4.6 VARIABLE ACCELERATION

Although most of our examples in this book will have constant acceleration segments, it is worthwhile to look at a case of arbitrarily varying acceleration as an important extension of the methods for one-dimensional constant acceleration presented above.

The basic idea for handling an arbitrarily varying acceleration is to replace the original acceleration by an approximation obtained by dividing up the time interval into smaller time segments. If the time segments are small enough, we can approximate the original arbitrarily varying acceleration by a step-wise varying acceleration, where each step is an average acceleration in the corresponding time segment.

Clearly the original situation of a continuously varying acceleration is not the same as its replacement by the constant acceleration steps. However, Sir Isaac Newton showed that if the intervals were allowed to be arbitrarily small, then the predictions of the final position and velocity, based on step-wise approximate acceleration, can be made arbitrarily close to the exact answer (see Fig. 4.10).

Recall that acceleration, velocity and position are vectors whose magnitudes and components can be written as functions and plotted in a graph. Since the kinematic equations separate into x-, y and z-components, it is sufficient to discuss one of the Cartesian components. To be specific, we will consider x-components of acceleration, velocity and position of an object in which x-component of the acceleration changes with time and we wish to find the change in x-components of velocity and position with time.

As shown in Fig. 4.10, we divide the full interval $[0, T]$ into N subintervals, which we will denote by $[0, \Delta t]$, $[\Delta t, 2\Delta t]$, $[2\Delta t, 3\Delta t]$, $\cdots$ $[(N-1)\Delta t, N\Delta t]$, with $T = N\Delta t$.

Let the x-components of the average acceleration be a_{1x}, a_{2x}, a_{3x}, $\cdots$, a_{Nx} in the N time segments respectively. Let v_{1x}, v_{2x}, v_{3x}, $\cdots$, v_{Nx} be the x-components of velocity at the end of the corresponding intervals. Of course v_{Nx} is same as $v_x(T)$ or simply v_x at the end of the final time of the net interval.

Now, we show that our procedure leads us to the prediction of the x-component of the final velocity from the x-component of the initial velocity v_{0x}, the velocity at time $t = 0$.

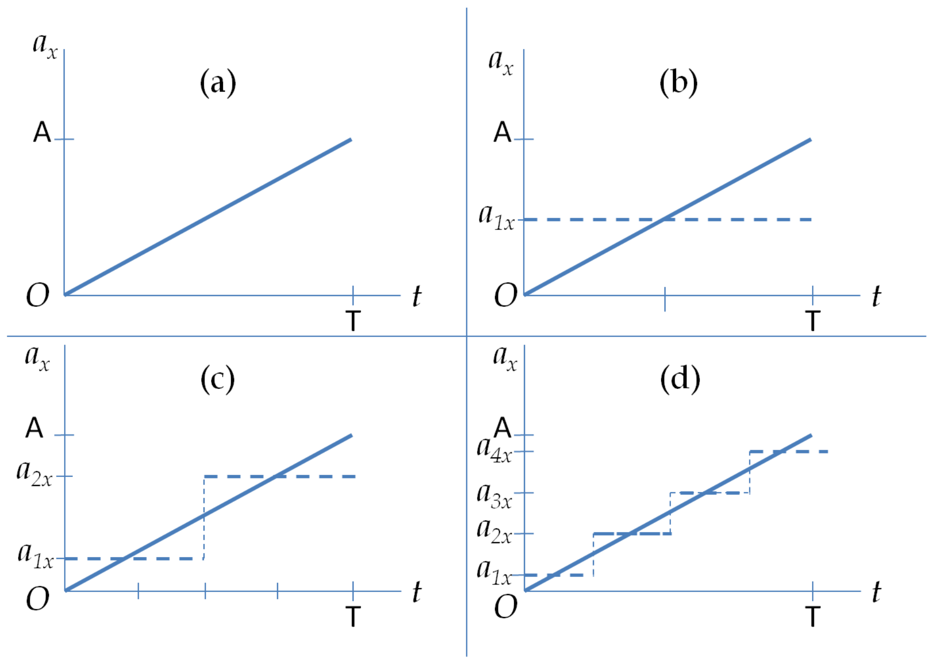


Figure 4.10: Successive approximations of a varying acceleration. The plots in this figure show the varying x-component of the acceleration. Fig. (a) contains the continuous function $a_x(t)$. In Fig. (b) the variable $a_x(t)$ is replaced by one constant step with the average x-component of the accelerationdenoted as a_{1x}. In Fig. (c) $a_x(t)$ is replaced by two segments of constant acceleration steps of a_{1x} and a_{2x}, and in Fig. (d) by four steps. The process can be continued ad-infinitum. As you decrease the step size, or equivalently increase the number of steps, the original acceleration is covered more accurately by the approximation.

