11.1 ELECTROMAGNETIC OSCILLATIONS - LC CIRCUIT

Equations of Motion in an LC circuit

The current in an electric circuit containing a capacitor and an inductor oscillates like a simple harmonic oscillator. A circuit containing a capacitor and and an inductor is also called an LC-circuit.

To study the oscillations of an LC-circuit consider connecting the plates of a charged capacitor of capacitance C to the ends of a coil of self-inductance L through a switch which is closed at time $t = 0$ as shown in Fig 11.1. We wish to determine the charge on the capacitor plates as a function of time.

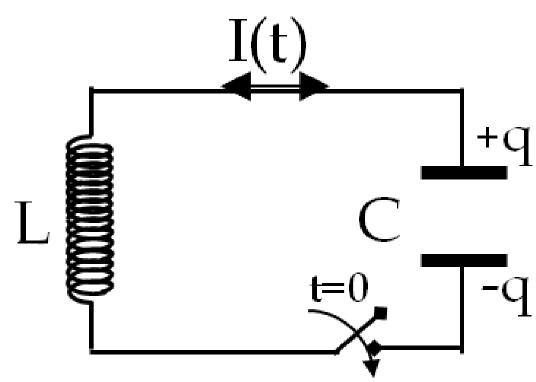

Figure 11.1: An LC- circuit. Initially the capacitor is charged when the switch is closed at $t = 0$. The current in the circuit flows from one plate to the other till the charges on the plate are neutralized and then the charges continue to flow resulting in reversal of the polarity of the capacitor. The plates are charged again and then the current flows in the opposite direction leading to the plated becoming charged in the same way as before. At the end of each cycle the net charge on the capacitor is same as at the beginning of the previous cycle.

Previously we have used the symbol $\mathcal{L}$ for the self-inductance since we were using letter L for the length of the solenoid coil. In this chapter and the next we will use the letter L for the self-inductance. We will assume that the electromagnetic induction of a circuit is mostly from the varying the magnetic field inside the solenoid coil in the circuit and the magnetic field in other parts of the circuit is negligible.

Let $\pm q(t)$ denote the charges on the plates at the time t. Since the situation is a dynamic situation we use Faraday's law to write the

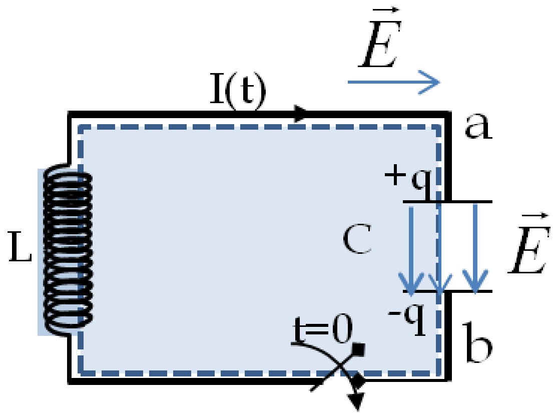

Figure 11.2: LC circuit. The Faraday's loop is calculated for the loop shown dashed. The magnetic flux through the area in the inductor dominates the magnetic flux. For the calculation of the loop integral of electric field we choose the loop direction the same as the direction of the electric field between the plates of the capacitor at the instant shown in this figure.

loop equation. To be consistent with the sign of various terms we will pick t be an instant when the charges on the plates are building up and the current is pointed to the positively charged plate as shown in Fig. 11.2. With this choice the electric current in the wires and the electric field between the plates will be in the same direction. The current $I(t)$ in the circuit at time t would be related to the charge on the positive plate by

$$I = \frac{dq}{dt}. \tag{11.1}$$

The Faraday's loop integral is taken along the electric field direction in the wires and the capacitor shown in Fig. 11.2. We will assume that the resistances of connecting wires and the solenoid wires are negligible. From Ohms's law this would mean that the electric field can be assumed to be negligible except between the plates of the capacitor. Therefore,

$$\oint \vec{E}\cdot d\vec{l} = -L\frac{dI}{dt} \implies \int_{\text{ab}} \vec{E}\cdot d\vec{l} = -L\frac{dI}{dt}, \tag{11.2}$$

where $\vec{E}$ is the electric field between the plates and $d\vec{l}$ is an element of a path from the positive plate to the negative plate. Assuming no magnetic field in the space between the capacitor plates we can say that the electric field between the plates is a conservative field and the line integral of the electric field from the positive plate to the negative plate will be equal to the the voltage V_C across the plates, which can be equated to $q(t)/C$ based on the fundamental capacitor relation.

