4.2 INTENSITY OF LIGHT

Traveling waves transport energy and momentum from one place to another without actually moving particles. Therefore, you can think of an electromagnetic wave in terms of the flow of the electromagnetic energy and momentum. Consider an electromagnetic wave in vacuum. The energy per unit volume, or energy density (u), contained in the electric and magnetic fields of the wave would be

$$u = u_E + u_B = \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}. \tag{4.10}$$

For a plane electromagnetic wave, we plug in the expressions for electric and magnetic waves given above and obtain the following for energy per unit volume.

$$\begin{aligned} u &= \frac{1}{2}\epsilon_0 E_0^2 \cos^2(kx - \omega t + \phi) + \frac{1}{2\mu_0}\left(\frac{E_0^2}{c^2}\right)\cos^2(kx - \omega t + \phi) \\ &= \epsilon_0 E_0^2 \cos^2(kx - \omega t + \phi) = \frac{1}{2}\epsilon_0 E_0^2 \cos(2kx - 2\omega t + 2\phi) + \frac{1}{2}\epsilon_0 E_0^2. \end{aligned} \tag{4.11}$$

Thus, the energy density moves with the wave. One might say that you have a wave of the electromagnetic energy density with half the wavelength and twice the frequency. We can figure out the rate of transport of energy by considering what happens over an area in a small time interval Δt as shown in Fig. 4.3.

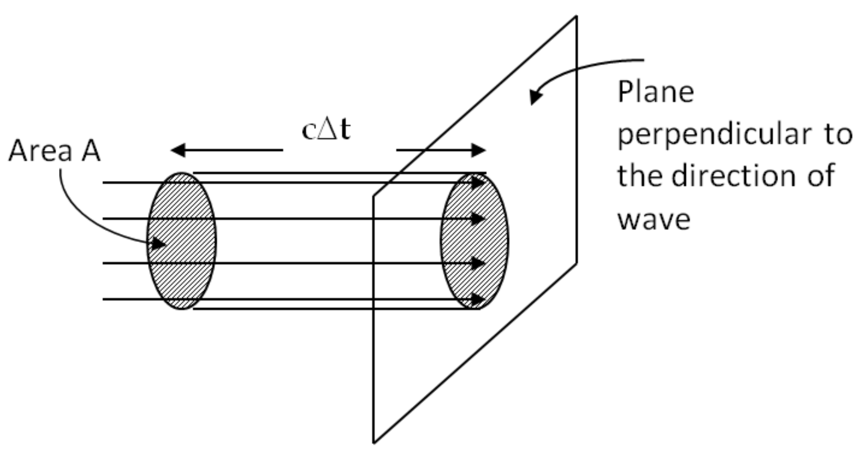


Figure 4.3: Energy $uAc\Delta t$ contained in volume $Ac\Delta t$ will pass the area A perpendicular to the direction of the wave in time Δt.

The energy contained in a box of length $c\Delta t$ and cross-sectional area A will pass through the cross-sectional plane in time Δt.

$$\text{The energy passing through area } A \text{ in time } \Delta t \;= u\times\text{volume} = uAc\Delta t. \tag{4.12}$$

The energy per unit area per unit time passing through a plane perpendicular to the wave is called the **power flux** and denoted by letter S.

$$S = \frac{\text{The energy passing area } A \text{ in time } \Delta t}{A\Delta t} = uc. \tag{4.13}$$

Using Eq. 4.12 we find that the power flux is related to the energy density as

$$S = uc. \tag{4.14}$$

The flux of power at any place also fluctuates in time as can be seen by substituting u from Eq. 4.11.

$$S = c\epsilon_0 E_0^2 \cos^2(kx - \omega t + \phi) \tag{4.15}$$

Since the frequency of the visible light is very high (of the order of 10^{14} cycles per second) the power flux will be an extremely rapidly varying quantity. Our eyes as well as most measuring devices cannot respond this fast. That means a detector will detect electromagnetic energy that would be averaged over many cycles. The quantity obtained by time-averaging the power flux is called the **intensity** or **irradiance** I of light.

$$\text{Intensity,}\quad I = \langle S \rangle_{\text{time-average}} \tag{4.16}$$

The time-averaging of the expression for the power flux given above can be easily performed by noting the following mathematical result for the time-average over one period $T = 2\pi/\omega$.

$$\frac{1}{T}\int_0^T \cos^2(kx - \omega t + \phi)dt = \frac{1}{2}.$$

Hence, the intensity of light moving at speed c in vacuum will be

$$I = \frac{1}{2} c\epsilon_0 E_0^2, \tag{4.17}$$

This can also be written in terms of the amplitude of the magnetic field in the wave.

$$I = \frac{1}{2} c\epsilon_0 E_0^2 = \frac{c}{2\mu_0} B_0^2. \tag{4.18}$$

If the wave is not in vacuum but in some other linear medium such as glass or air we will need to replace the speed c by the speed v in the medium, the permittivity ϵ_0 by ϵ, and the permeability μ_0 by μ.

$$I = \frac{1}{2} v\epsilon E_0^2 = \frac{v}{2\mu} B_0^2. \quad \text{(Linear medium)} \tag{4.19}$$

Note that the amount of energy crossing a given area also depends upon the orientation of the area relative to the direction of light. To take the direction into account a power flux vector $\vec{S}$, also called the **Poynting vector**, is introduced by the following defining equation.

