3.6 EXERCISES

Simple Harmonic Oscillator

Ex 3.6.1. A harmonic oscillator of mass 200 grams is attached to a spring of spring constant 100 N/m. Find the angular frequency, frequency and time period.

Ans: ang. freq. = 22.4 rad/sec, 3.56 Hz, 0.28 s.

Ex 3.6.2. Consider a simple harmonic oscillator of mass m, amplitude A and frequency f. (a) What fraction of the energy of the oscillator is in the potential energy when the oscillator's displacement is half the amplitude? (b) What fraction of the energy of the oscillator is in the kinetic energy when the oscillator's displacement is half the amplitude? (c) Where in the cycle of an oscillation does the oscillator have the lowest speed? Why? (d) Where in the cycle of an oscillation does the oscillator have the largest speed? Why? Ans: (a) 25%, (b) 75%, (c) Turning points: $x = \pm A$. (d) At equilibrium point, $x = 0$.

Ex 3.6.3. Two oscillators of masses 200 g and 400 g oscillate with the same frequency. They move such that their positions from their corresponding equilibrium points are given by the following two functions (t in seconds, x in cm) x_1 and x_2 respectively.

$$x_1 = 2\cos(t)$$
$$x_2 = 2\sin(t)$$

(a) What are the angular frequencies, amplitudes and phase constants of the two oscillators? (b) What were their positions and velocities at the initial time $t = 0$? (c) What are the kinetic and potential energies of the two oscillators at t=0? (d) Plot the positions of the two oscillators versus time on the same graph, and interpret which oscillator is ahead, and by how much.

Ans: (a) Ang. freq. = 1 rad/s, ampl. = 2 cm, phase const. = 0 and $-\pi/2$ radians; (b) $x_1(0) = 2$ cm; $x_2(0) = 0$; $v_{1x}(0) = 0$, $v_{2x}(0) = 2$ cm/s. (c) $K_1(0) = 0$, $K_2(0) = 800$ g cm^2/s^2 = 8.0 $\times 10^{-5}$ J, $U_1(0) = 400$ g cm^2/s^2 = 4.0×10^{-5} J, $U_2(0) = 0$.

Ex 3.6.4. Two oscillators of masses 100 g and 400 g oscillate with the same frequency. They move such that their positions from their corresponding equilibrium points are given by the following two functions (t in seconds, x in cm).

$$x_1 = 2\cos(t)$$
$$x_2 = 2\cos(t - \pi)$$

(a) What are the angular frequencies, amplitudes and phase constants of the two oscillators? (b) What were their positions and velocities at the initial time $t = 0$? (c) What are the kinetic and potential energies of the two oscillators at $t = 0$? (d) Plot the positions of the two oscillators with time and interpret which oscillator is ahead, and by how much.

Ans: (a) Ang. freq. = 1 rad/s, ampl. = 2 cm, phase const. = 0 and $-\pi$ radians; (b) $x_1(0) = 2$ cm; $x_2(0) = -2$ cm; $v_{1x}(0) = 0$, $v_{2x}(0) = 0$. (d) $K_1(0) = 0$, $K_2(0) = 0$, $U_1(0) = 4.0 \times 10^{-5}$ J $= U_2(0)$.

Ex 3.6.5. A block of mass 5 kg is hung from a spring of the original length 50 cm and the spring constant 100 N/cm. As a result of the weight of the block, the spring stretches to a different length at equilibrium. The spring is attached to a ceiling at a height 2 m from the ground. The block is then hit with a hammer giving it an instantaneous velocity of 4 cm/s downward. (a) When the block is hanging in equilibrium, where is the equilibrium position of the block with respect to the floor? (b) Pick the zero reference points for the potential energies due to the spring force and that due to the gravity, and determine the potential energy of the block at the time it was hit by the hammer? (c) What is the kinetic energy of the block immediately after being hit by the hammer? (d) How much work did the hammer do? (e) What is the frequency of the oscillations of the block?

Ans: (a) 149.51 cm from the floor. (b) Let the point 149.51 cm from the floor be the reference zero for both potential energy due to the spring force and due to the gravity. Then, the net potential energy the block at that point will be zero. (c) The kinetic energy 0.004 J. (d) The work by hammer = 0.004 J. (e) The frequency = 7.12 Hz.

