10.10 INDUCTANCE

10.10.1 Mutual Inductance

Two separate loops of wires carrying time-varying currents will induce EMF into each other since the time-varying magnetic field of each circuit will pass through the other circuit. To study this effect closely let us consider two circuits or loops C_1 and C_2 fixed in position relative to one another as shown in Fig. 10.33. Let a constant current I_1 flow through circuit C_1. This current will produce a position dependent magnetic field $\vec{B}_1$. Let Φ_{21} be the magnetic flux of field $\vec{B}_1$ through the area of an open surface S_2 attached to loop C_2.

$$\Phi_{21} = \iint_{S_2} \vec{B}_1 \cdot d\vec{A}_2. \tag{10.47}$$

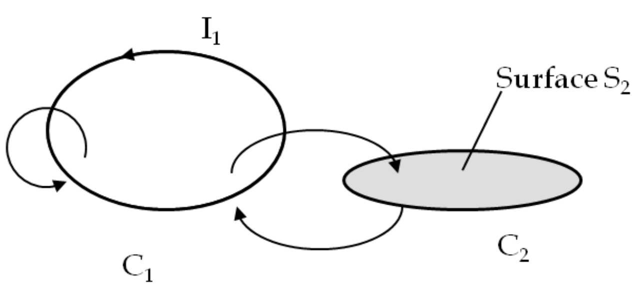


Figure 10.33: Mutual Inductance - Magnetic field of current in loop passes through the other loop.

Since $\vec{B}_1$ is proportional to I_1, we expect the flux of $\vec{B}_1$ to be also proportional to I_1.

$$\vec{B}_1 \propto I_1 \implies \Phi_{21} \propto I_1. \tag{10.48}$$

We can write this as an equality by introducing a proportionality constant which we will write as $\mathcal{M}$.

$$\boxed{\Phi_{21} = \mathcal{M} I_1}. \tag{10.49}$$

Constant $\mathcal{M}$ is called the coefficient of mutual inductance of the two circuits. Mutual inductance only depends on the geometry of the two circuits, their separation and the magnetic properties of the medium. The SI unit for $\mathcal{M}$ is known as Henry (H).

$$\text{Unit of } \mathcal{M} : \Omega.\text{sec} \equiv \text{Henry(H)}.$$

In the above we have defined the mutual inductance of two circuits from the magnetic flux through C_2 by the current in the circuit C_1. What would happen if we reversed the situation, i.e. pass a current I_2 in circuit C_2 and studied the magnetic flux of the magnetic field $\vec{B}_2$ through circuit C_1? In this case, we will have the flux Φ_{12} through a surface attached to the loop of circuit C_1 in terms of the current in circuit C_2 as

$$\Phi_{12} = \mathcal{M}' I_2, \tag{10.50}$$

where the mutual inductance is now written as $\mathcal{M}'$. We will not prove here, but you can try to prove that the two mutual inductances, $\mathcal{M}$ and $\mathcal{M}'$, must be equal, and we represent them by the same symbol $\mathcal{M}$.

$$\mathcal{M}' = \mathcal{M}. \tag{10.51}$$

To find a formula for the mutual inductance in a given situation, you can introduce a current in one of the loops and calculate the magnetic flux though the other loop. Then, Eq. 10.49 or Eq. 10.50 gives the expression for the mutual inductance. Since mutual inductance does not depend on which circuit we choose to send current through, we exploit this symmetry to do calculations for mutual inductance as illustrated below.

From Eq. 10.49, if current in circuit C_1 changes with time, then there will be a changing magnetic flux through the other circuit, namely C_2, even though the two circuits are not connected with each other. According to Faraday's flux rule, the changing magnetic flux will produce an induced EMF $\mathcal{E}_2$ in C_2. Using Eq. 10.49, Faraday's flux rule can be written as

$$\boxed{\mathcal{E}_2 = -\mathcal{M}\frac{dI_1}{dt}.} \tag{10.52}$$

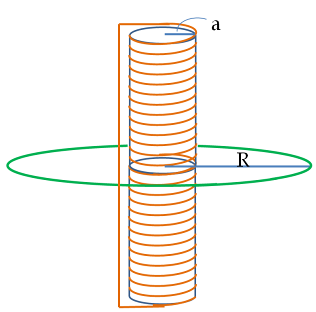


Figure 10.34: Example 10.10.1.

Example 10.10.1. Mutual inductance of two loops.

A solenoid loop C_1 consists of n circular loops of radius a per unit length. The length of the solenoid will be assumed to be much greater than the radius of the solenoid so that we can use the infinite solenoid approximation. Another circuit C_2 of a circular loop of radius R surrounds the solenoid such that the solenoid is at the axis of the second loop as shown in Fig. 10.34. Assume $a << R$ and find the mutual inductance of the two loops in two ways: (a) from the flux of current in C_1 through the area of a surface attached to the circuit C_2 and (b) from the flux of current in C_2 through the area of a surface attached to the circuit C_1, and demonstrate the two methods give rise to the same formula for the mutual inductance.

