8.3 POTENTIAL ENERGY FUNCTIONS

We saw in the last section that the work done by a conservative force on a particle is accounted by a change in the potential energy of the particle. The contribution of the work done by the conservative forces can be included either as work by the force or by the potential energy change.

To facilitate the use of conservation of energy given in Eq. 8.38 and other formulas containing potential energy terms, we now work out the formulas for the potential energies of point particles when they are subject to conservative forces. Since the work integral gives the potential energy difference, we usually write the potential energy as a function of coordinates of space by evaluating the potential difference between the space point of interest and a carefully chosen reference point in each case. These functions are called **potential energy functions** and usually denoted by letter U.

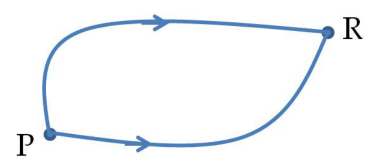

Figure 8.16: The value of the potential energy function at a space point P is obtained by work integral from P to a reference point R by any path.

Thus, the potential energy function for a point particle placed at the space point $P(x, y, z)$ due to a conservative force is be obtained by performing the work integral W_{PR} for moving the particle from point P to an arbitrarily chosen reference point R, and equating the work done to the change in the potential energy $U_P - U_R$.

$$U_P = U_R + W_{PR}, \tag{8.40}$$

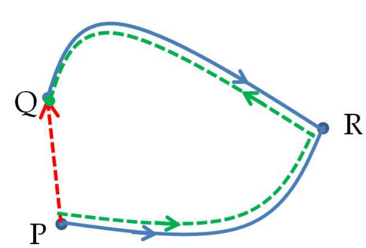

Figure 8.17: The difference in potential energy between points P and Q is independent of the choice of reference point R. Work from P to Q directly is equal to work from P to R then from R to Q.

The work integral W_{PR} can be done on any convenient path between points P and R as shown in Fig. 8.16 since the work integral for conservative forces are independent of the path. We choose the value of the arbitrary constant U_R and the location of the reference site R so that the potential energy function is well-defined in the region of interest and the formula is simplest as we will illustrate in the examples below. We will often write U_P as $U(x, y, z)$ to indicate the coordinates of point P in the formula for the potential energy function.

The arbitrariness of the reference energy U_R and the reference point R does not affect the difference in potential energy between any two points which is where the physical information resides. For instance, the difference $U(x, y, z) - U(x', y', z')$ of potential energies between points $P(x, y, z)$ and $Q(x', y', z')$ is independent of U_R and

R.

$$\begin{aligned} U(x,y,z) - U(x',y',z') &= [U_R + W_{PR}] - [U_R + W_{QR}] \\ &= W_{PR} - W_{QR} \\ &= \int_P^R \vec{F}_c \cdot d\vec{r} - \int_Q^R \vec{F}_c \cdot d\vec{r} \\ &= \int_P^Q \vec{F}_c \cdot d\vec{r} \ \ \text{(independent of } R \text{ and } U_R\text{)}. \end{aligned}$$

Each conservative force has its own formula for the potential energy function obtained in this manner as we will see below. In the following, you will study examples illustrating a general procedure for finding the formula for the potential energy function corresponding to a given force. The common steps in the derivation of a potential energy function can be summarized as follows.

1. Pick an inertial coordinate system.

2. Write the force law for the conservative force $\vec{F}_c$ in that coordinate system.

3. Pick two points in space, one for the reference (R) and the other an arbitrary point (P). Use symbolic coordinates for R and P, with coordinates (x_R, y_R, z_R) for R and coordinates (x, y, z) for P. Use the analytic form of $\vec{F}_c$ to decide if you need a one-dimensional, a two-dimensional, or a three-dimensional consideration.

4. Perform the integral to find the work done for a path from P to R. While doing the integral, pick the path that simplifies the integral by using segments where the path and the force directions are aligned either parallel, anti-parallel or perpendicular to each other. Call the result symbolically as $U_P - U_R$.

