1.3 ELECTRIC FIELD

1.3.1 Electric Field and Electric Force

The Coulomb's force law treats the interaction between charges as an action-at-a-distance. By action-at-a-distance we mean that a charge Q experiences force by another charge q which is not touching Q. One problem with the action-at-a-distance concept of a force is that, if you move q, the effect on Q is immediate regardless of their separation distance. This violates our notion of finite speed of travel of information.

The English physicist **Michael Faraday** (1791-1867) introduced the idea of the electric field in 1830's, which he called electric lines of force, in analogy with his earlier explanation of the arrangement of iron filings around a magnet in terms of the magnetic force lines. He believed that each charge created invisible electric lines of force that changed the properties of the space around the charge. According to Faraday's way of thinking, a force on a charge is a response to these invisible lines of forces passing through the space point where the charge is located at the moment. The force on a charge is a measure of the strength and directions of these electric field lines.

Suppose you place a positive charge Q at some space point P and notice that there is a force of magnitude F on the charge Q pointed towards the North. You will find that a charge 2Q will experience $2 \times F$ pointed towards the North. To defined a quantity independent of the arbitrary amount of charge that we can place at point P we can divide the magnitude of the force on a charge by the amount of the charge. Now, we look at F/Q or $2F/2Q$, both give us the information intrinsic to the point P independent of the amount of the test charge. We include the direction information by dividing the force vector $\vec{F}_{\text{on Q}}$ by the amount of the positive test charge Q to define this intrinsic electrical property of the space point. The intrinsic property discovered by the response of a test charged particle this way is called the **electric field** at that point. We will denote electric field at point P by $\vec{E}_P$ to further emphasize the local nature of electric field.

Thus, when a test charge Q is placed at a point P where the electric field is $\vec{E}_P$, the test charge will experience a Coulomb force equal to $Q\vec{E}_P$.

$$\boxed{\vec{F}_{\text{on Q}} = Q\vec{E}_P.} \tag{1.12}$$

The charge Q is only "feeling" or "discovering" the electric field that already exists due to other charged somewhere else. We call the "feeler" charge Q the **test charge**, and the other charges whose electric field the test charge is responding to are called the **source charge** or charges. The space point P is called the **field point**. **The electric field at the field point exists regardless of whether or not we test its existence by placing the test charge at that point.**

This relation 1.12 tells us that in the SI family of units, the unit of electric field will be N/C. With the viewpoint of electric force based on the electric field, there is no action at a distance. Instead, the electric force on the test charge Q is due to the local value of the electric field regardless of the source of electric field. When the source charge(s) of the electric field moves, it takes some time before the electric field at point P changes and the charge Q responds to the new electric field. From the dynamic theory of electric fields one can show that the information of movement of charges travels from the source to the field point P at the speed of light.

From the definition above, it is clear that the electric field at the space point P and the electric force on a positively charged test particle placed at P are in the same direction. That is, when $Q > 0$, the force on Q would be in the same direction as the electric field at that point.

The relation between the electric force on a test charge and the electric field at the site of the test charge, i.e. $\vec{F}_{\text{on Q}} = Q\vec{E}_P$, suggests a procedure for mapping out the electric field in a region of space by placing a test charge at various locations and experimentally determining the force per unit charge of the test charge. You have to make sure that the source charges do not change or move during your measurements. The result can be displayed either as a collection of vectors, one for each point in space, or as a vector function, $\vec{E}(x, y, z)$ which is a function of space coordinates x, y, and z.

Faraday's ideas about the electric lines of force were only pictorial, but they laid the foundation for field theory upon which modern theories of fundamental physics is based. In the following, we will use Coulomb's law to figure out the analytic expressions for the electric fields in various situations.

Example 1.3.1. Electric Force from Electric Field. A proton is placed at a point in space where electric field is 300 N/C pointed due North. Find the acceleration of the proton.

Solution. From the electric field at the site of the proton and the charge of the proton, we can obtain the force on the proton by using

Eq. 1.12. Once, we have obtained the force on proton, we can use Newton's second law to find the acceleration of proton.

