2.3 IMAGE FORMATION BY REFRACTION

2.3.1 Refraction at Plane Interface - Apparent Depth

When you look at a straight rod partially submerged in water, it appears bent at the interface as shown in Fig. 2.13. The reason behind this strange effect is that the image of the rod inside water is a little bit above the actual position of the rod, and does not line up with the part of the rod that is above water. A similar phenomenon is the reason when a fish in water appears to be closer to the surface than it actually is.

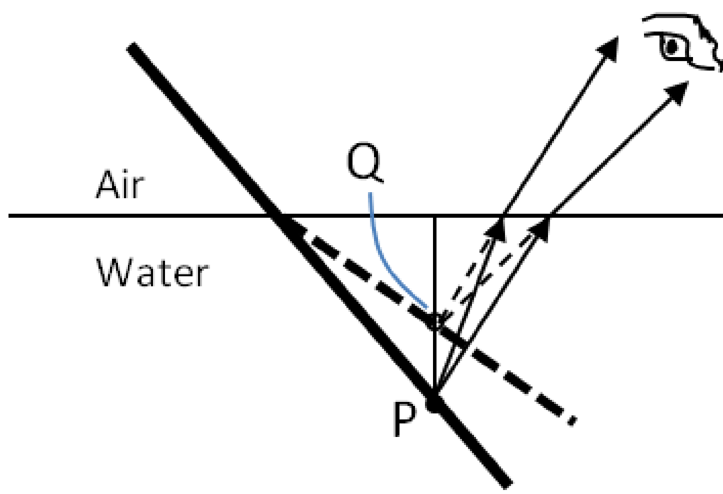

Figure 2.13: Bending of rod at air/water interface. The point P on the rod would appear at the point Q, the image of the point P due to refraction at the interface.

To study image formation as a result of refraction, we ask the following questions:

1. What happens to the rays of light when they enter or pass through another medium?
2. Do the refracted rays originating from a single point meet at some point or diverge away from each other?

To be concrete, let us consider a simple system consisting of two media separated by a plane interface (see Fig. 2.14). The object will be in one medium and the observer in the other. For instance, when you look at a fish from above the water surface - fish would be in

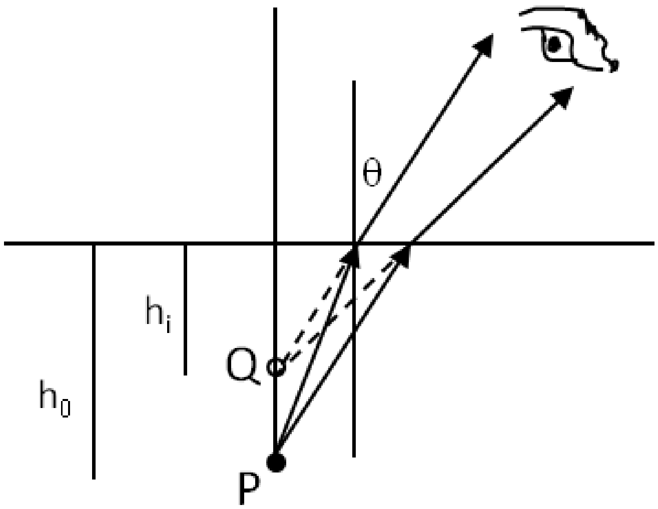

Figure 2.14: Apparent depth due to refraction. [Note: add labels air and water in the figure.]

medium 1 with refractive index 1.33, your eyes in medium 2 with refractive index 1.00, and the surface of water is the interface. The depth that you will "see" (h_i) will be called the **apparent depth**, and the actual depth of the fish will be denoted by h_o.

The apparent depth h_i depends on the angle at which you view the image. For a view from directly above, the so-called the normal view, we can approximate the refraction angle θ to be small, and replace $\sin\theta$ in Snell's law by $\tan\theta$. With this approximation, using the triangles $\triangle$OPR and $\triangle$OQR you can show that the apparent depth is given by the following formula.

$$\frac{h_i}{h_o} \approx \frac{n_2}{n_1}. \tag{2.13}$$

The derivation of this result is left as an exercise for the student. Thus, a fish will appear $\frac{3}{4}$ of the real depth when viewed from above.

2.3.2 Refraction at a Spherical Interface

Spherical shape plays an important role in optics primarily because high quality spherical shapes are far easier to manufacture than other specialized curved surfaces. A spherical shape can be convex if the material is bulging out or concave if the material has been carved out. To study refraction at one spherical surface we will assume that the medium with the spherical surface at one end continues indefinitely.