Solution. (a) **Calculation of mutual inductance by using current through the long solenoid.** Let us insert a battery in the circuit of the inductor so that a current I runs through the solenoid as shown in Fig. 10.35. Since, the solenoid is very long, the magnetic field is significant only inside the space of the solenoid. We use a cylindrical coordinate with the z-axis along the axis of the solenoid and the circular loop in the xy-plane. Let s be the distance of a point from the axis, i.e. the radial distance in the xy-plane. We know that the magnitude of the magnetic field of current in the solenoid is given by

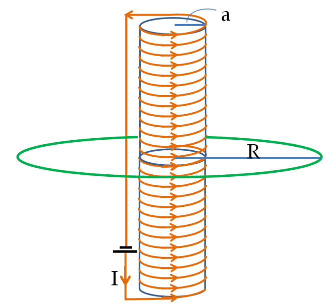


Figure 10.35: Example 10.10.1.

$$\vec{B}_{\text{solenoid}} = \begin{cases} \mu_0 n I \hat{u}_z & s < a \\ 0 & s \geq a \end{cases}$$

We attach a circular surface to the circular wire C_2 to find the flux through C_2. Let $\hat{u}_z$ be the normal to the surface. Since the magnetic field is zero outside the solenoid, the magnetic flux through C_2 comes only from the field in the area of the circular loop that is within the cross-section of the solenoid. This gives the following for the magnetic flux Φ_2 through C_2.

$$\Phi_2 = \mu_0 n I \times \pi a^2.$$

Dividing out the current gives the mutual inductance between the two circuits.

$$\mathcal{M} = \pi \mu_0 n a^2. \tag{10.53}$$

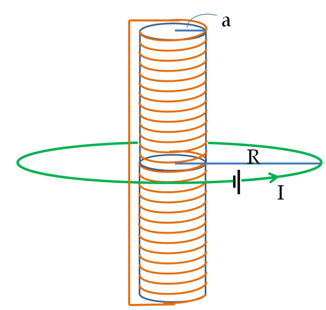


Figure 10.36: Example 10.10.1.

(b) **Calculation of mutual inductance by using current through the circular loop.** Let us insert a battery in the circuit of the loop around the solenoid so that a current I runs in circuit C_2 as shown in Fig. 10.36. The magnetic field will pass through a surface attached to the wires of the solenoid, whose flux can be calculated by summing over the flux through each ring.

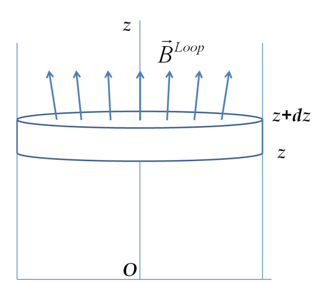


Figure 10.37: Example 10.10.1.

In the approximation $a << R$, the magnetic field at a point in the solenoid will be approximately the same as the value at the center of solenoid coil. Since the magnetic field of the current in the outer ring depends on the distance $|z|$ from the center of the ring, we can set up magnetic flux $d\Phi_1$ through loops of the solenoid between z and $z + dz$. Since, there are ndz loops of solenoid within a distance dz, the flux through the surface attached to this part of the solenoid will have ndz times flux through one loop at z. Therefore, $d\Phi_1$ is given as

$$d\Phi_1 = B_z^{\text{Loop}} \pi a^2 n dz, \tag{10.54}$$

where B_z^{Loop} is the z-component of the magnetic field of the current in the outer loop, which has been worked out in the section on Biot-

Savart law.

$$B_z^{\text{Loop}} = \frac{\mu_0}{2}\frac{IR^2}{\left(R^2+z^2\right)^{3/2}} \quad \text{(on axis.)} \tag{10.55}$$

Putting B_z^{Loop} from Eq. 10.55 into Eq. 10.54 and integrating from $z=-\infty$ to $z=\infty$ gives the following.

$$\Phi_1 = \pi\mu_0 n a^2 I. \tag{10.56}$$

Dividing by current gives the mutual inductance

$$\mathcal{M}' = \pi\mu_0 n a^2, \tag{10.57}$$

which is the same as the formula of mutual inductance found by sending a current through the solenoid instead of through the outer circular wire.

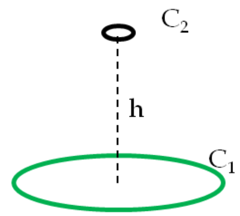


Figure 10.38: Example 10.10.2.

Example 10.10.2. Mutual inductance of two loops. A circuit C_1 consists of a circular loop of radius R_1. Another circuit C_2 of a circular loop of radius R_2 is placed so that their centres are on the same line and the two circles are in parallel planes as shown in Fig. 10.38. Let h be the distance between the two centres. Find their mutual inductance.

Solution. Note that to find the magnetic flux through any loop we need the magnetic field at all points of a surface attached to the loop. In the present situation, if we calculate flux Φ_2 of magnetic field of a current I_1 in C_1 through a flat surface attached to loop C_2, all the points of this surface will be close to the axis of loop C_1. Therefore, it is possible to assume that the magnetic field is uniform over the surface attached to the loop C_2. With this assumption, the flux Φ_2 is readily obtained without much effort.

Contrast this to the choice of a current in loop C_2 and trying to calculate the magnetic flux through the loop C_1. Since loop C_1 is much bigger than the loop C_2, we cannot assume that the magnetic field at all points of a surface attached to C_1 will be constant, which would make this calculation difficult.

