1.9 Relativistic Momentum

The momentum $\vec{p}$ of a particle of mass m and velocity $\vec{u}$ in Newtonian mechanics is given by

$$\vec{p} = m\vec{u}. \tag{1.102}$$

We will work with the symbols u and w for velocity of particles or objects and reserve the symbol V for the relative velocity of frames. In Newtonian mechanics this momentum is conserved in every inertial frame. Einstein's principle of relativity states that laws of physics should be same in all inertial frames. However, when you apply the definition of momentum in Eq. 1.102 to a collision process we find that momentum is not conserved if we use the velocity addition rule derived by using Einstein's relativity. We now ask the following question:

Is there some way we can modify the definition in Eq. 1.102 so that the law of conservation of momentum is obeyed by the new momentum in every inertial frame *and* the new definition reduces to the old definition when the speed of the particle is much less than the speed of light?

For the new definition of momentum a simple modification to try would be to leave the velocity vector alone and focus on the scalar quantity mass. Let mass m be (some as yet unknown) function of speed u of the particle and see if we can deduce what the mass function will be so that momentum in a collision is conserved in all inertial frames.

$$\text{Hypothesize momentum: } \vec{p} = m(u)\,\vec{u} \tag{1.103}$$

Here $m(u)$ is not mass times u but a function of u whose form we wish to figure. For simplicity in our analysis we will examine a very symmetric situation from the perspective of two frames. Consider a glancing elastic collision of two identical bodies A and B of mass $m(u)$ from the perspectives of two frames, frame A in which the x-component of the velocity of A is zero and frame B in which the x-component of the velocity of B is zero.

$$\begin{array}{ll} \text{Frame A:} & u_{Ax} = 0 \\ \text{Frame B:} & U_{Bx} = 0 \end{array} \tag{1.104}$$

Let us denote the velocities of the particles with respect to the A frame by u for quantities before the collision and by w for quantities after the collisions and with respect to frame B by U and W respectively.

$$\text{Notations:} \quad \begin{array}{lll} \text{Frame A:} & \text{Before: } \vec{u}_A, \vec{u}_B; & \text{After: } \vec{w}_A, \vec{w}_B \\ \text{Frame B:} & \text{Before: } \vec{U}_A, \vec{U}_B; & \text{After: } \vec{W}_A, \vec{W}_B \end{array} \tag{1.105}$$

Thus, in frame A, object A has only the y-component of velocity non-zero, and in frame B, object B has only the y-component of velocity non-zero. Let the y-component of the velocity of A in its own frame be $-u_0$ (since pointed towards negative y-axis). To keep the analytic calculations simple let the y-component of the velocity of B in its own frame be u_0 (since pointed towards positive y-axis).

$$\text{In frame A:} \quad u_{Ay} = -u_0 \tag{1.106}$$

$$\text{In frame B:} \quad U_{By} = u_0 \tag{1.107}$$

Let the x-component of B with respect to frame A be u_{Bx}. This is the velocity with which the frame B moves with respect to frame A. Let V be the relative velocity of the two frames. In the figure, B is moving towards the negative x-axis, therefore, we will have the following also.

$$\text{In frame A:} \quad u_{Bx} = -V \tag{1.108}$$

$$\text{In frame B:} \quad U_{Ax} = V \tag{1.109}$$

From the velocity addition law we can write the y-component of velocities of A and B given in their own frames to those in the other frames. Thus, the y-component of the velocity of A in the frame of B will be

$$\text{In frame B:} \quad U_{Ay} = \frac{-u_0}{\gamma(1 - U_{Bx}u_0/c^2)} = -\frac{u_0}{\gamma} \tag{1.110}$$

Similarly, the y-component of the velocity of A in the frame of B will be

$$\text{In frame A:} \quad u_{By} = \frac{u_0}{\gamma(1 + u_{Bx}u_0/c^2)} = \frac{u_0}{\gamma} \tag{1.111}$$

Now, we have accumulated the complete information on various velocities in the two frames before the collision. Let us summarize them here.

