9.1 KINEMATICS OF PURE ROTATION

Angle and Axis of Rotation

The most basic aspects of a rotating body are the axis of rotation and the amount of rotation about that axis. Look at a rotating body and you will find that the particles of the body go around in circles about a line in space. The line about which the body rotates is called the axis of rotation. A rotating body changes orientation with respect to some fixed direction in space and therefore, we use an angle to measure the amount of rotation. The angle between a line from the axis to a point in the body and a fixed direction in space changes as the body rotates as shown in Fig. 9.1. Therefore, we use this angle as

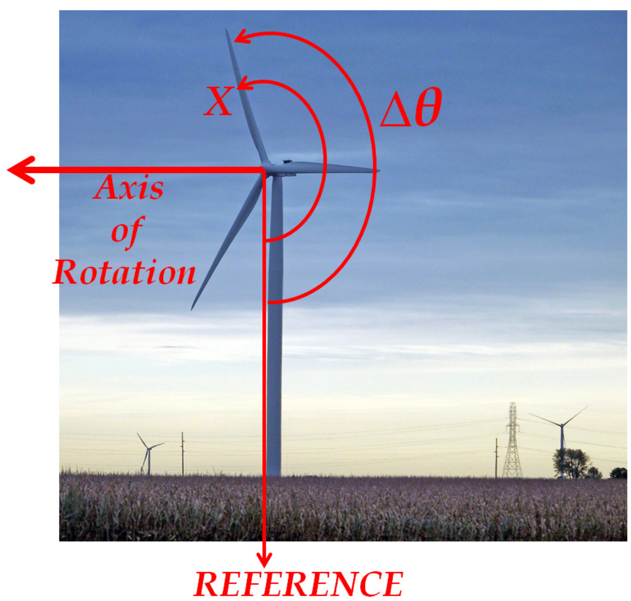

Figure 9.1: The axis and the angle of rotation. The angle of rotation is defined by the angle between a line from the axis of rotation to a point in the body and a fixed reference direction, which is shown as a vertical line in this figure. Since, every particle of a rigid body rotates at the same rate, you need only one angle to describe the rotation of the whole body.

a quantitative measure of the orientation of the entire body in space and describe the rotation of the body in terms of this angle.

Often we assign a sense of rotation as being clockwise or counterclockwise. When you observe the rotating body from one side of the axis, the body appears to rotate in a clockwise or a counterclockwise sense. Of course, if you look at the same body from the other side the sense of rotation will be opposite. A right-hand rule given in Fig. 9.2 illustrates the direction from which we look when we state

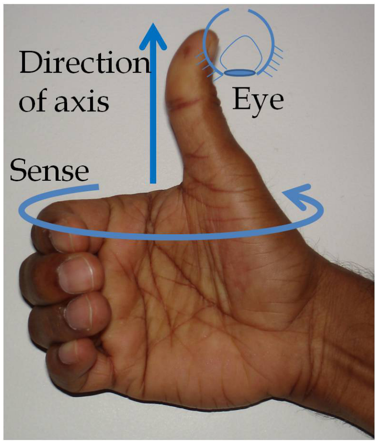

Figure 9.2: Right-hand rule. Sense of rotation and direction of the axis of rotation.

the sense of rotation. This is the same rule as the rule for a right-handed Cartesian coordinate system. Thus, if the rotation happens about the z-axis of a coordinate system in such a way that particles of the body go from positive x-axis towards positive y-axis, then the sense of rotation will be counterclockwise as observed from the side of positive z-axis.

The rotation angle can be expressed in various units as shown in Fig. 9.3. The most common ones being, the number of revolutions, degrees, arc-minutes, arc-seconds, and radians. We will be mostly using radians to express the rotation angle. For instance, if a body goes around five times then we say that it has rotated by 5 revolutions or 10π radians or $1800°$.

Angular Displacement

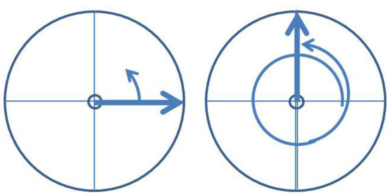

Figure 9.3: Angle of rotation $= 1.25$ rev, or $450°$, or $\frac{9}{4}\pi$ rad.