Electric force on the proton:

Magnitude:

$F_e = |q|E = 1.6 \times 10^{-19}\text{C} \times 300\ \text{N/C} = 4.8 \times 10^{-17}\ \text{N}$

Direction:

Due North

The gravitational force on the proton:

Magnitude:

$F_g = mg = 1.67 \times 10^{-27}\text{kg} \times 9.81\ \text{m/s}^2 = 1.6 \times 10^{-24}\ \text{N}$

Direction:

Towards the center of the Earth

Clearly $F_e >> F_g$. Therefore, we usually ignore the gravitational force. On dividing the magnitude of the net force by the mass of the proton, we obtain the resulting magnitude of the acceleration.

$$a \approx \frac{F_e}{m} = \frac{4.8 \times 10^{-17}\text{N}}{1.67 \times 10^{-27}\text{kg}} = 2.9 \times 10^{11}\text{m/s}^2.$$

Since proton is a positively-charged particle, the direction of the acceleration of the proton will be the same as the direction of the electric field. That is the acceleration of the proton would also be pointed towards the North.

1.3.2 Electric Field of a Point Charge

Faraday's picture says that each charge produces an electric field everywhere in space and Eq. 1.12 gives us an operational way of determining the electric field at an arbitrary point. To determine the electric field $\vec{E}_P$ at space point P(x, y, z), we would place a test charge at that point, determine the force on that charge, and then divide the force by the charge Q of the test particle. Now, we use this procedure to find a formula for the electric field of a point charge located at the origin of a coordinate system.

ELECTRIC FIELD OF A POINT CHARGE AT THE ORIGIN

Suppose a charge q is fixed to the origin of a Cartesian coordinate system. What is the electric field at point P a distant r from the charge? To find the electric field at P, let us place a positive "feeler" (test) charge Q at P, and deduce electric field by dividing the force on Q by the charge Q.

The magnitude of the force on Q is:

$$\text{Magnitude:} \quad F_{\text{on Q}} = \frac{1}{4\pi\epsilon_0}\frac{|Q|\,|q|}{r^2} \tag{1.13}$$

Now, we divide this force by the charge of the "feeler" charge Q to obtain the magnitude of the electric field at P.

$$\boxed{\text{Magnitude:} \quad E_P = \frac{F_{\text{on Q}}}{|Q|} = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2}.} \tag{1.14}$$

Electric field at point P
a distance r from charge q:

Magnitude : $\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$

Direction : (1) q to P if $q > 0$
(2) P to q if $q < 0$

The direction of the electric field is the direction of the force on a positively-charged test particle. Therefore, if the "source" charge q is positive, then the electric field at P would be pointed away from q and towards P, and when q is negative the electric field at P would have the direction from P towards q. Let me emphasize again that the electric field $\vec{E}_P$ exists at the space point P whether or not the test charge is placed there. The test charge Q only "discovers" an electric field by experiencing a force when placed at that point.

SUPERPOSITION OF ELECTRIC FIELD

Just as the electric force obeys a superposition principle, so does the electric field. Consider a system consisting of N charges q_1, q_2, $\cdots$, q_N. What will be the net electric field $\vec{E}_P$ at a space point P from these charges?

Each of these charges are "source" charges that would produce their own electric field at the point P, independent of whatever other changes may be doing. Let $\vec{E}_1$, $\vec{E}_2$, $\cdots$, $\vec{E}_N$ be the electric fields at P produced by charges q_1, q_2, $\cdots$, q_N, respectively. Then, the net electric field $\vec{E}_P$ at point P will be equal to the vector sum of these individual electric fields as you can easily show by placing a unit test charge at point P and adding up the forces on the test charge due to the multitude of electric fields at P.

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N. \tag{1.15}$$

Example 1.3.2. Electric Field of a Single Charge. Find the electric field of a proton in the hydrogen atom at a point P that is 5.29×10^{-11} m away from the proton.

Solution. From the definition of electric field we find the magnitude of the electric field at the field point P by

$$E_P = \frac{1}{4\pi\epsilon_0}\frac{e}{r^2} = 9 \times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \times \frac{1.6 \times 10^{-19}\text{C}}{(5.29 \times 10^{-11}\text{ m})^2} = 5.14 \times 10^{11}\text{N/C}$$

The electric field is directed from the proton to P since proton is positively charged.

Example 1.3.3. Superposition of Electric Field of Multiple Charges. Consider three charges q_1, q_2 and q_3 fixed at three consecutive corners of a square of side a. Find the electric field (a) at the fourth corner of the square, (b) at a point at height b above the center of the square, and (c) at a point b below the center of the square.

