9.5 ROTATIONAL WORK AND KINETIC ENERGY

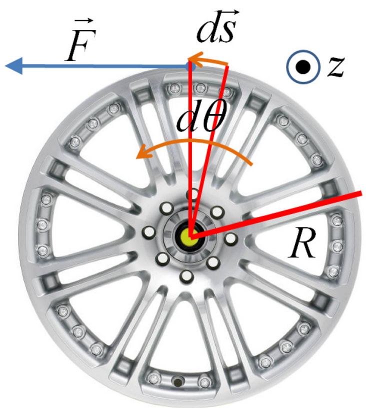

Figure 9.40: Rotational work by the force applied at the rim is $dW = \vec{F} \cdot d\vec{s}$.

Consider rotating a rigid wheel about its axle by applying a force $\vec{F}$ at the rim as shown in Fig. 9.40. In time dt, the force acts along the rim for a displacement ds. Therefore, the work done by the force will be

$$dW = F ds. \tag{9.86}$$

This work can be written in terms of the torque by this force about the axle and the angle $d\theta$ of rotation. If the radius of the wheel is R then, we will have $ds = R d\theta$, and the work will be given as

$$dW = F ds = F R d\theta. \tag{9.87}$$

The torque by the force applied at the rim is $\tau = FR$. Therefore, the work by the force in rotating the wheel by and angle $d\theta$ is

$$dW = \tau d\theta. \tag{9.88}$$

This work is called rotational work. We will write this work using vector notation so that the result can be applied more generally. We also add "rot" to the symbol of work to indicate that we are talking about rotational work.

$$dW^{\text{rot}} = \vec{\tau} \cdot d\vec{\theta}. \tag{9.89}$$

We can deduce an analogue of the work-energy theorem and discover the expression for the rotational kinetic energy by examining the work done over an interval. The work by all torques will be obtained by integrating $\vec{\tau} \cdot d\vec{\theta}$ from the initial time to the final time. Since we are dealing here with only a fixed-axis rotation about the z-axis, we can write the result as integral over the z-component of the net torque.

$$W_{fi}^{\text{rot,net}} = \int_i^f \tau_z^{\text{net}} d\theta_z \tag{9.90}$$

Using the equation of motion for rotation, we can replace the torque by the rate of change of angular momentum.

$$\tau_z^{\text{net}} = I_{zz} \frac{d\omega_z}{dt} \quad (\text{ rigid body}) \tag{9.91}$$

where we have assumed that the moment of inertia component I_{zz} is not a function of time as would be the case with a rigid body. Putting Eq. 9.91 in Eq. 9.90, and using the definition $\omega_z = d\theta_z/dt$, we find that the net rotational work on a body between two instances in time obeys the following relation.

$$W_{fi}^{\text{rot,net}} = \left(\frac{1}{2} I_{zz}\omega_z^2\right)_f - \left(\frac{1}{2} I_{zz}\omega_z^2\right)_i. \tag{9.92}$$

The quantity $\frac{1}{2} I_{zz}\omega_z^2$ is rotational kinetic energy of the body rotating around a fixed axis along the z-axis.

$$\text{Fixed axis rotation about } z\text{-axis:} \quad K_{\text{rot}} = \frac{1}{2} I_{zz}\omega_z^2, \tag{9.93}$$

which is written more compactly as $K_{\text{rot}} = \frac{1}{2} I\omega^2$. Thus, a rotational work gives rise to a change in the rotational kinetic energy. This is analogous to the work-energy theorem we have worked out before. The relation between the rotational work and the resulting change in rotational kinetic energy, often called the **rotational work-energy theorem**, is written as:

$$\boxed{W_{fi}^{\text{rot,net}} = (K_{\text{rot}})_f - (K_{\text{rot}})_i\,.} \tag{9.94}$$