3.3 ELECTRIC POTENTIAL OF CHARGE DISTRIBUTIONS

3.3.1 General Considerations

A charged object, such as a plastic rod rubbed with a rabbit fur, usually has a large number of charges. To handle this type of situation we have introduced the concepts of charge density. One method of finding electric field of a charge distribution starts by dividing up the charge distribution into cells. Then, each cell is treated as a point charge, and then electric fields from all cells are superimposed to obtain the electric field of the entire charge distribution.

The same will be done here for the electric potential of a charge distribution. We will divide up a continuous charge distribution into tiny cells, and collect the charges in each cell together into one point charge located at the site of the cell. This procedure will convert the problem of a continuous charge distribution to a problem of N point charges if there are N cells in all. Suppose the charge in the i^{th} cell is denoted by Δq_i and the distance from the i^{th} cell to the field point P is r_{Pi}, then the electric potential V_P at the space point P will be given by the following sum.

$$V_P = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{\Delta q_i}{r_{Pi}} \tag{3.35}$$

In the limit of infinitesimal cells, the sum becomes an integral, which we can write as a coneptual integral.

$$V_P = \frac{1}{4\pi\epsilon_0} \int_{\text{cells}} \frac{dq}{r_{\text{cell-to-P}}}, \tag{3.36}$$

where $r_{\text{cell-to-P}}$ is the distance to point P from an infinitesimal cell. To be concrete, let the position of the space point P be denoted as $\vec{r}$ and the position of a representative cell element be $\vec{r}\,'$, then the integral can be written more explicitly.

$$\boxed{V_P = \frac{1}{4\pi\epsilon_0} \int_{\text{cells}} \frac{dq}{|\vec{r} - \vec{r}\,'|},} \tag{3.37}$$

where dq refers to the charge in the cell at position $\vec{r}\,'$. As we have seen for electric field of charge distribution the formula for elemental charge dq depends on the dimensions over which the charges are distributed.

$$\boxed{dq = \begin{cases} \lambda dl & \text{(1-dim)} \\ \sigma dA & \text{(2-dim)} \\ \rho dV & \text{(3-dim)} \end{cases}} \tag{3.38}$$

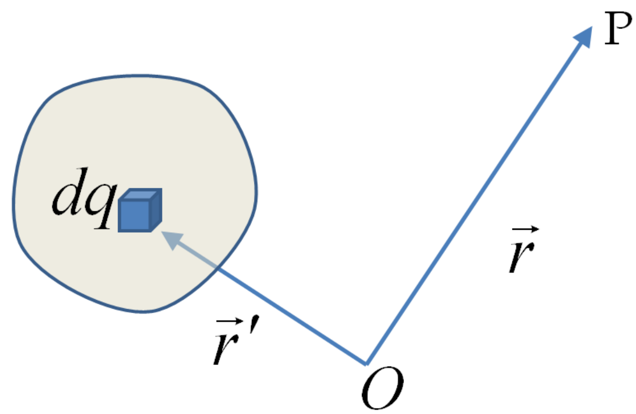

Figure 3.17: The position vectors $\vec{r}$ to the field point and $\vec{r}\,'$ to a cell of the charge distribution are illustrated. The potential at point P from the charge dq inside the cell depends on the direct distance $|\vec{r} - \vec{r}\,'|$.

Here λ is charge per unit length, σ charge per unit area and ρ charge per unit volume. In order to reduce the algebra, it is often helpful to pick the shape of the cell in accordance with the geometry of the charge distribution. The following examples illustrate calculations for electrical potential of continuous charge distributions.

3.3.2 Examples Of Potential Of Continuous Charges

The calculation of electric potential of charge distribution follows more or less the same steps as a calculation of electric field done previously except with one major difference - while we had to worry about the direction of electric field, we do not need to worry about directions at all here since electric potential is a scalar quantity. Another difference is that the denominator in the integrand has only one power of the distance which often makes the integral for potential easier to perform.

Example 3.3.1. Potential of a Uniformly Charged Wire. Find the electric potential of a uniformly charged non-conducting wire with linear density λ (Coulomb/meter) and length L at a point that lies on a line that divides the wire into two equal parts.