Solution. Let us draw axes so that the square is in the xy-plane with q_2 is at the origin, q_1 at the y-axis and q_3 at the x-axis as shown in Fig. 1.17.

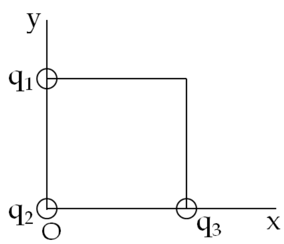

Figure 1.17: Example 1.3.3.

The positions of the three charges are $q_1(0,a,0)$, $q_2(0,0,0)$, and $q_3(a,0,0)$. We need to evaluate the net electric field at the following points (a) $P_1(a,a,0)$, (b) $P_2(a/2,a/2,b)$, and (c) $P_3(a/2,a/2,-b)$.

Electric field at P_1:

$$\begin{aligned}\vec{E}_{P_1} &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{a^2}\hat{u}_x + \frac{q_2}{(a^2+a^2)^{3/2}}\left(a\hat{u}_x + a\hat{u}_y\right) + \frac{q_3}{a^2}\hat{u}_y\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{a^2}\left[\left(q_1 + \frac{q_2}{\sqrt{8}}\right)\hat{u}_x + \left(q_3 + \frac{q_2}{\sqrt{8}}\right)\hat{u}_y\right]\end{aligned}$$

Electric field at P_2 is obtained similarly.

$$\begin{aligned}\vec{E}_{P_2} = \frac{1}{4\pi\epsilon_0}\Bigg\{ & \frac{q_1\left(0.5a\hat{u}_x - 0.5a\hat{u}_y + b\hat{u}_z\right)}{\left[(0.5a)^2 + (0.5a)^2 + b^2\right]^{3/2}} \\ & + \frac{q_2\left(0.5a\hat{u}_x + 0.5a\hat{u}_y + b\hat{u}_z\right)}{\left[(0.5a)^2 + (0.5a)^2 + b^2\right]^{3/2}} \\ & + \frac{q_3\left(-0.5a\hat{u}_x + 0.5a\hat{u}_y + b\hat{u}_z\right)}{\left[(0.5a)^2 + (0.5a)^2 + b^2\right]^{3/2}}\Bigg\}\end{aligned}$$

The electric field at P_3 is obtained from $\vec{E}_{P_2}$ by replacing b by $-b$.

Example 1.3.4. Electric Field of Two Equal Point Charges. Find the electric field of two charges, each of magnitude q, at a point that is equidistant from the two.

Solution. Let the charges be located on the x-axis at $x = \pm a$ and we find electric field at a point P on the y-axis as shown in Fig. 1.18. Since the two charges are equal in magnitude and are equidistant from the field point P, their electric field $\vec{E}_1$ and $\vec{E}_2$ will have an equal magnitude given by:

$$\text{Magnitude:}\quad E_1 = E_2 = \frac{1}{4\pi\epsilon_0}\frac{|q|}{(a^2+y^2)},$$

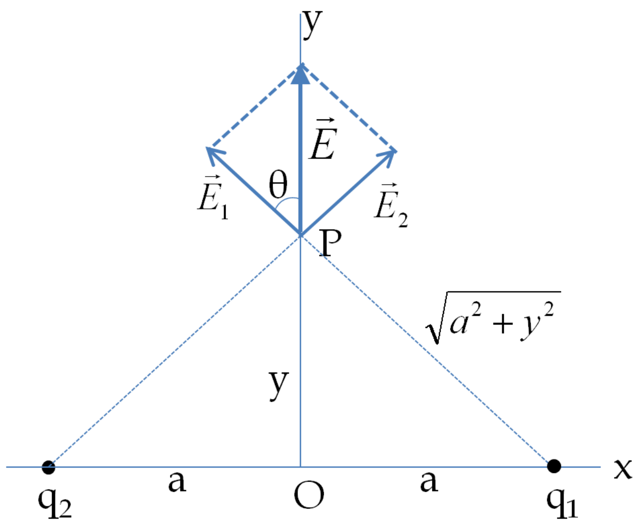

Figure 1.18: Example 1.3.4.

where $\sqrt{a^2+y^2}$ is the distance from each charge to the field point P. We will drop subscript P from electric field symbols to keep the notation simpler, but mind you that we are still dealing with electric field at a particular space point. The directions of the two fields are shown in Fig. 1.18. Adding the two electric field vectors will yield the net electric field at point P. From the symmetry of the situation, it is immediately clear that the x-component of the net electric field will be zero, and the y-component of the net electric field will be twice the y-component of one of the electric fields.

