10.2 UNDERSTANDING ELECTROMAGNETIC INDUCTION

Various experiments of Faraday show that an electricity is induced in a circuit when there is a magnetic field. These effects are collectively called the **electromagnetic induction**.

Faraday reasoned that the induced current in the loop must be due to some driving force in the wire, as is the case with any current in a wire. However, different experiments of Faraday appear to imply different mechanisms at work in different situations. We find two sources for the induced EMF in these experiments.

1. EMF in Moving Conductors

For instance, when the magnet is fixed and the loop is moving, the current may be said to arize from the magnetic force $\vec{F}_m = q\vec{v}\times\vec{B}$ on the conduction electrons of the wire. The conduction electrons are guided along the loop by the cumulative influence of this magnetic force. The induced EMF in this circuit will be equal to the line integral of the magnetic force per unit charge along the circuit.

$$\boxed{|\mathcal{E}_{\text{ind}}| = \left|\oint \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l}\right| \quad \text{(Moving conductor.)}} \tag{10.1}$$

Consider a moving cross-bar on a U-shaped metal placed in a uniform magnetic field as shown in Fig. 10.5. When you pull on the bar to the right so that the bar slides with a constant velocity, an EMF will be induced in the circuit which will be equal to the line integral given in Eq. 10.1.

$$\begin{aligned}
|\mathcal{E}_{\text{ind}}| &= \left|\int_{\text{abcda}} \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l}\right| \\
&= 0 + \left|\int_{\text{ab}} \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l}\right| \quad \text{(since } v = 0 \text{ in other parts.)} \\
&= vBw.
\end{aligned}$$

An intriguing feature of this result is that the line integral of $\vec{v}\times\vec{B}$ is actually equal to the rate at which magnetic flux through the loop abcda of the circuit changes since the area A enclosed by the loop is changing.

$$\left|\int_{\text{abcda}} \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l}\right| = \left|B\frac{dA}{dt}\right| = \frac{d\Phi_B}{dt} \quad \text{(since B is constant.)}$$

This important "coincidence" turns out to be a special case of the Faraday flux rule to be discussed below.

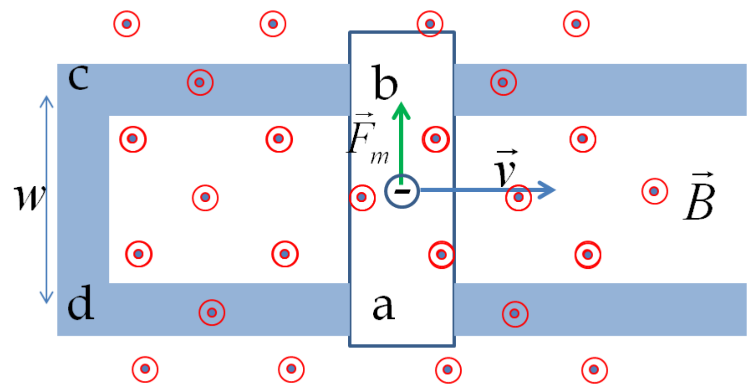

Figure 10.5: An EMF is induced in a circuit as a result of movement of a part of the circuit in a magnetic field. The induced EMF is equal to the line integral of the magnetic force per unit charge.

2. EMF in Changing Magnetic Field

When the wire is held fixed and the magnet is moving, there is no $q\vec{v}\times\vec{B}$ force on the charges. Faraday found that the induced EMF in these circuits is due a new effect. He found that when the circuit is not moving but instead the magnetic field through the space of the circuit is varying, then the induced EMF is proportional to the rate of change of the magnetic field.

$$|\mathcal{E}_{\text{ind}}| \propto \left|\frac{dB}{dt}\right| \qquad (\text{ Fixed circuit, Fixed orientation.}) \tag{10.2}$$

This kind of experiment is easily done in experiments like Fig. 10.4 where the magnetic field through the loop can be varied by turning the switch for the current in the solenoid on/off and/or moving the solenoid while holding the loop #2 fixed in position.

It is possible to state the two effects that lead to induced EMF in a loop of wire, viz. one due to the changing magnetic field and the other due to the motion of the wire into one relation. This relation is due to Faraday and is called the Faraday flux rule. The induced effect in places where there is no clear loop of wire, e.g. in a moving rod or a rotating disk the induced EMF is understood in terms of the magnetic force $\vec{v}\times\vec{B}$ on mobile charges in moving conductors.

