6.4 SIMPLE DC CIRCUITS

In this section we will consider a network of conductors connected to one or more **constant voltage sources**. A network of electrical devices connected together is called a **circuit**. A conductor used in a circuit that has a significant resistance is also called a **resistor**. When a circuit has only conductors and sources, then the circuit is also called a **resistive circuit**. In a circuit diagram a resistor is represented by a saw tooth line as shown in Fig. 6.14 with a value of the resistance R indicated either above or below the drawing.

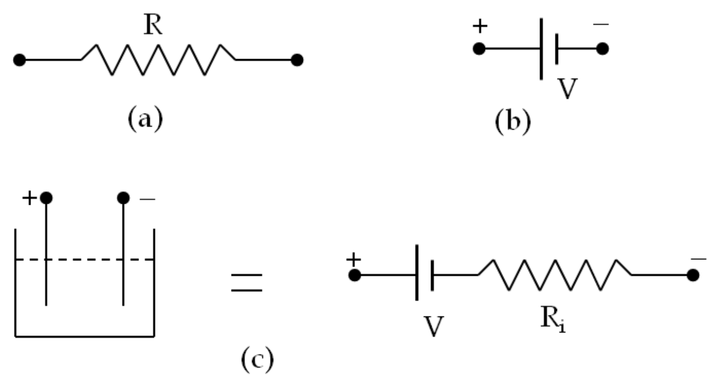


Figure 6.14: Models of circuit elements (a) resistor, (b) EMF or voltage source, and (c) a voltage source with an internal resistance.

In our circuits the EMF source will be a constant voltage source such as a battery. Such circuits are also called direct current (DC) circuits since the direction of current in these circuits does not change with time. The voltage source in a DC circuit will be represented by two vertical lines, one larger than the other as shown in Fig. 6.14. The larger line indicates the terminal at the higher electric potential and the shorter line denotes the terminal at lower electric potential. The terminal at the higher potential is also called the **positive terminal**, then the terminal at lower potential is called the **negative terminal**. The difference between the potentials at the terminal is called the **voltage** of the source.

Often batteries and other constant voltage sources also have significant resistance themselves, which adds to the total resistance of the circuit. The resitance of a source is also called the **internal resistance**. If the internal resistance is a significant factor in the circuit, we would include an internal resistance in the model as given in Fig. 6.14(c), otherwise we will tend to ignore them.

A SIMPLE CIRCUIT

The simplest circuit is obtained by connecting a resistor across a voltage source as shown in Fig. 6.15. We wish to analyze this circuit using Ohm's law.

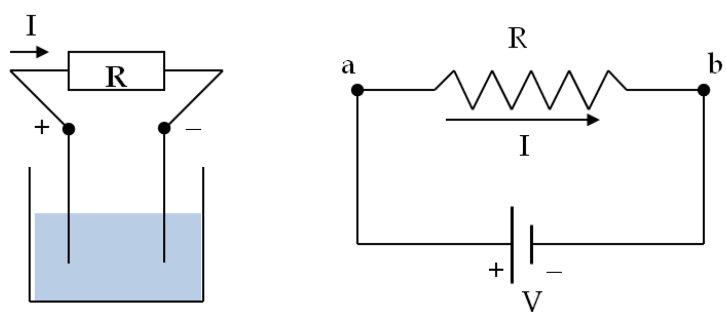


Figure 6.15: A resistor of resistance R connected across the terminals of a battery of voltage V.

Since voltages change only when the current goes through a circuit element or device, we start by labeling points of unique voltage values, also called **node voltages**. In Fig. 6.15, node points are labeled a and b. The connecting wires in a circuit diagram are supposed to be "ideal" with zero resistance. Therefore, even when current flows through them, there is no voltage drop across connecting wires.

$$\Delta V = 0 \quad \text{(Connecting wires)}.$$

This is illustrated in Fig. 6.16, where the potentials at point a is same as the potential at $+$-terminal and the potential at point b is equal to $-$-terminal.

