6.3 FRAUNHOFER DIFFRACTION THROUG A SINGLE SLIT

In this section we study the diffraction phenomenon in more detail. As our first example, we consider a light source in front of a narrow slit in an opaque material (Fig. 6.4). Suppose the light source is either far away or at the focal plane of a lens so that the waves are planar at the slit. We place a screen behind the slit to observe the diffraction pattern there when it is placed at different distances from the slit. If the screen is placed immediately behind the slit, we find a **shadow of the slit** on the screen. When we move the screen further out, the shadow develops fringes whose pattern changes as you move the screen further out. These patterns are called **Fresnel or near-field diffraction**. Moving the screen further away from the slit you reach a region where the patterns stabilize and although they spread out more as you move the screen further away, the diffraction pattern itself remains the same. We call this pattern the **Fraunhofer or far-field diffraction** after **Joseph von Fraunhofer**, who built the first diffraction grating consisting of 260 closely, spaced parallel wires in 1821, and measured wavelengths of colored lights.

Now, if you conducted the above set of experiments when the source of light was too close to the slit, then the wave would not be planar at the slit. Instead, the wave at the slit would be spherical. In that case, you only see the Fresnel diffraction on the other side at all distances. For Fraunhofer diffraction, you need the wave to be planar both at the slit and at the screen, i.e. both the screen and the source of light need to be far away from the slit. In this chapter we will study only the Fraunhofer diffraction as it uses a less complicated mathematics and has the most applications.

6.3.1 Minima in a Single-slit Diffraction

How can we predict the locations of bright and dark bands on the screen of a single-slit Fraunhofer diffraction? It turns out that the minima of the diffraction on the screen can be found by a clever procedure. We imagine the slit as providing an infinite number of coherent sources according to the Huygens-Fresnel principle. But, now we try to pair up secondary wavelet sources on the wavefront at the slit so that all the pairs interfere destructively at a point on the screen.

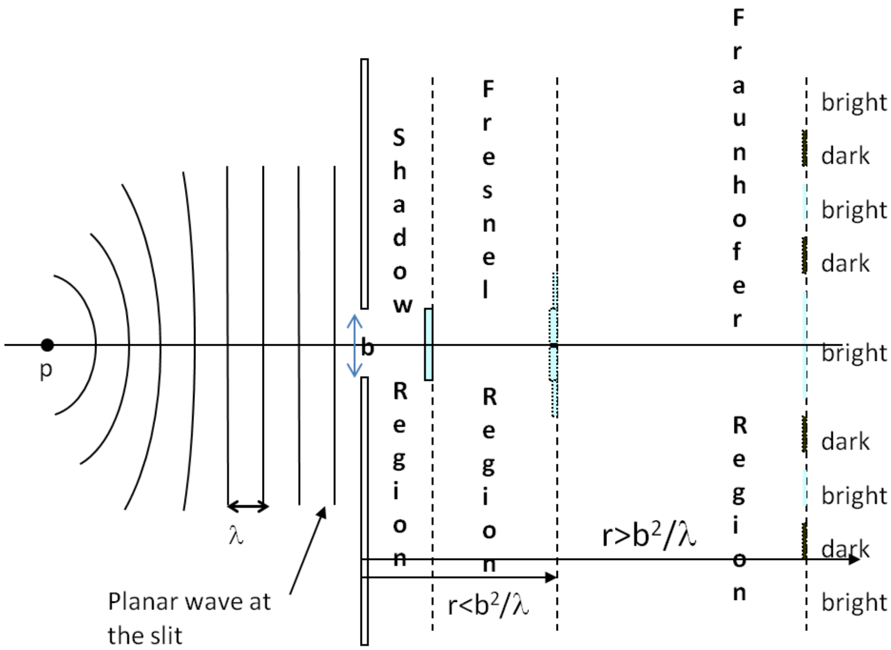


Figure 6.4: Shadow, Fresnel and Fraunhofer diffraction regions and patterns from a horizontal slit in front of a point source placed far away from the slit of width b.

For instance, by dividing up the slit into two equal zones, we obtain one set of pairs of sources which consist of a point at the top of the upper zone and a point at the top of the lower zone, the next point of the top zone is paired with the corresponding point of the lower zone etc (Fig. 6.5). As the screen is far away, all the rays will be treated approximately parallel so that the path difference between the members of a pair are same, regardless of their location in the zones. Thus if the path difference of one of these pairs is equal to half a wavelength, they will all interfere destructively at the screen.