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}. \tag{4.20}$$

The vector product of $\vec{E}$ and $\vec{B}$ makes it evident that the **Poynting vector** $\vec{S}$ points in the direction of the propagation of the electromagnetic wave. The power crossing any arbitrary surface is then obtained by dividing the surface into small patches, calculating the power through each patch, and then summing up the contributions. For this purpose we define an area vector of a plane surface by the magnitude equal to the area and direction perpendicular to the surface. Let $\vec{S}$ be the average Poynting vector over a patch of area vector $\Delta\vec{A}$, then power crossing area $\Delta\vec{A}$ will be given by the projection of $\vec{S}$ on $\Delta\vec{A}$ times the area which is equal to the scalar product of $\vec{S}$ and $\Delta\vec{A}$.

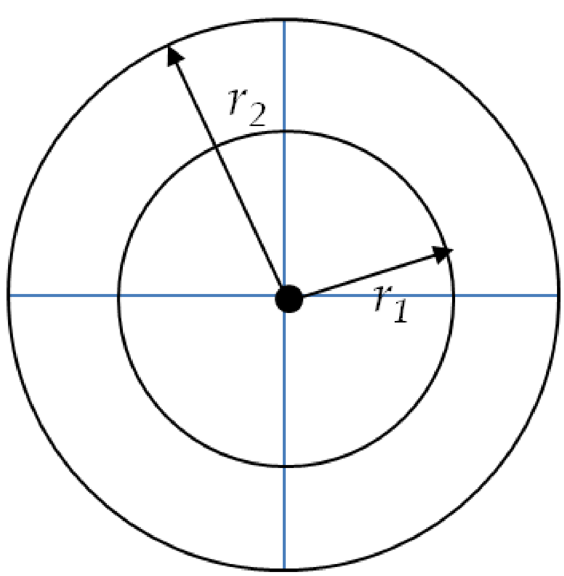
$$\text{Power moving past the area element } = \vec{S} \cdot \Delta\vec{A}. \tag{4.21}$$

The total power crossing any surface will be equal to the sum of the contributions from all patches. The sum over infinitesimal patches leads to the following integral formula.

$$\text{Power crossing any surface } = \iint \vec{S} \cdot d\vec{A}. \tag{4.22}$$

Example 4.2.1. Intensity of an isotropic source Energy conservation implies that the intensity from a source must drop off with distance. Find the relation between intensities at two distances from an isotropic source.

Solution. Consider a point source that emits light in a spherical symmetric way, e.g. in a spherical wave, as shown in the figure.



Energy passing through the spherical surface with radius r_1 in an interval Δt must equal the energy passing through the surface with radius r_2 in the same interval of time. Let the intensities at r_1 and r_2 be denoted by I_1 and I_2 respectively. As the product of intensity and surface area equals the total energy passing the surface per unit time we obtain the following equality based on the energy conservation in each unit time interval.

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2 \quad \Longrightarrow \quad \frac{I_2}{I_1} = \frac{r_1^2}{r_2^2} \tag{4.23}$$

This shows that the intensity of an isotropic source drops off as inverse square.

Example 4.2.2. Electric and magnetic field in a plane wave light Consider a plane wave light source that delivers 3 mW of power over a 0.5 cm^2 area. Find the (a) intensity, and (b) amplitudes of electric and magnetic fields.

Solution.

(a) The intensity is power per unit area.

$$I = \frac{3 \times 10^{-3}\ \text{W}}{0.5 \times 10^{-4}\ \text{m}^2} = 60\ \text{W/m}^2.$$

(b) The intensity is proportional to the square of the electric field amplitude.

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \frac{\sqrt{2 \times 60\ \text{W/m}^2}}{\sqrt{3 \times 10^8\ \text{m/s} \times 8.85 \times 10^{-11}\ \text{N.m}^2/\text{C}^2}} = 213\ \text{N/C}$$

The magnetic and electric fields in an electromagnetic wave are related as follows.

$$B_0 = \frac{E_0}{c} = \frac{213\ \text{N/C}}{3 \times 10^8\ \text{m/s}} = 7.09 \times 10^{-7}\ \text{T}.$$