9.3 DYNAMICS OF FIXED-AXIS ROTATION

The laws of the dynamics of rotation are deduced from the fundamental laws of mechanics as given by the Newton's laws of motion. In this section we will first derive the relation governing the change of angular momentum of a single particle and then apply the result to multiparticle systems. We will find that the angular momentum of a body is essential for understanding the rotational dynamics. We will also discover that in a fixed-axis rotation we need study only one component of the angular momentum and the problem of rotation of a rigid body becomes equivalent to a one-dimensional problem.

9.3.1 Rotational Dynamics of a Single Particle

The dynamics of a single particle is completely given by Newton's laws of motion. By rotational dynamics of a single particle, we mean the study of the angular momentum of the particle. The second law of motion gives the rate at which momentum of a particle changes. Here, we ask: how can the angular momentum of a particle change? Recall that the angular momentum of a single particle about a reference at the origin is given by

$$\vec{L} = \vec{r} \times \vec{p},$$

where $\vec{r}$ is the position vector of the particle and $\vec{p}$ the momentum of the particle. We start by taking the derivative of both sides of this equation with respect to time t.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \tag{9.49}$$

The first term on the right side is $\vec{v} \times \vec{p}$. Since the velocity and momentum vectors point in the same direction, their cross product is equal to zero.

$$\text{First term in Eq.9.49:} \quad \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m\left(\vec{v} \times \vec{v}\right) = 0. \tag{9.50}$$

Using Newton's second law, we can replace $d\vec{p}/dt$ in the second term on the right side of Eq. 9.49 by the net force $\vec{F}_{net}$. This gives us the law for the rate of change of the angular momentum of a particle.

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{\text{net}}.} \tag{9.51}$$

The right-hand side of this equation is the net torque $\vec{\tau}_{net}$ on the particle about the origin.

$$\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}_{\text{net}}. \tag{9.52}$$

Note that both the torque and the angular momentum depend upon $\vec{r}$, which depends upon the choice of the reference point. The unit of torque in the SI system is kg. m^2/s^2, or N.m, which is the same as the unit of energy, Joule (J). The usual practice for the unit of torque is to use N.m rather than Joule.

Various techniques for the evaluation of torques have been discussed in Chapter 5. A student should review them before proceeding further in this chapter.

9.3.2 Rotational Dynamics of Extended Bodies

By now you must already know that we examine the dynamics of extended bodies by representing them as collections of particles. This makes sense since all physical bodies are eventually made up of discrete particles. Suppose, there are N particles that make up an extended body. A separate consideration of the dynamics of each particle will generate N vector equations of motion, one per particle. Let us label the particles as 1, 2, $\cdots$, N. Let m_1, m_2, $\cdots$, m_N be the masses of the particles and $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_N$ their position vectors with respect to the origin. Each particle of the body will have external forces from outside the system and internal forces from other particles. Let us label the external forces on individual particles by $\vec{F}_1^{\text{ext}}$, $\vec{F}_2^{\text{ext}}$, etc, and the internal forces between particles by $\vec{F}_{12}$ for force of 2 on 1, $\vec{F}_{13}$ for force of 3 on 1, $\cdots$, $\vec{F}_{ij}$ for force of j on i, etc. The rate of change of the angular momenta about the origin for each particle are given by by the following N vector equations.

$$\begin{aligned}
\frac{d\vec{L}_1}{dt} &= \vec{r}_1 \times \vec{F}_1^{\text{ext}} + \vec{r}_1 \times \vec{F}_{12} + \vec{r}_1 \times \vec{F}_{13} + \cdots + \vec{r}_1 \times \vec{F}_{1N} \\
\frac{d\vec{L}_2}{dt} &= \vec{r}_2 \times \vec{F}_2^{\text{ext}} + \vec{r}_2 \times \vec{F}_{21} + \vec{r}_1 \times \vec{F}_{23} + \cdots + \vec{r}_1 \times \vec{F}_{2N} \\
&\vdots \\
\frac{d\vec{L}_N}{dt} &= \vec{r}_N \times \vec{F}_N^{\text{ext}} + \vec{r}_N \times \vec{F}_{N1} + \vec{r}_1 \times \vec{F}_{N2} + \cdots + \vec{r}_1 \times \vec{F}_{N-1,N}
\end{aligned}$$