Ex 3.6.6. An oscillator of frequency 20 cycles per second starts 3 cm from the equilibrium with an initial velocity of 10 cm/s pointed towards the equilibrium point. Draw a figure, showing the origin and the x-axis along the motion of the oscillator and find $x(t)$ for the oscillator.

Ans: $x(t) = (3\text{ cm})\ \cos(40\pi t) - \left(\dfrac{1\text{ cm}}{4\pi}\right)\ \sin(40\pi t).$

Ex 3.6.7. An oscillator of frequency 20 cycles per second starts 3 cm from the equilibrium with an initial velocity of 10 cm/s pointed away from the equilibrium point. Find the displacement as a function of time.

Ans: $x(t) = (3\text{ cm})\ \cos(40\pi t) + \left(\dfrac{1\text{ cm}}{4\pi}\right)\ \sin(40\pi t)$

Ex 3.6.8. The solution of a Simple Harmonic Motion can be written in three ways:

$$\begin{aligned} x(t) &= C_1\ \cos(\omega t) + C_2\ \sin(\omega t) \\ &= A\cos(\omega t + \phi_c) \\ &= B\sin(\omega t + \phi_s) \end{aligned}$$

Find the relations among C_1, C_2, A, ϕ_c, and ϕ_s.

Ans: $C_1 = A\cos(\phi_c) = B\sin(\phi_s)$; $C_2 = -A\sin(\phi_c) = B\cos(\phi_s)$; $A^2 = C_1^2 + C_2^2 = B^2$; $\tan(\phi_c) = -\frac{-C_2}{C_1} = -\tan(\phi_s)$.

Ex 3.6.9. Find C_1 and C_2 in terms of x_0, v_{0x} and ω when $x(t) = C_1\ \cos(\omega t) + C_2\ \sin(\omega t)$, where $x_0 = x(0)$ and $v_{0x} = v_x(0)$.

Ans: $C_1 = x_0$ and $C_2 = v_{0x}/\omega$.

The Damped Harmonic Oscillator

Ex 3.6.10. Decide what type of damping is in the following oscillators: (a) m = 0.2 kg, k = 10 N/m, b = 6 kg/s (b) m = 0.2 kg, k = 20 N/m, b = 8 kg/s (c) m = 0.8 kg, k = 32 N/m, b = 8 kg/s (d) m = 0.8 kg, k = 80 N/m, b = 16 kg/s

Ans: (a) Over; (b) Over; (c) Under; (d) Critically.

Ex 3.6.11. If $x(0) = 1$ m, and $v_x(0) = 0$ for the oscillators in Exercise 3.6.10, find $x(t)$ for each oscillator.

Ans: (a) $x(t) = -(0.068\text{ m})\ e^{-28t} + (1.07\text{ m})\ e^{-1.8t}$. (c) $x(t) = e^{-5t}\left[(1\text{ m})\ \cos(6.3t) + (0.79\text{ m})\ \sin(6.3t)\right]$.

Ex 3.6.12. Find the Q factor for the following underdamped oscillators. Here x is in cm and t in sec.

(a) $x(t) = 2\exp(-0.1t)\cos(2\pi t)$
(b) $x(t) = 2\exp(-0.02t)\cos(t)$
(c) $x(t) = 2\exp(-0.25t)\cos(200\pi t)$
(d) $x(t) = 2\exp(-0.015t)\cos\left(2000\pi t + \frac{\pi}{2}\right)$

Ans: (a) 31; (b) 25; (c) 1300; (d) 210,000.

Ex 3.6.13. The position of a 250-gram lightly damped oscillator is given by the following function of time, $x(t) = 2\exp(-0.01t)\cos(2\pi t)$, where t is in seconds and x in meters.

(a) Plot x vs t.
(b) How long does it take for the envelop of oscillations to drop by $\frac{1}{e}$?

(c) How long does it take the envelop to drop by a factor $\frac{1}{e^2}$?
(d) What is the Q factor of the oscillator?
(e) What is the rate at which the energy of the oscillator is dissipated at $t = 0$?
(f) What is the frequency of oscillations?
(g) How many oscillations will the oscillator make before 90% of the energy is dissipated?