The analysis of a resistive circuit is based on the Ohm's law for each resistor, which can be illustrated as follows in the simple circuit given above. Let V_a and V_b be potential at the two nodes. Then, according to Ohm's law, current I through the resistor is simply the electric potential drop across the resistor divided by the resistance R.

$$I = \frac{V_a - V_b}{R}. \tag{6.37}$$

The potential difference $V_a - V_b$ here is equal to the potential difference between the positive and negative terminals of the source as explaned above. Writing the difference in the positive and negative terminals

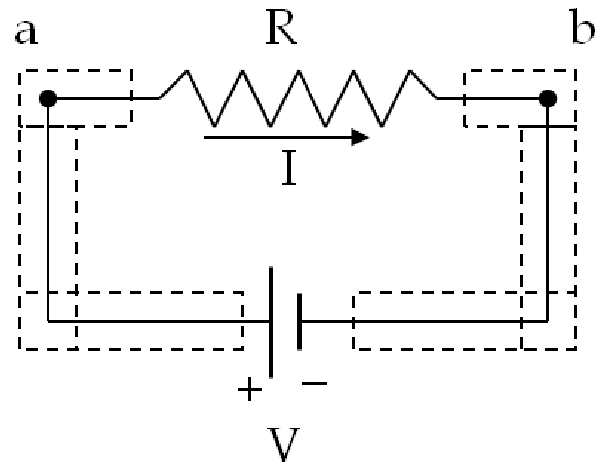


Figure 6.16: This figure illustrates that there are only two unique values of potential in this circuit, one at each end of the resistor. These potentials are same as the potentials at the ends of the terminals of the voltage source since they are directly connected by connecting wires. Points a and b are called nodes.

by the voltage V of the source the potential differences are often written as

$$V_a - V_b = V_+ - V_- \equiv V. \tag{6.38}$$

This notation is very dangerous since we lose the physically important concept of the difference in the potential which is the correct concept here. Anyhow, we will use the standard notation and note that one symbol V will represent the difference $V_+ - V_-$. Therefore, Eq. 6.37 can be written more simply as

$$\boxed{I = \frac{V}{R} \quad \text{(CAUTION!! V is a difference.)}} \tag{6.39}$$

Thus, if we connect a 120 Ω resistor across a 12 V battery, then the current through the resistor will be 12V/120Ω, or 0.1 Amp.

Since the reference of electric potential is arbitrary, we always assign a zero volt to the negative of a battery or the EMF or voltage source in the circuit. If you have more than one voltage source in a circuit, then the negative terminal of only one source can be set to zero. Then, the potential at the negative terminals of other sources may not necessarily be zero and needs to be calculated based on the workings of the circuit.

In the following we will use Ohm's law to analyze a few useful simple circuits that have only one voltage source. Basically, there are three types of simple resistive circuits with multiple resistors connected to a single voltage source - circuits with resistors connected in series, circuits with resistors connected in parallel, and circuits with resistors connected neither in series nor in parallel but in a mixed way. In each circuit, we will be interested in determining the current through each resistor and voltage values at all nodes. Once we have determined these quantities for each resistor, we can find the power dissipated in the resitors and the energy used by them.

Resistors in Series

When resistors are connected end-to-end such that the same current must flow through them, then we say that the resistors are **in series**. In Fig. 6.17, two resistors R_1 and R_2 are shown connected in series to each other and to an EMF source of voltage V. In order to apply Ohm's law in this circuit, first we will mark points on the circuit shown as a, b and c in the figure that have unique values of the potential, i.e. at voltage nodes.

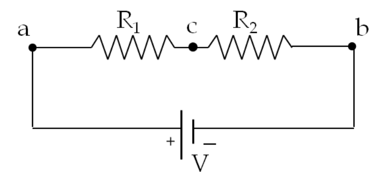


Figure 6.17: Resistors in series.

As usual, we will choose zero potential at the negative terminal of the source. This choice of zero gives the potentials at points a and b as V and zero. Let the potential at c be denoted by V_c as indicated in Fig. 6.18. The voltage and current situations for the two resistors are shown in Fig. 6.18, which we now use to write Ohm's law expression for each resistor.