$$E_y = E_{1y} + E_{2y} = 2E_1 \cos\theta \quad \text{or} \quad 2E_2 \cos\theta$$

where the angle θ is given by

$$\cos\theta = \frac{y}{\sqrt{a^2+y^2}}.$$

Therefore, the electric field at a point on the y-axis is pointed along the y-axis and is given by the following analytic expression for the electric field at point P in figure.

$$\vec{E}_P = 2\left(\frac{1}{4\pi\epsilon_0}\frac{q}{(a^2+y^2)}\right)\frac{y}{\sqrt{a^2+y^2}}\hat{u}_y.$$

That is, the electric field at point P in the figure has the following magnitude and direction.

Magnitude: $\dfrac{1}{2\pi\epsilon_0}\dfrac{|qy|}{(a^2+y^2)^{3/2}}$

Direction: (1) positive y-axis if the product, $q.y > 0$

(2) negative y-axis if the product $q.y < 0$.

1.3.3 The Electric Field of a Continuous Charge Distribution

Imagine putting some extra electrons on a metal plate. The added electrons will increase the population of the conduction electrons in the metal. After you place the charges, the electrons will redistribute over the metal. Even a small amount of charge corresponds to a large number of electrons. For instance, the charge on a comb that could pick 1 gram of paper would be of the order of a nano-Coulomb, which would correspond to $\sim 10^{10}$ electrons. Therefore, we speak in terms of the density of charges rather than the individual discrete charges.

The electric charge density is defined as the amount of electric charge per unit volume. Note that, unlike the mass density, electric charge density can be positive or negative, depending upon the type of charge present.

The electric charge density may vary from point to point depending on the distribution of charges in the material or the medium. Therefore, we define an **electric charge density** function. To get an estimate of the charge density $\rho(x, y, z)$ at a particular point (x, y, z), we choose a cell that is large enough to include many atoms, but small compared to the size of the object. Let the amount of net charge in a cell be Δq, and the volume be ΔV. The electric charge density in the cell with coordinates (x, y, z) is then defined by

$$\rho(x, y, z) = \frac{\Delta q}{\Delta V} \quad \text{[in the cell containing point } (x, y, z).] \tag{1.16}$$

in the limit when $\Delta V \to 0$. If the charge density throughout the body is same, we say that the object is uniformly charged.

$$\boxed{\rho(x, y, z) = \rho_0 \ \ \text{(constant)} \quad \Longleftrightarrow \quad \text{unifromly charged.}} \tag{1.17}$$

Note that the charges do not have to be spread in a three-dimensional space. They may be distributed over a surface. In that case, the density would be charge per unit area of the surface. This density is called **surface charge density**. While the volume charge density is denoted by the letter ρ, the surface charge density is often denoted by the letter σ (read: sigma). Another possibility is that charges could be distributed on a line or a curve, then we will have the density as charge per unit length. This density is called **line or linear charge density** . The line charge density is denoted by the letter λ.

Electric field of an object with densities $\rho(x, y, z)$, $\sigma(x, y, z)$, or $\lambda(x, y, z)$ is obtained by replacing the continuous charge distribution by a discrete collection of charges by "dividing up" the space of the distribution, whether a volume, an area or a line, into smaller cells. This procedure converts the problem of the electric field of a continuous charge distribution into that of the problem of the electric field of N discrete charges if there are a total of N cells. As we increase the number of cells, the superposition of electric fields from charges in different cells becomes an integral.

$$\begin{aligned}
\vec{E} &= \sum_i \frac{1}{4\pi\epsilon_0}\frac{q_i}{r_i^2}\hat{u}_i \quad \text{(discrete charges)} \\
&= \int \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{u}_r \quad \text{(continuous charges)}, \qquad (1.18)
\end{aligned}$$

where the formula for the the continuous charge element dq depends on the dimensions over which the charges are distributed.

$$dq = \begin{cases} \lambda dl & \text{(1-dim)} \\ \sigma dA & \text{(2-dim)} \\ \rho dV & \text{(3-dim)} \end{cases} \qquad (1.19)$$

The applications of these formulas are often simplified if the charge distribution has some type of symmetry. We will illustrate this point with a few examples below. In the next chapter, you will learn a more powerful method for exploiting the symmetry.