Ans: (b) 100 sec, (c) 200 sec, (d) 310, (e) 3.9×10^{-5} J/s, (f) 1 Hz, (g) 230.

The Forced Harmonic Oscillator

Ex 3.6.14. An underdamped oscillator with $\beta = \frac{\omega_0}{10} = 2$ rad/sec is driven by a harmonic driving force $\vec{F}_d$. The oscillator oscillates along the x-axis. The driving force has the following x-component: $F_x = (5N)\cos(\omega_d t)$. (a) Find the steady state amplitude of the oscillator when the driving frequency is equal to the resonance frequency. (b) By what factor the amplitude drops compared to the amplitude at the resonance, if the driving frequency is equal to $0.1\omega_R$. (c) At what frequencies would the amplitude be half of the amplitude at the resonance?

Ans: (a) 0.31 m. (b) 0.20. (c) $\omega = 5.76$ rad/s and $\omega = 27.4$ rad/s.

Ex 3.6.15. An underdamped oscillator with $\beta = \frac{\omega_0}{10} = 2$ rad/sec is driven by a harmonic driving force $\vec{F}_d$. The oscillator oscillates in x-axis. The driving force has the following x-component: $F_x = (5N)\cos(\omega_d t)$. (a) Find the phase difference between the driving force and the displacement, if the driving frequency is equal to (i) $0.1\omega_R$, (ii) $0.9\omega_R$, or (iii) $1.1\omega_R$. (c) At what frequency would the phase difference be $-\frac{\pi}{4}$ radian?

Ans: (a) (i) $\delta = 1.15°$, (ii) $\delta = 40.8°$, and (iii) $\delta = -49.5°$. (b) $\omega = 18.1$ rad/s.

Ex 3.6.16. A block of mass 300 grams is attached to a spring of spring constant 100 N/m. The block moves in an environment that damps its motion. The damping force is proportional to the speed of motion of the mass with the constant of proportionality given as 0.1 N.s/m. A sinusoidal driving force acts on the block with amplitude 10 N and a variable frequency. Find the following.

(a) The natural frequency of the oscillator
(b) The type of damping of the oscillator - under, over or critical
(c) The Q factor of the oscillator

(d) The resonance frequency of the oscillator
(e) The amplitude and phase of the oscillator in the steady state when the frequency of the driving force are: i. 0.25 ω_R, ii. 0.5 ω_R, iii. 0.75 ω_R, iv. ω_R, v. 1.25 ω_R, vi. 1.5 ω_R, vii. 1.75 ω_R, viii. 10 ω_R.

Ans: (a)ω_0 = 18.3 rad/s. (b) underdamped. (c) 54.8. (d) 18.3 rad/s.

Ex 3.6.17. Consider a one-dimensional damped oscillator of mass $m = 0.5$ kg, $\omega_0 = 10$ rad/sec, and $\beta = 1$ rad/sec. Suppose the oscillator can oscillate along the x-axis. A sinusoidal force with the x component $F_x = 100$ N $cos(\omega t)$ acts on the oscillator. (a) What should be the condition on time t so that the steady state for the oscillator can be assumed? (b) Suppose the steady state has been reached and the displacement of the oscillator can be written as $x = A_c \cos(\omega t) + A_s \sin(\omega t)$. Find the amplitudes A_c and A_s as a function of driving frequency ω. (c) Find the amplitude of the steady sate oscillations. (d) Find the phase lag? (e) What is the expression for the instantaneous power of the driving force? (f) Find the average power as a function of driving frequency. (g) At what frequency is the power maximum? (h) At what frequency is the amplitude of the oscillator's motion maximum? (i) What is the width at half height of the power versus driving frequency plot?

Ex 3.6.18. A harmonically driven lightly damped oscillator of mass 200-grams has a resonance frequency of 100 Hz. Its amplitude at the resonance frequency is 750 N/kg. The applied force has a peak amplitude of 2 N. (a) Find the quality factor (Q) of the oscillator. (b) What is the approximate value of the damping constant b of the damping force?

Ans: (a) 31,400. (b)0.004 N.s/m.