13.2 MATHEMATICS INTERLUDE

13.2.1 The Gradient of a Function

The derivative of a continuous, differentiable function $f(x)$ of one independent variable x tells us about its rate of change. Suppose the function f could represent a property that varies with the x-coordinates only, then

$$\text{Rate of change of } f \text{ along the } x\text{-axis } = \frac{df}{dx}. \tag{13.9}$$

Suppose we define a vector of magnitude $|df/dx|$ and the direction towards the positive or negative x-axis depending upon whether the derivative is positive or negative. Such a vector will capture the information about the rate at which the function f changes and the direction on the x-axis towards which the larger values of the function occurs. This vector is a representation of the **gradient** of the function $f(x)$ of one variable x. Fig. 13.3 shows this representation as a vector diagram for a function that has two constant segments between $x = a$ and $x = c$.

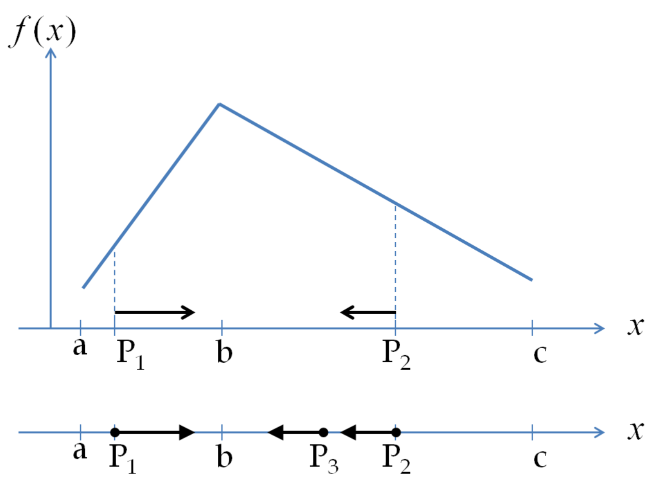

Figure 13.3: The gradient of the function of one variable at a point has the magnitude equal to the magnitude of the derivative there and the direction is towards the positive x-axis if the derivative with respect to x is positive and towards the negative x-axis if the derivative with respect to x is negative. Gradient of a function is a vector field which can be represented by vector maps as shown below the x axis for the function here. Note the gradient at any point is pointed towards larger value of the function.

In the example shown in Fig. 13.3 the gradients at every point between $x = a$ and $x = b$ are all equal to the vector shown at point

P_1. The gradients at every point between $x = b$ and $x = c$ are all equal to the vector shown at point P_2. The gradient of the function at any point is shown to be pointed towards the larger value. At point P_1, the larger value is towards $x = b$, which is the direction of the gradient there. Similarly, the gradient at P_2, the larger value is towards $x = b$, which is the direction of the gradient there.

The gradient of a function are vector fields, which can be represented by vector maps or vector line maps as shown in Fig. 13.3. Note that if the derivative is not defined at some point, such as the point $x = b$, the gradient would not be defined there either.

The situation for a function of two or more variables is more complicated. For simplicity in our treatment, we will define the gradient using the Cartesian coordinate representation of the gradient.

Suppose we want the gradient of the height from the sea-level in a flat-earth approximation. We will have a function $h(x, y)$ where x and y are the independent variables and h is the dependent variable. Now, when we take the derivative with respect to x, we need to decide if y remains constant or y is also allowed to change.

We introduce a new type of derivative, called the partial derivative, defined by keeping all other independent variables fixed. We will use the symbol ∂ in place of d for partial derivatives. The partial derivatives of h with respect to x will be just like the ordinary derivative when we treat y as a constant. Similarly for the partial derivative with respect to y. The two partial derivatives of h will be

$$\text{Patial derivative with respect to } x\text{:} \quad \frac{\partial h}{\partial x} \tag{13.10}$$

$$\text{Patial derivative with respect to } y\text{:} \quad \frac{\partial h}{\partial y} \tag{13.11}$$

Now, the partial derivative with respect to x multiplied by the unit vector $\hat{u}_x$ towards the positive x-axis would give the vector along the x-axis just as we have seen for the function of one variable $f(x)$. Similarly, for the y-derivative giving a vector along the y-axis.

