# 3.1 ELECTRIC POTENTIAL ENERGY

## 3.1.1 Electric Potential Energy

Consider an electric charge $q$ fixed at some place and move another charge $Q$ towards $q$ in such a manner that at each instant the applied force $\vec{F}$ exactly balances the electric force $\vec{F}_e$ on $Q$ as shown in Fig. 3.1. The work done by the applied force $\vec{F}$ on the charge $Q$ would go towards changing the potential energy of $Q$. We call this potential energy the electrical potential energy of $Q$.

Suppose the charge $Q$ goes from a point $P_1$ to another point $P_2$. How much would the potential energy of $Q$ change? To find this we can work out the work done $W_{12}$ by the applied force $\vec{F}$ when the particle moves from $P_1$ to $P_2$. This requires the evaluation of the following work integral along the path of movement of the charge $Q$.

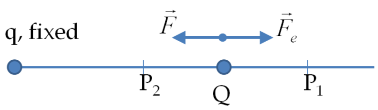


Figure 3.1: Displacement of "test" charge Q in the presence of fixed "source" charge q.

$$W_{12} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}. \tag{3.1}$$

Since the applied force $\vec{F}$ balances the electric force $\vec{F}_e$ on $Q$, the two forces would have equal magnitude and opposite directions. Therefore, the applied force will have the following analytic expression in spherical coordinates with the charge $q$ placed at the origin and $Q$ at the radial coordinate $r$ in the direction $\hat{u}_r$ from the origin.

$$\vec{F} = -\vec{F}_e = -\frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{u}_r. \tag{3.2}$$

The displacement vector element written in spherical coordinates is

$$d\vec{r} = \hat{u}_r dr.$$

Let $r_1$ and $r_2$ be the distances from $q$ to the points $P_1$ and $P_2$ respectively. Then, the work integral in Eq. 3.1 can be evaluated with the following result.

$$\begin{aligned} W_{12} &= -\frac{qQ}{4\pi\epsilon_0}\int_{r_1}^{r_2}\frac{1}{r^2}dr \\ &= \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_2} - \frac{1}{4\pi\epsilon_0}\frac{qQ}{r_1}. \end{aligned} \tag{3.3}$$

The work $W_{12}$ done against the electric force is independent of path between $P_1$ and $P_2$ as we now demonstrate by evaluating the work integral on another path between $P_1$ and $P_2$, say the path $P_1$-$P_3$-$P_4$-$P_2$ shown in Fig. 3.2. The segments $P_1$-$P_3$ and $P_4$-$P_2$ are arcs of circles of radii $r_1$ and $r_2$ respectively. Since the force on $Q$ is either

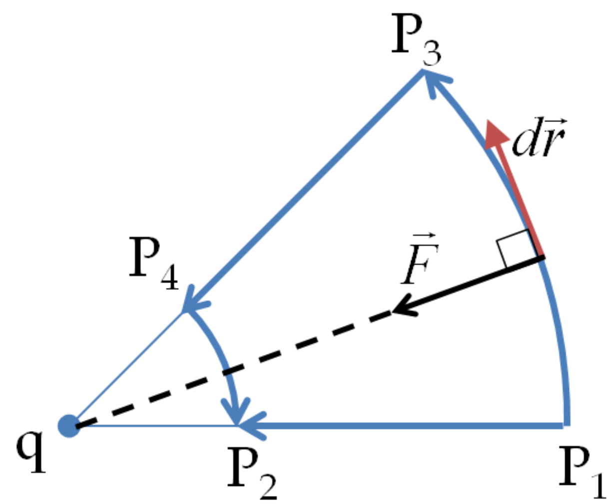


Figure 3.2: Two paths for displacement $P_1$ to $P_2$. The work on segments $P_1$-$P_3$ and $P_4$-$P_2$ are zero since electric force is normal to the displacment on these paths. Therefore work on paths $P_1$-$P_2$ and $P_1$-$P_3$-$P_4$-$P_2$ are equal.

