8.2 WORK-ENERGY THEOREM

8.2.1 Integrating Equation of Motion of One Particle in One Dimension

Consider a one-dimension motion of a particle of mass m restricted to the x-axis. Newton's second law says that the x-coordinate of the mass will obey the following equation.

$$F_x^{net} = m\frac{dv_x}{dt}, \tag{8.8}$$

where F_x^{net} is the x-component of the net force on the mass. Suppose we treat F_x^{net} as a function of the coordinate x of the particle and integrate with respect to x, corresponding to a particular path between x_i and x_f, so that the left side of this equation gives the work done by the net force or the net work done on the body.

$$\int_{x_i}^{x_f} F_x^{\text{net}} dx = m\int_{x_i}^{x_f} \frac{dv_x}{dt} dx. \tag{8.9}$$

The right side is a complicated integral. It appears that, to perform this integral, we need x-component of acceleration as a function of x. However, a change of variable from x to t transforms the integral into an integral over v_x as we will see now. We note that

$$dx = \left(\frac{dx}{dt}\right) dt = v_x dt. \tag{8.10}$$

Replacing dx on the right side of Eq. 8.9 by $v_x dt$ we find

$$\text{Right side of Eq.8.9 } = m\int_{t_i}^{t_f} \frac{dv_x}{dt} v_x dt, \tag{8.11}$$

where the limits on the integration have been changed to the corresponding time values. In Eq. 8.11, we can cancel out dt to obtain an integration over v_x instead of over dt, and the change in the limits accordingly.

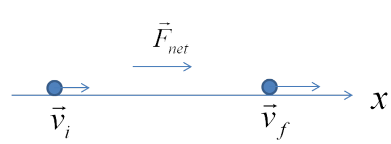

Figure 8.14: Work-energy theorem for a particle of mass says that the net work done by all forces equals the change in the kinetic energy of the particle, $W_{if}^{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$.

$$\text{Right side of Eq.8.9 } = m\int_{v_{ix}}^{v_{fx}} v_x dv_x, \tag{8.12}$$

which can be performed to yield.

$$\text{Right side of Eq.8.9 } = \frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2. \tag{8.13}$$

Therefore, an integration of the equation of motion Eq. 8.8 for a one-dimensional motion gives the following equality.

$$\int_{x_i}^{x_f} F_x^{\text{net}} dx = \frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2. \tag{8.14}$$

Since the motion is restricted to one dimension, the y and z-components are zero, and we can replace the square of the x-components of velocity by square of speed at those instants. The quantity $\frac{1}{2}mv^2$ is called **kinetic energy**, which will be denoted by K. In three-dimensions, kinetic energy can be written in terms of all three components of velocity, (v_x, v_y, v_z).

$$\boxed{K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right).} \tag{8.15}$$

The left side of Eq. 8.14 gives the net work done on the particle by all forces and the right side is the difference in kinetic energy, $K_f - K_i$.

$$\boxed{W_{if}^{\text{net}} = K_f - K_i, \quad \text{for a particle.}}, \tag{8.16}$$

This result is called the **Work-Energy Theorem**. This equation also shows that the units of work and energy are same. In the SI system of units the unit of work is $N.m$ and the unit of energy is $kg.m^2/s^2$ as seen from the definition of kinetic energy. The two units are equal and are also known by another name, Joule (J).

$$1 \text{ N.m} = 1 \text{ kg.m}^2/\text{s}^2 = 1 \text{ J}.$$

Therefore, we can say that a 1-kg object moving at speed 1 m/s has 1 J of kinetic energy. Also, if 1 N force acts in the direction of the displacement of 1 m, then the work done by that force will be 1 J.

Although, we have derived the work-energy theorem using a motion in one dimension, it is true more generally. We will next illustrate this assertion by explicitly working out the calculation in three dimensions.

8.2.2 Integrating Equation of Motion of One Particle in Three Dimensions

The general equation of motion of a particle of mass m subject a net force $\vec{F}$ is given as

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt}. \tag{8.17}$$

Let us think of $\vec{F}_{\text{net}}$ as a function of position $\vec{r}$ rather than a function of time t as we have done in previous chapters. Taking the dot product of both sides with $d\vec{r}$ and integrating from the initial position to the final position we notice that the left side corresponds to the net work done on the particle and the right side can be transformed into the change in kinetic energy as we have seen in the simpler situation of one-dimensional motion.

