2.1 FLUX OF ELECTRIC FIELD

The flux of electric field is a measure of the electric field lines passing through an area as illustrated in Fig. 2.1. If you think of the electric field as the velocity of a fluid, then the electric flux through an area would correspond to the volume of the fluid through that area. Just as the volume of fluid flowing through an area depends on the orientation of the area relative to the velocity of the flow, the electric flux also depends on the relative orientation of the area with respect to the direction of the electric field vector. To take into account the orientation of an area in space we introduce an area vector.

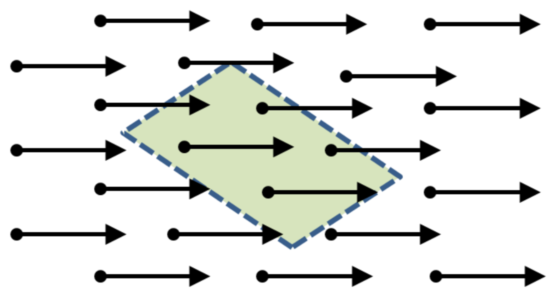

Figure 2.1: The flux of electric field through the shaded area captures the information about the "number " of electric field lines passing the area. The numerical value of the electric flux would depend on the magnitudes of the electric field and the area and the relative orientation of the area with respect to the direction of the electric field.

2.1.1 Area Vector

In order to discuss the flux of a vector field it is helpful to introduce an area vector $\vec{A}$. The area vector of flat surface of area A has the following magnitude and direction.

Area vector of a flat surface:

$$\text{Magnitude} = \text{Area}$$
$$\text{Direction} = \text{Along the normal to the surface.}$$

Since the normal to a flat surface can point in either direction from the surface the direction of the area vector of an open surface is ambiguous as illustrated in Fig. 2.2. However, if a surface is closed, then the surface would enclose a volume. In that case, the direction of the normal vector at any point to the surface points from the inside to the outside. OF course, in a closed surface the normal vectors would point at different directions in space at different points of the surface.

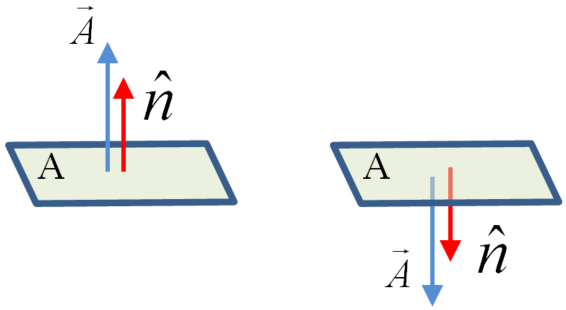

Figure 2.2: Area vector. The normal to an open area could be in either direction. This gives the direction of the area vector of an open surface ambiguous, it could be either of the two cases displayed in this figure. The area vector of a part of a close surface will be taken to point from the inside of the closed space to the outside. This rule will give a unique direction.

For instance, the normal at the two opposite faces of a cube will point exactly in the opposite directions to each other as illustrated in the Fig. 2.3.

2.1.2 Electric Flux

Basic Definition

The electric flux of a uniform electric field through a flat area is defined by the scalar product of the electric field and the area vector.

$$\boxed{\Phi_E = \vec{E} \cdot \vec{A} \quad (\text{uniform } \vec{E})} \tag{2.1}$$

Electric fluxes due to several electric fields passing through a given area can be added to obtain the net flux through that area. Suppose there are N sources of electric field with their electric fields given by $\vec{E}_1$, $\vec{E}_2$, $\cdots$, $\vec{E}_N$, respectively. The flux of each electric field through an area $\vec{A}$ can be added to obtain the net flux through the area.

$$\Phi_E^{\text{net}} = \vec{E}_1 \cdot \vec{A} + \vec{E}_2 \cdot \vec{A} + \cdots + \vec{E}_N \cdot \vec{A}. \tag{2.2}$$

We can simply the expression on the right by taking $\vec{A}$ common and replacing the sum of the electric fields by the net electric field using the superposition principle of the electric field.

$$\begin{aligned} \Phi_E^{\text{net}} &= \vec{E}_1 \cdot \vec{A} + \vec{E}_2 \cdot \vec{A} + \cdots + \vec{E}_N \cdot \vec{A} \\ &= \left(\vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N\right) \cdot \vec{A} \\ &= \vec{E}_{\text{net}} \cdot \vec{A} \end{aligned}$$