pointed towards $q$ or away from $q$, no work will be done by a force that balances the electric force on the segments $P_1$-$P_3$ and $P_4$-$P_2$ since force vector for this force would be perpendicular to the displacement vectors on the segments of the arcs. Therefore, the only work done by the applied force will come from the segments $P_3$-$P_4$. The work on the segment $P_3$-$P_4$ works out identically to the work on the path $P_1$-$P_2$. That is,

$$W_{12}(\text{Path } P_1\text{-}P_2) = W_{12}(\text{Path } P_1\text{-}P_3\text{-}P_4\text{-}P_1).$$

This work is the negative of the work by the electric force $\vec{F}_e$ on the charge $Q$. Since the work done $W_{12}$ is independent of the paths between $P_1$ and $P_2$, the electric force is a conservative force. Recall that the work by a conservative force is also expressed as the difference in the potential energy corresponding to that force. Therefore, the work $W_{12}$ can be written as as the difference in potential energy $(U_2 - U_1)$ of the test charge $Q$ between the two points.

$$W_{12} = U_2 - U_1.$$

Examining the expression for $W_{12}$ in Eq. 3.3, we can write the expressions for the electric potential energy of $Q$ when $Q$ is at points $P_1$ and $P_2$ up to a common additive reference value. Writing the potential energy $U(r)$ of $Q$ for an arbitrary point P at a distance $r$ away from the source charge $q$ we have

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r} - U_{\text{ref}} \tag{3.4}$$

where $U_{\text{ref}}$ is a constant and serves as the zero reference for the potential energy. You can put $r = r_1$ and $r = r_2$ in this expression to obtain $U_1$ and $U_2$ respectively and verify the $W_{12} = U_2 - U_1$ as given in Eq. 3.3 since $U_{\text{ref}}$ will cancel out.

A convenient reference that relies on our common sense is that when the two charges are infinitely apart, there would be no interaction between them. Taking the potential energy of this state to be zero gets rid of the term $U_{\text{ref}}$ from the equations, and the potential energy of $Q$ when it is separated from $q$ by a distance $r$ assumes the following simple form.

$$\boxed{U(r) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r} \quad \text{(Zero reference at } r = \infty.)} \tag{3.5}$$

The formula for the electrical potential energy for a two-charge system is symmetric in $q \leftrightarrow Q$ switch. If you work out the potential energy of $q$ by moving $q$ while keeping $Q$ fixed, you will get the same formula. Thus, it is better to say that $U(r)$ is the potential energy stored in the two-charge system.

Note that the electrical potential energy is positive if the two charges are of the same type, both positive or both negative, and negative if the two charges are of opposite types. This makes sense if you think of the change in the potential energy $\Delta U$ as you bring the two charges closer or move them further apart. Depending of the relative types for the two charges, you may have to work on the system or the system would do work on you, i.e. you work will be either positive or negative. If you have to do a positive work on the system, then the energy of the system should go up.

When you bring two positive charges or two negative charges closer, you would have to do a positive work on the system, which will raise their potential energy. Since potential energy is proportional to $1/r$, the potential energy goes up when $r$ goes down between two positive or two negative charges.

On the other hand, if you bring a positive and a negative charge nearer, you would have to do negative work on the system, which means you would take away energy from the system. This will reduce the potential energy. Since potential energy is negative in the case of a positive and a negative charge pair, the increase in $1/r$ would make the potential energy more negative, which is same as a reduction in potential energy.

### 3.1.2 The Superposition Principle of Electric Potential Energy

According to Coulomb's law, charges interact with each other directly via a two-particle interaction. Thus, when a test charge $Q$ is brought from infinity to some point P in the presence of two other charges $q_1$ and $q_2$, the potential energy of $Q$ will change from zero at infinity to some finite value (Fig. 3.3). The potential energy of $Q$ when it is at point P will be equal to the work done by an agent in bringing the charge from infinity to P if we take the zero of the potential energy at the point at infinity as we have done above.

