6.3 PRESSURE IN A STATIC FLUID NEAR EARTH

Pressure in a fluid near earth varies with depth due to the different weight of fluid above a particular level. We can understand this by imagining a surface at a given level and noting that at the deeper level, more weight of the fluid acts on the imagined surface. Therefore we expect a greater pressure at the deeper level. Note that if the same fluid is taken in outer space of zero gravity then this effect will not be present any more.

In this section we will derive a formula for the variation of pressure with depth in a tank containing a fluid of density ρ on the surface of Earth. Imagine a thin element of fluid at a depth h. Let the element have a cross-sectional area A and height Δy. Then, the forces on the element are from the pressures $p(y)$ above and $p(y+\Delta y)$ below it, and the weight of the element itself as shown in the free-body diagram in Fig. 6.5.

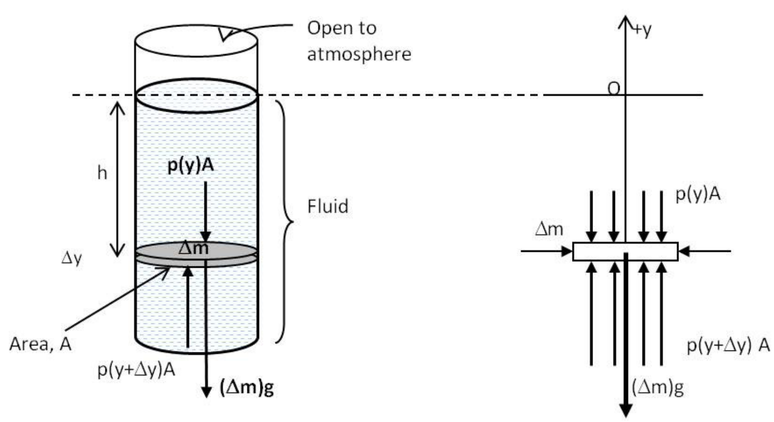

Figure 6.5: Forces on a mass element inside a fluid.

Since the element of fluid between y and $y+\Delta y$ is not accelerating, the forces are balanced. Using the Cartesian y-axis pointed up, we find the following equation for the y-component.

$$p(y + \Delta y) - p(y)A - g\Delta m = 0 \quad (\Delta y < 0) \tag{6.4}$$

Note that if the element had a non-zero y-component of the acceleration, the right side would not be zero, it would be the mass times the y-acceleration instead. The mass of the element can be written in terms of the density of the fluid and the volume of the elements.

$$\Delta m = |\rho A \Delta y| = -\rho A \Delta y \quad (\Delta y < 0). \tag{6.5}$$

Putting the expression for Δm from Eq. 6.5 into q. 6.4, and then dividing both sides by $A\Delta y$ we find the following.

$$\frac{p(y+\Delta y)-p(y)}{\Delta y} = -\rho g \tag{6.6}$$

Taking the limit of infinitesimally thin element $\Delta y \rightarrow 0$, we discover the following differential equation that gives the variation of pressure in a fluid.

$$\boxed{\frac{dp}{dy} = -\rho g.} \tag{6.7}$$

This is an important equation that tells us that the rate of change of pressure in a fluid is proportional to the density of the fluid. The solution of this equation will depend upon whether the density ρ is constant or changes with depth, i.e. the function $\rho(y)$.

To be sure, if the range of the depth being analyzed is not too great, we can assume the density to be constant. But, if the range of depth is great, such as in the case of the atmosphere, there would be significant change in density with depth. In that case, we cannot use the approximation of a constant density. In our first application of Eq. 6.7, we will work out a formula for the pressure in a tank of liquid such as water, where the density of liquid can be taken to be constant.

Example 6.3.1. Constant density fluid

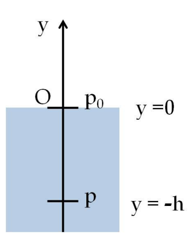

Figure 6.6: Example 6.3.1.

We now consider a simple application of Eq. 6.7 and find a formula for the pressure at a depth h from the surface in a fluid of constant density. We need to integrate the equation from $y = 0$, where the pressure is the atmospheric pressure (p_0), to $y = -h$, the y-coordinate of the depth.

$$\int_{p_0}^{p} dp = -\int_{0}^{-h} \rho g dy \implies p - p_0 = \rho g h. \tag{6.8}$$

Hence pressure at a depth of a fluid on the surface of earth is equal to the atmospheric pressure plus $\rho g h$ if the density of the fluid is constant over the height.

Note that the pressure in a fluid only depends on the depth from the surface and not on the shape of the container. Therefore if you construct a container where a fluid can freely move in various parts, the liquid will be at the same level in every part regardless of the shape as shown in Fig. 6.7.

Example 6.3.2. Variation of Atmospheric Pressure With Height

The change in atmospheric pressure with height is of particular interest. Assuming the temperature of air to be constant, and that ideal gas law of thermodynamics describes the atmosphere to a good approximation, find the variation of atmospheric pressure with height. [Note added: We will study the ideal gas law in more detail in a later chapter. Here we just make use of that law for illustrative purposes.]

Figure 6.7: If the fluid can flow freely, it rises to the same height in each part.

Solution. Let $p(y)$ be the atmospheric pressure at height y. The density ρ at y, the temperature T in the Kelvin scale (K), and the mass m of a molecule of air are related to the absolute pressure by the ideal gas law given as

$$p = \rho \frac{k_B T}{m} \quad \text{(atmosphere)} \tag{6.9}$$

where k_B is a constant called Boltzmann's constant that has a value of 1.38×10^{-23} J/K. You may have studied the ideal gas law in the "chemistry" notation, $PV = nRT$, where n is the number of moles, and R the gas constant. Here, the same law has been written in a different form, using the density ρ instead of volume V. Therefore, if pressure p changes with height so would the density ρ. Using density from the ideal gas law, we find that rate of variation of pressure with height is given as

$$\frac{dp}{dy} = -\left(\frac{mg}{k_B T}\right) p, \tag{6.10}$$

where constant quantities have been collected inside the parenthesis. Replacing the quantity within the parenthesis () by a single symbol α the equation looks much simpler.

$$\frac{dp}{dy} = -\alpha p. \tag{6.11}$$

Now, we can solve this equation by noticing that the pressure p is a function whose derivative is proportional to itself. Since the exponential functions are such functions we try an exponential function and sure enough find the solution immediately.

$$p(y) = p_0 \exp(-\alpha y) \tag{6.12}$$

Thus atmospheric pressure drops exponentially with the height since the y-axis is pointed up from the ground and y has positive values in the atmosphere above sea level. The pressure drops by a factor of $\frac{1}{e}$ when the height is $\frac{1}{\alpha}$, which gives us a physical interpretation for α: constant $\frac{1}{\alpha}$ is a length scale that characterizes how pressure varies with height.

An approximate value of α can be obtained by using mass of a nitrogen molecule as a proxy for an air molecule. At temperature 27°C or $300K$, we find

$$\alpha = -\frac{mg}{k_B T} = \frac{4.8 \times 10^{-26}\ \mathrm{kg} \times 9.81\ \mathrm{m/s}^2}{1.38 \times 10^{-23}\ \mathrm{J/K} \times 300\ \mathrm{K}} = \frac{1}{8,800\ \mathrm{m}}. \tag{6.13}$$

Therefore, for every 8,800 meters, the air pressure drops by a factor $\frac{1}{e}$ or approximately one-third of its value. Of course, this gives us only a rough estimate of the physical situation since we have assumed both the temperature and g constant over such great distances from the Earth, neither of which is correct in reality.