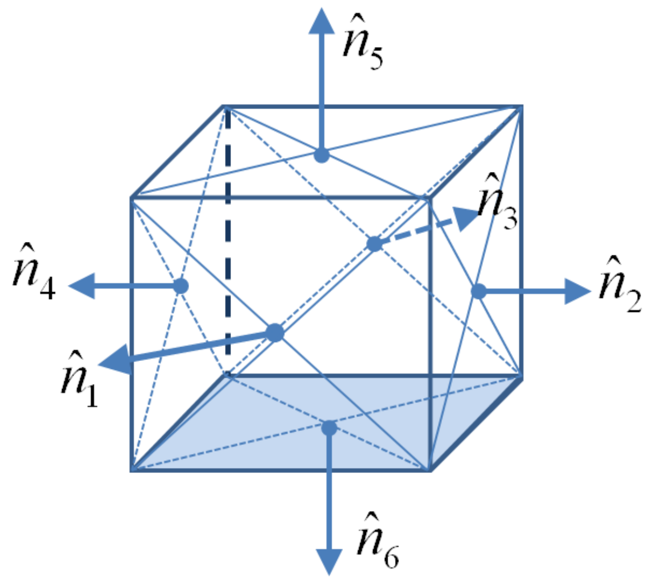

Figure 2.3: The normal to faces of a cube. Since the surface of the cube is a closed surface, normal vector to surface areas point from inside to outside.

That is, the electric flux from several sources add such that the net electric flux is given by the electric flux of the net electric field. Similarly, the electric flux of an electric field through various areas can be added to obtain the net electric flux through the composite of those areas. Suppose, flux of an electric field through the six sides of a rectangular box are $\Phi_E^{(1)}$, $\Phi_E^{(2)}$, $\cdots$, $\Phi_E^{(6)}$. Then, the net electric flux through all the sides of the box will be simply the sum

$$\Phi_E^{net} = \Phi_E^{(1)} + \Phi_E^{(2)} + \cdots + \Phi_E^{(6)}. \tag{2.3}$$

Figure 2.4 shows the electric field of a oppositely charged parallel plate system and an imaginary box between the plates. The electric field between the plates is uniform and points from the positive plate towards the negative plate. A calculation of the flux of this field through various faces of the box shows that the net flux through the box is zero. How does the flux cancel out here?

Recall that the area vector points from the inside of the box to the outside. In the face ABCD, the area vector is pointed down while the electric field is pointed up, therefore we get a negative electric flux through this face. In the face labeled FGHK the area vector is pointed up and so is the electric field. Therefore, the flux through the surface FGHK is positive. The fluxes through the other faces of the cube are all zero since their area vectors are perpendicular to the direction of the electric field. The magnitude of the fluxes through

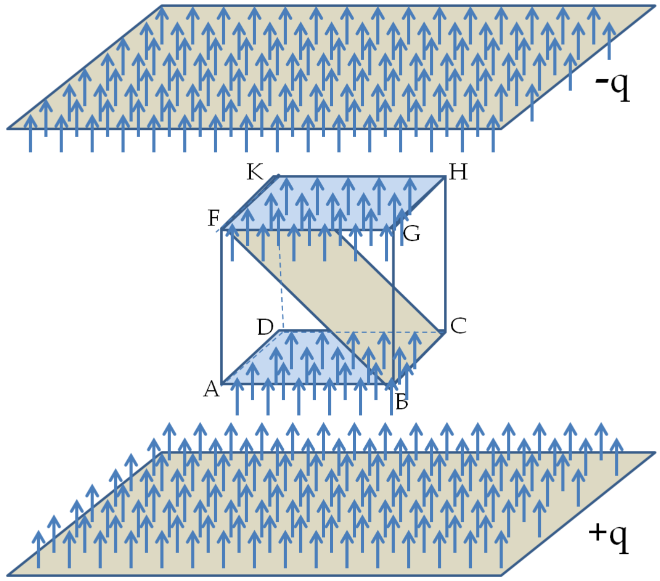

Figure 2.4: Electric flux through a cube. Electric flux through bottom face is negative because $\vec{E}$ is in opposite direction to the normal to the surface. The electric flux through the top face is positive since electric field and the normal are in the same direction. The electric flux through the other faces are zero since electric field is perpendicular to the normals to those faces. The net electric flus through the cube is the sum of fluxes through the six faces. Here the net flux through the cube is equal to zero.

