2.3 APPLICATIONS OF GAUSS'S LAW

Gauss's law is very helpful in determining the expressions for the electric field even though the law is about the electric flux and not directly about the electric field. It turns out that in situations that have a certain symmetry in the charge distribution we can deduce the electric field based on a knowledge of the electric flux. In these situations an application of the Gauss's law can make a complicated problem very simple. Therefore, it is worthwhile for you to learn when and how you may be able to use Gauss's law to find electric field.

Basically, there are only three types of problems that have the requisite symmetry so that Gauss's law can be fruitfully used to deduce the electric field. They are as follows.

- A charge distribution with a spherical symmetry
- A charge distribution with a cylindrical symmetry
- A charge distribution with a planar symmetry

In each case, the symmetry of the charge distribution dictates the particular directions and functional forms of the electric field. To exploit the symmetry we will perform the calculations in appropriate coordinate systems and imagine the right kind of Gaussian surface for that symmetry.

The symmetry leads to the transformation of the flux integral into a product of the magnitude of the electric field and an appropriate area as we will see below. When you use this flux in the expression for the Gauss's law you obtain an algebraic equation which can be easily solved for the magnitude of the electric field which looks like:

$$\boxed{E_P \sim \frac{Q_{enc}}{\epsilon_0 \times \text{Area}}.}$$

The direction of the electric field at the field point P is obtained from the symmetry of the charge distribution and the type of charge in the distribution.

For each of three type of symmetries we will study, I will present four separate steps in the argument. They are

- Consequences of symmetry
- Gaussian surface and flux calculation

- Using Gauss's law
- Computing enclosed charge

2.3.1 Charge Distribution with Spherical Symmetry

A charge distribution has a spherical symmetry if the density of charge depends only on the distance from a center and not on the direction in space. For instance, if a sphere of radius R is uniformly charged with charge density ρ_0 then the distribution has a spherical symmetry as shown in Fig. 2.14(a). On the other hand, if a sphere of radius R is charged so that top half of the sphere has uniform charge density ρ_1 and the bottom half a uniform charge density $\rho_2 \neq \rho_1$, then the sphere does not have a spherical symmetry because the charge density depends on the direction (Fig. 2.14(b)). So, it is not the shape of an object but rather the shape of the charge distribution that determines whether or not a system has a spherical symmetry!

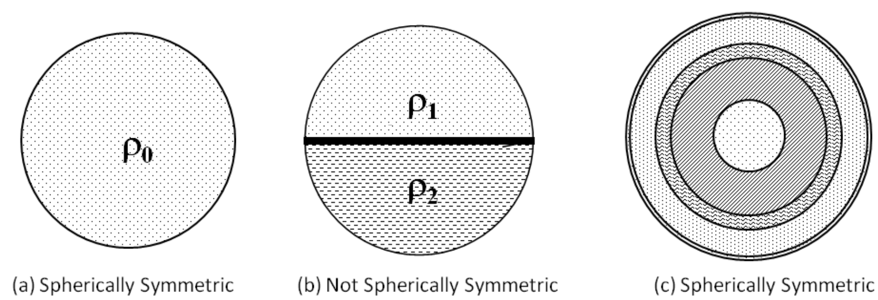

Figure 2.14: Illustrations of spherically symmetric and non-symmetric situations. Charges placed on spherical-shaped objects does not necessarily mean the charges are distributed with spherical symmetry. The spherical symmetry occurs only when the charge density does not depend on the direction. In (a) charges are distributed uniformly in a sphere. In (b), the upper half the sphere has a different charge density than the lower half. Therefore (b) does not have spherical symmetry. In (c) the charges are in spherical shells of different charge densities, which means that charge density is only a function of radial distance from the center, and therefore, has a spherical symmetry.

In Fig. 2.14(c), a sphere with four different shells, each with its own uniform charge density is shown. Although this is a situation where charge density in the full sphere is not uniform, but since the charge density function depends only on the distance from the center and not on the direction, this charge distribution does have a spherical symmetry.

One good way of telling whether or not you have a spherical symmetry in your problem is to look at the charge density function in the spherical coordinates, $\rho(r, \theta, \phi)$. If the charge density is only a function of r, that is $\rho(r)$, independent of θ and ϕ, then you have a spherical symmetry.

Consequences of Symmetry

In all spherically symmetric cases, the electric field at any point must be radially directed, either towards the center or away from the center, because there are no preferred directions in the charge distribution. Therefore, using the spherical coordinates the the origin at the center of the spherical charge distribution we can write down the expected form of the electric field at a space point P located at a distance r from the center.

$$\boxed{\text{Spherical symmetry: } \vec{E}_P = E_P(r)\hat{u}_r,} \tag{2.9}$$

where $\hat{u}_r$ is the unit vector pointed in the direction from the origin to the field point P. The radial component E_P of the electric field can be positive or negative. When $E_P > 0$, the electric field at P would be pointed away from the origin, and when $E_P > 0$, the the electric field at P would be pointed towards the origin.

Gaussian Surface and Flux Calculations

We can now use the form of the electric field given in Eq. 2.9 to obtain the flux of this electric field through the Gaussian surface. In the case of spherical symmetry, the Gaussian surface of choice is a closed spherical surface that contains the field point P and has the same center as the center of charge distribution. Then, the direction of the area vector of an area element on the Gaussian surface at any point is parallel to the direction of the electric field there, since they are both radially out as shown in Fig. 2.15.