$$\int_{\vec{r}_i,\text{path}}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} = m \int_{\vec{r}_i,\text{path}}^{\vec{r}_f} \frac{d\vec{v}}{dt} \cdot d\vec{r}. \tag{8.18}$$

The left side of this equation is W_{if}^{net}. Now, we perform a series of manipulations, similar to the one done for the one-dimensional motion, to show that the right side of Eq. 8.18 is actually equal to the change in kinetic energy.

$$\begin{aligned}
\text{Right side of Eq. 8.18} &= m \int_{\vec{r}_i,\text{path}}^{\vec{r}_f} \frac{d\vec{v}}{dt} \cdot d\vec{r} = m \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} dt \\
&= m \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_i^f \vec{v} \cdot d\vec{v} \\
&= m \left(\int_{v_{ix}}^{v_{fx}} v_x dv_x + \int_{v_{iy}}^{v_{fy}} v_y dv_y + \int_{v_{iz}}^{v_{fz}} v_z dv_z \right) \\
&= \frac{1}{2} m \left(v_{fx}^2 + v_{fy}^2 + v_{fz}^2 \right) - \frac{1}{2} m \left(v_{ix}^2 + v_{iy}^2 + v_{iz}^2 \right) \\
&= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = K_f - K_i
\end{aligned}$$

This shows that work-energy theorem given in Eq. 8.16 is applicable for a general motion of a particle in three dimensions.

8.2.3 Work-Energy Theorem Applied to Multiparticle Systems

A multiparticle system consists of two or more particles. Every macroscopic body, whether the body is a person, a car, a train, the entire planet or a star, is a multiparticle system. The work-energy theorem obtained in Eq. 8.16 apply to each particle of the multiparticle system. The work in the work-energy system consists of the work by every force on the particle, whether the force is by other particles of the same system or by some other external bodies.

In the case of a system consisting of two or more particles we have found that a separation of forces on the particles as internal forces and external forces is very helpful as far as the translation motion of the entire system is concerned. For instance, the acceleration of the center of mass depends on the external forces only. In the next chapter, we will see that the rotational motion of rigid bodies also depends on only the external forces.

Does a similar simplification occur when you combine the the work done on all the particles of a multiparticle system? We will answer this question here by actually writing out the work done on each particle and combining them.

Consider a system containing N particles of masses m_1, m_2, $\cdots$, N at positions $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_N$, respectively, with respect to a reference point O. Let $\vec{F}_1^{\text{net}}$, $\vec{F}_2^{\text{net}}$, $\cdots$, $\vec{F}_N^{\text{net}}$ be the net forces on the particles $1,\ 2, \cdots, N$ respectively. The net force on each particle is a sum of external forces and internal forces. For instance, the net force on particle #1 is sum of the external force $\vec{F}_1^{\text{ext}}$ and the internal forces $\vec{F}_{12}$, $\vec{F}_{13}$, $\cdots$, $\vec{F}_{1N}$, where $\vec{F}_{1k}$ is the force on particle #1 by particle #k.

$$\vec{F}_1^{\text{net}} = \vec{F}_1^{\text{ext}} + \left(\vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1N}\right) \tag{8.19}$$

The work done on each particle will separate into work by external forces and internal forces. The work done by all forces on particle #1 will be

$$W_1 = W_1^{\text{ext}} + W_1^{\text{int}}. \tag{8.20}$$

The work-energy theorem applied to particle #1 gives the net work on particle #1 equal to change in kinetic energy of particle #1. Let v_{1i} and v_{1f} be speed of particle #1 at the beginning and the end of the interval of interest. Then, we have

$$W_1^{\text{ext}} + W_1^{\text{int}} = \frac{1}{2}m_1 v_{1f}^2 - \frac{1}{2}m_1 v_{1i}^2. \tag{8.21}$$

The sum of work on by all forces, external as well internal give the net change in kinetic energy.

$$\sum_{k=1}^{N} W_k^{\text{ext}} + \sum_{k=1}^{N} W_k^{\text{int}} = \sum_{k=1}^{N} \frac{1}{2} m_k v_{kf}^2 - \sum_{k=1}^{N} \frac{1}{2} m v_{ki}^2. \tag{8.22}$$

The right side of this equation is the change in kinetic energy of all the particles and the left is the sum of **work done by the external and internal forces**. We can write this equation in simpler notation.

$$\boxed{W_{\text{net}}^{\text{ext}} + W_{\text{net}}^{\text{int}} = K_f - K_i.} \tag{8.23}$$

