6.7 DC CIRCUITS WITH CAPACITORS

When an uncharged capacitor is connected to a voltage source, electrons flow from the negative terminal of the voltage source to the one of the plates of the capacitor which repels electrons on the other plate causing a flow of electron from the other plate to the positive terminal of the voltage source. Consequently, two plates of the capacitor develop equal charges of opposite types. At any instant the amount of charge on any one of the plates $Q(t)$ is related to the voltage $V(t)$ across the plates at that instant by the usual capacitor formula.

$$\boxed{|Q(t)| = C|V(t)|} \tag{6.72}$$

Once the capacitor is fully charged, the current in the circuit stops and the entire voltage of the source falls across the capacitor. A charged capacitor provides a ready supply of separated charges.

To use charges on a charged capacitor one can connect the ends of the capacitor through a device R and a switch S as indicated in Fig. 6.43. When the switch is closed, current flows in the circuit until electrons from the negative plate neutralize the positive charges on the positive plate. The process is called discharging the capacitor. The time to reach $1/e$ of the original charge, or equivalently, to lose approximately $2/3^{rd}$ or $(1-e)/e$ of the original charge is an important characteristic of the circuit called time constant, which is usually denoted by the Greek letter τ.

Since the time constant of a capacitor circuit can be controlled a capacitor can be charged or discharged at a desired rate. We will now find an expression for the time constant in terms of the resistance and capacitance of the circuit by studying the discharging process mathematically. After we study the discharging process, we will show that charging process is also characterized by the same time constant.

6.7.1 Discharging A Capacitor

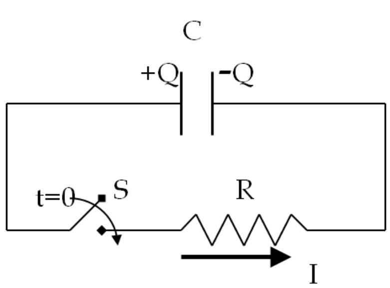

Figure 6.43: Discharging a capacitor.

Suppose we discharge a charged capacitor that has $+Q_0$ on the positive plate and $-Q_0$ on the negative plate at time $t = 0$, and we ask what is the rate at which the capacitor will discharge when connected in series with a resistor as shown in Fig. 6.43. When the switch is closed, electrons flow from the negative plate through the resistor and neutralize the excess positive charge on the other plate.

Let $\pm Q(t)$ be the charges on the plates, $I(t)$ be the current through the resistor, and $V_c(t)$ be the voltage across the capacitor plates at time t after the switch is closed. By Kirchhoff's Loop Rule on the loop shown in Fig. 6.44 we find the following equation for relation among these quantities at time t.

$$-I(t)R + V_c(t) = 0, \tag{6.73}$$

where current I is related to charge on the plate as

$$I(t) = -\frac{dQ}{dt} \tag{6.74}$$

since current is away from positive charge which is decreasing in time, and the potential across the capacitor

$$V_c(t) = \frac{1}{C} Q(t) \tag{6.75}$$

Substituting Eqs. 6.74 and 6.75 in Eq. 6.73 we obtain the following equation for charge $Q(t)$ remaining at the plates.

$$\frac{dQ}{dt} = -\frac{1}{RC} Q \tag{6.76}$$

Solution of this equation with $Q(0) = Q_0$ is

$$\boxed{Q(t) = Q_0 \exp\left(-t/RC\right).} \tag{6.77}$$

The decaying behavior of charges on plates in a discharging capacitor is shown graphically in Fig. 6.45. This relation says that charge on

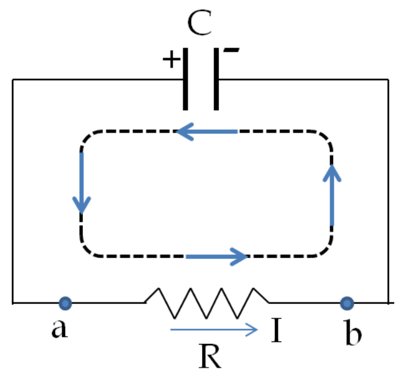

Figure 6.44: Kirchhoff's Loop Rule applied using dashed loop.