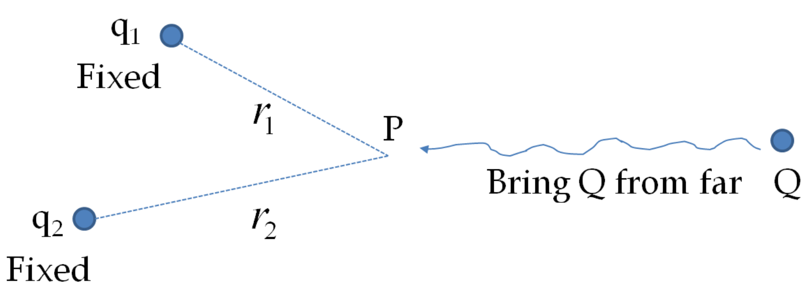


Figure 3.3: Potential energy of a test charge $Q$ when placed at point P is the work required to bring charge from a reference point located far away to the point P while keeping all other charges fixed in their places.

The external agent will do the work on $Q$ against the electric forces from charges $q_1$ and $q_2$. The total work done by the external agent will be the sum of the work against the two electric forces on $Q$, one by $q_1$ and the other by $q_2$. Let $r_1$ and $r_2$ be the direct distances between the point P and the locations of $q_1$ and $q_2$ respectively. Therefore, the potential energy $U_P$ of $Q$ when the charge is at point P will be given by

$$U_P = \frac{1}{4\pi\epsilon_0}\frac{Q\ q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{Q\ q_2}{r_2}. \tag{3.6}$$

You can easily extend the above result to the potential energy $U_P$ of the test charge $Q$ at the point P in the presence of an arbitrary number of charges, say $N$ charges, $q_1, q_2, \cdot, q_N$. Let the distance of the point P from these charges be denoted by $r_1, r_2, \cdot, r_N$, respectively. Then the formula for the potential energy of the charge $Q$ when placed at the point P will be given by the following sum.

$$U_P(\text{of } Q) = \frac{1}{4\pi\epsilon_0}\frac{Qq_1}{r_{01}} + \frac{1}{4\pi\epsilon_0}\frac{Qq_2}{r_{02}} + \cdots + \frac{1}{4\pi\epsilon_0}\frac{Qq_N}{r_{0N}}. \tag{3.7}$$

**Example 3.1.1. Work for Assembling a Charge Distribution.** Find the amount of work an external agent must do in assembling

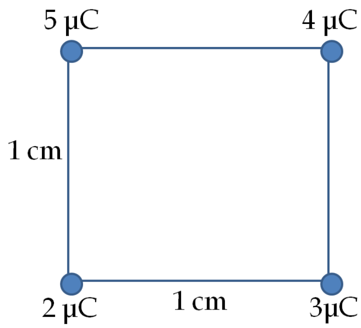


Figure 3.4: How much work is needed to assemble these charges?

four charges $+2\ \mu$C, $+3\ \mu$C, $+4\ \mu$C, and $+5\ \mu$C at the vertices of a square of side 1 cm, starting each charge from infinity.

**Solution.** To find the work needed to assemble the charge distribution we can calculate the work needed to bring one charge at a time while keeping the other charges fixed in their final positions.

Step 1: First bring $+2\ \mu$C charge to the origin. Since there are no other charges at a finite distance from this charge yet, no work will be done in bringing it from infinity.

$$W_1 = 0.$$

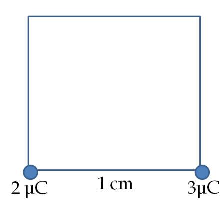


Figure 3.5: Step 2: Work $W_2$ to bring 3 $\mu$C charge from infinity.

Step 2: While keeping $+2\ \mu$C fixed at the origin, bring $+3\ \mu$C charge to $(x, y, z) = (1\ \text{cm}, 0, 0)$. Now, the applied force will do work against the force by the $+2\ \mu$C charge fixed at the origin. The work done will equal the change in the potential energy of $+3\ \mu$C charge.

$$\begin{aligned} W_2 &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}} \\ &= 9\times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \times \frac{2\times 10^{-6}\text{C}\times 3\times 10^{-6}\text{C}}{1\times 10^{-2}\text{m}} \\ &= 5.4\ \text{N.m} \end{aligned}$$

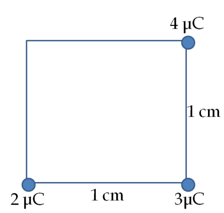


Figure 3.6: Step 3: Work $W_3$ to bring 4 $\mu$C charge from infinity.

