4.4 ONE DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

Suppose the constant acceleration is towards the positive x-axis, then, if the initial velocity has zero y and z-components, the motion will occur only along the x-axis. Even if you start with a zero x-component of velocity, the acceleration in the positive x-direction will change the x-component of the velocity. Since, there is no y- or z-component of acceleration, which happened as a result of the choice of coordinates, there is no way of changing the y- or z-component of velocity. That is, if v_y and v_z were zero at the beginning, they will remain zero throughout, keeping the motion entirely along the x-axis. In this case, we say that the motion is one-dimensional. In **one-dimensional motion**, only one component is needed.

1-D motion x-components only:

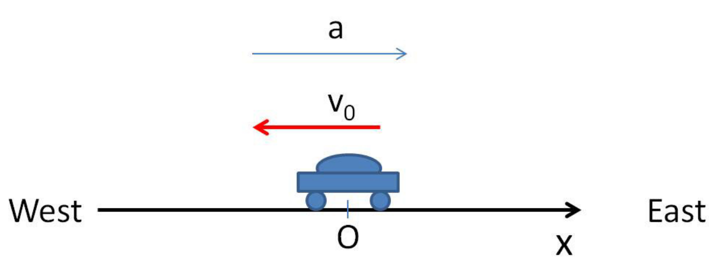
$$\boxed{x(t) = v_{0x}t + \frac{1}{2}a_x t^2}$$

$$\boxed{v_x(t) = v_{0x} + a_x t}$$

We now illustrate the use of these results with several examples.

Example 4.4.1. A Basic One-Dimensional Constant Acceleration Problem. A car moves on a horizontal road with a constant acceleration of 2 m/s^2 pointed towards the East. The velocity of the car at $t = 0$ is 15 m/s pointed towards the West. (a) Find the magnitude and direction of the velocity of the car at $t = 10$ sec. (b) Find the position of the car at $t = 10$ sec.

Solution. This problem exemplifies a typical scenario of a one-dimensional constant-acceleration kinematics problem. Note that the description of the problem does not tell us about the axes - we need to place one axis on the line of motion and choose the location of origin and also zero time reference. It is convenient to place x-axis on the line of motion with the positive x direction towards East so that the acceleration is towards the positive x-axis.



The origin is chosen to be where the car is at time $t = 0$. Since the direction of acceleration is towards the positive x-axis, therefore the value of x-component of acceleration will be positive. The direction of initial velocity is towards the negative x-axis, therefore the value of x-component of initial velocity will be negative. After we have settled on the axis and the positive and negative values of the given data, we summarize the known values.

$$t_0 = 0$$
$$x_0 = 0$$
$$v_{0x} = -15 \text{ m/s}$$
$$a_x = +2 \text{ m/s}^2$$
$$t = 10 \text{ s}$$

The list of unknowns are:

$$x$$
$$v_x$$

Now, it is simply a matter of choosing the appropriate equation from the list of the constant acceleration kinematics equations given in Table 4.1.

(a) In this part we need to find the velocity at $t = 10$ sec. Therefore, we see that Eq. 1 in Table 4.1 will immediately give the answer. All the lengths are in meters and times in seconds, so, we will suppress the units while doing the calculation. It is often a good idea to take care of unit conversions before you input numerical values in your calculations. Then you can simply put the units at the end of the calculations. Here v_x will turn out in m/s.

$$v_x = v_{0x} + a_x t = -15 + 2 \times 10 = 5 \text{ m/s}$$

Since we have solved for the x-component of the velocity here (the y and z-components being zero), the sign of the answer does correspond to the direction of the actual velocity. The positive value of the x-component of the velocity at $t = 10$ sec means that the velocity is pointed towards the East which is the positive direction of the x-axis we have chosen here.

(b) In this part we need to find the x-coordinate at $t = 10$ sec. We have already found v_x in part (a). So, we will use Equation 2 in the Table 4.1.

$$x = v_{0x} t + \frac{1}{2} a_x t^2 = -15 \times 10 + \frac{1}{2} \times 2 \times 10^2 = -50 \text{ m}.$$

Since the value of x is negative here, the car is 50 m from the starting place towards the West - direction of the negative x-axis.

Example 4.4.2. More Questions about the accelerating car. Let us analyze the accelerating car given in the last example, Example 4.4.1. (a) Note that velocity at initial time is in the opposite direction to acceleration. Therefore, the car will slow down and eventually come to rest. When does the car come to rest momentarily? (b) How far away from the starting place does the car come to rest?

