7.3 VISCOSITY

Real liquids slow down in their flow unless some energy is continuously pumped in the liquid. The intermolecular or **cohesive forces** responsible for the dissipation of energy of flow in liquids are called **viscous forces**.

It is possible to learn about the viscous forces inside the liquid by applying a shear force on the liquid. Unlike solids, fluids do not develop restoring shear force against the external force applied tangentially to the surface. The fluids flow when subjected to a tangential force. However, when you stop applying the external force, the flow slows down due to the work by the internal viscous forces. For instance, when a wind blows over the surface of a lake, the water flows, and when the wind stops, the flow also stops due to the dissipation of energy by way of heat.

There is another way we can access the internal viscous force in a fluid, which is suggested by the wind blowing over the lake. When the wind is blowing over a lake, the speed of flow at the surface is more than the flow deeper down in the water. If you assume a layer-by-layer model of a fluid, then we can study the flow of the fluid layer by layer. We find that the speed of flow of each layer is same. This type of flow is called **laminar flow**. The laminar flow with one surface fixed and the opposite surface subjected to a shear stress will give rise to varying speeds of the layers due to the viscous forces in the liquid as shown in Fig. 7.10.

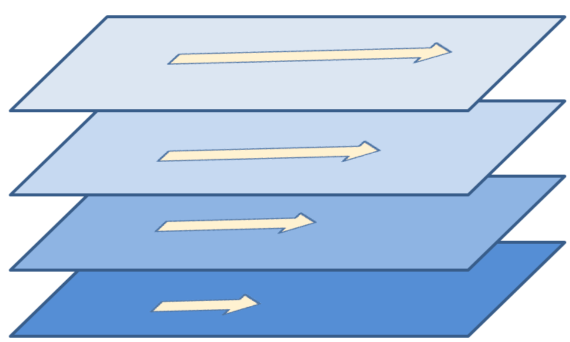

Figure 7.10: Different layers of a fluid may flow at different speeds. In this figure, the top layer is pushed along by a shear force and the bottom layer is next to a fixed plate.

Now, let us make our understanding a little more quantitative. Consider the following experiment. Place some liquid between two flat plates separated by a distance h. To describe the orientation and directions of vectors, let us introduce a coordinate system. Let the plates be parallel to the xy-plane as shown in the Fig. 7.11. Keeping the bottom plate fixed, let us apply a constant horizontal force F on the top plate to make the top surface move at a steady velocity $\vec{v}$.

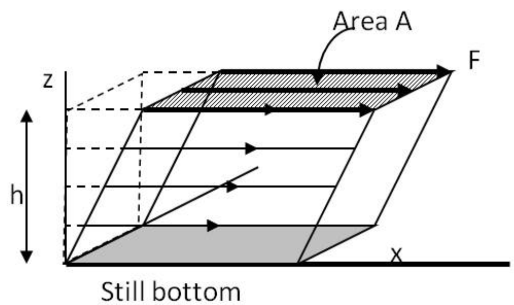

Figure 7.11: Fluid deformation under a shear stress in a duration Δt. Note the stress force $\vec{F}$ is at the top. The top layer moves a longer distance than all other layers. The layer in contact with the still bottom is not moving.

The liquid layer in contact with the bottom plate will stay at rest while the layer in contact with the top plate will move at the same velocity $\vec{v}$ as for the plate itself. As a first step, and the only step for us, we assume that the velocity of the fluid between the plates increases linearly from zero at the bottom and $\vec{v}$ at the top. This is also true experimentally when the velocity of flow is truly layer-like.

Experiments show that the horizontal force $\vec{F}$ needed to maintain a steady velocity $\vec{v}$ of the top plate is directly proportional to the contact area A and the velocity $\vec{v}$ of the top plate relative to the fixed plate, and inversely proportional to the distance h between the plates as long as the speed v is not too large.

$$\vec{F} = \eta A \frac{\vec{v}}{h} \tag{7.28}$$

where the constant of proportionality η (the Greek letter "eta") is variously called the **coefficient of viscosity**, **dynamic viscosity**, or simply viscosity. The velocity $\vec{v}$ is the velocity of flow of a layer at distance h from the fixed plate. Right next to the fixed plate, $\vec{v} = 0$. The velocity increases layer by layer, and the resulting flow is laminar.