Proceeding with the first choice, we now write down the magnetic field $\vec{B}_1$ at a point on the axis of C_1. This was calculated in an earlier chapter on Biot-Savart Law. For the current direction shown in figure and the z axis chosen, the magnetic field $\vec{B}_1$ at the center of loop C_2 is

$$\vec{B}_1 = \frac{\mu_0 I_1 R_1^2}{2\left(R_1^2+h^2\right)^{3/2}}\hat{u}_z.$$

Assuming $\vec{B}_1$ be the same at all points of the loop C_2 and taking the direction of vector area elements of a flat surface attached to loop C_2

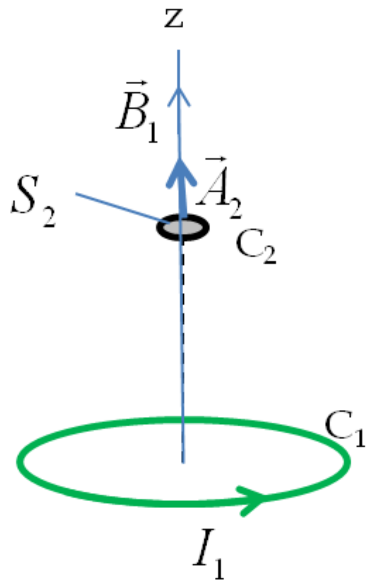


Figure 10.39: The magnetic flux of current I_1 in circuit C_1 through the flat surface S_2 attached to circuit C_2 is easier to computer because the magnetic field $\vec{B}_1$ at the axis of loop C_1 is known, which can be also be used for other points of the surface S_2 since they are are close to the axis.

in the positive z direction as well, we obtain the magnetic flux Φ_2 as

$$\Phi_2 \approx \frac{\mu_0 I_1 R_1^2}{2\left(R_1^2 + h^2\right)^{3/2}} \times \pi R_2^2.$$

Factoring out I_1 we find the mutual inductance

$$\mathcal{M} = \frac{\pi\mu_0}{2} \frac{R_1^2 R_2^2}{\left(R_1^2 + h^2\right)^{3/2}}.$$

Example 10.10.3. Mutual inductance of two solenoids, one inside the other. A small solenoid loop C_1 of length L_1, N_1 turns, and a circular cross-section of radius R_1 is placed inside a very long solenoid loop of length L_2, N_2 turns, and a circular cross-section of radius R_2. Calculate the mutual inductance of the two solenoids assuming $L_1 << L_2$.

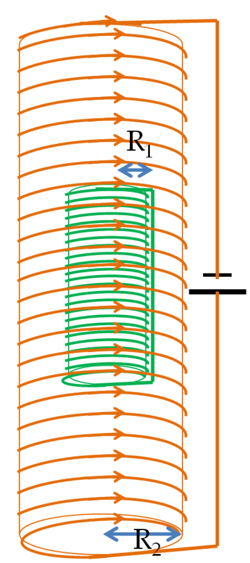


Figure 10.40: Example 10.10.3.

Solution. Since the magnetic field of an infinitely long solenoid is known, we will use current in the longer solenoid and calculate the magnetic flux of this current through the smaller solenoid to figure out the mutual inductance of the two. We introduce a battery in the solenoid loop C_2 so that a current I pass through this solenoid (Fig. 10.40). This will produce a uniform magnetic field $\vec{B}_2$ inside the solenoid that has the direction along the axis of the solenoid and the magnitude given by

$$\text{Magnitude: } B_2 = \mu_0 \frac{N_2}{L_2} I.$$

The magnetic flux Φ_1 through a surface attached to the coils of the solenoid loop C_1 can be approximated to equal N_1 times the magnetic flux through one coil of solenoid C_1.

$$|\Phi_1| = \mu_0 \frac{N_2}{L_2} I \times N_1 \pi R_1^2.$$

Dividing out I gives the following for the mutual inductance of the two solenoids.

$$\mathcal{M} = \pi \mu_0 \frac{N_1 N_2 R_1^2}{L_2}.$$

10.10.2 Self Inductance

We saw above that a changing current in one circuit induces currents in nearby loops. But, the changing current in a circuit must also change magnetic flux through itself as illustrated in Fig. 10.41. Since

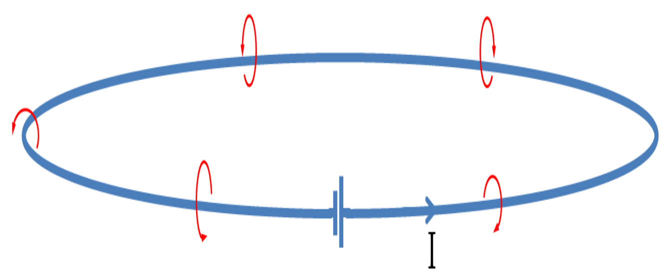


Figure 10.41: Magnetic field lines of the current in any circuit also pass through the circuit itself. The change in the flux of this magnetic field gives rise to a self-induced electromotive force, called back EMF, in the circuit, which drives an induced current in the same direction as the original current if the current in the circuit is decreasing and in the opposite direction if the current in the circuit is increasing.

Faraday's flux rule does not care about the source of magnetic field, an EMF will be induced in the circuit if the magnetic flux Φ_{self} from its own magnetic field s changing with time. This EMF is called self-induced or the back EMF. We will use the symbol $\mathcal{E}_{\text{back}}$ for the back EMF.

$$\boxed{\mathcal{E}_{\text{back}} = -\frac{d\Phi_{\text{self}}}{dt}.} \tag{10.58}$$

Since the magnetic field $\vec{B}$ of the current in the circuit is proportional to current I in the circuit, the magnetic flux Φ_{self} through an open surface attached to the circuit will be proportional to the current in the circuit.

$$|\Phi_{\text{self}}| \propto |I|.$$

We write this proportionality as equality by introducing a constant $\mathcal{L}$:

$$\Phi_{\text{self}} = \mathcal{L} I. \tag{10.59}$$

The proportionality constant $\mathcal{L}$ is called the self inductance, or, simply the inductance of the circuit. It depends on the geometry of the loop. The induced EMF due to the self induction is thus written as

$$\boxed{\mathcal{E}_{\text{back}} = -\mathcal{L}\frac{dI}{dt}.} \tag{10.60}$$

The negative of the time derivative of current in this equation tells us that the induced EMF, and hence, the induced current will be in a direction that opposes the change of current: if the current is increasing, then induced current will be in the opposite direction of the current, and, if the current is decreasing, then the induced current will be in the same direction as the current at the time. Of course, when the current is steady and not changing then $\mathcal{E}_{\text{back}}$ will be zero.

