6.3 OHM'S LAW

6.3.1 General Statement of Ohm's Law

In a wide variety of materials electric current is driven by the electric field pushing on charges such that the drift velocity $\vec{v}_d$ of charge is proportional to the electric field $\vec{E}_P$ on the particle.

$$\vec{v}_d \propto \vec{E}_P \tag{6.18}$$

We can multiply both sides by the conduction electron density on both sides and introduce a constant of proportionality to write this relation as a relation between the current density and the electric field.

$$\boxed{\vec{J}_P = \sigma \vec{E}_P.} \tag{6.19}$$

The constant of proportionality σ is called the **conductivity** of the substance. For isotropic materials, i.e. a material that does not have intrinsic preferred direction of flow for charges, σ is just a positive scalar quantity. Equation 6.19 is called Ohm's law, after German physicist Georg Simon Ohm (1789 - 1854). A more common form of Ohm's law between voltage drop ΔV and total current I will be derived below.

Caution:
Beware of the multiple uses of the symbols σ and ρ. We have used σ for surface charge density and now we are using σ for conductivity. Similarly, ρ was used for charge density and now we use it for resistivity.

Instead of the conductivity, we often use the inverse of conductivity, called the **resistivity** ρ.

$$\boxed{\vec{J}_P = \frac{1}{\rho}\vec{E}_P.} \tag{6.20}$$

The SI unit for resistivity is Volt $\times$ meter/Ampere as can be derived from units of J and $\vec{E}$. The ratio Volt/Ampere is also called Ohm, which is denoted by the Greek letter Ω.

$$[\rho] = \frac{[E]}{[J]} = \frac{\text{V/m}}{\text{A/m}^2} = \frac{\text{V.m}}{\text{A}} = \Omega\text{.m}$$

The resistivity of some common materials are given in Table 6.1. Note that metals have very low resistivities, of the order of $10^{-7}\Omega$.m, and insulators have high resistivities. There are materials, called semiconductors, whose resistivities fall between those of metals and insulators.

Temeprature-dependence of resistivity

The resistivity ρ (or equivalently the conductivity σ) of a material depends on the temperature. For most metals the dependence is nearly linear over narrow ranges of temperature. Using the value

Table 6.1: Resistivities at 20°C (Various sources)

Material	Resistivity $\rho(\times 10^{-8}$m)	Temperature coefficient $\alpha(\times 10^{-3})$C^{-1}
Typical Metals		
Silver	1.6	6.1
Copper	1.7	6.8
Aluminum	2.7	4.3
Gold	2.2	3.4
Tungsten	5.6	4.5
Nickel	7.0	6.0
Iron	9.8	5.0
Platinum	1.0×10^1	3.9
Lead	2.1×10^1	3.9
Manganin (Cu, Mn, Ni alloy)	4.8×10^1	2.0×10^{-3}
Mercury	9.8×10^1	9×10^{-1}
Nichrome (Ni,Fe,Cr alloy)	1.0×10^2	4×10^{-1}
Typical Semiconductors		
Pure Silicon	2.5×10^{11}	
n-type Silicon ($n = 10^{23} m^{-3}$)	8.7×10^4	
p-type Silicon ($p = 10^{23} m^{-3}$)	2.8×10^5	
Typical Insulators		
Pure water	2.5×10^{13}	
Glass	10^{18} - 10^{22}	
Polystyrene	$> 10^{22}$	
Hard rubber	10^{21} - 10^{23}	

of resistivity ρ_0 at some reference temperature T_0 in a range we express the resistivity ρ at some other temperature T near the reference temperature as follows.

$$\boxed{\frac{\rho - \rho_0}{\rho_0} = \alpha\,(T - T_0)} \tag{6.21}$$

The proportionality constant α is called the temperature coefficient. We have listed some typical values in Table 6.1 for reference.

