3.7 MOTION USING POLAR COORDINATES

Some planar motions are more effectively analyzed in a different coordinate system than the Cartesian coordinates illustrated above. Many coordinate systems exist that help one make use of particular symmetries of the motion. For instance, polar coordinates are more natural for circular and elliptical trajectories. In this section, we will introduce polar coordinates and define new unit vectors for analysis of vectors.

3.7.1 Polar Coordinates

A point P with coordinates (x, y) in the xy-plane of a Cartesian coordinate system is at a distance r from the origin and at an angle θ counter-clockwise from the the positive x-axis. The distance r and angle θ are two coordinates of the **polar coordinate** (r, θ) as shown in Fig. 3.34. The polar coordinate r is called the radial coordinate and the polar coordinate θ the angular coordinate. You can see that

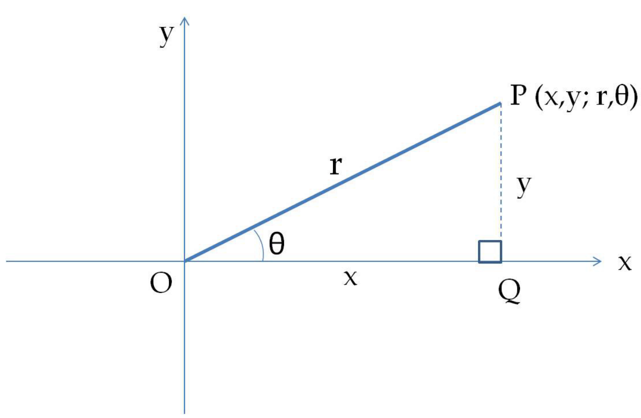

Figure 3.34: Polar coordinates r and θ. The right-angled triangle $\triangle OPQ$ shows the relation between the Cartesian coordinates (x, y) and polar coordinates (r, θ) for the same point P.

the polar coordinates are simply the magnitude (r) and direction (θ) of the position vector ($\vec{r} = x\hat{u}_x + y\hat{u}_y$). The relations between the polar coordinates (r, θ) and the Cartesian coordinates (x, y) for the same point are

$$\boxed{x = r\cos\theta} \tag{3.50}$$

$$\boxed{y = r\sin\theta} \tag{3.51}$$

$$\boxed{r = \sqrt{x^2 + y^2}} \tag{3.52}$$

$$\boxed{\theta = \arctan{(y/x)}\ \ \text{Beware of quadrants!}} \tag{3.53}$$

Polar coordinates in xy-plane together with the Cartesian z is called Cylindrical coordinates. Cylindrical coordinate system is useful for studying properties that have a symmetry about an axis, which is taken to be the z-axis.

3.7.2 Unit Vectors $\hat{u}_r$ and $\hat{u}_\theta$

Any vector in a plane can be written as a sum of two mutually independent vectors. Two vectors are mutually independent if their directions are different and one cannot be turned into the other by a multiplication with a scalar number. We have seen an application of this fact by constructing a vector in the xy-plane from the sum of a vector along the x-axis and another along the y-axis; the vector along the x-axis was obtained by multiplying the unit vector $\hat{u}_x$ pointed towards the positive x-axis with a real number, called the x-component of the vector, and the vector along the y-axis was obtained by multiplying the unit vector $\hat{u}_y$ pointed towards the positive y-axis with a real number, called the y-component of the vector.

Although, vectors along the Cartesian axes are very important for analysis of motion, they are not necessarily the only choice - any other mutually independent vectors would work also. In writing kinematic equations in polar coordinates, one often uses other sets of mutually independent vectors than vectors parallel to the x and y-axes.