The angular displacement $\Delta\vec{\theta}$ in a time interval Δt is defined as a vector whose magnitude is the angle of rotation and whose direction is the direction of the axis of rotation as given by the right-hand rule given in Fig. 9.2. Rather than denote the direction as along the axis or opposite to the axis, we also use the language of clockwise and counterclockwise senses of rotation using the right-hand rule.

$$\boxed{\Delta\vec{\theta} = \begin{cases} \text{magnitude} = & \Delta\theta, \\ \text{direction} = & \text{towards axis using right-hand rule.} \end{cases}} \tag{9.1}$$

When you use the direction of the axis of rotation as the direction of the angular displacement vector, then you can make use of an analytic description of rotation just as for other vectors we have studied in this book. Suppose the axis of rotation is pointed in an arbitrary direction with respect to a fixed Cartesian axes (see Fig. 9.4). Using the Cartesian axes, the angular displacement vector $\Delta\vec{\theta}$ pointed along the axis of rotation can be decomposed into the x, y and z-components similar to other vectors in the three-dimensional space.

$$\Delta\vec{\theta} = \Delta\theta_x\hat{u}_x + \Delta\theta_y\hat{u}_y + \Delta\theta_z\hat{u}_z. \tag{9.2}$$

Another way to write the displacement vector is to use a unit vector $\hat{u}$ in the direction of the axis of rotation. The product of the magnitude of the angular displacement $\Delta\theta$ and the unit vector denotes the angular displacement vector.

$$\Delta\vec{\theta} = \Delta\theta\hat{u}, \tag{9.3}$$

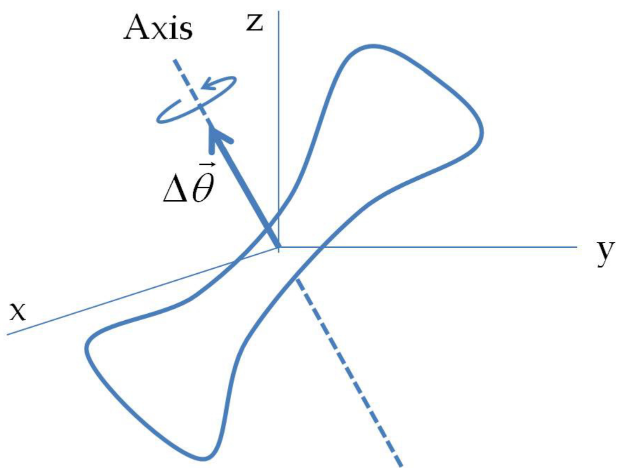

Figure 9.4: Angular displacement vector and Cartesian axes.

The two analytic forms given in Eqs. 9.2 and 9.3 are, of course, equivalent, although, the later gives the magnitude and direction of the vector directly.

The magnitude and direction of the angular displacement vector can be determined from its components in Eq. 9.2 as we have learned to do for other vectors. Thus, the magnitude of the angular displacement vector, which is equal to the angle of rotation about the axis, is related to the components as follows.

$$\text{Magnitude:}\ \ |\Delta\theta| = \sqrt{\Delta\theta_x^2 + \Delta\theta_y^2 + \Delta\theta_z^2}. \tag{9.4}$$

The direction of $\Delta\vec{\theta}$ is obtained from the angle the vector makes with respect to the Cartesian axes. If only one component is non-zero, we say that the situation is one-dimensional, and the sign of the component tells us the direction with respect to the axis for which the component is non-zero. If only two components are non-zero, then the situation is called two-dimensional, and the angle with respect to any one of the two axes can be worked out to determine the direction of the vector. If all three components are non-zero, then we need two angles in space to tell the direction - the two angles of a spherical coordinate system is often used for this purpose, although other choices are also possible.

For fixed-axis rotation, we can orient the fixed Cartesian axes so that one of the axes, usually the z-axis points in the direction of the axis of rotation. With the axis of rotation pointed towards the positive z-axis, the displacement vector has only the z-component non-zero. This simplifies the mathematical treatment of rotation considerably.

$$\text{Fixed-axis along } z\text{-axis:}\ \ \Delta\vec{\theta} = \Delta\theta_z \hat{u}_z. \tag{9.5}$$

Thus, the absolute value of $\Delta\theta_z$ is the angle rotated and the sign of $\Delta\theta_z$ together with the unit vector $\hat{u}_z$ gives us the direction of the rotation. If the rotation is counterclockwise as seen from the positive z-axis, then $\Delta\theta_z > 0$ and $\Delta\vec{\theta}$ will point towards the positive z-axis. Similarly, the clockwise rotation corresponds to the negative z-component. Since, a fixed axis rotation is only one-component problem, we often draw analogy to the one-component problems of translational motion. Also, many authors drop the vector character of the angular displacement altogether and treat the problems of fixed-axis rotation in a non-vector language. We will not use that language and continue to use the vector language since angular displacement, whether one-component type or three-components type are at the core vector quantities.