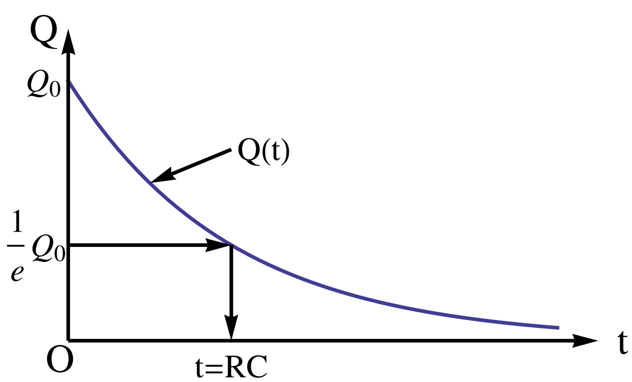

Figure 6.45: Charge on positive plate decays exponentially with time with a time constant $\tau = RC$. The time constant is equal to the time for charge to decay to $1/e$ of its value.

the plates decreases exponentially to zero. The time for charge Q to decay by $1/e$ of the original amount is called the time constant τ. From Eq. 6.77, the time constant τ is equal to RC.

$$\boxed{\tau = RC.} \tag{6.78}$$

The current and voltage in the circuit also decay similarly, given by the same time constant. Using equations given above we can write the following for current in the circuit and voltage across the capacitor. The derivation is left to the student.

$$\boxed{I(t) = I_0 \exp\left(-t/RC\right)} \tag{6.79}$$

$$\boxed{V_c(t) = V_0 \exp\left(-t/RC\right)} \tag{6.80}$$

where $I_0 = Q_0/RC$ and $V_0 = Q_0/C$.

Example 6.7.1. Discharging a Capacitor

Consider a capacitor of capacitance 2 μF. (a) Find the time constant of a circuit when the capacitor is connected to a 1 kΩ resistor. (b) Suppose the capacitor was charged to contain 5 μC at $t = 0$, at what time will there be 2 μC left on the capacitor?

Solution. (a) The time constant follows from the result $\tau = RC$ for this circuit given above.

$$\tau = RC = 1000\ \Omega \times 2 \times 10^{-6} F = 2 \times 10^{-3} sec = 2\ ms.$$

(b) Using Eq. 6.77, we set up the following equation for charge at an unknown time.

$$2\mu C = 5\mu C \times \exp\left(-\frac{t}{2\ ms}\right)$$

Therefore, we have the following exponential equation to solve.

$$\exp\left(-\frac{t}{2\ ms}\right) = 0.4.$$

Taking natural logarithm of both sides converts this equation into an polynomial equation for the unknown t.

$$-\frac{t}{2\ ms} = \ln(0.4),$$

which gives $t = 1.8\ ms$.

6.7.2 Charging A Capacitor

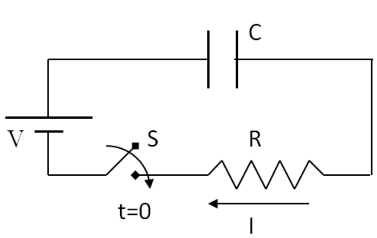

Figure 6.46: Charging a capacitor.

To charge a capacitor we make the circuit shown in Fig. 6.46 with a constant EMF source. In the diagram, a capacitor of capacitance C is in series with a voltage source of voltage V. The resistance R is the total resistance in the circuit and and a switch S is included to control the closing and opening of the circuit.

When switch S is closed at $t = 0$, a current flows in the circuit whose value decreases with time as charges develop on the capacitor.

Once the capacitor is fully charged the current in the capacitor stops and the entire voltage of the source drops across the capacitor. In this section we will study the charging process in detail and determine the rate of charging of the capacitor.

Let $Q(t)$ and $I(t)$ be the charge on the positive plate and current in the circuit at time t. We can use Kirchhoff's Loop Rule around the loop shown in Fig. 6.47 to write down the following equation.

$$V - V_c - IR = 0, \tag{6.81}$$

where

$$V_c = Q/C, \tag{6.82}$$

and

$$I = dQ/dt. \tag{6.83}$$

Here I is pointed towards positive plate where charges are increasing with time, unlike the case with discharging.

