4.5 TWO DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

We have established above that the most general constant acceleration will occur in a plane. Suppose the acceleration is pointed towards the positive x-axis, then the y-axis can be chosen so that the motion happens entirely in the xy-plane. If $v_y = 0$ at the beginning in this set-up, then the motion will be one-dimensional, only along x-axis.

On the other hand, if v_y is non-zero at the beginning, the motion will be two-dimensional. You will need both the x and y-components for constructing the position and velocity vectors in the xy-plane. Table 4.1 summarizes the full set of equations for the planar constant acceleration motion.

Note that, if you choose axes such that the constant acceleration happens along the y-axis and not along the x-axis as assumed for the equations in Table 4.1, then you will need to switch x and y in the table. Now, you will have a_y non-zero and a_x zero.

x-components:	$v_x(t) = v_{0x}$
	$x(t) = v_{0x}t$
y-components:	$v_y(t) = v_{0y} + a_y t$
	$y(t) = v_{0y}t + \frac{1}{2}a_y t^2$

4.5.1 Projectile Motion

The projectile motion of an object thrown sideways or at an angle from the vertical illustrates the general principles of the motion with constant acceleration well. Note again that, in order for the motion under study to have a constant acceleration, you must restrict the time domain to a time segment after the projectile has been launched and before the projectile lands or hits any other object. This is necessary because when the projectile is launched, it has a different acceleration than the acceleration due to gravity, and similarly on the other end, the acceleration is different after the projectile hits the ground.

With a proper choice of time domain and the assumption of negligible air resistance, the vertical motion of a projectile is subject to a constant acceleration pointed down, but the horizontal motion has constant velocity. Although the direction of the constant acceleration is down, the traditional choice of coordinates is to point the positive

y-axis up and to point the x-axis in the horizontal direction. With these choices for the coordinates the equations for the components of velocity and acceleration become

x-components	y-components
$v_x = v_{0x}$	$v_y = v_{0y} - gt$
$x - x_0 = v_{0x}t$	$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = \frac{v_y^2 - v_{0y}^2}{2g}$

When applying these equations to problems you will notice that the following steps are common to most problems.

- Step 1: Choose a Cartesian coordinate system.
- Step 2: List knowns and unknowns.
- Step 3: Work out components (this step and step 2 are often done together).
- Step 4: Set up equations and solve (what to solve for depends on particular question asked).
- Step 5: Present answer (often the numerical values need to be interpreted or translated into a descriptive language).

Example 4.5.1. Separating a planar motion into components

A cannonball is launched at a speed of 40 m/s speed at an angle of 30° from the horizontal. What will be the velocity of the cannonball after 3 sec?

Solution. I will use the present problem to illustrate the use of the systematic approach outlined above. Note that, although the statement of the problem does not make a reference to a coordinate system, we choose to work in the analytic approach for vectors and choose a convenient coordinate system first. We choose a coordinate system to perform the analysis. Here, as explained above, the acceleration is equal to (9.81 m/s^2, *down*). Therefore, one of the Cartesian axes will be vertical. It is a time-honored tradition to point the positive y-axis up. This makes the y-component of the acceleration negative. The horizontal direction is chosen to be the x-axis and the positive x direction is usually chosen in the direction of the increasing x-coordinate of the projectile under study. That is, if the projectile is going towards the North-east, we point the positive x-axis towards the Northeast, and if the projectile is going towards the North, then we point the positive x-axis towards the North, etc. These choices are shown in Fig. 4.4.

Step 1: Choose a Coordinate System

Step 2: List Knowns And Unknowns

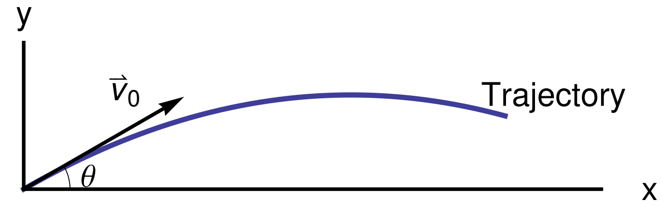

Figure 4.4: Example 4.5.1. The arrow labeled $\vec{v}_0$ is the velocity vector at time $t = 0$.

