9.4 CONSERVATION OF ANGULAR MOMENTUM

9.4.1 For One Particle

It was shown above that the rate of change of the angular momentum of a particle is equal to the net torque on the particle.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}.$$

Therefore, if the net torque on a particle is zero, then the angular momentum cannot change. That is, both the magnitude and the direction of the angular momentum of the particle will remain fixed in time if the net torque on the particle is zero. We say that the angular momentum of the particle is conserved if there is no torque on the particle.

$$\boxed{\text{If } \vec{\tau}_{\text{net}} = 0, \text{ then } \frac{d\vec{L}}{dt} = 0 \implies \Delta\vec{L} = 0.}$$

A consequence of this result is that a particle with zero net torque about some point O will remain confined to a plane. This can be seen by examining the basic definition of angular momentum. The angular momentum of a particle located at the position $\vec{r}$ and having a momentum $\vec{p}$ is given by $\vec{L} = \vec{r} \times \vec{p}$.

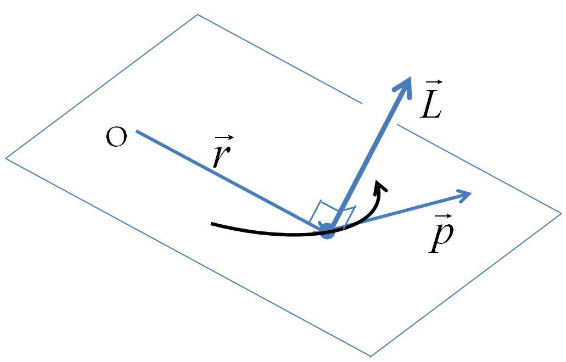

Figure 9.35: Angular momentum $\vec{L}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

When angular momentum is conserved, $\vec{r}$ and $\vec{p}$ are confined to a fixed plane in space since the direction of angular momentum, which is perpendicular to this plane, is constant. Therefore, a particle with a conserved angular momentum will remain confined to a planar motion.

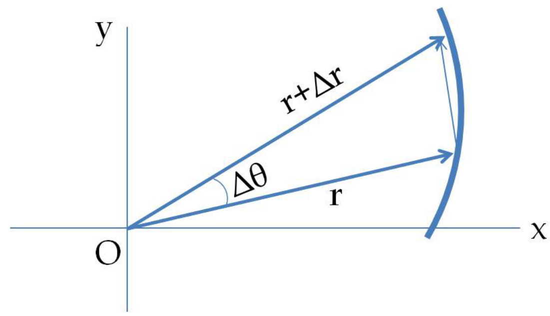

Figure 9.36: Set-up for equal area in equal time calculation.

Example 9.4.1. Equal area in equal time

The conservation of angular momentum also provides a fundamental reason for Kepler's second law of planetary motion, which states that the line from the sun to the planets sweep out equal area in equal time. Since the gravitational force on a planet is pointed towards the center of sun, the torque of this force on a planet about the center of sun is zero. Therefore, a planet's angular momentum will be conserved and not change with time. This is why the motion of planets occurs in planes. Kepler's second law holds true more generally whenever the torque on a particle vanishes as we show in the next example.

Example 9.4.2. Consider a particle that has zero torque about the origin. The position vector of the particle sweeps out an area with time. Show that the rate of the area swept is constant.

Solution. Since $\vec{L}$ is fixed in time due to zero torque, the motion of the particle will be in the plane of $\vec{r}$ and $\vec{p}$ of the particle. To be concrete, let the a particle be confined to the xy-plane. This will make $\vec{L} = L_z \hat{u}_z$. That is the constant angular momentum will mean constant L_z.

We will show that the rate of change of the area in xy-plane swept out by the position vector $\vec{r}$ from the origin to the position of the particle is proportional to the z-component of the constant angular momentum.

In this problem we work in the polar coordinate to make use of the radial distance of the position vector from the origin. Let the position of the particle at t and $t + \Delta t$ in the polar coordinates be (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$, respectively. The area ΔA swept out by the position vector $\vec{r}$ during this time interval can be approximated by the area of the triangle of base r and height $r\Delta\theta$ as shown in the Fig. 9.36.

$$\Delta A = \frac{1}{2}(r)\,(r\Delta\theta) = \frac{1}{2}r^2\Delta\theta.$$

To obtain the rate of change of the area we divide this result by Δt and take the $\Delta t \to 0$ limit. In this limit, the second term goes to zero, and we obtain.

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}. \tag{9.79}$$

This can be shown to be proportional to the z-component of the angular momentum as follows. Working in the polar coordinates and using the polar unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ in the xy-plane, the angular momentum of a particle confined to the xy-plane be written using the polar coordinates representation of vectors $\vec{r}$ and $\vec{v}$ given in section 3.7.