Solution. In the question, we have a different set of knowns and unknowns than was the case in the last example. We are told that the car comes to rest and we are asked about that instant in time. We will denote the time segment as $0 \leq t \leq T$. Let us list the new set of knowns, which will include $v_x = 0$.

$$\begin{aligned} x_0 &= 0 \\ v_{0x} &= -15 \text{ m/s} \\ a_x &= +2 \text{ m/s}^2 \\ v_x &= 0 \quad \text{(since at } t = T\text{, the car is at rest.)} \end{aligned}$$

The list of unknowns are:

$$\begin{gathered} x \\ T \end{gathered}$$

The tools are the same set of equations given in Table 4.1.

(a) For this part we need to find T. By looking at the equations in Table 4.1, it is clear that the first equation will readily give us the time.

$$v_x = v_{0x} + a_x T \Longrightarrow 0 = -15 + 2T$$

solving for T we find $T = 7.5$ sec. Thus, it will take 7.5 sec for the car to come to rest.

(b) For this part we need to find x. We know one more data now from the solution of part (a): $T = 7.5$ sec. Once again, the equations in Table 4.1show that the second equation in Table will immediately give us the value of x.

$$x = v_{0x}T + \frac{1}{2}a_x T^2 = 0 - 15 \times 7.5 + \frac{1}{2} \times 2 \times 7.5^2 = -56 \text{ m}.$$

Thus, the car will come to rest at the coordinate $x = -56$ m, which is 56 m from the starting point towards the West.

4.4.1 Free Fall: Application of Constant Acceleration

Free fall refers to an idealized vertical motion near earth when the air friction is negligible and the only force on the object is the gravitational force of earth. Although the unaided motion near the Earth is called free fall, the object itself may be either going up or coming down. For instance, when you throw a ball vertically up, it will be rising initially, and then, later it will be coming down.

Figure 4.2: Examples of free fall. The motions of arrow, cannonball and soccer can be approximated as free fall when we ignore air resistance. Picture credits: archer: APoincot, cannon: Alfo23 from Svizzera, and soccer: Rdikeman, all at www.creativecommons.org.

The entire motion of the ball, after leaving hand and before getting caught or hitting the ground, is called free fall. It is confusing that the term "free fall" refers to motion in both directions but that is how we will call the motion of an object upon which the only force is the force of gravity.

Falling objects have been studied for a long time. Aristotle believed that when an object is released from rest, it acquires speed instantaneously in proportion to its weight (or on some other quality, such as its "fiery" or "earthy" character). Galileo rejected this idea based on simple arguments such as the following.

According to Aristotle, a two-pound rock will fall faster than a one-pound rock since the two-pound rock has more "Earthiness". Now, if you split the two-pound rock and join the halves by a light string, then we have an inconsistency in the prediction - on the one hand, each half should fall at one speed, and on the other, the two together should fall at a higher speed. But, an object must have a unique speed. Hence, Galileo concluded that Aristotle's thinking must be wrong.

Galileo then set out to find a quantitative way to study the motion of falling bodies and in the process revolutionized science. Galileo is considered to be the father of modern science since he was the first person to successfully make a break with the Aristotelian way of thinking about the fundamental nature of motion, which had dominated the thinking of scientists for nearly two millennia. Galileo used quantitative methods and interplay between observations, theoretical predictions and experimental testing, much like modern physicists.

To cut down the enormous acceleration due to gravity and make the speeds in the experiments more manageable, Galileo conducted his observations of motion on inclined planes. Through experiments, Galileo established that the distance d travelled by an object during a time interval t varies as the square of time t when released from rest.

$$d = \frac{1}{2}at^2, \tag{4.17}$$

where the value of acceleration a depends on the angle of the incline. As you increase the angle of inclination, the acceleration changes, having the largest value when the ball is dropped vertically. The acceleration in the vertical free fall is called the **acceleration due**

to gravity, whose magnitude is commonly denoted by the symbol g, which has a standard value of 9.81m/s^2 although it varies from place to place on Earth.

$$\boxed{g = 9.81\ m/s^2 \ \ \text{(near the surface of the Earth)}} \tag{4.18}$$

The value of g varies with the latitude on earth and the altitude from earth. To keep calculations simple, we will use the standard value for g in this book as long as the distance of the object under study from the earth is much smaller than the radius of the Earth. Since radius of Earth is approximately 6.37×10^6 m or $6,370$ km and the thickness of the atmosphere above ground is approximately only 65 km, you could use the standard value of g for anything moving in the atmosphere without incurring much error. For larger distances, we will have to use a more exact law for for the gravitational force which you will study in a later chapter.

In our notation, the vertical motion is a one-dimensional motion along a Cartesian axis. It is a common practice to use the y-axis for the vertical motion with the positive y-axis pointed up.

Although nothing will go wrong if we call this direction the x or z-axis, but we will follow the standard practice and point the positive y-axis up. With this choice the equations of constant acceleration will have non-zero y-components, and the value of the constant y-component of the acceleration will be equal to $-g$ (Table 4.2).