The path difference can be worked out from the right-angled triangle in Fig. 6.5(b). Therefore we find that there should be a minimum at the screen in the direction if

$$\frac{b}{2}\sin\theta = \frac{\lambda}{2}, \tag{6.1}$$

which simplifies to

$$b\sin\theta = \lambda. \tag{6.2}$$

This is the $m = 1$ minimum for the single-slit diffraction. Similarly, if you divide up the wave at the slit into four equal zones as shown in

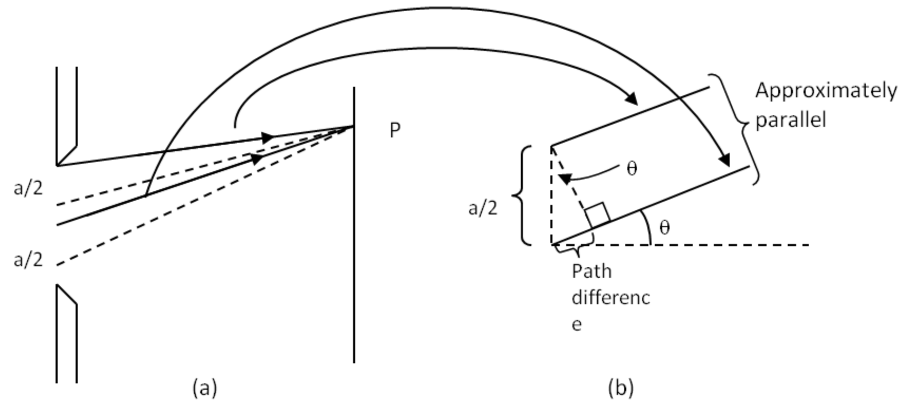


Figure 6.5: The first-order minimum - the path difference must equal half-wavelength for the paired waves to be out-of phase. Error: The labels a in the figure should be b.

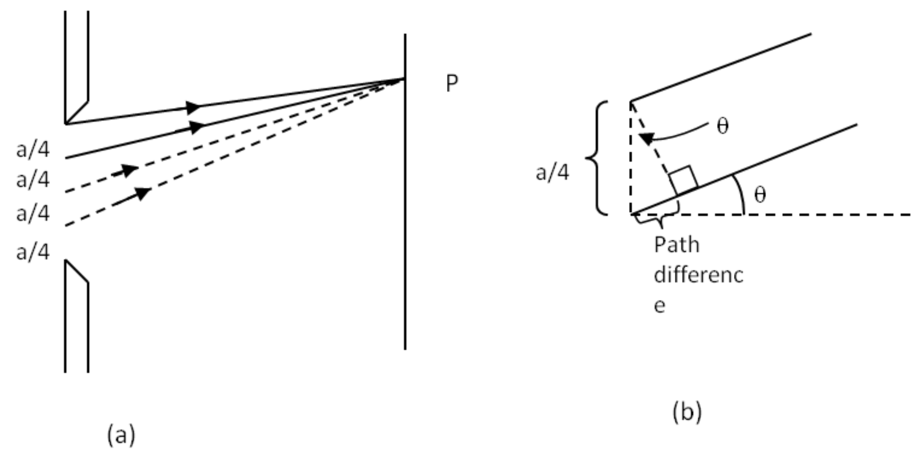


Figure 6.6: Second-order minimum - the path difference must equal half-wavelength for the paired waves to be out-of phase. Errors: the labels a should be b.

Fig. 6.6. Let us label the zones 1, 2, 3 and 4, and consider (1,2) and (3,4) pairs in a similar fashion as above. The path difference, which is now $\frac{b}{4}\sin\theta$, can be equated to $\lambda/2$ to find a minimum.

$$\frac{b}{4}\sin\theta = \frac{\lambda}{2}, \tag{6.3}$$

which gives

$$b\sin\theta = 2\lambda. \tag{6.4}$$

This is the $m = 2$ minimum for the single-slit diffraction. From the pattern of formulas, we easily expect the following for all the dark fringes.