Angular Speed

In a rotation of a rigid body about a fixed axis, every particle covers the same amount of angle in the same amount of time. Therefore, the rate of angle covered by each particle in their own circle of motion is a characteristic of the entire rigid body. This rate is called the angular speed of rotation.

As explained above, we measure the angle of rotation in a plane perpendicular to the axis of rotation. Let $\Delta\theta$ be the angle rotated in a time duration Δt, then the average angular speed ω_{ave} is given by

$$\boxed{\text{Average angular speed:} \quad \omega_{\text{ave}} = \frac{\Delta\theta}{\Delta t}.} \tag{9.6}$$

The commonly used units for angular speed are revolutions per minute (rpm), degrees per second (deg/sec), and radian per second (rad/sec). We will mostly used rpm and rad/sec.

In the limit of infinitesimal durations, we obtain the instantaneous speed ω.

$$\boxed{\text{Instantaneous angular speed:} \quad \omega = \frac{d\theta}{dt}.} \tag{9.7}$$

Example 9.1.1. Steady rotation of a computer hard disk.

A computer disk is rotating at a constant rate of 7200 rpm, where rpm stands for revolutions per minute. What is the angle in radians rotated in 2 msec?

Solution. Since the rate of rotation is constant, the angle rotated will simply be equal to the product of the speed and time. We do not need to place the axis of rotation along any Cartesian axis. We can work with the magnitudes alone in this problem. In this problem we need to convert units of the speed of rotation from revolutions per minute to radians per second.

$$\omega = 7200 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60 \text{ sec}} = 754 \text{ rad/sec}.$$

Now, we multiply the angular speed by the time (converted in sec) to obtain the angle rotated.

$$\Delta\theta = \omega t = 754 \frac{\text{rad}}{\text{sec}} \times 2 \times 10^{-3} \text{ sec} = 1.51 \text{ rad}.$$

Angular Velocity

The rate of change of angular displacement is called angular velocity. Thus, average angular velocity $\vec{\omega}_{ave}$ during an interval Δt in which angular displacement is $\Delta\vec{\theta}$ is given by the ratio of angular displacement to the interval.

$$\boxed{\vec{\omega}_{ave} = \frac{\Delta\vec{\theta}}{\Delta t}.} \tag{9.8}$$

The direction of the angular velocity is the same as the direction of the angular displacement. Thus, if the displacement vector is 200 rad in 20 sec in the counterclockwise sense, then the average velocity during this interval is 10 rad/sec in the counterclockwise sense.

In the limit of infinitesimal time interval, we obtain the instantaneous angular velocity $\vec{\omega}$ as the time derivative of the angular position vector, which is the angular displacement with respect to a fixed direction in a plane perpendicular to the axis of rotation.

$$\boxed{\vec{\omega} = \frac{d\vec{\theta}}{dt}.} \tag{9.9}$$

The magnitude of instantaneous angular velocity is equal to the angular speed of rotation and the direction is the direction of the axis of rotation as determined from using the right-had rule stated above.

Instantaneous angular velocity $\vec{\omega}$:

$$\boxed{\begin{array}{l}\text{Magnitude} = \text{Angular speed.} \\ \text{Direction} = \text{Towards the axis using the right-hand rule.}\end{array}} \tag{9.10}$$

Analytically, the angular velocity vector is treated in the same way as we have described for the angular displacement vector above. Suppose a unit vector $\hat{u}$ points towards the axis of rotation and the angular speed is ω, then analytically we write the angular velocity vector as

$$\vec{\omega} = \omega\hat{u}, \tag{9.11}$$

We can also express the angular velocity vector which is pointed in the direction of the axis of rotation in terms of its components along Cartesian axes.

$$\vec{\omega} = \omega_x\hat{u}_x + \omega_y\hat{u}_y + \omega_z\hat{u}_z. \tag{9.12}$$

The angular speed, ω, is as usual the magnitude of angular velocity vector,

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}. \tag{9.13}$$

For fixed-axis rotation, if we choose to orient the z-axis along the axis of rotation, then only the z-component will be non-zero.

