6.5 PROBLEMS CONCERNING MOTION OF EXTENDED BODIES

6.5.1 General Considerations

We have laid out the foundation for how to apply Newton's law for motion of individual particles. An extended body can be modeled as consisting of numerous particles, each obeying Newton's laws of motion.

Motion of a two particle system

To be specific, consider a simple system consisting of only two particles of masses m_1 and m_2 that interact with each other and with bodies external to the system. Let $\vec{F}_1$ and $\vec{F}_2$ be net forces on the two particles, and $\vec{p}_1$ and $\vec{p}_2$ be the corresponding momenta. Each particle may have more than one force on it, and the forces, $\vec{F}_1$ and $\vec{F}_2$ represent the vector sums of all forces on each particle. Then, assuming the masses to be constant, Newton's second law says that each particle will accelerate according to force on that particle.

$$\text{Particle 1:}\ \ \vec{F}_1 = m_1\vec{a}_1. \tag{6.24}$$

$$\text{Particle 2:}\ \ \vec{F}_2 = m_2\vec{a}_2. \tag{6.25}$$

Just as we had done in the last chapter, we separate the forces on each particle of the system into two categories: internal and external. For instance, the net force $\vec{F}_1$ on particle 1 is a vector sum of forces on particle 1 by particle 2, which we call internal force on particle 1 or $\vec{F}_1^{int}$, and forces on particle 1 from objects that are outside the system, which we call external force on particle 1 or $\vec{F}_1^{ext}$.

$$\vec{F}_1 = \vec{F}_1^{\text{int}} + \vec{F}_1^{\text{ext}}.$$

Then, equations of motion of the two particles become

$$\text{Particle 1:}\ \ \vec{F}_1^{\text{int}} + \vec{F}_1^{\text{ext}} = m_1\vec{a}_1. \tag{6.26}$$

$$\text{Particle 2:}\ \ \vec{F}_2^{\text{int}} + \vec{F}_2^{\text{ext}} = m_2\vec{a}_2. \tag{6.27}$$

Adding Eqs. 6.26 and 6.27 leads to the cancellation of internal forces because they are equal in magnitude and opposite in direction. We find that the net external force gives rise to mass weighted acceleration.

$$\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} = m_1\vec{a}_1 + m_2\vec{a}_2. \tag{6.28}$$

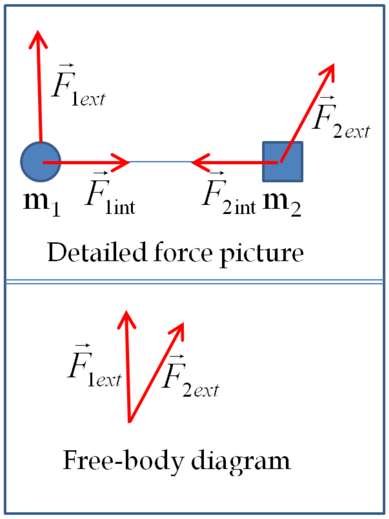

Figure 6.10: We collect all the external forces on various parts of the body in a separate diagram, called the free-body diagram, so that their vector sum can be worked out systematically.

The left side of this equation is the net external force $\vec{F}^{ext}_{net}$ on all particles of the system. The net external force is a vector sum of all forces regardless of which particle a particular force acts. To add the forces we draw all the external forces coming out of one point rather than the point they may actually be acting. The figure collecting all forces coming out of one point is called a **free-body diagram** of the system.

$$\vec{F}^{\text{ext}}_{\text{net}} = \vec{F}^{\text{ext}}_1 + \vec{F}^{\text{ext}}_2. \tag{6.29}$$

The right side of Eq. 6.28 is equal to sum of mass times accelerations of each particle. We can write the right side as total mass M $(m_1 + m_2)$ of the particles times acceleration of a space point, called center of mass of the system. We will have a more extended discussion of center of mass in the next chapter. We will be content that the right-side of equation, can be written as total mass times an acceleration simply on the dimensional ground.

$$M\vec{A} = m_1\vec{a}_1 + m_2\vec{a}_2. \tag{6.30}$$

The acceleration $\vec{A}$ is sum of the mass-weighted accelerations of the parts.

$$\vec{A} = \left(\frac{m_1}{M}\right)\vec{a}_1 + \left(\frac{m_2}{M}\right)\vec{a}_2. \tag{6.31}$$

The acceleration $\vec{A}$ is called the acceleration of the Center of Mass (CM) as we will see in the next chapter. Therefore, the net external force on a system of two particles is equal to the product of the total mass and the acceleration of the CM.

$$\vec{F}^{\text{ext}}_{\text{net}} = M\vec{A}. \tag{6.32}$$

Although we have deduced this equation from a consideration of a two-particle system, the result is valid for a system of any number of particles. Some special situations of interest are:

Case 1: All particles having the same acceleration

All particles of a rigid body that moves in a straight line will have the same acceleration, $\vec{a}$. In this case, the acceleration $\vec{A}$ of the center of mass will be same as $\vec{a}$ as we can prove easily by putting $\vec{a}_1 = \vec{a}_2 = \vec{a}$ in Eq. 6.30.

$$\begin{aligned}\vec{A} &= \left(\frac{m_1}{M}\right)\vec{a}_1 + \left(\frac{m_2}{M}\right)\vec{a}_2\\ &= \left(\frac{m_1}{M}\right)\vec{a} + \left(\frac{m_2}{M}\right)\vec{a}\\ &= \left(\frac{m_1+m_2}{M}\right)\vec{a} = \vec{a}.\end{aligned}$$

Therefore, if all points of a body have the same acceleration, then $\vec{A}$ will be the same as the acceleration of any of the particles.

