6.2 NEWTON'S SECOND LAW OF MOTION

Newton's second law addresses the central question of motion: what causes change in motion of an object? Before we can discuss the second law of motion, we need definitions of mass and momentum.

6.2.1 Definition of Mass

In *Principia* Newton gives a simple definition of mass based on the density and volume.

"The quantity of matter is the measure of the same, arising from its density and bulk conjunctly".

Newton went on to further clarify the definition, "Thus air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction; and of all bodies that are by any causes whatever differently condensed".

This definition of mass was common in Newton's time and he did not feel the need to define mass in other ways. Now a days, it is more common to define mass operationally by the response of different materials to the same force.

The definition of mass based on response to a known force is independent of the nature of the force and is called **inertial mass**. The inertial mass is sometimes distinguished from the mass used for the gravitational force; the later is called **gravitational mass**.

Although the definition of mass given in *Principia* is quite adequate for our present purpose of introducing the basic aspects of the second law of motion, we will present below the definition of mass based on the second law of motion later in this section along with the operational definition of force.

6.2.2 Definition of Momentum

Newton defined a measure of the quantity of motion we call momentum now.

"The quantity of motion, is the measure of the same, arising from the velocity and quantity of matter conjunctly".

Let m be the mass of a body and $\vec{v}$ the velocity, then the **momentum** $\vec{p}$ is defined as

$$\boxed{\vec{p} = m\ \vec{v}.} \tag{6.1}$$

Momentum has the same direction as velocity since mass is always a positive real number. The dimensions of momentum are:

$$[p] = \frac{[M][L]}{[T]}.$$

Therefore, the metric unit for momentum is kg.m/s. A momentum of a shot put of mass 5 kg thrown with speed 10 m/s is 50 kg.m/s in the direction of the velocity.

If a system consists of more than one particle, then the momentum of the whole system will be equal to the vector sum of the momenta of all of its parts. Let $\vec{p}_1$, $\vec{p}_2$, $\cdots$, $\vec{p}_N$ be the momenta of N parts of a system, then the momentum of the whole, the net momentum is given as

$$\boxed{\vec{p}_{net} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N.} \tag{6.2}$$

Note that the sum of momenta is a vector sum and one must be careful not to just add the magnitudes of the momenta since directions are equally important. For instance, the total momentum of two objects of equal mass and equal speed but moving towards each other in a straight line is equal to zero since the vector sum of two equal magnitude but oppositely directed vectors is zero.

Example 6.2.1. Momentum of one body. A car of mass 3000 kg is moving to the East at a speed of 30 m/s. How much momentum does the car have?

Solution. From the definition of the momentum, we can immediately state the magnitude and direction of the momentum of the car.

Magnitude $= mv = (3000 \text{ kg}) \times (30 \text{ m/s}) = 90,000 \text{ kg.m/s}.$

Direction: Towards the East.

Example 6.2.2. Total momentum of two bodies moving along the same line. A car of mass 3000 kg is moving to the East at a speed of 30 m/s and a truck of mass 6000 kg is moving to the West at a speed of 20 m/s. What is the net momentum of the two?

Solution. By applying the definition of momentum to the car and the truck separately, we find that the car has a momentum of $90,000$ kg.m/s pointed towards the East and the truck has a momentum of $120,000$ kg.m/s pointed towards West. Now, we need to add these two vectors. Since their directions are opposite, they will subtract, giving a net momentum of $30,000$ kg.m/s pointed towards the West.

You can also use the analytic method to add the two vectors by utilizing a Cartesian coordinate. To be concrete, let us point the x axis towards the East. Then, the two momentum vectors will have the following x-components (the y and z-components of both being zero).

$$\begin{aligned} p_{\text{car},x} &= 90,000 \text{ kg.m/s} \\ p_{\text{truck},x} &= -120,000 \text{ kg.m/s} \\ &\quad (\text{negative since } \vec{p}_{\text{ truck}} \text{ is pointed towards negative } x\text{-axis}) \end{aligned}$$

Therefore, the net momentum has the following components.

$$\begin{aligned} p_{\text{net},x} &= -30,000 \text{ kg.m/s} \\ p_{\text{net},y} &= 0 \\ p_{\text{net},z} &= 0 \end{aligned}$$

Therefore, the net momentum has the magnitude $30,000$ kg.m/s pointed towards the negative x-axis, which is towards the West.