3. Faraday's Flux Rule

The Faraday's flux rule states that the induced EMF in a circuit, whether due to the relative motion of the magnet and the loop or due to the changing magnetic field, can be combined into the rate at which the magnetic flux changes.

$$\boxed{|\mathcal{E}_{\text{ind}}| = \left|\frac{d\Phi_B}{dt}\right|}, \tag{10.3}$$

where the magnetic flux is through any open surface attached to the circuit.

$$\boxed{\Phi_B = \iint_{\text{open surface}} \vec{B}\cdot d\vec{A}.} \tag{10.4}$$

To illustrate the change in the magnetic flux consider a rectangular conducting loop of dimensions a and b in a uniform magnetic field $\vec{B}$ as shown in Fig. 10.6. We choose the area vector in the direction so that magnetic flux will be positive and denote the angle between the area vector and the magnetic field by θ. Then, magnetic flux Φ_B through the flat open surface attached to the loop will be

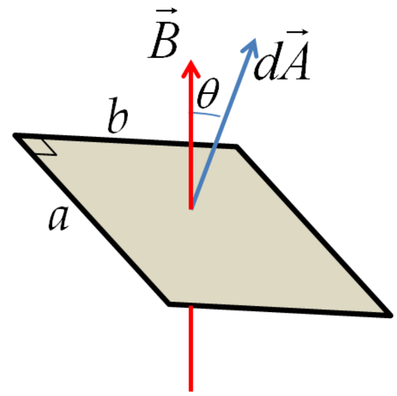

Figure 10.6: A rectangular loop of area ab in a magnetic field $\vec{B}$ at an angle θ between the area vector and the magnetic field.

$$\Phi_B = \int \vec{B}.d\vec{A} = abB\cos\theta = \mathrm{A}\times B\times\cos\theta, \tag{10.5}$$

where area $A = ab$. This says that magnetic flux can be changed by changing either the area of the closed loop, the magnetic field, or the orientation of the area of the closed loop with respect to the magnetic field.

$$\frac{d\Phi_B}{dt} = \frac{dA}{dt}B\cos\theta + A\frac{dB}{dt}\cos\theta + AB\frac{d\cos\theta}{dt}. \tag{10.6}$$

This illustrates that the change in magnetic flux can occur in three ways: (1) a change in the magnitude of the magnetic field, (2) a change in the loop area, or (3) a change in the orientation of the loop with respect to the magnetic field, which can occur as a result of a change in the direction of the magnetic field or a change in the orientation of the loop or both. We will study later in this chapter several examples of these three ways of changing magnetic flux.

The changing magnetic flux as a result of changing magnetic field alone, while keeping the loop fixed in space, will be

$$\frac{d\Phi_B^{\text{Field}}}{dt} = A\frac{d(B\cos\theta)}{dt} \quad \text{(Only magnetic field changing,)} \tag{10.7}$$

and the changing magnetic flux as a result of changing loop alone, while keeping the magnetic field fixed, will be

$$\frac{d\Phi_B^{\text{Loop}}}{dt} = B\frac{d(A\cos\theta)}{dt} \quad \text{(Only loop moving.)} \tag{10.8}$$

The EMF due to the moving loop, that is as a result of $d\Phi_B^{\text{Loop}}/dt$ requires a conductor to be at the site of the loop so that magnetic field can act on the charges. This flux change accounts for the electromagnetic induction as a result of $\vec{v} \times \vec{B}$ on the moving charges.

On the other hand, the EMF as a result of $d\Phi_B^{\text{Field}}/dt$ does not require the presence of any charge on the loop. Therefore, this type of electromagnetic effect will occur for any abstract or mathematical loop in space.

$$\left|\mathcal{E}_{\text{ind}}^{\text{Field}}\right| = \frac{d\Phi_B^{\text{Field}}}{dt} \quad \text{(Abstract Loop)} \tag{10.9}$$

Since the abstract loop may not have any electric charge, the electromotive force induced in the abstract loop by a changing magnetic field must be due to an induced electric field $\vec{E}_{\text{ind}}$. Thus, we come to a conclusion that changing magnetic field induces electric field.

$$\left|\oint_{\text{Loop}} \vec{E}_{\text{ind}} \cdot d\vec{l}\right| = \left|\frac{d\Phi_B^{\text{Field}}}{dt}\right|, \quad \text{(Abstract Loop)} \tag{10.10}$$

where I have placed the absolute signs so that we can focus on the magnitudes here and defer the discussion of direction of the electric field to a later section. This electric field is different in character than the static electric field $\vec{E}_{\text{stat}}$ of fixed charges since a line integral of the later around a closed loop gives a zero.