At each patch, the electric flux will simply be a product of the electric field and the area of the patch.

$$\Delta\Phi_{\text{patch}} = E_{\text{patch}} \times \Delta A. \tag{2.10}$$

But, the magnitude of electric field at all points of the Gaussian surface are equal since all points of this surface are at the same distance from the origin at the field point P. Therefore, the flux through any patch of the Gaussian surface can be written in terms of the magnitude of the electric field at the field point P and the area of the patch under consideration.

$$\Delta\Phi_{\text{patch}} = E_P \times \Delta A. \quad \text{since } E_{\text{any patch}} = E_P, \tag{2.11}$$

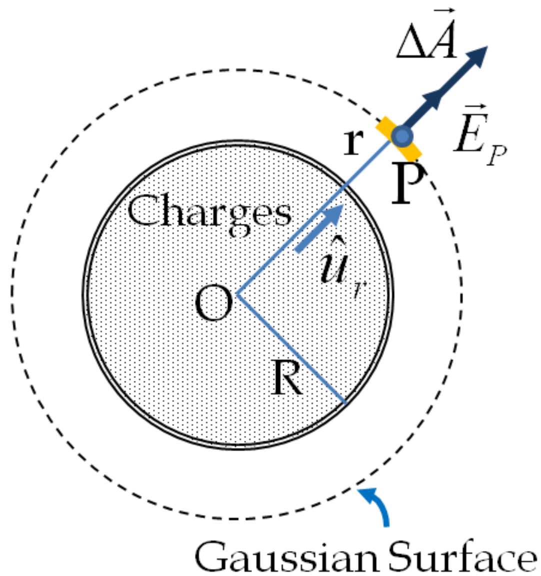

Figure 2.15: The electric field at any point of the spherical Gaussian surface for a spherical symmetric charge distribution is parallel to the area element vector at that point, giving flux as product of the magnitude of electric field and the value of the area. Note that the radius R of the charge distribution and the radius r of the Gaussian surface are different quantities.

where E_P is the magnitude of the electric field vector at point P. Hence, the electric field flux through the chosen spherical Gaussian surface will be

$$\Phi = E_P \times 4\pi r^2. \tag{2.12}$$

This shows that the electric flux through the entire Gaussian surface can be written in terms of the electric field at the field point P and an area of the surface.

Using Gauss's Law

According to Gauss's law, the flux through a closed surface is equal to the total charge enclosed within the closed surface divided by the permittivity ϵ_0 of vacuum. Let q_{enc} be the total charge enclosed inside the distance r of origin, which is the space inside the Gaussian spherical surface of radius r. This gives the following relation for the Gauss's law here.

$$4\pi r^2 E_P = \frac{q_{enc}}{\epsilon_0}. \tag{2.13}$$

Hence, the electric field at point P that is a distance r from the center of a spherically symmetric charge distribution has the following magnitude and direction.

$$\text{Magnitude:} \quad E_P(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{enc}}{r^2} \tag{2.14}$$

$$\text{Direction:} \quad \text{radial from O to P or from P to O} \tag{2.15}$$

The direction of the field at point P depends on whether the charge in the sphere is positive or negative. For net positive charge enclosed within the Gaussian surface, the direction is from O to P and for a net negative charge the direction is from P to O.

Computing Enclosed Charge

The charge of a spherical charge distribution enclosed q_{enc} depends on the distance r of the field point relative to the radius of the charge distribution R. If point P is located outside the charge distribution, that is, if $r \geq R$, then the Gaussian surface containing P will enclose all charges in the sphere. In this case, q_{enc} will equal the total charge in the sphere. On the other hand, if point P is within the spherical charge distribution, that is, if $r < R$, then the Gaussian surface encloses smaller sphere than the sphere of charge distribution. In this case, q_{enc} would be less than the total charge present in the sphere. Referring to Fig. 2.16 we can write q_{enc} as

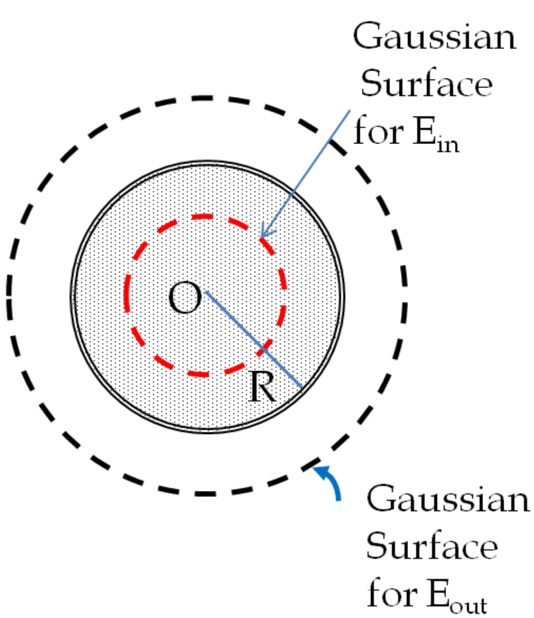

Figure 2.16: Gaussian surfaces for $\vec{E}_{out}$ and $\vec{E}_{in}$.

$$q_{enc} = \begin{cases} q_{\text{tot}}(\text{total charge}) \text{ if } r \geq R \\ q_{\text{within } r<R}(\text{only charge within } r < R) \text{ if } r < R \end{cases} \tag{2.16}$$

The field at a point outside the charge distribution is also called $\vec{E}_{out}$ and the field at a point inside the charge distribution is called $\vec{E}_{in}$.

Specializing to the two types of field points, either inside or outside the charge distribution, we can now write Eq. **??** as

$$\text{P outside sphere} \quad E_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}}}{r^2} \tag{2.17}$$

$$\text{P inside sphere} \quad E_{\text{in}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{within } r<R}}{r^2} \tag{2.18}$$

Note that the electric field outside a spherically symmetric charge distribution is identical with that of a point charge at the center that has a charge equal to the total charge of the spherical charge distribution. This is remarkable since the charges are not located at the center only.

We will now work out specific examples of spherical charge distributions, starting from the case of a uniformly charged sphere.

Example 2.3.1. Uniformly Charged Sphere. A sphere of radius R has a uniform volume charge density ρ_0. Find the electric field at a point outside the sphere and at a point inside the sphere.