$$\text{Variation parallel to the } x\text{-axis:} \quad \frac{\partial h}{\partial x}\hat{u}_x \tag{13.12}$$

$$\text{Variation parallel to the } y\text{-axis:} \quad \frac{\partial h}{\partial y}\hat{u}_y \tag{13.13}$$

Using these changes along the axes, we define the gradient for two dimensional function as

$$\text{Gradient of } h(x, y) : \quad \boxed{\textbf{grad } h = \frac{\partial h}{\partial x}\hat{u}_x + \frac{\partial h}{\partial y}\hat{u}_y.} \tag{13.14}$$

Figure 13.4 shows $h(x, y)$ in the shape of a pyramid whose sides are parallel to the axes. The gradient vector of different points in the xy-plane are shown on the right. At each point in the plane, the gradient vector is the sum of a vector that is the change as a function of x or y and a change in the perpendicular direction.

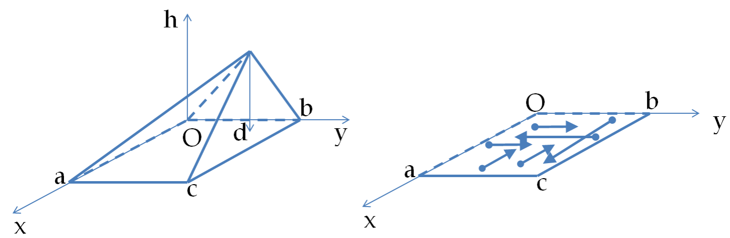

Figure 13.4: The gradient of the function of two variable $h(x, y$ at a point is a vector whose x-component is the partial derivative with respect to x and the y-component is the partial derivative with respect to y. The figure to the right shows the gradient vector at some representative points. Since the planes of the pyramids are parallel to the axes, the gradient vectors at different points are parallel to axes also.

In three space dimensions, we are usually interested in how a continuous, differentiable function $f(x, y, z)$ varies with respect to changes in x, y, and/or z. The variability of $f(x, y, z)$ at a particular point P will usually depend on the direction in space from P. We cannot easily display a function of three variables since we will need four axes, one each for the independent variables x, y, and z, and one for the dependent variable f. Often, the projections in planes are useful, for instance, you may try to study how things go in the xy-, yz-, and/or xz-plane. In the absence of a full pictorial view, we are left with algebraic definition to guide in the calculation and interpretations.

Once again, the gradient of a function will be defined in the Cartesian axis representation as an extension of the two-dimensional situation we have discussed above. The gradient of a function f is written as **grad** f or $\vec{\nabla} f$ or ∇f. We will use $\vec{\nabla} f$.

$$\boxed{\vec{\nabla} f = \hat{u}_x \frac{\partial f}{\partial x} + \hat{u}_y \frac{\partial f}{\partial y} + \hat{u}_z \frac{\partial f}{\partial z}.} \tag{13.15}$$

The symbol $\vec{\nabla}$ is also called the **del operator**.

Variation of a function in an arbitrary direction

The gradient operator is very helpful in writing the formula for a change of a function in an arbitrary direction. Let us again look at a two-dimensional function $h(x, y)$. What will be the rate of variation of the function if we go in the direction from point $P(x, y)$ towards $P'(x + \Delta x, y + \Delta y)$? The rate of variation, called the **directional derivative**, will be given by the change of the function over the direct distance,

$$\text{Rate of variation} = \frac{\text{Change}}{\text{Distance}} = \frac{h(x + \Delta x, y + \Delta y) - h(x, y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}. \tag{13.16}$$

This can be written using the gradient of h and the unit vector $\hat{u}$ in the direction P to P'. Since, the unit vector in this direction is given by

$$\hat{u} = \frac{\Delta x\ \hat{u}_x + \Delta y\ \hat{u}_y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, \tag{13.17}$$

the directional derivative will be

$$\boxed{\text{Rate of variation in the direction } \hat{u} = \hat{u} \cdot \vec{\nabla} h.} \tag{13.18}$$

Example 13.2.1. Variations in arbitrary directions. (a) Suppose we want to know how the function varies parallel to the x-axis as we walk towards the positive x-axis direction. In this case, $\hat{u} = \hat{u}_x$. This will give the rate of variation equal to $\hat{u}_x \cdot \vec{\nabla} h = \partial h/\partial x$ as expected.

