4.5 CONDUCTORS AS BOUNDARIES

4.5.1 General Considerations

We found above that electric field inside a conductor is zero in the case of electrostatic conditions, and consequently a conductor has the same potential throughout its body, including its surface. Hence, the surface of a conductor is a constant potential surface. A conductor having a particular amount of excess charge will be at a definite potential with respect to zero at infinity.

Equipotential surfaces naturally occur in space around charges and you could cover the surface with a conductor. Covering an equipotential surface with a metal will separate the inside space from the outside space. If you do that the outside world will not notice any difference.

For instance, an isolated charge has equipotential surfaces in the shape of spherical surfaces centered at the charge. If you cover a particular spherical equipotential surface around the point charge by a spherical conductor with enough charge to match the value of the potential for that equipotential surface, the electric potential and electric field outside the conductor will not change, although the electrical potential and electric field inside the conductor will be different than what had been in the case of the point charge.

Another more complicated example comes from a spherically-shaped zero potential surface when you have two unequal charges as shown in Fig. 4.23. This situation shows that the potential from an electric charge and a spherical conductor at zero potential is same as the potential of two unequal charges and no spherical conductor as long as we look at the field points that are outside the sphere.

The importance of conductors as equipotential bodies relies in the fact that, once you have set a conductor at some potential, all points of its surface act as a boundary condition of constant value for potential function outside the conductor. This is highly useful due to an important mathematical theorem obeyed by electric potential:

Uniqueness of potential: Two electric potential functions that satisfy the same boundary conditions must be identical.

This says that once we have picked a reference for electric potential, only one function of space can satisfy all boundary conditions.

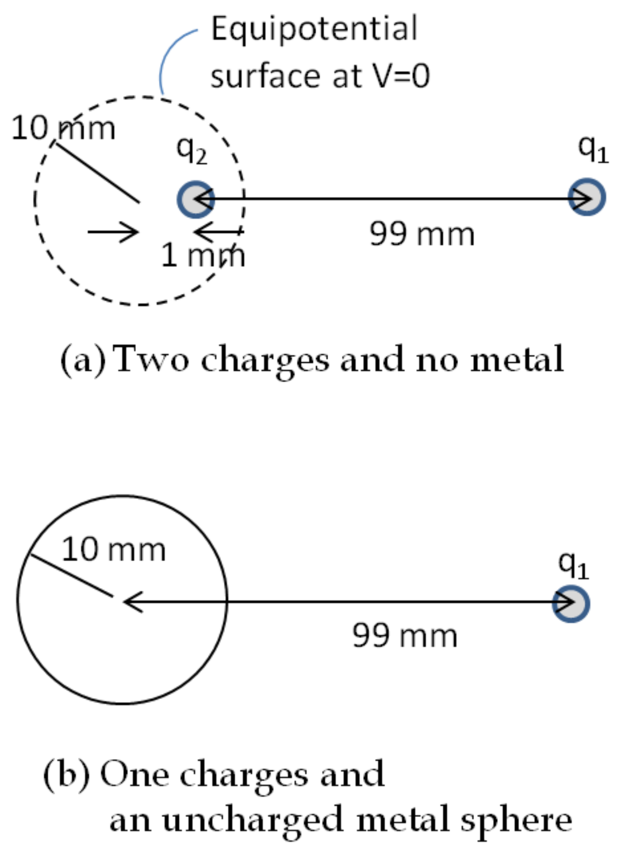

Figure 4.23: The spherical equipotential of a two charge system can be covered by a conductor leaving the potential at all points outside the sphere same as before. The two-charge system in (a) has same potential outside the sphere as a system consisting of one charge and an uncharged sphere have the same potential everywhere outside the sphere.

Therefore, if you can guess a function that satisfies the potentials at all boundaries in a particular situation, then that guess will be equivalent to the solution of the problem done by other methods. The uniqueness theorem gives a license to guessing - if two problems have the same potential values at all boundary points of a given region, then the two problems give same potentials at all points of the region.