Solution. Here the charge distribution has a spherical symmetry since the charge density does not depend on the direction. Therefore, we can use Eq. 2.17 and 2.18 for calculation of the electric field magnitudes at field points that are outside and inside the sphere, respectively. The answer for electric field amplitude can be written down immediately for a point outside the sphere, to be labeled E_{out} and a point inside the sphere, labeled E_{in}.

$$E_{out} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{r^2}, \quad q_{tot} = \frac{4}{3}\pi R^3\rho_0 \tag{2.19}$$

$$E_{in} = \frac{q_{enc}}{4\pi\epsilon_0 r^2} = \frac{\rho_0 r}{3\epsilon_0}, \text{ since } q_{enc} = \frac{4}{3}\pi r^3\rho_0 \tag{2.20}$$

It is interesting to note that the magnitude of the electric field increases inside the material as you go out since the amount of charge by the imagined spherical surface, i.e. the Gaussian surface, increases with the volume enclosed which goes as r^3 while the surface area increases only as r^2. The magnitude of the electric field outside the sphere decreases as you go away from the charges since the included charge remains the same but the distance increases. Fig. 2.17 displays the variation of the magnitude of the electric field with distance from the center of a uniformly charged sphere.

The direction of the electric field at any point P is radially outward from the origin if ρ_0 is positive, and inward, i.e. towards the center, if ρ_0 is negative. The electric field at some representative space points are displayed in Fig. 2.18 whose radial coordinates r are $r = R/2$, $r = R$, and $r = 2R$.

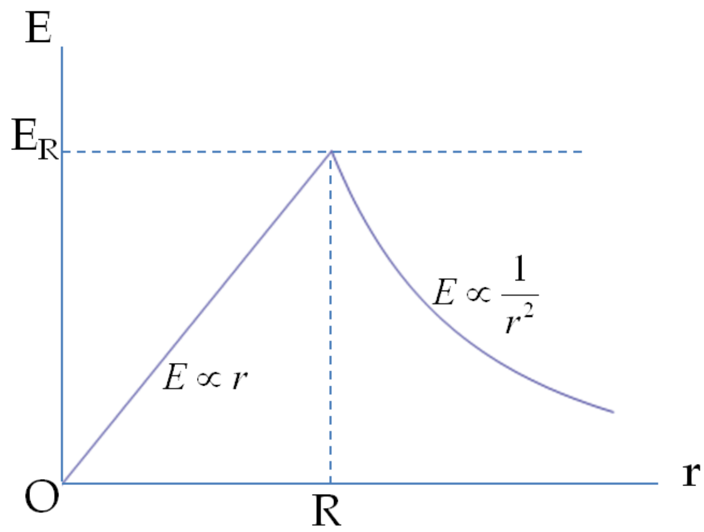

Figure 2.17: Electric field of a uniformly charged non-conducting sphere increases inside the sphere to a maximum at the surface, and then decreases as $1/r^2$. Here $E_R = \frac{\rho_0 R}{3\epsilon_0}$.

Example 2.3.2. Non-Uniformly Charged Sphere. A non-conducting sphere of radius R has a non-uniform charge density that varies with the distance from its center as given by the following function.

$$\rho(r) = ar^n, \quad (r \leq R;\ n \geq 0).$$

where a is a constant. We require $n \geq 0$ otherwise the charge density will be undefined at $r = 0$. Find the electric field at a point outside the sphere and at a point inside the sphere.

Solution. Since the given charge density function has only a radial-distance dependence and no dependence on direction, we have a spherically symmetric situation. Therefore, the magnitude of electric field at any point will be as given by Eq. 2.14 and the direction will be radial. We just need to find the enclosed charge q_{enc}, which will depend on the location of the field point.

A note about symbols: we will use r' for locating charges in the charge distribution and r for locating the field point(s) at the Gaussian surface(s). The letter R is used for the radius of the charge distribution.

As charge density is not constant here, we will need to integrate the charge density function over the volume enclosed by the Gaussian surface. Therefore we set up the problem for charges in one spherical shell, say between r' and $r' + dr'$ as shown in Fig. 2.19. The volume of the infinitesimal width shell of charges will be equal to the product of the area of surface $4\pi r'^2$ and the thickness dr'. Multiplying the

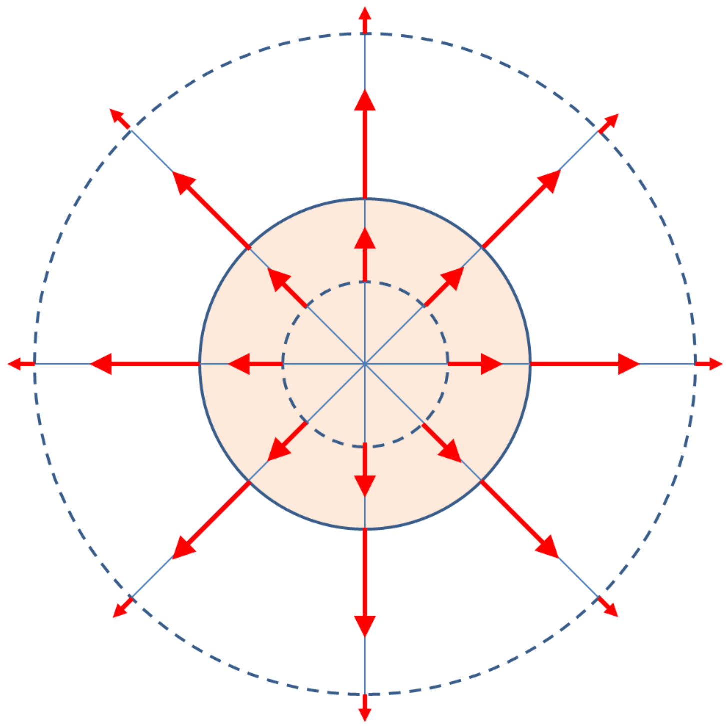

Figure 2.18: Electric field vectors inside and outside a uniformly charged sphere.

volume with the density at this location, which is ar'^n, gives the charge in the shell

$$dq = ar'^n \times 4\pi r'^2 dr'$$

Case: Field at a point outside the charge distribution.