$$\text{Fixed axis along } z\text{-axis:} \quad \vec{\omega} = \frac{d\theta_z}{dt}\hat{u}_z \equiv \omega_z\hat{u}_z, \tag{9.14}$$

where the z-component of the angular velocity vector is

$$\boxed{\omega_z = \frac{d\theta_z}{dt}.} \tag{9.15}$$

Equation 9.15 can be inverted giving the z-component of the angular displacement in terms of integration of the z-component of the angular velocity.

$$\theta_z(t) - \theta_{0z} = \int_0^t \omega_z(t)dt, \tag{9.16}$$

where $\theta_z(t) - \theta_{0z}$ is the z-component of the angular displacement.

Example 9.1.2. A merry-go-round.

A merry-go-round is rotating about an axis that is pointed vertically up. The line from the center of the merry-go-round to a rider makes an angle ϕ with respect to a line pointing East from the center of the merry-go-round. The angle ϕ changes with time according to $\phi(t) = bt^2$, where b is a positive constant. What is the angular velocity at $t = T$?

Solution. Using Cartesian coordinates whose z-axis is pointed up we see that the angle $\phi(t)$ gives the z-component of angular displacement of the rotation of the Merry-go-round. Therefore, the z-component of angular velocity is obtained by taking the derivative of $\phi(t)$ with respect to time. For the fixed-axis rotation described in the problem statement, the other components of the angular velocity are zero.

$$\omega_z = \frac{d\theta_z}{dt} = \frac{d\phi}{dt} = 2bt,$$

which is evaluated for $t = T$ to give $\omega_z = 2bT$. Therefore, the angular velocity has the magnitude $2bT$ and the direction is pointed up towards the z-axis as shown in the figure.

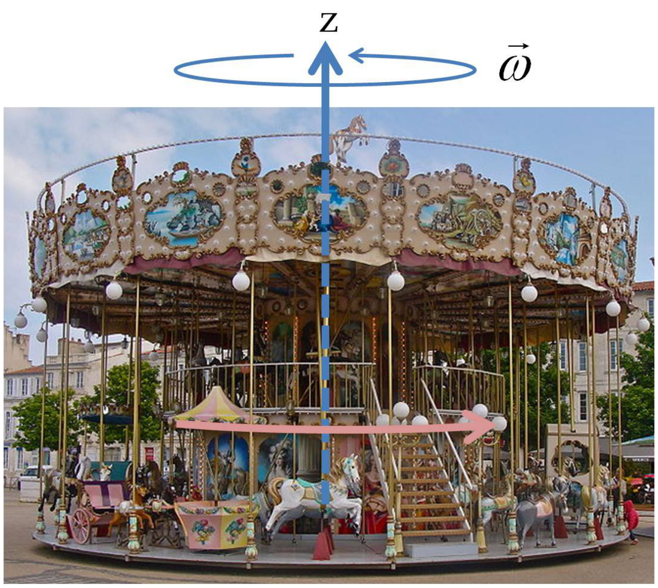

Figure 9.5: Merry-go-round. Credits: Jebulon, commons.wikimedia.org

Angular Acceleration

The rate of change of angular velocity vector defines the angular acceleration of the body. Thus, if the angular velocity changes by $\Delta\vec{\omega}$ in a time interval Δt, the average angular acceleration $\vec{\alpha}_{ave}$ will be

$$\boxed{\vec{\alpha}_{ave} = \frac{\Delta\vec{\omega}}{\Delta t}.} \tag{9.17}$$

When time duration is infinitesimal we obtain the instantaneous angular acceleration $\vec{\alpha}$.

$$\boxed{\text{Angular acceleration vector,} \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt}.} \tag{9.18}$$

Angular acceleration $\vec{\alpha}$ may or may not be in the same direction as the angular velocity $\vec{\omega}$. Suppose the axis of rotation is fixed and the body is rotating in a counterclockwise sense with increasing speed, then the angular acceleration would also be in the counterclockwise sense as shown in Fig. 9.6. However, if the body is rotating counterclockwise with decreasing speed, then the angular acceleration will be in the clockwise sense.

