6.6 THE ARCHIMEDES'S PRINCIPLE

It is said that once King Hiero of Syracuse, a port city in Greece, ordered a gold crown and gave the exact amount to a goldsmith. When the king receive the crown, it had the correct weight, but he suspected that some of the gold was replaced by some other metal. The king asked the famous geometer of the time, Archimedes (287? BC 212 BC), to find out if the crown was made of gold without destroying it. This was a difficult problem at the time and no one knew how to do it.

The idea came to Archimedes one day as he was entering his bathtub. He noticed that the amount of water that overflowed the tub was proportional to the amount of his body submerged. He also recognized that he could use this to solve the mystery of the crown. He was very excited by this discovery and ran through the streets of Syracuse shouting "Eureka! Eureka!" (I have found it!).

Archimedes's principle refers to the force of buoyancy that results when a body is submerged in a fluid, whether partially or wholly. The pressure of a fluid acts on a body perpendicular to the surface of the body, always pointed into the body. That means that the pressure at the bottom part is pointed up while at the top part is pointed down and the pressures at the sides are pointing into the body.

Since the bottom of the body is at a greater depth than the top of the body, the pressure at the lower part of the body would be higher than the pressure at the upper part as shown in Fig. 6.17. Therefore, there will be a net upward force on the body. This upward force is called the **force of buoyancy** or simply the **Buoyancy**. Archimedes found a quantitative rule for the magnitude of the buoyancy force, which is codified in the **Archimedes's principle**.

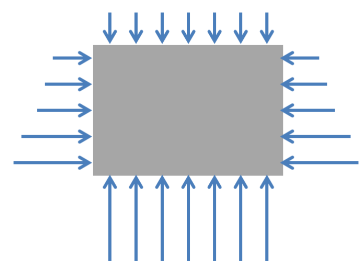

Figure 6.17: The cause of buoyancy is the greater pressure at the higher depth. the higher pressure at the lower part of the body gives a net upward force to the body.

The magnitude of the buoyancy force is equal to the weight of the displaced fluid and the direction of buoyancy force is opposite to the direction of the weight.

Let ρ_0 be the density of the fluid and V_{sub} the volume of a material submerged in the fluid, then according to Archimedes's principle, the force of buoyancy has the following magnitude and direction.

$$\text{Magnitude: } F_B = m_{\text{disp}} g = (V_{\text{sub}}\ \rho_0)\, g, \quad \text{Direction: } \uparrow \tag{6.21}$$

where m_{disp} is the mass of the displaced fluid. For the purposes of calculating torque on the body due to the force of buoyancy, the buoyancy force can be thought to act at the center of buoyancy (CB), which coincides with the center of mass (CM) of the displaced fluid

(Fig. 6.18) and not necessarily at the CM of the body if the body is only partially submerged. If the body is fully submerged, the CM of the body and CB coincide.

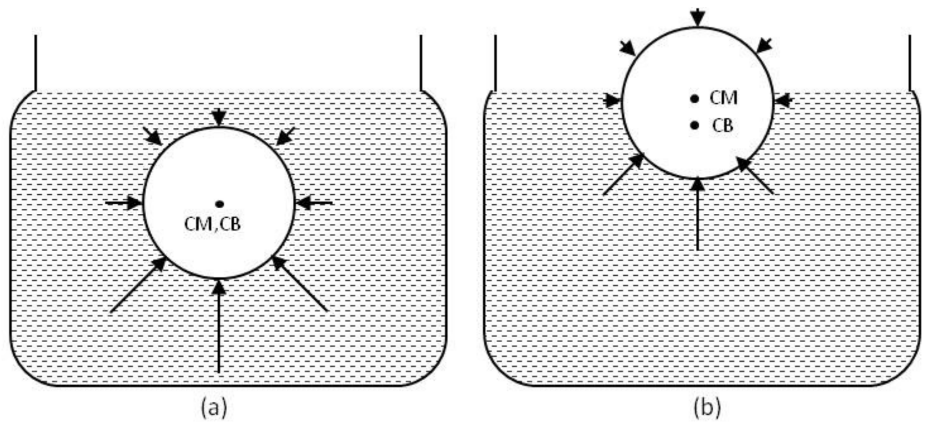

Figure 6.18: Buoyancy in (a) fully submerged and (b) partially submerged material arises due to the different pressure at different depths in the fluid. The center of buoyancy (CB) is at the center of mass of the displaced fluid which may not be at the center of mass of the body itself.