As it turns out, there are infinitely many pairs of perpendicular vectors, one pair for each ray from the origin in the xy-plane, are defined for writing kinematics in polar coordinates. Consider a particular ray from origin to infinity passing through point P with Cartesian coordinates (x, y) and polar coordinates (r, θ) shown in Fig 3.35. The vector quantities, such as the position, velocity, and acceleration vectors, for a particle whose position falls on this ray can be expressed using the following pair of two mutually perpendicular unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ as we will show below.

$$\hat{u}_r = \cos\theta\ \hat{u}_x + \sin\theta\ \hat{u}_y \tag{3.54}$$

$$\hat{u}_\theta = -\sin\theta\ \hat{u}_x + \cos\theta\ \hat{u}_y \tag{3.55}$$

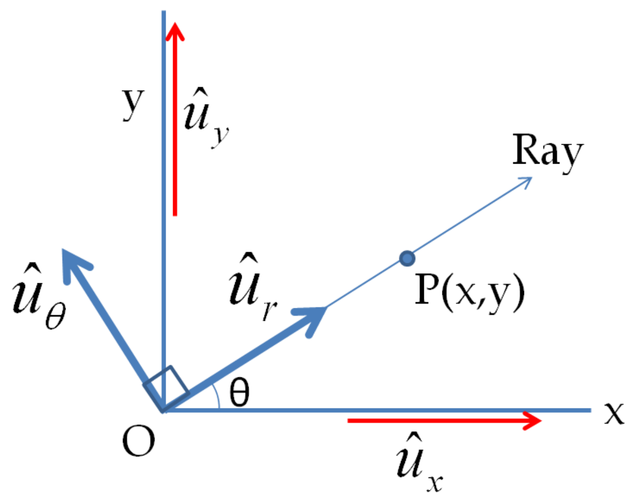

Figure 3.35: Polar unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ in the direction of the ray from origin to infinity at an angle θ with respect to the x-axis. Unlike $\hat{u}_x$ and $\hat{u}_y$ for the Cartesian axes directions, there are infinitely many polar vectors $\hat{u}_r$ and $\hat{u}_\theta$, one pair for each direction in the plane given by the angle θ. For instance, if P is on the x-axis, $\hat{u}_r = \hat{u}_x$ and $\hat{u}_\theta = \hat{u}_y$, and if P is on the y-axis, $\hat{u}_r = \hat{u}_y$ and $\hat{u}_\theta = -\hat{u}_x$.

It is easy to see that these vectors are mutually perpendicular by explicitly working out their dot product.

$$\hat{u}_r \cdot \hat{u}_\theta = 0. \tag{3.56}$$

You may find in other books that the vectors $\hat{u}_r$ and $\hat{u}_\theta$ are denoted as simply $\hat{r}$ and $\hat{\theta}$. For pedagogical reasons we will continue to practice our notation, although at times it may appear too cumbersome to write u all the time and you may want to use the simpler notation when there is no confusion.

Equations 3.54 and 3.55 show that the polar unit vectors depend on the angle θ, which can vary from 0 to 2π radians, but the unit vectors on the x and y-axes do not depend on θ. Therefore, there will be infinitely many $\hat{u}_r$ and $\hat{u}_\theta$ vector pairs, one for each value of θ, while the Cartesian unit vectors $\hat{u}_x$ and $\hat{u}_y$ are fixed once you have chosen the direction of the Cartesian axes.

Note that Eqs. 3.54 and 3.55 can be inverted so that we can write the unit vectors $\hat{u}_x$ and $\hat{u}_y$ in terms of $\hat{u}_r$ and $\hat{u}_\theta$.

$$\hat{u}_x = \cos\theta\ \hat{u}_r - \sin\theta\ \hat{u}_\theta \tag{3.57}$$

$$\hat{u}_y = \sin\theta\ \hat{u}_r + \cos\theta\ \hat{u}_\theta \tag{3.58}$$

How can we write a vector in terms of polar unit vectors if the Cartesian components of the vector are known? Suppose we have an arbitrary vector $\vec{A}$ which has Cartesian components A_x and A_y. Then, by using Eqs. 3.57 and 3.58 we can replace $\hat{u}_x$ and $\hat{u}_y$ in terms of $\hat{u}_r$ and $\hat{u}_\theta$.

$$\begin{aligned}
\vec{A} &= A_x\hat{u}_x + A_y\hat{u}_y \\
&= A_x(\cos\theta\ \hat{u}_r - \sin\theta\ \hat{u}_\theta) + A_y(\sin\theta\ \hat{u}_r + \cos\theta\ \hat{u}_\theta) \\
&= (A_x\cos\theta\ + A_y\sin\theta\ )\,\hat{u}_r + (-A_x\sin\theta\ + A_y\cos\theta\ )\,\hat{u}_\theta
\end{aligned}$$