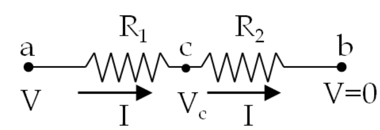


Figure 6.18: Resistors in series have same current through them although voltage drop across each resistor is different.

$$\text{Ohm's law for } R_1\text{:}\quad V_c - 0 = IR_2 \tag{6.40}$$

$$\text{Ohm's law for } R_2\text{:}\quad V - V_c = IR_1 \tag{6.41}$$

We can solve these two equations for the current I and the potential V_c. Adding the two equations, we get

$$V = I\,(R_1 + R_2)\,,$$

which gives

$$I = \frac{V}{R_1 + R_2}. \tag{6.42}$$

We can use this expression for I in Eq. 6.40 to obtain V_c:

$$V_c = \left(\frac{R_2}{R_1 + R_2}\right) V \tag{6.43}$$

When the two resistors are in series a part of the total voltage drops across one resistor and the rest across the other. Let V_1 and V_2 be voltage drops across R_1 and R_2. Then, our calculations above show that these voltage drops are given by

$$\boxed{V_1 = \left(\frac{R_1}{R_1 + R_2}\right) V} \tag{6.44}$$

$$\boxed{V_2 = \left(\frac{R_2}{R_1 + R_2}\right) V} \tag{6.45}$$

In a sense the total voltage V is divided into two parts, in proportions to the two resistance values. Hence a circuit with two resistors in series is also called a **voltage divider circuit**.

It is possible to replace the two resistors R_1 and R_2 by one resistor of resistance $R_1 + R_2$ without affecting the value of the current I that flows in the circuit outside the two resistors (Fig. 6.19). The resistance of the replacement resistor is called the **equivalent resistance** of the two resistors connected in series. If the *a-to-b* branch was part of a larger circuit, the outside elements will not notice this replacement! We will denote the equivalent resistance for resistors in series by R_S.

$$\boxed{R_S = R_1 + R_2} \tag{6.46}$$

$$\boxed{V = IR_S} \tag{6.47}$$

For N resistors in series, the equivalent resistance R_S is simply the sum of all them.

$$\boxed{R_S = R_1 + R_2 + \cdots + R_N.} \tag{6.48}$$

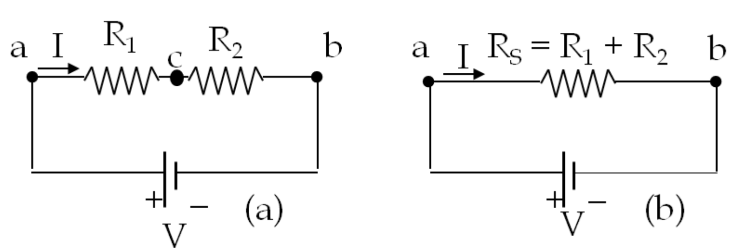


Figure 6.19: Resistors in series and their equivalent resistance R_S. Same current I flows through R_S as through R_1 and R_2.

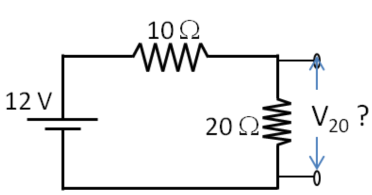


Figure 6.20: Example 6.4.1

Example 6.4.1. A Voltage Divider. Find the voltage drop across the 20 Ω resistor in the given circuit.

Solution. The two resistors are in series, and therefore, they act as a voltage divider.

$$V_{20} = \left(\frac{20\Omega}{10\Omega + 20\Omega}\right) \times 12 \text{ V} = 8\text{V}.$$

Resistors in Parallel

We say that two resistors R_1 and R_2 are connected in parallel if a current entering the combination splits and recombines after going through the two resistors as shown in Fig. 6.21. The two ends of the resistors in parallel fall on two nodes labeled a and b on figure. Note that just splitting of a current at a junction does not imply that you have a parallel connection; you must also have the currents come together.

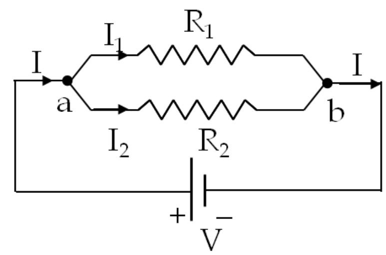


Figure 6.21: Two resistors in parallel connection.