Analytically, we represent angular acceleration with the components along Cartesian axes. In the case of fixed axis rotation about the z-axis, only the z-component of the angular acceleration will be non-zero.

$$\vec{\alpha} = \alpha_z \hat{u}_z, \tag{9.19}$$

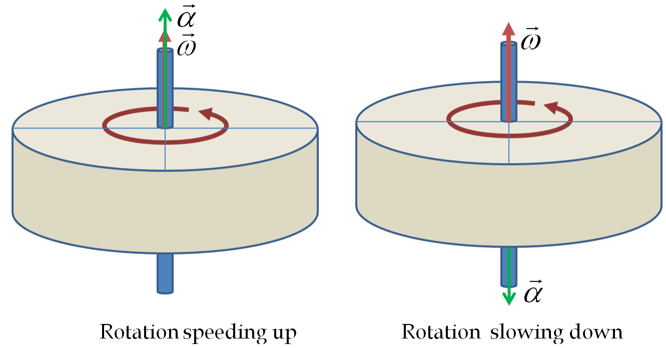

Figure 9.6: The angular acceleration and angular velocity may be in different directions. When the angular acceleration is in the same direction as the angular velocity, the rotational speed increases. When the angular acceleration is in the opposite direction to that of the angular velocity, the rotational speed decreases. In the later case, body will pick the rotational speed after the body has stopped rotating, but the rotation velocity thereafter will be in the same direction as the angular acceleration.

where z-component of angular acceleration is the rate of change of the z-component of the angular velocity.

$$\alpha_z = \frac{d\omega_z}{dt}. \tag{9.20}$$

The angular acceleration can also be written in terms of the angular displacement by replacing the z-component of the angular velocity by the time derivative of the z-component of the angular position vector, which is the angle rotated from the positive x-axis.

$$\vec{\alpha} = \frac{d^2\theta_z}{dt^2}\hat{u}_z. \tag{9.21}$$

Equation 9.20 can be inverted giving the change in the z-component of the angular velocity in terms of integration of the z-component of the angular acceleration.

$$\boxed{\omega_z(t) - \omega_{0z} = \int_0^t \alpha_z(t)dt,} \tag{9.22}$$

where $\omega_z(t)$ and ω_{0z} are z-component of angular velocity at time $t = t$ and $t = 0$ respectively.

Constant Angular Acceleration

Just as constant acceleration was an important case of study for the translational motion, constant angular acceleration case also plays an important role in the study of the rotational motion. Once again, let

z-axis to be the axis of rotation, and to emphasize constant angular acceleration, we place a bar over the symbol for the z-component of the angular acceleration.

$$\boxed{\vec{\alpha}(t) = \bar{\alpha}_z \hat{u}_z} \tag{9.23}$$

For constant angular acceleration, we expect the angular velocity to change linearly in time, which is validated by replacing $\alpha_z(t)$ by $\bar{\alpha}_z$ in Eq. 9.22 and carrying out the integration.

$$\boxed{\omega_z(t) = \omega_{0z} + \bar{\alpha}_z t} \tag{9.24}$$

Now, using this expression for ω_z in Eq. 9.16, and carrying out the integration gives the following angular displacement.

$$\boxed{\theta_z(t) = \theta_{0z} + \omega_{0z} t + \frac{1}{2}\bar{\alpha}_z t^2} \tag{9.25}$$

Equations 9.24 and 9.25 are the two main equations for the analysis of the constant angular acceleration motion. Just as in one-dimensional constant acceleration translational motion, we can eliminate time t from these two equations to obtain the following useful relation.

$$\boxed{\omega_z(t)^2 = \omega_{0z}^2 + 2\bar{\alpha}_z \left[\theta_z(t) - \theta_{0z}\right]} \tag{9.26}$$

This equation does not have any information that is not already contained in Eqs. 9.24 and 9.25, but sometimes provides a short-cut way of solving some problems.

Table 9.1: Constant angular acceleration, $\alpha_z = \bar{\alpha}_z$

$\omega_z = \omega_{0z} + \bar{\alpha}_z t$
$\theta_z = \theta_{0z} + \omega_{0z} t + \frac{1}{2}\bar{\alpha}_z t^2$
${\omega_z}^2 = \omega_{0z}^2 + 2\bar{\alpha}_z\left(\theta_z - \theta_{0z}\right)$

Example 9.1.3. Constant acceleration rotation A wheel is mounted on an axle and placed in a support so that the wheel can rotate about an axis through the axle. The wheel is then rotated from rest with a uniform angular acceleration of 3 rad/sec^2 pointed so that the wheel rotates counterclockwise. (a) Find the angular velocity at t = 10 sec. (b) Find the total rotation in 10 sec.

