4.4 CAPACITORS

A capacitor is a device used for storing electric charge and electric energy. The simplest capacitor consists of two conductors. The two conductors are called plates regardless of their shape. Moving charges from one plate to the other creates a situation where one plate is positively charged and the other negatively charged, and the work done is stored as the electrical energy in the capacitor.

Figure 4.17: Wilmhurst mill uses Leiden jars to store separated charges by rubbing off rotating metal strips. The wheel contains strips of metal that rubs against a metal wool when the wheel rotates. The separated charges are sent to Leyden jars, which are the two containers on the two sides.

Early versions of capacitors were invented by the Dutch physicist Pieter van Musschenbroek of the University of Leiden around 1746, and are called **Leiden jars**. Leiden jars were instrumental in early experiments on electricity. Fig. 4.17 shows a pair of Leiden jars used in a Wilmhurst mill for storing charges.

A Leiden jar consists of a glass jar with inner and outer metal coatings. The inner coating is in contact with a brass rod through a loose metal chain. The brass rod passes through a cork stopper and a metal knob is mounted on the rod. When an electric charge is put on the knob, it spreads to the inner metal which attracts opposite charges of the outer metal. By grounding the outer surface of the outer metal, you are left with only opposite charges on the inner and outer metal surfaces facing each other.

Leiden jars can be charged to fairly high voltage. When wires connecting the two plates of a charged Leiden jar are brought near each other spectacular sparks from one wire to the other are often visible. Leiden jars are sometimes also called condensers.

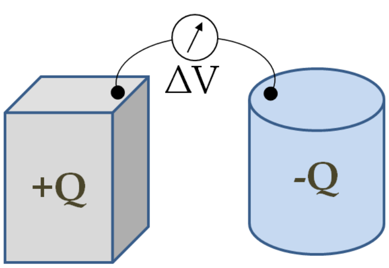

Figure 4.18: Two conductors having equal and opposite charges Q and with potential difference ΔV.

4.4.1 Capacitance

Consider two arbitrarily shaped conductors that have equal and opposite charges $+Q$ and $-Q$ on them as shown in Fig. 4.18. Let ΔV be the potential difference between the positive and negative plates. Independently of the shape, size, or separation of the two conductors, voltage between the plates will increase if we put more charges on the two.

$$Q \propto \Delta V \quad \Longrightarrow \quad Q = C\Delta V \tag{4.8}$$

The proportionality constant C is called capacitance. The capacitance refers to the capacity of a capacitor for charges. A capacitor with higher capacitance will hold more charge for the same potential difference than the one with a smaller capacitance.

The unit of capacitance in SI system of units is Coulomb/Volt or

Farad, named after the English physicist Michael Faraday.

$$1 \text{ Farad} = 1 \, \frac{\text{Coulomb}}{\text{Volt}} \;\Leftrightarrow\; 1 \text{ F} = 1 \, \frac{\text{C}}{\text{V}}.$$

It turns out that 1 Farad is too big a unit for the laboratory use. Usually one uses micro-farad (μF) or pico-farad (pF) in the lab.

You can also define the capacitance of a single isolated conductor if you send its partner to infinity. For instance, an isolated sphere of radius R with charge Q has the following potential ΔV with respect to infinity.

$$Q = 4\pi\epsilon_0 R \Delta V \tag{4.9}$$

Therefore the capacitance C_{sphere} sphere of a single sphere would given by the following formula.

$$C_{\text{sphere}} = 4\pi\epsilon_0 R \;\; \text{(the other partner at infinity)} \tag{4.10}$$

4.4.2 The Parallel Plate Capacitor

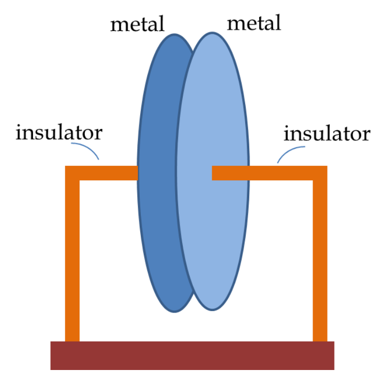

Figure 4.19: A parallel plate capacitor

A parallel plate capacitor consists of two large conducting plates of area A on each face separated by a gap d. In this section we will find a formula for the capacitance of this capacitor.

To find the capacitance of the capacitor, we place charges $+Q$ and $-Q$ on the two plates and find the expression for potential difference ΔV between the plates. The ratio of ΔV to Q is then calculated to find the formula for the capacitance of the parallel plate capacitor. The potential difference will be obtained from the expression of electric field between the plates. Therefore, we have the following steps in the calculation.

