13.4 EQUATION OF CONTINUITY AND CONSERVATION OF CHARGE

The principle of conservation of charge is automatically included in Maxwell's equations. To see it explicitly we can use the point form of Maxwell's equations. First, let us take the divergence of both sides of the Ampere-Maxwell's law and then use the Gauss's law for the electric field.

$$\begin{aligned}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) &= \vec{\nabla}\cdot\left(\mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\\ &= \mu_0\vec{\nabla}\cdot\vec{J}+\mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})\\ &= \mu_0\left(\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}\right)\end{aligned} \tag{13.48}$$

Since, the divergence of a curl is zero, the left side is identically zero. Therefore, we find that

$$\boxed{\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0.} \tag{13.49}$$

This equation is called the **equation of continuity**. The equation is a statement of conservation of charge at each space point. From our discussion on the divergence above, you know that the first term of this equation is equal to the charge per unit volume leaving a surface per unit time enclosing a point and the second term is the rate at which the density of charge in the volume changes with time. If no charge can be created or destroyed at any point, then the two terms must add up to zero as demanded by this equation.

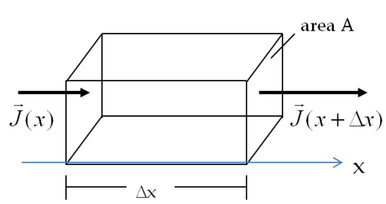

Figure 13.9: Charges flowing in and out of a box at different rate result in net accumulation in the box consistent with the conservation of charge.

Let us look at Eq. 13.49 more closely and see how the divergence of the current density gives us the rate at which charges leave a point. Consider a rectangular box in which the current enters at the left face (coordinate x) with the current density $\vec{J}(x)$, which may vary along the box as shown in Fig. 13.9. The current leaves the right face (coordinate $x+\Delta x$) with the current density there given by $\vec{J}(x+\Delta x)$. We now ask the question: how much charge accumulates in the box in a duration Δt? We can answer this question by the following step-by-step arguments. To simplify the discussion, we assume that the current density has only the x-component non-zero.

The amount of charge flowing in the box, at x, in time Δt, $= |J_x(x)A\Delta t|$.

The amount of charge flowing out of the box, at $x+\Delta x$, in time Δt, $= |J_x(x+\Delta x)A\Delta t|$.

Net charge accumulation due to flow in and out of the box $= |J_x(x) - J_x(x + \Delta x)|\, A\Delta t$.

Increase of charge density in the box in time Δt would be

$$\frac{|J_x(x) - J_x(x + \Delta x)|\, A\Delta t}{A\Delta x}.$$

Hence, the rate of increase of charge density in the box is equal to $|J_x(x) - J_x(x + \Delta x)|\, /\Delta x$. By the principle of the conservation of charge, this must be equal to the rate of increase of charge density in the box as accounted by the charge density itself.

$$\frac{\Delta\rho}{\Delta t} = \frac{|J_x(x)At - J_x(x + \Delta x)|}{\Delta x}$$

When we shrink the box by taking the limit $x \to 0$, we obtain the following rate of change of charge density at the point P in the box.

$$\frac{\partial\rho}{\partial t} = -\frac{dJ_x}{dx}$$

When you consider currents flowing in all directions you will get the full continuity equation.

$$\frac{\partial\rho}{\partial t} = -\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right)$$