This shows that every vector in the xy-plane can be written as sum of a real number times $\hat{u}_r$ and another real number times $\hat{u}_\theta$. The multiplying factor of these unit vectors are called r and θ components, although there will be infinitely many of them since there are infinitely many ($\hat{u}_r$,$\hat{u}_\theta$) pairs. We can denote these components with a similar notation as used for the Cartesian components, namely by attaching a subscript to the symbol for the vector without the arrow.

$$\vec{A} = A_r\hat{u}_r + A_\theta\hat{u}_\theta. \tag{3.59}$$

3.7.3 Velocity and Acceleration in Polar Coordinates

In this section we will deduce the formula for velocity and acceleration vectors written in polar coordinates. We start with noting that the position vector $\vec{r}$ is readily written in polar coordinates by simply substituting x and y in its definition in the Cartesian representation.

$$\begin{aligned}
\vec{r} &= x\hat{u}_x + y\hat{u}_y \\
&= r\cos\theta\,(\cos\theta\ \hat{u}_r - \sin\theta\ \hat{u}_\theta) + r\sin\theta\,(\sin\theta\ \hat{u}_r + \cos\theta\ \hat{u}_\theta) \\
&= \left(r\cos^2\theta\ + r\sin^2\theta\ \right)\hat{u}_r
\end{aligned}$$

Therefore,

$$\boxed{\vec{r} = r\hat{u}_r.} \tag{3.60}$$

To deduce the polar form of the velocity vector we recall the following two facts: (1) velocity is equal to the derivative of the position vector and (2) the Cartesian directions are constant. Expressing the position vector in terms of Cartesian coordinates and taking derivatives leads to

$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{dx}{dt}\hat{u}_x + \frac{dy}{dt}\hat{u}_y \quad (\text{since } d\hat{u}_x/dt = 0 = d\hat{u}_y/dt) \\
&= \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\hat{u}_x + \left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\hat{u}_y \\
&= \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta \quad (\text{replacing unit vectors and rearranging terms.})
\end{aligned}$$

Therefore, the velocity vector in polar coordinates is

$$\boxed{\vec{v} = \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta.} \tag{3.61}$$

The rate of change of the radial coordinate, i.e. the quantity dr/dt is called radial velocity, and the rate of change of the angular coordinate, i.e., the quantity $d\theta/dt$ is called the angular velocity. Note that neither dr/dt nor $d\theta/dt$ is a vector since they do not have directions. More appropriate names would be radial velocity component and angular velocity component, but we will stick with the standard names for these quantities and make a mental note that, despite their names, the rates of change of radial and angular coordinates are not vectors. The angular velocity $d\theta/dt$ is often denoted by another symbol, ω.

$$\boxed{\text{Magnitude of the angular velocity: } \omega = \frac{d\theta}{dt}.} \tag{3.62}$$

When we study rotation, we will encounter the true angular velocity vector which would have both a magnitude and direction. We will find that the direction of angular velocity vector of a rotating particle is perpendicular to the position vector and velocity vector.

Making use of ω simplifies the writing of the formulas for velocity and acceleration as we will see below. Thus, the velocity of an object in the xy-plane can be written as

$$\vec{v} = \frac{dr}{dt}\hat{u}_r + \omega r\hat{u}_\theta. \tag{3.63}$$

Therefore, the radial and angular components of the velocity vector $\vec{v}$ are

$$\boxed{v_r = \frac{dr}{dt}.} \tag{3.64}$$

$$\boxed{v_\theta = r\frac{d\theta}{dt} = r\omega.} \tag{3.65}$$

Similarly, starting with $\vec{r} = x\hat{u}_x + y\hat{u}_y$ and taking two time derivatives gives the following expression for the acceleration vector.

$$\boxed{\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{u}_r + \left[2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right)\right]\hat{u}_\theta.} \tag{3.66}$$

