4.2 NORMAL MODES OF A STRING

Newton's second law can be applied to oscillations of a string such as guitar string. Consider a string tied at both ends. First, we divide up the mass of the string in N parts and place them as beads on equal intervals as shown in Fig. 4.4. This converts the problem of vibration of a string to the vibrations of N beads, each of mass m, on a massless string which are coupled by the tension force in the string. For the sake of ease of visualization of the modes we consider the modes of the transverse motion of the string.

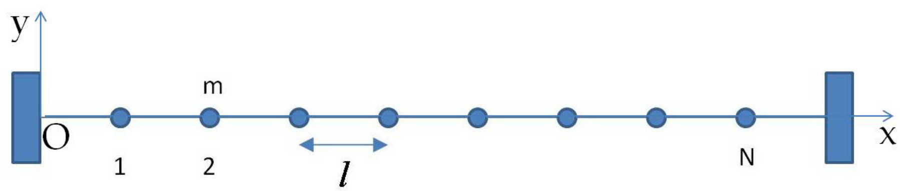

Figure 4.4: N beads on a taut string act as N coupled oscillators.

There are N normal modes of frequencies for this system.

$$\omega_n = \sqrt{\frac{4T}{ml}} \sin\left[\frac{n\pi}{2(N+1)}\right], \quad n = 1, 2, \cdots, N. \tag{4.13}$$

where T is the tension in the string, m mass of each bead, and l the separation between beads.

Let the string be along x-axis, then the transverse coordinates of the beads will be in the yz plane. Let us orient y-axis to be the axis in which the transverse motions of the beads occur. With this choice we display the y-displacements of the beads in different normal modes for some particular values of N in Fig. 4.5.

With the origin at one end, the modes can be written as a sine function of x where the x values are x coordinates of the beads, which can be worked out explicitly. For mode of frequency ω_n the pattern along the string of length L that is fixed at both ends is

$$y_n(x) = A_n \sin\left(\frac{\pi x}{L}\right), \quad 0 \le \ x \le L. \tag{4.14}$$

The oscillation properties of the continuous string would be obtained by taking $N \to \infty$ limit in Eqs. 4.13 and 4.14, which corresponds to decreasing indefinitely the distance l between elementary masses of the string. By taking this limit we obtain the following normal mode frequencies for a string of length L, mass per unit length μ which is under tension T.

$$\omega_n = n\omega_1, \quad n = 1, 2, 3, \cdots, \tag{4.15}$$

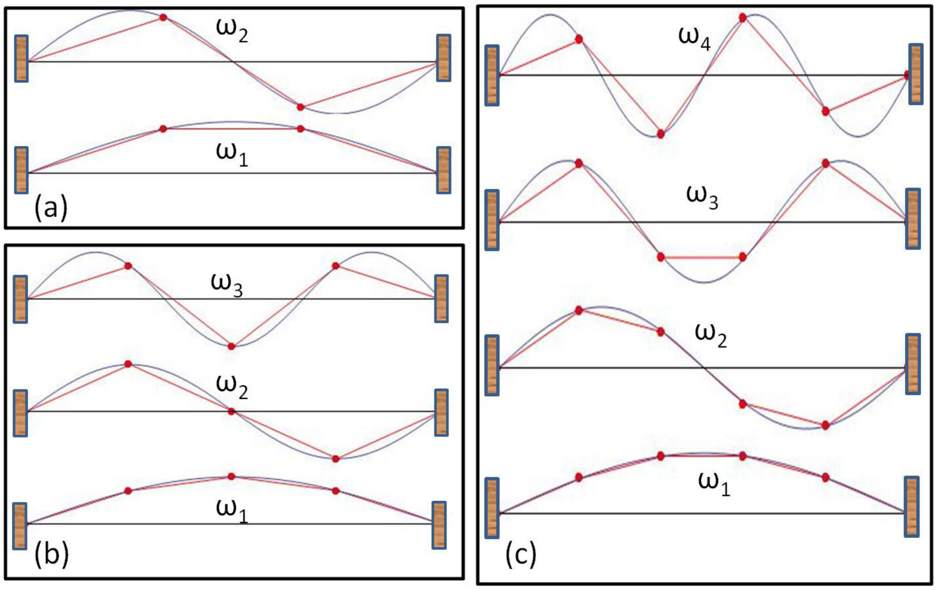

Figure 4.5: Transverse modes of (a) $N = 2$, (b) $N = 3$, and (c) $N = 4$. In each case ω_1 is the lowest frequency mode. The points at the nodes do not oscillate; other points of the string oscillate up and down with the frequency of the normal mode.

where

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\mu}} \tag{4.16}$$

is the frequency of the lowest mode, which is also called the **fundamental mode**, or simply the fundamental. The frequency of the upper modes are integral multiples of the frequency of the fundamental mode. Therefore, the upper modes are also called **overtones**.

4.2.1 Transverse Vibrations of a String

In a transverse vibration mode of a string, each part of the string vibrates up and down with the frequency corresponding to that of the mode. To get a feel for the oscillations of the string, we will plot some of the lowest frequency modes over one complete cycle. Figure 4.6 shows the lowest frequency mode. We find that the lowest frequency mode is symmetric about the middle of the string.

Suppose, we pull the string such that string takes the shape of the lowest mode, and release the string from rest. What will happen? Fig. 4.6 shows the position of the string at intervals of $\frac{1}{8}^{th}$ periods. The string moves slowly in the first $\frac{1}{8}^{th}$ period when it is furthest from the equilibrium than in the second and third $\frac{1}{8}^{th}$ periods when it passes through the equilibrium position, which is similar to the behavior of a mass attached to a spring. The particles near the middle of the string cover the largest distance in the same time than the particles near the fixed ends, and all particles of the string vibrate with the same period of oscillation. Each particle of the string "acts" as if the particle was attached to an "invisible" spring and vibrates up and own in a sinusoidal manner corresponding to the Simple Harmonic Motion at the frequency of the mode.

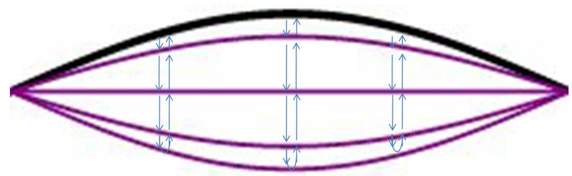

Figure 4.6: The fundamental mode of a string fixed at both ends. The string is plucked in the shape shown in thick black line and let go from rest. The string's location at successive $\frac{1}{8}$th period intervals are shown in the figure. The string vibrates such that the ratio of the vertical displacements at different horizontal locations remains independent of time.

The first overtone, i.e. mode $n = 2$, is shown in Fig. 4.7. The first overtone mode is antisymmetric about the middle position. When the left half of the string moves up, the right half moves down, and vice-versa with every particle oscillating with frequency ω_2. The point in the middle remains at rest and is called a **node**.

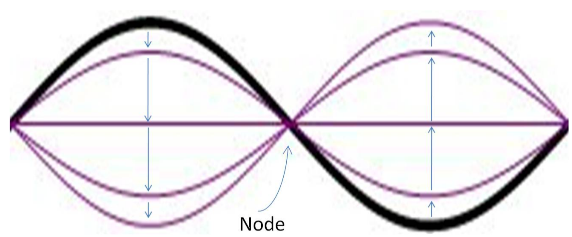

Figure 4.7: The second lowest frequency mode, $n = 2$, of a string fixed at both ends. The stretched string is pulled in the shape shown in thick black line and let go from rest. The string's location at successive $\frac{1}{8}^{th}$ period intervals are shown in the figure. The point labeled "Node" is stationary and does not move.