance. The sharpness of transition depends on the Q-factor of the oscillator as shown in another figure.

11.3.4 Resonance Of Steady Current

For times $t >> 2\tau_L$ the current in the circuit reaches a steady state, and oscillates at the same frequency as the driving frequency as we have seen above. Let us write the steady current in the circuit as

$$I_S(t) = I_0 \cos\left(\omega t + \phi + \frac{\pi}{2}\right), \tag{11.57}$$

where the amplitude I_0 and phase constant ϕ depend on the driving frequency ω as already given in the last subsection.

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}, \tag{11.58}$$

$$\tan(\phi) = \frac{1}{\omega \tau_C} - \omega \tau_L, \tag{11.59}$$

where $\tau_C = RC$ and $\tau_L = L/R$. In the following we analyze the consequences of these formulas for the amplitude I_0 and phase $\omega t+\phi+\pi/2$ of the current. The frequency dependence of ϕ here is the same as that of the voltage across the capacitor V_C which has already been discussed above. In Fig. 11.15, a phasor diagram has been drawn to illustrate the phases of the driving EMF, the steady capacitor voltage and the steady current.

Now, we look at the amplitude I_0 of the steady current as we change the angular frequency of the driving EMF. From the expression for I_0 given above, you can think of I_0 as a function of ω for fixed V_0, R, L and C. For a lightly-damped oscillator, the plot of I_0 versus ω shown in Fig. 11.16 clearly exhibits a maximum current when the driving EMF has a particular frequency called the resonance frequency of the current. We will use the symbol ω_I for the resonance frequency of the current in the circuit.

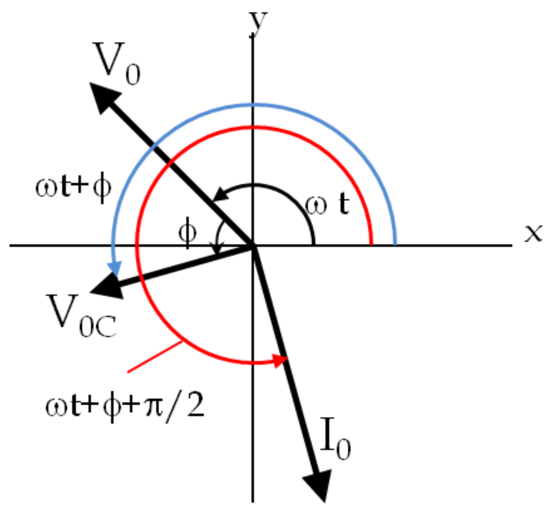


Figure 11.15: A phasor diagram showing the driving EMF, current in the circuit and the capacitor voltage. Note that phasors rotate counterclockwise with time. The length of each phasor corresponds to the amplitude of the quantity. Since phasors in this diagram correspond to different types of physical quantities we cannot compare their lengths. However, since all three phasors rotate at the same rate ω, the diagram provides a visual way to compare their phases or timings.

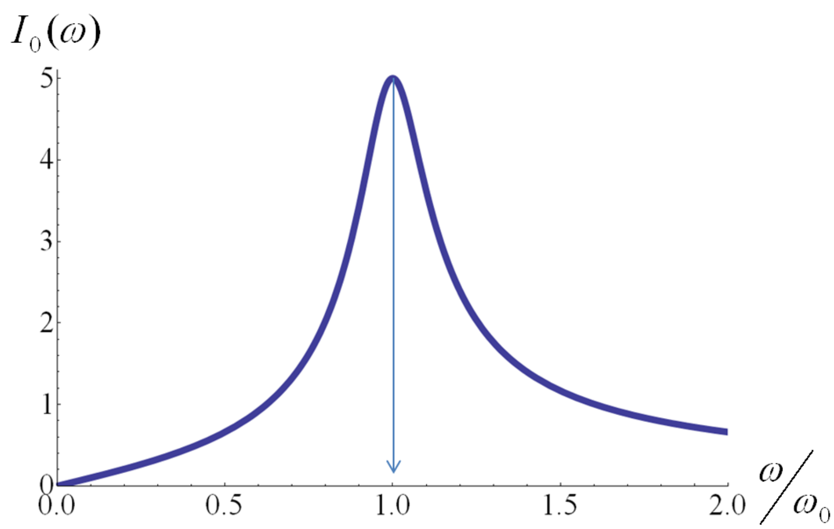


Figure 11.16: Resonance of amplitude of current for $R = 0.2\ \Omega$, $L = 1$ H, $C = 1$ F, and the peak voltage of the driving EMF set to $V_0 = 1$ V. The peak current in the circuit is maximum when the driving frequency ω is equal to the natural frequency ω_0 of the circuit.