Case 2: Different particles with different accelerations

In this general situation, different parts of the body will move with different accelerations, and depending upon whether the body was rigid or not the behavior will be different. For a rigid body, different accelerations of different parts will give rise to the rotation of the body as well the translational motion of the body as a whole. We will study this motion in the chapter on rotation.

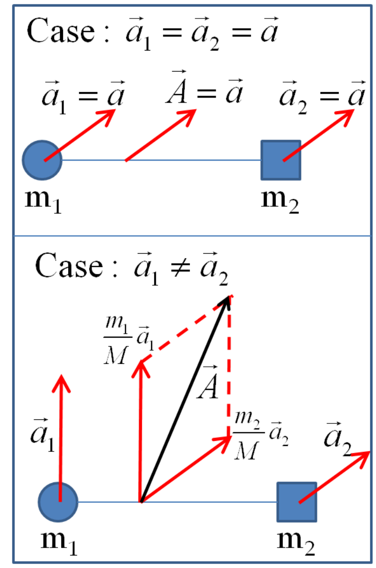

Figure 6.11: The upper figure shows that center of mass acceleration is same as the acceleration of the parts if all parts have the same acceleration. This would be the case of a rigid body executing only a translational motion. The lower figure illustrates the general case of unequal accelerations of the parts: the figure shows that the center of mass acceleration is mass-weighted acceleration of the parts.

6.5.2 Examples of Translational Motion of one Body

This section contains a variety of standard physics problems. Some books emphasize step-by-step approach to solving physics problems, implying that there is some standard method of solving problems, while there is no such thing. Every problem requires you to think through the problem and understand what is given and what is required. We also restrict our examples to the physical situations where the net torque about the center of mass is zero. With a balanced torque about the center of mass, the system under study will not start rotating about the center of mass or tumble over. We will discuss non-zero torque situations when we study rotation in a later chapter.

Example 6.5.1. A box pushed on a horizontal surface. A box of mass m is pushed with a constant horizontal force of magnitude F on a flat horizontal surface so that the box slides on a surface. If the coefficient of kinetic friction between the surface and the bottom of the box is μ_k, find the acceleration of the box in terms of m, g, F, and μ_k.

Solution. We draw a figure of the physical situation first, and identify all the external forces on the box. We find that the external forces on the box are its weight $\vec{W}$ with magnitude mg, the normal $\vec{N}$ from the table, whose magnitude is not known, the kinetic friction $\vec{F}_k$ from the table, whose magnitude will be written in terms of the magnitude of the normal force, $F_k = \mu_k N$, and the applied force $\vec{F}$ from an agent that has not been specified in the problem, as shown in Fig. 6.12. For brevity we will choose symbol N for the normal force rather than the symbol F_N.

Next, we draw a free-body diagram, where we indicate the expected direction of the acceleration which helps us pick the direction

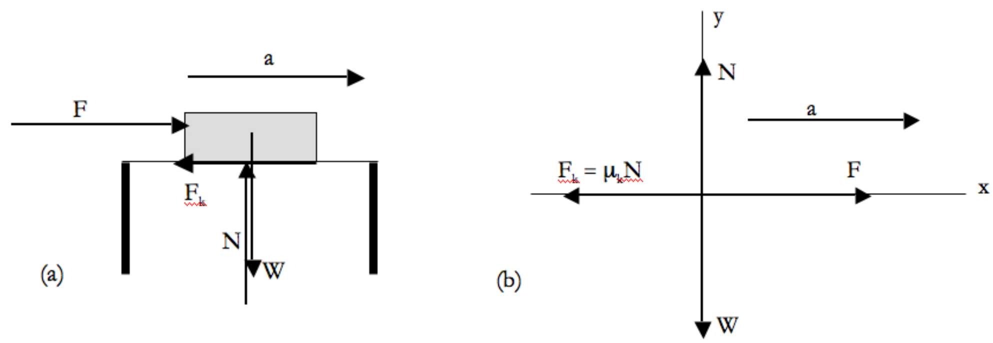

Figure 6.12: (a) Forces on a rectangular box pushed along on a horizontal surface and the resulting acceleration. (b) The free-body diagram, the direction of acceleration, and the Cartesian coordinates.

of one of the Cartesian axes. Note that, effectively, all the forces are in one plane, which we will choose to be the xy-plane with x-axis pointed towards the expected acceleration.

With this choice of coordinates, the z-components of the force will be zero, which would mean that if z-component of the velocity is zero at any time, it will remain zero all the time. This choice of coordinate system will let us ignore z-axis.

Next, we organize the information about the forces and their components in a table so that it is convenient for finding the net force on the box.

