1.1 ACCELERATING FRAME

1.1.1 Kinematics in Accelerating Frame

The position of a particle in an accelerating frame is defined in the same way as in any other frame. The position vector of a point particle is the displacement vector from the origin to the current position of the particle. The relation with a non-accelerating frame can be established by drawing the two frames and noting the triangle of vectors formed by the position of the particle $\vec{r}$ and $\vec{r}'$ in the non-accelerating and accelerating frames respectively and the vector $\vec{R}$ from the origin O of the non-accelerating frame to the origin O' of the accelerating frame (Fig 1.1).

$$\vec{r} = \vec{R} + \vec{r}\,' \tag{1.1}$$

The velocity $\vec{V}$ and acceleration $\vec{A}$ of the accelerating frame with

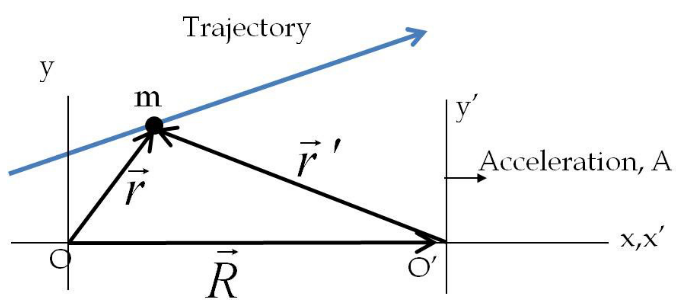

Figure 1.1: Position vectors of a point mass in two frames.

respect to the non-accelerating frame are simply the velocity and acceleration of origin O' of the accelerating frame, and can be obtained by successively taking time derivatives of vector $\vec{R}$.

$$\vec{V} = \frac{d\vec{R}}{dt} \tag{1.2}$$

$$\vec{A} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{R}}{dt^2} \tag{1.3}$$

The velocity $\vec{v}'$ of the particle with respect to the accelerating frame is simply the rate at which position $\vec{r}'$ of the particle changes with time.

$$\vec{v}\,' = \frac{d\vec{r}\,'}{dt} \tag{1.4}$$

Similarly, the acceleration $\vec{a}\,'$ of the particle with respect to the accelerating frame is simply the rate at which velocity $\vec{v}\,'$ of the particle with respect to the same frame changes with time.

$$\vec{a}\,' = \frac{d\vec{v}\,'}{dt} \tag{1.5}$$

Now, by taking successive derivatives of both sides of Eq. 1.1 with respect to time gives us the relation between velocity and acceleration of a particle in an in an inertial and a non-inertial frame.

$$\boxed{\vec{v} = \vec{V} + \vec{v}\,'} \tag{1.6}$$

$$\boxed{\vec{a} = \vec{A} + \vec{a}\,'} \tag{1.7}$$

1.1.2 Newton's Second Law in Accelerating Frame

Newton's second law of a point particle of mass m in an inertial frame is given by

$$m\vec{a} = \vec{F} \tag{1.8}$$

where $\vec{F}$ is the net force on the particle and $\vec{a}$ the acceleration. This equation remains the same in all inertial frames, which are frames that have constant velocity with respect to each other. Now, we wish to find the corresponding equation of motion if the particle's motion is observed from an accelerating frame. Let $\vec{a}\,'$ be the acceleration of the particle as observed from an accelerating frame. Let the acceleration of the accelerating frame with respect to the inertial frame be $\vec{A}$. By substituting Eq. 1.7 from the last section, we find that the following relation holds.

$$m\vec{a}\,' = \vec{F} - m\vec{A}. \tag{1.9}$$

Therefore, mass times acceleration of a particle with respect to an accelerating frame is not equal to the net force, but to the net force minus mass of the particle times acceleration of the frame itself. Thus, in an accelerating frame one needs to add $(-m\vec{A})$ to the real net force $\vec{F}$ to equate the "corrected" force to mass times acceleration.

The quantity $(-m\vec{A})$ acts like an additional force on the mass m, and is called a "**fictitious force**" or "**inertial force**" to distinguish it from the "real force" $\vec{F}$. In the accelerating frame the inertial forces are felt the same way as real forces except there is no agent that applies them. The inertial force in a uniformly accelerating frame acts just like the gravitational force since it is proportional to mass. This observation led Albert Einstein to develop an alternate theory of gravitation called the general theory of relativity.

Example 1.1.1. Apparent Weight in an Elevator. As an example of an accelerating frame, consider observation made by a person in an elevator which accelerates with respect to the ground. We treat the frame of a ground-based observer as an inertial frame for purposes of this example. Both the ground-based observer and the observer in the elevator have access to a reading of a weighing scale fixed to the elevator. A person stands on the weighing scale in the elevator. What will be the reading in the weighing scale as recorded by the two observers?