Example 10.10.4. Self inductance of a solenoid. Find the self inductance of a solenoid with length L and radius a with n turns per unit length.

Solution. First we connect the solenoid to a power source that drives a current I through the solenoid and then we determine the magnetic flux through the solenoid. We know that the current in the solenoid gives rise to the magnetic field that has the following magnitude.

$$\begin{aligned} B_{\text{in}} &= \mu_0 n I \\ B_{\text{out}} &= 0 \end{aligned} \tag{10.61}$$

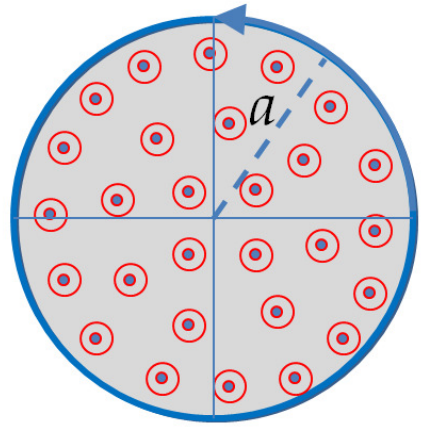
We neglect the magnetic field of the current in the external wire needed to complete the circuit. To compute the magnetic flux through the circuit of the solenoid, we attach an open surface that has the circuit at its boundary. Ignoring the magnetic flux through the part of the surface that is outside the loops, the magnetic flux Φ_{self} through the closed circuit will be approximately equal to the magnetic flux through one loop times the number of loops in the solenoid.

$$|\Phi_{\text{self}}| \approx \Phi \text{ through one loop } \times \text{ No. of loops} = \left[\mu_0 n I \times \pi a^2\right] \times nL,$$

Factoring out the current gives the desired self inductance $\mathcal{L}$ of a solenoid.

$$\boxed{\mathcal{L} = \pi \mu_0 n^2 L a^2.} \tag{10.62}$$

Therefore, a longer solenoid with more dense loops and larger area of cross-section will have a higher self-inductance. The self-inductance can also be increased by filling the space inside the solenoid with a



$$\Phi_{self}(\text{one loop}) = B_{in}\pi a^2$$

Figure 10.42: Example 10.10.4.

material that has an effective magnetic permeability larger than μ_0. For instance, when you wrap the solenoid wires around an iron core, the self-inductance increases by many fold.

10.10.3 Inductance in an electric circuit

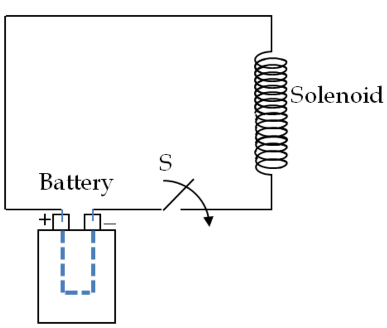


Figure 10.43: A solenoid connected to a battery and a switch. When switch is closed, the current rises from a value of zero to a steady value. At what rate does the current rise?

Every circuit with a changing current has an induced back EMF due to the self-inductance of the circuit. The self-inductance of a circuit is considerably enhanced, if there is a solenoid in the circuit due to the quadratic multiplicative effect of the density of turns as given in Eq. 10.62. Furthermore, if there is magnetizable material, such as iron, in the core of the solenoid, the self-inductance is further enhanced.

To study the effect of self-inductance, consider a circuit in which a source of constant EMF, such as a battery, is connected to a solenoid through a switch as shown in Fig. 10.43. Let there be no current in the circuit before the switch S is closed. The current in the circuit will increase after the switch is closed. The changing current will cause a change in the self magnetic flux through the circuit, which will induce a back EMF opposing the rising current. Eventually, the current will reach a maximum steady value. From that point on in time, the induced EMF will be zero since the self magnetic flux will be unchanging.

We are interested in understanding how the current develops from zero at the zero of time to the steady current value. Since we are interested in the time-changing aspect of the circuit, we will use Faraday's flux rule to write the loop equation as opposed to the Kirchhoff's loop equation, which is valid only in steady circuits. For Faraday's flux rule we will need to evaluate a line integral of the electric field around the circuit. This is aided by redrawing the physical circuit given in Fig. 10.43 as a conceptual circuit and displaying the resistance in the loop separately by a resistor of resistance R as shown in Fig. 10.44.

