## 1.7 Relativistic Transformation of Velocity

Suppose an object P is moving in the $S'$ frame with constant $x'$, $y'$, and $z'$ components of velocity $v'_x$, $v'_y$, and $v'_z$, respectively. Since $v'_x$ and $v'_x$ are constant, the position of P at time $t'$ is given as follows.

$$x' = v'_x t', \quad y' = v'_y t', \quad z' = v'_z t'. \tag{1.70}$$

Now, the question we ask is: What is the position of the particle in frame $S$ at time $t$, if $S'$ frame moves at constant speed $V$ with respect to the frame $S$ towards the positive $x$-axis? From Lorentz transformations we readily obtain the following for $t$, $x$, $y$, and $z$.

$$\left.\begin{array}{l} t = \gamma\left(t' + \dfrac{V}{c^2}x'\right) = \gamma\left(1 + \dfrac{V}{c^2}v'_x\right)\, t' \\ x = \gamma\,(x' + Vt') = \gamma\,(v'_x + V)\, t' \\ y = y' = v'_y t' \\ z = z' = v'_z t' \end{array}\right\} \tag{1.71}$$

From these equations we readily obtain

$$x = \left(\frac{v'_x + V}{1 + Vv'_x/c^2}\right)\, t, \quad y = \left(\frac{v'_y/\gamma}{1 + Vv'_x/c^2}\right)\, t, \quad z = \left(\frac{v'_z/\gamma}{1 + Vv'_x/c^2}\right)\, t. \tag{1.72}$$

Let $v_x$ and $v_y$ be the $x$ and $y$ components of the velocity of the object P in frame $S$.

$$\boxed{v_x = \frac{v'_x + V}{1 + Vv'_x/c^2}, \quad v_y = \frac{v'_y/\gamma}{1 + Vv'_x/c^2}, \quad v_z = \frac{v'_z/\gamma}{1 + Vv'_x/c^2}.} \tag{1.73}$$

This is the velocity addition law that replaces the Galilean velocity addition laws, $v_x = v'_x + V$, $v_y = v'_y$, and $v_z = v'_z$. We see that when the relative velocity of the frames is much smaller than the speed of light, i.e., when $V << c$, the special relativity velocity addition law reduces to the Galilean velocity law.

**Example 1.5.** A spaceship moving at a speed of $0.99\,c$ with respect to Earth sends light beam in the forward direction. The light beam travels with speed $c$ with respect to the spaceship. What will be the speed of the beam as observed by an observer on Earth?

**Solution.** We really need no formula to answer this question. Speed of light is same in all frames regardless of the state of motion. Let us check if this is borne out by the velocity addition formulas of this section. Let positive $x$-axis be pointed towards the forward direction of the spaceship. Then, we need to add

$$V = 0.99\, c \ \text{ and } \ v'_x = c.$$

Using the velocity addition formula we get the speed of light with respect to the Earth-based observer

$$v_x = \frac{v'_x + V}{1 + Vv'_x/c^2} = \frac{c + 0.99\, c}{1 + 0.99} = c.$$

**Example 1.6.** A spaceship moving at a speed of $0.9\,c$ with respect to Earth fires a rocket in the forward direction that has a speed of $0.9\,c$ with respect to the spaceship. What will be the speed of the rocket as observed by an observer on Earth?

**Solution.** Let the positive $x$-axis be pointed towards the forward direction of the spaceship. Then, we need to add

$$V = 0.9\,c \quad \text{and} \quad v'_x = 0.9\,c.$$

Using the velocity addition formula we get the speed of light with respect to the Earth-based observer

$$v_x = \frac{v'_x + V}{1 + V v'_x/c^2} = \frac{0.9\,c + 0.9\,c}{1 + 0.9^2} = 0.99\,c.$$