2.3 GRAVITATIONAL POTENTIAL ENERGY

Since the gravitational force is a conservative force, the work by this force can be expressed as a change in the potential energy of the object the force is acting. We have already derived the formula for the gravitational potential energy in the chapter on energy, Ch. **??**. Here, we present the derivation once again.

To find the expression for the gravitational potential energy we will calculate the work needed from an applied force $\vec{F}_{\text{appl}}$ to pull masses M and m apart from an initial distance of r_1 to the final distance of r_2. The work done gives the potential energy difference between the two states of the two-mass system: state (1) when the two masses are a distance r_1 apart and the state (2) when the two masses are a distance r_2.

$$\Delta U = -\int_i^f \vec{F}_{\text{appl}} \cdot d\vec{r} = \int_{r_1}^{r_2} \frac{G_N M m}{r^2} dr = -G_N M m \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

In the chapter on energy, we had also defined a potential energy function by introducing a reference state. The potential energy of a system in an arbitrary state was given with respect to the potential energy in reference state taken as zero.

In the present case, the usual practice is to use the state when the two masses are infinitely apart since then they would have zero interaction. Therefore, we set $r_1 = \infty$, and drop the subscript from r_2 to obtain the formula for the gravitational potential energy of two masses m and M when they are an arbitrary distance r apart.

$$\boxed{U(r) = -\frac{G_N M m}{r}.} \tag{2.3}$$

Note that gravitational potential energy is not defined when the two bodies are on top of each other, i.e., when their separation is zero. We do not worry about this mathematical problem since such a configuration is physically impossible.

The gravitational potential energy gives the potential energy of the two masses together. That is, Eq. 2.3 is not the potential energy of m or M. Rather, it is the potential energy in the state of the two masses separated by a distance r. We will use potential energy function in applying conservation of energy to various problems.

The total energy of the two masses interacting with each other through a gravitational force only will be obtained by adding the kinetic energies of the two masses and the gravitational potential energy given above. That is, if the mass M has a speed V and the mass m a speed v at the time they are separated by a distance r, the total energy E of the two will be

$$\boxed{E = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 - \frac{G_N M m}{r}.} \tag{2.4}$$

When the two masses move so that their separation distance changes, their speeds must also change so that remains unchanged. Note that if we were working only on mass m, then the energy of mass m will be

$$\boxed{E_m = \frac{1}{2}mv^2 - \frac{G_N M m}{r} \text{(Energy of } m \text{ only).}}$$

And, when we are working with only mass M, the energy of M will be

$$\boxed{E_M = \frac{1}{2}MV^2 - \frac{G_N M m}{r} \text{(Energy of } M \text{ only).}}$$

But their combined energy is as given in Eq. 2.4. The energy of the combined system is not equal to the sum of the energy of individual parts since the parts interact with each other and you must not count the energy of the same interaction twice.

Example 2.3.1. Placing a satellite in a geocentric orbit about the Earth. A satellite of mass 1000 kg is placed in a geocentric orbit at an altitude of approximately 30,000 km from Earth. How much energy was expended to accomplish that? Mass of Earth $= 5.97 \times 10^{24}$ kg.

Solution. The energy for sending satellites above the Earth is usually supplied by burning fuel as we have discussed in the chapter on impulse and momentum. The energy goes in lifting the material of the satellite from the surface of the Earth to the orbit. The burning of the fuel gives some initial kinetic energy to the satellite. We assume that the satellite starts out with some kinetic energy at the surface of the Earth which is converted to the greater potential energy when the satellite is in the orbit. We will calculate the required kinetic energy at the surface of the Earth as a measure of the energy needed to send the satellite to the orbit. Let us label quantities at the surface of the Earth by a subscript 1 and the corresponding quantities when the satellite is in the orbit by 2. Then, the required energy is

$$W_{\text{needed}} = K_1.$$

Now, from the conservation of energy of the satellite, we have

$$K_1 + U_1 = K_2 + U_2.$$

Therefore, we have

$$K_1 = K_2 + U_2 - U_1 = \frac{1}{2}mv_2^2 - \frac{G_N Mm}{r_2} + \frac{G_N Mm}{r_1}$$

From the given information in the problem, we can obtain the change in the potential energy, but we do not have the information for v_2 given in the problem. We can obtain v_2 from the equation of motion of the satellite when it is in the circular orbit. From the equation of motion of the satellite in a circular orbit at radius r_2 we have

$$m\frac{v_2^2}{r_2} = \frac{G_N Mm}{r_2^2}$$

Therefore, the energy equation becomes

$$K_1 = -\frac{G_N Mm}{2r_2} + \frac{G_N Mm}{r_1}$$

Now, we are ready to put in the numbers, and obtain the numerical answer.

$$\begin{aligned} W_{\text{needed}} &= \left(6.67 \times 10^{-11}\frac{\text{N.m}^2}{\text{kg}^2}\right) \times 5.97 \times 10^{24}\ \text{kg} \times 1000\ \text{kg}\times \\ &\quad \left(\frac{1}{6.37 \times 10^6\text{m}} - \frac{1}{2(6.37 \times 10^6\text{m} + 30 \times 10^6\text{m})}\right) \\ &= 5.7 \times 10^{10}\ \text{J}. \end{aligned}$$

How does this energy compare to the chemical energy in gasoline? Googling for the energy in gasoline we find that one kilogram of conventional gasoline contains approximately 4.4×10^7 J. Therefore, we would need energy in approximately 750 kg gasoline to put a 1000-kg satellite into the geosynchronous orbit 30,000 kg above the surface of Earth.