6.6.1 Using Archimedes's Principle to Find Density

Consider a material of an unknown density ρ. Let the mass of the sample be M. When the sample is dipped in a fluid of a known density ρ_0 its apparent weight is less than it weight Mg. Let us consider the case of $\rho > \rho_0$ so that the sample sinks in the fluid and can be fully submerged. The density of the unknown can be found from its apparent weight through the following procedure.

Tie a thread of negligible mass to the sample and hang it to a spring balance. Now, submerge the sample in the given fluid making sure not to touch the bottom of the container, and record the apparent weight reading W' in the spring balance (Fig. 6.19).

The forces on the mass when it is submerged are: weight Mg, buoyancy force F_B and tension $T = W'$, all of whom act in the vertical direction. Since the mass has no acceleration, Newton's second law of motion gives us the following relation for the vertical direction.

$$T + F_B - W = 0 \tag{6.22}$$

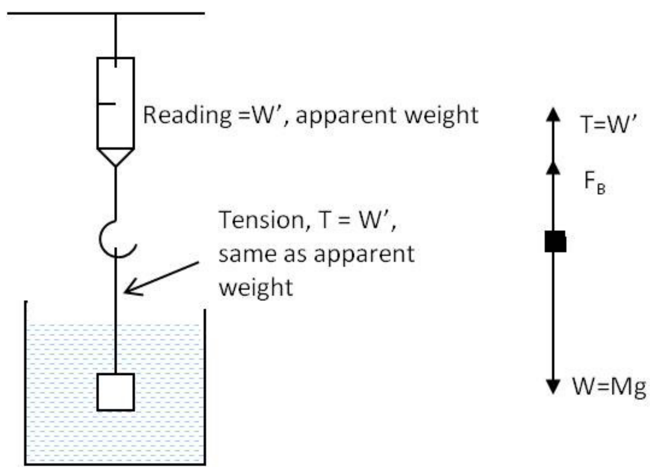

Figure 6.19: Finding density of a fully submerging substance in a liquid.

For the forces we have the following expressions.

$$
\begin{aligned}
& W = Mg \\
& T = W' \\
& F_B = V_{\text{submerged}} \times \rho_0 \times g = V_{\text{sample}} \times \rho_0 \times g \quad \text{(since fully submerged)} \\
& \quad = \left(\frac{M}{\rho}\right) \times \rho_0 \times g = \frac{\rho_0}{\rho} \times W
\end{aligned}
$$

After putting these equations in Eq. 6.22 we can solve for the density ρ of the sample in terms of the apparent weight W' while fully submerged in the fluid, weight W, and density ρ_0 of the fluid.

$$
\rho = \left(\frac{W}{W - W'}\right) \rho_0 \tag{6.23}
$$

Using this relation you could determine the density of the crown given to Archimedes without knowing the exact volume of the crown. Legend has it that the jeweler did cheat the king and he was beheaded when Archimedes found the cheat.

Example 6.6.1. Apparent Weight of Steel Ball in Oil.

A 100-g steel ball is submerged in oil of density 0.8 g/cc. Find the apparent weight if the density of steel is 8.0 g/cc.

Solution. Solving for W' in Eq. 6.23, we find the following.

$$
\begin{aligned}
W' &= \left(1 - \frac{\rho_0}{\rho}\right) W \\
&= \left(1 - \frac{0.8}{8.0}\right) \times 0.100 \text{ kg} \times 9.81 \text{ m/s}^2 \\
&= 0.882 \text{ N}.
\end{aligned}
$$

The weight is 0.981 N and the apparent weight 0.882 N.

6.6.2 Buoyancy for Partially Submerged Bodies

Case 1 ($\rho > \rho_0$)

A compact object whose density ρ is greater than the density of the fluid ρ_0 will sink in the fluid. To submerge it partially we can hang the object in the fluid by some mechanism. As more of the object is submerged the apparent weight decreases till when all of the body is inside the fluid. After that, the apparent weight does not change. We will find a formula that relates the apparent weight to the fraction of the volume submerged.