5. The potential energy U_P is then obtained by carefully studying the equation produced, and making a choice for the coordinates of R and the value of U_R so that you get the simplest or the desired expression for U_P, which is now written as $U(x, y, z)$. The usual choice for U_R is zero, but any finite value for U_R will do just as well. The reference point R is usually placed at origin or infinity. However, sometimes, neither of these standard choices are allowed, in that case we pick an arbitrary point at a finite distance from the origin as the reference point.

8.3.1 Potential Energy Associated with Gravity

Consider a particle of mass m near the surface of earth. Since the weight of magnitude mg acts vertically at the particle, force of gravity will do work only for a vertical displacement. That is, the work by the force $\{mg, \text{pointed down}\}$ from any point on $y = a$ plane to any point in $y = b$ plane will be the same (Fig. 8.18). From the work

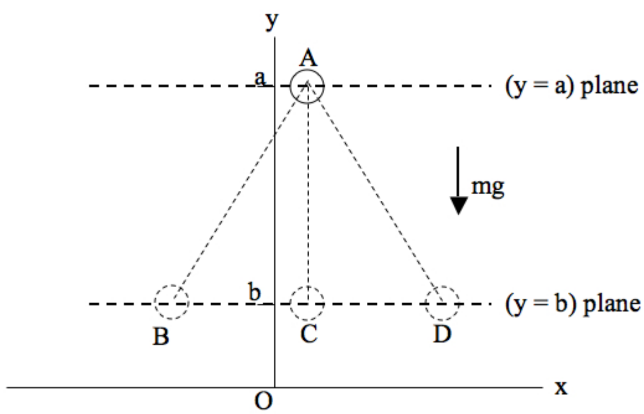

Figure 8.18: The work done by mg is same between points A and B, C or D since it depends only on the vertical displacement.

done by the force of gravity we will deduce a formula for the potential energy as a function of height $y = h$ with respect to a reference at $y = y_R$.

Since weight is a constant force, we can find work easily by the dot product of the displacement vector with the constant force. With the y-coordinates pointed up we have the following vectors for the displacement from P to R and the weight vector.

$$\text{Displacement:} \quad \Delta\vec{r} = (y_R - h)\hat{u}_y.$$

$$\text{Force:} \quad \vec{F} = -mg\hat{u}_y.$$

Therefore, work done on the object when it is displaced from P to R is

$$W = \int_h^{y_R} F_y dy = -mg\,(y_R - h)\,. \tag{8.41}$$

We now equate the work done to $U(h) - U(y_R)$ and obtain

$$U(h) = U(y_R) - mgy_R + mgh. \tag{8.42}$$

We can choose the reference point to be at the origin so that $y_R = 0$, and choose the reference value of potential energy at the origin to

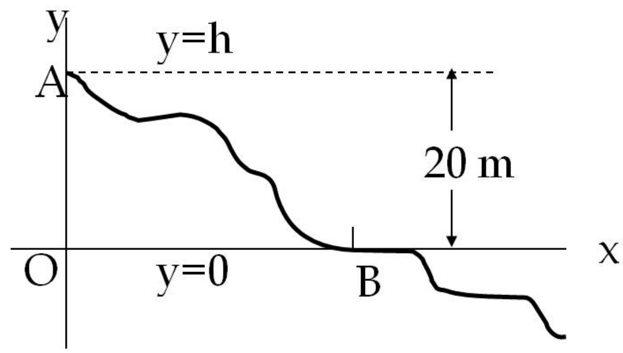

Figure 8.19: Example 8.3.1.

be zero, i.e. set $U(y_R) = 0$ to simplify the formula for the potential energy function corresponding to the force of gravity.

$$\boxed{U(h) = mgh, \quad \text{Reference at } h = 0.} \tag{8.43}$$

This result is often called the **gravitational potential energy** of an object of mass m at coordinate (x, h, z) with a reference point at $y = 0$. Note that $h < 0$ for points below the $y = 0$ plane so that gravitational potential energy is negative for $h < 0$ and positive for $h > 0$.