Now, we list the known and unknown quantities, usually in a table form, such as shown in Table 4.3. Unlike the one-dimensional problems, now we need to keep track of the x and y-components of all five vectors, namely, the initial position, the final position, the initial velocity, the final velocity, and the acceleration. The time domain of interest in this problem is from $t = 0$ to $t = 3$ sec.

Table 4.3: Known and unknown components of vectors

Vector Quantity	x-component	y-component
Knowns:		
$\vec{r}(0)$	$x(0) \equiv x_0 = 0$	$y(0) \equiv y_0 = 0$
$\vec{v}(0)$	$v_x(0) \equiv v_{0x}$	$v_y(0) \equiv v_{0y}$
$\vec{a}$	$a_x = 0$	$a_y = -9.81 \text{ m/s}^2$
Unknowns:		
$\vec{v}(t)$	$v_x(t)$	$v_y(t)$
$\vec{r}(t)$	$x(t)$	$y(t)$

Step 3: Work Out Components

The components of the velocity vector at $t = 0$ are found by drawing the projections of the vector on the corresponding axes. In a two-dimensional situation, you need to draw the projection onto only one of the axes and use trigonometric relations for the resulting right-angled triangle as shown in Fig. 4.5.

From Fig. 4.5 we find the components of the initial velocity vector to be

$$v_x(0) = 40\ \cos 30^\circ = 34.6 \text{ m/s} \equiv v_{0x}$$
$$v_y(0) = 40\ \sin 30^\circ = 20.0 \text{ m/s} \equiv v_{0y}$$

Now, we are ready to answer the question about the velocity vector at $t = 3$ sec by working out its x and y-components. The data on the x-component given in Table 4.3 tells us that the x-component of the velocity does not change. Therefore,

Step 4: Set Up Equations and Solve

$$v_x = v_{0x} = 34.6 \text{ m/s}.$$

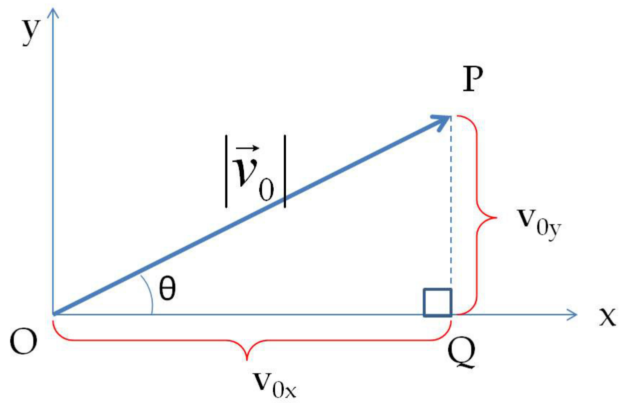

Figure 4.5: Components of initial velocity vector. The initial velocity has a magnitude $|\vec{v}_0|$ and direction O-to-P, which is at an angle θ counterclockwise from the positive x-axis. The right-angled triangle $\triangle OPQ$ is used to determine the components v_{0x} and v_{0y} of the initial velocity vector $\vec{v}_0$. Writing v_0 for the magnitude $|\vec{v}_0|$, we get $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$.

The data on the y-components given in Table 4.3 can be used to find v_y as

$$v_y = v_{0y} + a_y t = 20.0 - 9.81 \times 3 = -9.43 \text{ m/s}.$$

We find that $v_y < 0$, which means that the y-axis vector, $v_y \hat{u}_y$, obtained by multiplying the y-component with the unit vector of the y-axis is pointed towards the negative infinity on the axis. In this problem, since the positive y-axis pointed up, $v_y \hat{u}_y$ is pointed down. Therefore, at $t = 3$ sec the ball is coming down. However, since $v_x > 0$, the ball is not coming straight down, but at an angle to the vertical, with the direction being actually in the fourth quadrant of the Cartesian axes chosen above.

Step 5: Present Answer

How do we construct the velocity vector at $t = 3$ sec? We have found its components, but the actual velocity is a vector. We use the components to obtain the magnitude and direction of the velocity vector. The magnitude is easy to work out from the components. [Note the z-component here is zero.]

$$\text{Magnitude of velocity vector: } |\vec{v}| = \sqrt{v_x^2 + v_y^2} = 35.9\ m/s.$$

Work Out Magnitudes And Directions If Needed

How do we specify the direction of the velocity vector. There are two ways to do this here.