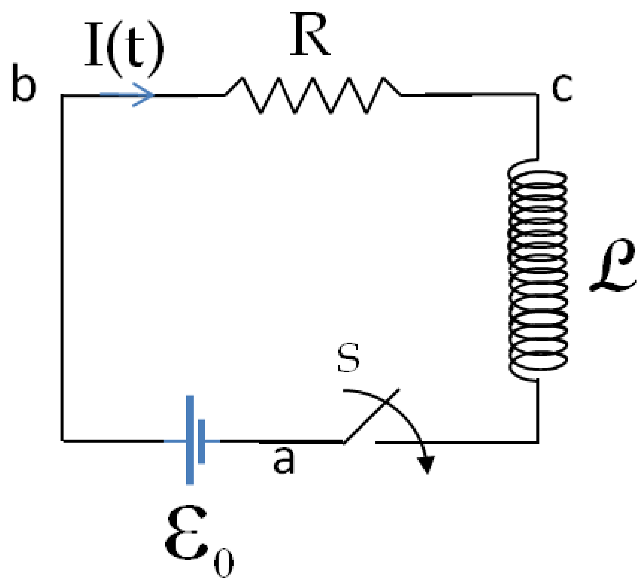


Figure 10.44: The equivalent circuit corresponding to the physical circuit in Fig. 10.43. The self-inductance of the circuit is displayed by the coil symbol for an inductor.

This resistance is the equivalent resistance of the resistance in the wires of the coils and the connecting wires and the internal resistance of the battery. Figure 10.44 also shows a circuit element, called inductor, which has a zero resistance and zero electric field, and represents the net self-inductance of the circuit, which is mostly the self-inductance of the solenoid. We perform the line integral of electric field in the direction of the current shown in Fig. 10.44 to obtain

$$\oint \vec{E} \cdot d\vec{l} = -\mathcal{E}_0 + I(t)R \ \ \text{(Going in the direction of the current).} \tag{10.63}$$

This must be equal to the negative of the rate of change of magnetic flux through the circuit, which is given by the following from the definition of self-inductance.

$$\frac{d\Phi_{\text{self}}}{dt} = \mathcal{L}\frac{dI}{dt}. \tag{10.64}$$

Therefore, we obtain the following equation for the current in the loop.

$$-\mathcal{E}_0 + I(t)R = -\mathcal{L}\frac{dI}{dt}. \tag{10.65}$$

The solution of Eq. 10.65 with $I(0) = 0$ will tell us how the current develops in the circuit. We have solved this type of differential equation in the context of charging of a capacitor. The answer can be obtained readily by analogy.

$$I(t) = I_{\max}\left(1 - e^{-\frac{R}{\mathcal{L}}t}\right), \tag{10.66}$$

where

$$I_{\max} = \frac{\mathcal{E}_0}{R}. \tag{10.67}$$

The solution can be plotted to gain a visual sense of the rise of current in the circuit. The plot of the solution in Fig. 10.45 shows that the current rises exponentially with a characteristic time, given by the ratio of self-inductance $\mathcal{L}$ and the resistance R.

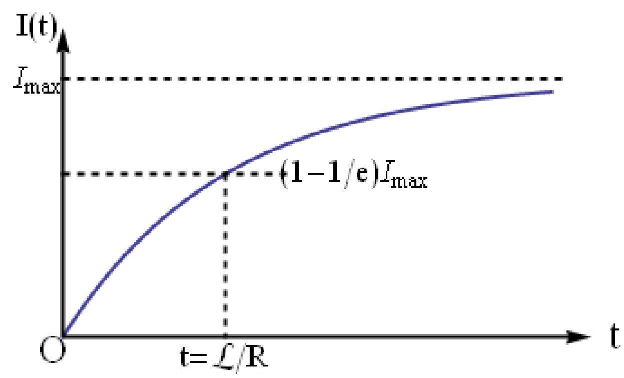


Figure 10.45: Time dependence of current after switch is closed in circuit given in Fig. 10.44.

In time $\mathcal{L}/R$ the current rises to approximately 63%, or more precisely, $(e-1)/e$ of the maximum. The characteristic time given by quantity $\mathcal{L}/R$ is called the **inductive time constant** τ of the circuit, and can be used to determine how quickly the circuit reaches the steady state.

$$\tau = \mathcal{L}/R. \tag{10.68}$$

For instance, in a circuit with 1 kΩ resistance, and 1 H self-inductance, the time-constant τ will be 1 msec. For R in Ohms and $\mathcal{L}$ in Henry, the time constant τ will be in seconds.

$$[\tau] = \frac{[\mathcal{L}]}{[R]} = \frac{\text{H}}{\Omega} = \frac{\Omega.\text{s}}{\Omega} = \text{s}.$$

Note that opening a switch in an inductive circuit can be dangerous and must be done with care, since the current in such a circuit cannot be turned off from some value $I_{\max}$ to zero instantaneously. If you open the circuit in Fig. 10.43, you will force the current to drop rapidly, inducing a large back EMF in a circuit with a gap. This will result in jumping of charges across the open switch contact leading to a spark, which may be fatal depending on the value of $\mathcal{L}$.

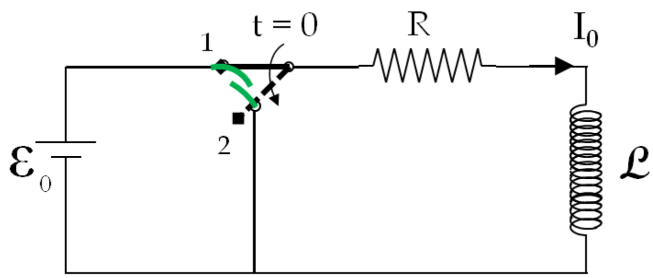


Figure 10.46: Opening an inductive circuit safely requires allowing time for current to decay in a circuit without the power source. The switch makes connection with the dissipation circuit before the connection to the power source is broken with a special switch called make-before-break switch.