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ &= r\hat{u}_r \times \left[m \left(\frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta \right) \right] \\ &= mr^2 \frac{d\theta}{dt}\hat{u}_z \equiv L_z \hat{u}_z. \end{aligned}$$

Therefore,

$$\frac{dA}{dt} = \frac{1}{2m} L_z. \tag{9.80}$$

Since L_z is constant, the rate of area covered by the position vector $\vec{r}$ is also constant. Therefore, if the torque on a particle about some point O is zero, then the trajectory of the vector from the point O to the position of the particle sweeps out equal area in equal time.

Since a point particle does not have structure, the consequences of the conservation of angular momentum are somewhat limited. We now examine the consequences of vanishing of net torque on an extended object, both rigid and non-rigid.

9.4.2 Conservation of Angular Momentum For Extended Bodies

It was shown above that the rate of change of the total angular momentum of an extended body depends on the net external torque on the body.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}}^{\text{ ext}}. \tag{9.81}$$

Therefore, if the external torque is zero, then the angular momentum of the body cannot change, i.e. angular momentum will be conserved.

$$\boxed{\text{If } \vec{\tau}_{\text{net}}^{\text{ ext}} = 0, \text{ then } \frac{d\vec{L}}{dt} = 0.} \tag{9.82}$$

This equation is true of all extended bodies, whether a body is rigid or deformable. Since this equation is a vector equation, it is also true for each component separately.

$$\text{If } \tau^{\text{ext}}_{\text{net,x}} = 0, \text{ then } \frac{dL_x}{dt} = 0.$$
$$\text{If } \tau^{\text{ext}}_{\text{net,y}} = 0, \text{ then } \frac{dL_y}{dt} = 0.$$
$$\text{If } \tau^{\text{ext}}_{\text{net,y}} = 0, \text{ then } \frac{dL_z}{dt} = 0.$$

Suppose we choose the positive z-axis direction to be the direction of the constant $\vec{L}$, then for a system of conserved angular momentum, we will have only the z-component non-zero.

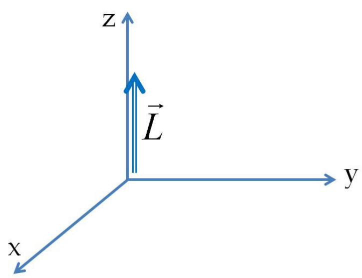

Figure 9.37: Choose z-axis to point in the direction of the conserved angular momentum.

$$L_x = 0$$
$$L_y = 0$$
$$L_z \text{ Non-zero, constant.}$$

Let us write the condition for the conservation of angular momentum using this coordinate choice. For teh fixed-axis rotation, we have found that the z-component of angular momentum is equal to the product of the zz component of the moment of inertia and the z-component of the angular velocity. Therefore Eq. 9.82 takes the following form.

$$\boxed{\text{If } \tau^{\text{ext}}_{\text{net,z}} = 0, \text{ then } \frac{d}{dt}\left(I_{zz}\omega_z\right) = 0,} \tag{9.83}$$

which says that the product $I_{zz}\omega_z$ is fixed in time when the z-component of the net external torque vanishes. If the extended body is a rigid body, then I_{zz} will not change with time. In that case, Eq. 9.83 says that the z-component of the angular velocity will not change with time. Let $\omega_z^{(1)}$ and $\omega_z^{(2)}$ be the z-component of angular velocity at times t_1 and t_2, respectively, then

$$\text{Net external torque zero, rigid body:} \quad \omega_z^{(1)} = \omega_z^{(2)}. \tag{9.84}$$

This statement is taken to mean that the rigid body with no external torque will rotate for-ever at the same angular velocity. This tendency of a rigid body is called **rotational inertia**.

On the other hand, if the extended body can change shape during some time without an application of external forces, i.e., purely by internal forces, then I_{zz} could be different at different times, which has important consequences for the motion of the body. If I_{zz} changes, ω_z must change also even when external torque is zero in order for the product $I_{zz}\omega_z$ to remain unchanged.

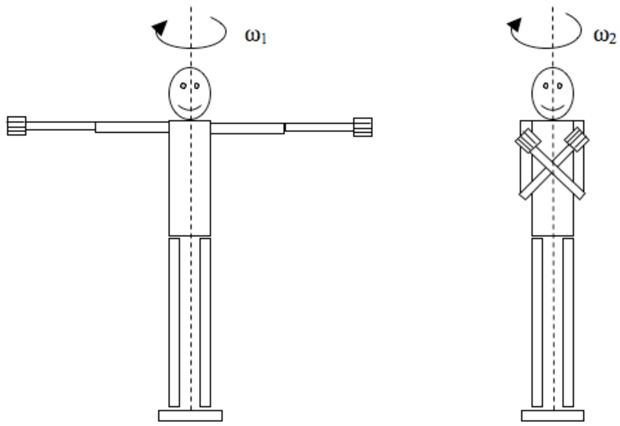

Figure 9.38: With negligible external torque on the skater the angular momentum of the skater about the vertical axis (z-axis) is conserved. The zz component of moment of inertia is larger when the arms are stretched out than when the arms are closer to the axis. Since $I_1 > I_2$, $\omega_1 < \omega_2$. The skater spins faster with arms closer to the body.