$$\int_{\text{ab}} \vec{E}\cdot d\vec{l} = V_C = \frac{q(t)}{C}. \tag{11.3}$$

Table 11.1: Analogy between harmonic oscillator and oscillating circuit

Mechanical System	Electrical System
Mechanical, m	Inductance, L
Spring constant, k	Inverse Capacitance, $1/C$
x-component of displacement, x	Charge on capacitor, q
x-component of velocity, v_x	Current, I
Kinetic energy,$\frac{1}{2}mv^2$	Magnetic energy, $\frac{1}{2}LI^2$
Potential energy,$\frac{1}{2}kx^2$	Electric energy, $\frac{1}{2}\frac{1}{C}q^2$
Damping constant, b	Resistance, R

From 11.3 and Eq. 11.2 we obtain the following relation between $q(t)$ on the positive plate and the rate at which the current in the circuit is changing.

$$\frac{q(t)}{C} = -L\frac{dI}{dt}. \tag{11.4}$$

Taking the derivative of both sides with respect to time and using the relation between the current and charge given in Eq. 11.1 we obtain the "equation of motion" for the current I in the LC-circuit.

$$\frac{d^2I}{dt^2} = -\frac{1}{LC}\ I. \tag{11.5}$$

This equation is analogous to the equation of motion of a simple harmonic oscillator. Therefore, just as the displacement of a simple harmonic oscillator oscillates in time we expect the current in the circuit to oscillate in time also.

Replacing I by dq/dt in Eq. 11.4 we obtain the equation of motion for the charge q.

$$\frac{d^2q}{dt^2} = -\frac{1}{LC}\ q. \tag{11.6}$$

Since the voltage across the pates of a capacitor V_C is proportional to the charge, the voltage V_C will follow a similar equation of motion.

$$\frac{d^2V_C}{dt^2} = -\frac{1}{LC}\ V_C. \tag{11.7}$$

The equations of motion for q, I, and V_C are analogous to the equation of motion for a simple harmonic oscillator as displayed in Table 11.1. Thus, q, I, and V_C would oscillate with time just as the displacement of a simple harmonic oscillator oscillates about the equilibrium.

Let us take one of these three equations, say the voltage across the plates V_C, and see how the solution works out. From the analogy of the equation for the voltage V_C to the equation of motion of the harmonic oscillator, we can write the general solution as

$$\boxed{V_C(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t),} \tag{11.8}$$

with the angular frequency ω given by

$$\boxed{\omega = \frac{1}{\sqrt{LC}},} \tag{11.9}$$

where constants A_1 and A_2 are determined from the initial conditions. The initial conditions here can be in terms of the voltage V_C and the current at $t = 0$. Suppose the voltage V_C at $t = 0$ was V_0 and the current was zero, that is

$$V_C(0) = V_0 \tag{11.10}$$
$$I(0) = 0. \tag{11.11}$$

Then we can solve for the constants A_1 and A_2 and find that

$$A_1 = V_0 \tag{11.12}$$
$$A_2 = 0. \tag{11.13}$$

Therefore, in the case of the initial conditions given in Eqs. 11.10 and 11.11 the voltage across the plates will change in time as

$$V_C(t) = V_0 \ \cos(\omega t). \tag{11.14}$$

You can show that in the present case the charge on a plate of the capacitor and the current into that plate will change as

$$q(t) = C\, V_0 \ \cos(\omega t) \tag{11.15}$$
$$I(t) = -\omega\, C\, V_0 \ \sin(\omega t). \tag{11.16}$$

Energy in an LC-Circuit

The analogy between the electrical and mechanical systems also extends to the energy in the circuit with the magnetic field energy being analogous to the kinetic energy and the electric field energy to the potential energy. The conservation of energy leads to the constancy of the sum of the magnetic and electric energies. That is,

$$\frac{1}{2}LI^2 + \frac{1}{2}CV_C^2 = \text{Constant.} \tag{11.17}$$

In the example initial condition of voltage V_0 and zero current at time $t = 0$, the constant is known from the values of the voltage and current at zero time.

$$\frac{1}{2}CV_C^2 + \frac{1}{2}LI^2 = \frac{1}{2}CV_0^2. \quad \text{(Here)} \tag{11.18}$$