Example 1.3.5. Electric Field Of A Line of Charges. A straight non-conducting plastic wire of length L has a uniform line charge density λ. Find the electric field of these charges at an arbitrary point a distance s from the wire on the plane that divides the wire in half and is perpendicular to the wire.

Solution. Notice that this problem is for a non-conducting wire. For a conducting wire, the charges will move away from each other and will not be distributed in a uniform manner assumed here. On a non-conducting wire, they will not move from where they are placed, and therefore can be distributed uniformly.

For analytic purposes, we choose the axes so that the line of charge falls on the y-axis and the point P is located on the x-axis as showing in Fig. 1.19 so that the coordinates of point P are $(s, 0, 0)$. Now, we can proceed to set up the problem in either a brute-force approach or an approach that makes use of the physical symmetry of the charges.

Brute force:

In this approach we do not worry about simplifying the situation much more than making a judicious choice of the Cartesian axes and then we carry out a straight-forward calculation. Let us divide the charges on the line into small cells. We will first write the electric field by charges in one cell. Let one element of the wire be between y and $y + dy$. This element has charge $\lambda \times dy$. We can treat this element as a point charge since the element is infinitesimal in size. Now, using the electric charge formula for a point charge case, we

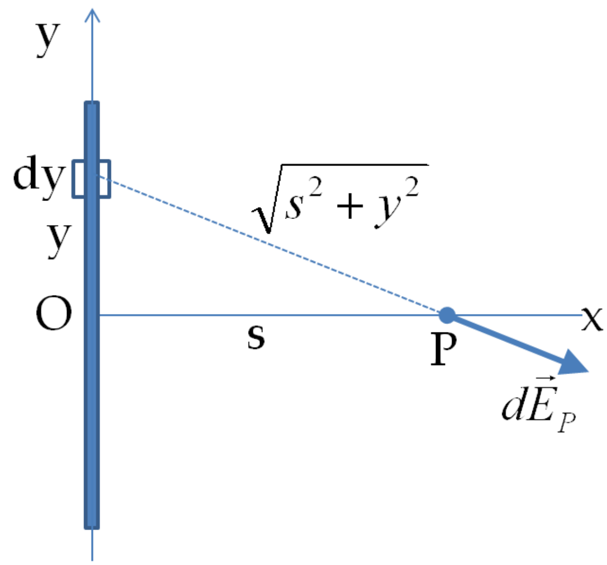

Figure 1.19: Example 1.3.5.

can almost immediately write the electric field at P from the charges in this element as

$$d\vec{E}_P = \frac{1}{4\pi\epsilon_0}\frac{\lambda dy}{\left(s^2+y^2\right)^{3/2}}\left(s\hat{u}_x - y\hat{u}_y\right) \quad \text{[P on the } x\text{-axis]}$$

Separately writing the x-and y-components we have

$$dE_x = \frac{\lambda}{4\pi\epsilon_0}\frac{sdy}{\left(s^2+y^2\right)^{3/2}}$$
$$dE_y = \frac{\lambda}{4\pi\epsilon_0}\frac{-ydy}{\left(s^2+y^2\right)^{3/2}}$$

I have dropped the subscript P to simplify the notation. To get the contribution from all charges, we need to integrate these equations over the integration variable y, from $-L/2$ to $L/2$. By symmetry the integral for the y-component computes to zero.

$$E_x = \int_{-L/2}^{L/2}\frac{\lambda}{4\pi\epsilon_0}\frac{sdy}{\left(s^2+y^2\right)^{3/2}} = \frac{\lambda}{4\pi\epsilon_0}\frac{L}{s\left[s^2+(L/2)^2\right]^{1/2}} \qquad (1.20)$$
$$E_y = 0$$

Therefore, the electric field on the points on the x-axis is pointed along the x-axis as expected from the symmetry in the situation, and has the magnitude E_x. If $\lambda > 0$, then $\vec{E}$ field is pointed away from the charged wire and if $\lambda < 0$, the $\vec{E}$ field is pointed towards the wire as shown in Fig. 1.20.