This gives the radial and angular components of the acceleration

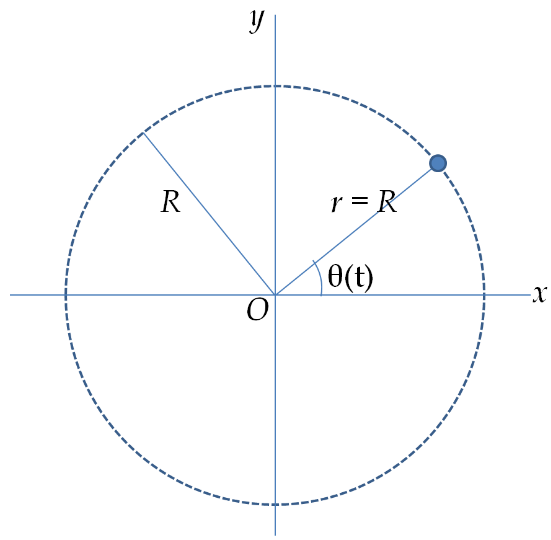

Figure 3.36: Radial coordinates in circular motion.

vector $\vec{a}$ as

$$\boxed{a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.} \tag{3.67}$$

$$\boxed{a_\theta = 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right).} \tag{3.68}$$

In the case of circular motions, these components, a_r and a_θ with their corresponding unit vectors are also called centripetal and tangential accelerations respectively.

3.7.4 Circular Motion

Speed and Velocity

Polar coordinates are natural for an object moving in a circle. Let us denote the radius of the circle by the capital letter R. With origin at the center of the circle, the radial polar coordinate r is equal to the radius of the circle and remains constant in time (see Fig. 3.36). Incidentally, note that if the origin is not at the center of the circle, then the radial coordinate will change with time even for a particle moving in a circle.

$$\text{Radial coordinate: } r = R, \text{ constant.} \tag{3.69}$$

Therefore, the only coordinate that changes with time is the angular coordinate of the object, $\theta(t)$. Recall that the rate of change of the angular coordinate is given by its own symbol, $\omega = d\theta/dt$. Hence, for a circular motion, the position and velocity vectors take the following simpler forms. **Circular Motion:**

$$\boxed{\vec{r} = R\hat{u}_r.} \tag{3.70}$$

$$\boxed{\vec{v} = R\omega\hat{u}_\theta.} \tag{3.71}$$

The magnitude of the velocity is equal to the product of the radius of the circle and the angular velocity. This makes sense if you think about the distance on the circle being covered with time. If in time Δt, the angle coordinate changes by $\Delta\theta$, distance moved on the arc of the circle would be equal to $R\Delta\theta$. Therefore, your average speed will be

$$\text{Average speed} = \frac{R\Delta\theta}{\Delta t}.$$

In the limit of infinitesimal time, we find the formula for instantaneous speed as in terms of the derivative of the angular coordinate.

$$\boxed{\text{Circular motion instantaneous speed} = R\left|\frac{d\theta}{dt}\right| = R\left|\omega\right|,}$$

where I have used the absolute sign so that speed is positive regardless of which way the object moves in the circle - movement in one direction on the circle leads to increasing angle values, giving a positive ω, and in the opposite direction on the circle corresponds to decreasing angle, and hence negative ω.

Eq. 3.71 for velocity tells us that the velocity of an object moving in a circle is tangential to the circle, i.e. in the direction of the unit vector $\hat{u}_\theta$ or in the opposite direction to $\hat{u}_\theta$. Since the tangential direction in space changes from place to place in the circle, the velocity of a particle moving in a circle always changes with time as shown in Fig. 3.37.

Tangential and Centripetal Accelerations

For circular motion we set $r = R$, a constant, in the definition of acceleration in Eq. 3.66 to find

$$\boxed{\textbf{Circular motion:}\ \ \vec{a} = -R\omega^2\hat{u}_r + R\left(\frac{d\omega}{dt}\right)\hat{u}_\theta,} \tag{3.72}$$

where we find that the acceleration at any time is a sum of a radially pointed vector, the first term, and a tangentially pointed vector, the second term. These two contributions to acceleration are called **centripetal and tangential accelerations**, respectively.