Force name	x-component	y-component
$\vec{W}$	0	$-m\ g$
$\vec{N}$	0	N
$\vec{F}_k$	$-\mu_k\ N$	0
$\vec{F}$	F	0
$\vec{F}_{net}$	$F - \mu_k\ N$	$N - mg$

Let a be the magnitude of the acceleration, then the acceleration vector has the following components: $a_x = a$, $a_y = 0$ in the present case. Now, equating each component of the net force to mass times the corresponding acceleration we find

$$F - \mu_k\ N = m\ a$$
$$N - m\ g = 0.$$

We can easily solve these equations for the magnitude a of the acceleration.

$$a = \frac{F}{m} - \mu_k \, g.$$

The magnitude of the acceleration is given by this equation and the direction of the acceleration is towards the positive x-axis as shown in the figure.

Example 6.5.2. A box pushed on a frictionless surface.

A 5-kg box is pushed with a constant horizontal force of 10 N on a flat frictionless surface. Find the displacement of the box in 5 sec starting from rest.

Solution. Since forces are constant, the acceleration will be constant also. Hence, we can use the constant acceleration formulas given in the chapters on kinematics to determine the displacement in 5 sec if we know the acceleration of the box. Therefore, our first task is to find the acceleration of the box, and then use one-dimensional kinematics to find the distance.

We start by listing the forces acting on the box. We find the following external forces on the box: the weight $\vec{W}$ of the box of magnitude mg, the normal force $\vec{N}$ from the floor which has unknown magnitude N, and the push $\vec{F}$ of magnitude 10 N as shown in Fig. 6.13.

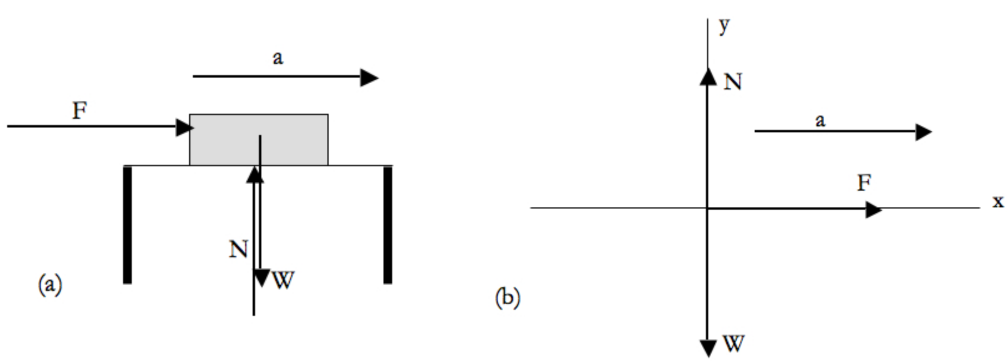

Figure 6.13: (a) Forces on a rectangular box pushed along on a frictionless horizontal surface and the resulting acceleration. (b) The free-body diagram, the direction of acceleration, and the Cartesian coordinates.

Since the acceleration is in the horizontal direction as indicated in the figure, the net force on the box must be in the horizontal direction also. The only force among the forces on the box that has any horizontal component is the horizontal push $\vec{F}$, therefore the acceleration of the box is from the horizontal component of this force alone.

$$x\text{-component:} \;\; a_x = \frac{F_{\text{net},x}}{m} = \frac{F_x}{m} = \frac{10\text{ N}}{5\text{ kg}} = 2\text{ m/s}^2.$$

Now, using the kinematics of 1-dimensional constant acceleration along x-axis we find the displacement of the box over the 5 *sec* interval.

$$\begin{aligned} x &= v_0\, t + \frac{1}{2} a_x\, t^2 \\ &= 0 + \frac{1}{2} \times (2\ \text{m/s}^2) \times (5\ \text{s})^2 \\ &= 25\ \text{m}. \end{aligned}$$

Since the box was at rest initially, the displacements along y and z-axes are zero. Therefore, the net displacement of the box is towards the positive x-axis with a magnitude 25 m.

Example 6.5.3. Sliding on an inclined plane.

A hockey puck of mass 160 grams is shot upwards on a slanted flat icy surface inclined at 10° from the horizontal direction. Immediately after leaving the stick, the puck has a speed of 20 m/s. If the coefficient of kinetic friction between the puck and the ice is 0.03, how far the hockey puck will go before coming to rest? Assume zero air resistance.

Solution. This example is similar to the last example except that we now have an incline and the frictional force to worry about. The trick here is to isolate the time interval of interest. The problem gives us the speed after the puck leaves the hockey stick, so we start the time from there. This means that the force of the hockey stick will not be on the puck during the time interval of our interest. The only external forces acting on the puck are the weight $\vec{W}$, the normal $\vec{N}$ from the incline, and the kinetic friction force $\vec{F}_k$, whose magnitude will be written in terms of the magnitude of the normal force as $F_k = \mu_k N$ as before, where μ_k is the coefficient of kinetic friction between the incline and the puck. In Fig. 6.14 we draw a free-body diagram of forces on the hockey puck after the puck has left the stick and before it has come to rest. In this diagram, we also show the direction of the velocity and acceleration of the puck at a representative instant on the motion of the puck up the incline.