$$\frac{dQ}{dt} + \frac{1}{RC}\left(Q(t) - CV\right) = 0 \tag{6.84}$$

We need to solve this equation with initial condition

$$\text{Initial Condition: } Q(0) = 0 \tag{6.85}$$

The mathematics is easier if we change variable. Let

$$f(t) = Q(t) - CV. \tag{6.86}$$

Then,

$$\frac{df}{dt} + \frac{1}{RC} f = 0 \tag{6.87}$$

It is the same equation in f as the equation for Q in the discharging case we solved above. Therefore the general solution is

$$f(t) = f_0 \exp\left(-t/RC\right), \tag{6.88}$$

where f_0 is $f(0)$. Rewriting this solution in terms of Q, and using the initial condition on Q, $Q(0) = 0$, it is easily seen that

$$Q(t) = CV\left[1 - \exp\left(-t/RC\right)\right]. \tag{6.89}$$

Therefore, the maximum charge Q_∞ on the plates occurs as $t \to \infty$.

$$Q_\infty = CV. \tag{6.90}$$

The current at time t is obtained by taking the derivative of Q.

$$I(t) = \frac{V}{R} \exp\left(-t/RC\right). \tag{6.91}$$

The maximum current occurs at $t = 0$ when the charging begins. The maximum current, denoted as I_0 is found from Eq. 6.91 by setting $t = 0$.

$$I_0 = \frac{V}{R}. \tag{6.92}$$

At time zero, all potential drops across the resistor R, and at $t = \infty$, the current stops when the potential is fully charged and all the potential at that time drops across the capacitor as shown in Fig. 6.48.

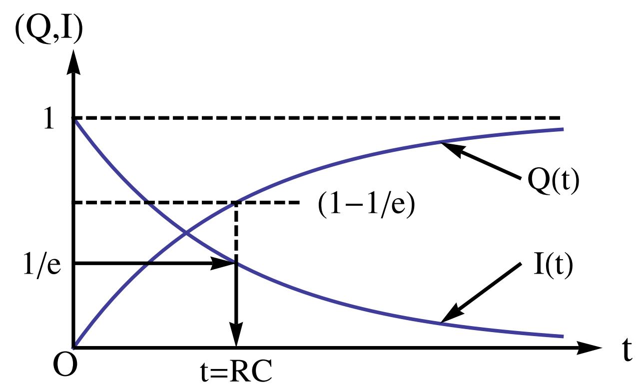

Figure 6.48: Charging a capacitor. The current in the circuit drops exponentially as the charge on the capacitor grows exponentially. The plots show $I(t)/I_0$ and $Q(t)/Q_\infty$ where I_0 is the current at time $t = 0$ and Q_∞ is the final charge.

Example 6.7.2. Charging a Capacitor

A capacitor of capacitance 5 nF is connected to a 1.5-volt battery. If the net resistance in the circuit is 1 MΩ in series with the capacitor, find the time it will take to charge up the capacitor to 90% of maximum. Also find the maximum charge on the positive plate once fully charged.

Solution. First, we find the time constant to be used in the time development of charge.

$$\text{Time constant, } \tau = RC = 10^6\Omega \times 5 \times 10^{-9}F = 5\ ms.$$

Let t be time for 90% charging. Then, we have the following equation for charging.

$$\frac{Q}{Q_\infty} = 0.9 = 1 - \exp\left(-t/\tau\right) = 1 - \exp\left(-t/5ms\right)$$

Solving for t in ms we find $t = 11.5\ ms$.

When the capacitor is fully charged, current goes to zero and all the voltage of the source drops across the capacitor. Using the capacitor formula for this voltage gives the maximum charge on the plates to be

$$Q_\infty = CV = 5 \times 10^{-9} F \times 1.5 V = 7.5 \times 10^{-9} C.$$

6.7.3 Energy Stored in a Capacitor

A capacitor is an energy-storing device. By storing charges separated by a distance, the capacitor essentially stores energy in the potential energy of the charges, or equivalently in the electric field of the space between plates. One way to easily figure out the energy stored in a capacitor is to use energy conservation in the discharging circuit. Connect a charged capacitor to a resistor R and let current flow in the simple RC-circuit and determine the net energy dissipated in the resistor. When current $I(t)$ passes through a resistor, moving charges are transferred at a rate of dQ/dt through a voltage drop of $V(t)$ across the resistor. The loss in energy of the moving charges heats up the resistor at the instantaneous power-dissipation rate, $P(t) = I(t)V(t)$, as we have seen before.

$$P(t) = I(t)V(t) = I(t)^2 R. \tag{6.93}$$

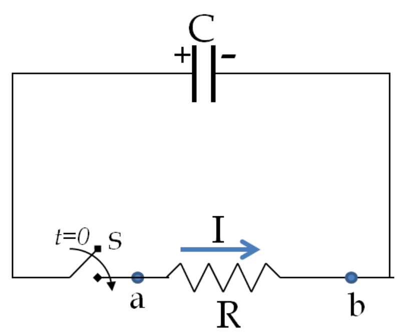

Figure 6.49: Discharging a capacitor through a resistor leads to heating of resistor. The energy dissipated in the resistor equals the energy stored in the capacitor.

