3.6 ACCELERATION

You know that objects can pick up speed after being at rest, or can slow down and come to rest. One example of speeding up happens at the beginning of a race as shown in Fig. 3.25.

Figure 3.25: Accelerating situation # 1: At the start of a race, the runner picks up speed due to acceleration. Photo Credits: Peter Griffin at www.publicdomainpictures.net.

Besides speeding up or slowing down, acceleration takes place when the direction of motion changes with time as shown for the cars as they round a bend in Fig. 3.26.

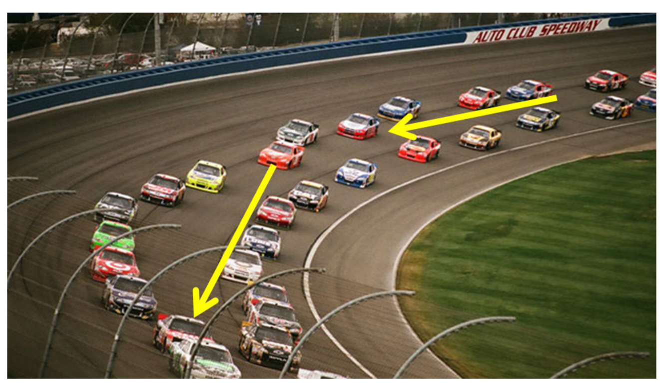

Figure 3.26: Accelerating situations #2: The direction of the velocities of the race cars change as a result of acceleration of the cars. Photo Credits: Michael Miloserdoff at www.publicdomainpictures.net.

These examples illustrate the fact that the magnitude as well the direction of the velocity can change with time. The rate of change of velocity is called **acceleration**; the change in velocity due to a change in either magnitude or direction or both.

We have already worked out a procedure for calculating the rate of change of a vector when we defined instantaneous velocity as the rate of change of the position vector. Briefly, to obtain the average rate of change of the position vector you subtract the position vector at time t from the position vector at time $(t + \Delta t)$ and then divide the result by the duration Δt. When the duration Δt was made arbitrarily small, the method led to an exact value of the rate of change at the instant t.

This procedure should work for the rate of change of any other vector. Therefore, we compute the rate of change of the velocity vector in an analogous way. Thus, average acceleration ($\vec{a}_{ave}$) is obtained by subtracting the velocity vector at time t from the velocity vector at time $(t + \Delta t)$, and then dividing the result by the duration Δt.

$$\boxed{\vec{a}_{ave} = \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t},} \tag{3.37}$$

which can be rearranged to obtain the change in velocity in a finite time interval as

$$\Delta\vec{v} = \vec{a}_{ave}\ \Delta t, \tag{3.38}$$

where $\Delta\vec{v}$ stands for the change in velocity vector over the interval from t to $(t + \Delta t)$.

$$\Delta\vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t).$$

Making the time interval Δt arbitrarily small gives us the instantaneous rate of change, and consequently, we obtain the instantaneous acceleration, i.e. acceleration <u>at time</u> t. Formally,

$$\vec{a} = \lim_{\Delta t \to 0} \left[\frac{\Delta\vec{v}}{\Delta t}\right], \tag{3.39}$$

which is also written as

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt},} \tag{3.40}$$

Similar to other vectors, there are two useful viewpoints for looking at the acceleration vector, namely geometric and analytic. We now study them in detail.

Geometric Viewpoint

The average acceleration is equal to the change in velocity divided by the time interval. To compute the change in velocity, we recall that in the geometric viewpoint, in order to subtract a vector $\vec{B}$ from

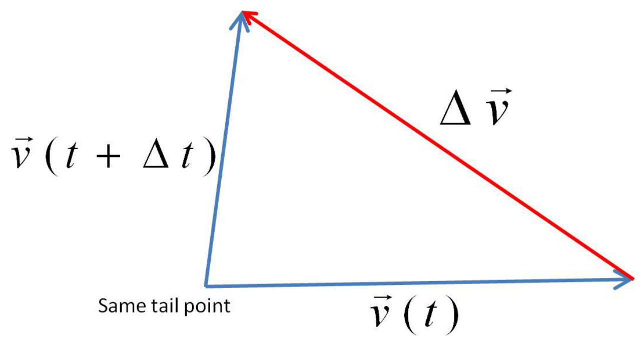

Figure 3.27: Change in velocity. When velocity vectors at instants $t + \Delta t$ and t are drawn with their tails at the same point, the vector from the tip of the velocity vector at t to the tip of the velocity vector at $t + \Delta t$ gives the change in velocity during the interval.

another vector $\vec{A}$, we start with drawing them so that their tails are at the same point. Then, the vector from the tip of $\vec{B}$ to the tip of $\vec{A}$ is the vector $(\vec{A} - \vec{B})$.