We have also chosen Cartesian coordinates with the x-axis pointed in the direction of the acceleration. By choosing the x-axis in the direction of acceleration, we have made sure that y-component of acceleration is zero and the x and y-components of the second law for the hockey puck are

$$F_x^{net} = ma_x \tag{6.33}$$

$$F_y^{net} = 0 \tag{6.34}$$

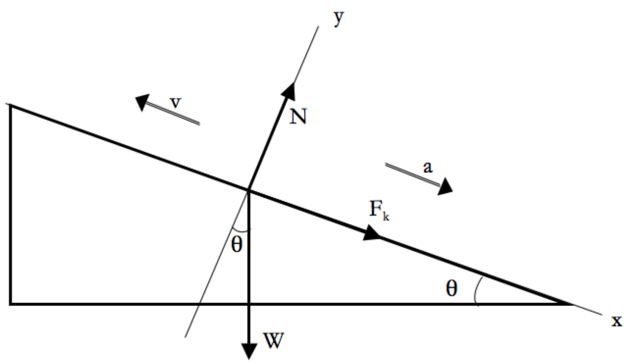

Figure 6.14: The free-body diagram of forces on a sliding box on an inclined plane. The kinetic friction force is pointed opposite to the velocity which is pointed up the incline since the puck is moving up. The expected direction of the acceleration is used to choose x-axis towards the direction of the acceleration.

Referring to Fig. 6.14, we can replace the left side of these equations with the components of the forces using their magnitudes and angles in the xy-plane.

$$\mu_k N + W \ \sin\theta = m a_x, \tag{6.35}$$

$$N - W \ \cos\theta = 0, \tag{6.36}$$

where we have used $F_k = \mu_k N$. Solving for N in Eq. 6.36 and substituting it in Eq. 6.35 we find a_x to be:

$$a_x = (\mu_k \cos\theta + sin\theta) g = 2.0 \text{ m/s}^2.$$

Along the x-axis, the motion is that of constant acceleration in a straight line with zero final velocity. Therefore, we immediately obtain the distance traveled before stopping.

$$x = \frac{v_x^2 - v_{0x}^2}{2a_x} = \frac{0 - 20^2}{2 \times 2} = -100 \text{ m}.$$

The answer for the x-component of the displacement vector is negative since the puck has climbed up the incline in the negative x-axis direction. Since y and z-components are zero, the distance d traveled up the incline is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = 100 \text{ m}.$$

6.5.3 Coupled Systems

Often movements of two or more objects are linked in the sense that the coordinates of one are related to the coordinates of others. For

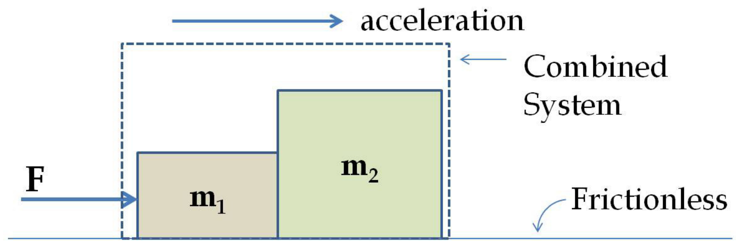

Figure 6.15: Coupled system of two blocks. Blocks m_1 and m_2 move together. The combined system moves with the same acceleration.

instance, when you hang two masses on the two sides of a pulley connected by an inextensible cord, the distance traveled by one mass must equal the distance traveled by the other. Similarly, the motion of earth is coupled to the motion of sun because the gravitational force on the two bodies depends on the distance between the two.

These relations are called constraint equations, which usually relate displacements of objects in a coupled system. These constraints create a relation among the accelerations, which must be taken into account when we solve equations resulting from an application of Newton's second law to the individual objects in coupled systems. In this section, we will examine simple coupled systems, where we will see that the constraints and equations of motion of the individual masses can be solved simultaneously to gain information about the dynamics of a coupled system.

Example 6.5.4. Moving objects together. Perhaps the simplest coupled system consists of two masses that move together in the same direction. Consider two blocks of masses m_1 and m_2 that are next to each other on a table, which, we will assume to be frictionless for the sake of simplicity. What happens when a force $\vec{F}$ pushes on mass m_1?

Note that, in the physical situation shown in Fig. 6.15, mass m_1 cannot move without also moving m_2. Therefore, the accelerations of the two masses for a right-moving horizontal motion must be equal. In this sense, we can treat both m_1 and m_2 as one system of total mass $m_1 + m_2$. On the combined system, there is only one horizontal force, $\vec{F}$. Therefore, the acceleration of both masses is simply given as

$$\vec{a} = \frac{\vec{F}}{m_1 + m_2}.$$

We can obtain the same result from examining one mass at a time. By treating one mass at a time, we obtain additional information than just an overall acceleration as we will see below. We will write equations of motion for the two masses separately and then, we will see what we can do with them to obtain the accelerations of the two.

Let us do the problem in a particular coordinate system. We choose x-axis to point in the direction of the external force $\vec{F}$. Let x_1 and x_2 be the x-coordinates of the center of masses of the two bodies so that the separation of the CM's of the two bodies is a constant D.

$$x_2 - x_1 = D. \tag{6.37}$$

This equation expresses the constraint in the motion of the two bodies as given by the coupling of their x-coordinates. Now, looking at the horizontal forces on m_1 alone, we find that there are two forces on m_1, the applied force $\vec{F}$ and a force on m_1 by m_2 as shown in Fig. 6.16. Similarly, there is one force on m_2, which is the force from m_1 on m_2.

