5.1 SUPERPOSITION PRINCIPLE AND INTERFERENCE OF WAVES

5.1.1 Interference Of Two Waves

According to the **superposition principle** when two or more waves overlap, the combination of the two waves results in another wave whose amplitude is equal to the sum of the amplitudes of the constituent waves. This principle applies to all waves, whether they are sound, light or matter waves. Although, the amplitudes of waves add simply, their intensities add in a more complicated way because intensity of a wave is proportional to the square of the amplitude.

To illustrate the effect of squaring of the amplitude to obtain the intensity consider two waves of amplitudes a and b. The net wave will have the amplitude $a + b$ and the net intensity will be proportional to the square of this sum.

$$\text{Intensity of combined wave} \quad \propto \quad (a+b)^2 = a^2 + b^2 + 2ab. \tag{5.1}$$

Note that the square of a sum contains a cross term in addition to the squares of each term. The cross term can be positive or negative depending upon the relative phases of the two waves. We will see below that the cross term is positive at a place where the two waves are in-phase and negative where they are out-of-phase as illustrated in Fig. 5.1. The result is an enhancement of intensity in places where the two waves are in-phase and a diminution where they are out-of-phase with each other. Consequently, we find a pattern of bright and dark fringes called the **interference pattern**. The phenomenon is called interference, and the cross-term responsible for it is called the interference term. When the interference term adds to the other terms, we say that there is a **constructive interference**, and when it subtracts, we refer to it as a **destructive interference**.

5.1.2 The Young's Double-Slit Experiment

A simple illustration of the interference phenomenon is provided by the Young's double-slit experiment, where a monochromatic source of light is shone at two narrow slits, and the emerging waves are then made to interfere on a far away screen (Fig. 5.2).

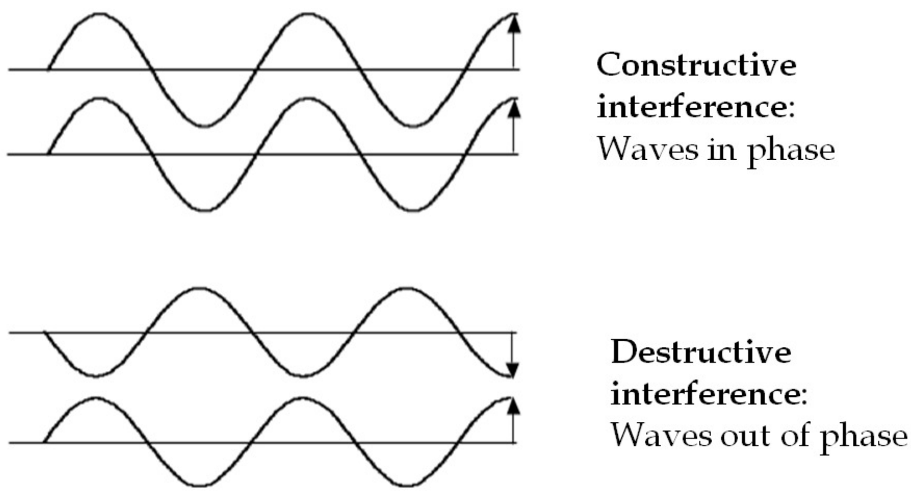

Figure 5.1: Constructive and destructive interference.