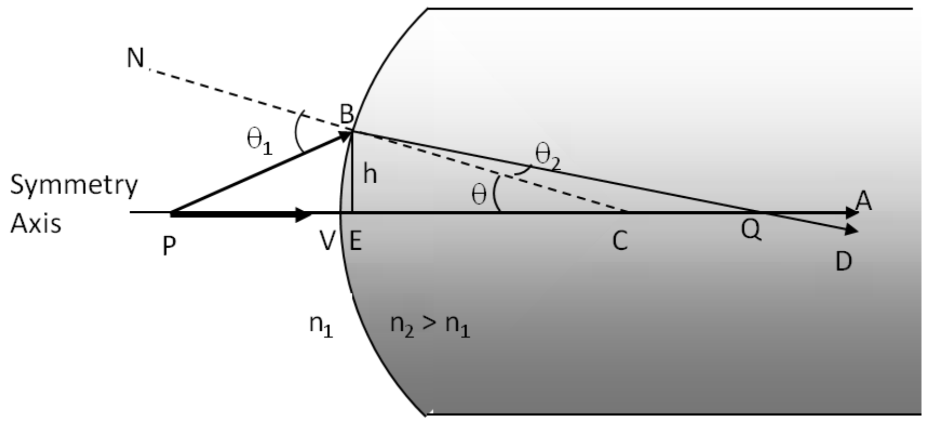

Figure 2.15: Refraction at a convex surface ($n_2 > n_1$).

Refraction at convex surface

For concreteness, consider a point source of light at point P in front of a convex surface made of glass. Let R be the radius of curvature, n_1 the refractive index of the medium in which object point P is located and n_2 the refractive index of the medium with the spherical surface. We are not concerned with the other face of medium n_2 since we have assumed the second face of the medium to be infinitely far away. We want to know what happens as a result of refraction on this face only.

Because of the symmetry in the physical setting here it is sufficient to examine the rays in only one plane. Fig. 2.15 shows two rays of light PVCA and PBQD that start at the point object P and refracted at the interface. In the following we derive a formula relating the object distance VP denoted by the letter p, image distance VQ denoted by the letter q , and the radius of curvature VC denoted by the letter R.

We will do our calculations in the paraxial approximation, i.e., for rays that make small angles with the symmetry axis. Paraxial rays make small angle θ, therefore we make the approximations $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. We also find that

$$EV \approx 0, \tag{2.14}$$

$$n_1\theta_1 \approx n_2\theta_2. \tag{2.15}$$

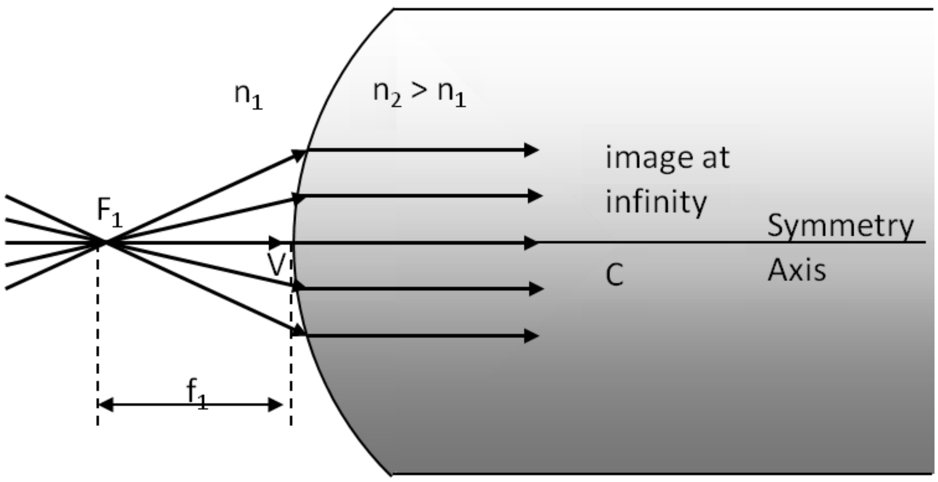

Figure 2.16: First (or object) focus for refraction at a convex surface.

The following geometric relations are also easily seen from the figure.

$$\text{In } \triangle PBE, \quad \angle BPE = \theta_1 - \theta \approx \frac{BE}{VP} = \frac{h}{p}. \tag{2.16}$$

$$\text{In } \triangle BEC, \quad \angle BCE = \theta \approx \frac{BE}{VC} = \frac{h}{R}. \tag{2.17}$$

$$\text{In } \triangle QBE, \quad \angle BPE = \theta - \theta_2 \approx \frac{BE}{VQ} = \frac{h}{q}. \tag{2.18}$$

Equations 2.15-2.18 yield the following relation.

$$\boxed{\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}.} \tag{2.19}$$

If the object is placed at a special point called the first focus or the object focus F_1 then image will be formed at infinity as shown in Fig. 2.16.

We can find the location of the **first focus** F_1 by setting $q = \infty$ in Eq. 2.19.

$$\frac{n_1}{f_1} + \frac{n_2}{\infty} = \frac{n_2 - n_1}{R}. \tag{2.20}$$

Solving for the first focal length f_1, we get the following.

$$\boxed{f_1 = \frac{n_1 R}{n_2 - n_1}.} \tag{2.21}$$

Similarly, one can define a **second focus** or **image focus** F_2 where the image is formed for a far away object (Fig. 2.17).