From the expression for the peak current given in Eq.. 11.58 we can prove analytically that the resonance frequency ω_I is equal to ω_0, the natural frequency. To deduce the expression for the resonance frequency of current we treat I_0 as a function of ω, set the derivative with respect to ω to zero, and solve the equation for ω. This gives the following for the resonance frequency of the current in the circuit.

$$\boxed{\omega_I = \omega_0 = \frac{1}{\sqrt{LC}}.} \tag{11.60}$$

Notice that the resonance frequency of the current in the circuit is different from the resonance frequency of the capacitor voltage. Other aspects of the resonance of I are similar to the resonance of V_C and we will not discuss them.

11.3.5 Power Delivered to the Circuit by the Source

The practical importance of the resonance phenomenon is seen dramatically in the choice of the frequency of the driving power source in delivering power to the rest of the circuit. The energy at any instant is stored in the capacitor and the inductor, but only the resistor in the circuit dissipates energy. Therefore, the average energy used by the circuit during an interval will be equal to the product of the instantaneous power into the resistor and the interval of time. Thus, the energy deposited in the resistor in an interval of time between t to $t + dt$ will be given by

$$P(t)dt = I(t)V_R(t), \tag{11.61}$$

where $I(t)$ is the current through the resistor and V_R is the voltage drop across the resistor. From Ohm's law we can replace V_R by IR and obtain

$$P(t)dt = I_0^2 R \cos^2(\omega t + \delta), \tag{11.62}$$

where δ is the phase of the current relative to the phase of the driving EMF. The energy deposited in the resistor per cycle is obtained by integrating this expression over a period, $T = 2\pi/\omega$.

$$\text{Energy(one cyle)} = \int_0^T P(t)dt = \frac{1}{2} I_0^2 RT. \tag{11.63}$$

Therefore, the average rate at which the energy is dissipated in the resistor will be $\frac{1}{2} I_0 R^2$. This is the average power dissipated.

$$P_{\text{ave}} = \frac{\text{Energy(one cyle)}}{T} = \frac{1}{2} I_0^2 R. \tag{11.64}$$

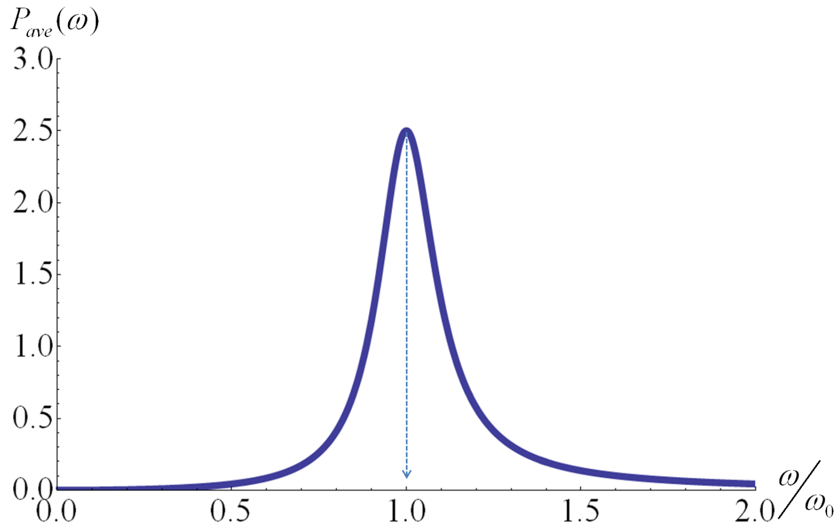


Figure 11.17: The resonance curve of the average power dissipated in the resistor for $R = 0.2\ \Omega$, $L = 1$ H, $C = 1$ F, and the peak voltage of the driving EMF set to $V_0 = 1$ V. The peak power is delivered when the driving frequency ω is equal to the natural frequency ω_0 of the circuit.