Example 4.5.1. Potential Between Two Metallic Spherical Shells. A spherical metallic shell of outer radius R_1 is surrounded by another metallic spherical shell of radius R_2. The two shells are maintained at different potentials, V_1 and V_2 respectively? What is the potential at a point P between the two spherical shells?

Solution. There are three conditions for potential function $V(r)$ to satisfy here, two at the boundaries and one at the reference point at infinity.

$$V(r) = V_1 \quad \text{when} \quad r = R_1$$
$$V(r) = V_2 \quad \text{when} \quad r = R_2$$
$$V(r) = 0 \quad \text{when} \quad r = \infty$$

The following function would satisfy these conditions and hence gives potential at any point between the two shells.

$$V(r) = \left(\frac{R_1 R_2}{R_2 - R_1}\right)\left[V_1\left(\frac{1}{R_1} - \frac{1}{r}\right) + V_2\left(\frac{1}{r} - \frac{1}{R_2}\right)\right]$$

4.5.2 Method Of Images

The method of images utilizes the uniqueness theorem of potentials. Essentially it is a method for guessing an equivalent problem for which the equipotential surface will coincide with the given conductor's surface. The guessing of an equivalent problem in the case of planar conductors is especially easy since it is similar to finding images in plane mirrors. Therefore, we illustrate the method of images for a simple case of a point charge in front of a plane conductor at zero potential.

Example 4.5.2. Point charge in front of a grounded conductor. Consider a point charge +q a distance h in front of an infinite planar conductor at zero potential. Find electric field and electric potential in the space that is on the same side of the planar conductor as charge +q.

Solution. Fig. 4.24(a) shows charge $+q$ in front of a grounded conductor plate. Let the conducting plane be the xy plane and the point charge $+q$ be on the z axis. We look for a problem that has charge $+q$ at $z = h$ and some other charge or charges arranged in such a way that the potential is zero for all point in the z=0 plane.

As shown in Fig. 4.24(b), there is a problem indeed that has the desired characteristics: two charges, $+q$ at $z = h$ and $-q$ at $z = -h$. We replace the original problem of one charge in front of a $V = 0$ conducting plane with a problem of two charges $+q$ and $-q$ separated by a distance $2h$ and no conducting plane as displayed in Fig. 4.24(b). The electric potential of the problem in Fig. 4.24(b) for $z \geq 0$ is identical to the electric potential of the problem in Fig. 4.24(a).

The potential at point P of the two charge system will be same as the potential for one charge and a $V = 0$ infinite plane for $z \geq 0$

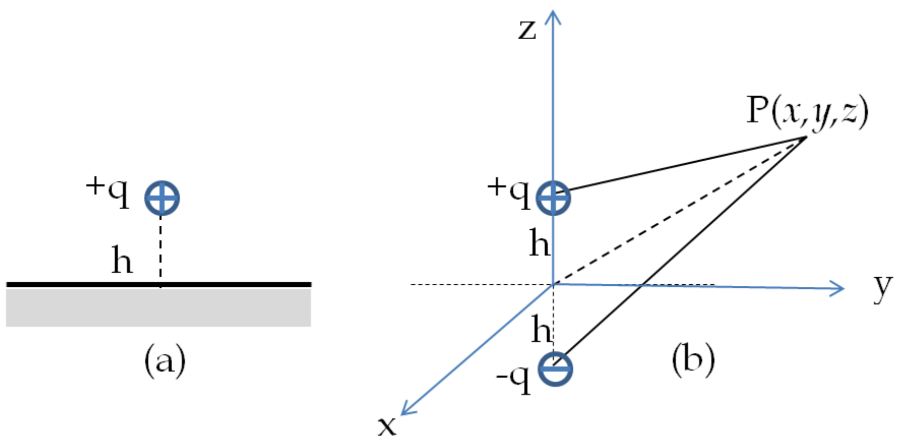

Figure 4.24: Method of images. (a)A point charge in front of a grounded infinite planar conductor. (b) Two point charges which have a zero potential surface that falls in the same place as the planar conductor's surface. The potential at point P with coordinates (x,y,z) is easily written from the "equivalent problem".