$$b\sin\theta = m\lambda \quad (m = \pm 1, \pm 2, \pm 3, \cdots .) \tag{6.5}$$

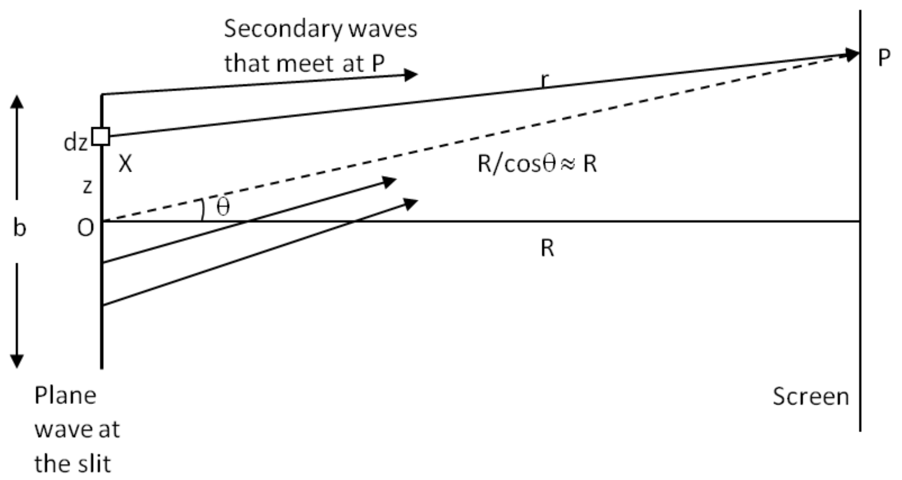


Figure 6.7: Geometry for the calculation of Fraunhofer diffraction from plane wave at a slit.

6.3.2 Intensity in Diffraction Pattern

Figuring out the intensities of the bands in the diffraction pattern is not so easy as finding the locations of the minima. We must develop a procedure for adding up the wave amplitudes, and then find the time-average of the square of the sum. We again start with imagining the plane wave at the slit to be made up of infinitely many points, each a source of spherical wave in accordance with the Huygens-Fresnel principle. Then, the pattern at the screen must be the result of interference of these infinitely many secondary waves (Fig. 6.7).

How can we perform the addition of spherical secondary waves emanating at points of the plane wave at the slit? First, we need to know how to write a spherical wave. A spherical wave is given by an amplitude that decreases as 1/distance from its point of origin, and moves radially outward equally in all directions. Therefore, a secondary spherical wave emanating from an infinitesimal element between z and dz of the plane wave at the slit has the following electric field at point P on the screen.

$$dE = \frac{(Adz)}{r} \sin(kr - \omega t). \tag{6.6}$$

where A is electric field amplitude of the wave at a point in the slit, ω the angular frequency, k the wave number, and r the distance to point P on the screen from the element dz. The coefficient $\frac{Adz}{r}$ is the amplitude of the net electric field at P from the wavefront in the element z to $z + dz$ of the slit. Since we are interested in the Fraunhofer diffraction, therefore the distance between the slit and the screen, $R >> b$, which implies that angle θ(in radians) ¡¡ 1. Therefore, the distance from the center of the slit O to the point P on the screen can be approximated to

$$\mathrm{OP} = \frac{R}{\cos\theta} \approx R. \tag{6.7}$$

Now using the law of cosines in $\triangle$OXP gives us the following for r^2.

$$r^2 = R^2 + z^2 - 2Rz\cos\left(\frac{\pi}{2} - \theta\right). \tag{6.8}$$

Since $z < b$ and $R >> b$, we have $R >> z$. We can approximate r in the present case by first taking a square root of both sides, and then, expanding the result in powers of z/R, and keeping only the leading order in z/R.

$$\begin{aligned} r &= \left(R^2 + z^2 - 2Rz\sin\theta\right)^{1/2} = R\left[1 + \left(\frac{z}{R}\right)^2 - 2\frac{z}{R}\sin\theta\right]^{1/2} \\ &\approx R\left[1 - \frac{z}{R}\sin\theta\right] = R - z\sin\theta. \end{aligned} \tag{6.9}$$

The distance r appears in the amplitude $\frac{Adz}{r}$ and in the phase $kr - \omega t$. Since $|z\sin\theta| << R$ we may be tempted to drop $z\sin\theta$ altogether. However, kr in the phase has a large k and the result is acted on by a sine function; small changes in kr make a large impact on the sine of kr. Therefore, in the phase we must keep $z\sin\theta$ with R. Dropping $z\sin\theta$ in the the amplitude $\frac{Adz}{r}$ does not have much impact. Therefore, we put r from Eq. 6.9 into Eq. 6.6 in the phase and $r \approx R$ in the amplitude to obtain the following electric field at the point P of the screen from the wavefront at z to $z + dz$ at the slit.