Example 6.2.3. Total momentum of two bodies moving in a plane. A car of mass 3000 kg is moving to the East at a speed of 30 m/s and a truck of mass 6000 kg is moving to the North at a speed of 20 m/s. What is the net momentum of the two?

Solution. By applying the definition of momentum to the car and the truck separately, we find that the car has a momentum of $90,000$ kg.m/s pointed towards the East and the truck has a momentum of $120,000$ kg.m/s pointed towards the North. Now, we need to add these two vectors. Since the vectors are not in one line, we need to employ the parallelogram law of addition of vectors, either geometrically or analytically. We have seen that analytical method is often easier to implement.

To use the analytical method, as usual we start with a choice of coordinate system. Let us point the x-axis towards the East, the y-axis towards the North and the z-axis vertically up. Then, we see that the two momenta have the following representation in components.

$$\begin{aligned} \vec{p}_{\text{car}} &= (90,000 \text{ kg.m/s}, 0, 0) \\ \vec{p}_{\text{truck}} &= (0, 120,000 \text{ kg.m/s}, 0) \end{aligned}$$

where x, y and z-components of each vector have been listed in order. Therefore, the net momentum has the following representation in the given coordinate system.

$$\vec{p}_{\text{net}} = (90,000 \text{ kg.m/s}, 120,000 \text{ kg.m/s}, 0)$$

From the components of the net momentum vector, we can easily determine the magnitude and direction.

$$\text{Magnitude} = \sqrt{(90,000)^2 + (120,000)^2} = 150,000 \text{ kg.m/s}$$
$$\text{Direction: } \arctan\left(\frac{120,000}{90,000}\right) = 53^\circ \text{ from the East towards the North.}$$

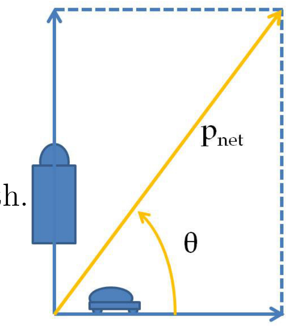

Figure 6.2: The net momentum of car and truck is the vector sum of their separate momenta.

6.2.3 The Second Law

The second law of motion addresses how the state of motion of an object changes when an external force acts on the object. In Newton's own words from *Principia*,

" The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed".

Newton explains the physical content of the law as follows.

"If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subdued from the former motion, accordingly as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both".

The alteration of motion here is meant to be the rate of change of momentum. We have seen above the special role of the inertial frames. We will recast Newton's second law using modern terminology.

Observed from an inertial reference frame, the rate of change of momentum of a body at an instant is proportional to the net force on the body acting at that instant.

Therefore, if a force $\vec{F}$ acting on a body for a time Δt causes a change in momentum $\Delta\vec{p}$, then Newton's second law says that

$$\frac{\Delta\vec{p}}{\Delta t} \propto \vec{F}. \tag{6.3}$$

We can write this statement as equality as follows.

$$\frac{\Delta\vec{p}}{\Delta t} = k\vec{F}, \tag{6.4}$$

where k is a proportionality constant. We fix the value of the proportionality constant k to 1 by choosing the units of p, F and m. Therefore, we define 1 N to be a force that will change the velocity of a 1 kg object by 1 m/s if the force acts on the body for 1 s. With these units we have the following statement of Newton's second law.

$$\frac{\Delta\vec{p}}{\Delta t} = \vec{F}. \tag{6.5}$$

Now, taking the limit $\Delta t \rightarrow 0$ we obtain instantaneous relationship between the rate of change of momentum and the force at that instant.

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{p}}{\Delta t} = \vec{F}, \tag{6.6}$$

which is written as

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}.} \tag{6.7}$$

Therefore, a more rapid change of momentum, whether direction or magnitude would require a larger force. Since, the momentum is a product of mass and velocity, you could change the momentum by either changing the mass of your system or the velocity. Rewriting Eq. 6.5 we obtain the change $\Delta\vec{p}$ in momentum in an interval Δt as

$$\Delta\vec{p} = \vec{F}\Delta t, \tag{6.8}$$

which says that the longer a force acts on a body, the larger the change in momentum, and larger the force the larger the change in momentum.