In this case, the Gaussian surface, which contains the field point P, has a radius r that is greater than the radius R of the charge distribution, $r > R$. Therefore, all charges of the charge distribution will be enclosed within the Gaussian surface. Note the space between $r' = R$ and $r' = r$ is empty of charges and therefore does not contribute to the integral over volume enclosed by the Gaussian surface.

$$q_{enc} = \int dq = \int_0^R ar'^n \times 4\pi r'^2 dr' = \frac{4\pi a}{n+3} R^{n+3}.$$

This is used in Eq. 2.14 to obtain the electric field at a point outside the charge distribution as

$$\vec{E}_{out} = \left[\frac{aR^{n+3}}{\epsilon_0 (n+3)} \right] \frac{1}{r^2} \hat{u}_r,$$

where $\hat{u}_r$ is a unit vector in the direction from the origin to the field point at the Gaussian surface.

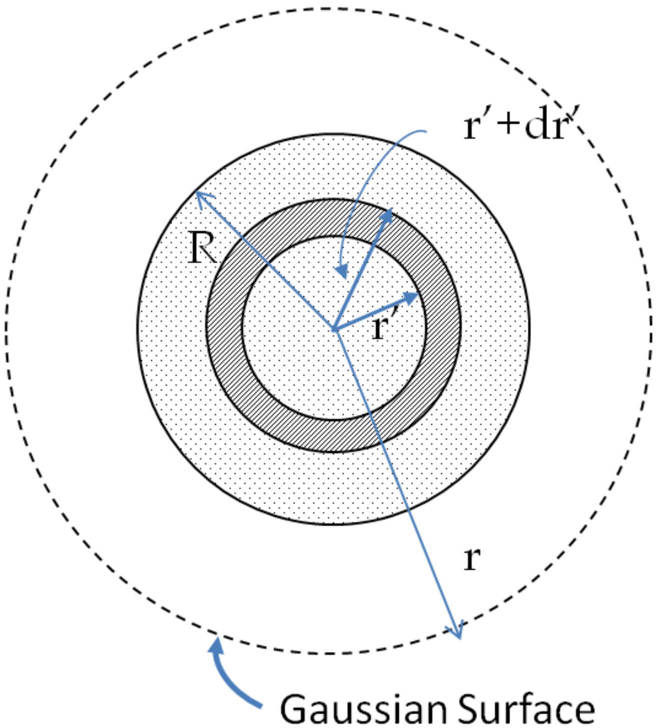

Figure 2.19: Spherical symmetry with non-uniform charge distribution. In this type of problems, we need four radii: R the radius of the charge distribution, r the radius of the Gaussian surface, r' the inner radius of the spherical shell and $r' + dr'$ the outer radius of the spherical shell. The spherical shell is used to calculate the charge enclosed within the Gaussian surface. The range for r' will be from 0 to r for the field at a point inside the charge distribution and from 0 to R for the field at a point outside the charge distribution. If $r > R$, then the Gaussian surface encloses more volume than the charge distribution, but the additional volume does not contribute to q_{enc}.

Case: Field at a point inside the charge distribution

The Gaussian surface will now be buried inside the charge distribution, with $r < R$. Therefore, only those charges in the distribution that are within a distance r of the center of the spherical charge distribution will count in q_{enc}.

$$q_{enc} = \int_0^r ar'^n \times 4\pi r'^2 dr' = \frac{4\pi a}{n+3} r^{n+3}.$$

Now, using Eq. 2.14, we find the electric field at a point inside a point which is a distance r from the center and lies within the charge distribution as

$$\vec{E}_{in} = \left[\frac{a}{\epsilon_0 (n+3)}\right] r^{n+1} \hat{u}_r,$$

where the direction information is included using the unit radial vector.

2.3.2 Charge Distribution With A Cylindrical Symmetry

A charge distribution has a cylindrical symmetry if the following two conditions are satisfied:

1. The charge is distributed in a cylindrical shape, either as a volume of a cylinder or as a cylindrical shell, extending to infinity in both directions along its axis.

2. the charge density may only depend upon the distance s from the axis and must not vary along the axis or with direction in the plane perpendicular to the axis.

Figure 2.20 shows four situations in which charges are distributed in a cylinder. A uniform charge density ρ_0 in a straight infinitely wire

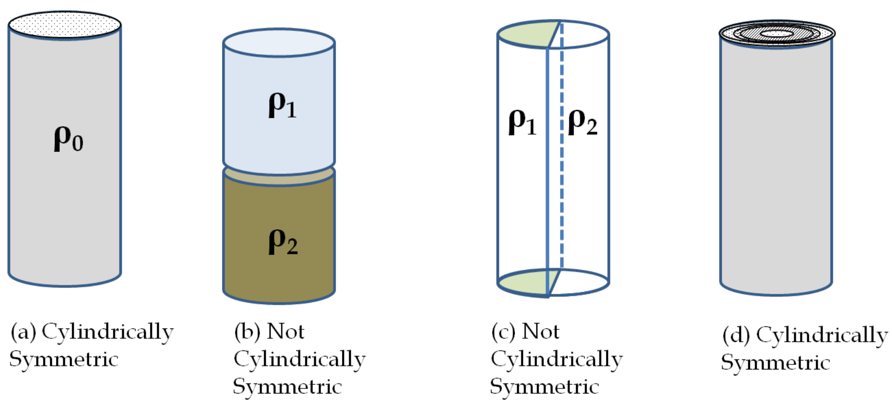

Figure 2.20: Cylindrical symmetry. To determine if a given charge distribution has a cylindrical symmetry, you look at the cross-section of an "infinitely long" cylinder. If the charge density does not depend on the polar angle of the cross-section or along the axis, then you have a cylindrical symmetry. In (a) charge density is constant in the cylinder, in (b) upper half of the cylinder has a different charge density that the lower half, in (c) left half of the cylinder has a different charge density than the right half, and in (d) charges are constant in different cylindrical rings but the density does not depend on the polar angle. Cases (a) and (d) have cylindrical symmetry and (b) and (c) do not.

has a cylindrical symmetry, and so does an infinitely long cylinder with constant charge density ρ_0. An infinitely long cylinder that has different charge densities along its length, e.g. a charge density ρ_1 for $z > 0$ and $\rho_2 \neq \rho_1$ for the $z < 0$, does not have a cylindrical symmetry, neither does a cylinder in which charge density varies with the direction, e.g. a charge density ρ_1 for $0 \leq \theta < \pi$ and $\rho_2 \neq \rho_1$ for $\pi \leq \theta < 2\pi$. A system with concentric cylindrical shells each with uniform charge densities, albeit different in different shells, as in Fig.