Example 8.3.1. Gravitational potential energy of a skier A skier starts from rest and slides downhill as shown in Fig. 8.19. What will be the speed of the skier if he drops by 20 meters in going from A to B? Ignore any air resistance (which will, in reality, be quite a lot), and any friction between the skis and the snow.

Solution. The motion of the skier occurs on a complicated surface. To apply $\vec{F} = m\vec{a}$ we would need the force on the skier at each instant on the path, which is impossible to obtain in this case. However, since the force of gravity is a conservative force, we can use the principle of the conservation of energy, which can relate quantities such as the position and speed at one instant to the position and speed at another instant.

When applying the principle of the conservation of energy to problems, the algebra is simpler if we choose the initial or the final point to be the reference point. In the case of gravity, it is customary to choose the lowest point in the motion as the reference point. This choice makes the gravitational potential energies at all other points of interest come out positive.

Therefore, we will choose point B as the reference point here, rather than the ground. Then, the height h above B in the gravitational potential energy formula will be positive for points above B and negative for points below B. The conservation of energy between points A and B takes the following form here.

$$K_A + U_A = K_B + U_B,$$

which gives the following equation

$$0 + mgh = (1/2)mv_B^2 + 0$$

solving for v_B we obtain

$$v_B = \sqrt{2gh},$$

where we keep only the positive root since the speed is a positive number. Now, we can plug in the values given in the problem to obtain the value of v_B:

$$v_B = \sqrt{2 \times (9.8\ m/s^2) \times (20\ m)} = 20\ m/s.$$

Note that, the mass of the skier canceled out in this problem. This happened because the gravitational potential energy U was proportional to mass. Therefore, all skiers, regardless of their masses, will have the same motion if we ignore the air resistance and ground friction on them.

8.3.2 Potential Energy Associated with the Spring Force

To find the formula for the potential energy of a particle subject a spring force we will carry out the same steps of calculations as we did for the particle subject to gravity. Consider a block of mass m attached to a spring of spring constant k. Without any loss of generality we can choose the origin of the coordinate system at the point in Fig. 8.20 when the spring is neither stretched nor compressed from its original length. We will also choose the x-axis to be the line in which the mass moves as shown in the figure. Let $x(t)$ be the x-coordinate of the block at an arbitrary time t. The spring force on the block is given by the Hooke's law whose mathematical expression written using the coordinate system given in Fig. 8.20 is as follows.

$$\vec{F} = -k\ x\ \hat{u}_x. \tag{8.44}$$

The infinitesimal displacement vector along the x-axis is

$$d\vec{r} = dx\ \hat{u}_x, \tag{8.45}$$

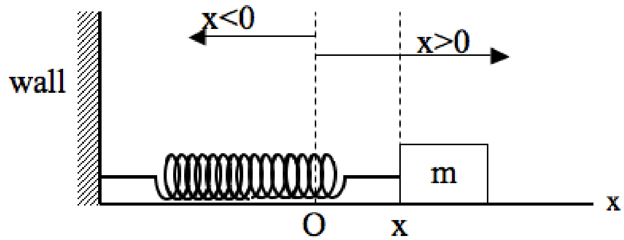

Figure 8.20: The set-up for calculation of the potential energy stored in the spring.

which is obtained by setting $dy = dz = 0$ in $d\vec{r} = dx\ \hat{u}_x + dy\ \hat{u}_y + dz\ \hat{u}_z$. The work integral from point $P(x, 0, 0)$ to a reference point $R(x_R, 0, 0)$ gives

$$\begin{aligned} W &= \int_P^R \vec{F} \cdot d\vec{r} = -\int_x^{x_R} kx dx \\ &= \frac{1}{2}kx^2 - \frac{1}{2}kx_R^2 \end{aligned} \tag{8.46}$$

This is the change in the potential energy, $U(x) - U(x_R)$, of the block due to the spring force.

$$U(x) - U_R = \frac{1}{2}k\ x^2 - \frac{1}{2}k\ x_R^2. \tag{8.47}$$

Suppose, we choose the reference of the potential energy to be zero when $x = 0$, i.e. when the spring is neither stretched nor compressed.