Solution. First, we note that the reading in the scale is that of the normal force between the scale and the person. Since the force is a real force, the two frames will have the same values for the normal force, although the calculations in the two frames will differ. In the following we present the two calculations.

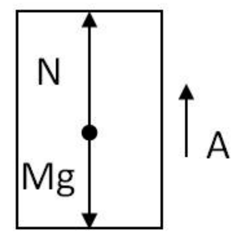

Figure 1.2: The free-body diagram of forces on a person in an accelerating elevator drawn from the perspective of the ground-based frame. The person accelerates with the elevator.

Ground-based frame.

This is the non-accelerating frame, therefore there will be no inertial forces in this frame. The person on the scale has acceleration $\vec{A}$ with respect to this frame since the acceleration of the person is same as that of the elevator. The forces on the person are (1) gravity (Mg, pointed down) and (2) normal (unknown, pointed up). The free-body diagram is shown in Fig. 1.2.

Let y-axis be pointed up. Now, we write down the y-component of Newton's second law in this inertial frame.

$$N - Mg = MA$$

Therefore, the reading on the scale will be:

$$N = M(g + A).$$

This result says that, when the accelerator is accelerating up (meaning the direction of the acceleration is pointed up), the scale will give a higher reading than Mg. If the acceleration of the elevator is pointed down, then A will be negative. In that case the scale will read less than Mg. Note that the reading on the scale does not depend upon the direction of the motion of the elevator but rather the direction of the acceleration. Therefore the scale reading is more than Mg if the acceleration is pointed up regardless of whether the elevator is going up or going down.

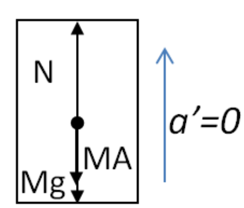

Figure 1.3: The free-body diagram of forces on a person in an accelerating elevator drawn from the perspective of the elevator frame. Since the person accelerates with the elevator, his acceleration with respect to the elevator is zero.

Elevator frame.

This is non-inertial frame when the elevator has non-zero acceleration $\vec{A}$ with respect to the ground. In this frame the real forces on the person are the same two force, viz. the forces of gravity (Mg, down) and normal (N unknown, up). But the statement of the second law is different. We need to subtract $M\vec{A}$, from the real forces to get the net force on the person as shown in the free-body diagram of the person in the elevator frame given in Fig. 1.3.

The person on the scale does not have any acceleration with respect to this frame since the person moves with the elevator.

$$\vec{a}\,' = 0.$$

Taking the vertically up direction as the positive y-axis, the y-component of the modified equation of motion is

$$N - Mg - MA = 0$$

Therefore, we find $N = M(g + A)$, which is the same conclusion for the reading on the scale as we found by working in the ground-based inertial frame.

Example 1.1.2. Pendulum In An Accelerating Train.

As a second example, consider a pendulum of mass M hanging from the ceiling of a train. When the train is at rest or coasting at a constant velocity, the pendulum hangs vertically. But when the train is accelerating the bob hangs at an angle θ to the vertical. We wish to find this angle when the train's acceleration is $\vec{A}$. We will do this problem in two frames to illustrate a non-accelerating frame and an accelerating frame.

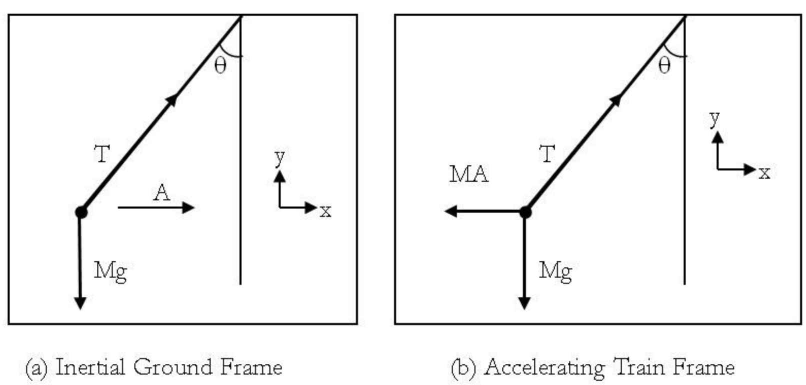

Figure 1.4: Example 1.1.2. Free-body diagrams in (a) inertial and (b) non-inertial frames. Note the additional inertial force on the bob in the accelerating frame.

Equations of motion in the ground-based frame ($\vec{a} \neq 0$):

$$\text{x component: } T\sin\theta = MA$$
$$\text{y component: } T\cos\theta - Mg = 0$$

Equations of motion in the train-based frame ($\vec{a}' = 0$):

$$\text{x component: } T\sin\theta - MA = 0$$
$$\text{y component: } T\cos\theta - Mg = 0$$

Note that the two frames yield identical relations for the equations of motion. Both frames will yield the same value for the real forces such as the tension in the string.