2.20(d) does have a cylindrical symmetry if they are infinitely long.

The best way to tell whether a system has a cylindrical symmetry is to first make sure that the charge density does not vary along the axis of the cylinder, and then you look at the cross-section map of the charge density. If the charge density is a function only of the distance from the axis, then the system has a cylindrical symmetry.

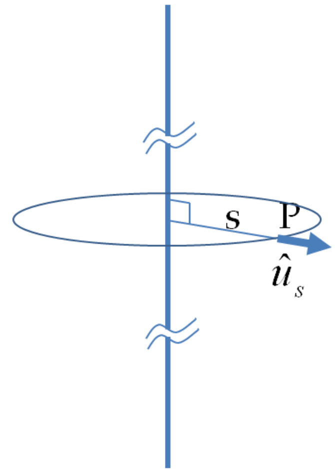

Figure 2.21: Electric field in a cylindrically symmetric situation depends only on the distance from the axis. The direction of the electric field is pointed away from the axis for positively charged and towards the axis for negative charges.

Consequence of Symmetry:

In all cylindrically symmetric cases, the electric field $\vec{E}_P$ at any point P must be directed perpendicular to the axis of the cylinder, and can only depend on the distance from the axis.

$$\boxed{\text{Cylindrical symmetry: } \vec{E}_P = E_P(s)\hat{u}_s.} \tag{2.21}$$

where s is the distance from the axis, and $\hat{u}_s$ is a unit vector directed perpendicular away from the axis as shown in Fig. 2.21.

Gaussian Surface and Flux Calculation:

To make use of the direction and functional dependence of electric field, we choose a closed Gaussian surface in the shape of a cylinder around the axis of the cylinder of the charge distribution. The flux through a cylindrical Gaussian surface of radius s and height L, having the same axis as the axis of the cylinder of charge distribution, is easy to compute if we divide our task into two parts: (a) a flux through the flat ends and a flux through the curved surface as displayed in Fig. 2.22.

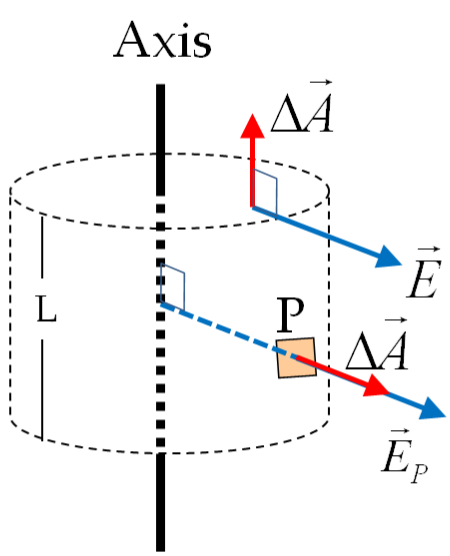

Figure 2.22: Gaussian surface in the case of a cylindrical symmetry. Electric field at a patch is either parallel or perpendicular to the normal to the patch of the Gaussian surface.

The flux through the flat ends must be zero since electric field is perpendicular to the area vectors of the patches on the ends.

$$\Phi_E^{\text{ends}} = \sum_{\text{patches}} \vec{E} \cdot \Delta\vec{A} = 0 \quad (\text{since } \vec{E} \perp \Delta\vec{A}) \tag{2.22}$$

The flux through the curved side of the Gaussian surface is simply the product of the magnitude of electric field $E(s)$ at any point of this surface, including point P, and the area of the surface, since $\vec{E} \parallel \Delta\vec{A}$ at all points.

$$\Phi_E^{\text{side}} = \sum_{\text{patches}} \vec{E} \cdot \Delta\vec{A} = E_P(s) \times 2\pi s L. \tag{2.23}$$

Therefore, the flux through the closed cylindrical surface is given by the following.

$$\Phi_E^{\text{closed surface}} = \Phi_E^{\text{ends}} + \Phi_E^{\text{side}} = E_P(s) \times 2\pi s L. \tag{2.24}$$

Using Gauss's Law:

According to the Gauss's law, this must equal the amount of charge within the volume enclosed by this surface divided by the permittivity of free space. When you do the calculation for a cylinder of length L, you will find that q_{enc} of the Gauss's law would be directly proportional to L. Let us write it as charge per unit length (λ_{enc}) times length L.

$$q_{enc} = \lambda_{enc} L. \tag{2.25}$$

Hence, Gauss's law for any cylindrically symmetric charge distribution yields the following result for the magnitude of the electric field a distance s away from the axis as:

$$\text{Magnitude:} \quad E_P(s) = \frac{\lambda_{enc}}{2\pi\epsilon_0}\frac{1}{s}. \tag{2.26}$$

The charge per unit length λ_{enc} will depend on whether the field point is inside the cylinder of charge distribution or outside the charge distribution, just as we have seen for the spherical distribution above.

Computing Enclosed Charge:

Let R be the radius of the cylinder within which charges are distributed in a cylindrically symmetric way. Let the field point P be at a distance s from the axis. Note the side of the Gaussian surface includes the field point P. For a point P outside the charge distribution, i.e. when $s > R$, the Gaussian surface will include all the charge in the cylinder of radius R and length L. On the other hand, if point P is located inside the charge distribution, i.e. when $s < R$, then only the charged within a cylinder of radius s and length L will be enclosed by the Gaussian surface.

$$\lambda_{enc}L = \begin{cases} \text{(total charge) if } s \geq R \\ \text{(only charge within } s < R\text{) if } s < R \end{cases} \tag{2.27}$$

Example 2.3.3. Uniformly Charged Wire. A thin straight wire has a uniform linr charge density λ_0. Find the electric field at a distance d from the wire.