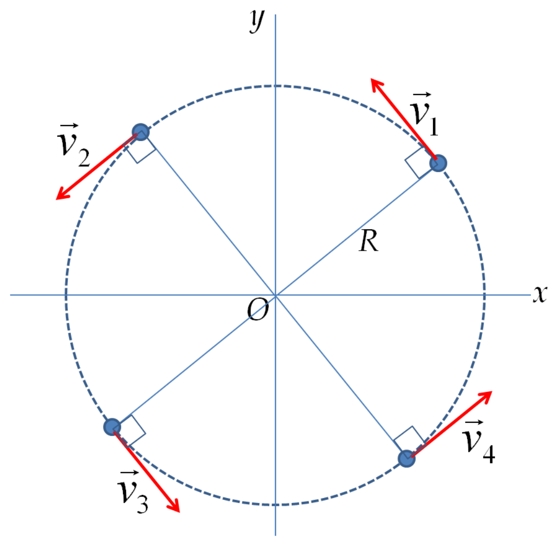

Figure 3.37: The direction of the velocity of a particle at a particular point in a circular motion is along the tangent to the circle at that point. Since tangent is changing, the direction of the velocity is always changing. Therefore, a particle in circular motion will always have non-zero acceleration.

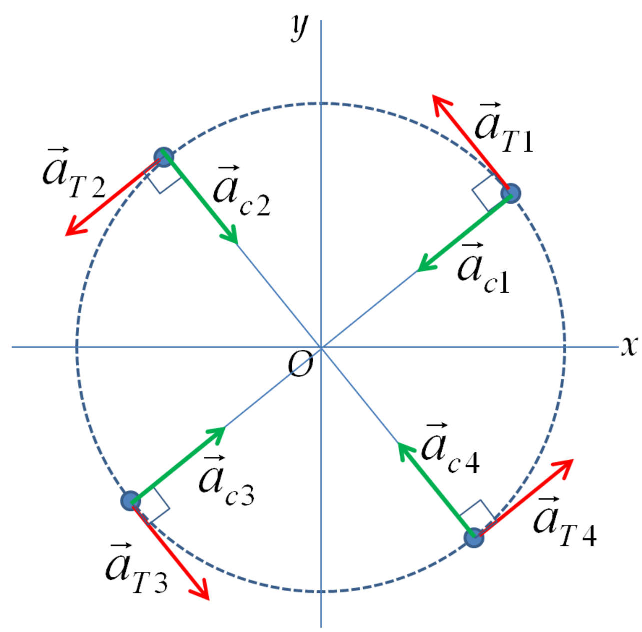

Figure 3.38: Tangential and centripetal accelerations of a particle in a circular motion are pointed in different directions at different points on the circle . The tangential acceleration is the acceleration along the tangent to the circle and the centripetal acceleration is the acceleration towards the center of the circle.

$$\boxed{\text{Centripetal acceleration: } \vec{a}_c = -R\omega^2 \hat{u}_r.} \tag{3.73}$$

$$\boxed{\text{Tangential acceleration: } \vec{a}_T = R\left(\frac{d\omega}{dt}\right)\hat{u}_\theta.} \tag{3.74}$$

Figure 3.38 shows $\vec{a}_c$ and $\vec{a}_T$ at four points in a circle. Figure shows that the two parts of the acceleration of an object moving in circle are perpendicular to each other at all time and their directions change throughout the motion of the object. Often the magnitudes of $\vec{a}_c$ and $\vec{a}_T$ are themselves called centripetal and tangential accelerations with the tacit understanding the direction will be stated by drawing arrows on a circle at the instant of interest.

$$\text{Magnitude of Centripetal acceleration: } a_c = R\omega^2. \tag{3.75}$$

$$\text{Magnitude of Tangential acceleration: } a_T = R\left(\frac{d\omega}{dt}\right) \tag{3.76}$$

Uniform circular motion

Since the velocity of a circular motion is tangential, only a non-zero tangential acceleration can change the magnitude of the velocity. Therefore, if $a_T = 0$, then speed does not change. We see from the formula for the tangential acceleration, that if $d\omega/dt = 0$, then $a_T = 0$. The circular motion of constant speed or constant angular speed is called **uniform circular motion**.