$$\text{Choice of reference:} \quad x_R = 0 \text{ and } U_R = 0.$$

Then, the potential energy of the block due to the spring force when the spring is stretched or compressed by an amount $|x|$ is given by a simple formula.

$$\boxed{U(x) = \frac{1}{2}k\ x^2, \quad \text{Reference at } x = 0.} \tag{8.48}$$

The reference is the state when the spring is neither stretched nor compressed. Sometimes, this potential energy is called the **spring energy** since this energy is contained in the spring since the spring is deformed compared to the original length. When the spring comes back to the original configuration, the energy in the spring goes towards speeding up the objects connected at the two ends. In the case of one end connected to an immovable support and the other end to a movable block, the energy in the spring foes towards changing the kinetic energy of the block.

Example 8.3.2. Mass attached to a spring on a frictionless table. Consider a block of mass 0.2 kg attached to a spring of spring constant 100 N/m. The block is placed on a frictionless table, and the other end of the spring is attached to the wall so that the spring is level with the table. The block is then pushed in so that the spring is compressed by 10 cm. Find the speed of the block as it crosses (a) the point when the spring is not stretched, (b) 5 cm to the left of point in (a), and (c) 5 cm to the right of point in (a).

Solution. Since there is no friction, only the spring force does work on the mass and since spring force is a conservative force, the mechanical energy is conserved. Here the potential energy is due to the spring force only. Therefore,

$$(K+U)_i = (K+U)_f \,.$$

$$\Longrightarrow \quad \left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)_i = \left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)_f .$$

(a) In this part, the initial speed $v_i = 0$ and the initial position $x_i = -10\ cm = -0.1\ m$, and the final speed v_f is unknown and the final position $x_f = 0$. Therefore,

$$0 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + 0$$

Ignoring the negative root, since v is speed and it is positive, we obtain

$$v_f = \sqrt{\frac{k}{m}}|x_i| = \sqrt{\frac{100\ N/m}{0.2\ kg}} \times 0.1\ m = \sqrt{5}\ m/s.$$

(b) In this part, we can choose i and f as follows. $x_i = 0$ (at origin); $x_f = -5$ cm to the left of origin. We know the following: $v_i = \sqrt{5}$ m/s, $x_i = 0$, $v_f =$ unknown, and $x_f = -5$ cm. Therefore, writing the conservation equation in symbols before we plug in numbers.

$$\frac{1}{2}mv_i^2 + 0 = \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2$$

We solve this equation for v_B.

$$v_B = \sqrt{v_A^2 - \frac{k}{m}x_B^2} = \sqrt{3.75}\ \text{m/s}.$$

(c) This part will give the unknown $v = \sqrt{3.75}\ m/s$. The detail is left as an exercise for the student.

8.3.3 Potential Energy Associated With the Universal Gravitational Force

The universal gravitational force between any two objects is given by the Newton's law of gravitation. The law states that the gravitational force between two objects is proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force on the two objects are determined by the attractive nature of the force. Thus, if we place mass M at the origin, then force on mass m at a distance r is

$$\vec{F}_{on\ m} = G_N \frac{mM}{r^2}(-\hat{u}_r), \tag{8.49}$$

where $\hat{u}_r$ is a unit vector radially outward from the origin, and the negative sign reflects the fact that the force on m is towards M. Newton's gravitational constant G_N has value $6.67\times 10^{-11}\ N.m^2/kg^2$.