(b) What will be the variation in the direction 45° from the positive x-axis in the xy-plane? In this case, the unit vector will be

$$\hat{u} = \frac{\hat{u}_x + \hat{u}_y}{\sqrt{2}}$$

This will give the rate of variation in this direction to be

$$\hat{u} \cdot \vec{\nabla} h = \frac{1}{\sqrt{2}} \left(\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \right).$$

13.2.2 The Divergence of a Vector Field

The divergence of a vector field is related to the concept of flux we have studied in this book. The divergence of a vector field $\vec{F}(x, y, z)$, which is sometimes indicated by $\mathbf{div}(\vec{F})$, is defined by the following

$$\text{Divergence of } \vec{F}\text{:} \quad \boxed{\mathbf{div}(\vec{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.} \tag{13.19}$$

The divergence can be written more compactly by using the del operator $\vec{\nabla}$ acting on the vector function $\vec{F}$ through a dot product,

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{13.20}$$

The divergence of a vector field $\vec{F}$ is indicated as $\mathbf{div}(\vec{F})$ or $\vec{\nabla}\cdot\vec{F}$. We will use the notation $\vec{\nabla}\cdot\vec{F}$. Let us do a few examples of calculations and then discuss the physical meaning of the divergence.

WARNING!!

Beware of the common mistake of confusing the label of component, as x in F_x with the functional dependence of F_x. The x-component F_x could be a function of x, y and z. Sometimes we explicitly state this as $F_x(x, y, z)$. Keep in mind that the labeling for components is independent of the functional dependence.

Example 13.2.2. Calculating Divergence. Calculate the divergence of (a) $\vec{F_1} = x^2y\,\hat{u}_x + x^2y^3\,\hat{u}_y$, (b) $\vec{F_2} = x\,y\,z\,\hat{u}_x + cos(x\,y\,z)\,\hat{u}_y + (x^2y^2z^3)\,\hat{u}_z$.

Solution. (a)

$$\begin{aligned}\vec{\nabla}\cdot\vec{F_1} &= \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z}\\ &= \frac{\partial(x^2y)}{\partial x} + \frac{\partial(x^2y^3)}{\partial y} + \frac{\partial(0)}{\partial z}\\ &= 2xy + 3x^2y^2\end{aligned}$$

Note the divergence itself is a scalar field, i.e. the result is a function of position. Note also that when we take the partial derivative with respect to x we keep y and z constant. Similarly for other partial derivatives.

(b)

$$\begin{aligned}\vec{\nabla}\cdot\vec{F_2} &= \frac{\partial F_{2x}}{\partial x} + \frac{\partial F_{2y}}{\partial y} + \frac{\partial F_{2z}}{\partial z}\\ &= \frac{\partial(xyz)}{\partial x} + \frac{\partial(cos(xyz))}{\partial y} + \frac{\partial(x^2y^2z^3)}{\partial z}\\ &= yz - xz\sin(xyz) + 3x^2y^2z^2\end{aligned}$$

Physical Meaning of Divergence

Divergence refers to the spread of a vector field. That is, when we calculate the divergence of a vector field at a point P, the result tells us how the quantity represented by the vector field spread away from point P. You can measure the divergence of a vector field at a point P by the flux out of a closed surface such a box or a sphere about the point.

Figure 13.5 shows a situation where vectors for a vector field and a sphere are shown. The point P is at the center of the sphere. You can see that in this example the vector field is coming out of the center. This is a situation where the vector field is spreading out of P, and this gives a positive value for the divergence at points of the sphere.