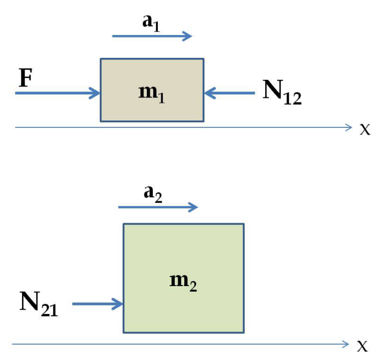

Figure 6.16: Coupled system of two blocks - set up for separate equations of motion.

Note that the force $\vec{F}$ does not act on m_2- this force acts on m_1, not on m_2. When we look at the combined system, that is one containing both m_1 and m_2, then an external force acting on either m_1 or m_2 would also act on the combined system also. But since now, we are looking at masses separately, the force $\vec{F}$ acts only on m_1 and not on m_2.

Let $\vec{a}_1$ and $\vec{a}_2$ be the accelerations of the two masses. Writing out the x-components of equations of motion of the two masses we obtain the following equations.

$$F - N_{12} = m_1 a_{1x} \tag{6.38}$$

$$N_{21} = m_2 a_{2x} \tag{6.39}$$

where the normal forces from 2 on 1 has magnitude N_{12} and the normal forces from 1 on 2 has magnitude N_{12}. Note that this normal force is different from the normal force between the blocks and the table - the normal forces in these equations are the normal forces between the two blocks. The two normal force magnitudes are equal. Therefore, we can use one symbol N_h for both of them. We use subscript h to distinguish this normal force from the normal force between the blocks and the table.

$$F - N_h = m_1 a_{1x} \tag{6.40}$$

$$N_h = m_2 a_{2x} \tag{6.41}$$

Taking two time derivatives of both sides in Eq. 6.37 we immediately see that x-components of the accelerations of the two blocks are equal, which we can replace with a simpler symbol, a_x.

$$a_{1x} = \frac{d^2 x_1}{dt^2} = \frac{d^2 x_2}{dt^2} = a_{2x} \equiv a_x. \tag{6.42}$$

Replacing the acceleration components in Eqs. 6.40 and 6.41 and adding the two equations we obtain.

$$a_x = \frac{F}{m_1 + m_2}$$

which gives the same acceleration as before since the y and z-components are zero.

$$\vec{a} = \frac{\vec{F}}{m_1 + m_2}.$$

However, we can get more information about the system from a treatment of the two masses separately - we can also obtain the force of one block on the other by calculating the normal forces in Eqs. 6.40 and 6.41. Thus, by using a_x we just found into any one of these equations, we find

$$N_h = \left(\frac{m_2}{m_1 + m_2} \right) F.$$

This shows that if $m_2 >> m_1$, then the normal force N_h will be equal to the external force applied, which would balance the force $\vec{F}$ on m_1 resulting in zero acceleration of block m_1.

Example 6.5.5. Atwood machine.

As our first example of coupled systems consider two masses m_1 and m_2 tied with an unstretchable string that goes over a massless and frictionless pulley. We will find the acceleration of the masses and tension in the string.

Solution. Since all the forces are in the vertical direction, we can point one of the Cartesian axes vertically. It is customary to take y-axis pointed vertically upwards (Fig. 6.17).

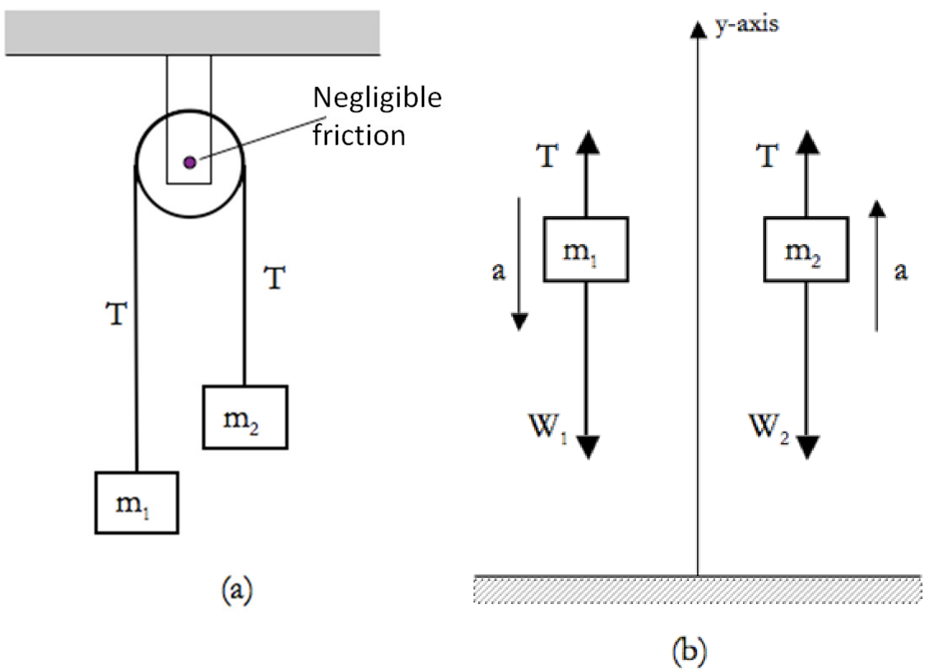

Figure 6.17: (a) An Atwood machine has two masses hanging on two sides of a pulley. Here we assume that the mass of the pulley negligible compared to the two masses, and the axle of the pulley is frictionless. (b) Separate free-body diagrams of the two masses, the directions of their accelerations, and the direction of the y-axis.