Summing these equations gives an equation for the rate of change of the total angular momentum of the whole body, which can be separated into the torques from the external forces and the internal forces.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}^{\text{ ext}} + \vec{\tau}_{\text{net}}^{\text{ int}} \tag{9.53}$$

where

$$\vec{\tau}_{\text{net}}^{\text{ ext}} = \vec{r}_1 \times \vec{F}_1^{\text{ ext}} + \vec{r}_2 \times \vec{F}_2^{\text{ ext}} + \cdots + \vec{r}_N \times \vec{F}_N^{\text{ ext}},$$

and $\vec{\tau}_{\text{net}}^{\text{ int}}$ is the sum of the torques from the internal forces. These internal torques have the following form for a force between arbitrary particle i and j.

$$\text{One term in } \vec{\tau}_{ij}^{\text{ int}}: \quad (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}. \tag{9.54}$$

If the internal forces between the particles of the system act only along the line joining the two particles, then these terms will all be zero since then $\vec{F}_{ij}$ will be either parallel to $(\vec{r}_i - \vec{r}_j)$ vector or antiparallel to it. This will leave only the torques from the external forces to generate the change in the net angular momentum of the body. Therefore, we find that the rate of change of the net angular momentum of an extended body is equal to the net external torque on all particles of the body.

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}^{\text{ ext}}.} \tag{9.55}$$

This derivation is completely general and applies to the motion of any system as long as the forces between the particles act along the line between the particles. Care must be taken into computing the torque from external forces since each external force acts on individual particles, and the torque must be evaluated with respect to the same reference point as the angular momentum of the body.

9.3.3 The Law for Fixed Axis Rotation

In a fixed-axis rotation, the direction of the axis of rotation does not change with time. This simplifies the equations derived above since we do not need to be as general as Eq. 9.55. Suppose the fixed-axis of rotation is along the z-axis of a Cartesian coordinate system. Then, the angular momentum has only the z-component non-zero. Therefore, we need only the z-component of Eq. 9.55:

Simpler Notation:
$\frac{d}{dt}(I\omega) = \tau$
(Components along fixed axis understood)

$$\boxed{\text{Fixed Axis:} \quad \frac{dL_z}{dt} = \tau_{\text{net,z}}^{\text{ext}}} \tag{9.56}$$

which can also be written in terms of the zz component of moment of inertia and the z-component of the angular velocity.

$$\boxed{\text{Fixed Axis:}\quad \frac{d}{dt}\left(I_{zz}\omega_z\right) = \tau^{\text{ext}}_{\text{net,z}}.} \tag{9.57}$$

If the body does not change shape, then its moment of inertia remains unchanged, so that this equation can be written using z-component of the angular acceleration.

Simpler Notation: $I\alpha = \tau$ **(Components along fixed axis understood)**

$$\boxed{\text{Fixed Axis/Rigid Body:}\quad I_{zz}\alpha_z = \tau^{\text{ext}}_{\text{net,z}}.} \tag{9.58}$$

Equations 9.57 and 9.58 are analogous to the following two equations of the center of mass of an extended body.

$$\frac{d}{dt}\left(MV_z^{\text{CM}}\right) = F^{\text{ext}}_{\text{net,z}} \tag{9.59}$$

$$MA_z^{\text{CM}} = F^{\text{ext}}_{\text{net,z}} \tag{9.60}$$

where V_z^{CM} and A_z^{CM} are the z-components velocity and acceleration of the center of mass, respectively. Beware of the pitfalls of pushing the analogy between a translational motion in one dimension and a rotational motion about a fixed-axis too far since this superficial analogy does not go over to a more general motion of the bodies.