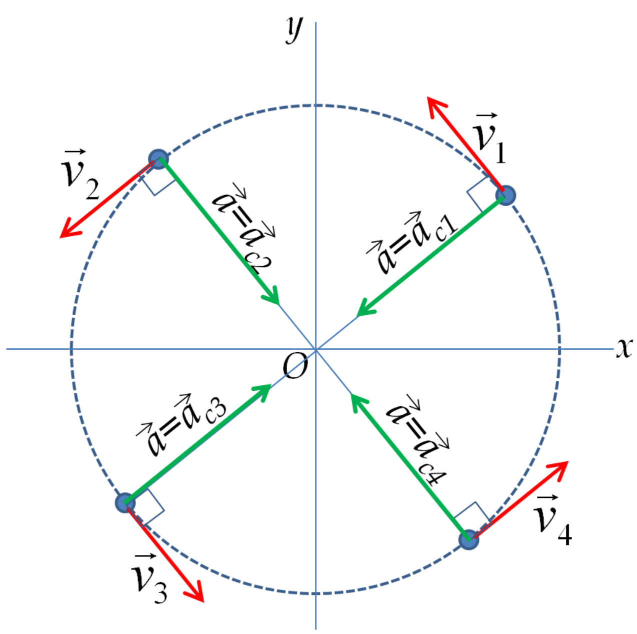

Figure 3.39: Uniform circular motion. The tangential acceleration is zero so that speed is constant although velocity is not constant, $|\vec{v}_1| = |\vec{v}_2| = |\vec{v}_3| = |\vec{v}_4|$ at points around the circle. The acceleration is entirely centripetal at each point.

The acceleration vector for the uniform circular motion points towards the center of the circle. Since ω for a uniform circular motion

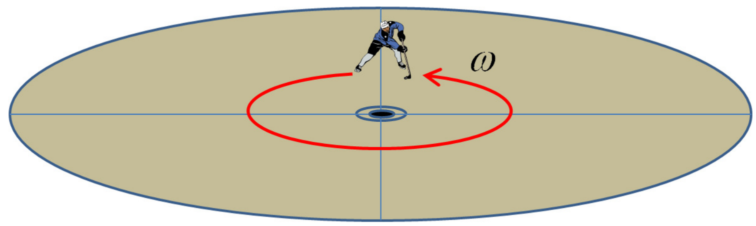

Figure 3.40: Example 3.7.1. A rotating platform.

does not change with time, the magnitude of the centripetal acceleration is constant. However, that does not mean that the acceleration vector is constant. For an object moving in a circle the radially inward direction ($-\vec{u}_r$) is pointed in different directions in the plane for different points on the circle (see Fig. 3.39).

For instance, when the object is at the x-axis, the acceleration vector in Eq. 3.73 is pointed towards the negative x-axis, and when the object is at the y-axis, the acceleration vector is pointed towards the negative y-axis. Therefore, the acceleration of an object in circular motion, always changes with time, even when the magnitude of the acceleration may be constant.

The magnitude of the centripetal acceleration can also be written in terms of the constant speed in the uniform circular motion by replacing ω^2 with $(v/R)^2$.

Centripetal acceleration for uniform circular motion:

$$\boxed{\text{Magnitude: } |\vec{a}| = |\vec{a}_c| = \frac{v^2}{R} \quad (\text{since } a_T = 0).} \tag{3.77}$$

Now, I will illustrate the use of polar coordinates with numerical and non-numerical problems.

Example 3.7.1. A rotating platform. Consider a rotating platform of radius 5 meters that is rotating at a constant angular velocity of 0.5 radians per second. (a) What will be your speed when you are standing at the edge? (b) What will be your acceleration? (c) Find your speed and acceleration when you are standing 2 meters from the center.

Solution. Note that parts (a) and (b) are about the circular motion when the radius of the circle is $R = 5$ m and part (c) is for a circular motion in a different radius, viz., $R = 2$ m. Using the formulas presented above for uniform circular motions, we immediately find the following answer. (a) Speed, $v = R\omega = (5 \text{ m}) \times (0.5 \text{ rad/s}) = 2.5$ m/s. Note that m $\times$ rad $=$ m, since the radian is dimensionless as it is the ratio of arc length to radius, both of dimension length.

(b) Now, since the question is about a vector, we must find the magnitude and direction.

Magnitude of acceleration $= R\omega^2 = (5 \text{ m}) \times (0.5 \text{ rad/s})^2 = 1.25 \text{ m/s}^2$.

