1.6 DIMENSIONAL ANALYSIS

Dimensional analysis is a powerful tool for studying the dependence of a physical quantity on the dimensions and properties of the system. In the most basic form, dimensional analysis is about checking the units in a physics equation: **you should get the same units on both sides of an equation if the equation is valid**. But, you can often go beyond just checking the units, and discover interdependence among physical quantities if you apply dimensional analysis to a given physical situation. In this application, dimensional analysis often helps you guess the right physics as we will see in the example below.

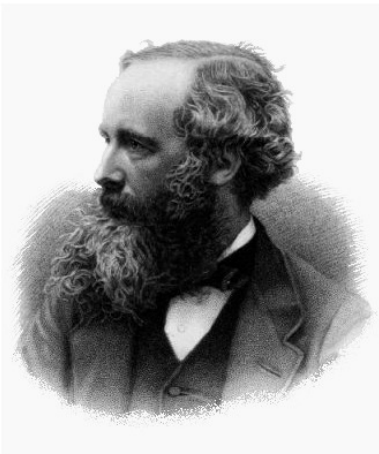

Figure 1.9: James Clerk Maxwell (1831 - 1879) made invaluable contributions to the theories of electricity and magnetism.

Following the Scottish physicist James Clerk Maxwell (1831 - 1879), we denote the dimension of a physical quantity by enclosing its symbolic name in square brackets. Thus, dimension of time is denoted by $[T]$, length by $[L]$, and mass by $[M]$. There are five base dimensional quantities corresponding to five base units: length $[L]$, time $[T]$, mass $[M]$, temperature $[\theta]$, and electric current $[I]$. The angle in radian does not have any dimensions, as you can easily see from the angle subtended by an arc of a circle: the angle in radian subtended by an arc is equal to the ratio of arc length to the radius of the circle.

The dimensions of other physical quantities are determined by first expressing their units in base units, and then by replacing the unit names by the corresponding base physical quantities. For instance, the unit of speed is meter per second, which means that its dimension is $[L]/[T]$, i.e length over time. Similarly, the dimensions of density are $[M]/[L]^3$, and electric charge is $[I][T]$. The dimensions of some commonly encountered mechanical quantities are listed in Table 1.3. You do not need to memorize these dimension yet. You will encounter all of these quantities in future chapters. Table 1.3 is included here as a reference that can be used to solve problems in this chapter to get a feel for how dimensional analysis works in calculations.

Example 1.6.1. Guessing the formula for frequency of a pendulum. The frequency of the pendulum is the number of cycles it makes in unit time. How does the frequency of a pendulum depend upon its length, mass, angle of swing and force of gravity?

Solution. The dimensional analysis can be used to find a formula for the frequency f of a pendulum of length l, and mass m, which swings between angles $\pm\theta$ radians of the vertical axis. Because pendulum swings as a result of gravity, we need to include the acceleration due

Table 1.3: Dimensions and SI units of common mechanical quantities

Quantity	Dimension	SI unit
Mass	[M]	kg
Time	[T]	s
Velocity	$[L][T]^{-1}$	$\mathrm{m.s}^{-1}$
Acceleration	$[L][T]^{-2}$	$\mathrm{m.s}^{-2}$
Angle	Dimensionless	rad*
Angular velocity	$[T]^{-1}$	s^{-1}, or, $\mathrm{rad.s}^{-1}$
Density	$[M][L]^{-3}$	$\mathrm{kg.m}^{-3}$
Momentum	$[M][L][T]^{-1}$	$\mathrm{kg.m.s}^{-1}$
Force	$[M][L][T]^{-2}$	$\mathrm{kg.m.s}^{-2}$ = Newton = N
Work, Energy	$[M][L]^2[T]^{-2}$	$\mathrm{kg.m}^2.\mathrm{s}^{-2}$ = Joule = J
Torque	$[M][L]^2[T]^{-2}$	$\mathrm{kg.m}^2.\mathrm{s}^{-2}$ = N . m
Power	$[M][L]^2[T]^{-3}$	$\mathrm{kg.m}^2.\mathrm{s}^{-3}$ = Watt = W
Pressure, Stress	$[M][L]^{-1}[T]^{-2}$	$\mathrm{kg.m}^{-1}.\mathrm{s}^{-2}$ = Pascal = Pa

*rad or radian is not a unit since it comes from the ratio of two lengths, the arc length and the radius, the result of which is a unitless quantity we call radian.

to gravity g as one of the possible variables. We assume that the mass of the string is negligible compared to the mass of the pendulum bob. Our task is to find the frequency, f, as a function of m, l, θ, and g.

$$frequency,\ f = f(m, l, \theta, g).$$

Now, we anticipate that frequency would go as some power of each of the physically relevant variables.

$$[f] = h(\theta) \times [l]^a \times [m]^b \times [g]^c. \tag{1.8}$$

where $h(\theta)$ is dimensionless since angle θ is dimensionless, and exponents a, b, and c are to be determined. Dimensional analysis would not help us with the form of the function $h(\theta)$ since θ is dimensionless. Now, let us write out the dimensions of all physical quantities I have listed.

$$[f] = 1/[T];\ [l] = [L];\ [m] = [M];\ [g] = [L]/[T]^2;\ [\theta] = Dimensionless.$$

Now, putting the dimensions in Equation 1.8 we find

$$\frac{1}{[T]} = h(\theta) \times [L]^a \times [M]^b \times \frac{[L]^c}{[T]^{2c}}.$$

Equating the exponents of $[L]$, $[T]$ and $[M]$ on the two sides of the equation gives us the following relations among a, b and c. If a particular dimension is missing on one side of the equation, e.g. $[M]$ is missing on the left side, then the exponent of that quantity would be zero on that side of the equation.

$$a + c = 0;\ b = 0;\ 2c = 1, \implies a = -\frac{1}{2};\ b = 0;\ c = \frac{1}{2}.$$

Therefore, we find that the frequency of a pendulum is

$$f = h(\theta)\sqrt{\frac{g}{l}}.$$

To appreciate the power of dimensional analysis, compare the following exact answer for the frequency of a pendulum obtained from a more difficult calculation based on Newton's second law of motion and small angle approximation.

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}.$$

The dimensional analysis gave us the important part of physics that frequency of a pendulum does not depend on its mass, and it is inversely proportional to the square root of length of the pendulum with very little effort on our part.