9.3.4 Practice With Torque Calculations

See section 5.5.

9.3.5 Example Problems - Single Rigid Bodies

Example 9.3.1. Torque on a rotating a wheel.

A wheel of mass 2 kg and radius 30 cm is rotating counterclockwise when viewed from one end of the axle. It is seen that the rotation speed is increasing uniformly by 1.5 rad/sec for every second. Find the net torque on the wheel. For the purposes of the moment of inertia calculations, assume the wheel to be a uniform thin disk.

Solution. The data given in the problem shows that the angular acceleration has magnitude 1.5 rad/sec^2 and direction same as the direction of the angular velocity since the rotation speed is increasing. The direction is given as counterclockwise sense of rotation when looked from a particular side of the axle. Suppose that side of the axle is pointed towards positive z-axis. Then we have the z-component of the angular acceleration as

$$\alpha_z = 1.5\ \text{rad/sec}^2$$

Now, we need the zz component of the moment of inertia. Using the disk formula for a disk for an axis through the center and perpendicular to the disk we find that,

$$I_{zz} = \frac{1}{2}MR^2 = \frac{1}{2}(2\ \text{kg})(0.30\ \text{m})^2 = 0.09\ \text{kg.m}^2.$$

Using the equation of motion for the rotation we find the z-component of the torque as

$$\tau_z = I_{zz}\alpha_z = 0.135\ \text{kg.m}^2/\text{s}^2.$$

Other components of the torque are zero here. Therefore, the torque has the magnitude $0.135\ \text{kg.m}^2/\text{s}^2$ or 0.135 N.m, pointed towards the positive z-axis, which is the direction of the particular side of the axle described in the question. Note: when the "unit" radian is multiplied by a dimensionful number such as a meter, the result is not radian-meter, but just meter since radian is a ratio of two lengths, and therefore, does not have a dimension.

Example 9.3.2. A simple plane pendulum.

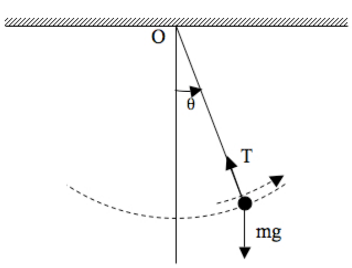

Figure 9.28: A plane pendulum.

In a simple pendulum a mass m is attached to the end of a "massless" string of length L which is then tied to a post so that the mass can swing in a vertical arc. In this example, we will work out the equation of motion of the pendulum. We will also solve the equation of motion by making an approximation for small angular displacement from the vertical.

From Fig. 9.28 of the pendulum, we note that the axis of the rotation of the pendulum passes through the suspension point O. The axis is fixed in time and perpendicular to the plane of the drawing. We have learned above that the signs of various quantities are easier to take into account if we work with with components of vectors in a particular coordinate system. Therefore, we start with a choice of coordinate directions. Let the out of page be positive z direction and vertically up be positive y direction. The choices of coordinate system are shown in Fig. 9.29.

As shown in the figures, there are two forces on the mass: the tension from the string and the force of gravity from the Earth. The torque of the tension force about any axis through O is zero since the line of force goes through this point and the lever arm for this force will be zero. But the torque from the weight is not zero. Therefore, the net torque on the mass comes from only one force - its weight. At the instant of the motion shown in the drawing, the z-component of the torque has a clockwise sense, which means that the torque is

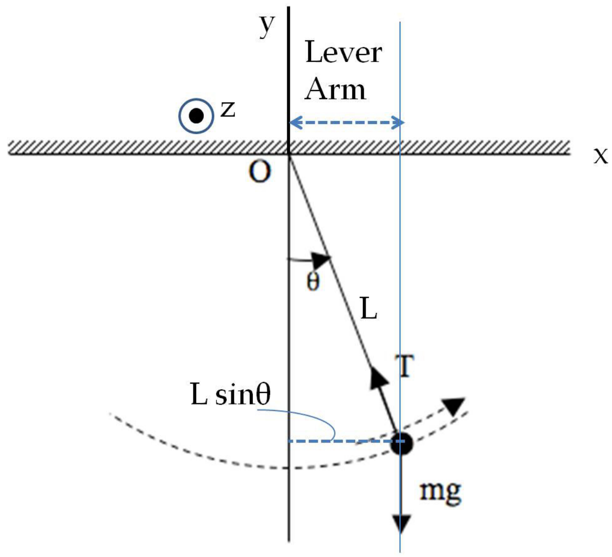

Figure 9.29: Calculation of torque about suspension point O. Note: z-axis is coming out of page.