From the conservation of charge at the nodes, you can prove that the sum of currents I_1 and I_2 must equal the entering current I at junction a in Fig. 6.22. Similarly, for junction b.

$$I = I_1 + I_2. \tag{6.49}$$

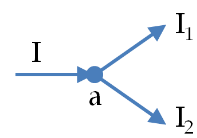


Figure 6.22: Two resistors in parallel connection.

The potential differences between the ends of each of the resistors R_1 and R_2 are same, each equal to $V_a - V_b$, which is equal to the voltage of the source in this circuit as shown in Fig. 6.23. Now, we can write Ohm's law for the two resistors as follows.

$$\text{Resistor } R_1 \text{ has voltage drop } V \text{ and current } I_1: \quad V = I_1 R_1 \tag{6.50}$$
$$\text{Resistor } R_2 \text{ has voltage drop } V \text{ and current } I_2: \quad V = I_2 R_2 \tag{6.51}$$

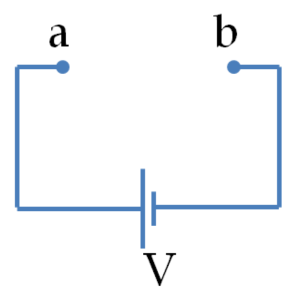


Figure 6.23: The voltage difference between nodes a and b is equal to the the voltage V of the battery.

Hence,

$$I_1 R_1 = I_2 R_2. \tag{6.52}$$

From Eqs. 6.49 and 6.52 we find the resistors in parallel act as a **current divider**.

$$I_1 = \left(\frac{R_2}{R_1 + R_2}\right) I \tag{6.53}$$
$$I_2 = \left(\frac{R_1}{R_1 + R_2}\right) I \tag{6.54}$$

Clearly more current goes through the branch that has less resistance. If two resistors 2Ω and 4Ω are connected in parallel, then 2/3rd of the current will pass through the 2Ω and 1/3rd through the 4Ω resistor.

Just as we were able to define an equivalent resistance of resistor in series by asking the following question: what equivalent resistor could replace the existing resistors in the circuit so that the current and voltages outside the replaced circuit are not affected by the replacement? To determine the equivalent resistance of two resistors in parallel, we think of a resistance R_P that will not change the total

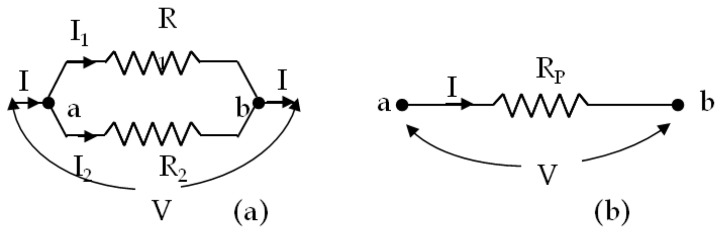


Figure 6.24: Two resistors in parallel and their equivalent resistance R_P. The current I through the equivalent resistor must be equal to the current entering or exiting the parallel combination.

current I that flows through all the branches of the replaced resistors (Fig. 6.24).

Therefore for the resistor R_P in the equivalent circuit we have the following Ohm's law equation.

$$\boxed{\text{Ohm's law for } R_P: \quad I = \frac{V}{R_P}.} \tag{6.55}$$

Replacing I by $I_1 + I_2$ in this equation we get

$$I_1 + I_2 = \frac{V}{R_P}. \tag{6.56}$$

Now, from Eqs. 6.50 and 6.51 we have $I_1 = V/R_1$ and $I_2 = V/R_2$. Therefore,

$$\frac{V}{R_1} + \frac{V}{R_2} = \frac{V}{R_P}, \tag{6.57}$$

where we can cross out the common factor V to obtain the inverse of the equivalent resistance in terms of the inverses of the individual resistances.

$$\boxed{\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2}.} \tag{6.58}$$

Note that R_P is less than both R_1 and R_2. For instance, if $R_1 = 1\ \Omega$, and $R_2 = 2\ \Omega$, then $R_P = 2/3\ \Omega$. When you connect resistors in parallel, you provide additional route for the charges to flow, and therefore, additional resistors in parallel lower overall resistance. This is consistent with our understanding that a thicker wire has less resistance than a thinner wire since the thicker wire can be thought of as made up of many thinner wires put in parallel (Fig. 6.25).

