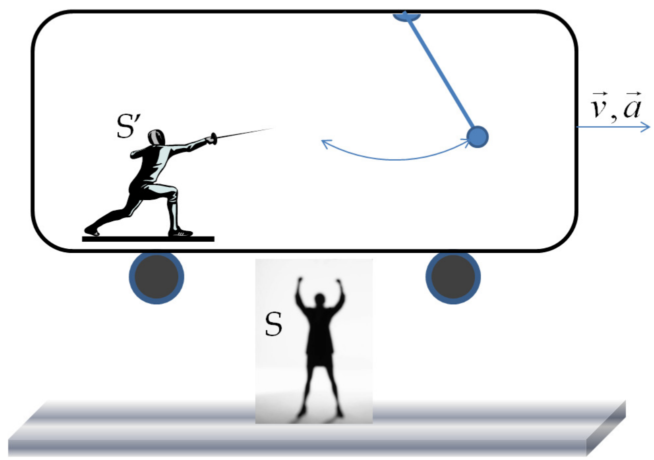

Figure 4.15: Observers S and S' give different values to the position, velocity and acceleration of the pendulum bob. What are their relations?

4.7 RELATIVE MOTION

Have you ever wondered whether physics is same from the perspectives of two observers who are in a relative motion with respect to each other, for instance one inside a moving train and another on a fixed railway platform? In this section you will study the relation between the kinematics quantities the two observers assign to the same object. To simplify our discussion we will orient axes for the two observers such that their axes have the same directions in space and one observer moves along the x-axis of the other observer. This movement will be followed by the displacement of the origin of one coordinate system with respect to the origin of the other coordinate system. The two coordinate systems are called frames.

We will also synchronize their clocks so that the two frames set their clocks to read $t = 0$ when they are on top of each other and their origins and axes coincide. We will also assume that the clocks run at the same rate in the two frames. This assumption turns out to lead to trouble in physics as shown by Albert Einstein in 1905 in his theory of special relativity.

The mistake caused by the assumption of equal rate of time elapse in the two moving frames, however, is very small for ordinary velocities, and becomes significant only when the relative speed of the two frames approaches the speed of light which is approximately 3×10^8 m/s. Most situations encountered in everyday life have speeds that are much smaller than the speed of light, and hence we will not bother about the difference in the rates of clocks in the two frames until we come to study motions with speeds near the speed of light. The study of relative motion of frames in this limit is called **Galilean**

relativity.

4.7.1 Observers Moving at Uniform Velocity

In Fig. 4.16, I have drawn two frames S and S' in which S is at rest and S' moves towards the positive x-axis with speed V. The relative velocity $\vec{v}_{rel}$ has only x-component in the two frames. In frame S, the relative velocity is $\vec{v}_{rel} = V\hat{u}_x$ and in frame S', the relative velocity is $\vec{v}'_{rel} = -V\hat{u}'_x$. We wish to follow the motion of a point particle P with respect to the two frames.

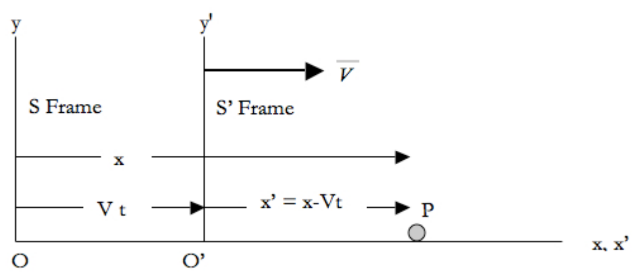

Figure 4.16: The position of an object P with respect to two frames S and S' which are in a uniform relative motion with respect to each other.

With $t = 0$ when the origins O and O' are on top of each other, the x-coordinate of the particle P at time t in the frames S and S' will be x and $x - Vt$ respectively. I will denote position, velocity and acceleration in the two frames by the same symbols but add a prime for quantities in S' frame.

$$x' = x - Vt \tag{4.33}$$

The y and z-coordinates will be same in the two frames since there is no relative motion along y- or z-axis.

$$y' = y; \quad z' = z. \tag{4.34}$$

The relation between velocities in the two frames can be obtained by taking derivatives of position with time. Therefore, the components of the velocity of the particle P in the two frames are related as follows.

$$\begin{aligned} v'_x &= \frac{dx'}{dt} = \frac{d(x - Vt)}{dt} = \frac{dx}{dt} - \frac{d(Vt)}{dt} = v_x - V \\ v'_y &= \frac{dy'}{dt} = \frac{dy}{dt} = v_y \\ v'_z &= \frac{dz'}{dt} = \frac{dz}{dt} = v_z \end{aligned} \tag{4.35}$$

We find that the x-component of the velocity of the particle is different in the two frames, but the y and z-velocities are the same. Since the relative speed V is constant, the x-acceleration, which is equal to the derivative of the x-component of the velocity with time, will turn out to be same in the two frames. The y and z-accelerations are, of course, same in the two frames because the y and z-motion of the particle is unaffected by the relative motion along x-axis of the train and the platform.

$$\begin{aligned} a'_x &= \frac{dv'_x}{dt} = \frac{d(v_x - V)}{dt} = \frac{dv_x}{dt} = a_x \\ a'_y &= a_y \\ a'_z &= a_z \end{aligned} \tag{4.36}$$

Example 4.7.1. Relative speed of a runner. Three runners A, B, and C are running on a straight road that run East to West. The runners A and B have velocities {5 m/s, East} and {2 m/s, West}, respectively, with respect to the runner C. What is the relative velocity of the runner A with respect to the runner B?