Solution. Since the angular acceleration is constant, we can use the constant angular acceleration formulas. From the right-hand rule, we know that, if z-axis is the axis of rotation, then the counterclockwise corresponds to the direction of the vector towards positive z-axis. We will work with the components of the vectors in this problem.

(a) The z-component of the constant angular acceleration is $\alpha_z =$ 3 rad/sec^2. The z-component of the angular velocity changes linearly for a constant angular acceleration. Therefore, we find the z-component of the angular velocity to be

$$\omega_z = \omega_{0z} + \bar{\alpha}_z t = 0 + 3\frac{\text{rad}}{\text{sec}^2} \times 10\ \text{sec} = 30\frac{\text{rad}}{\text{sec}}.$$

Since, other components of the angular velocity are zero, the magnitude of the angular velocity would be 30 rad/sec and the direction would be towards positive zR axis, or counterclockwise sense.

(b) The z-component of the net rotation angle in the given time is also readily obtained from the given formulas for the constant angular acceleration case.

$$\Delta\theta_z = \omega_{0z} t + \frac{1}{2}\bar{\alpha}_z t^2 = 0 + \frac{1}{2} 3 \frac{\text{rad}}{\text{sec}^2} \times (10\text{ sec})^2 = 150\text{ rad}.$$

This gives the total rotation angle to be 150 rad, which is approximately 24 revolutions.

Example 9.1.4. Constant acceleration rotation. A bicycle wheel rotates by an angle 2000 radians in 5 sec before coming to rest with a fixed axis of rotation. Assuming constant angular acceleration, what is the angular acceleration?

Solution. Just as the last example, we again work with the components of vectors along the axis of rotation. By using the right-hand rule on the initial motion, let us define the positive direction of the z-axis to be along the direction of the initial angular velocity. From the given information, we can set up the following two equations since the final angular velocity is zero.

$$\begin{aligned} 0 &= \omega_{0z} + \bar{\alpha}_z t \\ \Delta\theta_z &= \omega_{0z} t + \frac{1}{2}\bar{\alpha}_z t^2 \end{aligned}$$

In these equations, we have only two unknowns, ω_{0z} and $\bar{\alpha}_z$. We can solve for ω_{0z} in the first equation and then substitute in the second equation. The resulting equation can be solved for $\bar{\alpha}_z$. Once we have solved for $\bar{\alpha}_z$ we can put the numerical values to find the value of the z-component of the angular acceleration. At that point we will state the magnitude and direction of this vector. From the first equation, $\omega_{0z} = -\bar{\alpha}_z t$, therefore,

$$\Delta\theta_z = -\bar{\alpha}_z t^2 + \frac{1}{2}\bar{\alpha}_z t^2 = -\frac{1}{2}\bar{\alpha}_z t^2.$$

This gives

$$\bar{\alpha}_z = -2\frac{\Delta\theta_z}{t^2} = -2 \times \frac{2000\text{ rad}}{(5\text{ sec})^2} = -160\text{ rad/sec}^2.$$

Since the other components of the angular acceleration are zero, the negative sign of the z-component of the angular acceleration means

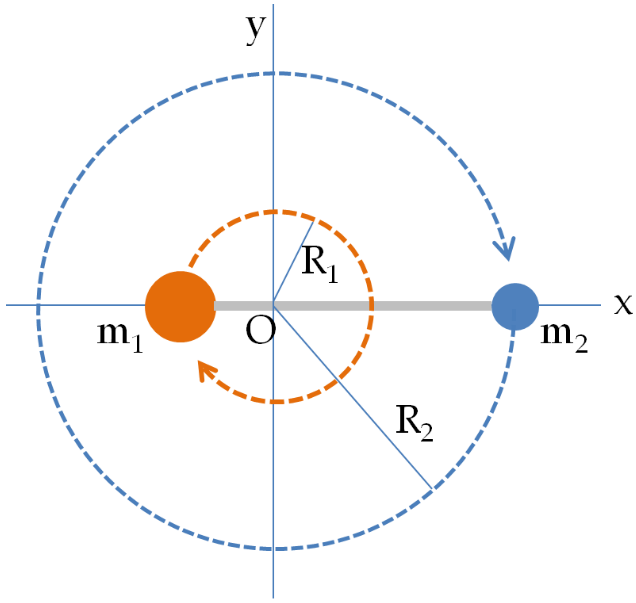

Figure 9.7: Two masses of a dumb-bell move in different circles when the dumb-bell rotates about an off-center point. The spatial speeds of masses are $v_1 = R_1\omega$ and $v_2 = R_2\omega$ if angular speed ω in rad/sec.

that the angular acceleration is pointed towards the negative z-axis. Recall that the initial velocity was used to define the direction of the positive z-axis. Therefore, angular acceleration has the magnitude 160 rad/sec^2 and has the direction opposite to the direction of the initial angular velocity.