To safely open a circuit with a large inductance, you will need to give the circuit some time for the current to decay. This is usually done by leaving a resistor in series with the inductor when switching off the power supply as shown in Fig. 10.46. The current is diverted to a circuit without the power source. Faraday flux rule for the new loop of current which does not have source is

$$RI(t) = -\mathcal{L}\frac{dI}{dt}, \tag{10.69}$$

with initial condition $I(0) = I_0$. The solution of this equation is readily obtained.

$$I(t) = I_0 e^{-\frac{R}{\mathcal{L}}t}. \tag{10.70}$$

The solution tells us that the current will decease in the circuit with the same time constant $\tau = \mathcal{L}/R$ once the power source is disconnected as we found for the rise in the current when the power is turned on.

Example 10.10.5. Analysis of an RL-Circuit. A 12-V battery is connected to a 10-mH inductor and a 50-Ω resistor at $t = 0$. Find the following: (a) the inductive time constant, (b) the current at $t = 0$, (c) the current as $t \to \infty$, (d) the time for the current to reach quarter of the maximum.

Solution.

(a) The time constant of an RL-circuit is $\tau = \mathcal{L}/R = 10\text{mH}/50\Omega = 0.2$ msec.

(b) Since a current through an inductor cannot change abruptly as that would imply infinite back EMF. That means that current through an inductor is a continuous function of time. Therefore, $I = 0$ for $t < 0$ implies that $I = 0$ at $t = 0$.

(c) Current reaches a steady state as $t \to \infty$, and there will be no induced EMF in the circuit. Setting dI/dt in Faraday's flux rule yields, $I_{\text{max}} = \mathcal{E}_0/R$, where $\mathcal{E}_0$ is the EMF of the battery. Therefore, $I_{\text{max}} = 12\text{ V}/50\ \Omega = 0.24$ A as $t \to \infty$.

(d) To find the time to reach one-fourth of the maximum current we set $I(t)$ to I_{max} and solve the resulting equation for t.

$$\frac{I_{\text{max}}}{4} = I_{\text{max}}\left(1 - e^{-t/\tau}\right).$$

Simplifying this equation gives $e^{-t/\tau} = 3/4$. Taking the natural logarithm of both sides yields $t = -\tau \ln(0.75) = 58\ \mu$s.

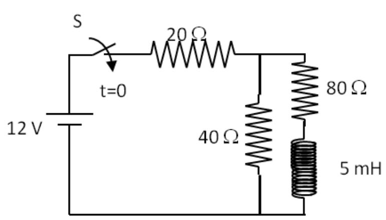


Figure 10.47: Example 10.10.6.

Example 10.10.6. Current in an RL circuit. The switch in the circuit given in figure is closed at $t = 0$. Determine current through each resistor as $t \to \infty$.

Solution. With time the circuit approaches steady state. In the steady state, the current does not change, which means that there would be no self-inductance as $t \to \infty$. If there is no self-inductance, we can ignore the presence of inductor in the circuit. The circuit simplifies to the resistors only as shown in Fig. 10.48.

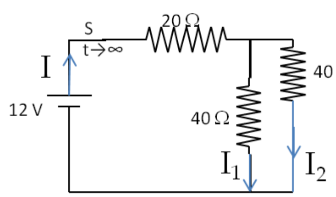


Figure 10.48: Circuit at large time for circuit in Fig. 10.47.

The circuit in Fig. 10.48 can be readily solved for currents. The current through the battery can be found by dividing the voltage of the battery by the equivalent resistance of the circuit.

$$R_{eq} = 20\Omega + \frac{40\Omega \times 80\Omega}{40\Omega + 80\Omega} = 46.7\Omega.$$

The current through the battery is

$$I = \frac{12\text{ V}}{46.7\ \Omega} = 0.257\text{ A}.$$

The parallel circuit is a current divider. Therefore, currents in the two branches are

$$I_1 = \left(\frac{80\Omega}{40\Omega + 80\Omega}\right) 0.257\text{ A} = 0.171\text{ A}$$
$$I_2 = \left(\frac{40\Omega}{40\Omega + 80\Omega}\right) 0.257\text{ A} = 0.086\text{ A}$$

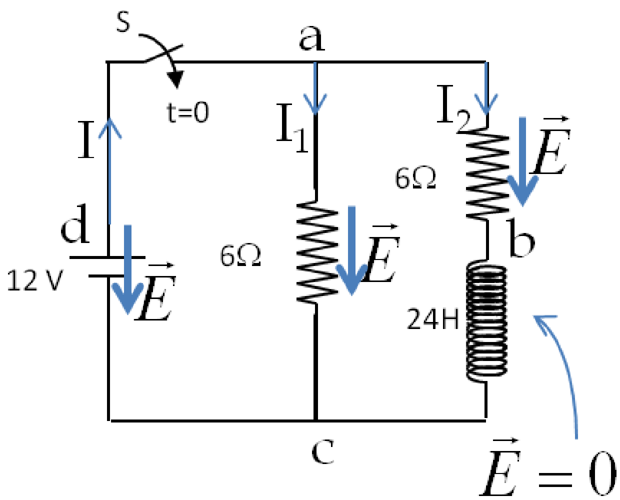


Figure 10.50: Example 10.10.7.

Example 10.10.7. Different time constants in same circuit. The switch in the given circuit in Fig. 10.49 is closed at $t = 0$. Find currents in the circuit as functions of time and describe how different currents in the circuit develop with time.