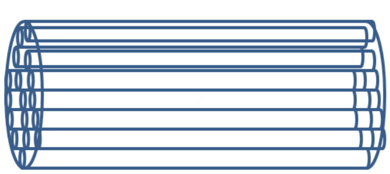


Figure 6.25: A thick wire is equivalent to many think wires in parallel connection.

For N resistors in parallel, you can show that

$$\boxed{\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}.} \tag{6.59}$$

Conductance (G) of a resistor is defined to be the inverse of its resistance (R).

$$\boxed{G = \frac{1}{R}.} \tag{6.60}$$

Hence, conductances add in a parallel circuit.

$$\boxed{G_P = G_1 + G_2 + \cdots + G_N.} \tag{6.61}$$

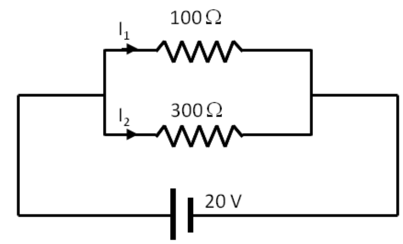


Figure 6.26: Example 6.4.2.

Example 6.4.2. A Current Divider. Find the percent of current that passes through each resistor in the given circuit.

Solution. In this example, current I_1 will be more than current I_2 since I_1 passes through the smaller resistance.

$$I_1 = \left(\frac{300\Omega}{100\Omega + 300\Omega}\right) I = \frac{3}{4} I.$$

$$I_2 = \left(\frac{100\Omega}{100\Omega + 300\Omega}\right) I = \frac{1}{4} I.$$

Hence 75% of the total current passes through the 100Ω resistor and 25% through the 300Ω resistor.

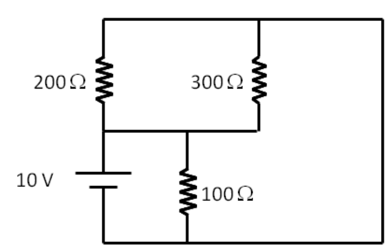


Figure 6.27: Example 6.4.3.

Example 6.4.3. A parallel circuit in disguise. Find the current through each resistor in the given circuit in Fig. 6.27.

Solution. As before, we will choose the negative terminal of the battery as zero volt reference. Then, the positive terminal of battery is at 10 volts. We see from the diagram in Fig. 6.28 that the voltage situation across each of the resistors is same: Each resistor is actually connected across the two terminals of the battery. The given circuit

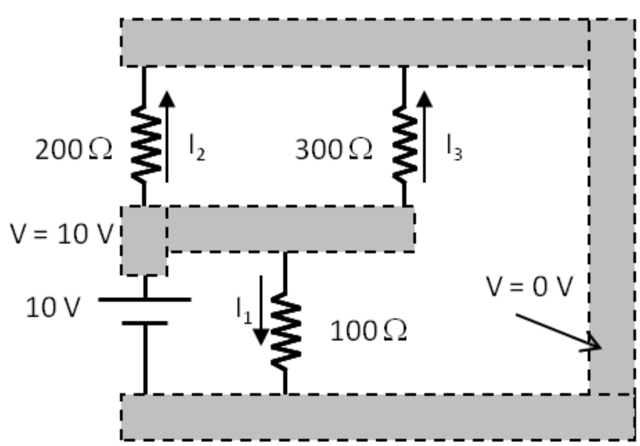


Figure 6.28: Example 6.4.3.

is identical with the three resistors connected parallel to the battery. Hence the currents are

$$I_1 = \frac{10V}{100\Omega} = \frac{1}{10}A; \quad I_2 = \frac{10V}{200\Omega} = \frac{1}{20}A; \quad I_3 = \frac{10V}{300\Omega} = \frac{1}{30}A.$$

Simplifying Complicated Circuits

There are a number of circuits that can be simplified using the rules of equivalent resistance for series and parallel resistors presented above. The basic idea is to successively use equivalent resistance rules for series and parallel resistors in a sequence of steps. First we replace all the resistors in series by their equivalent resistances. This step usually leaves some resistors in parallel. Then we replace the resistors that are in parallel by their equivalent resistances. This step generates new resistors in series, which can then be replaced by their equivalent resistances. This process is continued until we simplify the circuit to one net equivalent resistor connected to one source. We will illustrate these circuits with an example. This simplifying process does not always work. In that case we require more powerful methods of Kirchhoff's rules to be described below.

