13.1 AMPERE'S LAW DOES NOT WORK ALL THE TIME

The four basic laws of electricity and magnetism were discovered experimentally. While studying them **James Clerk Maxwell** (1831-1879) discovered some logical inconsistencies. The source of inconsistencies turned out to be the incompleteness of Ampere's law. Recall that according to Ampere's law the circulation of magnetic field around a loop C is proportional to the current passing through any surface (S) whose boundary is the loop C itself.

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \quad \text{(independent of surface)} \tag{13.1}$$

Although, there are infinitely many surfaces that can be attached to any loop, it is important to realize that Ampere's law stated in Eq. 13.1 is independent of the choice of surface as illustrated in Fig. 13.1.

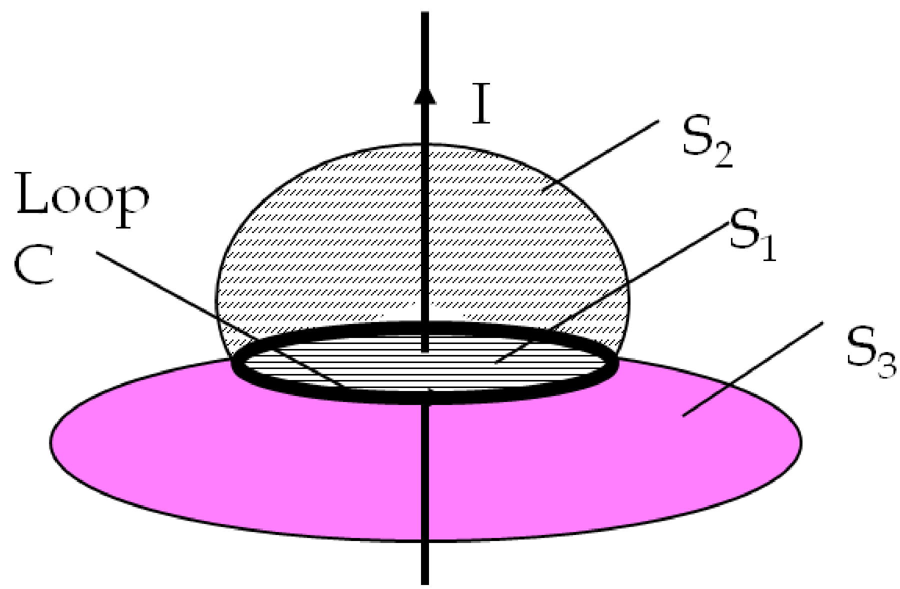

Figure 13.1: Three open surfaces S_1, S_2, S_3 attached to the same loop C. For steady current in the wire all three surfaces will give the same value for the enclosed enclosed current consistent with the Ampere's law.

Now, consider the following experiment shown in Fig. 13.2. Connect a parallel plate capacitor to an EMF source V. When the switch is closed, a time-dependent current develops in the wire. Let us now apply Ampere's law to loop C shown at a time before the capacitor is fully charged such that $I \neq 0$. With the surface S_1, you get a non-zero value for the enclosed current I, while with surface S_2 you

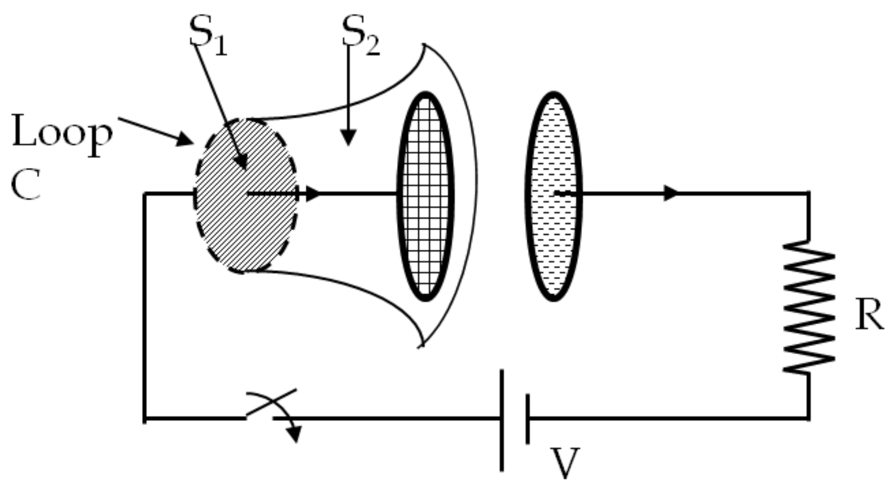

Figure 13.2: Applying Ampere's law in a non-steady situation results into inconsistent results! The currents through S1 and S2 are not equal although they have the same boundary loop C.

get a zero for the enclosed current since no current passes through it.

$$\oint_C \vec{B} \cdot d\vec{l} = \begin{cases} \mu_0 & \text{if you use surface } S_1 \\ 0 & \text{if you use surface } S_2 \end{cases} \tag{13.2}$$

Clearly, Ampere's law does not work here. You might have already suspected this because Ampere's law requires a steady current, while the current in this experiment is changing with time and not steady at all.