ABCD and FGHK are equal since the electric field has a constant value and the projections of the two area vectors on the electric field directions are equal in magnitude. Calculationally this explains why we get a zero flux through the box between the two plates.

A more fundamental reason is that the sources of the electric field are outside the box. Therefore, if any electric field line enters the volume of the box, then it must also exit somewhere on the surface since there is no charge inside for the lines to land on. Therefore, quite generally, electric flux through a closed surface will be zero if there are no sources of electric field, whether positive or negative charges, inside the enclosed volume.

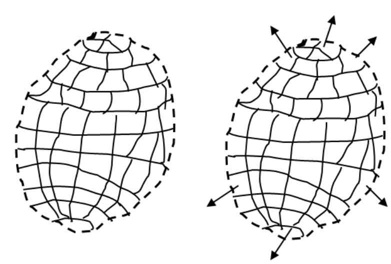

Figure 2.5: Left: A surface is divided up in patches to find the flux. Right: Normal to a few patches shown by convention the normal to a closed surface points from inside to outside.

General Considerations

Any smooth non-flat surface can be replaced by a collection of tiny approximately flat surfaces as shown in Fig. 2.5. If we divide a surface S into small patches, then we notice that, as the patches become small, they can be approximated by flat surfaces.

To keep track of patches, we can number them, from 1 through N. Now, the area vector for each patch can be defined as the area of the patch pointed in the direction of the normal. Let us denote the area vector for the i^{th} patch by $\Delta\vec{A}_i$. We have used the symbol Δ to remind us that the area is of an arbitrarily small patch. Beware that this Δ does not refere to any change in the area, it refers to just a small area. Since the electric field is in general non-uniform, we will have different directions and/or different magnitudes of the electric field at the sites of different patches. Let us denote the average electric field at the location of the i^{th} patch by $\vec{E}_i$.

$$\vec{E}_i = \text{Average electric field over the } i^{th} \text{ patch.}$$

Therefore, we can write the electric flux Φ_i through the area of the i^{th} patch as

$$\Phi_i = \vec{E}_i \cdot \Delta\vec{A}_i \quad (i^{th} \text{ patch}).$$

The flux through each of the individual patches can be constructed in this manner, and then added up to give us an estimate of the net flux through the entire surface S, which we will denote simply as Φ.

$$\Phi = \sum_{i=1}^{N} \Phi_i = \sum_{i=1}^{N} \vec{E}_i \cdot \Delta\vec{A}_i \quad (N \text{ patch estimate})$$

This estimate of the flux gets better as we decrease the size of the patches. However, when you use smaller patches, you need more of them to cover the same surface. In the limit of infinitesimally small patches, you would need infinitely many patches, and the sum would become a surface integral.

$$\boxed{\Phi = \iint_{\text{Surface}} \vec{E} \cdot d\vec{A}.} \tag{2.4}$$

If the surface is closed, another symbol is used to denote the result.

$$\Phi = \oiint \vec{E} \cdot d\vec{A} \quad \text{(closed surface)} \tag{2.5}$$

Example 2.1.1. Flux of a Uniform Electric Field # 1. There exists a constant electric field of magnitude E_0 in the direction of the positive z-axis. What will be the electric flux through a rectangle $(a \times b)$ in the xy-plane?

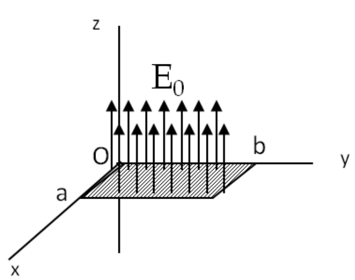

Figure 2.6: Example 2.1.1.

Solution. The electric field is given to be $\vec{E} = E_0\hat{u}_z$. The area vector has the magnitude $a.b$, but the direction could be either the positive z-axis or the negative z-axis. Therefore, the flux will be $\pm$(Electric Field) $\times$ (Area), i.e. $\pm E_0.a.b$.

Example 2.1.2. Flux of a Uniform Field #2. There exists a constant electric field of magnitude E_0 in the direction of the positive z-axis. What is the electric flux through a rectangle $(a \times b)$ in the xz-plane.