Let $L_z^{(1)}$ and $L_z^{(2)}$ be the z-component of the angular momentum at times t_1 and t_2. Let $I_{zz}^{(1)}$ and $\omega_z^{(1)}$ be the zz component of moment of inertia and the z-component of the angular velocity at the time t_1 and $I_{zz}^{(2)}$ and $\omega_z^{(2)}$ be the corresponding quantities at the time t_2, then

Memory tool:
$I^{(1)}\omega^{(1)} = I^{(2)}\omega^{(2)}$
if $\tau^{\text{ext}} = 0$.

$$\text{Net external torque zero, general: } L_z^{(1)} = L_z^{(2)}$$
$$\Longrightarrow \boxed{I_{zz}^{(1)}\omega_z^{(1)} = I_{zz}^{(2)}\omega_z^{(2)}.} \tag{9.85}$$

Thus, even though the angular momentum of a system cannot change under a condition of the zero external torque, the angular velocity of the body can change if the body is not a rigid body with a change in the moment of inertia of the body: if I_{zz} increases, then ω_z would decrease, and vice-versa.

This is what happens in figure skating. The ice skater has a smaller moment of inertia when the arms are closer to the body and so the skater spins faster. As the arm is moved outward, away from the body, the moment of inertia increases, which is accompanied by slower rotation. This is due to the fact that external torques on the skater from the ice and air resistance are negligible, and therefore, the product of the moment of inertia and the angular velocity about the axis must be conserved (Fig. 9.38). To stop the rotation, the skater needs an external torque which is supplied by the reaction force from the ice when she pushes into the ice by her skate.

Example 9.4.3. Walking on a rotating platform. Consider a platform of mass M and radius R with a person of mass m standing

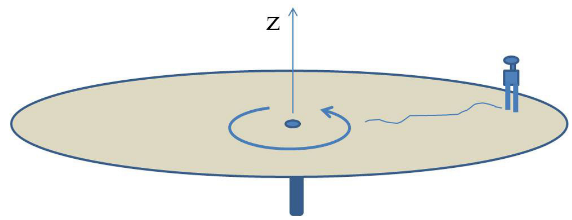

Figure 9.39: Example 9.4.3.

at a distance a from the center. The platform is initially rotating at angular speed ω. When the person on the platform walks to another place on the platform with radial distance $b < a$, the platform rotates faster. Find the new rotation speed.

Solution. We consider the person and the platform as one system. This system is not rigid since the relative positions of the person and the platform can change.

Notice that during the walk there is no external torque. There is only torque from the person on the platform and vice-versa. Therefore, the net angular momentum of the combined system is conserved.

We choose z-axis to pass through the center of the platform in the direction of the axis of rotation. The angular speed is ω when the person is at a radial distance a from the center of the platform, and let ω' be the angular speed when the person is at radial distance b.

We will assume that distances a and b are much greater than the diameter of the person's body so that all the mass of the person is at a distance a or b from the center in the two situations. Therefore, I_{zz} of the person will be ma^2 when he is a distance a from the axis and mb^2 when he is a distance b from the axis. As for the platform, we will assume that the density is uniform so that I_{zz} of the platform is $\frac{1}{2}MR^2$, where R is the radius of the platform.

Now, we can write the z-component of the angular momentum in the two situations, before and after the walk, as follows.

$$\text{Before walk:}\quad L_z^{(1)} = \left(L_z^{(1)}\right)_{\text{platform}} + \left(L_z^{(1)}\right)_{\text{person}} = \frac{1}{2}MR^2\omega + ma^2\omega$$

$$\text{After walk:}\quad L_z^{(2)} = \left(L_z^{(2)}\right)_{\text{platform}} + \left(L_z^{(2)}\right)_{\text{person}} = \frac{1}{2}MR^2\omega' + mb^2\omega'$$

Equating the angular momenta at two instants gives us the following relation.

$$\frac{1}{2}MR^2\omega + ma^2\omega = \frac{1}{2}MR^2\omega' + mb^2\omega',$$

which can be solved for the rotation speed after the person has reached the second spot.

$$\omega' = \left(\frac{MR^2 + 2ma^2}{MR^2 + 2mb^2}\right)\omega.$$