Therefore, we draw the velocity vectors $\vec{v}(t)$ and $\vec{v}(t + \Delta t)$ from the same point, although on the trajectory of the motion the vector $\vec{v}(t)$ is tangent to the trajectory at time t and the vector $\vec{v}(t+\Delta t)$ is tangent to the trajectory at time $(t + \Delta t)$. Once we have drawn the velocity vectors corresponding to two nearby instants we obtain the vector for the change in velocity as shown in Fig. 3.27.

The change in velocity obtained by following the graphical procedure shown in Fig. 3.27 is divided by the time interval Δt to compute the average acceleration in the interval from t to $(t + \Delta t)$. One can compute average velocity for successively smaller intervals generating a sequence of average accelerations whose limit would give the instantaneous acceleration at time t.

Analytic Viewpoint

We now look at acceleration from analytic viewpoint by setting up a Cartesian coordinate system first. This makes computing the change in velocity simple. The change in velocity vector amounts to changes in the x, y and z-components separately. Therefore, average acceleration in the interval t to $(t + \Delta t)$ can be written explicitly in terms of the components.

$$\vec{a}_{ave} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{u}_x + \frac{\Delta v_y}{\Delta t}\hat{u}_y + \frac{\Delta v_z}{\Delta t}\hat{u}_z.$$

This shows that the components of the average acceleration vector $\vec{a}_{ave}$ are just the average rates of change of components of the velocity vector.

$$a_x^{ave} = \frac{\Delta v_x}{\Delta t}$$
$$a_y^{ave} = \frac{\Delta v_y}{\Delta t}$$
$$a_z^{ave} = \frac{\Delta v_z}{\Delta t}$$

The components of instantaneous acceleration at t is obtained by letting the interval near t become infinitesimally small, which says that the components of the instantaneous acceleration are simply time derivatives of the components of instantaneous velocity.

$$\boxed{a_x = \frac{dv_x}{dt}} \tag{3.41}$$

$$\boxed{a_y = \frac{dv_y}{dt}} \tag{3.42}$$

$$\boxed{a_z = \frac{dv_z}{dt}} \tag{3.43}$$

The instantaneous acceleration vector is the sum of vectors along the x, y, and z-axes obtained by multiplying the components with the unit vectors along the axes.

$$\boxed{\vec{a} = a_x\ \hat{u}_x + a_y\ \hat{u}_y + a_z\ \hat{u}_z = \frac{dv_x}{dt}\ \hat{u}_x + \frac{dv_y}{dt}\ \hat{u}_y + \frac{dv_z}{dt}\ \hat{u}_z.} \tag{3.44}$$

The magnitude of the acceleration vector is obtained in the usual way from the components.

$$\boxed{\text{Magnitude: } a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}} \tag{3.45}$$

As usual, the direction of acceleration vector requires two angles if the motion is in three dimensional space, one angle if the motion is in a plane, and sign of the vector component if the motion is along one of the Cartesian axes.

Recall that the derivative of a function $f(t)$ evaluated for some instant $t = t_1$ is also equal to the slope of the tangent to f vs t curve at $t = t_1$. Therefore, the components of acceleration can be obtained from slopes of respective plots of velocity components versus time. That is, a plot of v_x versus t can be used to determine the x-component of acceleration, and similarly for other components.

1. a_x = slope of tangent to v_x vs t curve.

Finally, since the velocity components are rates of changes of position coordinates it is also possible to write acceleration components in terms of two derivatives of the position coordinates.

$$\boxed{a_x = \frac{d^2x}{dt^2}} \tag{3.46}$$

$$\boxed{a_y = \frac{d^2y}{dt^2}} \tag{3.47}$$

$$\boxed{a_z = \frac{d^2z}{dt^2}} \tag{3.48}$$

Example 3.6.1. Average acceleration in one dimension. A runner runs on straight East-West road. At time $t = 0$, her velocity is 10 m/s towards East. After running for 1000 seconds, her velocity is still pointed towards East but she has slowed to a speed of 5 m/s. At a later time at $t = 5000$ sec, she is back to a speed of 10 m/s but moving in the opposite direction, i.e. towards West. What are her average accelerations between the following intervals of time: (a) $t = 0$ to $t = 1000$ sec, (b) $t = 1000$ sec to $t = 5000$ sec, and (c) $t = 0$ to $t = 5000$ sec?