$$\left|\oint_{\text{Loop}} \vec{E}_{\text{stat}} \cdot d\vec{l}\right| = 0, \quad \text{(Static Field; Any Loop)} \tag{10.11}$$

NOTE:

Although, Faraday's flux rule provides a satisfactory explanation of most of the situations that result in an induced EMF, it does not explain the induced EMF in every situation. For instance, there are situations in which the magnetic flux does not change and yet there is an induced EMF as we will see below. These situations occur in experiments where the EMF is generated by the magnetic force $q\vec{v}\times\vec{B}$ alone such that neither the area of the loop nor the orientation of the loop changes. Therefore, it is important to keep in mind the two different mechanisms for the induced EMF. We need both the flux rule and the Lorentz force to understand the full range of phenomena of electromagnetic induction.

Faraday's paradox Faraday also stressed on the limitations of the flux rule in providing an explanation of all electromagnetic phenomenon. Figure 10.7 shows an adaptation of Faraday's disc problem in a practical setting. A conducting disc is connected to a galvanometer with one contact at the shaft and the other at the edge of the disc so that the contacts rub the metal shaft and the metal edge when

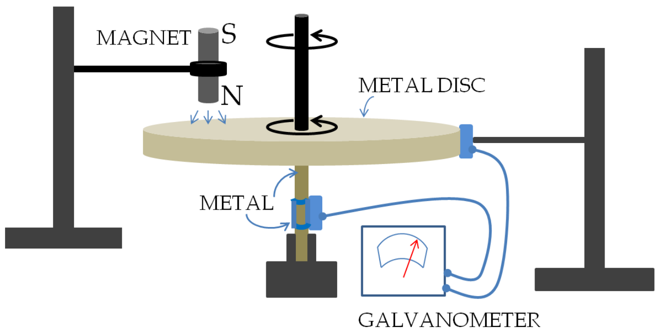

Figure 10.7: A metal disc is rotated in the presence of a fixed magnet. When the disc rotates, there is no change in magnetic flux through any choice of line on the disc to complete the loop through the galvanometer. Yet there is an induced current through the galvanometer. The EMF for the current through the galvanometer can be understood in terms of $\vec{v} \times \vec{B}$ on the charges moving at the points on the disc with non-zero magnetic field. Adapted from Feynman's Lectures in Physics, Vol II.

the disc rotates. A magnet is fixed to provide a magnetic field perpendicular to the surface of the disc.

First note that a unique circuit is not clear here since you can draw infinitely many lines on the disc between the contact at the edge and the contact at the shaft. The magnetic flux through any circuit you draw in space would not change with time. However, when the disc rotates a current is detected by the galvanometer.

A naive application of the flux rule gives us a contradiction here: the flux rule says that no flux change would mean no induced current. However, a closer look at the experimental setting shows that the magnetic force, $q\vec{v} \times \vec{B}$, on the moving charges will cause a radial component of the velocity which will result in an overall drift of charges towards the shaft. The EMF in this setting develops neither from a changing magnetic field nor from a changing area of a loop or a changing orientation of a loop. A student may wish to consult Feynman's Lectures in Physics, Vol II, where another paradox has been illustrated that shows that there are situations where no current is induced even when the magnetic flux is changing. Granted that these paradoxical situations are unusual circumstances, they nevertheless illustrate that, as a matter of principle, a student must be aware to think both in terms of the changing of flux and the magnetic force.

Flux Rule and Choice of Surface

Faraday's flux rule given in Eq. 10.3 is applicable to any loop/surface combination in space. As a matter of fact, the loop does not even have to be a physical loop, although in experiments we usually have a physical circuit in which we want to induce current.

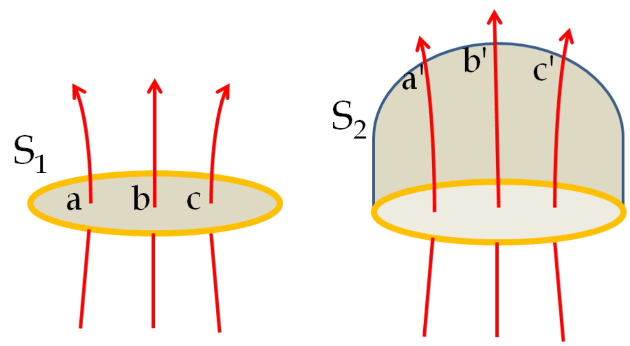

Figure 10.8: Magnetic fluxes through all open surfaces that share the same boundary loop are equal since the magnetic field lines that enter the area of the loop will pierce all surfaces attached to the loop.