We find that, unlike the combining of forces on all particles, the separation of forces in the external and internal forces does not remove the necessity of knowing the internal forces. Therefore, we can conclude the it is much harder to use the work-energy theorem for calculations if internal forces can also do work on the system. There are two special cases in which internal work may vanish and then we will need to bother with external work only.

$$\boxed{W_{\text{net}}^{\text{ext}} = K_f - K_i, \quad \text{if } W_{\text{net}}^{\text{int}} = 0} \tag{8.24}$$

Case 1: $\vec{F}_{ij} \perp \Delta\vec{r}_i$

If internal forces are perpendicular to the displacements of the particles, then, the work done by the internal forces will be zero.

$$W_{\text{net}}^{\text{int}} = 0, \text{ if } \vec{F}_{ij} \perp \Delta\vec{r}_i, \text{ for all}\{ij\}$$

Case 2: $\vec{r}_i$ independent of {i}

If the displacement of every particle is same, then, the the work done by the internal forces will also be zero as we will show with a two-particle system.

$$W_{\text{net}}^{\text{int}} = 0, \text{ if } \Delta\vec{r}_1 = \Delta\vec{r}_2 = \cdots .$$

Two-particle System

Let us apply these equations for an N-particle system to a two-particle system to gain a better understanding of their content. The work by external forces and change in kinetic energy do not require more discussion since they are similar to the terms for a one-particle system. The new term is the work by internal forces. In the case of a two-particle system, the internal forces are a force on particle #1 by particle #2, denoted by $\vec{F}_{12}$ and a force on particle #2 by particle #1, denoted by $\vec{F}_{21}$. The force $\vec{F}_{12}$ will do work on particle #1 and the force $\vec{F}_{21}$ will do work on particle #2. Their works in a time interval when the displacements of the two particles are $\Delta\vec{r}_1$ and $\Delta\vec{r}_2$ respectively are given by

$$W_1^{\text{int}} = \vec{F}_{12} \cdot \Delta\vec{r}_1 \tag{8.25}$$

$$W_2^{\text{int}} = \vec{F}_{21} \cdot \Delta\vec{r}_2 \tag{8.26}$$

Therefore, the net work by internal forces is

$$W_{\text{net}}^{\text{int}} = \vec{F}_{12} \cdot \Delta\vec{r}_1 + \vec{F}_{21} \cdot \Delta\vec{r}_2 \tag{8.27}$$

From the third law of motion , the two forces have equal magnitude but are in opposite direction. That is,

$$\vec{F}_{12} = -\vec{F}_{21}. \tag{8.28}$$

Using this in Eq. 8.27 we obtain the following simpler result for the net work done by the internal forces.

$$W_{\text{net}}^{\text{int}} = \vec{F}_{12} \cdot (\Delta\vec{r}_1 - \Delta\vec{r}_2) \tag{8.29}$$

We find that, although the net internal force, $\vec{F}_{\text{net}}^{\text{int}} = \vec{F}_{12} + \vec{F}_{21} = 0$, the net internal work is not zero unless the displacements of the two particles are equal.

$$W_{\text{net}}^{\text{int}} = 0, \quad \text{if} \quad \Delta\vec{r}_1 = \Delta\vec{r}_2. \tag{8.30}$$

We can generalize this result systems having any number of particles. We note that if all particles of a system move with the same displacement, then, we need work by only the external forces.

8.2.4 Uses of Work-Energy Theorem

The work-energy theorem equates the change in kinetic energy of a system to the net work done by all forces on all particles of the system. One practical use of the work-energy theorem is to find change in speed from given forces and displacements.

But, we have a major problem here: in the section on work we found that work integral may sometimes depend on the path of the particle. This means that, in order to find the change in kinetic energy, we need to know the actual path of the particle and the force at each instant on the path. This kind of detailed information required for using the work-energy theorem makes the theorem useless for general applications.

However, there are many important forces, such as gravity, spring, etc, whose work integral does not depend on path but only on the end points of the motion. These forces are called **conservative forces**. For conservative forces, the work-energy theorem becomes very useful for relating the positions and speeds at different points in time without having to know anything about the full motion as we will see below.

$$\boxed{W_{if}^{\text{Conservative force}}\ :\ \text{Independent of path.}} \tag{8.31}$$

There is another situation where work-energy theorem becomes useful. Suppose the object is constrained to move in a pre-determined trajectory. For instance, when a ball is forced to move in a track, we know the CM of the ball must follow the path of the track. To constrain an object to a particular trajectory one requires a force that acts on the body to change the direction of the velocity by acting perpendicular to the velocity. Since constraining forces are directed perpendicular to the velocity, they are also perpendicularly to the infinitesimal displacement. This makes the work integral for constraining forces identically zero.