Solution. Although the motion is in one straight line, the direction of motion reverses in the duration of interest. In such situations, it is very helpful to place the motion on one of the Cartesian axes and work with the corresponding components of vectors. To be concrete, we will work with the x-axis such that the positive x-axis points towards East. Now, we can translate the given information into x-component of velocity at different times.

$$\begin{aligned} &v_x(0) = +10 \text{ m/s}. \\ &v_x(1000 \text{ s}) = 5 \text{ m/s}. \\ &v_x(5000 \text{ s}) = -10 \text{ m/s} \quad (\text{since } \vec{v} \text{ towards negative } \ x\text{-axis}). \end{aligned}$$

Now, we can use the definition of the x-component of average acceleration for each interval.

(a) Note that when doing numerical calculations, it is often helpful to do calculations on the units separately from the calculations on the numbers as illustrated here.

$$\begin{aligned} a_x^{ave} &= \frac{v_{2x} - v_{1x}}{t_2 - t_1} \\ &= \frac{5 - 10}{1000 - 0} \left[\frac{\text{m/s}}{\text{s}}\right] = -5.0 \times 10^{-3} \text{ m/s}^2 \end{aligned}$$

Therefore, average acceleration is $\vec{a}_{ave} = \left(-5.0 \times 10^{-3} \text{ m/s}^2\right) \hat{u}_x$, or 5.0×10^{-3} m/s^2 towards West.

(b)

$$a_x^{ave} \;=\; \frac{-10-5}{5000-1000}\left[\frac{\text{m/s}}{\text{s}}\right] = -3.8\times 10^{-3}\ \text{m/s}^2$$

Therefore, average acceleration is $\vec{a}_{ave} = \left(-3.8\times 10^{-3}\ \text{m/s}^2\right)\hat{u}_x$, or $3.8\times 10^{-3}\ \text{m/s}^2$ towards West.

(c)

$$a_x^{ave} \;=\; \frac{-10-10}{5000-0}\left[\frac{\text{m/s}}{\text{s}}\right] = -4.0\times 10^{-4}\ \text{m/s}^2$$

Therefore, average acceleration is $\vec{a}_{ave} = \left(-4.0\times 10^{-4}\ \text{m/s}^2\right)\hat{u}_x$, or $4.0\times 10^{-4}\ \text{m/s}^2$ towards West.

Example 3.6.2. Acceleration and velocity from a given position as a function of time. A missile is fired in a straight line such that its position with respect to time in a particular coordinate system is given by only its x-coordinates in meters as $x(t) = 3\,t+2\,t^3$, where t is in seconds. Find its velocity and acceleration at (a) $t = 1$ sec, and (b) $t = 2$ sec.

Solution. First we find the x-components of velocity and acceleration for an arbitrary time t from the given $x(t)$, and then find the values for the particular instants by substituting the specific values of time given in (a) and (b). The velocity and acceleration will be obtained by multiplying the values of the x-components by the unit vector $\hat{u}_x$ along the x-axis, which will show whether the vector is in the direction of the positive x-axis or in the direction of the negative x-axis. Since the motion is in a straight line which coincides with the x-axis, the vector uses only the base vector for the x-axis.

$$v_x(t) = \frac{dx}{dt} = \frac{d}{dt}\left[3\,t + 2\,t^3\right] = 3 + 6t^2.\ \ [\text{m/s}]$$
$$a_x(t) = \frac{dv}{dt} = \frac{d}{dt}\left[3 + 6\,t^2\right] = 12t.\ \ [\text{m/s}^2]$$

Now we use $t = 1$ sec and $t = 2$ sec in these equations to find answers for parts (a) and (b) respectively.

(a) $t = 1$ sec:

$$v_x(1) = 3 + 6(1)^2 = 9\ \text{m/s}.$$
$$a_x(1) = 12(1) = 12\ \text{m/s}^2.$$
$$\vec{v}(1) = (9\ \text{m/s})\hat{u}_x;\quad \vec{a}(1) = (12\ \text{m/s}^2)\hat{u}_x;$$

(b) $t = 2$ *sec*:

$$v_x(2) = 3 + 6(2)^2 = 27\ \text{m/s}.$$
$$a_x(2) = 12(2) = 24\ \text{m/s}^2.$$
$$\vec{v}(2) = (27\ \text{m/s})\hat{u}_x;\quad \vec{a}(2) = (24\ \text{m/s}^2)\hat{u}_x;$$

Note: Since we do not know which way in space the x-axis is pointed, we cannot go further than the analytic representation of the final answer.