The relation between W, W' and F_B given in equation 6.22 still holds. Let x be the fraction of volume of the sample submerged, then the vertical component of forces on the body give the following relation.

$$\begin{aligned} & T + F_B - W = 0 \\ & \quad \Longrightarrow \quad W' + x\frac{\rho_0}{\rho}W - W = 0 \\ & \quad \Longrightarrow \quad W' = \left(1 - x\frac{\rho_0}{\rho}\right)W \end{aligned} \tag{6.24}$$

Case 2 ($\rho \leq \rho_0$)

An object whose density is less than that of the fluid will float in the fluid. Here, we will find a formula for the fraction of the volume that will be submerged when a body of density less than the density of the fluid floats on the surface of the fluid. You may think of this as an application of the partially submerged case where $W' = 0$ since if you were to use a string to lay the sample on the surface of the fluid there will be zero tension in the string when you let go of it. Setting $W' = 0$ in Eq. 6.23 we find the fraction of volume submerged is given by the ratio of the densities.

$$x = \frac{\rho}{\rho_0} \tag{6.25}$$

Of course, you can also do this problem directly. Refer to Fig. 6.20 for the free-body-diagram. Since vertical acceleration of the mass is zero, the second law of motion gives us the following relation.

$$xV\rho_0 g - V\rho g = 0 \quad \Longrightarrow \quad x = \frac{\rho}{\rho_0} \tag{6.26}$$

Even an object whose density is more than that of fluid can be made to float in the fluid if it is shaped such that the displaced fluid has more weight than the weight of the object itself. This is the principle by which a boat made of steel floats in water even though the density

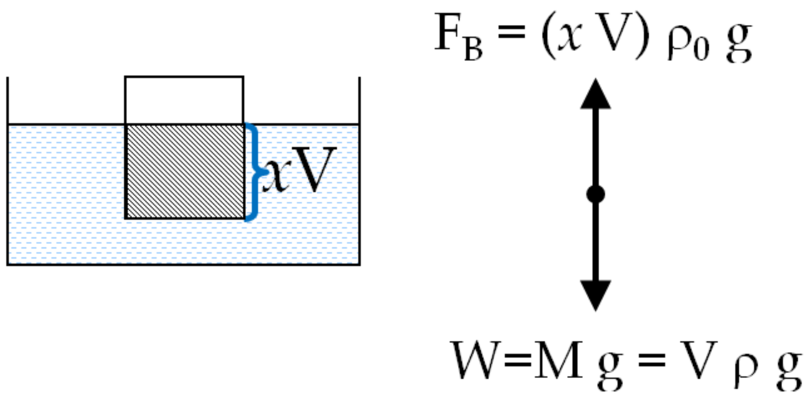

Figure 6.20: Partially submerged object.

of steel is approximately eight times that of water. The boat has a shape that lets it displace more volume of fluid than the volume of the steel itself. For a steel boat to float it must displace at least eight times its volume.

Example 6.6.2. Density of Wood Floating in Water.

While freely floating, sixty five percent of the volume of a block of wood is found to be submerged in salt water of density 1.2 g/cc. What is the density of the wood?

Solution. Since freely floating, we can use x = ratio of densities.

$$\rho = x\rho_0 = 0.65 \times 1.2 \text{ g/cc} = 0.78 \text{ g/cc}.$$

Example 6.6.3. Partially Submerged Hollow Steel Ball.

A hollow spherical steel ball with inner radius 2 cm and outer radius 2.1 cm floats freely in saltwater of density 1.1 g/cc. What is the percentage of total volume submerged under saltwater? Assume density of steel to be 8 g/cc.

Solution. The hollow steel ball has an effective density much less than the density of compact steel. What matters for buoyancy is the displaced fluid which depends on the outer surface of the sample. Therefore we calculate the effective density of the sample from the mass of the given sample and the volume of the outer surface.

$$M = \frac{4}{3}\pi\left(r_{out}^3 - r_{in}^3\right)\rho_{steel} = 46.5 \text{ g}.$$
$$V_{out} = 44.6 \text{ cc}.$$

The effective density of the sample is:

$$\rho_{eff} = \frac{M}{V_{out}} = \frac{46.5\ g}{44.6 \text{ cc}} = 1.04 \text{ g/cc}$$

Percentage of volume under the given saltwater

$$x = \frac{\rho_{eff}}{\rho_0} = \frac{1.043 \text{ g/cc}}{1.1 \text{ g/cc}} = 0.95.$$

This says that 95% of the steel ball will be submerged.