pointed into the page or towards the negative z-axis. Therefore the z-component of torque will be negative

$$\text{Torque, } \tau_z = -mgL\sin\theta,$$

where $L\sin\theta$ is the lever arm of the weight about O. The zz component of moment of inertia of the mass m about O is mL^2. Now, the equation of motion for the rotation can be written to obtain the z-component of the angular acceleration from

$$mL^2\alpha_z = -mgL\sin\theta.$$

Writing the z-component of the angular acceleration in terms of the angle of rotation, this equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.$$

This relation describes the dynamics of a plane pendulum completely. However, this equation is difficult to solve in this form, but becomes easily solvable when we look at the motion for small angles. For small angle oscillations, we can make a linear approximation of $\sin\theta$ as $\sin\theta \approx \theta$ when the angle is expressed in radians. Therefore, the equation of motion takes the following simpler form.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta. \tag{9.61}$$

The solution of this equation is oscillatory in time and is given by a combination of sine and cosine functions of time with a well-defined period T.

$$\theta(t) = A\cos\left(\frac{2\pi t}{T}\right) + B\sin\left(\frac{2\pi t}{T}\right), \tag{9.62}$$

where A and B are constants of motion to be determined from the initial values of the angle θ and the z-component of the angular velocity ω_z. The time period T of the pendulum in this solution is given by

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

This formula for the time period can be verified by inserting the solution given in Eq. 9.62 into the equation of motion, Eq. eq:simp-pendulum-1. This step is left an exercise for the student.

Example 9.3.3. Physical pendulum

A rigid body hung from a post swings just like a pendulum. Such oscillating bodies are called **physical pendulums**. Almost anything can be a physical pendulum. Find an expression for the time period of the oscillation of a physical pendulum.

Solution. An illustration of a physical pendulum and the forces on the body are shown in Fig. 9.30. Here the axis of rotation passes through the suspension point. Let M be the mass and D the distance between the axis of rotation and the center of mass (CM) of the body.

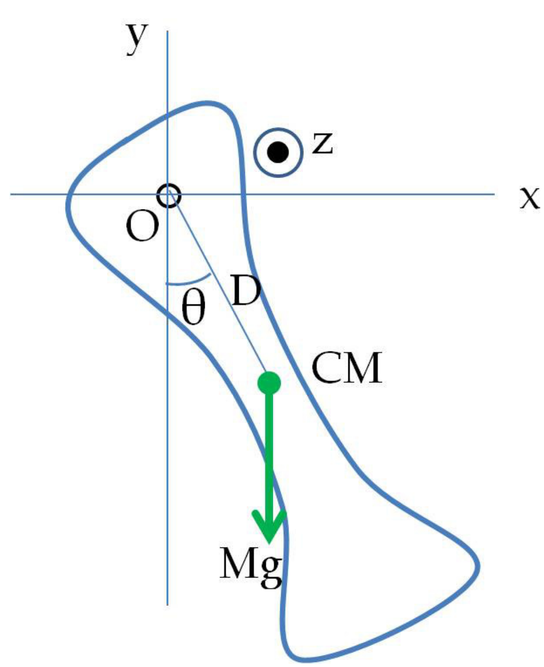

Figure 9.30: A physical pendulum can be any rigid body that can swing about an axis. Note z-axis is pointed out of page.