Further Remarks:

For non-isotropic materials, e.g. crystals, where charges may have preferred direction(s) of flow such that the same electric field may give rise to different currents in different directions in these materials. Consequently, the volume current density vector $\vec{J}$ may not be parallel to the electric field $\vec{E}$. We express this complication by treating conductivity σ as a tensor and write the Cartesian components of Ohm's law using nine components of σ.

$$\begin{aligned} J_x &= \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z \\ J_y &= \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z \\ J_z &= \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z \end{aligned} \tag{6.22}$$

As is evident non-isotropic materials require a more complicated analysis. Since, in this book we will focus exclusively on isotropic materials, we will not discuss the conductivity tensor any further.

6.3.2 Ohm's Law Applied to a Cylindrical Wire

We now work out an important consequence of Ohm's law that is very useful in the study of electric circuits. Consider a uniform metallic wire of cross-section area A and length L that carries current I (Fig. 6.9).

Let $\vec{J}$ be the volume current density at a point in the wire and $\vec{E}$ the electric field there. According to Ohm's law

$$\vec{E} = \rho\vec{J}. \tag{6.23}$$

The magnitude of the current density is given in terms of current I and the area of cross-section A as

$$J = \frac{I}{A} \tag{6.24}$$

Using the direction of the x-axis in Fig. 6.9 we have the following for the current density vector.

$$\vec{J} = \frac{I}{A}\hat{u}_x. \tag{6.25}$$

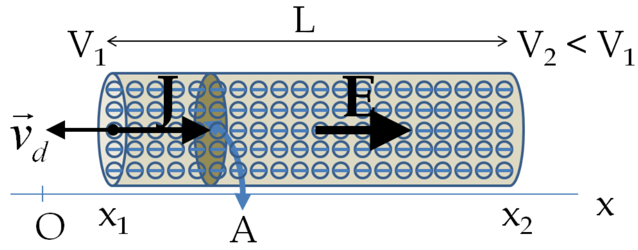

Figure 6.9: Applying Ohm's law to a cylindrical metallic wire. Note the current density vector is in the opposite direction of the electron flow and in the same direction as the electric field. The directions of flow and electric field line are shown for the case $V_1 > V_2$.

Therefore, Ohm's law for the wire now reads as

$$\vec{E} = \rho \frac{I}{A} \hat{u}_x, \tag{6.26}$$

which shows that electric field in the wire is independent of the position along the x-axis. We can integrate this electric field from one end of the wire to the other end to relate to the electric potential difference across the wire.

$$\begin{aligned} V_2 - V_1 &= -\int_{(1)}^{(2)} \vec{E} \cdot d\vec{r} = -\int_{x_1}^{x_2} E_x dx \\ &= -\rho \frac{I}{A}(x_2 - x_1) = -\rho \frac{I}{A} L, \end{aligned}$$

which we can rearrange to write as

$$\boxed{V_1 - V_2 = \left(\frac{\rho L}{A} \right) I.} \tag{6.27}$$

The quantity in parenthesis is a composite property of the wire, consisting of the intrinsic property of resistivity ρ and two geometric properties, viz. the length L and the area of cross-section A. This composite property is called resistance of the wire. We denote the **resistance** by letter R.

$$\boxed{R = \frac{\rho L}{A}.} \tag{6.28}$$

This relation says that a thicker wire with larger area of cross-section will have lower resistance and a thinner wire of the same material and same length will have larger resistance. Additionally, a longer wire will have larger resistance than a shorter wire of the same material and the same thickness. One can understand this by noting that in a thicker wire there are more charge carriers in the cross-section, and therefore, there are more paths for carrying the current, and hence less resistance.

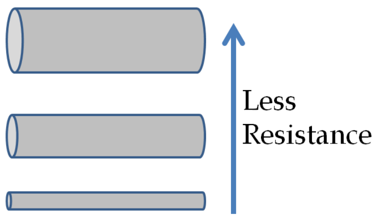

Figure 6.10: Thicker wire of the same material has less resistance.

We write Eq. 6.27 the following simple form of Ohm's law for a cylindrical wire.

$$\boxed{\Delta V = RI,} \tag{6.29}$$

where current is positive when $\Delta V = V_1 - V_2 > 0$. This relation is often taken to be the statement of Ohm's law. According to this relation, a wire with higher resistance will allow smaller current for the same potential difference. Although we have derived the relation $\Delta V = IR$ for a cylindrical wire, this equation is applicable much more generally and can be applied to other material shapes.