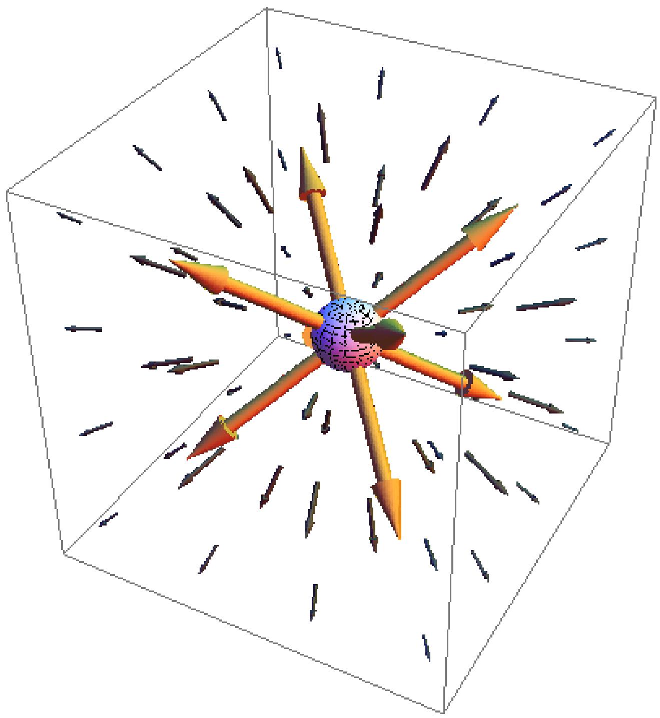

Figure 13.5: The divergence of a vector field at a point tells us about the way the vector field is spreading out from that point. In this illustration, the divergence at the center of the sphere is positive.

You might think of Fig. 13.5 as the electric field $\vec{E}$ of a positive charge at the center of the sphere. The divergence of the electric field at any point of the sphere will be positive. Now, if you had a negative charge at the center, then all the fields will be pointed in. In that case, the divergence of the electric field at any point of the sphere will be negative. the divergence of $\vec{E}$ at the point where a point charge is located is not defined since the derivatives of electric field blow up there.

A more visual interpretation is possible if you think of Fig. 13.5 as the velocity field $\vec{v}$ of a gas. Then, the divergence $\vec{\nabla} \cdot \vec{v}$ represents the rate of expansion per unit volume. Since the fluid would be expanding out if $\vec{v}$ is given by Fig. 13.5, the divergence will be positive. On the other hand, if the gas is compressing, then the divergence of $\vec{v}$ would be negative at that point.

Suppose you have a steady flow of water in a tube. What will be the divergence at any point P? In this case, you will have as much water flowing as flowing out through any small closed surface around the point P. That means, the divergence will be zero.

Therefore, we can summarize the physical meaning of divergence of a vector field $\vec{F}$ at a point P as

1. if $\vec{\nabla} \cdot \vec{F} > 0$ at point P, then there is a net outflow of "something" through a surface enclosing the point P. In the case of the velocity of a fluid as the field that "something" is the fluid. In the case of electric field, we do not have a simple answer for what is flowing. You can think of this "something" as "field lines", but even that is an abstract quantity.

2. if $\vec{\nabla} \cdot \vec{F} < 0$ at point P, then there is a net inflow through a surface enclosing the point P.

3. if $\vec{\nabla} \cdot \vec{F} = 0$ at point P, then the outflow and inflow through a surface enclosing the point P cancel each other out.

Divergence and Flux

We saw pictorially that the divergence is related to the flow. Now, we make it more quantitative. Let us work in a two-dimensional space so we can draw the diagrams and write the equations more simply. After we have worked out the relations in two dimensional space, we will will write the general formula for the three-dimensional space. Let us also use the language of the fluid flow since that helps visualize the flow better.

Consider the flow through a rectangular space abcd around point P(x, y) in the xy-plane shown in Fig. 13.6. I have placed the rectangle symmetrically so that the bottom left vertex is at $(x-\Delta x/2, y-\Delta y/2)$ and the top right vertex is at $(x + \Delta x/2, y + \Delta y/2)$. The velocity field at the point P with coordinates (x, y) in the xy-plane will be given by $\vec{v} = v_x(x, y)\hat{u}_x + v_y(x, y)\hat{u}_y$. Note the x- and y-components of the velocity field are each dependent on both coordinates of the position (x, y) in the fluid. That is v_x is a function of two independent variables (x, y) and so is the function v_y.

Beware of the common mistake of thinking the label of component, as x in v_x as somehow also telling the functional dependence of v_x - that is not true. As pointed out above, the labeling for components is independent from the functional dependence.

