2.2 GAUSS'S LAW

Motivating Gauss's law

Gauss's law states a general property of the flux of the electric field through a closed surface. We found above that if a closed surface does not have any charge inside where an electric field line can terminate then any electric field line entering the surface at one point must necessarily exit at some other point of the surface. Therefore, if a closed surface does not have any charges inside the enclosed volume, then the electric flux through the surface will be zero. Now, what happens to the electric flux if there are some charges inside the enclosed volume? Gauss's law gives a quantitative answer to this question.

To get a feel for what to expect, let us calculate the electric flux through a spherical surface around a positive point charge q since we already know the electric field in such a situation. Recall that when we place the point charge at the origin of a coordinate system, the electric field at a point P that is at a distance r from the charge at the origin is given by

$$\vec{E}_P = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^2}\,\hat{u}_r,$$

where $\hat{u}_r$ is the radial vector from charge at the origin to the point P. Now, we use this electric field to find the flux through the spherical surface of radius r. The procedure for calculation of flux from a given electric field calls for dividing up the area into patches and calculating the flux through each patch. Notice that the area vectors of the patches are all radially outward, given by $\hat{u}_r$. Therefore, if q is a positive charge, then the electric field and the area vectors of the patches will be parallel as shown in Fig. 2.9, and we would simply get the following for the flux through any patch.

$$\Delta\Phi = E_{\text{at patch}}\Delta A_{\text{patch}}.$$

Furthermore, notice that the magnitude of the electric field on every patch is same as that of $\vec{E}_P$ since every point of the surface is the same distance r away from the charge q.

$$E_{\text{at patch}} = q/4\pi\epsilon_0 r^2. \quad \text{(all patches)}$$

Therefore, we obtain the following for the flux of the electric field:

$$\Phi = q/4\pi\epsilon_0 r^2 \times \sum_{\text{patches}} \Delta A_{\text{patch}}.$$

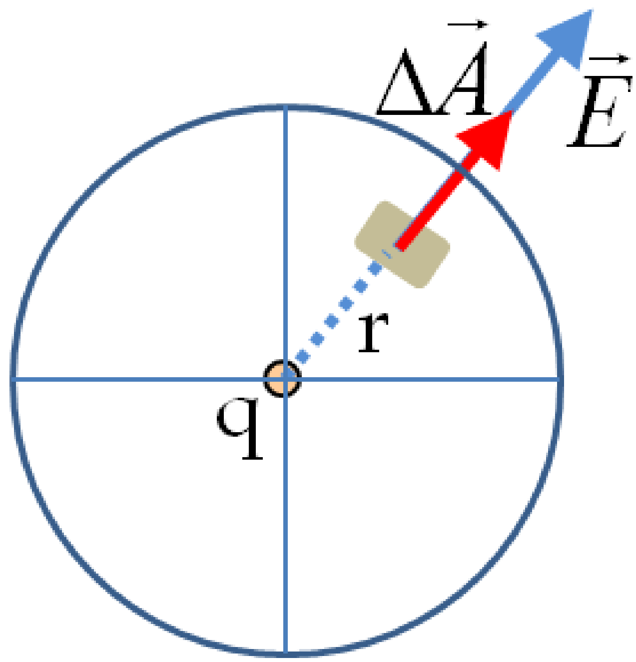

Figure 2.9: Gauss's law applied to a point charge q. The flux through the patch $d\Phi_E = E\Delta A$. The flux through the entire surface $\Phi_E = q/\epsilon_0$, proportional to the charge enclosed.

The sum of area of all patches is equal to the total surface area of the spherical surface, $4\pi r^2$. This gives the flux through the closed spherical surface at radius r to be

$$\Phi = \frac{q}{\epsilon_0} \tag{2.6}$$

A remarkable fact about Eq. 2.6 is that the flux is independent of the size of the spherical surface. This can be directly attributed to the fact that electric field of a point charge decreases as $1/r^2$ with distance, which just cancels the r^2 rate of increase of the surface area.

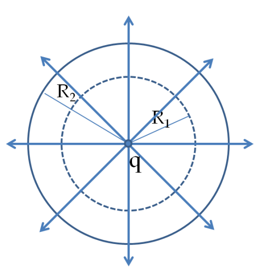

Figure 2.10: Flux through spherical surfaces of radii R_1 and R_2 enclosing the charge q are equal, independent of the size of the surface since all E-field lines that piece one surface from inside to outside direction pierce the other surface also in the same direction.

Electric Field Lines Picture

Another way to see why the flux through a closed spherical surface is independent of the radius of the surface is to look at the electric field lines. Note that every filed line from q that pierces the surface at radius R_1 also pierces the surface at R_2 as shown in Fig. 2.10.

Therefore, the net "number of E-field lines" passing through the two surfaces from inside to outside direction are equal. The net number of electric field lines passing from inside direction to outside direction, which is obtained from subtracting the number of lines from outside to inside direction from the number in the direction of inside to outside, gives a pictorial measure of the electric flux through the surfaces.