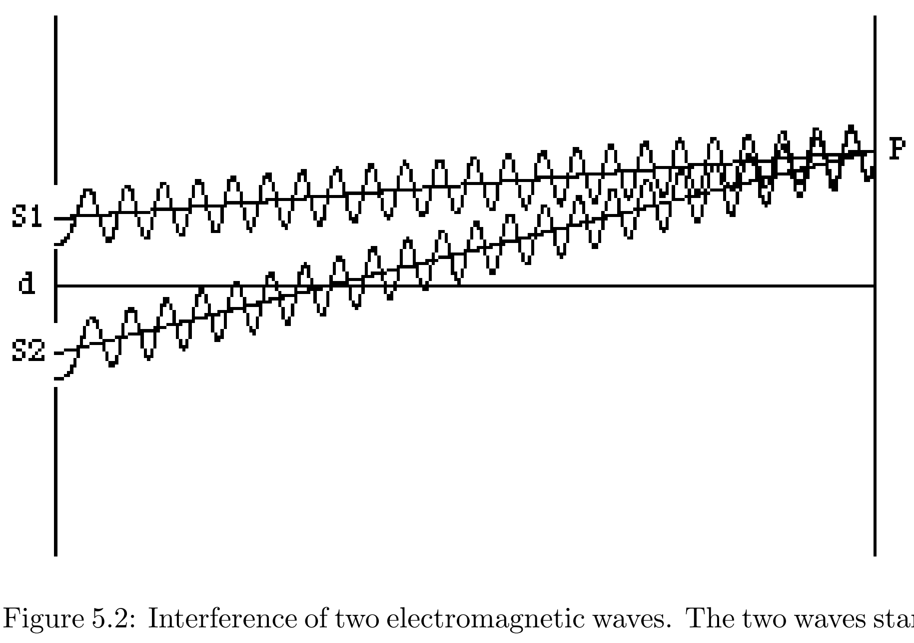

Figure 5.2: Interference of two electromagnetic waves. The two waves start out in-phase at the two slits, and travel different distances and interfere at points P on the screen. In this picture, the wave from S1 makes goes 23 wavelengths and that from S2 goes 24 wavelengths to interfere at P constructively there.

The emerging waves near the slits are spherical but their curvature drops as we move farther away. Since the screen is far away, the waves can be approximated by plane waves. Therefore, we can represent the waves arriving at the screen by the following analytic expressions stated in the last chapter for plane waves.

$$\vec{E}_1 = \vec{E}_0 \cos\left(2\pi f t - \frac{2\pi}{\lambda} l_1\right) \tag{5.2}$$

$$\vec{E}_2 = \vec{E}_0 \cos\left(2\pi f t - \frac{2\pi}{\lambda} l_2\right) \tag{5.3}$$

where $\vec{E}_0$ is equal amplitude of the waves, and l_1 and l_2 are the distances traveled by each wave, which are S_1P and S_2P respectively in the figure. Here f and λ denote the frequency and wavelength respectively.

The net electric field at point P will be the sum of the two.

$$\vec{E} = \vec{E}_1 + \vec{E}_2. \tag{5.4}$$

Net intensity is obtained from the net electric field by time averaging.

$$I = c\epsilon_0 \langle \vec{E} \cdot \vec{E} \rangle_{\text{time average}}. \tag{5.5}$$

Using the net electric field given in Eq. 5.4 we find that the net intensity at any point is the sum of individual intensities I_1 and I_2 for the two waves separately and an additional term denoted by I_{12} called the **interference term**.

$$I = I_1 + I_2 + I_{12}, \tag{5.6}$$

where

$$I_1 = c\epsilon_0 \langle E_1^2 \rangle_{\text{time average}} = \frac{1}{2} c\epsilon_0 E_0^2 \tag{5.7}$$

$$I_2 = c\epsilon_0 \langle E_2^2 \rangle_{\text{time average}} = \frac{1}{2} c\epsilon_0 E_0^2 \tag{5.8}$$

$$I_{12} = 2c\epsilon_0 \langle \vec{E}_1 \cdot \vec{E}_2 \rangle_{\text{time average}} \tag{5.9}$$

Evaluating the interference term explicitly we find the following.

$$I_{12} = \left(c\epsilon_0 E_0^2\right) \cos\left[\frac{2\pi}{\lambda}\left(l_1 - l_2\right)\right] \tag{5.10}$$

Note that unlike the intensities I_1 and I_2, the interference term varies depending upon the location of P from S_1 and S_2. We will denote the phase difference that is the argument of the cosine in the interference term by Δ_{12}.

$$\Delta_{12} = \frac{2\pi}{\lambda}\left(l_1 - l_2\right). \tag{5.11}$$

Clearly, there will be points on the screen, where the path difference $|l_1 - l_2|$ from the two slits will be an integral multiple of the wavelength. At those points on the screen, the phase difference will be $2m\pi$ with m an integer. Since $\cos(2m\pi) = 1$, the interference term I_{12} will add to $I_1 + I_2$ at those points. Therefore, we will get a constructive interference when this condition holds.