Solution. It is best to perform the calculation analytically in a Cartesian coordinate system. Let the positive x-axis of the runner C be pointed towards East. Then, we can write the velocities of the runners A and B in this frame.

$$\begin{aligned} \vec{v}_A &= (5\ \text{m/s})\hat{u}_x \\ \vec{v}_B &= (-2\ \text{m/s})\hat{u}_x \end{aligned}$$

The relative velocity of A with respect to B will be $\vec{v}_A - \vec{v}_B$ will be

$$\vec{v}_A - \vec{v}_B = (5\ \text{m/s})\hat{u}_x - (-2\ \text{m/s})\hat{u}_x = (7\ \text{m/s})\hat{u}_x.$$

Therefore, the velocity of A is 7 m/s towards the East with respect to B.

4.7.2 Observers Moving at Uniform Acceleration

Now, consider observing the motion of a particle P from two frames S and S' which have a relative acceleration. Let the frame S' have a constant acceleration of magnitude A directed towards the positive x-axis with respect to the frame S as shown in Fig. 4.17. Once again, we set $t = 0$ at the instant when the two origins coincide. Let V_0 be the relative speed at time $t = 0$.

The position of the origin O' of frame S' at time t is obtained from the relative acceleration $A\hat{u}_x$ and the initial velocity $V_0\hat{u}_x$. We

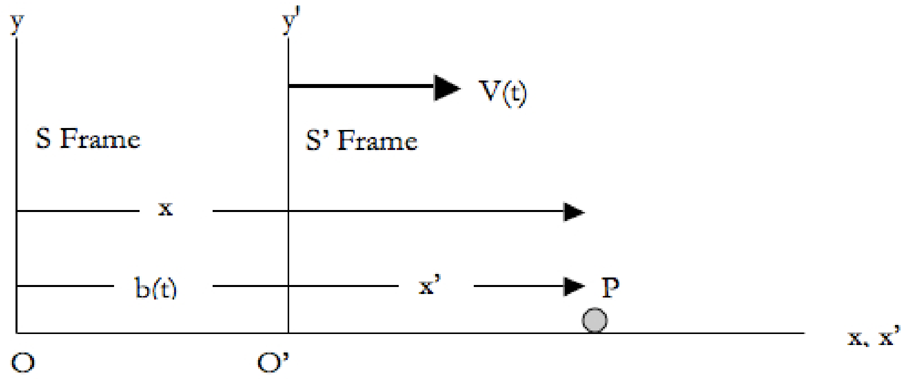

Figure 4.17: Observations on a particle P from a stationary frame S and an accelerating frame S'.

note that O' moves at a constant acceleration, hence $b(t)$ is obtained by using the kinematics equations for constant acceleration in one-dimension along x-axis.

$$b(t) = V_0 t + \frac{1}{2}At^2$$

Therefore, the coordinates of the particle P in the two frames are related as follows.

$$\begin{aligned} x' &= x - b(t) = x - V_0 t - \frac{1}{2}At^2 \\ y' &= y \\ z' &= z \end{aligned} \tag{4.37}$$

Now we find the relation between the velocities of the particle P with respect to the two frames.

$$\begin{aligned} v'_x &= \frac{dx'}{dt} = \frac{d(x - b(t))}{dt} = \frac{dx}{dt} - \frac{db}{dt} = v_x - V_0 - At \\ v'_y &= \frac{dy'}{dt} = \frac{dy}{dt} = v_y \\ v'_z &= \frac{dz'}{dt} = \frac{dz}{dt} = v_z \end{aligned} \tag{4.38}$$

Acceleration of particle P is now also different in the two frames.

$$\begin{aligned} a'_x &= \frac{dv'_x}{dt} = \frac{d(v_x - V_0 - At)}{dt} = \frac{dv_x}{dt}\frac{d(At)}{dt} = a_x - A \\ a'_y &= a_y \\ a'_z &= a_z \end{aligned} \tag{4.39}$$

Example 4.7.2. Freely falling marble in an accelerating elevator. A marble is dropped in an elevator that is accelerating at 3 m/s^2 upward with respect to the ground. Find the acceleration of the marble with respect to the ground and with respect to an observer in the elevator.

Figure 4.1

Solution. We draw a picture of the situation given in the problem in Fig. 4.18 where we denote the ground-based observer as O and the observer in the elevator as O'. The accelerations of the marble with respect to the observers O, and O' are related as follows.

$$\vec{a}_0 = \vec{A} + \vec{a}_0{}'$$

The vector relation here has only vertical component. Therefore, we obtain the following for the magnitude of $\vec{a}_0{}'$.

$$-9.81 = 3 + a_0',$$

which give the following value for the acceleration of the marble for the observer in the elevator.

$$a_0' = -12.81\ m/s^2.$$

Note that the acceleration observed in the elevator frame is independent of the velocity of the accelerator. Therefore, the acceleration of the marble will be -12.8 m/s^2 in the elevator frame, regardless of whether the elevator is moving up or down and the value of the speed of the elevator as long as the acceleration of the elevator with respect to ground is 3 m/s^2 pointed up.

We see that the "effective g" for an observer in an elevator with acceleration pointed up is higher than 9.81 m/s^2. You can also deduce that if the acceleration of the elevator were pointed down then the value of the acceleration would be less than 9.81 m/s^2. For instance, if the elevator itself has an acceleration of 9.81 m/s^2, pointed down, the acceleration of a freely falling object as seen by the a person moving with the elevator will be zero, i.e., the freely falling objects with respect to a freely falling elevator is weightless!