Although, in reality each charge produces infinite number of lines, we can draw a finite number of electric field lines to represent electric field of a charge faithfully to count number of lines to get a qualitative picture of flux through surfaces. This interpretation can be extended to deduce that electric flux through all surfaces, regardless of the shape, whether spherical or some arbitrary shape, will be same as long as they all enclose the charge as indicated in Fig. 2.11.

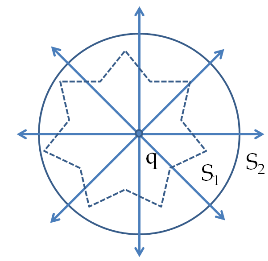

Figure 2.11: Flux of electric field of charge q through two surfaces of different shapes are equal as long as the two surfaces enclose the charge.

Counting of electric field lines is helped by the following notation: at each instance, when a field line crosses the surface from inside to the outside, it will contribute plus one to the flux, and when a field line crosses the surface from outside to the inside, it will contribute minus one to the flux. For example, a field line that enters a closed surface at one point (count = -1), and then later exits it at another (count =+1), will contribute zero to the flux. With this understanding, you can immediately see that if no charges are included within a closed surface, then the electric flux through it must be zero since as many lines cross the surface in one direction as in the other direction (Fig. 2.12(a)). The same will happen if the charges of equal and opposite sign are included inside the closed surface, so that the total charge included is zero (Fig. 2.12(b)). All surfaces that include the same amount of charge have the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surfaces enclose the same amount of charge (Fig. 2.12(c)).

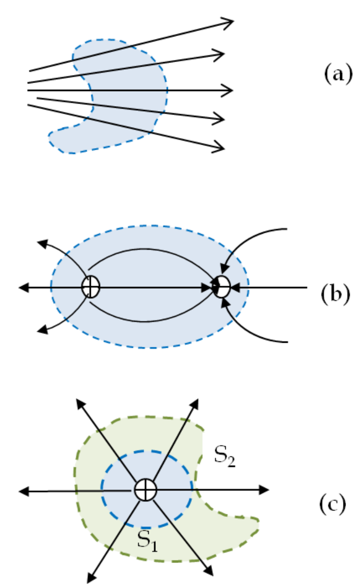

Figure 2.12: Understanding the flux in terms of field lines. (a) No charges included leads to zero flux through the closed surface. (b) Charges enclosed, but the net charge included =0 makes the net flux through the closed surface zero. (c) The shape and size of the surfaces that enclose a charge does not matter since all surfaces enclosing the same charge have the same flux.

Statement of Gauss's Law

Gauss's law generalizes the result obtained above to the case of any number of charges and any location of the charges in the space inside the closed surface. According to Gauss's law, the flux of electric field $\vec{E}$ through any closed, also called a **Gaussian surface**, is equal to the net charge enclosed (q_{enc}) divided by the permittivity of free space (ϵ_0).

$$\boxed{\Phi_{\text{Closed Surface}} = \frac{q_{enc}}{\epsilon_0}.} \tag{2.7}$$

Note that the charge enclosed (q_{enc}) is the sum total of all charges in the volume bounded by the closed Gaussian surface. If the charges are discrete point charges, then we just sum over them, and if they are described by a continuous charge distribution, then we would need to integrate appropriately to find the total charge that resides inside the enclosed volume.

You might be wondering if other $\frac{1}{r^2}$ force laws, most notably the Newton's law of gravitational force, would also have a Gauss's law. The answer is yes as you can see from Fig. 2.13 for gravitational field flux. There we define a gravitational field $\vec{g}$ as the gravitational force on a test object per unit mass of the test object, in analogy with the electric force per unit charge for the electric field. This gives the gravitational field $\vec{g}$ at a distance r of an isolated point mass M located at the origin to be equal to $G_N M/r^2$ in magnitude, and the direction pointed radially towards the mass M, the source of the gravitational field. Since the patch area vector is pointed radially outwards here, the normal to the are makes an angle of 180° with respect to the direction of the gravitational field $\vec{g}$. Therefore, the flux of the gravitational field through the surface under consideration will be a negative number, and we would get the following for the flux of the gravitational field.

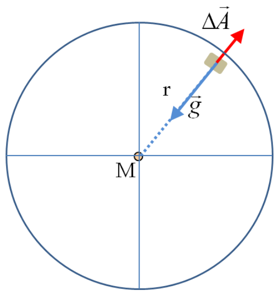

Figure 2.13: Gauss's law for Newtons gravitational force. The flux through the patch $d\Phi_G = -g\Delta A$. The flux through the entire surface $\Phi_G = -4\pi G_N M$, proportioanal to the mass enclosed.

$$\begin{aligned}\Phi_G &= -\frac{G_N M}{r^2} \sum_{\text{patches}} \Delta A \\ &= -\frac{G_N M}{r^2} \times 4\pi r^2 = -4\pi G_N M \end{aligned} \tag{2.8}$$

Again, the flux is independent of the size of the spherical surface. Equations 2.6 and 2.8 demonstrate that the flux of all vector fields that drop off as $\frac{1}{r^2}$ from the source will be independent of the size of the spherical surface. They also show that, if the source of the corresponding field is not in the enclosed space of the closed surface, then the flux will be zero.