In the SI system of units, the unit of resistance is ohm (denoted by the Greek letter "Ω"). According to Ohm's law in the form given in Eq. 6.29, a current of one Ampere will flow through a wire of resistance one Ohm if a potential drop of one Volt is maintained between the ends.

$$1\ \text{Ohm}(\Omega) = 1\ \frac{\text{Volt(V)}}{\text{Ampere(A)}}$$

Example 6.3.1. Resistances of different metal wires. (a) Find the resistances of one meter copper wire and one meter nichrome wire, each of thickness 0.02 mm. (b) Find the resistance of one meter copper wire but of thickness 0.04 mm. (c) How long a copper wire of the same thickness would you need that has the same resistance as a 1-meter nichrome wire of the same thickness?

Solution. We look up the resistivity table to look up the resistivities of the two conductors. Now a days you can even google for this information. Here they are:

$$\rho_{\text{copper}} = 1.7 \times 10^{-8}\Omega.\text{m}$$
$$\rho_{\text{nichrome}} = 100 \times 10^{-8}\Omega.\text{m}$$

Now, we make use of the fundamental relation between the resistivity and the resistance of a wire to find the resistances of the wires, $R = \rho L/A$.

(a) Resistances of the two wires are:

$$R_{\text{copper}} = 1.7 \times 10^{-8}\Omega.\text{m} \times \frac{1\ \text{m}}{\pi(0.01 \times 10^{-3}\text{m})^2} = 54\ \Omega$$
$$R_{\text{nichrome}} = 100 \times 10^{-8}\Omega.\text{m} \times \frac{1\ \text{m}}{\pi(0.01 \times 10^{-3}\text{m})^2} = 3200\ \Omega$$

(b) When the thickness goes up 2-fold, the area of cross-section will go up 2^2 or 4-fold. This will make the resistance $\frac{1}{4}$-times as much.

$$R = \frac{54\ \Omega}{4} = 13.5\ \Omega.$$

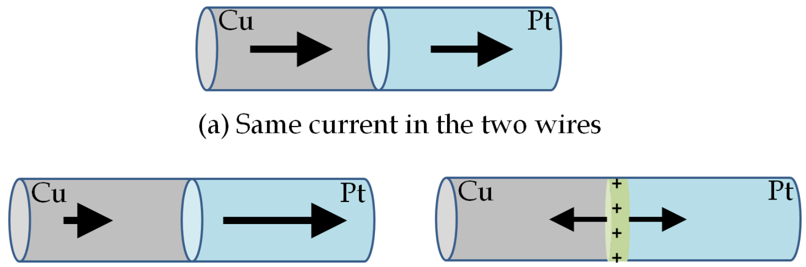

Figure 6.11: Junction of two conductors develops a layer of charge when steady current flows through the two wires. The electric field of charges at the junction add to the electric field from the potential difference for points on the right of the junction and subtract at points on the left. This gives rise to higher electric field in platinum than in copper.

(c) Since the two wires have the same resistance and the same thickness we have the following condition, $R = \rho_1 L_1/A = \rho_2 L_2/A$. Therefore,

$$\frac{L_1}{L_2} = \frac{\rho_2}{\rho_1} = \frac{100 \times 10^{-8}\Omega.\text{m}}{1.7 \times 10^{-8}\Omega.\text{m}} = 59.$$

This says that we would need 59 meters of copper wire for 1 m of a nichrome wire of the same thickness in order for the two wires to have the same resistance.

6.3.3 Current Through a Junction of Two Materials

Suppose you weld together a copper wire and a platinum wire with the same cross-section and pass a steady current through the composite conductor (Fig. 6.11). Since the same current I would flow through the two conductors of the same cross-sectional area, their volume current densities J would be equal. But the two metals have different resistivities with $\rho_{Pi} \approx 6 \times \rho_{Cu}$. Therefore, by Ohm's law, we can conclude that the electric field in platinum will be more than the electric field in copper.

$$E_{\text{in Cu}} \approx \frac{1}{6} E_{\text{in Pt}}.$$

How can the electric field be different in the two metals that are welded together? You can trace the difference in the electric field to an accumulation of charges at the interface layer. As shown in Fig. 6.11, if the current goes from copper into platinum a layer of positive charges will build up at the interface. The excess charges at the interface create electric field that adds to the external electric field and opposes inside copper.