Let the z-axis be pointed out of the page. Since the force of gravity is proportional to the mass of the particles of the extended body, it can be shown that the torque on an extended body will equal to the entire weight acting on the CM only. The net torque on the physical pendulum is from the weight at the CM only. Let I_O is I_{zz} about the axis through the suspension point, i.e. the moment of inertia about an axis through O and perpendicular to the plane of the drawing. Therefore, the z-component of the equation of motion of angular momentum of the body gives the following equation for the angle θ made by the line between the suspension point and the CM with the vertical direction.

$$I_O \frac{d^2\theta}{dt^2} = -MgD\sin\theta.$$

To deduce the formula for the time period we need only the oscillations for small oscillations. For small oscillations, we use $\sin\theta \approx \theta$ to yield the following.

$$\frac{d^2\theta}{dt^2} = -\frac{MgD}{I_O}\theta$$

This equation is similar to the equation for a plane pendulum, and can be solved by analogy. Therefore, we find that the period of oscillation T of a physical pendulum is given by:

$$T = 2\pi\sqrt{\frac{I_O}{MgD}}.$$

Suppose the physical pendulum is a rod of length L and mass M. Then, we will have

$$I_O = \frac{1}{3}ML^2$$

and

$$D = \frac{L}{2}$$

Therefore, the period of oscillation of the rod of length L suspended from one end would be

$$T = 2\pi\sqrt{\frac{I_O}{MgD}} = 2\pi\sqrt{\frac{2L}{3g}}.$$

9.3.6 Example Problems - Coupled Systems

Example 9.3.4. Unwinding a tape on an anchored wheel. Consider a wheel of mass M and radius R that is free to rotate about an axle through its center. Several turns of a thin light non-sticky tape, whose mass can be neglected, are wound at the edge and

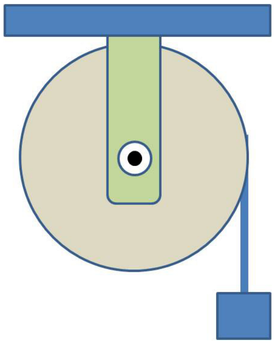

Figure 9.31: Example 9.3.4.

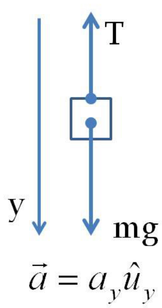

Figure 9.32: Free-body diagram of block.

a block of mass m is attached to the free end. When the block is released, the tape unwinds smoothly and the wheel rotates. Since the tape is thin, assume that the distance of the tape from the center of the wheel is approximately equal to the radius R of the wheel. Assuming the friction at the axle to be negligible, find the angular acceleration of the wheel and the tension in the tape.

Solution. Here the translational motion of the block and the rotational motion of the wheel are coupled. For every radian of rotation, the tape unwinds by a distance R as given by the arc-radius-angle formula of a circle. Therefore, every radian of rotation is accompanied by a vertical displacement of the mass m by R. More generally, for an angular displacement of θ radians of the wheel, the vertical displacement of the block is $R\theta$.

Let the z-axis be point into the page and the y-axis pointed down as shown in Figure 9.33. Then, torque and angular acceleration of the wheel will be along the z-axis while the forces and acceleration of the block will be along the y-axis.

The vertical motion of the block

We start by drawing the free-body diagram for the translational motion of the block as shown in Fig. 9.32. From the free-body diagram, we get the following equation for the y-component of Newton's second law of motion for the block.

$$mg - T = ma_y \tag{9.63}$$

The rotational motion of the wheel

For the rotational motion of the wheel, we examine torques on the wheel. Fig. 9.33 shows various forces that act on the wheel and we need to determine their torques about the center of the wheel.

The force F_A is a net force from the axle on the wheel, which acts all along the contact surface between the wheel and axle. We assume that the torque from this force is negligible as given in the problem statement.

The forces shown as F_{tp} are the forces from the tape pressing on the wheel. These forces act normally to the surface and can be assumed to have direction such that they go through the axis of rotation. This will give zero torque about the axis of rotation.