How can we modify Ampere's law so that it will work in all situations? This question was answered by James Clerk Maxwell. The experiment with the capacitor suggests a solution for this problem. In Fig. 13.2, you will notice that, while current passes through surface S_1, a time-dependent electric field passes through S_2. So, it would be tempting to correct Eq. 13.1 by adding the flux of E-field through a surface in addition to I_{enc}. But this will not work because when the capacitor is fully charged you get another inconsistency: although there will be no current through S_1, there will still be a non-zero flux of E-field through S_2. Maxwell suggested that rather than adding the electric flux, you will need the rate of change of electric flux to correct the Ampere's law. He suggested the form of the additional contribution to the real current I called the displacement current I_d.

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{\text{enc}} + I_{\text{d,enc}} \right) \quad \text{(independent of surface)} \tag{13.3}$$

The displacement current I_d is supposed to be the contribution from the changing electric flux through the surface attached to the loop for the circulation integral on theleft side of this equation. Specifically, the displacement current is defined to be

$$\boxed{I_d = \epsilon_0 \frac{d\Phi_E}{dt},} \tag{13.4}$$

where ϵ_0 is the permittivity of free space, and Φ_E the electric flux defined as

$$\Phi_E = \iint_{\text{Surface,S}} \vec{E} \cdot d\vec{A}. \tag{13.5}$$

Note that since surfaces attached to the loop are not closed the electric flux is not equal to the Q_{enc}/ϵ_0 as would be expected for the electric flux through a closed surface according to Gauss's law.

Recall the relationship of current I to the current density vector .

$$I = \iint_S \vec{J} \cdot d\vec{A}. \tag{13.6}$$

Similarly, we also define a displacement current density vector $\vec{J}_d$.

$$I_d = \iint_S \vec{J}_d \cdot d\vec{A}. \tag{13.7}$$

Putting Eq. 13.7 in Eq. 13.4 and using Eq. 13.5 we deduce

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{13.8}$$

The modified Ampere's law, Eq. 13.3 also explains correctly even the time-dependent situations depicted in Fig. 13.2 - the surface S_1 has contribution from current and the surface S_2 has equal contribution from the displacement current of the time-varying electric field. We will call the modified Ampere's law as Ampere-Maxwell's law.

Example 13.1.1. Displacement current in a charging capacitor.

A parallel-plate capacitor C whose plates are separated by a distance d is connected to a resistor R and a battery of voltage V. The current starts to flow at $t = 0$. (a) Find the displacement current density between the capacitor plates at time t. (b) From the displacement current between the plates find the current in the circuit, and compare the answer to the expected current in the wires of the RC-circuit.

Solution. (a) The voltage between the plates at time t is given by

$$V_C = \frac{1}{C} q(t) = V_0 \left(1 - e^{-t/RC}\right).$$

Let the z-axis point from the positive plate to the negative plate. Then, the z-component of the electric field between the plates as a function of time t is

$$E_z(t) = \frac{V_0}{d}\left(1 - e^{-t/RC}\right).$$

Therefore the z-component of the displacement current density between the plates is

$$J_{dz} = \epsilon_0 \frac{\partial E_z(t)}{\partial t} = \epsilon_0 \frac{V_0}{d} \times \frac{1}{RC}\, e^{-t/RC} = \frac{V_0}{RA}\, e^{-t/RC}.$$

(b) Integrating J_d over the area of cross-section, which is an area equal to the area of the plate, will yield the current.

$$I_d = \int\int \vec{J}_d \cdot d\vec{A} = \frac{V_0}{R}\, e^{-t/RC}.$$

(c) From the analysis of an RC-circuit in a precious chapter, the current into the capacitor is found decrease with time after the circuit is closed as

$$I = \frac{V_0}{R}\, e^{-t/RC}.$$

This current is same as I_d we found in (b).