Solution. To set up the problem analytically we choose Cartesian coordinates such that we exploit the symmetry in the problem as much as possible. We will place origin at the center of the wire and orient the y-axis along the wire so that the ends of the wire are at $y = \pm L/2$. The field point P will be in the xy-plane and since choice of axes is up to us, we will choose the x-axis to pass through the field point P as shown in Fig. 3.18.

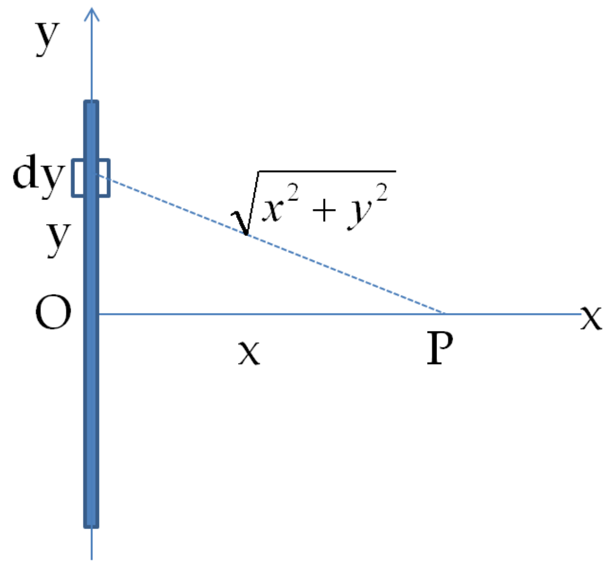

Figure 3.18: Electric potential of a line charge.

Consider a small element of the charge distribution between y and $y + dy$. The charge in this cell is $dq = \lambda dy$ and the distance from the cell to the field point P is $\sqrt{x^2 + y^2}$. Therefore, potential dV_P at x from charges in the cell between y and $y + dy$ will be

$$dV_P = \frac{1}{4\pi\epsilon_0} \frac{\lambda dy}{\sqrt{x^2 + y^2}}$$

We write the potential as dV_P rather than V_P to indicate that the potential here is that of an infinitesimal charge. Now, we must superpose the contributions from all cells, which can be done here by integrating from $y = -L/2$ to $y = L/2$.

$$\begin{aligned} V_P &= \frac{\lambda}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\lambda dy}{\sqrt{x^2 + y^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{L + \sqrt{L^2 + 4x^2}}{-L + \sqrt{L^2 + 4x^2}}\right] \qquad (3.39) \\ & \qquad (3.40) \end{aligned}$$

Example 3.3.2. Potential of a Uniformly Charged Ring. A ring has a uniform charge density λ Coulomb per unit meter of arc. Find the electric potential at a point on the axis of the ring.

Solution. We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle. and use polar coordinates shown in Fig. 3.19. A general element of the arc between θ and $\theta + d\theta$ will be of length $Rd\theta$ and therefore contain

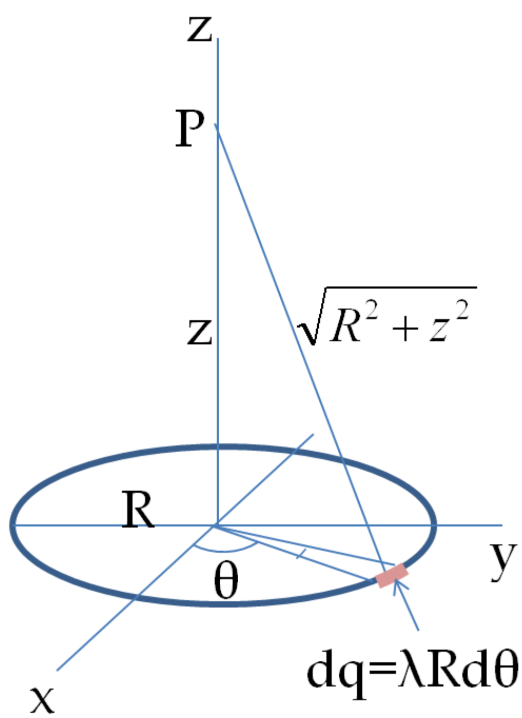

Figure 3.19: Electric potential of a ring of charge.

