8.2 CIRCULATION OF MAGNETIC FIELD

Circulations of magnetic and electric fields play important role in understanding electric and magnetic phenomena fully. Circulation of a field combines the strength and direction properties of a field around a directed closed loop in space into one scalar quantity.

Length Vector

The definition of circulation requires a length vector on the directed loop in space. The length vector on the directed loop is defined by its magnitude and direction as follows.

Length Vector

Magnitude = Length

Direction = Direction of the loop in space

The length vector is same as the displacement vector on the imagined loop. On a straight segment of a directed loop, length vector will have one direction. When the loop turns a corner, the length vector must also turn the corner. For instance, as shown in Fig. 8.13, four directed lines are needed to describe a rectangular directed loop. Since the loop is a closed path, the sum of all length vectors making the path will equal zero, although the length of the loop itself is not zero.

If a loop is curved at some point, then the length vector at that point would be along the tangent to the loop as shown in Fig. 8.13. For general purposes, we will denote infinitesimal element length vector on the loop by $d\vec{l}_A$ or $\Delta\vec{l}_A$, where the subscript A is attached here to remind us about its use for Ampere's law stated below. Sometimes, the infinitesimal length vector is given other symbols such as $d\vec{r}$ and $d\vec{s}$.

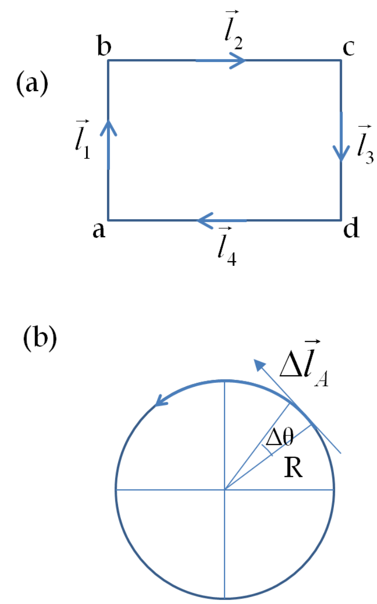


Figure 8.13: Length vectors of two "mathematical" directed loops in space. (a) The loop containing four straight segments have four length vectors. The closed loop means sum of the four segments equal zero: $\vec{l}_1+\vec{l}_2+\vec{l}_3+\vec{l}_4 = 0$. (b) The curved loop such as a circle has infinitely many vectors; the vector element subtending angle $\Delta\theta$ at the center of a circle has length $R\Delta\theta$ and direction given by the arrow along the tangent.

Circulation of Magnetic Field

Basis Definition

The circulation of magnetic field $\vec{B}$ around a closed loop in space is defined as the sum of projection of magnetic field on the length vectors $\Delta\vec{l}_A$ of the loop.

$$\text{Circulation of } \vec{B} \text{ around a loop} = \sum_{\text{Closed Path}} \vec{B}\cdot\Delta\vec{l}_A. \tag{8.18}$$

Operationally, we divide up a directed loop into a sum of length vectors $\Delta\vec{l}_i$, $i = 1, 2, \cdots, N$, calculate the scalar product of local magnetic field and the corresponding length vector. The sum total of the scalar products gives the circulation of the magnetic field. When the elements of the loop are made infinitesimally small, the procedure can be written as a conceptual integral.

$$\text{Circulation of } \vec{B} \text{ around a loop} = \oint_{\text{Closed Path}} \vec{B}\cdot d\vec{l}. \tag{8.19}$$

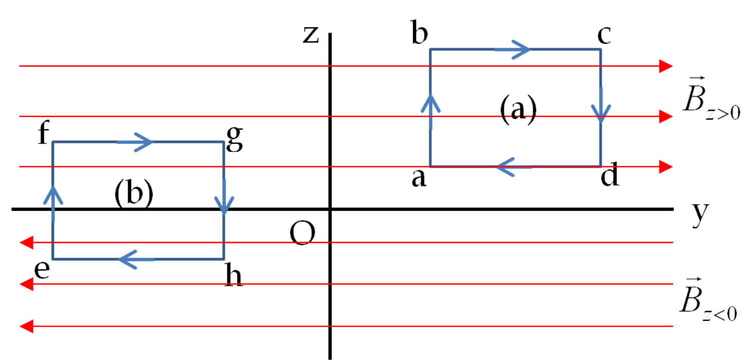
This integral is the same type of integral that you have practiced when calculating work of a force, except that now, you need to perform the integral for a closed path in space.

Circulation can be calculated for any vector field and any loop. For instance, if we take the scalar product of electric field with length vector elements and sum them over all the elements of a loop in space, we will obtain the circulation of electric field around that loop. Note that unlike the

Example 8.2.1. Calculation of Circulation # 1 The magnetic field in a region of space is given by

$$\vec{B}(x,y,z) = \begin{cases} B_0\hat{u}_y & z>0 \\ -B_0\hat{u}_y & z<0 \end{cases} \tag{8.20}$$

That is, there is a uniform magnetic field of magnitude B_0 towards the positive y-axis in the half-space when $z > 0$ and a uniform magnetic field of magnitude B_0 towards the negative y-axis in the half-space when $z < 0$. Calculate the circulation of the magnetic field for the space loops (a) and (b) in the yz-plane shown in figure. Here $bc = ad == fg = eh = l_1$ and $bc = ad == fg = eh = l_2$.