The law makes sense when you think about your own experience with forces and motion. For instance, when you push a cart gently, it picks up a low velocity, and when you push the cart harder, the corresponding pick up in velocity is larger. Also, longer you push on a cart consistently the more speed it picks up. These qualitative observations let us verify the essential point of Eq. 6.7 that velocity changes as we exert a force on the system. controlled experiments in laboratory have demonstrated the correctness of the proportionality of force and change in momentum.

Other forms of second law

Newton wrote his second law of motion in the form we have presented above, namely in terms of the rate of change of momentum.

$$\frac{d\vec{p}}{dt} = \vec{F},$$

which is also written as

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{6.9}$$

In most elementary textbooks, we find another form of Newton's second law, which is obtained when mass of the system does not change with time. Let us deduce the form of second law for constant mass case.

We replace momentum in Eq. 6.9 by its definition, $\vec{p} = m\vec{v}$ to get

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right).$$

Now, the chain rule of differentiation gives two terms on the right side.

$$\vec{F} = \left(\frac{dm}{dt}\right)\vec{v} + m\left(\frac{d\vec{v}}{dt}\right),$$

where the first term is the rate of change of momentum due to changing mass, as in the case of a rocket when it is releasing burnt fuel from the exhaust, and the second term is the rate of change of momentum due to acceleration. For most systems, the mass does not change, and therefore the simpler equation results.

$$\vec{F} = m\left(\frac{d\vec{v}}{dt}\right),$$

which is the same thing as

$$\boxed{\vec{F} = m\vec{a}. \quad \text{(mass constant)}} \tag{6.10}$$

You should also note that Newton's second law, whether in the form given in Eq. 6.7 or in Eq. 6.10, are **instantaneous statements**. That is, the second law is a statement about the relation between the force acting on a body and the resulting rate of change of momentum or mass times acceleration at the same instant.

Another point to note is that Newton's second law gives **relation among vectors**, the force and the rate of change of momentum. Thus, equations 6.7 and 6.10 are **vector equations**. You must be very careful when making algebraic manipulations with them since the laws of vector addition and multiplication are very different from those for scalar numbers.

Algebraic manipulations are often easier in analytic picture. In the analytic picture, we work with components of vectors in a coordinate system of our choice. The components separate into separate equations as usual.

Component	Always correct	Correct when mass constant
x component	$F_x = dp_x/dt$	$F_x = m\ a_x$
y component	$F_y = dp_y/dt$	$F_y = m\ a_y$
z component	$F_z = dp_z/dt$	$F_z = m\ a_z$

If more than one force acts on a particle, each force contributes to the change in momentum. The net acceleration of the particle subject to more than one force is the vector sum of the contributions of each force.

$$\text{Multiple forces on a single particle:} \left\{ \begin{array}{l} \vec{F}_1 = m\vec{a}_1 \\ \vec{F}_2 = m\vec{a}_2 \\ \quad \cdots \\ \vec{F}_N = m\vec{a}_N \end{array} \right\} \implies \vec{F}_{\text{net}} = m\vec{a}_{\text{net}}.$$

6.2.4 Operational Definition of Force and Mass

Newton's second law of motion transforms an intuitive notion of force into a quantifiable entity. The quantifiable definition of force also helps us develop an operational definition of mass. Let us first look at how mass is given a more fundamental meaning by second law of motion than just the notion of mass being the"quantity" of matter.

Operational definition of mass

Imagine pulling two objects on a frictionless table with equal forces. For example, you could attach identical springs to different blocks, and pull on them so that the stretch is the same in the two so as to make sure equal forces pull on them as shown in Fig. 6.3.