a charge equal to $\lambda R d\theta$. The element is at a distance $\sqrt{z^2+R^2}$ from the field point P is as shown in the figure. Therefore, the potential at the space point P by this element of the ring is

$$dV_P = \frac{1}{4\pi\epsilon_0}\frac{\lambda R d\theta}{\sqrt{z^2+R^2}}$$

Integrating from $\theta = 0$ to 2π takes care of contributions from all the charges on the ring. Since R and z do not depend on θ, they come out of the integral leaving a simple integral to perform.

$$V_P = \frac{\lambda R}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+R^2}}\int_0^{2\pi} d\theta.$$

This gives the potential at the field point P to be

$$V_P = \frac{2\pi R\lambda}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{\sqrt{z^2+R^2}}.$$

This is expected because every element of the ring is at the same distance from point P. The net potential at P is that of the total charge placed at the common distance, $\sqrt{z^2+R^2}$.

Example 3.3.3. Potential of a Uniformly Charged Disk. Find the electric potential of a uniformly charged nonconducting disk with charge density σ (Coulomb/square meter) and radius R at a point on the axis.

Solution. We divide the disk in ring-shaped cells and make use of the electric field of a ring worked out in Example 3.3.2. An infinitesimal width cell between cylindrical coordinates r and $r + dr$ shown in the Fig. 3.20 will be a ring of charges whose electric potential dV_P at the field point has the following expression.

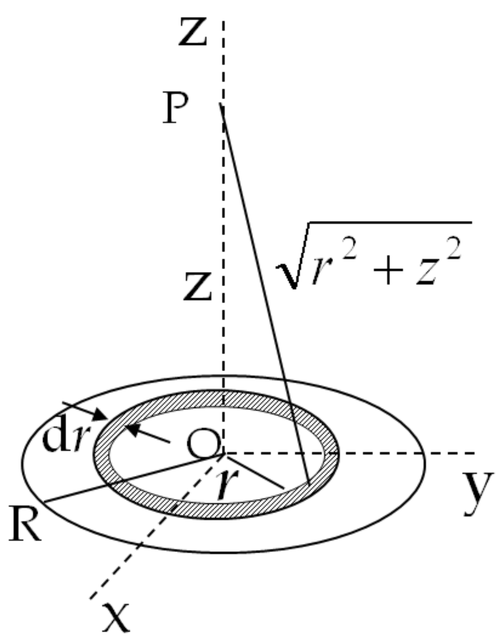

Figure 3.20: Example 3.3.3.

$$dV_P = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{z^2 + r^2}},$$

where

$$dq = \sigma \times 2\pi r dr.$$

The superposition of potential of all the infinitesimal rings that make up the disk gives the net potential at point P. This is accomplished by integrating from $r = 0$ to $r = R$.

$$\begin{aligned} V_P &= \frac{2\pi\sigma}{4\pi\epsilon_0}\int_0^R \frac{rdr}{\sqrt{z^2 + r^2}} \\ &= \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2 + R^2} - \sqrt{z^2}\right) \end{aligned} \tag{3.41}$$

The potential is largest at $z = 0$. Near the surface of the disk, $z << R$, and therefore, we can use an approximate expression for the potential. To obtain the approximate expression of V_P when $z << R$ we expand the first term in parenthesis and keeping up to the first term in the series that contains z.

$$\sqrt{z^2 + R^2} = R\left(1 + \frac{z^2}{R^2}\right)^{1/2} \approx R + \frac{z^2}{2R}. \tag{3.42}$$

Putting the approximation in Eq. 3.42 in Eq. 3.41, we can rewrite the potential at P as

$$V_P \approx \frac{\sigma R}{2\epsilon_0} - \frac{\sigma |z|}{2\epsilon_0} \quad \text{(dropping the quadratic term)} \tag{3.43}$$