Let us first discuss the constraint in the motions of the two masses. The motions of m_1 and m_2 are constrained so that the change Δy_1 in y-coordinate of m_1 is negative of the change Δy_2 in y-coordinate of m_2.

$$\Delta y_2 = -\Delta y_1.$$

This means that y-components of their velocities are equal in magnitude but opposite in sign.

$$v_{2y} = -v_{1y}.$$

By taking time derivatives of both sides, we find that the y-component of their accelerations are equal in magnitude and opposite in sign.

$$a_{2y} = -a_{1y}. \tag{6.43}$$

Let a denote the common magnitude of the accelerations of the two blocks. Let $m_1 > m_2$, then we expect a_{2y} to be positive and a_{1y} negative. Equation 6.43 can be replaced by:

$$a_{1y} = -a \tag{6.44}$$
$$a_{2y} = a \tag{6.45}$$

Note: The tension in the string on the two sides of a massless and frictionless pulley have equal magnitude. If either of the two requirements, i.e. massless or frictionless, is not satisfied the tensions on the two sides may be different!

Before we write the separate equations of motion of the two masses it is worth noting another simplifying feature of the problem. When the mass of the pulley is negligible compared to m_1 and m_2, and the friction at the axle of the pulley has negligible torque compared to the torques from the tensions in the strings, then we can show that the tensions in the two parts of the string - tension $\vec{T}_1$ between the pulley and m_1 and tension $\vec{T}_2$ between pulley and m_2 have the same magnitude, and therefore, we can denote them by the same symbol T.

$$T_1 = T_2 = T \quad \text{(magnitude only)}. \tag{6.46}$$

Now, we draw free-body diagram of the two masses and write down Newton's second law of motion for each (Fig. 6.17(b)) where we have already incorporated requirements of the constraint given in Eqs. 6.44 and 6.45.

$$y\text{-}equation\ for\ m_1: \quad F_{1y}^{net} = m_1 a_{1y} \implies T - m_1 g = -m_1 a. \tag{6.47}$$
$$y\text{-}equation\ for\ m_2: \quad F_{2y}^{net} = m_2 a_{2y} \implies T - m_2 g = m_2 a. \tag{6.48}$$

Solving Eqs. 6.47 and 6.48 for a and T we find

$$a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g. \tag{6.49}$$
$$T = \left(\frac{2 m_1 m_2}{m_1 + m_2}\right) g. \tag{6.50}$$

The magnitude of the acceleration $a > 0$ if $m_1 > m_2$ as assumed. If $m_2 > m_1$, then we will find that $a < 0$, which would reverse the direction of the acceleration assumed in the calculation above.

Let us look at what these results say about a system where you hang two equal masses, $m_1 = m_2 = m$. As expected, Eq. 6.49 says that $a = 0$, and Eq. 6.50 says that the magnitude of the tension force in the string is $T = mg$, not $2mg$. Thus, even when you hang two

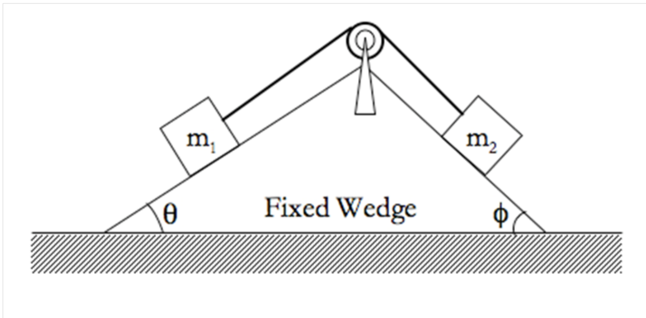

Figure 6.18: Two blocks hanging on two sides of a pulley on a fixed wedge. Here we assume that the mass of the pulley is negligible compared to the two masses, and the axle of the pulley is frictionless.

masses on two sides of a pulley, the magnitude of the tension is equal to the weight of only one of the masses hanging from the ceiling. This can be seen from the second law on any one of the mass - since the tension force in a balanced situation is required to balance only the weight of one mass. Suppose you give an initial velocity, say velocity v_0 up to m_1 and v_0 down to m_2, the two masses will continue to move at constant velocity since the net force on each mass will be zero.

Example 6.5.6. Two masses on different inclines linked together. Two blocks of masses m_1 and m_2 are tied to a cord that goes over a "massless and frictionless" pulley. The blocks move on two inclines of a fixed wedge as shown in the Fig. 6.18. Let the coefficient of kinetic friction for m_1 and m_2 be μ_1 and μ_2 respectively. Find the accelerations of the two blocks and the tension in the cord.