Before Collision:

$$\text{Frame A:} \quad u_{Ax} = 0,\ u_{Ay} = -u_0,\ u_{Bx} = -V,\ u_{By} = \frac{u_0}{\gamma} \tag{1.112}$$

$$\text{Frame B:} \quad U_{Ax} = V,\ U_{Ay} = -\frac{u_0}{\gamma},\ U_{Bx} = 0,\ U_{By} = u_0 \tag{1.113}$$

Speeds of the particles before the collision are

Before Collision:

$$\text{Frame A:} \quad u_A = u_0,\ u_B = \sqrt{V^2 + \frac{u_0^2}{\gamma^2}} \tag{1.114}$$

$$\text{Frame B:} \quad U_A = \sqrt{V^2 + \frac{u_0^2}{\gamma^2}},\ U_B = u_0. \tag{1.115}$$

From the symmetry we can assert that if the y-component of velocity of A after the collision in its frame (i.e. the A frame) equals u', then the y-component of velocity of B after the collision in its frame (i.e. the B frame) equals $(-u')$. Furthermore, the x-components of the velocities will not change since the glancing collision will change the y-velocities only. Thus, we will have the following after the collision.

After Collision:

$$\text{Frame A:} \quad w_{Ax} = 0,\ w_{Ay} = u',\ w_{Bx} = -V,\ w_{By} = -\frac{u'}{\gamma} \tag{1.116}$$

$$\text{Frame B:} \quad W_{Ax} = V,\ W_{Ay} = \frac{u'}{\gamma},\ W_{Bx} = 0,\ W_{By} = -u' \tag{1.117}$$

Speeds of the particles after the collision will be

After Collision:

$$\text{Frame A:} \quad w_A = u',\ w_B = \sqrt{V^2 + \frac{u'^2}{\gamma^2}}, \tag{1.118}$$

$$\text{Frame B:} \quad W_A = \sqrt{V^2 + \frac{u'^2}{\gamma^2}},\ W_B = u'. \tag{1.119}$$

Now, we are ready to find out what needs to happen so that momentum is conserved in both frames. The conservation of x-component of momentum in the A frame gives the following relation.

$$\text{Frame A:}\ \ x\text{-component:} \quad -m(u_B)\,V = -m(w_A)\,V, \tag{1.120}$$

Therefore,

$$m(u_B) = m(w_A). \tag{1.121}$$

Assuming m to be a monotonic function of speed, this equation will mean

$$u_B = w_A, \tag{1.122}$$

which can be shown to give

$$u_0 = u'. \tag{1.123}$$

Now, the conservation of momentum in the y-direction in the A-frame yields

$$\text{Frame A:}\ \ y\text{-component:} \quad -m(u_0)u_0 + m(u_B)\frac{u_0}{\gamma} = m(u')u' + m(w_B)\frac{u_0}{\gamma} \tag{1.124}$$

Using $u_0 = u'$ and $u_B = w_B$ this equation turns into

$$m(u_B) = \gamma\, m(u_0) = \frac{m(u_0)}{\sqrt{1 - V^2/c^2}}. \tag{1.125}$$

From Eq.. 1.114 we note that in the limit $u_0 \to 0$, $u_B \to V$. Therefore, we obtain

$$m(V) = \frac{m(0)}{\sqrt{1 - V^2/c^2}}. \tag{1.126}$$

The mass $m(0)$ is the mass of the object in its rest frame and is called the **rest mass**. The rest mass is often denoted by the symbol m_0.

$$\boxed{m(V) = \frac{m_0}{\sqrt{1 - V^2/c^2}} = \gamma m_0.} \tag{1.127}$$

Thus, the relativistic momentum $\vec{p}$ of a particle of mass m in a frame in which it has velocity u is give by

$$\boxed{\vec{p} = \frac{m_0}{\sqrt{1 - u^2/c^2}}\, \vec{u},} \tag{1.128}$$

which is often written as

$$\boxed{\vec{p} = m\vec{u} = \gamma m_0 \vec{u}, \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}.} \tag{1.129}$$

It is left for the student to verify the x- and y-components of momentum defined by Eq. 1.129 are conserved in frme B as well. Eq. 1.129 shows that the relativistic momentum is similar to the regular momentum except now we have a speed-dependent mass. The speed-dependent mass reduces to the regular mass when speed is zero and as the speed increases the speed-dependent mass also increases until $u = c$ beyond which mass is not defined. For $u > c$ mass is actually an imaginary number, which does not make sense. We say that the expression for the relativistic momentum in Eq. 1.129 should be used for particles at speeds less than the speed of light in vacuum.

What happens to $\vec{F} = m\vec{a}$?