1. Draw the velocity vector in space.

2. Provide an angle with respect to one of the coordinate directions. Since we have a two-dimensional situation, we need to specify only one angle. The angle can be with respect to any

of the four directions: (a) the positive x-axis, (b) the negative x-axis, (c)the positive y-axis, or (d) the negative y-axis. We also need to specify whether the angle is in the clockwise or counter-clockwise from the axis when we look down from the side of the positive z-axis.

Drawing the vector is straight forward. Mark the scale on the axes according to the quantity being drawn. Here, we mark the axes by the scale of velocity for drawing approximately 35 m/s along x-axis and -10 m/s along the y-axis . We represent the velocity vector by drawing an arrow from the origin to the point (v_x, v_y) as shown in Fig. 4.6.

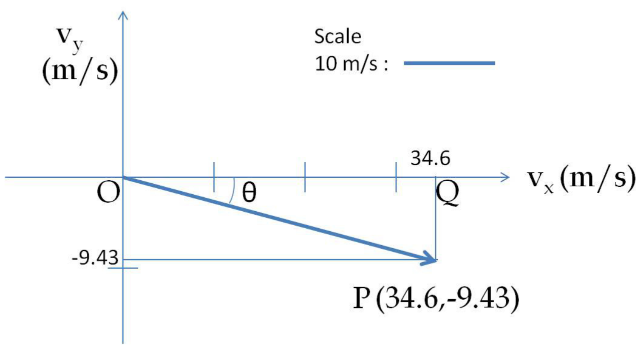

Figure 4.6: Graphically displaying direction of velocity vector. The x-component of the vector is plotted along x-axis and the y-component along the y-axis using the same scale, and the vector is then drawn from the origin to the point (v_x, v_y). Either angle θ clockwise from the positive x-axis or its complement angle counterclockwise from the negative y-axis are also used to indicate the direction.

In Fig. 4.6 the angle θ clockwise from the positive x-axis can be read off by a protractor or calculated from the components of the vector. If you use trigonometry on the right-angled triangle $\triangle OPQ$, you find that you get a negative value.

$$\theta = \arctan\left(\frac{v_y}{v_x}\right) = \arctan\left(\frac{-9.43}{34.6}\right) = -15.2^\circ.$$

The usual practice is to state the angle as a positive number and say whether you have to go clockwise or counter-clockwise from the axis. Here, we will say that the direction is 15.2° clockwise from the positive x-axis. Sometimes, one also says 15.2° below the positive x-axis.

Beware of the value from your calculator when evaluating Arctan. Be mindful of the quadrant!

Example 4.5.2. Projectile motion problem with unknown time A projectile is fired with speed 60 m/s at an angle of 40° with respect to the ground. Assume that the projectile has a constant acceleration of g (9.81 m/s^2) pointed down and lands on the ground that is at same height as the place of launch. (a) Find the horizontal distance the projectile flies before landing. (b) What is the hang time, i.e., the time between launching and landing? (c) What is the velocity with which the projectile will strike the ground?

Solution. Constant acceleration problems in two dimensional motion in which the time interval is not given often require passing the time variable between the x and y-components. In some problems, it may be helpful to eliminate the variable t in the equations for the x and y-components. In some other problems, you may be able to find t in x or y alone and you will need to pass the value of t to the other component.

All these problems start with first sketching the anticipated trajectory and a choice of the coordinate system as shown in Fig. 4.7. Then, we set up the table of knowns and unknowns and the two

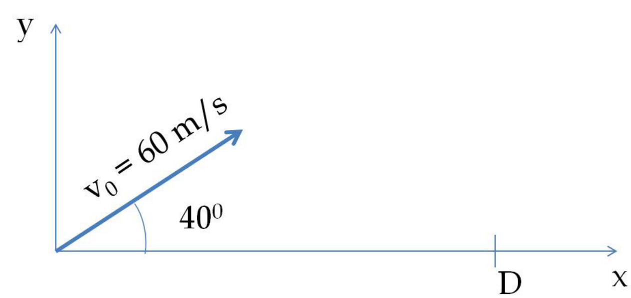

Figure 4.7: Physical setting and choice of coordinates for Example 4.5.2.

equations for each axis as shown in detail in the last example. We have once again placed the origin at the place of launch, and point the positive y-axis vertically up and the x-axis horizontally as in the previous problem. Let us denote the final time by capital letter T. Let D denote the horizontal distance traveled before landing.