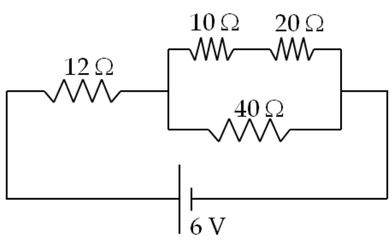


Figure 6.29: Example 6.4.4.

Example 6.4.4. Circuits with Series and Parallel Connections. Consider four resistors connected to a voltage source as shown in Fig. 6.29. Find current through each resistor.

Solution. To find current through the resistors, we need voltage drop across each. Therefore, we start with labeling the node voltage points on the circuit as shown in the Fig. 6.30.

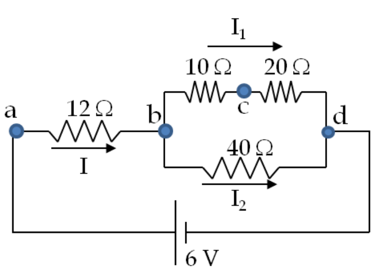
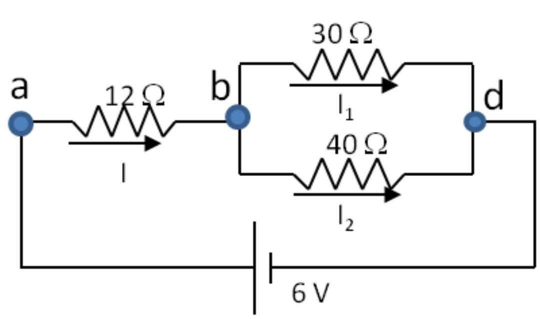


Figure 6.30: Labeled circuit diagram.

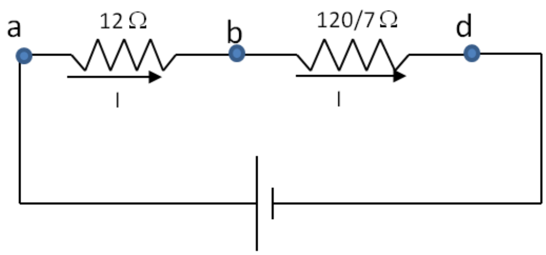
As explained above, the node points represent points of unique voltage values. There are four nodes in the given circuit, which are labeled a, b, c, and d. Node d is connected to the negative of the source and is usually set zero volt as reference, $V_d = 0$. Since voltage of the battery is 6 volts, the node voltage at a is 6 V, that is, $V_a = 0$. Now, we follow step-by-step procedure of simplifying the circuit to find the node voltages V_b and V_c.

Step 1: Replace all resistors in series by equivalent resistors. Here only 10 Ω and 20 Ω resistors are in series. Therefore we replace them by a 30 Ω resistor. It is best to keep the same labels here as in the previous diagram.

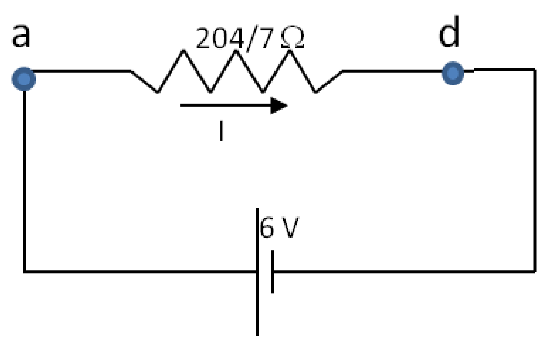


Step 2: Replace parallel resistors by equivalent resistors. Here only 30 Ω and 40 Ω resistors are in parallel. Therefore we replace the

combination by their equivalent resistance, which is (120/7) Ω.