Solution. The motions of the two blocks are linked here such that the two blocks cover the same distance in the same time. Therefore, the magnitude of their velocities and accelerations are equal. Let us denote the magnitude of their accelerations by common symbol a. Note that their accelerations are in different directions. We also have another element that simplifies the situation: the tension in the entire cord on both sides of the pulley has the same magnitude since the pulley is assumed to be "massless and frictionless," although the direction of the tension force changes, being different on the two sides of the pulley. We draw the free-body diagrams for the two blocks separately in Fig. 6.19, and then write out the equations of motion for each of the masses separately since we have already determined the quantities for the two that are the same.

mass m_1:

$$T - \mu_1 N_1 - W_1 \sin\theta = m_1 a \quad (\text{using } F_{1k} = \mu_1 N_1) \tag{6.51}$$

$$N_1 - W_1 \cos\theta = 0. \tag{6.52}$$

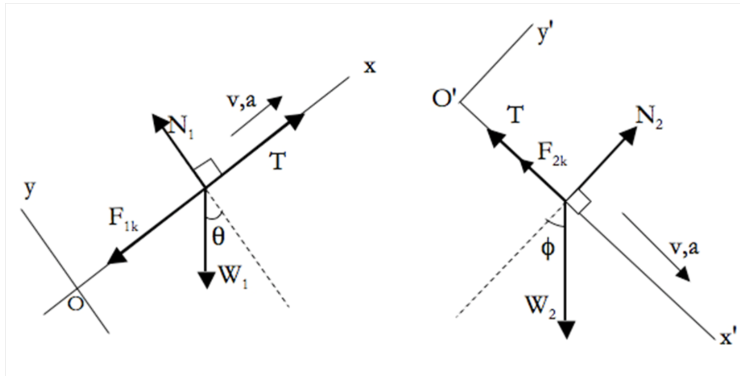

Figure 6.19: The free-body diagrams of the two masses, the assumed directions of their velocities and accelerations, and their separate Cartesian coordinates. Note that we do not need to use the same Cartesian axes to resolve the vector equations into components. The relations obtained are relations among magnitudes and angles which would be same in every coordinate system and therefore, the relations obtained from different coordinate systems can be solved together.

mass m_2:

$$-T - \mu_2 N_2 + W_2 \sin\phi = m_2 a \quad (\text{using } F_{2k} = \mu_2 N_2) \tag{6.53}$$

$$N_2 - W_2 \cos\phi = 0. \tag{6.54}$$

We have four equations, Eq. 6.51 to Eq. 6.54, and four unknown a, T, N_1, and N_2. Now, we solve Eq. 6.52 for N_1, and Eq. 6.54 for N_2, and substitute them in Eq. 6.51 and Eq. 6.53, which can be solved for a and T.

$$a = \frac{W_2 (\sin\phi - \mu_2 \cos\phi) - W_1 (\sin\theta + \mu_1 \cos\theta)}{m_1 + m_2}. \tag{6.55}$$

$$\begin{aligned} T &= m_1\ a + W_1 (\sin\theta + \mu_1 \cos\theta) \\ &= \frac{m_1 W_2 (\sin\phi - \mu_2 \cos\phi) + m_2 W_1 (sin\theta + \mu_1 \cos\theta)}{m_1 + m_2} \end{aligned} \tag{6.56}$$

These results give the magnitudes of the acceleration of the blocks and the tension in the string joining them. The directions of the accelerations of the two masses and the tension force on the masses are as indicated in Fig. 6.19. Numerical values of W_1, W_2, ϕ and θ will give $a > 0$ or $a < 0$. If $a < 0$ then we will reverse the direction for the acceleration given in the figure.

Equations 6.55 and 6.56 are quite complicated, and it is instructive to check some limits. When both angles θ and ϕ are 90^0, then the system should become the same as the Atwood machine discussed above.

$\theta = 0, \phi = 0$:

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g.$$

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2} \right) g.$$

Example 6.5.7. Block on a moving wedge. A block of mass m that slides on the frictionless incline of a wedge of mass M and angle of inclination α, which itself slides on a frictionless horizontal table. Find the accelerations of the block and wedge at an arbitrary time before the block reaches the bottom of the incline.

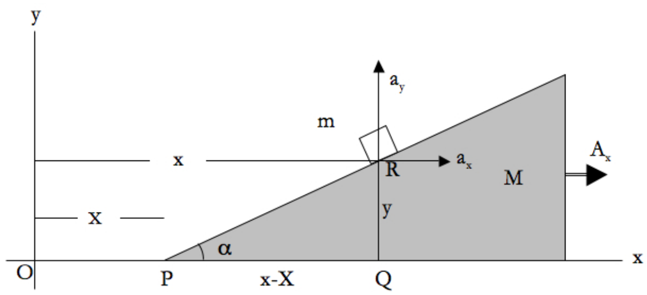

Figure 6.20: A block sliding on a wedge placed on a frictionless table. The block as well as the wedge will accelerate. A coordinate system attached to the wedge will not be an inertial frame. We will use a coordinate system attached to the fixed table as shown.

Solution. We first draw a figure to understand the physical situation better as shown in Fig. 6.20. Note that the wedge also accelerates in this problem. Therefore, we do not use a coordinate system attached to the wedge since we need an inertial frame for Newton's equation to be written as $\vec{F} = m\vec{a}$. A coordinate system fixed to the table as shown in Fig. 6.20 is convenient for this problem. We will generate equations of motion for the block and wedge and their components with respect to this coordinate system.

The position of the wedge requires assignment of only one point on the wedge since every point of the wedge has the same movement on the table. We will choose the left corner of the wedge whose x-coordinate will be indicated by X.