Now, the fluid can enter or leave the rectangle from either of the four sides. The flow into or out of the rectangle on each side will come from the normal component of the vector field on that side. Recall

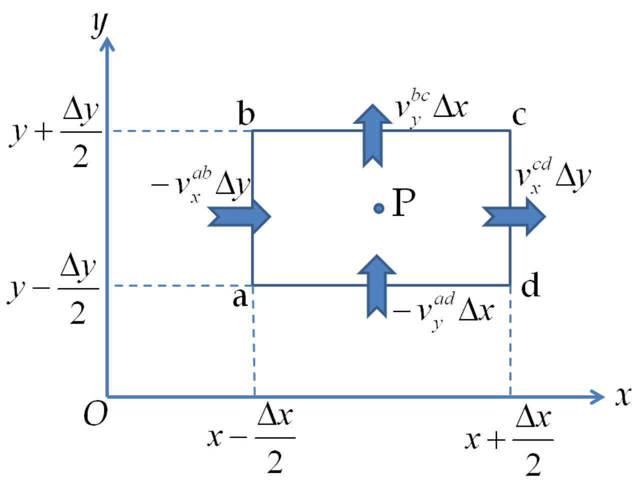

Figure 13.6: Calculating the flux through the rectangle around point P shows that flux per unit area is equal to the divergence of the field at the point P.

that the flows from the inside to the outside leads to the positive divergence. Therefore, we will point the normal at any side from the inside to the outside. For instance the normal direction on the left side ab will be pointed towards the negative x-axis and on the right side cd will be towards the positive x-axis. Therefore, the flow across the left side will be equal to the product of the negative of the x-component of the field at the left side and the length of the left side. That is,

$$\text{Flow across ab } = -v_x(x-\Delta x/2, y)\Delta y, \tag{13.21}$$

where I have used the value of the x-component of the velocity field at the mid-point of the side ab, and the flow across cd will be

$$\text{Flow across cd } = v_x(x+\Delta x/2, y)\Delta y \tag{13.22}$$

Similarly, the flow across ad and bc are

$$\begin{align} \text{Flow across ad } &= -v_y(x, y-\Delta y/2)\Delta x \tag{13.23}\\ \text{Flow across bc } &= v_y(x, y+\Delta y/2)\Delta x \tag{13.24}\\ & \tag{13.25} \end{align}$$

Adding up all the flows, we obtain the net flow outward from the rectangle as

$$\begin{multline} \text{Net flow out} = \left[v_x(x+\Delta x/2, y) - v_x(x-\Delta x/2, y)\right]\Delta y \\ + \left[v_y(x, y+\Delta y/2) - v_y(x, y-\Delta y/2)\right]\Delta x. \tag{13.26} \end{multline}$$

Now, divide both sides by the area of the rectangle to obtain

$$\frac{\text{Net outflow}}{\Delta x \Delta y} = \frac{v_x(x+\Delta x/2, y) - v_x(x-\Delta x/2, y)}{\Delta x} + \frac{v_y(x, y+\Delta y/2) - v_y(x, y-\Delta y/2)}{\Delta y}. \tag{13.27}$$

In the limit of the infinitesimally small rectangle the right side of this equation becomes equal to the divergence of the field $\vec{v}$ at point P and the left side is an outflow per unit area at point P.

$$\text{Net outflow per unit area} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}. \tag{13.28}$$

In a three-dimensional setting, the same arguments will give us

$$\boxed{\text{Net outflow per unit volume} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \vec{\nabla} \cdot \vec{v}.} \tag{13.29}$$

13.2.3 The Curl of a Vector Field

The divergence of a vector field was defined by the operation of the del operator $\vec{\nabla}$ on a vector field $\vec{F}$ through the dot product. The curl of a vector field is defined by the operation of the del operator on the vector field through the cross product. Formally, we write the curl of vector field $\vec{F}$ as,

$$\text{Curl of } \vec{F}: \boxed{\mathbf{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F}.} \tag{13.30}$$