The force of gravity acts on all the particles of the wheel individually, but the torque from force of gravity on various particles is equal to the torque of the entire weight of the wheel placed at the CM of the wheel, which is a point on the axis of rotation here. Therefore,

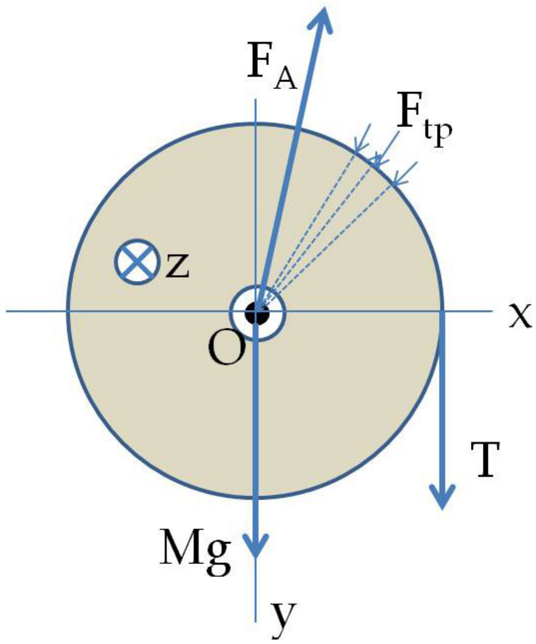

Figure 9.33: Forces on the wheel for torque calculation about O.

the weight also has a zero torque about the center of the wheel.

Thus, the tension in the tape is the only force that has a non-zero torque on the wheel about the z-axis through the center of the wheel. Since, the moment of inertia of the wheel about the axis is $I_{zz} = \frac{1}{2}MR^2$, the z-component of the rotational equation of motion of the wheel is as follows.

$$TR = \frac{1}{2}MR^2\alpha_z, \tag{9.64}$$

where α_z stands for the z-component of the angular acceleration of the wheel.

The constraint on the system

Finally, the coupling of the rotational motion of the wheel and the translational motion of the block gives rise to a relation between distance travelled by the block and angle of rotation of the wheel. In time dt the vertical displacement dy of the block and the angular displacement $d\theta_z$ of the wheel are related since the tape unwinds at the circumference of the wheel:

$$dy = Rd\theta_z$$

Dividing this relation by dt we find that the velocity of the block and the angular velocity of the wheel are related.

$$v_y = R\omega_z.$$

Taking a time derivative of this equation shows that the acceleration of the block and the angular acceleration of the wheel are related.

$$a_y = R\alpha_z. \tag{9.65}$$

Now, we can solve Eqs. 9.63, 9.64 and 9.65 together to obtain expressions for T and α_z in terms of the masses and the acceleration due to gravity.

$$T = \left(\frac{M}{M+2m}\right) mg.$$

$$\alpha_z = \left(\frac{2m}{M+2m}\right)\frac{g}{R}.$$

The acceleration of the block can be obtained by multiplying the angular acceleration by R.

$$a_y = \left(\frac{2m}{M+2m}\right) g$$

Example 9.3.5. Atwood machine. In an Atwood machine two blocks connected by a light string hang from the two sides of a pulley. Suppose the pulley is frictionless but the mass of the pulley is not negligible compared to the masses of the blocks. Let M and R be the mass and radius of the pulley, which, for purposes of moment of inertia, can be assumed to be a uniform disk. Let m_1 and m_2 be the masses of the two blocks. What would be tensions in the string on the two sides and the acceleration of the masses and the angular acceleration of the pulley?

Solution. We have solved a similar problem before by assuming that the pulley was massless and frictionless. These assumptions about the pulley let us assert that the tensions on the two sides had the same magnitude. Here, we cannot make this assumption. We will see below that the equation of rotation of the pulley will become inconsistent if we assume that the tensions on the two sides have the same magnitude.

Let T_1 and T_2 be magnitudes of tension force on the two sides of the pulley.