We can use Gauss's law to deduce the charge density at the interface. Let σ_{12} be the charge density at the interface, and E_1 and E_2 be the magnitudes of the electric fields in the two wires. Now imagine a Gaussian surface in the shape of a pill box about the interface as shown in Fig. 6.12.

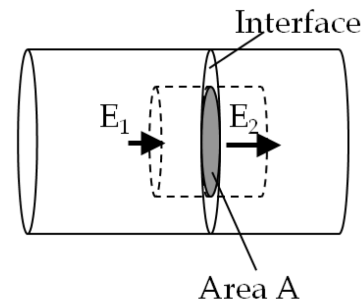

Figure 6.12: Gaussian pillbox straddling the interface used to find the excess charge density at the interface.

Note the flux is non-zero through the ends while zero through the side. Let A be the area of cross-section of the pill box at the interface. Then from Gauss's law we obtain the following.

$$E_2 A - E_1 A = \frac{\sigma_{12} A}{\epsilon_0} \tag{6.30}$$

Hence, the charge density at the interface is proportional to the difference in the magnitudes of the electric field in the two wires.

$$\sigma_{12} = \epsilon_0 \left(E_2 - E_1\right). \tag{6.31}$$

Question: What would you expect the electric fields if you weld platinum to nichrome? See if you can figure it out.

Example 6.3.2. Charge Density at the Interface of Two Metals. What would be the excess charge density at the interface of copper and aluminum rods of circular cross-section of diameter 2 mm carrying 12 A current?

Solution. Equation 6.31 tells us the charge density in terms of electric fields. We can use Ohm's law to express the electric fields in terms of the corresponding current densities, and then look up the resistivities from Table 6.1.

$$\begin{aligned}
\sigma_{12} &= \epsilon_0 \left(E_2 - E_1\right) = \epsilon_0 \left(\rho_2 J_2 - \rho_1 J_1\right) = \epsilon_0 \left(\rho_2 - \rho_1\right) \times \frac{I}{A} \\
&= 8.9 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N.m}^2} \left(2.7 - 1.7\right) \times 10^{-8} \Omega\mathrm{.m} \times \frac{12\ \mathrm{A}}{\pi \left(0.001\ \mathrm{m}\right)^2} \\
&= 3.4 \times 10^{-13} \frac{C^2}{\mathrm{N.m}^2} \times \Omega\mathrm{.m} \times \frac{\mathrm{A}}{\mathrm{m}^2} \\
&= 3.4 \times 10^{-13} \frac{\mathrm{C}}{\mathrm{m}^2} \quad \left(\text{using A.}\Omega = \mathrm{V} = \frac{\mathrm{N.m}}{\mathrm{C}}\right)
\end{aligned}$$

6.3.4 Power Dissipation in a Resistor

Electrons in motion collide with fixed nuclei, other electrons, and vibrational excitations called phonons. This situation is similar to a ball moving in a viscous medium. As a result the material heats up when a current flows through it. The heating of resistors is responsible for such useful applications of electricity as the conventional electric oven and the light bulb.

The rate of heating of a resistor can be calculated by using the principle of conservation of energy. Suppose current I flows through a cylindrical wire across which a voltage difference $\Delta V = V_1 - V_2$ is maintained (Fig. 6.13). Let N be the number of conduction electrons per unit volume and v_d their average drift speed. Let A be the area of cross-section and R the resistance of the wire. As a result of the current in the wire, electrons in the wire will move a distance $v_d\Delta t$ in duration Δt.