Step 3: While keeping $+2\ \mu$C and $+3\ \mu$C fixed in their places, bring in the $+4\ \mu$C charge to $(x, y, z)$=$(1\ \text{cm}, 1\ \text{cm}, 0)$. Work done in this step is

$$\begin{aligned} W_3 &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}} \\ &= 9\times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \times \\ &\quad \left[\frac{2\times 10^{-6}\text{C}\times 4\times 10^{-6}\text{C}}{\sqrt{2}\times 10^{-2}\text{m}} + \frac{3\times 10^{-6}\text{C}\times 4\times 10^{-6}\text{C}}{1\times 10^{-2}\text{m}}\right] \\ &= 15.9\ \text{N.m} \end{aligned}$$

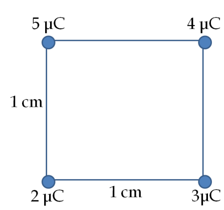


Figure 3.7: Step 4: Work $W_4$ to bring 5 $\mu$C charge from infinity.

Step 4: Finally, while keeping $+2\ \mu$C, $+3\ \mu$C and $+4\ \mu$C charges in their places, bring the $+5\ \mu$C charge to $(x, y, z)$=$(0, 1\ \text{cm}, 0)$. Work done here will be

$$\begin{aligned} W_4 &= \frac{q_4}{4\pi\epsilon_0}\left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right] \\ &= 9\times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \times 5\times 10^{-6}\text{C} \\ &\quad \left[\frac{2\times 10^{-6}\text{C}}{1\times 10^{-2}\text{m}} + \frac{3\times 10^{-6}\text{C}}{\sqrt{2}\times 10^{-2}\text{m}} + \frac{4\times 10^{-6}\text{C}}{1\times 10^{-2}\text{m}}\right] \\ &= 36.5\ \text{N.m} \end{aligned}$$

Hence, the total work done by the applied force in assembling the four charges is equal to the sum of the work in bringing each charge from infinity to its final position.

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + 5.4 \text{ N.m} + 15.9 \text{ N.m} + 36.5 \text{ N.m} \\ &= 57.8 \text{ N.m} \ \text{ or } \ 57.8 \text{ J.} \end{aligned} \tag{3.8}$$

**Example 3.1.2. Electrostatic Energy Stored in a Charge Distribution.** Find work needed to assemble an $N$-charge system, or, in other words, find the potential energy stored in an $N$-charge system.

**Solution.** Imagine assembling a system consisting of $N$ charges, $q_1$, $q_2$, $\cdots$ , $q_N$, located at positions $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_N$, respectively. To find the total work required we will bring one charge at a time from infinity and add up the work from each step. Let $r_{ab}$ denote the direct distance between charges $q_a$ and $q_b$ in the final configuration. When the first charge $q_1$ is brought no work would be done since there are no other charges to oppose it yet. Therefore, the work for this step will be

$$W_1 = 0.$$

Now, when we bring the second charge $q_2$ from infinity to its location $\vec{r}_2$ there will be work against the force by the first charge whose position is not allowed to change now. Let $r_{12}$ be the direct distance from where $q_1$ is located and where charge $q_2$ will end up. The work done on $q_2$ in this step will be

$$W_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

This is the energy stored in the two-charge system as found earlier. Now, bring the charge $q_3$ from infinity while holding the charges $q_1$ and $q_2$ in their places. The work needed will be

$$W_3 = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

Thus the energy stored in the three charge system will be

$$W_{123} = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

You can see the pattern in this calculation emerge nicely. It shows that there will be one term for each pair of charges. For an $N$-charge system we can write the general formula using a double sum.

$$\boxed{W_{12\cdots N} = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N-1}\sum_{i>j}^{N}\frac{q_i q_j}{r_{ij}}}$$

This is the energy stored in an $N$-charge system. The double sum can be written more symmetrically, with each index going over all values, but we need to exclude the terms when the two indices are same.

$$\boxed{W_{12\cdots N} = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{2}\sum_{i=1, i\neq j}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{r_{ij}}\right]}$$