$$dE \approx \frac{(Adz)}{R}\sin\left[k(R - z\sin\theta) - \omega t\right]. \tag{6.10}$$

The net electric field at point P is then the sum of contributions from all point of the slit, which is obtained by performing the integration of Eq. 6.10 from$z = -b/2$ to $+b/2$.

$$E_P = \frac{Ab}{R}\left[\frac{2\sin\left(\frac{kb}{2}\sin\theta\right)}{kb\sin\theta}\right]\sin\left[kR - \omega t\right]. \tag{6.11}$$

Intensity is obtained by time averaging the square of this rapidly fluctuating field. The time averaging of $\sin^2(\omega t)$ or $\cos^2(\omega t)$ gives $\frac{1}{2}$. Therefore, the intensity in the direction θ is found to be

$$I(\theta) = \epsilon_0 c\left(\frac{bA}{R}\right)^2\left[\frac{\sin\beta}{\beta}\right]^2, \tag{6.12}$$

where the symbol β stands for

$$\beta \equiv \frac{\pi b \sin\theta}{\lambda}. \tag{6.13}$$

The quantity $\epsilon_0 c \left(\frac{bA}{R}\right)^2$ is the intensity in the direction $\theta = 0$, which is also the maximum intensity since all the secondary wavelets interfere constructively in that direction. We may simplify the formula for intensity further by casting the general formula in terms of the intensity for $\theta = 0$. Then the intensity in the direction θ is more compactly written as

$$\frac{I(\theta)}{I(0)} = \left[\frac{\sin\beta}{\beta}\right]^2. \tag{6.14}$$

When $\beta = 0$, $\sin\beta/\beta$ is not zero, but 1. For $\beta = m\pi$ for non-zero integer m, $\sin\beta/\beta$ is zero.

$$\sin\beta/\beta = 0 \text{ when } \beta = m\pi \text{ for } m \text{ non-zero integer.} \tag{6.15}$$

Therefore, the intensity at P will be zero for the following conditions corresponding to the destructive interferences of all secondary wavelets from the slit.

$$\beta = m\pi \quad (m = \pm1, \pm2, \pm3, \cdots) \quad (\textit{destructive}), \tag{6.16}$$

which is same as

$$b\sin\theta = m\lambda \quad (m = \pm1, \pm2, \pm3, \cdots) \quad (\textit{destructive}). \tag{6.17}$$

We had found the same result earlier by a much simpler procedure. The intensity maxima do not occur at points halfway between the minima. Instead they can be obtained by treating the intensity I as a function of β and using calculus for finding the extrema of a function by setting the derivative to zero.

$$\frac{dI}{d\beta} = 0. \tag{6.18}$$

This condition will include both the minima and maxima. Carrying out the derivation explicitly we obtain two conditions on β.

$$\sin\beta = 0. \tag{6.19}$$

$$\tan\beta = \beta. \tag{6.20}$$

The first condition corresponds to one maximum at $\beta = 0$, and minima at other solutions of $\sin\beta = 0$ as discussed above. The second equation corresponds to all other maxima. The second equation is a