1. Find an expression for electric field between plates.
2. Find an expression for the potential difference ΔV.
3. Read off $\frac{Q}{\Delta V}$ for the formula for the capacitance C.

To find the electric field between plates, let us examine what happens when you place charges $\pm Q$ on the plates. Charges in the two plates will be attracted towards each other and the entire charge on the positive plate will be facing the entire charge on the negative plate as shown in Fig. 4.20, and there will be no charges on the back faces of the plates. Therefore, we have the problem of two large sheets with charge densities Q/A with A being the area of one face of the plate. The electric fields from these sheets add in the space between the plates and cancel elsewhere. The result is a constant electric field given by:

$$\vec{E} = \frac{Q}{\epsilon_0 A} \hat{u}_n \tag{4.11}$$

where $\hat{u}_n$ is a unit vector pointed from the positive plate to the negative plate.

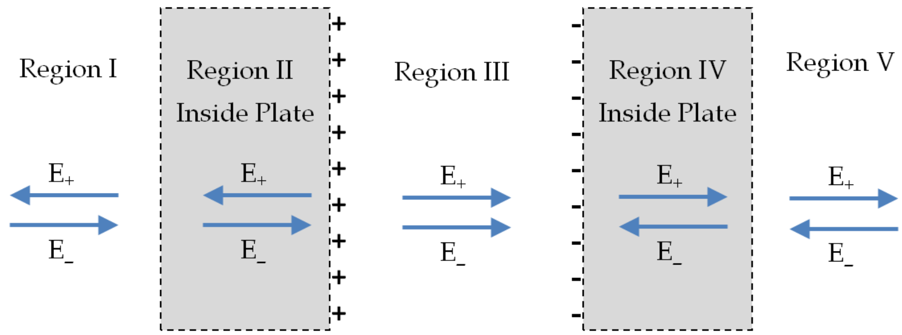

Figure 4.20: Two parallel conducting plates charged with equal and opposite charges. Note the charges are on the inside surface facing the other plate. If A is the area of one side of each plate and $\pm Q$ are charges on the plates, the charge densities are $\sigma = \pm Q/A$. The magnitude of the electric fields from the two plates are equal, $E_+ = E_- = Q/2A\epsilon$. The net electric field between the plates has the magnitude $E_{net} = Q/A\epsilon$ and has the direction from the positive plate to the negative plate.

The potential difference between the plates can be obtained by a line integral of electric field from point B on the negative plate to point O on the positive plate shown in Fig. 4.21.

$$V_O - V_B = -\int_B^O \vec{E}\cdot d\vec{r}. \tag{4.12}$$

Using the coordinate system shown in Fig. 4.21 and using the electric field between plates given in Eq. 4.11, this integral takes the following form.

$$V_O - V_B = -\int_d^0 \frac{Q}{\epsilon_0 A} dx, \tag{4.13}$$

The integrand is independent of x, therefore, the potential difference $\Delta V = V_+ - V_- = V_O - V_B$.

$$\Delta V = \frac{Qd}{\epsilon_0 A}. \tag{4.14}$$

We see that the potential difference ΔV is proportional to the charge Q. The proportionality constant is equal to $1/C$ by definition of capacitance given in Eq. 4.8. Hence, the capacitance of a parallel plate capacitor is

$$\boxed{C = \frac{\epsilon_0 A}{d}, \quad \text{Parallel-plate capacitor.}} \tag{4.15}$$

This shows that the capacitance of a capacitor depends on the geometrical features of the plates, such as surface area over which charges

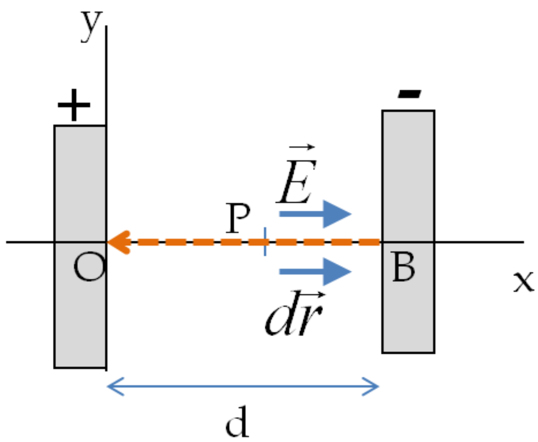

Figure 4.21: Coordinates for calculating potential difference between the plates. Potential difference between plates is obtained by integrating from a point B on negative plate to a point O on the positive plate on the path shown as dashed.

can spread, the separation between the plates, and the electric permittivity of the medium between the plates, which was vacuum in this example.