$$\boxed{W_{if}^{\text{Constraining force}} = 0.} \tag{8.32}$$

The forces which are neither conservative nor constraining are called non-conservative forces. The friction force is an example of a **non-conservative force**, so is the force of pull and push. In setting up

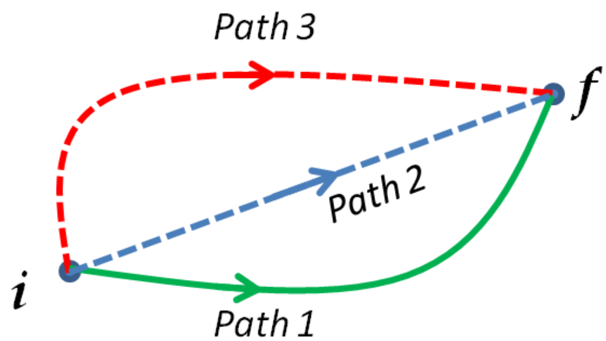

Figure 8.15: The work done by conservative force for displacement of the system between two points in space is independent of the path. The work integral $\int_i^f \vec{F}_c \cdot d\vec{r}$ for a conservative force $\vec{F}_c$ is same for all paths between points marked i and f.

problems that involve non-conservative forces, we need either a complete information about the path or some simplifying assumptions, such as constancy of force, if we wish to apply work-energy theorem to that problem.

8.2.5 Work-Energy Theorem For Conservative Forces

The work done by many commonly encountered forces such as push, pull, friction, etc, tend to depend on the actual path the system takes between the initial and final positions. However, work by many other forces, such as gravity, spring, and all fundamental forces, are path independent - the work by these forces depend only on the initial and final positions of the path. These forces are called **conservative forces**. Work-energy theorem is a very useful tool for an analysis of the motion when only conservative forces act on the body as we will see in this section.

Since the work done by a conservative force does not depend on the path but depends only on the position of the end points of the path, the result of the work integral can also be written as a difference of a function evaluated at the end points of the path. Denoting the conservative force by $\vec{F}_c$, we write the results of the work integral for this force as

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_c \cdot d\vec{r} = \text{function of } (\vec{r}_f) - \text{function of } (\vec{r}_i), \qquad (8.33)$$

Note that we have omitted mentioning the path for the work integral here since, for a conservative force, the integral does not depend on the path. The function on the right side of Eq. 8.33 is usually designated by letter U. We also include a negative sign for future convenience.

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_c \cdot d\vec{r} = -\left[U(\vec{r}_f) - U(\vec{r}_i)\right]. \tag{8.34}$$

For the sake of brevity we will write this equation as

$$\boxed{\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_c \cdot d\vec{r} = -U_f + U_i.} \tag{8.35}$$

The individual quantities U_i and U_f are called the **initial and final potential energies** of the body that is subject to the conservative force $\vec{F}_c$. At each point in space the body will have potential energy contributions from every conservative force acting on the body. For instance, if two conservative forces, say gravity and spring force act on a body, then the body will have two contributions to the potential energy, one due to the gravity force and another due to the spring force.

$$\boxed{U^{\text{net}} = U^{\text{gravity}} + U^{\text{spring}} + \cdots}$$

Supposing only conservative and constraint forces act on the body, the net work will equal the work by conservative forces only because the work by constraint forces are zero as explained above. Therefore, we can write the net work on the body as a change in the net potential energy from all conservative forces. We denote the net potential energy by the same symbol, U_i and U_f.

$$W_{if}^{\text{net}} = U_i - U_f \quad \text{(Conservative and constraint forces only.)} \tag{8.36}$$

Now, when we replace W_{if}^{net} in the work energy theorem by $U_i - U_f$, we obtain the following important relation.

$$U_i - U_f = K_f - K_i \quad \text{(Conservative and constraint forces only.)} \tag{8.37}$$

This equation can be rearranged so that all quantities for the initial state are one side of the equation and those of the final state on the other side.

$$\boxed{K_i + U_i = K_f + U_f \quad \text{(Conservative and constraint forces only.)}} \tag{8.38}$$

The combination $(K + U)$ is called mechanical energy, or simply energy E of the system.

$$\boxed{E = K + U.} \tag{8.39}$$

Eq. 8.38 says that mechanical energy is conserved if only conservative and constraint forces act on the system. This statement is referred to as the principle of conservation of energy.