Example 3.6.3. Components of acceleration from plots of components of velocity. The x-component of velocity of an object changes with time according to the graph shown in Fig. 3.28. (a) From this graph find the x-component of acceleration at $t = 0$, $t = 1$ sec, $t = 2$ sec, $t = 3$ sec and $t = 4$ sec. (b) What can you say about the velocity and acceleration of the object?

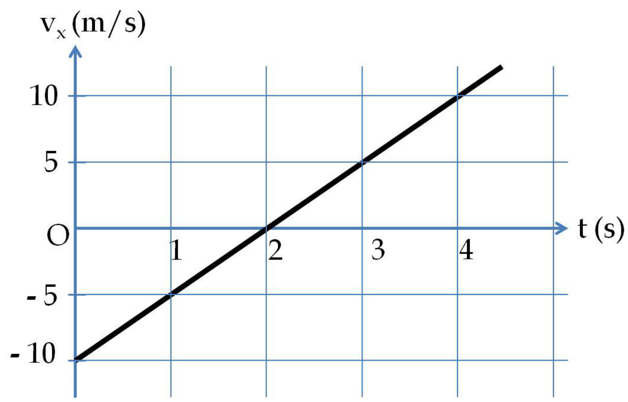

Figure 3.28: Example 3.6.3. Calculating the x-component of the acceleration from the rate of change of the x-component of velocity.

Solution. (a) From the definition of acceleration we know that each Cartesian component of acceleration is equal to the time derivative of corresponding component of velocity. Since the slope of the tangent to a curve when a quantity is plotted against the independent variable equals the derivative, we can use the given data to calculate the slope and find the derivative that way.

As the curve of the plot of v_x versus t is a straight line, the tangent to the curve is the line itself. Therefore, the slope of the tangent is the same regardless of the instant in time: there is only one tangent, and so there is only one slope to work out.

The slope of the line is

$$a_x = \frac{\text{Rise}}{\text{Run}} = \frac{20 \text{ m/s}}{4 \text{ s}}. \quad \text{Note: units in the rise and run!}$$

This gives the x-component of the acceleration at any point in time to be 5 m/s^2.

(b) In part (a) we found that the x-component of the acceleration was constant. We do not have data for y and z-components of velocity, so we cannot find the y and z-components of acceleration. Without all the three components of velocity and acceleration, we cannot find their magnitudes and directions of the velocity and acceleration.

Example 3.6.4. Acceleration from slope of velocity. An object moves in a straight line. A coordinate system is chosen so that the x-axis coincides with the line of motion. The x-component of velocity of an object varies as a function of time as shown in Fig. 3.29. Find the x-component of the instantaneous acceleration at the following instants (a) $t = 1$ sec, (b) $t = 3$ sec, and (c) $t = 5$ sec.

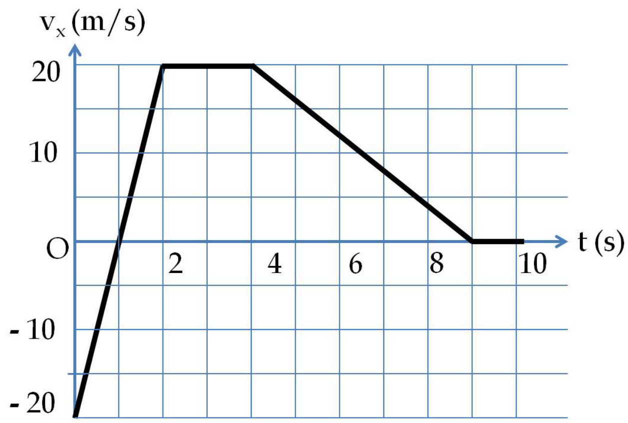

Figure 3.29: Plot of the x-component of velocity as a function of time.

Solution. Since the plot has only straight line segments, the tangents to the curves coincide with the segments. Therefore, here, it is relatively easy to find the slopes at different instants. At $t = 1$ sec, the slope can be calculated by noting that the velocity in this segment increases by 40 m/s in 2 sec interval at a constant rate. The slope is 40(m/s)/2s, or 20 m/s^2. The slope at $t = 3$ sec is clearly zero since velocity is not changing in this segment.