$$\text{Constructive: } \Delta_{12} = 2m\pi \quad (m \text{ integer}). \tag{5.12}$$

Similarly, there will be places on the screen, where the path difference $|l_1 - l_2|$ will be an odd-multiple of one-half wavelength. At those points on the screen, the phase difference will be odd multiples of π radians. The waves will be completely out of phase at these locations, resulting in the maximum negative value for the interference term. We will have the destructive interference at those points.

$$\text{Destructive: } \Delta_{12} = m'\pi \quad (m' \text{ odd integer}). \tag{5.13}$$

The conditions for the constructive and destructive interferences in the Young's double-slit experiment are summarized in terms of the phase difference as follows.

$$\Delta_{12} = \frac{2\pi\,|S_1P - S_2P|}{\lambda} = \begin{cases} 2m\pi & (m = 0, \pm 1, \pm 2, \cdots) \quad \text{Constructive} \\ m'\pi & (m' = \pm 1, \pm 3, \pm 5, \cdots) \quad \text{Destructive} \end{cases} \tag{5.14}$$

The integers m and m' are called the **orders of constructive and destructive interferences**. Solving for the path difference of a point on the screen from the slits, we find the conditions for constructive and destructive interference in terms of the path lengths as

$$|S_1P - S_2P| = \begin{cases} m\lambda & (m = 0, \pm 1, \pm 2, \cdots) \quad \text{Constructive} \\ m'\frac{\lambda}{2} & (m' = \pm 1, \pm 3, \pm 5, \cdots) \quad \text{Destructive} \end{cases} \tag{5.15}$$

This makes sense when you look at what happens to the waves. They start out at the slits in-step with each other, and when they travel to different distances, they will tend to go out of step by varying amounts. They will be completely in-step at places where path difference is an integer multiple of one wavelength, and they will be completely out of step, meaning if one is at the crest, then the other would be at the trough there, if the difference in path is off by an odd multiple of half a wavelength.

Denoting the intensity I_1 and I_2 by I_0, we find that the intensity at point P on the screen can be given by a simpler formula.

$$I = 2I_0 + 2I_0\cos\Delta_{12} = 4I_0\cos^2\left(\frac{\Delta_{12}}{2}\right). \tag{5.16}$$

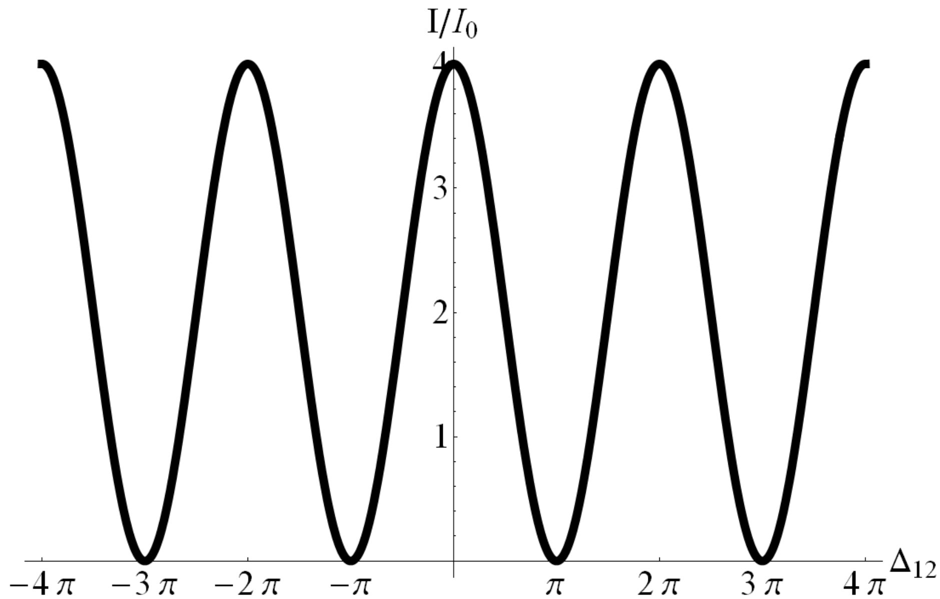

Figure 5.3: Intensity versus phase difference Δ_{12}.