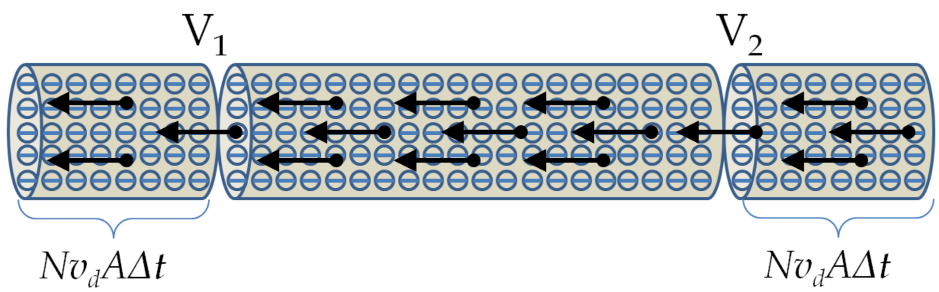

Figure 6.13: Electrons at the point where potential is V_2 enters the wire and exits at a point where potential is V_1, which is larger than V_2. With N conduction electrons per unit volume, area of cross-section A, and drift speed v_d, the number of electrons entering and exiting the wire between V_1 and V_2 in duration Δt is $Nv_dA\Delta t$.

Consider now the change in energy of the segment of wire between V_1 and V_2 in this interval. During this interval, $Nv_dA\Delta t$ electrons will enter the wire at potential V_2, which would bring in the potential energy equal to $-eNv_dA\Delta tV_2$ since the potential energy of a charge q at a point of potential V is equal to qV. During the same interval, $Nv_dA\Delta t$ electrons will exit the wire at potential V_1, which would take away potential energy equal to $-eNv_dA\Delta tV_1$.

Since the entering and exiting electrons move at the same drift speed, there is no change in the kinetic energy of the electrons. Therefore, any change in the energy of the wire between V_1 and V_2 comes from the difference in the potential energy of the entering and exiting electrons. The difference in the potential energy of the entering and exiting electrons is transformed into the thermal energy of the wire, and we find the following expression for the balance of energy,

$$\text{Change in thermal energy of wire} = -eNv_dA\Delta tV_2 - (-eNv_dA\Delta tV_1)\,. \tag{6.32}$$

Therefore, the rate of heating of the wire segment, also called power dissipated in the wire, will be given by

$$P = \frac{-eNv_d A\Delta t V_2 + eNv_d A\Delta t V_1}{\Delta t}, \tag{6.33}$$

which can be written as

$$P = eNv_d A \left(V_1 - V_2\right). \tag{6.34}$$

The quantity eNv_dA is the rate of net charge flow in the wire, which is equal to the current I in the wire. Therefore, we write power dissipated in a metal wire between two point with potentials V_1 and V_2 as

$$\boxed{P = I\Delta V}, \tag{6.35}$$

with $\Delta V = (V_1 - V_2)$. By using Ohm's law $\Delta V = IR$ we can rewrite power in alternate forms.

$$\boxed{P = I\Delta V = I^2 R = \frac{(\Delta V)^2}{R}.} \tag{6.36}$$

The SI unit of power is Watts (W), which is same as Joule/sec (J/s). In electrical units, 1 W of power is equal to 1 V.A, or, 1 $\mathrm{A}^2\Omega$, or, 1 V^2/Ω.

$$1\ \mathrm{W} = 1\ \mathrm{V.A} = 1\ \mathrm{A}^2\Omega = 1\ \mathrm{V}^2/\Omega = 1\ \mathrm{J/s}.$$

Example 6.3.3. Energy Used by Light Bulb

A 60-W light bulb has 120-V drop across its terminals. (a) How much energy is used by the bulb in 10 hours? (b) Determine the resistance of the filament. (c) Determine the height H that you could lift a 50-kg person with the same amount of energy.

Solution. (a) Energy Used $= P\Delta t = 60\ \mathrm{W} \times 10\ \mathrm{h} \times 3600\ \mathrm{s/h} = 2.2 \times 10^6\ \mathrm{J}$.

(b) The resistance R of the filament, $R = P/(\Delta V)^2 = 60\ \mathrm{W}/(120\ \mathrm{V})^2 = 4.2 \times 10^{-3}\Omega$.

(c) The given condition on energy gives $mgH = 2.2 \times 10^6$ J. Therefore, we obtain $H = 2.2 \times 10^6\ \mathrm{J}/\left(50\ \mathrm{kg} \times 9.81\ \mathrm{m/s^2}\right) = 4.4 \times 10^3$ m.