Recall that the cross-product can be done by placing the components of the vectors in a determinant, but you will need to maintain the order of operation since now the multiplication of elements is replaced by the operation of taking derivatives.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}, \tag{13.31}$$

$$= \Big[\mathbf{Curl}(\vec{F})\Big]_x \hat{u}_x + \Big[\mathbf{Curl}(\vec{F})\Big]_y \hat{u}_y + \Big[\mathbf{Curl}(\vec{F})\Big]_z \hat{u}_z, \tag{13.32}$$

where the x-, y-, and z-components of the curl are

$$\Big[\mathbf{Curl}(\vec{F})\Big]_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \tag{13.33}$$

$$\Big[\mathbf{Curl}(\vec{F})\Big]_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \tag{13.34}$$

$$\Big[\mathbf{Curl}(\vec{F})\Big]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{13.35}$$

Let us practice a few examples.

Example 13.2.3. Computing Curl. Calculate the curls of the following vector fields. (a) $\vec{F}_1 = y^2\hat{u}_x$, (b) $\vec{F}_2 = y\hat{u}_x - x\hat{u}_y$, (c) $\vec{F}_3 = x\hat{u}_x + y\hat{u}_y$(d) $\vec{F}_4 = x\,y\,z\,\hat{u}_x + cos(x\,y\,z)\,\hat{u}_y + (x^2y^2z^3)\,\hat{u}_z$

Solution. For each of these calculations, we can either start with the determinant notation or directly go to the formulas for the components of the curl. The common practice is to start with the determinant notation. This helps to organize the components in a nice way. So, I will do that here even though at times this does not give us any advantage over just computing the components directly.

(a)

$$\begin{aligned}
\vec{\nabla}\times\vec{F}_1 &= \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 0 & 0 \end{vmatrix}, \\
&= -\hat{u}_z\frac{\partial y^2}{\partial y}, \quad \text{(All other terms are zero.)} \\
&= -2y\hat{u}_z.
\end{aligned}$$

(b)

$$\begin{aligned}
\vec{\nabla}\times\vec{F}_2 &= \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}, \\
&= \hat{u}_z\left[\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right], \quad \text{(All other terms are zero.)} \\
&= -2\hat{u}_z.
\end{aligned}$$

(c)

$$\begin{aligned}
\vec{\nabla}\times\vec{F}_3 &= \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}, \\
&= 0. \quad \text{(All terms are zero.)}
\end{aligned}$$

(d)

$$\vec{\nabla}\times\vec{F}_4 = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & \cos(xyz) & x^2y^2z^3 \end{vmatrix},$$

Now, we do not seem to have any advantage of the determinant notation. We will write the components directly.

$$\left[\mathbf{Curl}(\vec{F}_4)\right]_x = \frac{\partial(x^2y^2z^3)}{\partial y} - \frac{\partial\cos(xyz)}{\partial z} = 2x^2yz^3 + xy\sin(xyz)$$
$$\left[\mathbf{Curl}(\vec{F}_4)\right]_y = \frac{\partial(xyz)}{\partial z} - \frac{\partial(x^2y^2z^3)}{\partial x} = xy - 2xy^2z^3$$
$$\left[\mathbf{Curl}(\vec{F}_4)\right]_z = \frac{\partial\cos(xyz)}{\partial x} - \frac{\partial(xyz)}{\partial y} = -yz\sin(xyz) - xz$$

Physical Meaning of Curl

The curl of a vector field at a point is related with its circulation around that point P in space as we will see now. To get a feel for the physical meaning of the curl let us draw the vector diagrams of the vector fields whose curls were calculated in parts (a), (b), and (c) of the Example 13.2.3 above. The vector diagrams shown in Fig. 13.7 show that, if the vector field was a force field, then the vector fields with non-zero curls will cause some kind of rotational motion. The