Hence, the intensity on the screen varies from 0 when they are out of phase to $4I_0$ when they are in phase. Plotting the net intensity I at point P vs the phase difference Δ_{12} shows an interference pattern of maximum and minimum brightness (Fig. 5.3). When the phase difference is 0, $\pm 2\pi$, $\pm 4\pi$, $\pm 6\pi$, etc, the interference term adds to the other terms and constructive interference results, and the brighter spots are observed on the screen. When the phase difference is $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, etc, the interference term subtracts from the other terms, and a destructive interference takes place that results in a dark or less bright spot.

INTERFERENCE AT SMALL ANGLES

Some simplification in the interference formula given in Eqs. 5.14 and 5.19 results when we look at the interference near the center of the screen. The angles made by S_1P or S_2P with the horizontal direction are small for these points. In Fig. 5.4 we redraw the two-slit interference set up in a way that is suitable for the analysis at small angle θ, which will be used to state the interference conditions in another way.

Note that in the small angle approximation $\angle\mathrm{PO'C} \approx \angle\mathrm{POC} = \theta$.

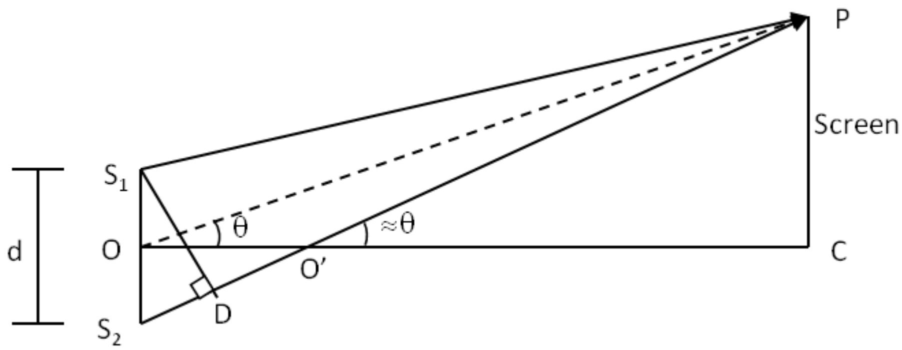

Figure 5.4: Geometry for calculation of interference in Young's double-slit experiment.

Now, from the triangles $\triangle \mathrm{S_2OO'}$ and $\triangle \mathrm{S_1S_2D}$ we can conclude that

$$\angle \mathrm{S_2S_1D} \approx \theta. \tag{5.17}$$

The path difference then becomes equal to $\mathrm{S_2D}$.

$$\text{Path difference} = |\mathrm{S_1P} - \mathrm{S_2P}| \approx \mathrm{S_2D} = d\sin\theta. \tag{5.18}$$

where d is the distance between the slits. We now rewrite the interference conditions for Young's double-slit experiment given in Eq. 5.19 for small angles as

$$d\sin\theta = \begin{cases} m\lambda & (m = 0, \pm1, \pm2, \cdots) \quad \text{Constructive} \\ m'\frac{\lambda}{2} & (m' = \pm1, \pm3, \pm5, \cdots) \quad \text{Destructive} \end{cases} \tag{5.19}$$

As the sine of an angle cannot be greater than 1, d should be larger than $m'\lambda/2$ in case of the destructive interferences. Similarly, d must be larger than $m\lambda$ for constructive interferences. Therefore, there is a maximum order for fringes possible for a given d.

$$\frac{m\lambda}{d} = \sin\theta \leq 1 \implies m_{\max} = \mathrm{int}\left\lfloor \frac{d}{\lambda} \right\rfloor, \tag{5.20}$$

where int stands for the integer obtained when the floor function is evaluated on the ratio of d to λ. For instance, if the slits are separated by 3.5 times the wavelength of light, then you will observe up to three bright fringes around the bright maximum in the middle.