Finally, the slope at $t = 5$ sec is seen to be negative since the velocity goes down by 20 m/s in about 4 sec interval in this segment, from $t = 4$ sec to $t = 8$ sec. This yields a slope of -5 m/s^2. Summarizing the results:

(a) $a_x = 10$ m/s^2 at $t = 1$ s.

(b) $a_x = 0$ at $t = 3$ s.

(c) $a_x = -5$ m/s^2 at $t = 5$ s.

Further Remarks: Unlike velocity, acceleration can change abruptly. Therefore, the acceleration after an instant may not equal the acceleration before that instant. Thus, corners are allowed in v_x or v_y or v_z vs t plots. Of course, these plots are not allowed to have discontinuities since velocity cannot change abruptly.

Example 3.6.5. Acceleration from slope of velocity - two dimensional case. An object moves in a plane. A coordinate system is chosen so that the xy-plane coincides with the line of motion. The x-component of velocity of an object varies as a function of time as shown in Fig. 3.30. Find acceleration at the following instants (a) $t = 2$ sec, and (b) $t = 6$ sec.

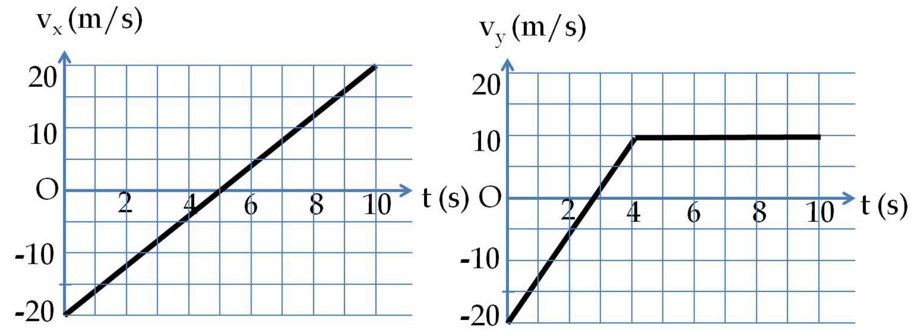

Figure 3.30: Plots of the x and y-components of velocity as a function of time.

Solution. From the plots of v_x and v_y versus t we find the following values of the components of acceleration. Note the z-component of acceleration is zero and therefore ignored.

(a) $t = 2\ sec$:

$a_x =$ Slope of v_x vs $t = 4.0$ m/s^2
$a_y =$ Slope of v_y vs $t = 7.5$ m/s^2
Magnitude: $a = \sqrt{a_x^2 + a_y^2 + 0} = 8.5$ m/s^2;
Direction: $\arctan(4.0/7.5) = 62^\circ$ counterclockwise from the positive x-axis.

(b) $t = 6\ sec$:

$a_x =$ Slope of v_x vs $t = 4.0$ m/s^2
$a_y =$ Slope of v_y vs $t = 0$
Magnitude: $a = 4.0$ m/s^2;
Direction: towards the positive x-axis. ;

Example 3.6.6. Acceleration and Deceleration in One Dimension. In one-dimensional motion, acceleration can be either in the same direction as the velocity or in the opposite direction as illustrated in Fig. 3.31.

If acceleration has the same direction as the velocity, it leads to an increase in the magnitude of the velocity vector. This situation is referred to as accelerating the motion.

If acceleration is in the opposite direction to the velocity, the immediate result is a slowing of motion, the object eventually coming to a rest, and finally, a reversal of the direction of motion occurs. The initial slowing effect of velocity and acceleration being in the opposite directions is also referred to as decelerating the motion.

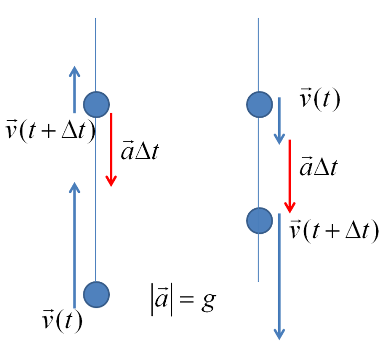

Figure 3.32: The acceleration of freely falling object is same whether ball goes upward or downward. When the ball is going up, the acceleration is in the opposite direction of mo-