Furthermore, the surface does not have to be a flat surface for a loop in a plane, but can be any arbitrary surface whatsoever as long as the surface ends at the same loop.

We can understand why all open surfaces that share the same edge will have the same magnetic flux if we think in terms of magnetic field lines. Since there are no magnetic charges, the magnetic field lines that enter the area of the loop will have to emerge somewhere as illustrated for the two surface in Fig. 10.8, the "number" of magnetic field lines that enter the area of the loop will also pass through any other surface that has the same edge.

Flux Rule and Stacking Loops

Faraday used stacked coils to enhance the induced EMF in the loop. A long wire is twisted into coils and the ends of the wire are connected to complete the loop. We can now understand why coils enhance the electromagnetic effect.

Consider two circuits labelled $\mathcal{C}_1$ and $\mathcal{C}_2$ in Fig. 10.9 which are made of identical material and have the same size loops. In the circuit $\mathcal{C}_1$ there is only one loop and in the circuit $\mathcal{C}_2$ there are two loops which are connected so that the induced current would flow parallel in the two loops. Note that, to make a two-loop circuit, you would start with a wire that is a little longer than twice the length used for making one-loop circuit. The wire will have to be twisted to connect the ends so that current flows in the same direction in the two loops.

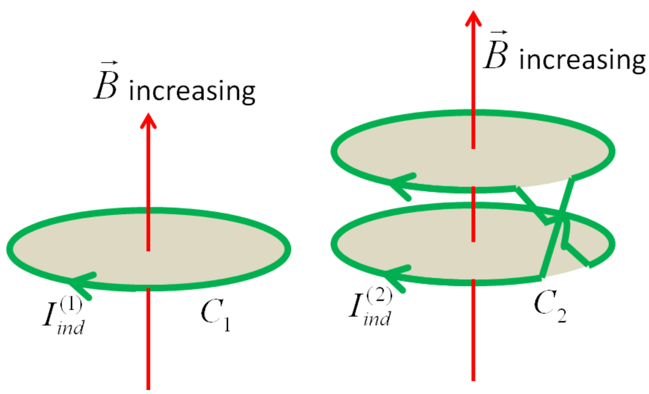

Figure 10.9: The rate of change of magnetic flux through the complicated open surface attached to two-loop circuit $\mathcal{C}_2$ will be approximately equal to twice through the surface of one-loop circuit $\mathcal{C}_1$. Therefore, induced EMF in the two-loop circuit will be twice the EMF induced in the one-loop circuit. Since, the length of $\mathcal{C}_2$ is twice that of $\mathcal{C}_1$, twice the EMF in $\mathcal{C}_2$ would imply that induced electric fields and induced currents in the two circuits are equal. The directions of the induced currents in the two figures are for the instant when the magnetic field through the loop(s) is increasing.

Now, we place the two circuits in identical magnetic field that changes with time identically. For illustrative purposes, in Fig. 10.9 the magnetic field is pointed up and *increasing*. The induced current in this situation flows clockwise when looked from above in the figure. The current in the connecting wires in the two-loop circuit is up in one wire and down in the other wire.

Now, we ask: how do various quantities in the two circuits compare? In particular, we compare the rates of the change magnetic fluxes and the induced EMFs in the two circuits. Let us use superscripts (1) and (2) for the quantities in the two circuits.

We see that a simple surface can be attached to calculate the magnetic flux through the one-loop circuit $\mathcal{C}_1$, but no simple surface can be attached to all the wires of circuit $\mathcal{C}_2$ due to the connecting wires on the side. For simplicity the extra surface that is needed to attach the open surface to all wires of the loop on the sides is not shown in Fig. 10.9, and we will assume that the magnetic flux through the extra surface is negligible. With this assumption, we see that the magnetic flux through $\mathcal{C}_2$ is twice the magnetic flux through

$\mathcal{C}_1$.

$$\left|\Phi_B^{(2)}\right| \approx 2 \times \left|\Phi_B^{(1)}\right|$$

We will now ignore the small error we make in neglecting the extra surface on the side of the loops and will write the relations as equality below. Thus, the rate of change of the magnetic flux through $\mathcal{C}_2$ is twice that through $\mathcal{C}_1$.

$$\left|\frac{d\Phi_B^{(2)}}{dt}\right| = 2 \times \left|\frac{d\Phi_B^{(1)}}{dt}\right|,$$

which implies that the induced EMF in $\mathcal{C}_2$ would be twice that in $\mathcal{C}_1$

$$\mathcal{E}^{(2)} = 2 \times \mathcal{E}^{(1)}$$