Since the peak current I_0 is a function of the driving frequency ω, the average power delivered would also be a function of the frequency ω of the EMF source. Thus, the average power is maximum at the same frequency at which the peak current is maximum. We saw above that the peak current was maximum when the source EMF was oscillating at the same frequency as the natural frequency of the circuit.

Figure 11.17 shows the resonance curve of average power. The curve shows that the maximum power is delivered to the circuit when the circuit is driven at the natural frequency. Denoting the resonance frequency of power as ω_R we have

$$\boxed{\omega_R = \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(Resonance of Power.)}} \tag{11.65}$$

This formula for the average power in the resistor differs from the power for the constant current circuit we had found in an earlier chapter. There, we had found that the power dissipated in a resistor is I^2R, but here we find that the average power is $\frac{1}{2}I_0^2R$ where I_0 is the amplitude of the oscillating current, which changes between $-I_0$ and $+I_0$ sinusoidally in time. It is possible to define an averaged current, called the **root-mean squared current**, or the **RMS-current**, we can rewrite the average power P_{ave} as a formula that is similar to a DC circuit.

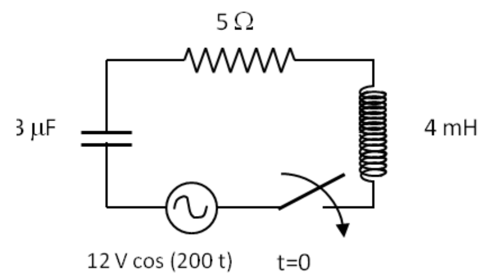
$$\boxed{I_{\text{rms}} = \frac{I_0}{\sqrt{2}}} \tag{11.66}$$

This will give the average power dissipated as

$$\boxed{P_{\text{ave}} = R\, I_{\text{rms}}^2.} \tag{11.67}$$

Example 11.3.1. Current and voltages in a series RLC circuit. Consider the circuit to the right. (a) Find the steady current and voltages across the resistor and the capacitor in the circuit for $t >> 2L/R$.

(b) Find the resonance frequency of current. (c) Compare the energy dissipated in the resistor at the frequency shown in the figure to the energy dissipated if driven at the resonance frequency.



Solution. (a) The steady current is $I = I_0\, \cos(\omega t + \phi + \pi/2)$, where the amplitude and phase constants are as follows in terms of R, L and C of the circuit and ω of the driving source. But before you put in numbers, it often pays to simplify the formulas. Here $1/\omega C$ dominates R and ωL.

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \approx \omega C V_0 = 7.2 \text{ mA}.$$
$$\phi = \arctan\left(\frac{1}{\omega RC} - \frac{\omega L}{R}\right) = 1.6 \text{ rad}.$$

The voltage across the resistor and the capacitor are

$$V_R = RI = RI_0\, \cos(\omega t + \phi + \pi/2) = (35 \text{ mV})\, \cos(\omega t + \phi + \pi/2)$$
$$V_C = \frac{I_0}{\omega C} \sin(\omega t + \phi + \pi/2) = (12 \text{ V})\, \cos(\omega t + \phi)$$

(b) The resonance frequency of the current is equal to the natural frequency, $\omega_I = \frac{1}{\sqrt{LC}} = 9129$ rad/sec.

(c) In 1 sec the circuit goes through many cycles, and therefore we use the average power. The average energy dissipated per second = $P_{\text{ave}} \times \Delta t = \frac{1}{2} I_0^2 R \times 1\text{sec} = 130\mu\text{J}$.

The energy dissipated at the resonance frequency would be obtained by using the value of I_0 when $\omega = \omega_I$, the resonance frequency of the current in this circuit. This gives the following value.

$$I_0(\omega = \omega_I) = \frac{V_0}{R} = \frac{V_0}{5\Omega} = 2.4 \text{ A}.$$

Using this current in the power dissipation formula we find that the average energy dissipated at the resonance frequency in one second =

$\frac{1}{2}(2.4\text{ A})^2 \times 5\Omega \times 1\text{ sec} = 14.4\text{ J}$. This is way more than the 130μJ given to the resistor when driven at $\omega = 200$ rad/sec instead of $\omega_I = 9129$ rad/sec. Clearly, the most effective way to put energy in a circuit is to drive the circuit at the resonance frequency of the circuit.