Table 4.2: Free fall with the positive y-axis pointed up - one-dimensional motion

$(t_0 = 0)$
1. $v_y = v_{0y} - gt$
2. $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$
3. $v_y^2 = v_{0y}^2 - 2g\,(y - y_0)$

Note that these equations are equally valid for free rise as well as for free fall. With the positive y-axis pointed up, the y-component of the velocity will be positive when the object is rising freely, and negative when the object is falling freely, but the y-component of the acceleration will always be negative, $a_y = -g$.

Example 4.4.3. A Freely Falling Ball. A ball is let go from rest (not thrown) from a tall building. (a) What is the velocity of the ball at $t = 5$ *sec*? (b) How far has the ball fallen during this interval?

Solution. Assume that the ball has enough room to fall during the 2 sec interval so that ball does not hit the ground in that time. Then, the acceleration will have the magnitude of the free fall. We can use the equations given in Table 4.2 to solve this problem.

(a) The y-component of the initial velocity is zero, $v_{0y} = 0$. Therefore, the velocity at $t = 5$ sec is

$$v_y = a_y t = -9.81\ m/s^2 \times (5\ s) = -49.05\ m/s.$$

Therefore, the velocity at this instant is 49.05 m/s pointed down.

Figure 4.3: The CN tower in Toronto is 446.5 m. A pebble falling from the peak of the CN tower will hit the street at approximately 94 m/s or 337 km/h. Credits: publicdomainpictures.net, Bobby Mikul.

(b) The distance fallen can be found by evaluating the change in the y-coordinate.

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = 0 - \frac{1}{2} \times 9.81\ \mathrm{m/s^2}(5\ \mathrm{s})^2 = -122.7\ \mathrm{m}.$$

Therefore, the ball would fall a distance of 122.7 m. The CN tower in Toronto is 446.5 m. The tallest building in the world in 2012, Burj Khalifa in Dubai, is 829.84 m. So, this experiment could be done from these tall skyscrapers. Can you calculate how fast a small pebble will be moving if dropped from the top of Burj Khalifa?

Example 4.4.4. A Typical Free Fall Problem. A baseball is thrown vertically up. Immediately after leaving the hand its speed was 20.0 m/s. The ball is caught on its way down 5 meters above the place it left the hand. (a) How long does it take to reach the maximum height? (b) To what height does the ball rise before turning back? (c) How long does it take before it is caught? (d) What was its velocity immediately before it was caught? Assume air resistance negligible.

Solution. Before I present the solution of the problem at hand, I want to discuss two fundamental considerations that goes into almost every problem that has a freely falling body.

Issue 1: The time domain of free fall

Note that you must be sure that you are looking at only that part of the motion which is truly freely falling. In this problem, only the motion of the ball after leaving the hand and before it is caught is freely falling motion. When the ball is touching the hand, the hand gives the ball a push up so as for the ball to gain the initial y-component of the velocity to go up. The y-component of the acceleration in that part of the motion of the ball is different than g downward as assumed for the free fall motion.

The same type of problem occurs when the ball is caught. Just before the ball is caught, the ball is moving downward with y-component of the acceleration $-g$ and a negative value of y-component of the velocity. When the ball touches the hand, the hand pushes the ball upward and provides an upward acceleration, which reduces the downward velocity to zero. When the ball is in touch with the hand, y-component of the acceleration of the ball is no longer $-g$.

Therefore, the only part of motion we can call freely falling is between the time after the ball has left the hand initially and before it touches the hand again. Note that in every problem concerning the one-dimensional kinematics of constant acceleration, you will need to pay attention to these details to properly understand the problem.

Issue 2: The time segment of interest

Now that you have determined the total time interval, say from $t = t_i$ to $t = t_f$ that the ball is freely falling, a particular part of the question may ask you to find things within a segment of time that falls within this time interval. Different parts of the same problem may refer to different time segments within the total time interval, after leaving the ground and before getting caught.

For instance, parts (a) and (b) refer to the motion after the ball leaves the hand and when it comes to rest at the top of the path, while parts (c) and (d) are concerned with the interval between the top of the path where it was momentarily at rest and immediately before the ball was caught, where it was still moving down. Transferring values of various quantities between parts must be done very carefully. Use physics to decide if a particular quantity in one part has the same or related value to a similar quantity in another part. Using formulas for transferring quantities from one part to another will most likely result in stupid mistakes.

Now, we are ready to discuss the solution.

(a) Let us list various known and unknown quantities for the segment of motion under consideration for this part of the problem. Basically, you must decide on four quantities: (1) the time interval in which acceleration is constant, (2) the positions at the two ends of the time interval, (3)the velocities at the two ends of the time interval, and (4) the value of acceleration.