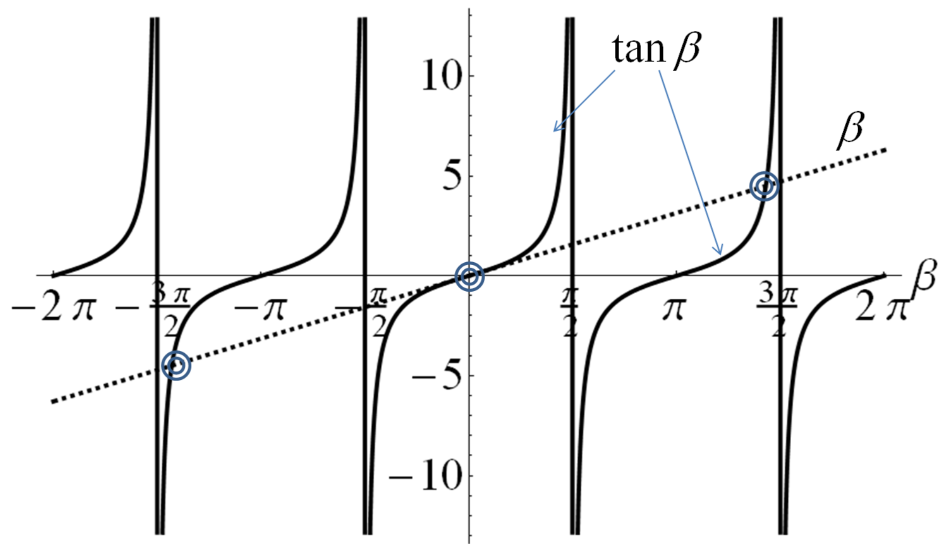


Figure 6.8: The graphical solution of (MATLAB).

transcendental equation and can be solved graphically. In graphical method, you would plot the two sides of Eq. 6.20 independently on the same graph against the same range for β and the place where the graphs intersect coorespond to the solution of the equation.

The solution of Eq. 6.20 is obtained graphically by plotting $\tan\beta$ and β against β as shown in Fig. 6.8. The solution for the maxima can be read off from the graphs. The approximate values are $\beta \approx 0$, $\pm\frac{3}{2}\pi$. A more precise and expanded graph gives $\beta = 0$, $\pm 1.4303\pi$, $\pm 2.4590\pi$, $\pm 3.4707\pi$, $\cdots$. The locations of the constructive diffraction, viz $\beta = 0$, $\pm 1.4303\pi$, $\pm 2.4590\pi$, $\pm 3.4707\pi$, $\cdots$ are referred to as $m = 0, \pm 1, \pm 2, \pm 3, \cdots$ constructive diffraction orders in analogy to the language for the interference in the double-slit experiment discussed in the last chapter.

The diffraction patterns are usually displayed in terms of the direction angle θ with respect to the horizontal direction. However, if you plot intensity as a function of the parameter β rather than angle θ, you would find that the centeral maximum lies between the values $\beta = -\pi$ and $\beta = \pi$ while other maxima are between $\beta = n\pi$ and $\beta = (n+1)\pi$. Thus, when plotted against β, the central diffraction maximum is twice as wide.

Intensity of the peaks decrease very rapidly. In Fig. 6.9, the intensity is plotted versus β which is related to the direction θ. The

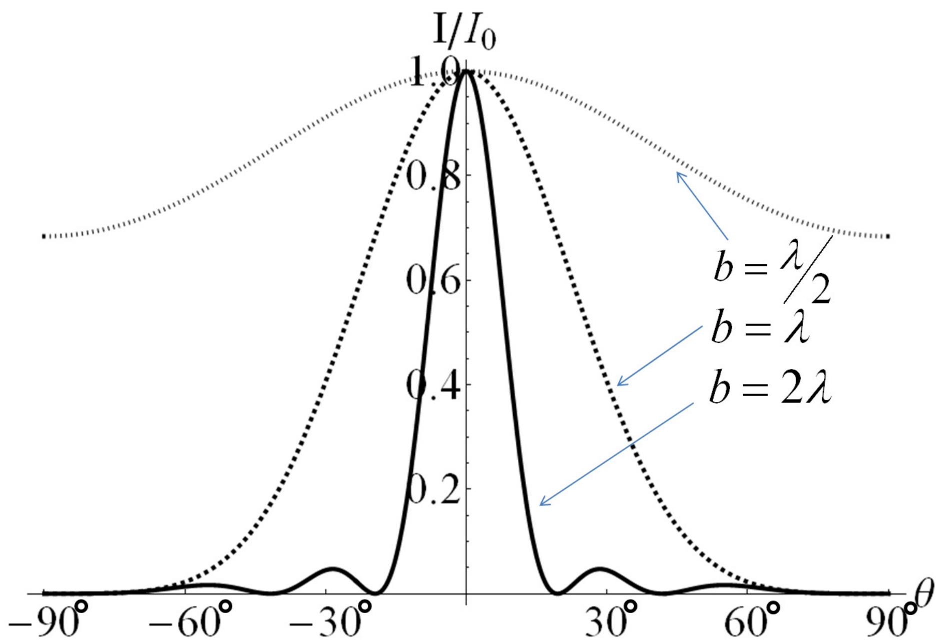


Figure 6.9: A plot of diffraction pattern for various slit widths.