How does the velocity vary inside?

To study the motion at a point inside the fluid, we examine a thin layer of fluid and apply Eq. 7.28 to this layer. There would be no external shear force inside the liquid. However, the shear will be provided for the flow by changing magnitudes of the viscous force between layers of the fluid since the viscous force is proportional to the velocity of flow.

For analytic treatment, let us choose Cartesian coordinates to discuss the layer-by-layer change in the velocity. Let z-axis be pointed from the fixed plate to the moving plate and x-axis point in the direction of the flow. Let the x-component of the velocity of liquid particles with the z-coordinate z be v_x and those with z-coordinate $z + \Delta z$ be $v_x + \Delta v_x$.

The layer at $z + \Delta z$ has a force $F_x^{above}\hat{u}_x$ from the layer above the $z + \Delta z$ layer and layer at z has a force $F_x^{below}\hat{u}_x$ from the layer below the z layer. The difference of the two forces must be balanced by inter-molecular forces so that the fluid molecules between z and $z + \Delta z$ do not accelerate. We call this force the viscous force or force of viscosity. Let us denote this force by $\vec{F}_{visc}$. In the present experiment, we get the x-component of the the viscous force to be

$$F_x^{visc} = F_x^{above} - F_x^{below} = \eta A \frac{\Delta v_x}{\Delta z} \tag{7.29}$$

Here the z-coordinate of the layer called "above" is $z + \Delta z + \epsilon$ with $\epsilon > 0$ and the z-coordinate of the layer called "below is $z - \epsilon$. Taking $\Delta z \to 0$ limit we find the following dependence of viscous force $\vec{F}_{visc}$ on the flow velocity $\vec{v}$ and the height z from a static layer.

$$F_x^{visc} = \eta A \frac{dv_x}{dz} \tag{7.30}$$

Many fluids, such as water, ethyl alcohol, glycerin, and most gases exhibit this simple linear dependence of shear on the gradient of velocity. They are called **Newtonian fluids**. When a fluid does not behave as postulated above, it is called a non-Newtonian fluid.

The viscosity coefficient has dimensions of $[M]/[L][T]$ as determined from the dimensions of other quantities in Eq. 7.28.

$$[\eta] = \frac{[F][h]}{[A][v]} = \frac{[M]}{[L][T]}.$$

The unit of the coefficient of viscosity in SI-system is Pa.s, which is also called poiseuille (PI) after Jean Louis Marie Poiseuille (1797-1869). A more common unit is the unit in the cgs-system of units called **poise**, which is equal to 1 dyne.sec/cm^2. The relation between the two system of units are:

$$1\text{poise} = 0.1 \text{ Pa.s} = 0.1 \text{ PI}$$

The viscosity of many common fluids are usually tabulated in centipoise as in Table 7.1.

Table 7.1: **Viscosity of some common liquids at** $20°C$

(To convert to the SI unit of viscosity Pa.s divide the numbers in the table by 1000)

Liquid	Viscosity centipoise(cP)
Benzene	0.652
Blood	~ 2.7
Carbon tetrachloride	0.969
Castor oil	984
Diethyl ether	0.233
Ethyl alcohol	1.200
Glycerol	1490
Mercury	1.554
Olive oil	84
Water	1.002

Example 7.3.1. Flow through a cylinder - Poiseuilli's law

Consider a viscous fluid flow through a cylindrical pipe of radius R placed horizontally. Let there be a difference of pressure p maintained across its ends separated by a distance L. The external force supplied by the difference of pressure at the ends is needed to overcome the effects of drag due to viscosity. The fluid layers of laminar flow in the pipe will be in the shape of concentric cylindrical shells. The fluid at the center moves fastest while the fluid next to the non-moving wall is at rest.