in order to coincide with the boundary condition.

$$V_P = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-h)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+h)^2}}\right]. \quad (4.22)$$

From the electric potential for a point charge in front of a plane conductor found above we can determine the electric field. The components of the electric field are

$$E_x = \frac{\partial V_P}{\partial x}; \quad E_y = \frac{\partial V_P}{\partial y}; \quad E_z = \frac{\partial V_P}{\partial z} \quad (4.23)$$

In Fig. 4.25, we show the equipotential lines and the electric field lines in the yz plane of Fig. 4.24(b). We see that the field lines bend and land perpendicular to the conducting plate where negative charges on the conducting plate is induced. The negative charges of the plate are attracted by the presence of the positive point charge, and there is an excess of negative charge on the plate near the fixed point charge. Of course, there is an equal positive charge at far away points of the plate. The distribution of negative charges on the plate can be determined from the difference of the normal component of the electric field across the surface.

$$\Delta E_\perp = \frac{\sigma}{\epsilon_0} \quad \Longrightarrow E_z|_{z>0} - E_z|_{z<0} = \frac{\sigma_{ind}(x,y,0)}{\epsilon_0}$$

where $\sigma_{ind}(x,y,0)$ is the induced charge density distribution on the plate. Using the z component of the electric field, we find the follow-

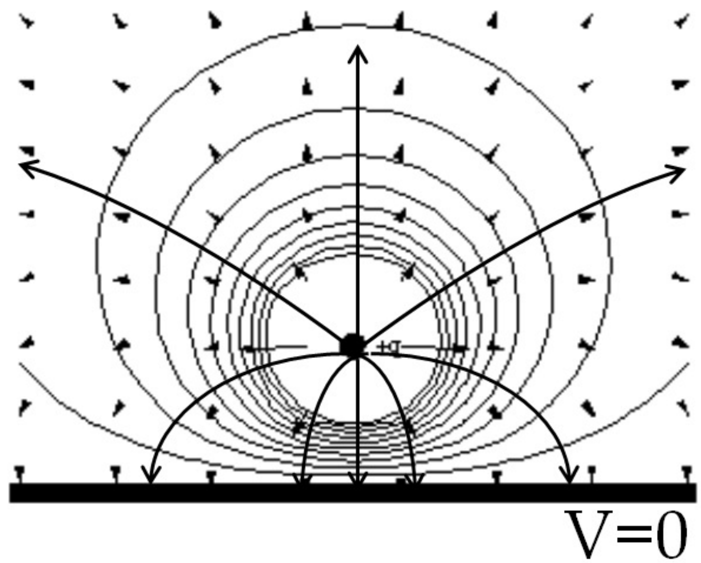

Figure 4.25: Equipotential and electric field lines for a point charge near a conducting plate.

ing for the surface charge density on the plate.

$$\sigma_{ind}(x, y, z) = \frac{-qh/2\pi}{(x^2 + y^2 + h^2)^{3/2}} \tag{4.24}$$

The total induced negative charge can be obtained by integrating ind over the plate, which goes to infinity. It is best to do this in the polar coordinates, (s, ϕ, z). The induced charge on the plate is function of only the radial coordinate s, which is integrated of the surface area of xy plane to yield that the induced charge is equal to the negative of the point charge of the point charge in front of the grounded plane.

$$\sigma_{ind}(s) = \frac{-qh/2\pi}{(s^2 + h^2)^{3/2}}$$

We can integrate the surface charge density over the plane and obtain the net charge induced in the plane.

$$q_{ind} = \int \sigma_{ind} 2\pi s ds = -q$$

Due to the induced charge on the conductor, there is a force of attraction between the point charge and the conducting plate, which we can determine fro

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4h^2} (-\hat{u}_z) \tag{4.25}$$