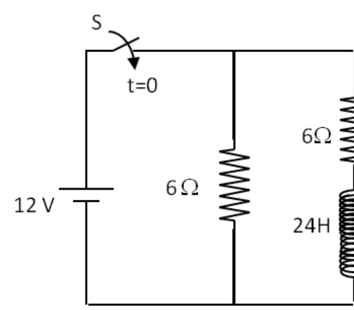


Figure 10.49: Example 10.10.7.

Solution. We will apply Kirchhoff's Current Law (KCL) to the nodes and Faraday's flux rule to the loops since we have time-changing current and there is an inductor in the circuit. If a loop does not contain an inductor then we will ignore the induced EMF due to the circuit since the induced EMF would be negligible for those circuits.

We start with labelling nodes of the circuit, which are a, b, and c here as shown in Fig. 10.50. We have added one more label, d, to help describe the loop on the left side. We then assign currents in the branches beginning with the current in the source, which we take from negative to the positive.

We will need the directions of the electric field in various elements of the circuit. As shown in the figure, electric field is in the direction of the current in the resistors and opposite to the direction of current in the battery. We assume that the solenoid is ideal in the sense that it has a zero resistance, or, the resistance of the solenoid is lumped with the other resistance in its branch.

Now, we can write relations among currents as follows.

$$\text{Currents at node a:} \quad I = I_1 + I_2 \tag{10.71}$$

$$\text{Loop a-c-d-a:} \quad +6I_1 - 12 = 0 \tag{10.72}$$

$$\text{Loop a-b-c-a:} \quad +6I_2 - 6I_1 = -24\frac{dI_2}{dt} \tag{10.73}$$

Note the last equation includes the induced EMF due to the changing of current I_2 through the inductor. These equations have to be solved together to find the three currents. Fortunately, current I_1 is independent of time as given by Eq. 10.72.

$$I_1 = 2 \text{ A}.$$

This says that current I_1 comes on instantaneously when the switch is turned on. That is, the time constant for this current is zero and there is no delay on reaching the maximum current in this branch since we have ignored minute self-inductance in the loop without the coils.

To find I_2, we replace I_1 in Eq. 10.73 and rearrange the result to

$$\frac{dI_2}{dt} + \frac{1}{4} I_2 = \frac{1}{2}.$$

Note that this equation would have been obtained also from the loop a-b-c-d-a. The solution of this equation is

$$I_2 = 2A\left(1 - e^{-t/4}\right).$$

The current I_2 in the branch on the right side in the circuit does not attain the maximum value of 2 A instantaneously, but it takes time due to the inductor in this branch. Initially, current is zero in this branch at $t = 0$. When the source attempts to force current through this branch, the back EMF opposes the rise in the current. The time constant for this current is 4 sec.

Finally, the current through the battery is obtained by using Eq. 10.71.

$$I = 2A + 2A\left(1 - e^{-t/4}\right)$$

The current through the battery rises instantaneously to 2 A, but takes time to get to the final value of 4 A due to the inductor in the circuit.

10.10.4 Energy in Magnetic Field

Energy in an Inductor

Recall that in a circuit that has a solenoid connected to a battery the current continues to flow in the circuit even after the battery is disconnected from the circuit. This is most readily seen by using a switch that works on the make-before-break switching mechanism shown in Fig. 10.46. Suppose the circuit with a battery was connected and kept running for a while so that the steady current $I_{\max}$

flows through the circuit at time $t = 0$. Now, when the switch is directed to remove the battery from the circuit. The current in the inductor is allowed to flow through a loop that has a resistance R. The current dies out in the circuit according to:

$$I(t) = I_{\text{max}} e^{-t/\tau},$$

where $\tau = \mathcal{L}/R$. Since a flow of current through a resistor causes the resistor to heat up. But since there is no battery in the circuit, the question arises:

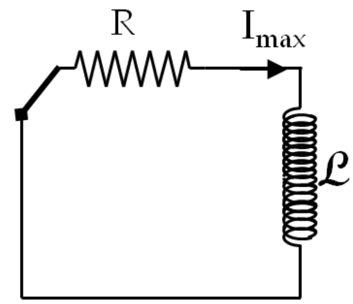


Figure 10.51: Situation at $t = 0$. The current decays to zero with time constant $\tau = R/\mathcal{L}$.

Who or what is supplying the energy that is showing up as the heating of the resistor?

The only other element in the circuit is the inductor. Therefore, from the principle of conservation of energy, we must conclude that the energy dissipated in the resistor must be present in the inductor at the instant the switch was redirected away from the battery in the circuit. This circuit demonstrates that an inductor with current stores energy.

However, unlike the separated charges in a capacitor, there is no obvious material in the inductor that we might find as storing energy. So, how does an inductor store energy? A clue comes from the fact that an inductor will store energy only when there is a current through it. Therefore, the energy stored in an inductor may be thought to be in the movement of charges comprising the current.

Now, note that whenever there is current, there is also a magnetic field and when the current has dissipated the magnetic field also disappears. Therefore, we could conclude that the energy stored in the inductor could be in the magnetic field or in the current. This is similar to the energy stored in a charged capacitor - there were two ways of looking at the energy: energy stored in the charges or energy stored in the electric field. Here, we also have two ways of looking at the energy - the energy in the currents or energy in the magnetic field.

With the two viewpoints we can work out a formula for the energy in magnetic field as follows. We will use the energy dissipated in the resistor as a proxy for finding formulas for the energy stored in an inductor, as current or as magnetic field.