Solution. (a) We note that magnetic field is either parallel or perpendicular to the direction of loop segments of the rectangular loop $abcda$. Therefore, the scalar products with length elements is easy to work out for the four segments with the following result.

$$\text{ab-segment: } \vec{B}\cdot d\vec{l}_A = 0 \;\; (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$
$$\text{bc-segment: } \int \vec{B}\cdot d\vec{l}_A = B_0 l_1 \;\; (\vec{B} \text{ parallel to } d\vec{l}_A)$$
$$\text{cd-segment: } \vec{B}\cdot d\vec{l}_A = 0 \;\; (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$
$$\text{da-segment: } \int \vec{B}\cdot d\vec{l}_A = -B_0 l_1 \;\; (\vec{B} \text{ anti-parallel to } d\vec{l}_A)$$

Adding up the contributions the circulation around the loop is equal to zero.

$$\oint_{\text{abcda}} \vec{B}\cdot d\vec{l}_A = 0.$$

(b) On loop $efghe$ magnetic field is either parallel or perpendicular to different segments of the loop. Working out the scalar products of $\vec{B}$ with loop elements $d\vec{l}_A$ on each segment we find the following.

$$\text{ef-segment: } \vec{B}\cdot d\vec{l}_A = 0 \;\; (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$
$$\text{fg-segment: } \int \vec{B}\cdot d\vec{l}_A = B_0 l_1 \;\; (\vec{B} \text{ parallel to } d\vec{l}_A)$$
$$\text{gh-segment: } \vec{B}\cdot d\vec{l}_A = 0 \;\; (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$
$$\text{he-segment: } \int \vec{B}\cdot d\vec{l}_A = B_0 l_1 \;\; (\vec{B} \text{ parallel to } d\vec{l}_A)$$

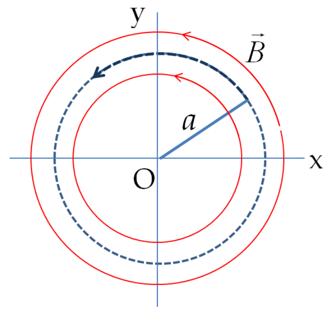
Summing up the parts we find that circulation of the magnetic field around the loop $efghe$ is non-zero.

$$\oint_{\text{efghe}} \vec{B}\cdot d\vec{l}_A = 2B_0 l_1.$$

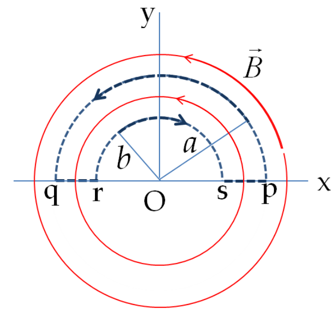
Example 8.2.2. Calculation of Circulation # 2 The magnetic field in a region of space is given by

$$\vec{B}(x,y,z) = \frac{a}{r}\hat{u}_\phi, \tag{8.21}$$

$r = \sqrt{x^2+y^2}$ and $\hat{u}_\phi$ is a unit vector in the direction perpendicular to radial vector from the z-axis. Calculate the circulation of the magnetic field for the space loops (a) and (b) in the xy-plane shown in figure. Loop (a) is a circle of radius a and loop (b), given as pqrsp, consists of two semicircular and two straight segments.



(a)



(b)

Solution. (a) The magnetic field $\vec{B}$ at all points of the loop is parallel to the direction of loop elements $d\vec{l}_A$. Therefore, the dot product between $\vec{B}$ and $d\vec{l}_A$ will just equal the product of the magnitude of the magnetic field and the length of the elements.

$$\vec{B} \cdot d\vec{l}_A = B\ dl_A$$

The magnitude of the magnetic field has same value everywhere on the loop, given by setting $r = a$ in the given formula for the magnetic field.

$$B = \frac{\alpha}{a}$$

Therefore, the circulation becomes simply the product magnitude of the magnetic field and the total length of the loop, which is just the circumference of the circle of radius a.

$$\text{Circulation} = \oint \vec{B} \cdot d\vec{l}_A = \frac{\alpha}{a} \times 2\pi a = 2\pi\alpha.$$

The circulation around the loop is independent of the radius of the loop. So, every closed circular loop around origin will give the same circulation for the given magnetic field.

(b) Loop pqrsp for this part consists of four segments, two semicircular parts and two straight parts. The magnetic field on the two semicircular segments are either parallel or antiparallel to the length vectors, but the magnetic field is perpendicular to length vectors on the straight segments. Using the arguments of part (a) we can find the integral of $\vec{B} \cdot dl_A$ on the two semicircular segments. The results for the four segments of the loop are:

$$\text{pq-segment: } \int \vec{B} \cdot d\vec{l}_A = \pi\alpha \ \ (\vec{B} \text{ parallel to } d\vec{l}_A)$$
$$\text{qr-segment: } \vec{B} \cdot d\vec{l}_A = 0 \ \ (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$
$$\text{rs-segment: } \int \vec{B} \cdot d\vec{l}_A = -\pi\alpha \ \ (\vec{B} \text{ parallel to } d\vec{l}_A)$$
$$\text{sp-segment: } \vec{B} \cdot d\vec{l}_A = 0 \ \ (\vec{B} \text{ perpendicular to } d\vec{l}_A)$$

Summing up the parts we find that circulation of the magnetic field around the loop $pqrsp$ is zero.

$$\oint_{\text{pqrsp}} \vec{B} \cdot d\vec{l}_A = 0.$$