8.6 ELASTIC COLLISIONS

Collisions are typically categorized as elastic or inelastic depending upon whether or not kinetic energy is conserved in the collision. If total kinetic energy is conserved, then the collision is said to be elastic. This can happen if the colliding bodies do not stick together, change shape, or break apart, or lose energy in heat and vibration. For instance, the collision of billiard balls and metal balls are elastic to a large extent. The motion of planets around the sun can be examined from this perspective also.

In an elastic collision, you have two conservation conditions, the conservation of momentum and the conservation of kinetic energy, which must hold independently of each other.

Elastic Collisions:

$$\boxed{(\vec{p}_{\text{total}})_{\text{before}} = (\vec{p}_{\text{total}})_{\text{after}}\,.} \tag{8.79}$$

$$\boxed{(K_{\text{total}})_{\text{before}} = (K_{\text{total}})_{\text{after}}\,.} \tag{8.80}$$

Note that, while the conservation of momentum is a vector equation, the conservation of kinetic energy is a scalar equation.

The elasticity of collision between two objects in a one-dimensional collision is also expressed by a quantity called the restitution. In a one-dimensional collision, the restitution is defined as the ratio of relative speed of separation after collision to the relative speed of approach before collision.

$$\boxed{\epsilon = \frac{(v_{\text{rel}})_{\text{after}}}{(v_{\text{rel}})_{\text{before}}}} \tag{8.81}$$

For a perfectly inelastic collision the two particles will stick to each other upon collision, then the restitution will equal zero. For a perfectly elastic collision the relative speed of separation after the collision is equal to the relative speed of approach before the collision, and therefore restitution is equal to 1.

Perfectly inelastic, $\epsilon = 0$; Perfectly elastic, $\epsilon = 1$.

Example 8.6.1. One-dimensional elastic collision. Consider a ball of mass m_1 traveling at speed v_1 towards another ball of mass m_2 at rest. As a result of the collision ball 1 is found to bounce back with speed v_1' and ball 2 moves with speed v_2' in ball 1's original direction. If the collision is elastic, find v_1' and v_2' in terms of m_1, m_2 and v_1.

Solution. We have two conditions here: (1) Conservation of momentum and (2) conservation of kinetic energy due to elasticity of the collision. To proceed analytically, we choose axis system so that the original direction of ball 1's motion be along the positive x-axis.

Then, x-component of momentum will be conserved.

$$m_1 v_1 + 0 = -m_1 v_1' + m_2 v_2'$$
$$\frac{1}{2} m_1 v_1^2 + 0 = \frac{1}{2} m_1 {v_1'}^2 + \frac{1}{2} m_2 {v_2'}^2$$

Solving these two equations for v_1' and v_2' we find the following.

$$v_1' = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_1$$
$$v_2' = \left(\frac{2m_1}{m_1 + m_2} \right) v_1$$

Note that when $m_1 = m_2$, then after the collision ball 1 comes to rest and ball 2 moves with the speed of ball 1. On the other hand, if m_2 is very large, as would be the case when you bounce a ball off the floor, then m_1 bounces off m_2 with the same speed as before, unless it is deformed during the collision and the collision is not perfectly elastic.

Example 8.6.2. Two dimensional elastic collision. A particle A of mass m_1 is subject to a repulsive force from another particle B of mass m_2. Initially particle A is traveling towards particle B which is at rest. After interacting with particle B, particle A is found to move with speed v_1' at an angle θ with its original direction, and B with speed v_2' at an angle ϕ from the original direction of A. Angle θ is called the angle of scattering of particle A while angle ϕ is called the angle of recoil of the target particle. Assuming the motion to be in the xy-plane and the collision elastic, find speeds v_1', v_2' and the scattering angle θ in terms of m_1, m_2, v_1, and ϕ.

Solution. We draw a figure to make the situation clearer and choose the Cartesian axes for calculations. As shown in Fig. 8.22, particle 1 is approaching particle 2 on the x-axis. After collision, the particle 1 is scattered in the direction specified by counterclockwise angle θ from the positive x-axis and the particle 2 recoils in the direction given by the clockwise angle ϕ from the positive x-axis.

Since, the motion of the particles is confined to a plane, we will get two conditions, one for the x-components and the other for the y-components, from the conservation of momentum across the collision process. In addition, since the collision is assumed to be elastic, we will get one more condition from the conservation of kinetic energy.

x-components of momentum: $m_1 v_1 + 0 = m_1 v_1' \cos\theta + m_2 v_2' \cos\phi$

y-components of momentum: $0 + 0 = m_1 v_1' \sin\theta - m_2 v_2' \cos\phi$

Elastic collision: $\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 {v_1'}^2 + \frac{1}{2} m_2 {v_2'}^2$

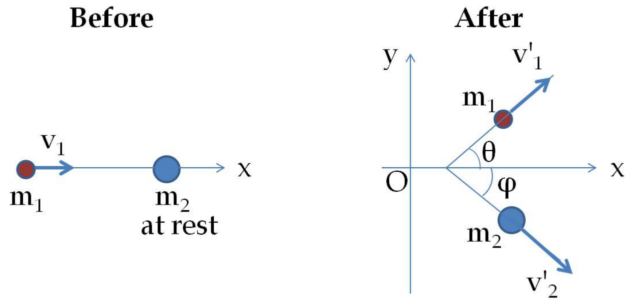

Figure 8.22: Particle 1 in motion collides with particle 2 at rest. After the collision particle 1 moves in the direction at angle θ while particle 2 recoils in the direction ϕ.

One can solve these three equations for at most three unknowns. Suppose m_1, m_2, v_1 and one of the after-collision variables, say the recoil angle ϕ are given, then one can solve for v_1', v_2', and ϕ in terms of the given quantities. Let us write c for m_1/m_2 to simplify the answer.

$$v_1' = \left(\frac{\sqrt{1 - 2c\cos(2\phi) + c^2}}{1+c}\right) v_1$$

$$v_2' = \left(\frac{2\cos\phi}{1+c}\right) v_1$$

$$\theta = \arctan\left(\frac{\sin(2\phi)}{c - \cos(2\phi)}\right)$$

The calculation is tedious and not very illuminating. To facilitate the calculation, it helps to introduce the ratios of speeds and masses, namely $a = v_1'/v_1$, $b = v_2'/v_1$ and $c = m1/m2$, and make use of $\sin^2\theta + \cos^2\theta = 1$ to eliminate θ and solve for a and b. A student is encouraged to carry out the necessary calculations as an exercise.