We would like to find the velocity profile, i.e., velocity as a function of the distance from the center for steady state flow. Towards that end consider a cylindrical shell of inner radius r and outer radius $r + \Delta r$ about the central axis. The cylindrical shell is taken to be thin enough that we can assume the fluid in the shell to move with the same speed v. The fluid just outside the shell moves at a lower speed and therefore has a retarding effect on the cylindrical shell under consideration, while the fluid just inside moves at a higher speed, and therefore tends to accelerate the fluid in the shell.

The layer immediately outside of radius $r + \Delta r$ applies a force of magnitude $F_{r+\Delta r}$ on the fluid in the cylindrical shell that attempts to slow the fluid in the shell while the force F_r from layer immediately inside radius r attempts to accelerate the fluid in the shell. In addition to these viscous forces, the force from the pressure difference at the ends attempts to accelerate the flow. In a steady state the three forces must be balanced so that there is no acceleration. Let

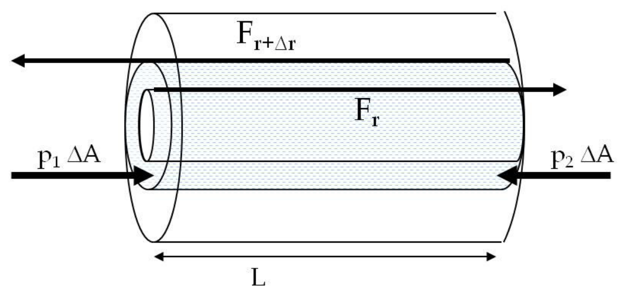

Figure 7.12: Example 7.3.1. Forces on the fluid in the cylindrical shell of inner radius r and outer radius $r + \Delta r$. Here the flow is to the right with $p_1 > p_2$. In the text we use Δp for $p_1 - p_2$.

the z-axis be pointed along the tube. Then, the z-component of the force-balancing equation gives the following.

$$p_1 \Delta A - p_2 \Delta A + F_z^r - F_z^{r+\Delta r} = 0.$$

Now, replacing the viscosity forces with their expressions in terms of the coefficient of viscosity and speeds of flow with respect to the wall, where the speed is zero, we obtain

$$\Delta p \times (2\pi r \Delta r) = -\eta 2\pi L \left[\left(r \frac{dv_z}{dr} \right)_{r+\Delta r} - \left(r \frac{dv_z}{dr} \right)_r \right],$$

where we have also replaced $p_1 - p_2$ by Δp. Note that dv_z/dr is negative so we need a minus sign on the right side to give the correct direction of the forces $\vec{F}_{r+\Delta r}$ and $\vec{F}_r$. Dividing both sides by Δr and taking $\Delta r \to 0$ limit, we find the following differential equation.

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = -\frac{\Delta p}{\eta L} = \text{constant} \tag{7.31}$$

We need to solve this differential equation in order to find the z-component of velocity v_z as a function of distance r from the center. Let us try a polynomial function for v_z as a trial solution.

$$\text{Try:} \quad v_z(r) = ar^2 + br + c.$$

If we find a, b and c then we have the desired answer. Now, since $v_z(r)$ is a maximum for $r = 0$, the coefficient of r^1 must be zero as can be seen by applying elementary result of Calculus: at a maximum or minimum the derivative is zero.

$$\left. \frac{dv_z}{dr} \right|_{r=0} \implies (2ar + b)|_{r=0} = 0 \implies b = 0.$$

Furthermore, at the wall of the pipe, i.e. when $r = R$, the velocity of flow is zero.

$$v_z(R) = aR^2 + c = 0 \implies c = -aR^2.$$

Hence, the velocity has a quadratic profile with a maximum at $r = 0$ and zero at $r = R$ as shown in Fig. 7.13.