Table 4.4: Known and unknown components of vectors

Vector Quantity	x-component	y-component
Knowns		
$\vec{r}(0)$	$x(0) = 0$	$y(0) = 0$
$\vec{v}(0)$	$v_x(0) \equiv v_{0x}$	$v_y(0) \equiv v_{0y}$
$\vec{a}$	$a_x = 0$	$a_y = -9.81$ m/s^2
Unknowns		
$\vec{r}(T)$	$x(T) = D$	$y(T) = 0$
$\vec{v}(T)$	$v_x(T) \equiv v_x$	$v_y(T) \equiv v_y$

Note that $\vec{v}(T)$ is not zero, since we are interested in the velocity an instant before the projectile lands. Once the projectile hits the ground its acceleration is no longer equal to the assumed constant value of (9.81 m/s^2, *down*).

Now, let us write down two independent constant acceleration kinematic equations for each axis.

x-component equations	y-component equations
$v_x = v_{0x}$	$v_y = v_{0y} + a_y T$
$D = v_{0x} T$	$0 = v_{0y} T + \frac{1}{2} a_y T^2$

So, we have four equations among four unknowns, D, T, v_x, and v_y. Since, we already have numerical values of some of the symbols in these equations, we will rewrite them with their specific values. First, note that we have the following for the x and y-components of the initial velocity vector.

$$v_{0x} = 60 \text{ m/s} \cos 40^\circ = 46 \text{ m/s}.$$
$$v_{0y} = 60 \text{ m/s} \sin 40^\circ = 39 \text{ m/s}.$$

The kinematic equations with numerical values include units, but they are cumbersome to carry around in calculations, so we will leave out the units. We will put the units back in at the end.

$$v_x = v_{0x} = 46 \qquad (4.19)$$
$$D = v_{0x} T = 46\ T \qquad (4.20)$$
$$v_y = 39 - 9.81\ T \qquad (4.21)$$
$$0 = 39\ T - 4.91\ T^2 \qquad (4.22)$$

The last equation, Eq. 4.22, can be solved for T which gives two values for T since T obeys a quadratic equation.

$$T = 0,\ 8 \text{ sec}.$$

The value of $T = 0$ refers to the initial time. Since, in our problem, the projectile has the y-coordinate equal to zero at the starting place and at the landing place, the equation includes the solution $T = 0$. We need to interpret the mathematical solution often to select the solution we seek. Therefore, we keep $T = 8$ *sec* for the final time.

Now, putting the value of T in Eq. 4.20 we obtain the distance D.

$$D = 366 \text{ m}.$$

Substituting the value of T in Eq. 4.21 gives us the last unknown, v_y.

$$v_y = -39.4 \text{ m/s}.$$

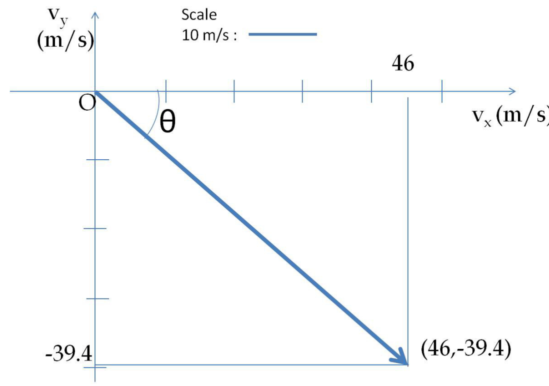

Figure 4.8: Graphically displaying direction of velocity vector. The x-component of the vector is plotted along x-axis and the y-component along the y-axis using the same scale, and the vector is then drawn from the origin to the point (v_x, v_y). Either angle θ clockwise from the positive x-axis or its complement angle counter-clockwise from the negative y-axis are used to indicate the direction.

Let us now summarize our results.

$$D = 366 \text{ m.}$$
$$T = 8 \text{ sec}$$
$$v_x = 46 \text{ m/s}$$
$$v_y = -39.4 \text{ m/s.}$$

The values of D and T give the answers to parts (a) and (b). To obtain the answer to part (c), we need to construct the velocity vector from the components v_x and v_y. The procedure is already explained in the last example. Here we give the answer.

