12.5 CIRCUIT ANALYSIS USING COMPLEX NUMBERS

The equations resulting from the application of Kirchhoff's rules to an AC circuit results in the summation of sine and cosine functions. These equations are solved for amplitudes and phases of currents and voltages in the circuit. Since we study the steady state in a sinusoidally driven circuit, the time dependences of the currents and voltages in the AC circuit are expressible in sines and cosines as well.

These calculations using the trigonometric functions are usually messy and an alternative method is desirable. It turns out that the algebra of complex numbers is better at these calculations than the sine and cosine functions. Therefore in this section we present the use of complex numbers for analyzing AC circuits. After this section, you will have a choice - to continue to do the calculations in sines and cosines or use the methods of complex numbers. A brief review of complex numbers is presented in the Appendix at the end of this book. If you are not sure about the complex algebra you should study the Appendix first before reading the rest of this section.

The application of complex numbers to the AC circuits comes from the equivalence of phasors to complex exponentials. Each phasor represented above by a two-dimensional vector is analytically represented by a complex number, whose real part corresponds to the physical quantity. Thus, we think of voltages and currents as complex numbers. I will use a tilde sign over the symbol to indicate the corresponding complex representation. For instance,

$$V(t) = V_0 \cos(\omega t + \phi) \quad \Longrightarrow \quad \tilde{V}(t) = V_0 \, e^{i(\omega t+\phi)}. \tag{12.54}$$

The physical quantity is the real part of the complex representation.

$$V(t) = Re[\tilde{V}(t)], \tag{12.55}$$

which follows from the Euler's relation for complex numbers

$$e^{i\theta} = \cos(\theta) + i \, \sin(\theta), \tag{12.56}$$

where $\cos(\theta)$ is the real part and $\sin(\theta)$ is the imaginary part of the complex number $e^{i\theta}$. You may know that any complex number z can be represented either as $x + iy$ or as $re^{i\theta}$.

Currents are similarly replaced by complex currents.

$$I(t) = I_0 \cos(\omega t + \phi) \quad \Longrightarrow \quad \tilde{I}(t) = I_0 \, e^{i(\omega t+\phi)}. \tag{12.57}$$

The physical current is the real part of the complex current.

$$I(t) = Re[\tilde{I}(t)], \tag{12.58}$$

One big advantage of using the complex representation is that the operations of derivation and integration does not change the function. In the real representation, these operations turn a cosine into a sine and vice versa, but in the case of the complex representation the exponential stays the same.

$$\frac{d}{dt}\tilde{V}(t) = i\omega\tilde{V}(t) \tag{12.59}$$

$$\int \tilde{V}(t)dt = \frac{1}{i\omega}\,\tilde{V}(t) \tag{12.60}$$

As a result the complex voltage $\tilde{V}$ across an element and complex current $\tilde{I}$ through it are related by the complex version of Ohm's law, we can call **Complex Ohm's Law**.

$$\boxed{\tilde{V} = \tilde{Z}\tilde{I},} \tag{12.61}$$

where the complex "resistance" denoted by $\tilde{Z}$ is called the **impedance** or the **complex impedance** of the element. In the following we will work out the expressions for the complex impedance of the three passive circuit elements, the resistor, the inductor and the capacitor.

The absolute value of the impedance is equal to the amplitude of the impedance we have defined above.

$$|Z| = |\tilde{Z}|. \tag{12.62}$$

The inverse of the impedance is called admittance and is denoted by $\tilde{Y}$.

$$\tilde{Y} = \frac{1}{\tilde{Z}}. \tag{12.63}$$

We will see that the complex impedance allows one to treat resistors, capacitors and inductors in an AC circuit on an equal footing.

Complex Impedance of a Resistor

To find the impedance of a resistor consider a circuit with a resistor driven by an alternating EMF. Here, we know that the current is in phase with the voltage across the resistor. Therefore $\tilde{Z}_R$ is simply the resistance R, a real number.

$$\tilde{Z}_R = \frac{\tilde{V}}{\tilde{I}} = R \quad \text{(Resistor)} \tag{12.64}$$

Complex Impedance of an Inductor

If the real voltage across an inductor is $V_L(t)$ and current through the inductor L is $I(t)$, then we have found above that they are related as follows

$$V_L = L\frac{dI}{dt} \tag{12.65}$$

Now, we relpace $V_L(t)$ by $\tilde{V}_L(t)$ and $I(t)$ by $\tilde{I}(t)$ we find

$$\tilde{V}_L = L\frac{d\tilde{I}}{dt} \quad \Longrightarrow \quad \tilde{V}_L = i\omega L\tilde{I}. \tag{12.66}$$

Therefore, the impedance of an inductor is purely imaginary.

$$\tilde{Z}_L = \frac{\tilde{V}_L}{\tilde{I}} = i\omega L = \omega L\, e^{i\pi/2} \quad \text{(Inductor)} \tag{12.67}$$

Thus, the amplitude of the impedance of an inductor is ωL and the phase is $\pi/2$. The following notation is also used sometimes to keep track of the two aspects of the complex impedance.

$$\tilde{Z}_L = \omega L\ \angle(\pi/2). \tag{12.68}$$

This notation is referred to as the phasor notation since this keeps track of the two information we tend to follow in a phasor diagram.

Complex Impedance of a Capacitor

The relation between the voltage V_C across a capacitor and the current into the capacitor was found to be

$$\frac{dV_C}{dt} = \frac{1}{C}\, I \tag{12.69}$$

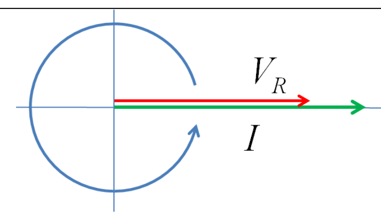
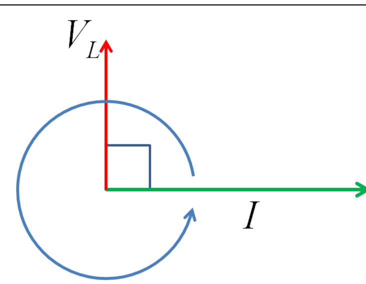
Replacing $V_C(t)$ by $\tilde{V}_C(t)$ and $I(t)$ by $\tilde{I}(t)$, and carrying a calculation similar to what we did in the case of an inductor, we find that the impedance of a capacitor is

$$\tilde{Z}_C = \frac{\tilde{V}_C}{\tilde{I}} = \frac{1}{i\omega C} = \frac{1}{\omega C}\, e^{-i\pi/2} \quad \text{(Capacitor)} \tag{12.70}$$

In the phasor notation the impedance of a capacitor is

$$\tilde{Z}_L = \frac{1}{\omega C}\ \angle(-\pi/2). \tag{12.71}$$

Table 12.1: Impedances and admittances of resistor, inductor and capacitor

Element	Impedance	Admittance	Phasor Relations
Resistor	R	$\frac{1}{R}$	
Inductor	$i\omega L$	$1/i\omega L$	
Inductor	$1/i\omega C$	$i\omega C$