$$v_z(r) = a\left(r^2 - R^2\right). \tag{7.32}$$

The remaining undetermined constant a can be found by substituting

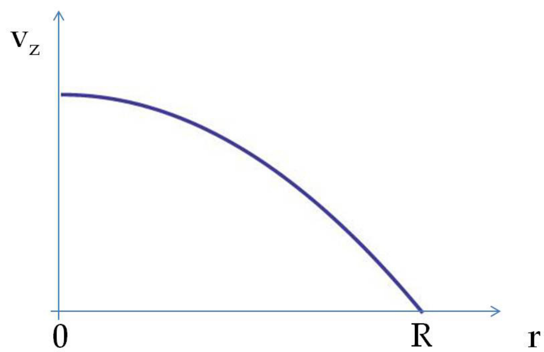

Figure 7.13: Velocity profile in a tube.

the expression for v_z in Eq. 7.31 giving

$$a = -\frac{\Delta p}{4\eta L}$$

Therefore, the velocity of flow has the following dependence on r.

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right). \tag{7.33}$$

The volume flow rate through the entire tube can be obtained by summing up the volume flow in each cylindrical shell. The volume rate of flow in the cylindrical shell is equal to the product of its cross-sectional area and the velocity of flow.

$$\begin{aligned}\text{Volume rate of flow in shell} &= \text{velocity} \times \text{cross-sectional area} \\ &= \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right) \times (2\pi r \Delta r)\end{aligned}$$

Summing over infinitesimally thick shells is done by replacing Δr by dr and integrating from $r = 0$ to $r = R$.

$$\frac{d(\text{Volume})}{dt} = \frac{\pi R^4}{8\eta L}\Delta p. \tag{7.34}$$

This relation is called the **Poiseuille's law**, named after the French physician Jean Poiseuille, who studied fluid flow experimentally in 1840's. Note the very strong dependence of the volume rate of flow on the radius of the pipe; if the radius is doubled then the volume flow rate increases 16-fold! The Poiseuille's law is useful in the study of flow of blood in the body and other engineering applications.

Example 7.3.2. Falling sphere in a viscous fluid - Stokes's law.

Stoke's law for viscous force on a falling spherical ball is an important result that finds use in many areas of physics and engineering. Consider dropping a spherical ball in a still fluid, which can be liquid or gas. As the ball falls, it experiences a drag or resistive force $\vec{F}_{\text{drag}}$ depending upon the viscosity of the fluid. Sir George Stokes found the characteristics of the drag force, which is called the **Stokes's Law**. Let R be the radius of the ball, $\vec{v}$ its velocity, and η the viscosity of fluid. Stokes's law states the following for the drag force $\vec{F}_{\text{drag}}$.

$$\vec{F}_{\text{drag}} = -6\pi\eta R\vec{v} \tag{7.35}$$

Note that the drag force by Stokes's law is directly proportional to the radius rather than the area of cross-section as one might expect. The formula of the drag force on a sphere can be guessed at by dimensional analysis.

The argument for a dimensional analysis goes as follows. We assume that the magnitude of the drag force will depend on the velocity, $F_{\text{drag}} \sim v^m$, for some power m to be determined by the dimensional argument here. We also know that it will involve the size of the ball, although we don't know if it will be R or R^2, or some other power of R, say R^n. The drag force should also be proportional to the viscosity coefficient. Writing out the dimension of the product of these dependencies of the drag force, we find

$$[\eta][v^m][R^n] = \frac{[M][L]^{n+m-1}}{[T]^{m+1}},$$

which should equal the dimension of force, $\frac{[M][L]}{[T]^2}$. Therefore, $m = 1$ and $n = 1$. The dimensional analysis helps us guess the form of the magnitude of the drag force on a spherical particle of radius R up to an undetermined constant C.

$$F_{drag} = C\eta Rv.$$

The calculations that shows that the constant C is equal to 6π is beyond this textbook. The direction of the drag force is opposite to the direction of the velocity of the ball since it is a resistive force for the motion of the ball.