Solution. The charge distribution has a cylindrical symmetry of the simplest kind. To apply Gauss's law, we imagine a cylindrical surface passing through P as the closed Gaussian surface. Then the enclosed charge is $\lambda_0 L$, which means that enclosed charge per unit length is:

$$\lambda_{enc} = \frac{\lambda_0 L}{L} = \lambda_0.$$

It was kind of silly to do this calculation for the charge per unit length, since we already knew it was λ_0 all the time. We did this so that we will use the same procedure in the next example. Now, that we know λ_{enc}, we can write electric field at P right away.

$$\vec{E}_P = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{d}\hat{u}_s, \tag{2.28}$$

where $\hat{u}_s$ is a unit vector, perpendicular to the axis and pointing away from the axis as shown in Fig. 2.23. The electric field at P is pointed in the direction of $\hat{u}_s$ given in Fig. 2.23 if $\lambda_0 > 0$ and in the opposite direction to $\hat{u}_s$ if $\lambda_0 < 0$.

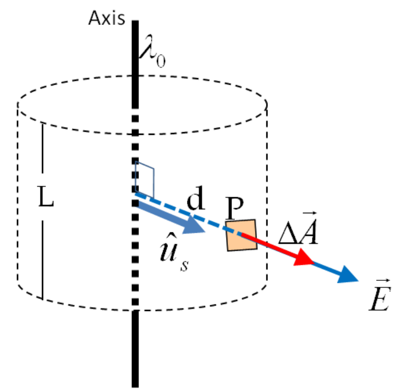

Figure 2.23: Uniformly charged infinite wire.

Example 2.3.4. Uniformly Charged Cylindrical Shell. A non-conducting cylindrical shell of radius R has a uniform surface charge density $sigma_0$. Find the electric field (a) at a point outside the shell and (b) at a point inside the shell.

Solution. (a) **Electric field at a point outside the shell**

For a point outside the cylindrical shell, the Gaussian surface will be the surface of a cylinder of radius $s > R$ and length L as shown in Fig. 2.24. The charge enclosed by the Gaussian cylinder is equal to the charge on the cylindrical shell of length L. Therefore, λ_{enc} is given by

$$\lambda_{enc} = \frac{\sigma_0 \times 2\pi RL}{L} = 2\pi R\sigma_0.$$

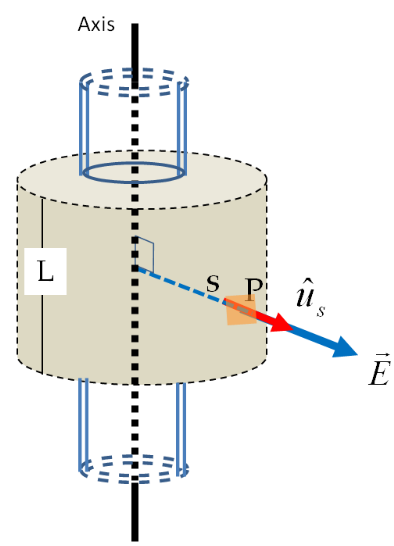

Figure 2.24: Example 2.3.4(a)

Hence, the electric field at a point P outside the shell at a distance s away from the axis is:

$$\vec{E}_P = \frac{2\pi R\sigma_0}{2\pi\epsilon_0}\frac{1}{s}\hat{u}_s \quad (s > R),$$

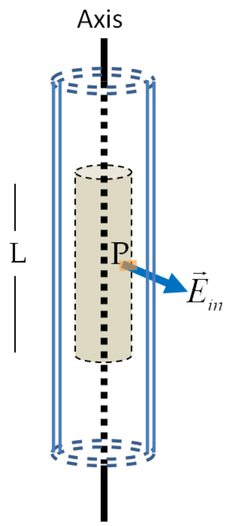

Figure 2.25: Example 2.3.4(b)

where $\hat{u}_s$ is a unit vector, perpendicular to the axis and pointing away from it as shown in the figure. The electric field at P will be pointed in the direction of $\hat{u}_s$ given in Fig. 2.24 if $\sigma_0 > 0$ and in the opposite direction to $\hat{u}_s$ if $\sigma_0 < 0$.

(b) **Electric field at a point inside the shell**

For a point inside the cylindrical shell, the Gaussian surface will be a cylinder whose radius s is less than R. That means, no charges will be included inside the Gaussian surface.

$$\lambda_{enc} = 0.$$

Since we have the cylindrical symmetry in the charge distribution of the cylindrical shell, the factorization of the magnitude of the electric field and the surface area takes place. This gives the following equation for the magnitude of the electric field E_{in} at a point whose s is less than R of the shell of charges.

$$E_{in} \times 2\pi s L = 0 \quad (s < R),$$

which gives

$$E_{in} = 0 \quad (s < R).$$

Example 2.3.5. Non-Uniformly Charged Cylinder. An infinite cylinder of radius R with a nonuniform charge density ρ which is a function of s, the distance from the axis of the cylinder. The charge density varies inside the cylinder but depends only upon the distance from the axis and not along the cylinder or on the direction perpendicular to the axis. The distance dependence of the charge density is given by $\rho(s) = \rho_0 \left(s/R\right)^n$, where ρ_0 is a constant and $n \geq 0$. Find the electric field (a) at a point outside the cylinder and (b) at a point inside the cylinder.

Solution. This problem will be done using the results of Example 2.3.4 by dividing the charge distribution in cylindrical shell-shaped elements whose radii will be labeled with s' and thickness denoted with ds' as shown in Figs. 2.26 and 2.27.

We will set up the problem of the cylinder in terms of a problem of electric field from one of these cylindrical shells and then add up the contributions from all shells that make up the cylinder. The summation will be done as an integral over the variable s'.