To solve this problem for accelerations and tensions we need to write out the equations of motion of the blocks and the pulley. We will not go into as much detail as was presented in the last example. A student who has skipped Example 9.3.4 given above is advised to go back and study that example first. Suffice to say that we need to identify the forces and relevant torques.

In Fig. 9.34 you will find the free-body diagrams for the forces on the two masses and the torque diagram for the pulley that helps

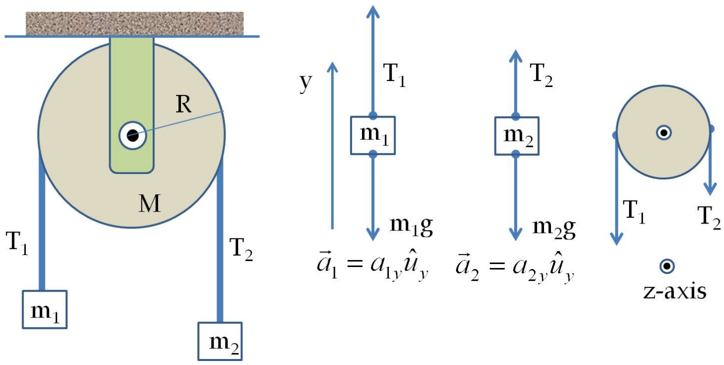

Figure 9.34: Atwood's machine. Free-body diagrams for the two masses and axes for computing the components. The two forces on the pulley that have non-zero torques about the center. Other forces on the pulley are not shown in this figure.

us obtain the following components of the equations of motion in the coordinate system shown in the figure.

$$\text{Translation of } m_1: \; T_1 - m_1 g = m_1 a_{1y} \tag{9.66}$$

$$\text{Translation of } m_2: \; T_2 - m_2 g = m_2 a_{2y} \tag{9.67}$$

$$\text{Rotation of pulley}: \; T_1 R - T_2 R = \frac{1}{2} M R^2 \alpha_z \tag{9.68}$$

The coupling of motion of the three moving bodies leads to the change in coodinates in time dt as follows.

$$\text{Translation of } m_1 \text{ and } m_2: \; dy_1 = -dy_2 \tag{9.69}$$

$$\text{Translation of } m_1 \text{ and rotation of pulley}: \; dy_1 = -R d\theta_z \tag{9.70}$$

The relative signs are very important in these relations. You can check the signs as follows: let m_1 moving up, then $dy_1 > 0$. This will make m_2 go down, which makes $dy_2 < 0$ whose magnitude is equal to that of dy_1. When $dy_1 > 0$, the wheel rotates clockwise, which makes the angular displacement pointed towards the negative z-axis, and hence $d\theta_z < 0$. The string is at the edge of the pulley which is at the circumference of a circle of radius R, therefore, the absolute value of dy_1 must be R times absolute value of $d\theta_z$. These relations give rise to the following relations among the accelerations.

$$\text{Translation of } m_1 \text{ and } m_2: a_{1y} = -a_{2y} \tag{9.71}$$

$$\text{Translation of } m_1 \text{ and rotation of pulley}: a_{1y} = -R\alpha_z \tag{9.72}$$

Therefore,we can write Eqs. 9.66-9.68 replacing all accelerations in terms of α_z.

$$T_1 - m_1 g = -m_1 R\alpha_z \tag{9.73}$$

$$T_2 - m_2 g = m_2 R\alpha_z \tag{9.74}$$

$$T_1 - T_2 = \frac{1}{2} M R\alpha_z \tag{9.75}$$

It is elementary to solve these equations for α_z, T_1 and T_2. The results are:

$$R\alpha_z = -\left(\frac{m_1 + m_2}{m_2 - m_1 + M/2}\right) g \tag{9.76}$$

$$T_1 = m_1 g - m_1 R\alpha_z \tag{9.77}$$

$$T_2 = m_2 g + m_2 R\alpha_z \tag{9.78}$$

The accelerations of the two blocks will be magnitude equal to R times the magnitude of α_z.