The potential approaches a constant value for $z = 0$, and decreases linearly (i.e. as one power of z) as you move away from the disk. The rate of decrease of potential near the center of the disk $-\sigma/2\epsilon_0$ is constant independent of the size of the disk.

$$\left.\frac{dV_P}{dz}\right|_{z<<R} \approx -\frac{\sigma}{2\epsilon_0}. \tag{3.44}$$

We will find below that negative of the z-derivative of potential is equal to the z-component of electric field. For points very near the disk, the disk appears infinite in extent, and therefore, the electric field at these points is same as that of an infinite sheet.

$$E_z \approx \frac{\sigma}{2\epsilon_0}.$$

Example 3.3.4. Potential of a Uniformly Charged Cylinder. Find the electric potential at a point on the axis by a uniformly charged nonconducting cylinder with charge density ρ, radius R and height H. Assume the field point to be outside the cylinder.

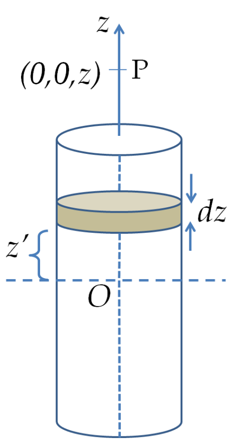

Figure 3.21: Example 3.3.4.

Solution. In this problem we will make use of the formula for the potential of the disk worked out in the last example which is given in Eq. 3.41. We orient the axes so that the z-axis is the axis of the cylinder and the origin is at the center of the cylinder as shown in Fig. 3.21.

Let the coordinate of the field point P be $(0, 0, z)$ with $|z| > H/2$. Now, consider a thin disk of charges between z' and $z' + dz'$ of the cylinder. [Note: it is important to use a different symbol for the z-coordinates for the charges and the z coordinate of the field point. Otherwise you will confuse yourself.]

The thin disk has the net charge equal to ρ times the volume $\pi R^2 dz$. When we divide this charge by the area πR^2, we get the σ of the Eq. 3.41 in terms of the volume charge density ρ of this problem. This calculation gives following equivalency: $\sigma \equiv \rho dz'$. The distance from the thin disk to the field point is $z - z'$ which will replace the z in the Eq. 3.41. Therefore, the electric potential at point P from the thin disk between z' and $z' + dz'$ will be

$$dV_P = \frac{\rho\, dz'}{2\epsilon_0}\left[\sqrt{(z-z')^2 + R^2} - \sqrt{(z-z')^2}\right].$$

Now, we need to add up the contributions from all charges in the disk. This can be done by integrating from $z' = -\frac{H}{2}$ to $z' = \frac{H}{2}$.

$$V_P = \int_{-H/2}^{H/2} \frac{\rho\, dz'}{2\epsilon_0} \left[\sqrt{(z-z')^2 + R^2} - \sqrt{(z-z')^2} \right].$$

This looks like a difficult integral to do. The integrand is different for $z > z'$ and $z < z'$ due to the square-root of the square or the absolute value in the second term. So, you would have to be careful there. I will leave the integral for you to practice upon and give you the answer I got by plugging into the Mathematica®. [Note: the log in this answer can be written in other ways.]

$$V_P = \frac{\rho}{2\epsilon_0} \Bigg[\sqrt{R^2 + (u-z)^2}(u-z) - \frac{u^2\sqrt{(u-z)^2}}{(u-z)} + \frac{2u\sqrt{(u-z)^2}z}{u-z} + R^2 \mathrm{Log}\left[u + \sqrt{R^2 + (u-z)^2} - z \right] \Bigg]_{-H/2}^{H/2} \tag{3.45}$$

Note that the answer is ugly because the cylinder has a finite size and the charge distribution does not have the cylindrical symmetry we have talked about when we discussed the Gauss's law. Actually, the integrals for the field points not on the axis are even more messy. Try to see if you can set up the problem for finding the electric potential at a point in the plane perpendicular to the cylinder.

When you have the symmetries such that you can use Gauss's law to figure out the electric field, then you would use another method to be described next. In this "superior" method you will be able to find the electric potential formula from the known electric field formula.