Spatial Velocity Versus Angular Velocity

You know that particles of a rotating rigid body move in circles about the axis of rotation. This is shown schematically in Fig. 9.7 for two masses of a dumb-bell that rotate about an axis through some point O in the rod that joins the two masses.

The mass m_1 moves in a circle of radius R_1 and the mass m_2 in a circle of radius R_2. In one revolution, the two masses move different distances, one moves a distance equal to $2\pi R_1$ and the other $2\pi R_2$. Therefore, although, the two masses possess the same angular speed, they have different "real" or "spatial" speeds.

Let the common angular speed of the two masses in Fig. 9.7 be ω in radians/sec. Then, the time for one revolution T in sec will be

$$T = \frac{2\pi}{\omega}.$$

Now, we can deduce the spatial speeds v_1 and v_2 of the masses in

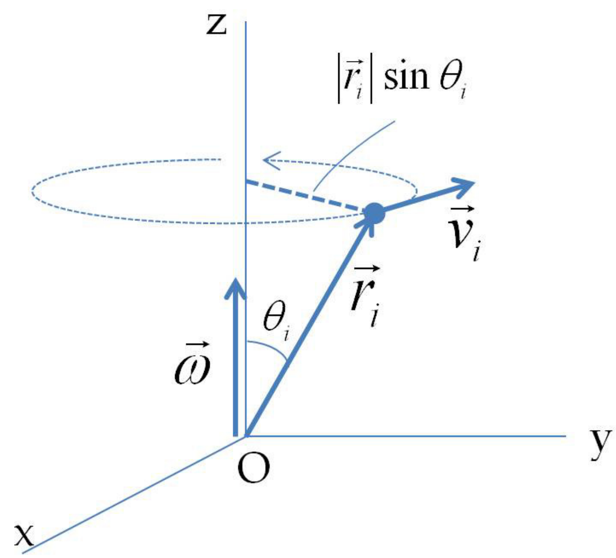

Figure 9.8: The spatial velocity $\vec{v}_i$ of a particle located at position $\vec{r}_i$ is equal to the vector product of the angular velocity $\vec{\omega}$ and the position vector, $\vec{v}_i = \vec{\omega} \times \vec{r}_i$.

terms of the radii R_1 and R_2 and the common angular speed ω.

$$v_1 = \frac{2\pi R_1}{T} = R_1\omega$$
$$v_2 = \frac{2\pi R_2}{T} = R_2\omega$$

The relation between the position and spatial velocity of a point of a rotating body are more generally related by a cross product as we show now. Consider a point particle at position $\vec{r}_i$ which is located at an angle θ_i from the axis of rotation, which will be taken to be the z-axis as shown in Fig. 9.8.

The point particle will move in a circle of radius $|\vec{r}_i| \sin\theta_i$ as shown. The physical distance covered by the particle will be equal to the product of the angle of rotation, expressed in radians, and the radius of this circle, as given from the arc-length and angle formula for a circle.

$$\boxed{\text{Arc length} = \text{Radius } \times \text{Angle in radians.}} \tag{9.27}$$

Therefore, the magnitude of the spatial velocity will be equal to the product of the magnitude of the angular velocity and the radius of the circle of rotation, and the direction of the spatial velocity will be in the direction of the tangent to the circle at that point in time.

Instantaneous spatial velocity $\vec{v}_i$:

$$\boxed{\begin{array}{l} \text{Magnitude} = \omega|\vec{r}_i| \sin\theta_i \\ \text{Direction} = \text{Tangent to the circle.} \end{array}} \tag{9.28}$$

We can write the definition of the spatial velocity of the particle at $\vec{r}_i$ more compactly using the vector product notation.

$$\boxed{\vec{v}_i = \vec{\omega} \times \vec{r}_i.} \tag{9.29}$$