Visit the following web site for pictures of real patterns.

http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/fraungeo.html#c1

Figure 6.10: Actual diffraction pattern of single slit.

intensity of the central peak is about 20 times the intensity of the second peak.

If the slit is horizontal, then the diffraction pattern will be spread out vertically, and if the slit is vertical then the diffraction pattern will be spread out horizontally. Visit the website given in Fig. 6.10 to see a diffraction pattern from experiment.

IMPORTANT RESULTS FOR SINGLE-SLIT DIFFRACTION

The intensity at an angle θ from diffraction of a monochromatic wave of wavelength λ incident on a slit of width b.

$$\frac{I(\theta)}{I(0)} = \left[\frac{\sin\beta}{\beta}\right]^2,$$

where

$$\beta \equiv \frac{\pi b \sin\theta}{\lambda}.$$

The diffraction minima occur when

$$\beta = m\pi \quad (m = \pm 1, \pm 2, \pm 3, \cdots) \quad (destructive),$$

which corresponds to

$$b \sin\theta = m\lambda \quad (m = \pm 1, \pm 2, \pm 3, \cdots) \quad (destructive).$$

The diffraction maxima occur when

$$\beta = 0, \pm 1.4303\pi, \pm 2.4590\pi, \pm 3.4707\pi, \cdots .$$

These maxima correspond to

$$b \sin\theta = 0, \pm 1.4303\lambda, \pm 2.4590\lambda, \pm 3.4707\lambda, \cdots ,$$

which are referred to as $m = 0, \pm 1, \pm 2, \cdots$ constructive diffraction orders.

Example 6.3.1. Single slit diffraction

Light of wavelength 550 nm passes through a slit of width 2 μm. Find the location of four minima on a screen about the central bright location (a) in terms of the angle subtendend with the horizontal direction from the center of the slit and (b) the positions of the dark bands if the screen is 50 cm away.

Solution. (a) The minima are given by the following condition

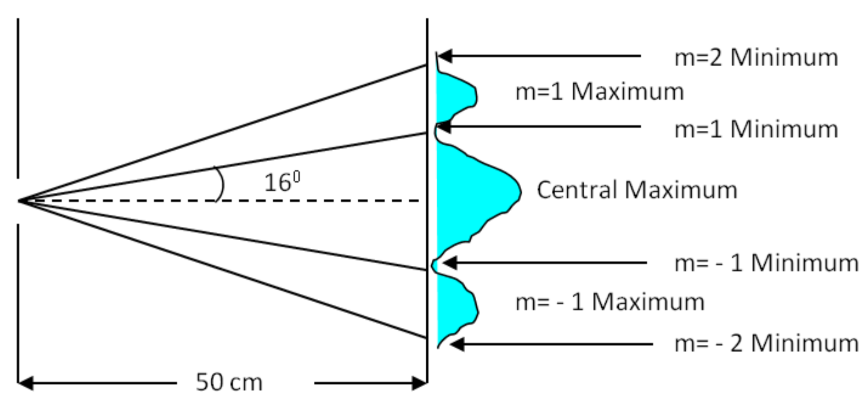
$$b \sin\theta = m\lambda \quad (m = \pm 1, \pm 2, \pm 3, \cdots) \quad (destructive).$$

Hence, the four minima around the central maxima will have $m = \pm 1, \pm 2$. The corresponding directions from the slit are

$$\theta_{\pm 1} = \pm \sin\left(\frac{\lambda}{b}\right) = \pm 16^\circ$$

$$\theta_{\pm 2} = \pm \sin\left(\frac{2\lambda}{b}\right) = \pm 33.4^\circ$$

Pictorially, the directions from the slit for the minima are:



(b) The location of the diffraction minima on the screen can be deduced from the right-angled triangles. Let the y-axis be pointed up on the screen, then we will have

$$y = (50 \text{ cm}) \ \tan\theta.$$

Denote the positions of the four minima by y_{+1}, y_{-1}, y_{+2} and y_{-2}.

$$y_{\pm 1} = (50\text{cm}) \ \tan\theta_{\pm 1} = \pm 14 \text{ cm}$$
$$y_{\pm 2} = (50\text{cm}) \ \tan\theta_{\pm 2} = \pm 33 \text{ cm}$$