Interval	Velocity at the end of the interval
0 to Δt	$v_{1x} = v_{0x} + a_{1x}\Delta t$
Δt to $2\Delta t$	$v_{2x} = v_{1x} + a_{2x}\Delta t$
$2\Delta t$ to $3\Delta t$	$v_{3x} = v_{2x} + a_{3x}\Delta t$
$\vdots$	$\vdots$
$(N-2)\Delta t$ to $(N-1)\Delta t$	$v_{N-1,x} = v_{N-2,x} + a_{N-1,x}\Delta t$
$(N-1)\Delta t$ to $N\Delta t = T$	$v = v_N = v_{N-1,x} + a_{Nx}\Delta t$
	Summing, $v_x = v_{0x} + \sum_{i=1}^{N} a_{ix}\Delta t$.

Summing the velocity change equation in each interval gives us the following equation for the change in velocity from v_{0x} to v over the entire time $[0, T]$.

$$v_x = v_{0x} + \sum_{i=1}^{N} a_{ix}\Delta t. \tag{4.27}$$

The approximation becomes better as Δt is made smaller. We denote the limit of the summation as Δt approaches zero by another symbol, called the definite integral of $a_x(t)$ from $t = 0$ to $t = T$.

Area under a_x vs t gives the change in velocity

$$v_x = v_{0x} + \int_0^T a_x(t)dt. \tag{4.28}$$

If the interval is other than $[0, T]$, e.g. from $t = t_1$ to $t = t_2$, then the integration will have to be done over the corresponding interval.

$$v_{2x} = v_{1x} + \int_{t_1}^{t_2} a_x(t)dt, \tag{4.29}$$

where v_{1x} is the velocity at t_1 and v_{2x} the velocity at t_2.

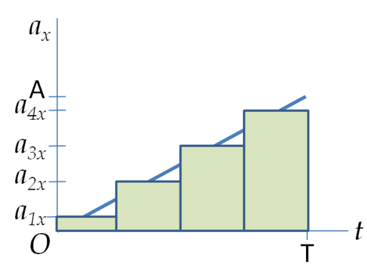
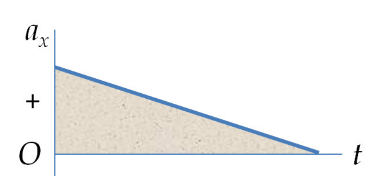


Figure 4.11: The change in velocity is equal to the area under the curve, approximated by the rectangles.

Another extremely useful interpretation of the result obtained in Eq. 4.27 is obtained by noting that $a_{1x}\Delta t$, $a_{2x}\Delta t$,$a_{3x}\Delta t$, $\cdots$, $a_{Nx}\Delta t$ are areas in the rectangle under the approximately constant x-component of the acceleration steps in the graph of a_x vs t plot as shown in Fig. 4.11 for four segments. The sum in Eq. 4.27 is the net area under the curve. Therefore, we can read Eq. 4.27 as follows.

$$v_x = v_{0x} + (\text{Area under the curve of } a_x \text{ vs } t). \tag{4.30}$$

The definite integral in Eq. 4.28 gives us an exact value of the area under the a_x vs t curve. Unlike the area of a polygon in space, though, the area under the curve can also be negative if the curve is below the horizontal since we would be multiplying a negative acceleration by a positive time segment resulting in a negative area (see Fig. 4.12.