Fig. 2.26 shows a cross-sectional view for the case of the electric field at a point outside the cylinder. The limits of the integration are chosen to find the charges in the volume enclosed by the Gaussian surface under study. If the Gaussian surface is outside the cylinder

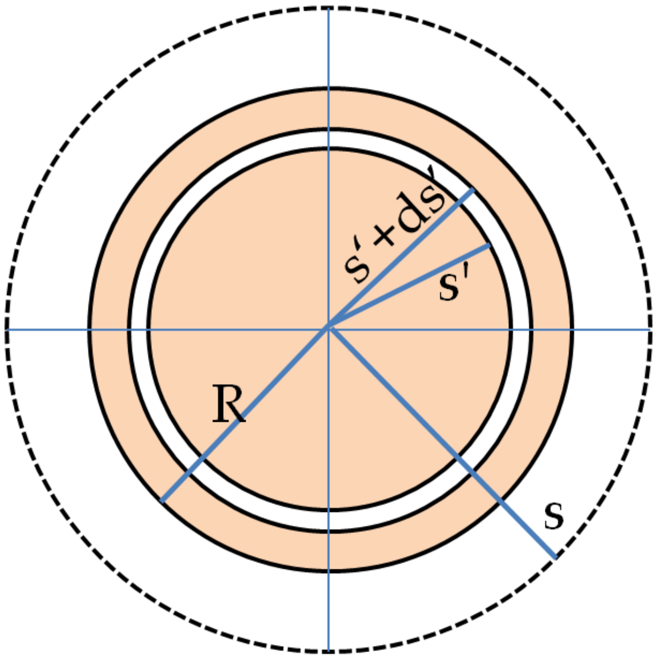

Figure 2.26: The cross-sectional view for the case of Gaussian surface outside the cylinder used for finding electric field at a point outside the cylinder.

of charges, then the integration of s' will be from 0 to R, but if the Gaussian surface falls within the cylinder of charges, then only charges up to a radius s of the Gaussian surface will be enclosed, with range of integration for s' from 0 to s.

Note that it is easy to get confused here since we have four radii in this problem: R the radius of the physical cylinder where charges exist, s the radius of the Gaussian surface which is where the field point P is located, s' the inner radius of the the cylindrical shell element and $s' + ds'$ the outer radius of the cylindrical shell element.

(a) Field at a point outside the charge distribution

The Gaussian surface will have $s \geq R$ and therefore all charges in the distribution will be enclosed within the Gaussian surface. This gives the following for the magnitude of electric field at a point outside the cylinder labeled as E_{out}.

$$E_{out} = \frac{1}{2\pi\epsilon_0}\frac{1}{L}\int_0^R \rho(s') \times 2\pi s' L ds'.$$

Substituting the given charge density $\rho(s') = \rho_0(s'/R)^n$ and carrying out this integration we find

$$E_{out} = \frac{\rho_0 R^2}{(n+2)\epsilon_0}\frac{1}{s}.$$

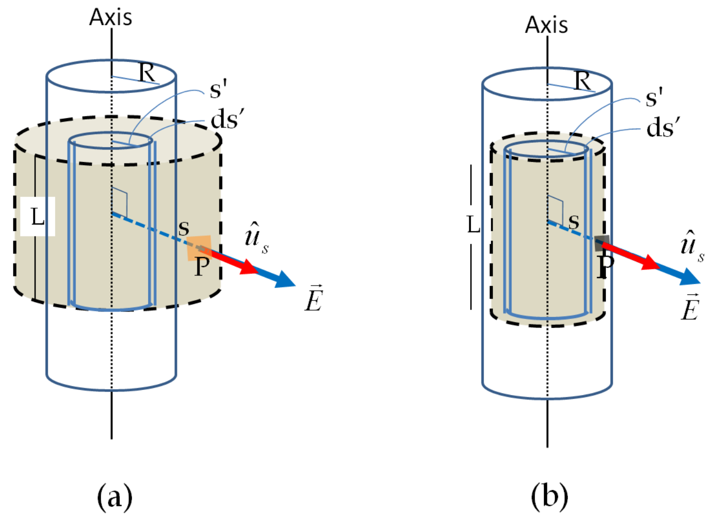

Figure 2.27: Gaussian surfaces for (a) a field point outside the cylinder and (b) a point inside the cylinder. To display the Gaussian surface and the cylindrical shell inside the cylinder, the physical cylinder has been shown with a different scale.

The direction of the electric field at point P is either parallel or anti-parallel to the radially outward vector $\hat{u}_s$ shown in Fig. 2.27(a) depending upon whether ρ_0 is positive or negative. Therefore, the electric field vector at a point P outside the cylinder is

$$\vec{E}_{out} = \frac{\rho_0 R^2}{(n+2)\epsilon_0} \frac{1}{s} \hat{u}_s.$$

(b) Field at a point inside the charge distribution

The radius of the Gaussian surface will be such that $s < R$ as shown in Fig. 2.27(b), and therefore, only the charges within a distance s of the axis will count. This gives the following for the magnitude of the electric field E_{in} at a point inside the cylinder.

$$E_{in} = \frac{1}{2\pi\epsilon_0} \frac{1}{L} \int_0^s \rho(s') \times 2\pi s' L ds'.$$

Once again, substituting for $\rho(s')$ and integrating we find the magnitude of the electric field. Putting in the direction information we obtain the electric field at a point inside the cylinder as

$$\vec{E}_{in} = \frac{\rho_0}{(n+2)\epsilon_0 R^n} s^{n+1} \hat{u}_s.$$

Further Remarks: Uniformly charged cylinder

From our results for $\vec{E}_{in}$ and $\vec{E}_{out}$, we can determine the electric field of a uniformly charged cylinder with charge density ρ_0 by setting

$n = 0$. This gives the following electric field for a uniformly charged cylinder of radius R. in the corresponding formulas.

$$\vec{E}_{in} = \frac{\rho_0}{2\epsilon_0} s\hat{u}_s \quad (s < R)$$
$$\vec{E}_{out} = \frac{\rho_0 R^2}{2\epsilon_0}\frac{1}{s}\hat{u}_s \quad (s \geq R)$$

2.3.3 Charge Distribution With A Planar Symmetry

Consequences of Symmetry

A planar symmetry of charge density is obtained when charges are uniformly spread over a large flat surface. In a planar symmetry, all points in a plane parallel to the plane of charge are in identical situation with respect to the charges.