In Newtonian mechanics, the second law of motion in an inertial frame is

$$\vec{F} = \frac{d\vec{p}}{dt}, \tag{1.130}$$

where $\vec{p} = m\vec{u}$, where $\vec{u}$ is the instantaneous velocity and m mass. We have already seen that momentum vector in relativity has a different expression. Suppose we want to keep the principle that momentum is conserved in the absence of an external force, we might keep the form the equation same as Eq. 1.130 but use the relativity formula for $\vec{p}$.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}\right], \tag{1.131}$$

where m_0 is the rest mass. This formula turns out to predict results that are experimentally correct. For instance, if you study the motion of high energy charged particles in electric and magnetic field the correct equation of motion of the particle is obtained when you use the Lorentz force on the left side of Eq. 1.131. This equation takes particularly simpler forms in the two cases: (1) $\vec{F} \parallel \vec{u}$ and (2)

$\vec{F} \perp \vec{u}$. For instance, when an electron is accelerated from rest in an electric field the force will be parallel to the velocity, and when an elctron is moving in a purely magnetic field force will be perpendicular to the velocity.

Example 1.8. An electron in a constant electric field. In a linear accelerator an electron is accelerated by a constant electric field $\vec{E}$ in the opposite direction to its velocity $\vec{u}$. Find the equation of motion of the electron.

Solution. Let the velocity of the electron be towards the positive x-axis. Then, electric field is towards the negative x-axis and force on the electron is towards the positive x-axis. Eq. 1.131 takes the following form.

$$eE = \frac{d}{dt}\left[\frac{m_0 u_x}{\sqrt{1-u_x^2/c^2}}\right], \quad u_y = 0, \quad u_z = 0.$$

Expanding the derivative on the right side we obtain

$$eE = m_0\gamma^3\frac{du_x}{dt}, \quad u_y = 0, \quad u_z = 0.$$

Example 1.9. An electron in a constant electric field. In a cyclotron an electron is accelerated by a constant magnetic field $\vec{B}$ perpendicular to the velocity $\vec{u}$ of the electron. The electron moves in a circle of radius R with a constant speed. Find the equation of motion of the electron.

Solution. Suppose the circle of motion is in the xy-plane and the magnetic field is towards the positive z-axis. The velocity of the electron will be in the $\hat{u}_\theta$ direction so that the force is radially in towards the center of the circle. Let us write the velocity of the electron as $\vec{u} = u\hat{u}_\theta$. Eq. 1.131 takes the following form.

$$euB(-\hat{u}_r) = \frac{d}{dt}\left[\frac{m_0 u\hat{u}_\theta}{\sqrt{1-u^2/c^2}}\right], \quad u_r = 0, \quad u_z = 0.$$

Expanding the derivative on the right side we obtain

$$euB(-\hat{u}_r) = \frac{m_0 u}{\sqrt{1-u^2/c^2}}\frac{d\hat{u}_\theta}{dt}, \quad u_r = 0, \quad u_z = 0.$$

The derivative of $\hat{u}_\theta$ will give a product of $d\theta/dt$ and $-\hat{u}_r$. Therefore, magnitude on the two sides will equal giving

$$eB = \frac{m_0}{\sqrt{1-u^2/c^2}}\frac{d\theta}{dt}, \quad u_r = 0, \quad u_z = 0.$$

Since the speed in the circle is uniform, we have $u = R d\theta/dt$ giving us the following equation.

$$eBR = p,$$

which has the same form as the equation $p = eBR$ in the non-relativistic case, except now, p is the relativistic momentum.

Example 1.10. Momentum of an electron. An electron is moving at a speed of $0.9\,c$, i.e. at 90% of the speed of light. Compare its momentum calculated from the relativistic formula to the one calculated from the Newtonian formula.

Solution. The mass of an electron is $m_0 = 9.11 \times 10^{-31}$ kg. Let us first calculate γ for the speed.

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - 0.9^2}} = 2.29.$$

The momentum of the electron from Newtonian expression is

$$p_{\text{Newton}} = m_0 u = 9.11 \times 10^{-31}\ \text{kg} \times 0.9 \times 3 \times 10^8\ \text{m/s} = 2.46 \times 10^{-22}\ \text{kg.m/s}.$$

The momentum of the same electron from relativistic expression is

$$p_{\text{Einstein}} = \gamma m_0 u = 2.29 \times 2.46 \times 10^{-22}\ \text{kg.m/s} = 5.63 \times 10^{-22}\ \text{kg.m/s}.$$

The correct momentum is given by the relativistic formula, which gives momentum to be 2.29 times the Newtonian value.