In the time interval from t to $t + dt$, the amount of energy dU dissipated in the resistor will be equal to the product of instantaneous power $P(t)$ and the interval dt.

$$dU = P(t)dt. \tag{6.94}$$

Total energy dissipated in the resistor can be obtained by integrating this from $t = 0$ to $t = \infty$. To perform the integration over time we use the solution given above for the discharging circuit.

$$U = \int_0^\infty P(t)dt = RI_0^2 \int_0^\infty \exp\left(-2t/RC\right)dt = \frac{CR^2 I_0^2}{2}, \tag{6.95}$$

where I_0 is the current at time $t = 0$, which was shown to be Q_0/RC when we solved the discharging circuit earlier. This must be the energy originally contained in the capacitor C charged with Q_0. Using the capacitor formula $Q = CV$, we can rewrite the result in terms of Q_0 or V_0 alone, which are initial charges on the capacitor and initial potential difference across capacitor plates.

$$\boxed{U = \frac{1}{2} Q_0 V_0 = \frac{1}{2} C V_0^2 = \frac{1}{2}\frac{Q_0^2}{C}.} \tag{6.96}$$

6.7.4 Capacitors in Series

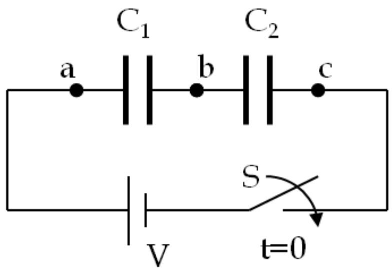

Figure 6.50: Capacitors in series.

Consider a circuit containing two capacitors C_1 and C_2 in series with a voltage source V (Fig. 6.50). When switch S is closed at time $t = 0$, electrons from the left plate of C_1 flow to the battery which does work on the electrons to put it on the right plate of C_2. The positively charged left plate of C_1 attracts electrons from the right plate of C_1 and left plate of C_2 conductor. At the same time, the negatively charged right plate of C_2 repels electrons from the left plate of C_2. The two tendencies act together, polarizing the right plate of C_1 and the left plate of C_2 by induction. As a result, both conductors are charged to the same amount. Let $\pm Q$ denote charges on various plates. Since the two capacitors have different capacitances yet charged to the same amount, voltage drops across them will be different. Let V_1 and V_2 be the voltages across C_1 and C_2 respectively. Then, we will have the following relations.

$$V_1 = \frac{Q}{C_1} \tag{6.97}$$

$$V_2 = \frac{Q}{C_2} \tag{6.98}$$

The voltages V_1 and V_2 drop in series and therefore add up to the voltage of the source V.

$$V = V_1 + V_2. \tag{6.99}$$

Hence,

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \tag{6.100}$$

This equation can be used to define an equivalent capacitance C_S of the two capacitors in series as

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2}. \tag{6.101}$$

The net charge on the equivalent capacitor is same as the charge Q on the original capacitors and the voltage drop across the equivalent capacitor will be the voltage V of the source. Therefore, the equivalent capacitor will have the following relation.

$$Q = C_S V. \tag{6.102}$$

Similar arguments lead to the equivalent capacitance of N capacitors connected in series.

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}. \tag{6.103}$$

We find that the formula for capacitors in series is similar to the formula for resistors in parallel. The net capacitance of capacitors in series is lower than the lowest capacitance in series. Hence, capacitance decreases when you connect capacitors in series.

This result makes sense when you think in terms of a parallel plate capacitor. The capacitance of a parallel plate capacitor decreases inversely with the distance between plates. By connecting plates in series, we are effectively increasing the distance between the plates, and hence a reduction in capacitance is expected here. When capacitors with very different capacitances are connected in series, you may be able to simply ignore the capacitors with high capacitance since they may contribute insignificantly compared to the effective capacitance of the group.