To be concrete we will take the plane of the charge distribution to be the xy-plane and we will find the electric field at a space point P with coordinates (x, y, z). Since charge density is same at all (x, y) coordinates in $z = 0$ plane, electric field at P cannot depend on the x or y coordinates of point P. Furthermore, the symmetry in the charge distribution tells us that electric field at point P will not have any components along the x or y-axis, because the contribution from charge at one place of the plane will cancel the contribution from another place situated symmetrically on the other side of the point underneath point P as shown in Fig. 2.28. Therefore the electric

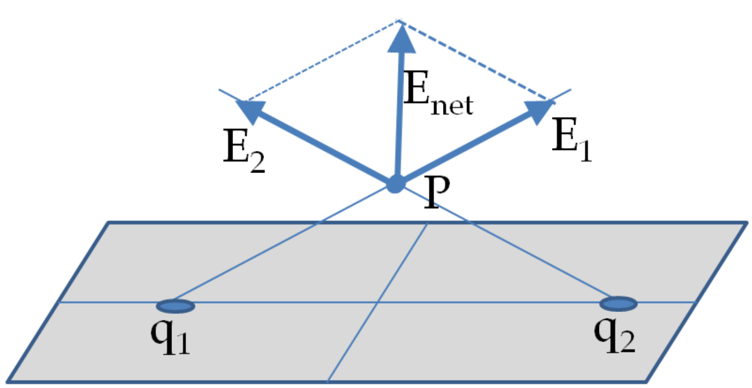

Figure 2.28: The components of the electric field parallel to a plane of charges cancel out from two charges located symmetrically from the field point P and therefore, the field at any point will be pointed vertically from the plane of charges.

field at P can only depend on the distance from the plane and has a direction either towards the plane or away from the plane. That is, the electric field at P will only have the z-component non-zero.

$$\boxed{\text{Uniform charges in } xy \text{ plane: } \vec{E}_P = E_P(z)\hat{u}_z.} \tag{2.29}$$

where z is the distance from the plane and $\hat{u}_z$ is the unit vector normal to the plane.

Gaussian Surface and Flux Calculation

In the present case, the Gaussian surface will need to be a box since the expected electric field is pointed in one direction only. To keep the Gaussian box symmetric about the plane of charges, the Gaussian box is taken to straddle the plane of the charges such that one face containing the field point P is taken parallel to the plane of the charges as shown in Fig. 2.29.

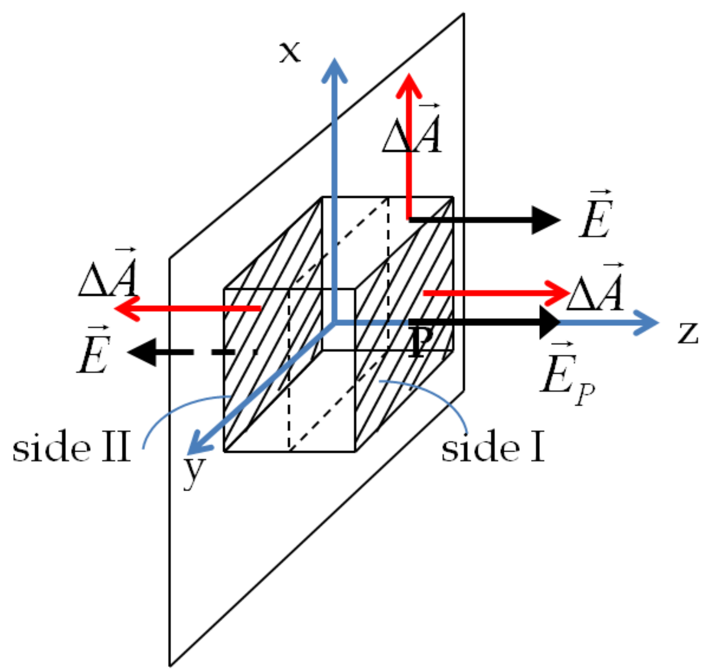

Figure 2.29: A thin charged sheet and the Gaussian box for finding electric field at the field point P. The normal to each face of the box is from inside the box to outside. On two faces of the box the electric fields are parallel to the normals and on the other four faces electric fields are perpendicular to the normals.

In Fig. 2.29 sides I and II of the Gaussian surface (the box here) that are parallel to the infinite plane has been shaded. They are the only surfaces that give rise to non-zero flux since the electric field and the normals to the surfaces to the other faces are perpendicular to each other.

Let A be the area of the shaded surface on each side of the plane and E_P be the magnitude of the electric field at point P. Since the sides I and II are at the same distance from the plane, the electric field will have the same magnitude at points in these planes, although the directions of the electric field at these points in the two planes are opposite of each other.

$$\text{Magnitude at I or II:} \;\; E(z) = E_P. \tag{2.30}$$

If the charge on the plane is positive then, the direction of electric field and the area vectors are as shown in Fig. 2.29. Therefore, we find the following for the flux of electric field through the box.

$$\Phi_E = E_P A + E_P A + 0 + 0 + 0 + 0 + 0 = 2E_P A. \tag{2.31}$$

where the zeros are for the flux through the other sides of the box. Note that if the charge on the plane is negative, the directions of electric field and normals to areas in planes I and II are opposite of each other and we will get a negative sign for the flux. I will work out the formula for the electric field based on positive charge on the sheet. The electric field would have the same magnitude if the charge on the sheet is negative but the direction would be opposite of those we will find below.

Using Gauss's Law

According to Gauss's law, this must equal q_{enc}/ϵ_0. From Fig. 2.29, we see that the charges inside the volume enclosed by the Gaussian box reside on an area A of xy plane. Hence,

$$q_{enc} = |\sigma_0| \times A, \tag{2.32}$$

where the absolute sign is placed around the surface charge densty so that the same formula will also give the magnitude if the charge was negative. Using Eqs. 2.31 and 2.32 in Gauss's law, we can immediately determine the magnitude of the electric field at a point at height z from a uniformly charged plane in xy plane.

$$\text{Magnitude:} \quad E_P = \frac{\sigma_0}{2\epsilon_0}. \tag{2.33}$$

The direction of the field depends on the sign of the charge on the plane and the side of the plane the field point P is located. The direction of electric field can be given in terms of the normal to the plane of charges - the direction of the field is along the normal away from the charged plane if the plane is positively charged and towards the plane if the plane is negatively charged.