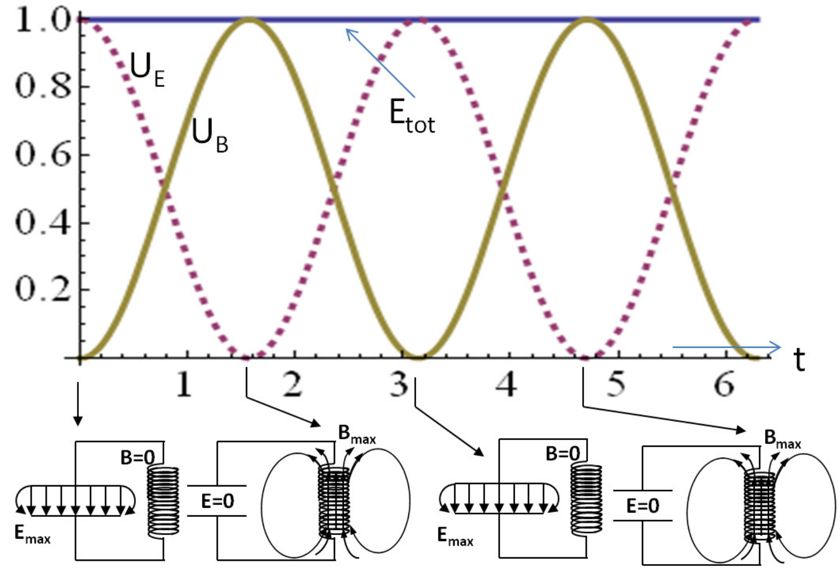

Figure 11.3: The total energy in an LC circuit oscillates between completely in the electric field between the plates of the oscillator to completely in the magnetic field. The curve with a solid line is the magnetic energy and the one with dashed line the electrical energy. The horizontal line the total energy which does not change with time.

Just as the energy of a simple harmonic oscillator oscillates between fully kinetic and fully potential, the energy in the LC-circuit oscillates from being completely in the electric field between the plates of the capacitor when $I = 0$ to completely in the magnetic field inside the inductor when the current has the largest value, $I = \omega\, C\, V_0$. In Fig. 11.3, I have plotted the energies in the electric field and the magnetic filed for the following values of constants, $C = \frac{1}{2}$ F, and $L = 2$ H, and $V_0 = 1$ V. We see that sum of the energies is constant but at sometime all of the energy is in electric field, at some other time all the energy is in the magnetic field, and the rest of the time the energy is in both fields.

Example 11.1.1. Oscillations of an LC-circuit #1. A 3-F capacitor is charged so that it contains $\pm 30\ \mu$C on its plates. It is then connected in series to a 2-H inductor through a switch. The switch is closed at t = 0. Find (a) the frequency of oscillations of the circuit, and (b) the voltage across the capacitor at t = 0.3 sec.

Solution. This example is just an application of the formulas obtained in this section.

(a) The frequency f is related to the angular frequency ω by $2\pi f = \omega$. Therefore,

$$f = \frac{1}{2\pi\sqrt{LC}} = 6.5 \times 10^{-2}\ \text{Hz}.$$

(b) The initial voltage will be related to the initial charge by $V_0 = Q_0/C = 10$ V. Therefore, the voltage at 0.3 sec will be

$$V(0.3 \text{ sec}) = (10 \text{ V}) \ \cos(2\ \pi \times 0.065 \text{ Hz} \times \ 0.3 \text{ sec}) = 9.9 \text{ V}.$$

Example 11.1.2. Oscillations of an LC circuit # 2. A 40-μF capacitor is connected across a 10-V battery till it is fully charged. It is then disconnected and connected across an inductor. The current in the LC circuit so formed oscillated with a period of 5 msec. Determine (a) the inductance of the inductor, (b) the total energy of the circuit, and (c) the maximum current in the circuit. (d) If at some instant, the current is half of the maximum, what is the amount of charge at that instant on the positive plate of the capacitor?

Solution. This example is use of formulas derived in this section.

(a) From $\omega = 1/\sqrt{LC}$ we obtain the following for L in terms of the period T of the oscillations.

$$L = \frac{1}{\omega^2 C} = \frac{T^2}{4\pi^2 C} = \frac{(0.005 \text{ s})^2}{4\pi^2 \times 40 \times 10^{-6}\text{F}} = 0.016 \text{ H}.$$

(b) The total energy in the circuit will be equal to the energy at any time, say at $t = 0$, which is

$$E(t=0) = \frac{1}{2}CV_0^2 = \frac{1}{2} \times 40 \times 10^{-6}\text{F} \times (10 \text{ V})^2 = 2 \text{ mJ}.$$

(c) Use conservation of energy to figure out the maximum current in the circuit. That will happen when all the energy is in the magnetic field.

$$\frac{1}{2}LI_{\text{max}}^2 = 2 \text{ mJ}$$

This gives $I_{\text{max}} = 0.5$ A. (d) Use conservation of energy for arbitrary time. Writing the energy in the capacitor in terms of the charge on the capacitor instead of the voltage across the capacitor we have

$$\frac{1}{2}LI(t)^2 + \frac{1}{2C}q(t)^2 = 2 \text{ mJ} \equiv E_{\text{tot}}.$$

Since the current at this time is given to be $I = 0.25$ A, we can find q at that instant.

$$q = \sqrt{2CE_{\text{tot}} - I^2/\omega^2} = 35\ \mu\text{C}.$$