You will find the accelerations of the two objects will in general be different, but the ratio of the magnitudes of the accelerations for two masses subject to equal forces is a constant.

$$\frac{a_1}{a_2} = \text{constant} \quad \text{(same force on 1 and 2.)} \tag{6.11}$$

The constant varies among different pairs of objects. The unequal accelerations for two different blocks corresponds well to the "quantity" of material in the two blocks and gives an operational way for defining mass. The acceleration for the objects with a higher "quantity" of matter gets less acceleration than the object with a lower

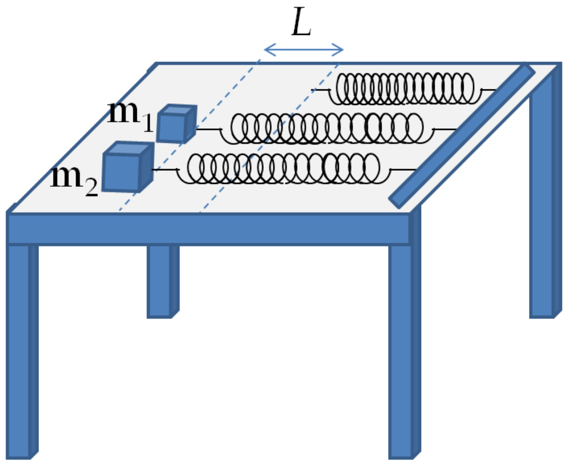

Figure 6.3: Two objects pulled with equal force on a frictionless surface have unequal accelerations.

"quantity" of matter. Therefore, mass can be defined by comparing the accelerations of two objects when subjected to the same force.

$$\frac{m_2}{m_1} = \frac{a_1}{a_2} \quad \text{(same force on 1 and 2.)} \tag{6.12}$$

Now, if we choose one of the masses as a standard mass, the masses of all other objects can be determined by comparing their accelerations. The standard mass sanctioned by the second Conférence Générale des Poids et Mesures (CGPM) conference in 1889 is a 1-kilogram cylinder of platinum-iridium alloy kept at the Bureau International des Poids et Mesures (BIPM) in Paris in France. This is the only standard unit still being represented by a man-made sample rather than naturally occurring phenomenon or unit such as mass of a particular atom.

Operational definition of force

Our experience with push and pull give us an intuitive notion of force. But, how do we compare the magnitudes of two forces quantitatively? The second law of motion provides an operational way to define a force. Once again, it is easier to illustrate the idea of force using spring force, which is proportional to the stretch. Attach a spring to a mass m and pull it on a frictionless surface such that the spring is stretched by some length L.

Measure the acceleration of the mass as you let go of it. Now, if you pull twice as hard such that spring is stretched by twice as much, 2 L, and let go of the mass. This time, you will find that the resulting acceleration is twice as much with the stretch 2 L of the spring as when the stretch was only L. Thus, acceleration is directly

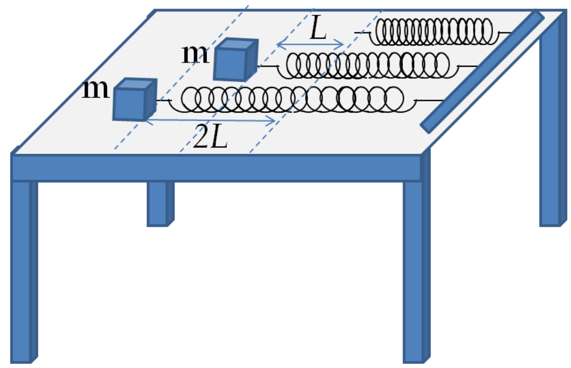

Figure 6.4: Twice the force on the same mass causes twice the acceleration.

proportional to the applied force.

Changing mass but keeping the force same gave us the conclusion that acceleration produced by the same force is inversely proportional to mass, while changing force but keeping the mass same shows that the magnitude of acceleration is proportional to the magnitude of force. Putting these two together, we find that acceleration a , force F and mass m are related as enunciated in the second law.

$$a \propto \frac{1}{m} \text{ and } a \propto F \quad \Longrightarrow a = k\frac{F}{m}$$

where k is a proportionality constant, which is absorbed in the units for force. For instance, if we call the magnitude of force that causes 1-kg mass to accelerate at 1 m/s^2 the unit force, we will be able to set $k = 1$. The unit so defined is called 1 Newton (N).