- **Interval:**
  - After leaving the hand and when it comes to rest at the top.
  - Set $t = 0$ at the instant after the ball leaves the hand. This makes the use of equations in the Table 4.2 possible without any modifications. Note the time t in equations in Table 4.2 is interval from the starting time.
  - Set $t = T$ at the instant the ball arrives at the top. This time is not known.
- **Positions:**
  - Set $y_0 = 0$ at the instant after the ball leaves the hand. This is the choice for the origin.
  - Set $y = H$ at the instant the ball arrives at the top. This is not known.
- **Velocities:**

  - Note $v_{0y} = +20.0$ m/s at the instant after the ball leaves the hand. The y-component of the velocity is positive since the ball's velocity is pointed in the positive y direction.
  - Note $v_y = 0$ since the ball is at rest at the instant the ball arrives at the top.

- **Acceleration:**
  - The y-component of the acceleration is $a_y = -g$, where the value of $g = 9.81$ m/s^2, the standard value for free fall.

Now, in this part of the problem, we need to find T. Looking at the set of equations in Table 4.2, it is readily apparent that the Equation 1 in the Table would give the value of T in one step.

$$v_y = v_{0y} - gt \implies 0 = 20 - 9.81T, \ or, \ T = 2.04 \text{ sec.}$$

When I used the calculator, I got $20/9.81 = 2.03873598369$. Should I present all the digits from the calculator? No. I used the rules of rounding off at the least significant digit. The data used in the calculation has three significant digits in g and three in v_{0y}. Therefore, I rounded off the answer to keep up to three digits. that is why I didn't write 2.0 or 2.039.

(b) This part also deals with the same interval as part (a). Therefore, we do not need to write out the table of values once again. We use the same table as in part (a). Furthermore, now we can also use the result $T = 2.04$ sec from part (a). We need to find $y = H$ in this part. It is clear that with the calculated T, equation number 2 in Table 4.2 will yield the value of the height H.

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \implies H = 0 + 20 \times 2.04 - \frac{1}{2} \times 9.81 \times 2.04^2 = 20.4 \text{ m}.$$

Once again,in the calculator I got a number with many digits, which I have rounded to three significant digits.

(c) This part and the next are about a different time interval. Therefore, we need to build another table. The table in part (a) does not apply anymore.

- **Interval:**
  - From the time the ball comes to rest at the top to the instant immediately before it is caught. Note: the instant the ball is caught is not a part of the interval.

  - Set $t = 0$ at the instant the ball comes to rest at the top. This makes the use of equations in the Table 4.2 possible without any modifications. Note the time t in equations in Table 4.2 is interval from starting time of the interval.
  - Set $t = T$ at the instant immediately before the ball is caught. This is not known.

- **Positions:**
  - Set $y_0 = 20.4$ m at the instant the ball leaves the top. We can leave the origin at the same place as in part (a). Although, leaving the origin alone is not required, it is often helpful to leave the origin alone. Sometimes, you may find it helpful to choose different origins for different parts.
  - Leaving the origin in the same location as above gives $y = 5$ m at the instant immediately before the ball is caught.

- **Velocities:**
  - Note $v_{0y} = 0$ at the instant the ball is at the top; this is the beginning of this interval. The velocity is zero since the ball is not moving at this instant.
  - Set $v_y = V$ at the instant immediately before the ball is caught. It is unknown. Note the ball is not caught yet. So V is not zero. This is one of the most common mistakes. We know that the value for V will be negative since the positive y-axis is pointed up and the velocity is pointed down.

- **Acceleration:**
  - The y-component of acceleration is $a_y = -g$, where $g = 9.81$ m/s^2.

In this part we need to find the value of T. Of course, this value of T is not the same as what we found in part (a). This T is for the present interval. So, using the same symbol should not confuse us as long as we know that each interval is a separate problem. Equation 2 of Table 4.2 appears to be the equation of choice since it will have only one unknown once we have substituted all the knowns into the equation.

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \implies 5 = 20.4 + 0 - \frac{1}{2} \times 9.81T^2 \implies T = 1.77\,\text{sec.}$$

(d) This part has the same interval as part (c). Therefore, we can use the results of part (c) and we do not need to make another table of values. Here we seek the value of the velocity just before the ball is caught. Equation 1 of Table 4.2 with the T value found in part (c) can help us find the required velocity.

$$v_y = v_{0y} - gt \implies V = 0 - 9.81 \times 1.77 \implies V = -17.4 \text{ m/s}.$$

The negative value of the velocity indicates the ball is moving towards the negative y-axis, which is pointed down, as expected since the ball is falling down at the moment.