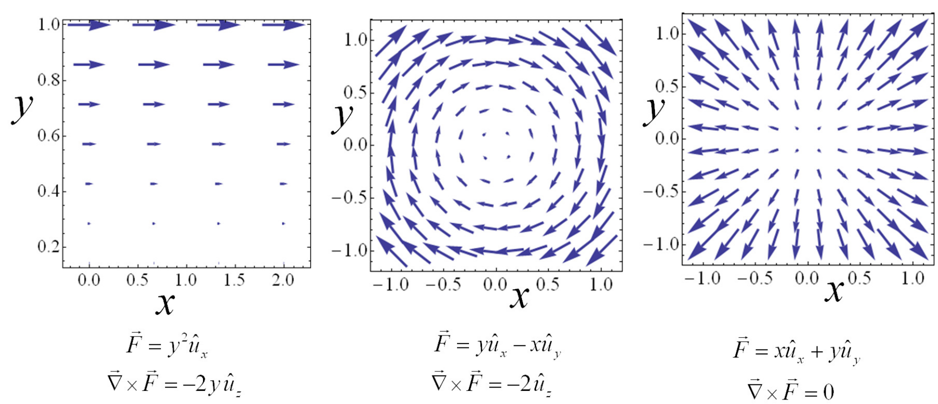

Figure 13.7: Demonstrating examples of vector fields for physical interpretation of the curl of a vector field.

vector field that spreads out uniformly in all directions such as the third diagram on the right in Fig. 13.7 does not give rise to rotational motion and has a zero curl. Another aspect of the curl is that the curl vector is perpendicular to the plane in which the vector field rotates.

We will now show that curl of a vector field at a point P is related to the **circulation** of the vector field in a loop around that point. Recall that we have encountered the property of circulation of a vector field when we studied the Ampere's law for the magnetic field, where we gave the following definition of the circulation of the magnetic field $\vec{B}$ in a closed path or loop C as

$$\boxed{\text{Circulation of } \vec{B} \text{ around closed path } C = \oint_C \vec{B}\cdot d\vec{l},} \qquad (13.36)$$

where $d\vec{l}$ is a vector element of the path whose direction is tangent to the path. Clearly, the circulation about a point P in space will depend on the orientation and size of the loop.

To find the relation between the curl of a vector field and the circulation of the vector field we will now calculate the circulation of a vector field $\vec{v}$ around a rectangle in the xy-plane shown in Fig. 13.8. The line integral on various segments are shown in the figure. Note

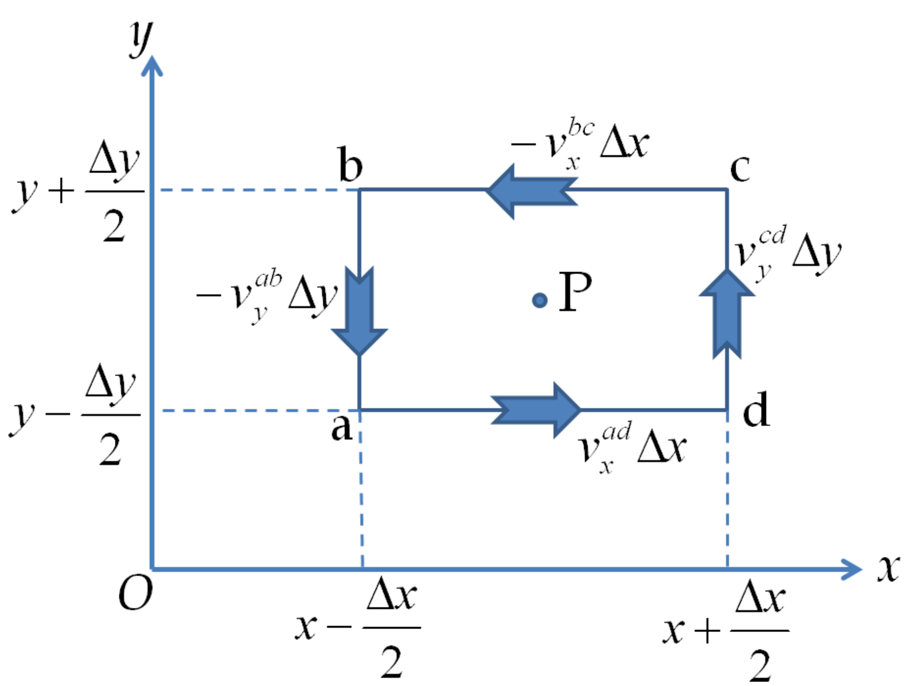

Figure 13.8: The curl around the rectangle is made up of four segments. In each segment, the segment vector is parallel to the axes. If the segment vector is pointed towards the negative x-axis or the negative y-axis, then the dot product with the vector field picks up a minus sign. On each segment the value of the vector field is set to the value of the field at the midpoint of the segment. The circulation around the loop is the sum of the contributions on each segment.

that the positive z-axis is pointed out-of-page in this figure. We have chosen to work with the circulation counterclockwise when observed from the side of the positive z-axis.