How much work does this force do when the distance between the masses change from r_0 to r_R? To find this work, we will imagine a process in which mass M is fixed at the origin and mass m is moved "virtually" by exerting an external force $\vec{F}_{appl}$ that balances the gravitational force at each point.

$$\vec{F}_{appl} = -\left[G_N \frac{mM}{r^2}(-\hat{u}_r)\right], \tag{8.50}$$

The work done by $\vec{F}_{appl}$ will be equal to the negative of the work done by the gravitational force on m. By the radial nature of the force, it is clear that work done on m between any points of a sphere of radius r_0 to any point on a sphere of radius r_R will be same. To be concrete, let $r_R > r$ and let us choose point P on the spherical surface of radius r_0 and point R on the spherical surface of radius r_R that are on the same radial line from the origin (Fig. 8.21). Then an infinitesimal displacement between $\vec{r}$ and $\vec{r} + d\vec{r}$ on this path is

$$d\vec{r} = dr\ \hat{u}_r, \tag{8.51}$$

From Eqs. 8.50 and 8.51 we find the work done by the applied force for the infinitesimal displacement to be

$$dW = \vec{F}_{appl}\cdot d\vec{r} = G_N \frac{mM}{r^2} dr, \tag{8.52}$$

where we have used the identity $\hat{u}_r \cdot \hat{u}_r = 1$, since $\hat{u}_r$ is a unit vector. Performing the integral we obtain the work done by the applied force to be

$$W = G_N \frac{mM}{r_0} - G_N \frac{mM}{r_R}. \tag{8.53}$$

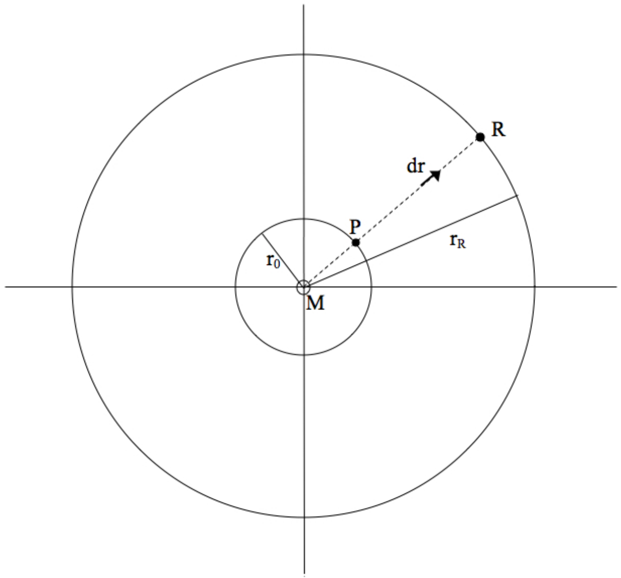

Figure 8.21: The set-up for calculation of the gravitational potential energy.

This is equal the negative of the work done by the gravitational force on m and can be equated to the potential energy difference $U(r_0)-U_R$ for the gravitational force, as we have done for other forces.

$$U(r_0) - U_R = G_N\frac{mM}{r_R} - G_N\frac{mM}{r_0}. \tag{8.54}$$

Now, we choose the values of r_R and U_R so that the potential energy $U(r_0)$ is simplest and has a well-defined value. Clearly $r_R = 0$ will not be an acceptable reference point since the first term on the right side of Eq. 8.54 is not defined for $r_R = 0$. A good reference here is $r_R = \infty$ and the choice of $U_R = 0$ further simplifies the formula. With these choices the gravitational potential energy of m when the separation between the masses is r_0 is given by

$$U(r_0) = -G_N\frac{mM}{r_0}\ \left[\text{Reference: } r_R = \infty \text{ and } U_R = 0\right]. \tag{8.55}$$

Since there is nothing special about the distance r_0, we use Eq. 8.55 to write the gravitational potential energy function $U(r)$ for m when it is at a distance r from M.

$$\boxed{U(r) = -G_N\frac{mM}{r}\ \left[\text{Reference: } r_R = \infty \text{ and } U_R = 0\right].} \tag{8.56}$$

Note that gravitational potential energy of m is a negative number for all finite separation of the two masses. That is, the gravitational potential energy of m increases as the distance between the masses increases. This is consistent with our intuition that to pull m away from M an agent will have to do work on m thereby increasing the energy of the later. Note also that the gravitational potential energy is not defined when the masses are on top of each other.

Example 8.3.3. Escape speed If an object is fired with large enough speed it will escape the gravitational pull of the Earth. The minimum speed needed to escape the pull of the Earth is called the escape velocity or more appropriately **escape speed**. Find the escape speed from the Earth. Assume the Earth to be a sphere of mass M_E and radius R_E.