Example 6.7.3. Capacitors in Series

Evaluate the equivalent capacitances of two capacitors connected in series (a) $C_1 = 2$ μF, and C_2= 2 μF, (b) $C_1 = 2$ μF, and C_2= 20 μF, and (c) $C_1 = 2$ μF, and C_2= 2000 μF

Solution. This example shows that the equivalent capacitance of two capacitors connected in series is dominated by the capacitor with the least capacitance. (a) $\frac{1}{C_S} = \frac{1}{2\mu F} + \frac{1}{2\mu F} = \frac{1}{1\mu F} \Longrightarrow C_S = 1\mu F$ (b) $\frac{1}{C_S} = \frac{1}{2\mu F} + \frac{1}{20\mu F} = \frac{11}{20\mu F} \Longrightarrow C_S = \frac{20}{11}\mu F \approx 1.82\mu F$ (c) $\frac{1}{C_S} = \frac{1}{2\mu F} + \frac{1}{2000\mu F} = \frac{1001}{2000\mu F} \Longrightarrow C_S = \frac{2000}{1001}\mu F \approx 1.998\mu F$.

6.7.5 Capacitors in Parallel

The physical effect of connecting capacitors in parallel is to increase the effective area available for charges to spread over. Therefore we expect the net capacitance to go up when we connect capacitors in parallel. In this section we will figure out a formula for the equivalent capacitance. Towards that end consider a circuit with two capacitors C_1 and C_2 in parallel as shown in Fig. 6.51.

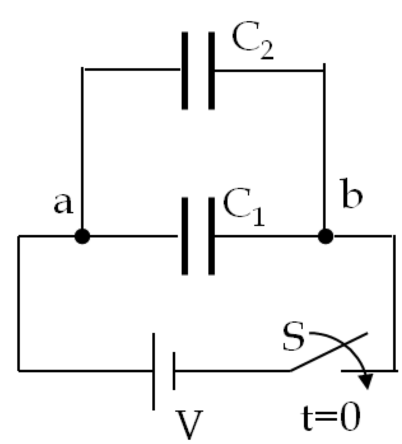

Figure 6.51: Capacitors in parallel .

Note that since the sides of the two capacitors are connected to the same node points, the capacitors connected in parallel, just as the resistors connected in parallel, experience the same potential difference V across them. Since their capacitance is different and potential difference same, they will store different amounts of charge as given by $Q = CV$. Let Q_1 and Q_2 be the charges on the two capacitors, then we will have

$$V = \frac{Q_1}{C_1} \tag{6.104}$$

$$V = \frac{Q_2}{C_2} \tag{6.105}$$

The equivalent capacitor must store a total charge $\pm Q = \pm(Q_1 + Q_2)$ on its plates when the same potential difference V is applied across it so that the equivalent capacitor appears identical in its effect to the rest of the circuit. This gives the following requirement for the capacitance C_P of the equivalent capacitor.

$$V = \frac{Q_1 + Q_2}{C_P} \tag{6.106}$$

These equations give us the following identities.

$$\frac{Q_1}{C_1} = \frac{Q_1 + Q_2}{C_P} \tag{6.107}$$

$$\frac{Q_2}{C_2} = \frac{Q_1 + Q_2}{C_P} \tag{6.108}$$

Eliminating Q_1 and Q_2 from these equations give us the following for the equivalent capacitance.

$$C_P = C_1 + C_2. \tag{6.109}$$

The equivalent capacitance of N capacitors in parallel connection is similarly found.

$$\boxed{C_P = C_1 + C_2 + \cdots + C_N.} \tag{6.110}$$

This says that the capacitance of the largest capacitor in parallel controls the overall capacitance of the group. If the capacitances of capacitors vary over a wide range, then it may be possible to ignore the capacitors with lower capacitances when in parallel to a capacitor with much higher capacitance as we will see in the following example.

Example 6.7.4. Capacitors in Parallel

Evaluate the equivalent capacitance of the following capacitors connected in parallel. (a) $C_1 = 1\ \mu$F, and $C_2 = 1\ \mu$F, (b) $C_1 = 1\ \mu$F, and $C_2 = 1000\ \mu$F, and (c) $C_1 = 1\ \mu$F, and $C_2 = 10^6\ \mu$F.

Solution. (a) $C_P = 1\mu F + 1\mu F = 2\mu F$. (b) $C_P = 1\mu F + 1000\mu F = 1001\mu F$. (c) $C_P = 1\mu F + 10^6\mu F \approx 10^6\mu F$.