Since the line integral for the circulation has a dot product between the segment vector $d\vec{l}$ and the vector field, you will get the contribution only from the x-axis of the field on that segment if the segment is parallel to the x-axis. Similarly for the segments that are parallel to the y-axis. If the direction of the segment is towards the negative x- or the negative y-axis, you will pick up a minus sign from the dot product. On each segment we use the value of the vector field at the center of the segment. For instance, on the segment ad we have the segment parallel to the x-axis and pointed towards the positive x-axis, therefore the contribution to the circulation integral will be

$$\text{Contribution from ad} = v_x(x, y - \Delta y/2)\Delta x.$$

You can work out the contributions from the other segments in a similar way. Adding the four contributions we obtain the following for the circulation $\mathcal{C}_{xy}$ around the loop adcba in the xy-plane as

$$\begin{aligned}\text{Circulation, } \mathcal{C}_{xy} &= \left[v_y\left(x + \frac{\Delta x}{2}, y\right) - v_y\left(x - \frac{\Delta x}{2}, y\right)\right]\Delta y \\ &\quad - \left[v_x\left(x, y + \frac{\Delta y}{2}\right) - v_x\left(x, y - \frac{\Delta y}{2}\right)\right]\Delta x\end{aligned}$$

Now, we divide both sides by the area of the rectangle $\Delta x \Delta y$ and take the infinitesimal area limit to obtain

$$\frac{\text{Circulation, } \mathcal{C}_{xy}}{\text{Area}} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}, \tag{13.37}$$

which is the z-component of $\textbf{Curl}(\vec{v})$ at point P.

$$\frac{\text{Circulation, } \mathcal{C}_{xy}}{\text{Area}} = \left[\vec{\nabla} \times \vec{v}\right]_z \tag{13.38}$$

This shows that the z-component of the curl of a vector field at a point P is equal to the circulation per unit area in a loop about the point P in the xy-plane. Clearly, similar calculations can be carried out for the loops in the yz- and xz-planes around the point P. These calculations will show that the x and y-components of the curl are equal to the circulation per unit area in loops in the yz- and xz-planes respectively. For the signs of the terms to be consistent with the right-handed coordinate system, the direction of the loops would be counter-clockwise when looked from the corresponding axes. Thus, the loop in the yz-plane would be in the counter-clockwise direction when observed from the positive x-axis. Similarly for the loops in the xy- and xz-planes.

13.2.4 Useful Identities

Now, I will give you some identities that you can prove based on the representations of the gradient, divergence and curl you have seen above. We will use these identities in calculations below. In these identities the fields ϕ and ψ are scalar fields, which is the same as regular functions, and the fields $\vec{A}$ and $\vec{B}$ are vector fields, which you can think as three functions corresponding to their components.

Identities with a single $\vec{\nabla}$:

$$\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$$
$$\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$$
$$\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}(\phi\psi) = \psi\vec{\nabla}\phi + \phi\vec{\nabla}\psi$$
$$\vec{\nabla}\cdot(\phi\vec{A}) = \vec{A}\cdot\vec{\nabla}\phi + \phi\vec{\nabla}\cdot\vec{A}$$
$$\vec{\nabla}\times(\phi\vec{A}) = \psi\vec{\nabla}\times\vec{A} + \vec{\nabla}\phi\times\vec{A}$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A})$$
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$
$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B}$$

Identities with two $\vec{\nabla}$s:

$$\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{A}$$

We often replace $\vec{\nabla}\cdot\vec{\nabla}$ by another symbol called the **Laplacian operator**, $\nabla^2 \equiv (\vec{\nabla}\cdot\vec{\nabla})$, which is a sum of second order partial derivatives. In the Cartesian coordinates the Laplacian has the following form.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$