Similar arguments will lead you to analogous relation between the change in the x-coordinate and the x-component of the velocity. We give the result for future reference and leave the derivation as an exercise for the student. Denoting the average velocities in segments $1, 2, 3, \cdots, N$ as $\bar{v}_{1x}, \bar{v}_{2x}, \bar{v}_{3x}, \cdots, \bar{v}_{Nx}$ we will obtain the following change in position $x - x_0$ for the total time $T = N\Delta t$. [We put a bar over the symbol of the average x-component of the velocity to distinguish the notation for the x-component of the velocity at the end of the segments.]

$$x - x_0 = \sum_{i=1}^{N} \bar{v}_{ix}\Delta t$$
$$\Longrightarrow x - x_0 = \int_0^T v_x(t)dt, \tag{4.31}$$
$$\text{(or, area under the curve of } v_x \text{ vs } t).$$

For an arbitrary interval $t = t_1$ to $t = t_2$, the integration will be done accordingly.

$$x_2 - x_1 = \int_{t_1}^{t_2} v_x(t)dt. \tag{4.32}$$

Figure 4.13: A US Marine Corp Paratrooper. Credits: Wikicommon.

Example 4.6.1. Area under the curve of acceleration as change in velocity. A skydiver drops off an air plane at zero speed and opens her parachute after 10 seconds. We will assume that parachute opens instantaneously at 10 second mark. For the first 10 seconds the acceleration of the paratrooper is 7m/s^2 pointed down. After the parachute opens at the 10-sec mark, the magnitude of the acceleration drops steadily to zero in another 20 seconds as shown in Fig. 4.14 in a coordinate system in which the positive y-axis is pointed up. Find the velocity of the paratrooper at (a) $t = 10$ sec, (b) $t = 30$ sec?

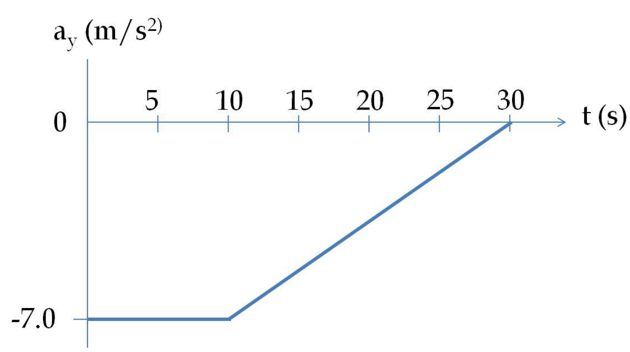


Figure 4.14: Example 4.6.1. Variable acceleration of a parachuter.

Solution. The area under the curve method for the changing acceleration can be applied to determine the change in velocity over any period. Part (a) is actually over an interval where the acceleration is constant. Therefore, for part (a) we could use either the constant acceleration equations or the area under the a_y vs t method. For part (b) we cannot use the constant acceleration formulas and must resort to the variable acceleration method.

(a) The area under the curve method can be applied to the segment of curve from $t = 0$ to $t = 10$ sec. The area is that of a rectangle with one side equal to -7 m/s^2 and the other side equal to 10 sec. Therefore, the area is $-70\ m/s$, which is the change in the y-component of the velocity of the paratrooper. Since the initial velocity of the paratrooper is given to be zero, the velocity at 10-sec mark is -70 m/s. The negative sign correctly gives the direction as pointed down since pointed up has been taken to be the positive direction of the y-axis. We can verify that we get the same result from using constant acceleration equations. Here $v_{0y} = 0$, $t = 10$ sec, $a_y = -7$ m/s^2. Therefore $v_y = v_{0y} + a_y t$ gives $v_y = -70$ m/s, same as the area under the curve method.

(b) This part has a variable acceleration, so it would be a mistake to use the constant acceleration formulas. The area under the y-component of the acceleration versus time curve is that of a triangle of height -7 m/s^2 and the base equal to 20 sec. This gives the change in the y-component of the velocity to be

$$\Delta v_y = \frac{1}{2}\left(-7\ m/s^2\right)(20\ sec) = -70 \text{ m/s}.$$

Since the y-component of the velocity at the beginning of this interval was also -70 m/s, the y-component of the velocity at the end of the interval will be $(-70$ m/s$) + (-70$ m/s$)$, or -140 m/s. Thus, the velocity at time $t = 30$ sec is 140 m/s pointed down.