Direction of acceleration: It is pointed towards the center. The physical direction in space depends on where on the circle the object is at the time.

(c) Similar calculations as those of parts (a) and (b) give the following answer. Speed $= 1.0$ m/s and magnitude of acceleration 0.5 m/s^2. The direction of the acceleration is pointed towards the center whose orientation in plane changes with time.

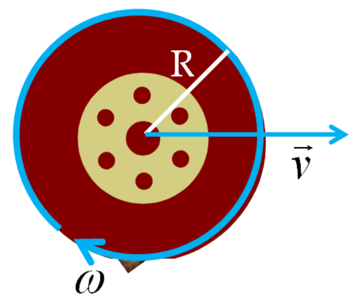

Figure 3.41: Example 3.7.2. Pebble stuck at the edge of a tire.

Example 3.7.2. Pebble on a rotating tire. A pebble is stuck in the grooves of a rotating tire of radius 25 cm. If the car moves at a constant velocity of 20 m/s what is the angular velocity of the pebble?

Solution. Notice that when the center of the tire moves a distance $2\pi R$, i.e. a full circle of the tire, the pebble rotates by an angle of 2π radians in a circle about the center of the tire whose radius is equal to the radius of the tire R. Since the center of the tire moves 20 m in 1 sec, the pebble rotates an angle of $20\text{m}/R$ radians in that time. Therefore, the angular speed of the pebble is 80 rad/s, since $R = 0.25$ m.

Example 3.7.3. Particle in uniform circular motion. A particle moves in a circle of radius a with a uniform circular motion of constant angular velocity $\omega = \omega_0$. The particle moves in the xy-plane in a counter-clockwise manner when observed from the positive z-axis with the center of the circle of motion at the origin. At $t = 0$ the particle is at the point whose Cartesian coordinates are $(a, 0, 0)$ with respect to origin at the center of the circle of motion. Where will the particle be at an arbitrary time T?

Solution. Since the particle is moving in a circle, it is advantageous to work in polar coordinates. We have the radial coordinate given to be $r = a$, which is independent of time. Therefore, we need to find the angular coordinate at time T. The angular coordinate changes with angular velocity ω, which is written in terms of derivative of the angular variable.

$$\frac{d\theta}{dt} = \omega_0.$$

This equation can be solved for general time t.

$$\theta(t) = \theta(0) + \omega_0 t.$$

The angle at zero time is zero since the particle was located on the x-axis at that instant. Therefore, the angle coordinate at the desired time is

$$\theta(T) = \omega_0 T.$$

We have found the polar coordinates at the desired time to be $(r, \theta) = (a, \omega_0 T)$. We can convert these into Cartesian coordinates if we so desire.

Example 3.7.4. Particle in a non-uniform circular motion A particle moves in a circle of radius a with a varying angular velocity $\omega = bt$ counterclockwise, where b is constant. At $t = 0$ the particle has Cartesian coordinates are $(a, 0, 0)$ with respect to the origin at the center of the circle of motion which is at the origin. Where will the particle be at an arbitrary time T?

Solution. In this example we have kept the data similar to the last example, except that the angular velocity is not constant any more. Since the particle is moving in a circle, it is again advantageous to work in polar coordinates. We have the radial coordinate given to be $r = a$, which is independent of time. Therefore, we need to find the angular coordinate at time T. Now, making use of the definition of angular velocity in terms of the derivative of the angular coordinate we find that the angular coordinate changes according to the following equation.

$$\frac{d\theta}{dt} = bt.$$

This equation can be solved for general time t.

$$\theta(t) = \theta(0) + \frac{1}{2} b t^2.$$

The angle at $t = 0$ is zero since the particle was located on the x-axis. Therefore, the angular coordinate at the desired time is

$$\theta(T) = \frac{1}{2} b T^2.$$

We have found that the polar coordinates at the desired time is $(r, \theta) = (a, 1/2\ bT^2)$. We can convert these into Cartesian coordinates by standard procedure with the result $x = a\cos\left(\frac{1}{2} bT^2\right)$ and $y = a\sin\left(\frac{1}{2} bT^2\right)$ ..