Magnitude of the velocity: $\sqrt{46^2 + (-39.4)^2} = 60$ m/s.
Direction of the velocity: As shown in the Fig. 4.8, $\theta = 40.3°$, clockwise from the positive x axis.

Example 4.5.3. Projectile motion problem with different heights. An air plane flying horizontally at a speed of 400 km/h drops off a package form an altitude of 200 m from the ground. How far does the package travel horizontally from the point of its release?

Solution. Note that in the time interval after the package is let go and the instant before the package lands on the ground, the package is falling freely with the constant acceleration of 9.81 m/s^2 pointed down if we neglect the air resistance. We follow the tradition of choosing a coordinate system with the positive y-axis pointed up as shown in Fig. 4.9. This will make a_y negative. The x-component

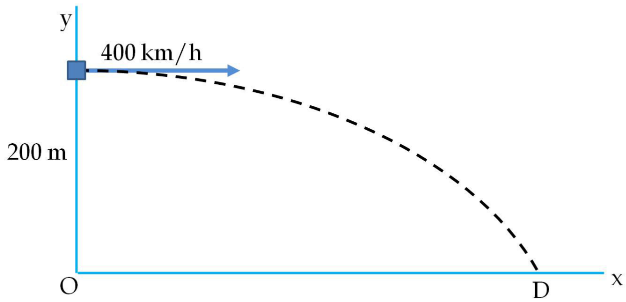

Figure 4.9: Example 4.5.3. Coordinate system used for the problem. The initial position of the package is at coordinate $(0, 200\ m)$ and the final position at $(D, 0)$.

of acceleration is, of course, zero. Let the value of time be $t = 0$ when the package is at point $(0, 200\ \text{m})$ and $t = T$ when the package is at $(D, 0)$. We now collect the known and unknown quantities for the time segment $0 < t < T$ in Table 4.5, where we also include the kinematic equations for the corresponding axes.

Table 4.5: Known and unknown components of vectors

Quantity	x-component	y-component
Known:		
$\vec{r}(0)$	$x(0) = 0$	$y(0) = 200$ m
$\vec{v}(0)$	$v_{0x} = 400$ km/h $= 111$ m/s	$v_{0y} = 0$
$\vec{a}$	$a_x = 0$	$a_y = -9.81\ \text{m/s}^2$
Unknown:		
$\vec{r}(T)$	$x(T) = D$	$y(T) = 0$
$\vec{v}(T)$	$v_x(T) \equiv v_x$	$v_y(T) \equiv v_y$

With the values from the table, we obtain the following set of four equations, two from each axis (leaving the units out):

$$v_x = v_{0x} = 111 \tag{4.23}$$
$$D = v_{0x}T = 111\ T \tag{4.24}$$
$$v_y = -9.81\ T \tag{4.25}$$
$$-200 = -4.91\ T^2 \tag{4.26}$$

In this question, we need to find the value of D only. Therefore, we focus on those equations that would give us the value of D with least additional algebra. We see that, if we get T from Eq. 4.26 and plug into Eq. 4.24, we will get the value of D. Let us implement this strategy.

Solving Eq.4.26 for T we obtain two values of T.

$$T^2 = \frac{200}{4.91} = 40.7 \quad \Longrightarrow \quad T = \pm\sqrt{40.7} = \pm 6.38 \text{ sec.}$$

We get two values of T since the equation is quadratic in T. What are the meanings of the two values and which one would I need for the point $(D, 0)$ in the question? The negative value of T refers to a time before the package was released at $t = 0$. So, it is clearly not meaningful for the given physical setting. The negative T says that if the package was launched with appropriate initial velocity so that it has a horizontal velocity of magnitude 111 m/s at the top of the trajectory, then it would have taken 6.28 sec to reach the top of the trajectory. Our physical setting verifies that $T > 0$ at $(D, 0)$, the point where the package lands. Therefore, we use the positive value of T in Eq. 4.24 to obtain the required value of D.

$$D = 111 \times 6.38 \quad \Longrightarrow \quad D = 708 \text{ m.}$$