Solution. Here the direction of the area vector is either along the positive y-axis or towards the negative y-axis. Therefore, the scalar product of the electric field with the area vector will be zero, giving zero flux.

Example 2.1.3. Flux Through a Slanted Area. There exists a constant electric field of magnitude E_0 in the direction of the positive z-axis. Find the electric flux through the surface (PQRS) inside the cube of side a as shown in the Fig. 2.7.

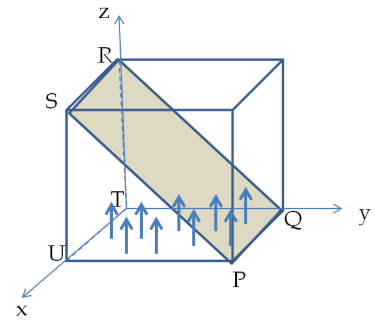

Figure 2.7: Example 2.1.3.

Solution. The flux is easy to work out here if you think conceptually in terms of the electric field lines. Since the electric field lines are parallel to the z-axis, the field lines that pass through the slanted area must also pass through the bottom of the box, i.e. $PQTU$. Therefore, the flux through the slanted area is equal to the flux through $PQTU$, which is found easily to be $\pm E_0 a^2$. The sign of the flux cannot be determined since the surface through which we seek the flux, viz. $PQRS$, is not a closed surface, which makes the direction of the normal ambiguous.

Example 2.1.4. Flux of a Non-homogeneous Field. Consider an electric field of a point charge q situated at the origin. Find the electric field of the point charge through a circular area of radius R about the z axis. The center of the circle coincides with the z axis and is a distance h from the origin.

Solution. Note that the electric field of a point charge is inhomogeneous, meaning different at different point in space. Making use of the symmetry of the situation we will work with flux through the ring-shaped element of the disk at $z = h$ as shown in Fig. 2.8. The electric flux through the ring-shaped area element with the inner radius r' and the outer radius $r' + dr'$ is equal to the product of the z-component of the electric field and the area of the ring with the sign ambiguous since the direction of the normal could be as shown in the figure or in the opposite direction.

$$d\Phi = \pm E_z 2\pi r' dr',$$

where

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\frac{h}{r},$$

where $r = \sqrt{r'^2 + h^2}$. Now, we can integrate over r' from 0 to R to obtain the flux through disk shaped area at $z = h$.

$$\Phi = \pm\frac{qh}{2\epsilon_0}\int_0^R \frac{1}{(r'^2 + h^2)^{3/2}} r' dr'.$$

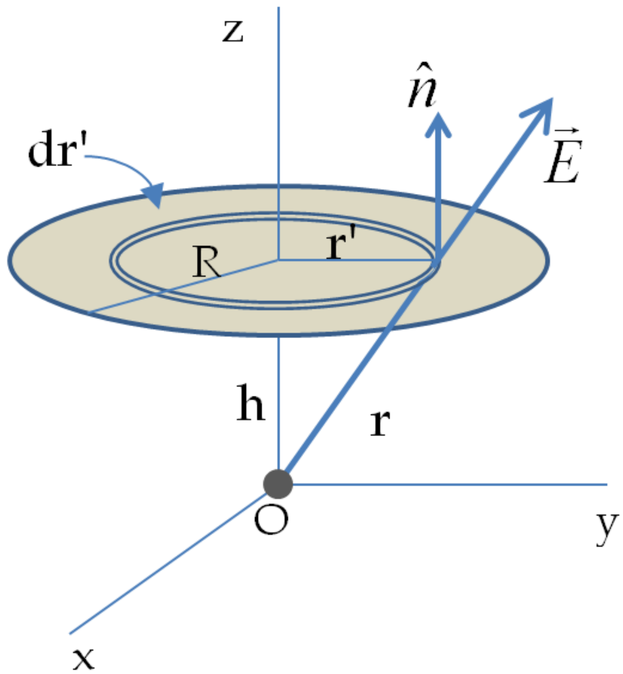

Figure 2.8: Example 2.1.4.

The integral over r' is easiy done by a change of variable $x = r'^2$ with the following result for the flux.

$$\Phi = \pm \frac{q}{2\epsilon_0}\left(1 - \frac{h}{\sqrt{h^2 + R^2}}\right).$$