Step 3: After replacing the parallel resistors by (120/7) Ω, we find that it is in series with the 12 Ω resistor. Therefore current through the (120/7) Ω resistor must be same as that goes through the 12 Ω resistor. Now the two resistors in series can be replaced by a (204/7) Ω resistor resulting in a really simple circuit. The current through the (204/7) Ω must be same as the 12 Ω or (120/7) Ω resistor since it is replacing resistors in series.



Step 4: Now, we can use Ohm's law to find the current through (204/7) Ω resistor.

$$I = \frac{6V}{(204/7)\Omega} = \frac{7}{34}\ \text{A}.$$

Step 5: We then trace our path backwards towards the original circuit. The current through the 12Ω resistor is also 7/34 A. Therefore voltage drop across the 12Ω resistor is

$$V_a - V_b = \frac{7}{34}\ \text{A} \times 12\ \Omega = \frac{42}{17}\ \text{V}.$$

The voltage drop across the (120/7) Ω resistor is (60/17)V.

$$V_b - V_d = \frac{7}{34}\ \text{A} \times \frac{120}{7}\ \Omega = \frac{60}{17}\ \text{V} \implies V_b = \frac{60}{17}\ \text{V}.$$

Going back one more step we find that voltage drop across the 30 is same as that across the (120/7) Ω worked out. Hence, current through the 30 Ω resistor is simply

$$I_1 = \frac{(60/17)\ \text{V}}{30\ \Omega} = \frac{2}{17}\ \text{A}$$

Therefore, voltage V_c is obtained from the voltage drop across the 20 Ω resistor as given by

$$V_c - V_d = \left(\frac{2}{17}\text{ A}\right) \times 20\ \Omega = \frac{40}{17}\text{ V}.$$

Since $V_d = 0$, we obtain $V_c = \frac{40}{17}$ V. We can use the node voltages to obtain the current through various resistors. Alternately, since the total current I was split into I_1 and I_2, we can find I_2 by subtracting I_1 from I.

$$I_2 = I - I_1 = \frac{7}{34}\text{ A} - \frac{2}{17}\text{ A} = \frac{3}{34}\text{ A}$$

Summarizing the answer:

$$V_d = 0;\ V_a = 6\text{ V};\ V_b = \frac{60}{17}\text{ V};\ V_c = \frac{40}{17}\text{ V}.$$
$$I_1 = \frac{2}{7}\text{ A};\ I_2 = \frac{3}{34}\text{ A};\ I = \frac{7}{34}\text{ A}.$$

Example 6.4.5. Dependence of the terminal voltage on the load.

Every battery has an internal resistance. As a result the voltage measured at the terminals is not equal to the EMF of the battery. Unlike the EMF of the battery, the terminal voltage depends upon the resistance of the load connected across the terminals. To examine this effect consider a battery with EMF V and internal resistance R_i connected to a load of resistance R_L.

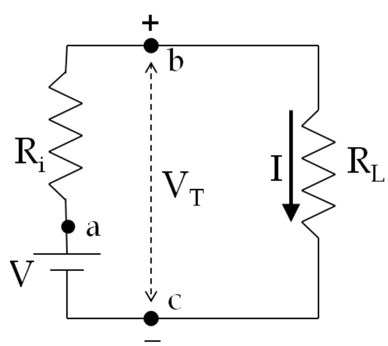


Figure 6.31: A resistor R_L connected across the terminals of a battery of internal resistance R_i. Load R_L "sees" the terminal voltage V_T.

$$I = \frac{V}{R_L + R_i} \tag{6.62}$$

$$V_T = \left(\frac{R_L}{R_L + R_i}\right) V \tag{6.63}$$

Note that the voltage across the terminals of a battery depends upon the load resistance R_L. The relative values of R_L and R_i determine the terminal voltage seen at the terminals of a battery as opposed to the full EMF of the battery. The entire EMF is "seen" by the load when $R_L >> R_i$ and only part of the EMF is used by the load if $R_L << R_i$.

$$V_t = \begin{cases} V & R_L >> R_i \\ 0 & R_L << R_i \end{cases} \tag{6.64}$$

This can be illustrated for a voltaic cell of EMF 1.5 volts and internal resistance 1Ω. With a 1000Ω resistor load, the terminal voltage will be approximately 1.5 volts, but with a 0.5Ω load resistor, the voltage across the same terminals will be only 0.5 volt!