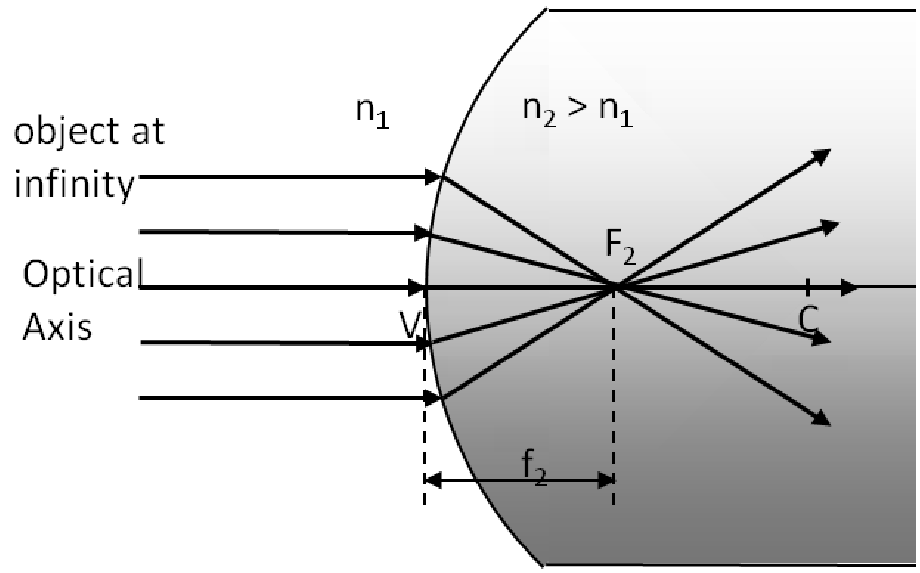

Figure 2.17: Second (or image) focus for refraction at a convex surface.

The location of second focus F_2 is obtained from Eq. 2.19 by setting $p = \infty$.

$$\frac{n_1}{\infty} + \frac{n_2}{f_2} = \frac{n_2 - n_1}{R}. \tag{2.22}$$

Solving for the second focal length f_2, we get the following.

$$\boxed{f_2 = \frac{n_2 R}{n_2 - n_1}.} \tag{2.23}$$

Note that the object focus is at a different distance from the vertex than the image focus since $n_1 \neq n_2$.

Refraction at a concave surface

Consider a point source P in front of a concave surface as shown in Fig. 2.18. You may think of the point P being in air and the concave surface of glass if you like. The refracted rays BD and VA diverge away from each other. Therefore they will not pass through any real image point. When refracted rays VA and BD are extended backwards they meet at point Q which is the virtual image of point P. Once again, we measure all distances from the vertex V. Thus VP is the object distance p, VQ the image distance q and VC the radius of curvature R.

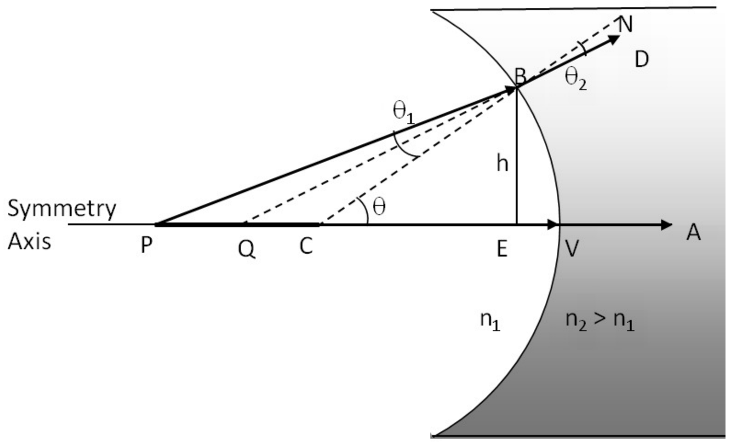

Figure 2.18: Refraction at a concave surface ($n_2 > n_1$).

Using the geometry given in Fig. 2.18, and the paraxial approximation with $EV \approx 0$, one can derive the following relation. The derivation is left as an exercise for the student to complete.

$$\frac{n_1}{p} + \frac{n_2}{-q} = \frac{n_2 - n_1}{-R}. \tag{2.24}$$

This equation for a concave surface is similar in form to Eq. 2.19 of the convex surface. The only differences are the negative signs with q and R. If we assume the following sign convention then we can write the same equation for both cases.

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \quad \text{(Sign Convention Required!)} \tag{2.25}$$

Sign convention for refraction at spherical surfaces.

1. p positive if object to the left of the surface otherwise negative.
2. q positive if image to the right of the surface otherwise negative.
3. R positive if center of circle falls on the right and negative if center on the left.