Fig. 5.5 shows a simulation of two slits separated by 4-wavelengths. One can see the bright and dark fringes result from their interference on the screen at the right.

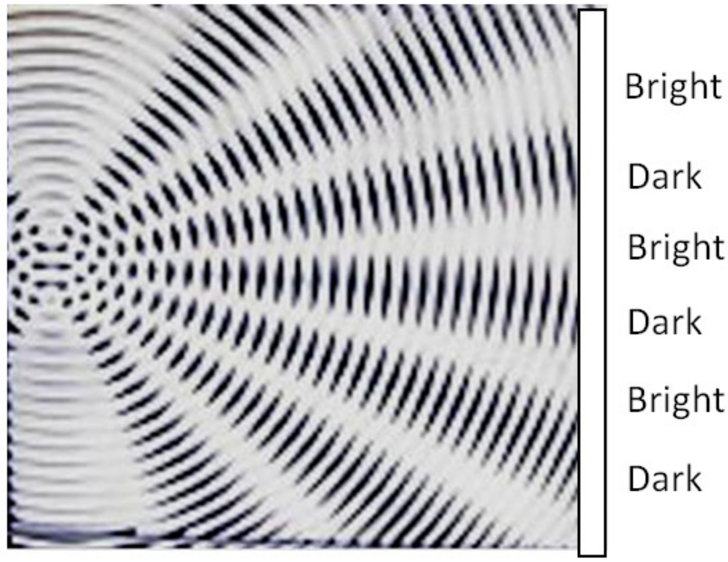

Figure 5.5: Simulation of interference pattern from an Young's double-slit experiment.

Example 5.1.1. Interference of laser light in a double-slit experiment A red light of wave length 630 nm is incident on a double-slit with slit width 300 nm each and separated by 1500 nm. A screen is place at a distance of $\frac{1}{2}$ m. Find the locations of the screen where you will find (a) bright fringes and (b) dark fringes.

Solution. We have $d = 1500$ nm and $\lambda = 630$ nm. Hence the maximum order of constructive interference we shall observe is

$$m_{\text{max}} = \text{int}\left\lfloor \frac{d}{\lambda} \right\rfloor = \text{int}\left\lfloor \frac{1500\text{ nm}}{630\text{ nm}} \right\rfloor = 2.$$

Therefore, we will have constructive interferences only for $m = 0,\ \pm 1,$ and $\pm$ 2. The angular directions of the constructive and destructive interferences are:

Constructive interferences:

(1) $m = 0: \quad \theta = 0$
(2) $m = 1: \quad 1500\text{ nm}\ \sin\theta = 630\text{ nm} \quad \Longrightarrow \quad \theta = 23^\circ.$
(3) $m = -1: \quad \theta = -23^\circ.$
(4) $m = 2: \quad 1500\text{ nm}\ \sin\theta = 2 \times 630\text{ nm} \quad \Longrightarrow \quad \theta = 52.6^\circ.$
(5) $m = 2: \quad \theta = -52.6^\circ.$

The angles for $m = \pm 2$ are too large for small angle approximation to be much accurate. However, the values do give an indication of the angles.

Destructive interferences:

(1) $m' = 1: \quad 1500 \text{ nm } \sin\theta = \dfrac{630 \text{ nm}}{2} \quad \Longrightarrow \quad \theta = 11°.$

(2) $m' = -1: \quad \theta = -11.2°.$

(3) $m' = 2: \quad 1500 \text{ nm } \sin\theta = \dfrac{3 \times 630 \text{ nm}}{2} \quad \Longrightarrow \quad \theta = 36°.$

(4) $m' = 2: \quad \theta = -36°.$

Let us denote the position on the screen by the y-coordinates with the origin at the $m = 0$ constructive interference. The locations on the screen are obtained by using the trigonometry of right angle with $y = R\tan\theta$, where R is the distance to the screen. Here $R = \frac{1}{2}$ m. Hence, the constructive interferences are at $y = 0$, ± 21 cm, and ± 65 cm, and the destructive interferences at $y = \pm 9.9$ cm and ± 36 cm.