Example 4.4.1. Capacitance of a Parallel Plate Capacitor

(a) Find the capacitance of a parallel plate capacitor that has two 50 cm $\times$ 50 cm square aluminum foils separated by a distance of 3 mm. (b) How much charge will be stored on the positive plate when the capacitor is connected across a battery that can maintain a potential difference of 12 V between the plates?

Solution. (a) We use the formula for capacitance of a parallel plate capacitor derived above. The unit of ϵ_0 can be converted into F/m as follows.

$$\frac{\text{C}^2}{\text{N.m}^2} = \frac{\text{C}}{\left(\frac{\text{N}}{\text{C}}.\text{m}\right).\text{m}} = \frac{\text{C}}{\text{V.m}} = \frac{\text{F}}{\text{m}}.$$

Therefore, we will have

$$C = \frac{(8.9 \times 10^{-12}\text{F/m})\ (0.5^2\ \text{m}^2)}{0.003\ \text{m}} = 7.4 \times 10^{-10}\text{F}.$$

(b) We use the defining equation of capacitance to figure out the charge Q.

$$Q = C\Delta V = 7.4 \times 10^{-10}\text{F} \times 12\ \text{V} = 8.9 \times 10^{-9}\text{C}.$$

4.4.3 Energy Stored in a Capacitor

Charging a capacitor involves work from an external agent. To figure out the energy stored in a charged capacitor we envision a process

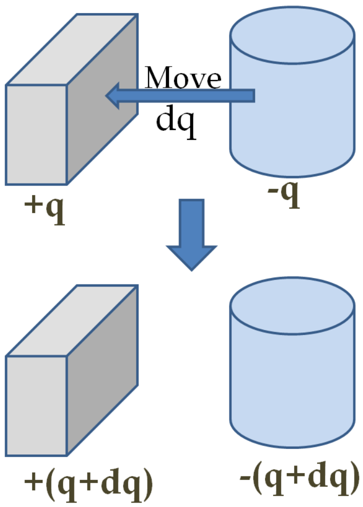

Figure 4.22: Charging a capacitor by transferring charge from one uncharged conductor to the other.

which starts with two uncharged plates and ends up with one plate having a net positive charge $+Q$ and the other plate a net negative charge $-Q$. One way we could do it move charges from one plate directly to the other plate.

As the work needed will depend on the charge already present on the plates, we would carrying out the process in infinitesimal steps (Fig. 4.22). At some intermediate step, let there be $+q$ on the positive plate and $-q$ on the negative plate, and let ΔV_q be the potential difference at that stage of charging when the charges on the plates are $\pm q$, which is not yet the full charge denoted by $\pm Q$. By the capacitor formula we know that ΔV_q and q are related as follows.

$$\Delta V_q = \frac{q}{C} \tag{4.16}$$

Transfer of an infinitesimal amount of charge $+dq$ from the negative plate to the positive plate will require the following work dW.

$$dW = \Delta V_q dq \tag{4.17}$$

Use Eq. 4.16 to replace ΔV_q in Eq. 4.17.

$$dW = \frac{q}{C} dq \tag{4.18}$$

Since capacitance C is independent of charge q on the capacitor, this equation can be integrated from $q = 0$ to $q = Q$ to obtain the net work needed for charging a capacitor to a total charge of $\pm Q$ on the plates.

$$W = \frac{1}{2}\frac{Q^2}{C} \tag{4.19}$$

Therefore, the energy U stored in the capacitor is given as

$$U = \frac{1}{2}\frac{Q^2}{C}, \tag{4.20}$$

which can be written in alternative forms using the capacitor equation, Eq. 4.8 using the final potential difference between plates as ΔV.

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}Q\Delta V.} \tag{4.21}$$

Example 4.4.2. Energy stored in a charged capacitor. A parallel plate capacitor made of two circular aluminum plates of radius 10 cm separated by 2 mm is charged by connecting to a 3-V battery. (a) Find the energy stored in the capacitor. (b) How much charge there is on the positive plate of the charged capacitor?

Solution. (a) First we find the capacitance,

$$C = \frac{\epsilon_0 A}{d} = 8.85 \times 10^{-12}\mathrm{Fm}^{-1} \times \frac{\pi(0.1\ \mathrm{m})^2}{0.002\ \mathrm{m}} = 1.4 \times 10^{-10}\mathrm{F}.$$

The energy stored $= \frac{1}{2}CV^2 = \frac{1}{2} \times 1.4 \times 10^{-10}\mathrm{F} \times (3\ \mathrm{V})^2 = 6.3 \times 10^{-10}\mathrm{J}.$

(b) To find the charge stored, we can use $Q = CV$.

$$Q = 1.4 \times 10^{-10}\mathrm{F} \times 3\ \mathrm{V} = 4.2 \times 10^{-10}\ \mathrm{C}.$$