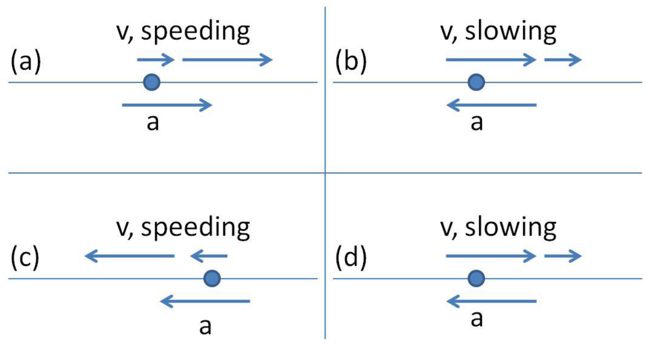

Figure 3.31: Example 3.6.6. Relative directions of velocity and acceleration determine if the motion will speed up or slow down. If acceleration is in the same direction as the velocity, as in (a) and (c), the magnitude of the velocity increases, and if the acceleration is in the opposite direction to the velocity as in (b) and (d), the magnitude of the velocity decreases.

We will see below that if acceleration is neither in the direction of the velocity nor opposite to it, the direction of the velocity changes.

Example 3.6.7. Acceleration for Vertically Falling or Rising Objects. If air resistance can be ignored, then a vertical motion of an object near Earth's surface has a constant acceleration of magnitude 9.81 m/s^2 and direction downward, called the acceleration due to gravity. The symbol g is used to denote this acceleration value.

$$g = 9.81\ \text{m/s}^2.\ \text{(Acceleration due to gravity)} \tag{3.49}$$

Note that the acceleration for a freely falling object is g pointed down regardless of whether the object is moving up or moving down or momentarily at rest. You should expect this to be the case since **acceleration is related to the change in velocity and not to the value of velocity at any instant**. To illustrate this case, we consider a ball thrown vertically up so that its initial velocity is pointed up as shown in Fig. 3.32.

Initially, as the ball goes up, its velocity and acceleration vectors are in opposite directions. As a result the ball slows down and comes to rest momentarily at the very top of the trajectory. After that, acceleration changes the velocity from zero at the resting point to non-zero value in the next instant, but now, the velocity is pointed down. On the way down, the ball picks up speed as it falls since velocity and acceleration are in the same direction.

This example shows that the same object may decelerate, meaning decreasing speed, in one time domain and accelerate, meaning increasing speed, in another time segment, all without a change in the acceleration. Therefore, you need to be careful when you see a

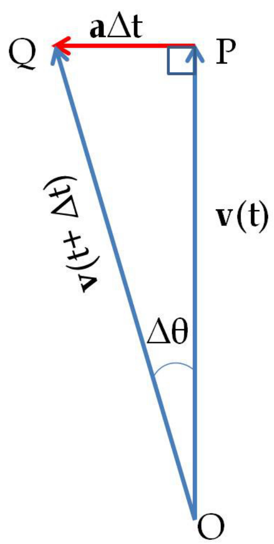

Figure 3.33: Example 3.6.8. Here $\vec{v}(t+\Delta t) = \vec{v}(t) + \vec{a}\Delta t$. When Δt becomes infinitesimal, dt, we get $|\vec{v}(t+dt)| = |\vec{v}(t)|$, but the direction of $\vec{v}(t+dt)$ is different from that of $\vec{v}(t)$. In this figure, vectors are indicated with bold letters.

decelerating motion since a decelerating motion does not mean the object will not pick up speed later after it has come to rest.

Example 3.6.8. Acceleration Perpendicular to Velocity If velocity and acceleration are collinear, the magnitude of the velocity either increases or decreases and the direction of the velocity either remains unchanged or reverses by 180°. What happens when the acceleration is not collinear with velocity? For instance, the acceleration of a moving charged particle caused by a magnetic force is perpendicular to the velocity of the particle.

In Fig. 3.33, the change in velocity as a result of an acceleration that is perpendicular to the velocity is shown. The velocity at time $t + \Delta t$ is equal to a vector sum of velocity at time t and the change in velocity $\vec{a}\Delta t$.

As we see from the figure, the direction of the velocity changes. The magnitude of the velocity at time $t + \Delta t$ is given by the hypotenuse of the triangle $\triangle OPQ$. From the triangle $\triangle OPQ$ you can deduce that the relative change in speed is of order $(a\Delta t/v)^2$ while the change in the direction of the velocity as given by the angle $\Delta\theta$ which goes as $(a\Delta t/v)$.

When we reduce the time interval, we find that the magnitude of the velocity does not change as quickly as this angle, i.e. the direction. As a result, in the limit $\Delta t \to 0$, only the direction changes if $\vec{a}$ is always perpendicular to $\vec{v}$. Note that, in order for the acceleration to be perpendicular to the velocity at all times, the acceleration must also rotate along with the velocity vector.