The position of the block on the wedge requires both x and y-coordinate. By our choice, the z-coordinate is zero all the time. Let (x, y) be the x and y-coordinates of the block at a representative instant in its motion down the incline.

We note that X of the wedge and (x, y) of the block are not all independent variables. We will now work out a relation between them based on the geometry here. This is the constraint equation in this problem. From the triangle $\triangle PQR$ we have

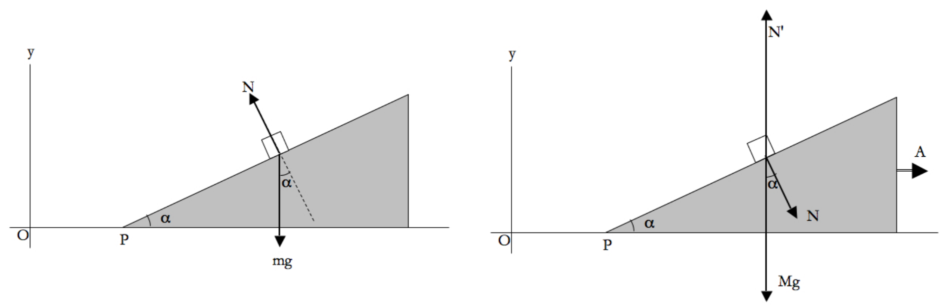

Figure 6.21: The free-body diagram of the block on frictionless incline (left) and the free-body diagram of the wedge (right). Here $\vec{N}$ is the normal force at the surface between the block and the wedge and $\vec{N}\,'$ is the normal at the surface between the wedge and the table.

$$\text{Constraint:}\quad \tan\ \alpha = \frac{y}{x-X} \quad\Longrightarrow\quad y = (x-X)\tan\alpha. \tag{6.57}$$

Taking successive derivatives with respect to time t, we conclude that the components of velocities and accelerations of the block are related to the corresponding quantities for the wedge. We continue to use small letters for the quantities of the block and capital letters for the wedge.

$$v_y = (v_x - V_x)\ \tan\alpha. \tag{6.58}$$
$$a_y = (a_x - A_x)\ \tan\alpha. \tag{6.59}$$

A point of confusion in this problem is that there are two surfaces of contact: the surface between the block and the wedge and the surface between the wedge and the table. Therefore, we will have two different normal forces and two different frictional forces in this problem. Frictional forces have been assumed to be negligible for both surfaces. Let N and N' be the magnitudes of the normal force at the surface between the wedge and the block and the wedge and the table respectively. As shown in the free-body diagrams, the block has one normal force but the wedge has two normal forces. From the free-body diagrams for the block and the wedge we find the following equations of motion.

Block:

$$F_x^{net} = m\ a_x \quad\Longrightarrow\quad -N\ \sin\alpha = m\ a_x \tag{6.60}$$
$$F_y^{net} = m\ a_y \quad\Longrightarrow\quad N\ \cos\alpha - m\ g = m\ a_y \tag{6.61}$$

Wedge:

$$F_x^{net} = M\ A_x \quad \Longrightarrow \quad N\ \sin\alpha = M\ A_x \tag{6.62}$$
$$F_y^{net} = M\ A_y \quad \Longrightarrow \quad N' - M\ g - N\ \cos\alpha = 0 \quad (\text{since } A_y = 0.) \tag{6.63}$$

From Eqs. 6.60 and 6.62 we have

$$A_x = -\left(\frac{m}{M}\right)\ a_x. \tag{6.64}$$

Using N from Eq. 6.60 into Eq. 6.61 we have

$$a_y = -a_x\ \cot\ \alpha - g. \tag{6.65}$$

Now, substituting A_x and a_y from Eqs. 6.64 and 6.65 into the constraint equation, Eq. 6.59 we find

$$a_x = \frac{-g\ \cot\ \alpha}{\frac{m}{M} + \text{cosec}^2\ \alpha}. \tag{6.66}$$

Then, by substituting a_x into Eqs. 6.64 and 6.65 we find

$$a_y = -\left(\frac{1 + \frac{m}{M}}{\frac{m}{M} + \text{cosec}^2\ \alpha}\right)\ g \tag{6.67}$$
$$A_x = \left(\frac{\frac{m}{M}\ \cot\ \alpha}{\frac{m}{M} + \text{cosec}^2\ \alpha}\right)\ g. \tag{6.68}$$

From the components of the acceleration of the block we determine the magnitude and direction of the acceleration of the block in the usual way.

$$a = \sqrt{a_x^2 + a_y^2} \tag{6.69}$$
$$\tan\theta = \frac{a_y}{a_x}, \tag{6.70}$$

where angle θ is measured counter-clockwise from the positive x-axis Once again, it is helpful to check some useful limits on our equations. One such limit is $M \to \infty$. In this limit, the wedge becomes fixed, and then the problem becomes that of a block sliding on a frictionless incline of angle α. In this limit, we find

$$a \xrightarrow{M\to\infty} g\ \sin\alpha, \text{ and} \tag{6.71}$$
$$\theta \xrightarrow{M\to\infty} \pi + \alpha, \tag{6.72}$$

which says that the acceleration is down the incline with magnitude $g\ \sin\ \alpha$ as expected.