Suppose that at time $t = 0$ the current in the inductor is I_{max} when the circuit is diverted to go through a resistor only. With time the current goes to zero and the energy is dissipated into the resistor. In a resistor the power dissipated, i.e. the rate of dissipation of energy is given by I^2R if current flow is I. The energy dissipated between time t and $t+dt$ will be

$$dW = P(t)dt = I(t)^2 R dt = I_{\text{max}}^2 e^{-2\frac{R}{\mathcal{L}}t} R dt, \tag{10.74}$$

which should be integrated from $t=0$ to $t=\infty$ to obtain the energy of the system at $t=0$.

$$\begin{aligned} W &= I_{\text{max}}^2 R \int_0^\infty e^{-2\frac{R}{\mathcal{L}}t} dt \\ &= I_{\text{max}}^2 R \frac{\mathcal{L}}{2R} \\ &= \frac{1}{2}\mathcal{L} I_{\text{max}}^2. \end{aligned} \tag{10.75}$$

Therefore, the energy stored in the inductor when it carries a current I is

$$\boxed{U_{\text{inductor}} = \frac{1}{2}\mathcal{L} I^2.} \tag{10.76}$$

Energy in Magnetic Field

We can now show that this energy can be written in terms of magnetic field at $t=0$. This will give a formula for the energy stored in the magnetic field. Since, the magnetic field is mostly inside the solenoid, we will ignore other places for magnetic field. In Eq. 10.76, we can replace I by B_{in} at time $t=0$ by

$$B_{\text{in}} = \mu_0\, n\, I,$$

where n is the number of turns per unit length in the solenoid. We can also write the self-inductance of solenoid $\mathcal{L}$ in terms of the length L of the solenoid and radius a of each coil.

$$\mathcal{L} = \pi \mu_0 n^2 L a^2.$$

Therefore, the energy U contained in the solenoid at time $t=0$ is

$$U = \frac{1}{2\mu_0} B_{\text{in}}^2 \left(L\pi a^2\right),$$

which can be written as an energy density times volume.

$$U = \frac{1}{2\mu_0} B_{\text{in}}^2 \times \text{Volume of solenoid} \tag{10.77}$$

Therefore, we say that magnetic field contains energy given by the following formula for energy per unit volume, u_B.

$$\boxed{u_B = \frac{1}{2\mu_0} B^2 \quad \text{(Energy density)}} \tag{10.78}$$

If magnetic field is not uniform, then the energy in the magnetic field in a particular space volume will be given by an integral over the volume.

$$U_B = \frac{1}{2\mu_0}\int_{\text{Volume}} B^2 dV \quad \text{(Energy in magnetic field)} \tag{10.79}$$

The formula for the energy in the magnetic field is similar to that of energy in the electric field.

$$U_E = \frac{1}{2}\epsilon_0 \int_{\text{Volume}} E^2 dV. \tag{10.80}$$

The sum of energy in electric and magnetic field is called the electromagnetic energy, or the energy in the electromagnetic field, U_{em}.

$$U_{em} = U_E + U_B. \tag{10.81}$$

Energy in Interacting Inductors

The arguments for the energy in an inductor carrying a current can be extended to obtain energy in two coupled circuits. Suppose one circuit of self-inductance $\mathcal{L}_1$ carries a current I_1 and another circuit of self-inductance $\mathcal{L}_2$ carries a current I_2. Suppose their mutual inductance is $\mathcal{M}$. Then, the energy stored in the two circuits will be

$$U = \frac{1}{2}\mathcal{L}_1 I_1^2 + \frac{1}{2}\mathcal{L}_2 I_2^2 + \mathcal{M} I_1 I_2. \tag{10.82}$$

The reference value of this energy is zero when both currents are zero. At any other value of the currents we require that U be positive.

$$U \geq 0. \tag{10.83}$$

Now, we write the energy expression in Eq. 10.82 as follows.

$$U = \frac{1}{2}\mathcal{L}_1\left(I_1 + \frac{\mathcal{M}}{\mathcal{L}_1} I_2\right)^2 + \frac{1}{2}\left(\mathcal{L}_2 - \frac{\mathcal{M}^2}{\mathcal{L}_1}\right) I_2^2. \tag{10.84}$$

We observe that if the first term is zero, then the second term will place restrictions on $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{M}^2$. The first term will be zero if currents have the following relation.

$$I_1 + \frac{\mathcal{M}}{\mathcal{L}_1} I_2 = 0. \tag{10.85}$$

In this case, the positivity of U requires that the second term in Eq. 10.84 be positive. Therefore,

$$\mathcal{L}_2 - \frac{\mathcal{M}^2}{\mathcal{L}_1} \geq 0. \tag{10.86}$$

Rearranging, we find that

$$\mathcal{M}^2 \leq \mathcal{L}_1 \mathcal{L}_2. \tag{10.87}$$

Since the mutual inductance can be positive or negative depending upon the sign convention for the currents, we write the absolute value of the mutual inductance as

$$|\mathcal{M}| \leq \sqrt{\mathcal{L}_1 \mathcal{L}_2}. \tag{10.88}$$

Therefore, the magnitude of the mutual inductance must always be less than the geometric mean of the self-inductances. We often write the mutual inductance in terms of the self-inductances by introducing a constant k called the coefficient of coupling.

$$\boxed{|\mathcal{M}| = k\sqrt{\mathcal{L}_1 \mathcal{L}_2}.} \tag{10.89}$$

In experimental situations, the absolute value of the coefficient of coupling is near 1, $|k| \sim 1$, if the magnetic flux from one circuit goes through the other circuit and vice-versa, as in an ideal transformer. If the two circuits are so arranged that the magnetic flux from one does not completely go through the other circuit, then $|k| < 1$.