Solution. Let v_e denote the escape speed of an object of mass m when the object is at the surface of the Earth. When the object has escaped far away, the speed there will be taken to be zero since we are looking for the minimum speed of escape. If the air resistance can be ignored, the energy of the fired object will be conserved since it has only the gravitational force of the Earth acting on it, which is a conservative force. The conservation of energy of the object between when it is at the surface of the Earth and when it at a far away point, which will be taken to be $r = \infty$, yields the following equation.

$$E_{\text{on earth}} = \frac{1}{2}mv_e^2 + \left(-G_N\frac{mM_E}{R_E}\right)$$

$$E_{\text{far away}} = 0$$

$$\text{Therefore, } \frac{1}{2}mv_e^2 + \left(-G_N\frac{mM_E}{R_E}\right) = 0$$

$$\Rightarrow \quad v_e = \sqrt{\frac{2G_N M_E}{R_E}} \quad \text{(positive root since its a speed).}$$

Since the mass of the fired object cancels out from the equation, the escape speed would be same for all objects regardless of the mass. Putting the values for G_N, M_E and R_E, the value of the escape speed from the Earth turns out to be $v_e \approx 112$ km/s, which is approximately $400,000$ km/h or $250,000$ mph.

Note that the calculation given above is with respect to a frame fixed to the center of Earth. A frame on Earth will be moving towards East with a speed equal to product of the angular rotation speed of the Earth and the distance from the axis of rotation of the Earth. Therefore, firing the projectile horizontally towards East will add the velocity of Earth-surface based frame to the launch velocity. This will result in a speed of launch needed to reach the escape speed to be considerably less than 112 km/s if you launch the projectile towards East. Also, firing from a place near the equator will give the largest distance from the axis of rotation, and hence the Earth-surface based frame with the largest speed with respect to the Earth-center based frame. That is why the space rocket launches in the United States are done from the state of Florida and not from the state of Alaska.

8.3.4 Force and Potential Energy Function

We have found above that the change in potential energy of an object is given by the negative of the work done by the conservative force $\vec{F}_c$. That is, the difference of potential energy between when the object is at P and when it is at Q is

$$U(\text{at Q}) - U(\text{at P}) = -\int_P^Q \vec{F}_c \cdot d\vec{r}. \tag{8.57}$$

It is possible to invert this equation and obtain the conservative force if we know the potential energy function U. Consider a one-dimensional situation, and let us apply Eq. 8.57 to an infinitesimal displacement between x and $x + \Delta x$.

$$U(x+\Delta x) - U(x) = -\int_x^{x+\Delta x} F_x dx \approx -F_x \Delta x, \tag{8.58}$$

where F_x is the x-component of the conservative force $\vec{F}_c$. Now divide both sides of Eq. 8.58 by Δx and take $\Delta x \rightarrow\ 0$ limit to obtain

$$F_x = -\lim_{\Delta x \rightarrow\ 0} \frac{U(x+\Delta x) - U(x)}{\Delta x} = -\frac{dU}{dx} \tag{8.59}$$

Thus, for a one-dimensional displacement along the $x$axis, the x-component of the force is equal to the negative of the x-derivative of the potential energy function $U(x)$.

$$F_x = -\frac{dU}{dx} \tag{8.60}$$

We now state without proof that if you have a three-dimensional situation, then the x, y, and z-components of the force will be obtained from the partial derivatives of the potential energy function.

$$\boxed{F_x = -\frac{\partial U(x,y,z)}{\partial x};\quad F_y = -\frac{\partial U(x,y,z)}{\partial y};\quad F_z = -\frac{\partial U(x,y,z)}{\partial z}} \tag{8.61}$$

In these equations we have used partial derivatives. Partial derivatives of a function of more than one independent variable refers to the derivative with respect to one variable while keeping the other variables fixed. That is, while performing the partial derivative with respect to x, we will keep y and z fixed here. Similarly, for y and z derivatives.