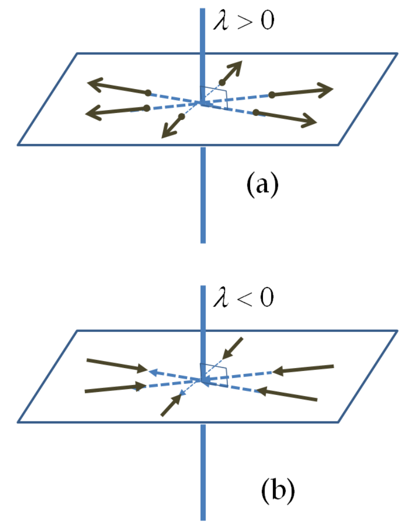

Figure 1.20: The electric field of a line of charges shown for (a) $\lambda > 0$ and (b) $\lambda < 0$. The electric field at some points in one plane are shown.

Now, we will see that the same result can be obtained much more easily if we exploit the symmetry present in the situation.

Use of symmetry:

From the symmetry of the charge distribution, we can see that at any point on the x-axis, there will be no net y- or z-component of the electric field. Therefore we will have only the x-component.

$$dE_x = (dE_P)\cos\theta = \left(\frac{1}{4\pi\epsilon_0}\frac{\lambda dy}{s^2+y^2}\right)\cos\theta \quad \text{[P on the } x\text{-axis]}$$

where dE_P stands for the magnitude of electric field at P from the element charge at dy and θ is the angle electric field of the element dy makes with the x-axis. The expression obtained is the same as the one found by the brute-force method. The multiplier $\cos\theta$ is equal to $s/\sqrt{s^2+y^2}$. Putting this in the expression and integrating over the source charges gives the same result as found by the brute force method albeit with much less effort.

Further Remarks:

Does this result make sense? It is a good idea to check what happens as we go very far away from the charged rod. Physically, we expect the rod to act like a point charge of total charge $q_{tot} = \lambda L$ at points where $s >> L$. Let us see if we get this result when we take $s >> L$ limit in the final answer given in Eq. 1.20. Sure enough, the electric field reduces to that of a point charge at the origin as expected.

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{s^2}; \quad E_y = 0. \quad \text{[P on the } x\text{-axis]}$$

where I have replaced λL by the total charge q_{tot} on the wire.

Example 1.3.6. Electric Field of a Ring of Uniform Charge Density. A metal ring of radius R but negligible thickness carries a uniform line charge density λ. Find the electric field at a point on the axis of the ring.

Solution. Here the ring can be a metal conductor where the charges will spread out uniformly due to symmetry. Let us divide the ring in cells each of arc length $Rd\theta$, where $d\theta$ is the angle the arc makes at the center of the ring. The amount of charge in the element will be $\lambda Rd\theta$. This element will have the electric field $d\vec{E}_P$ in the direction

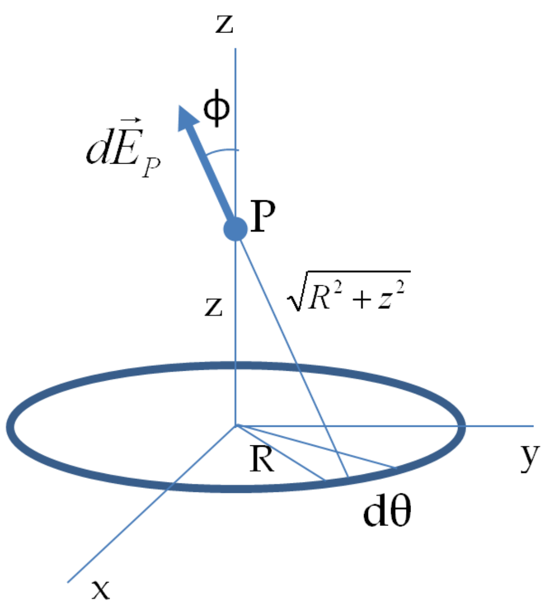

Figure 1.21: Example 1.3.6.

shown in Fig. 1.21 and with following magnitude:

$$dE_P = \frac{1}{4\pi\epsilon_0}\frac{\lambda R d\theta}{R^2+z^2}$$

From the symmetry, we assert that only the z-component will be non-zero. The z-component of $d\vec{E}_P$ is given from the magnitude dE_P and the cosine of the angle ϕ that the vector $d\vec{E}_P$ makes with the z-axis.

$$dE_z = dE_P\cos\phi,$$

where the angle ϕ shown in the figure is given by

$$\cos\phi = \frac{z}{\sqrt{R^2+z^2}}$$

Therefore, the z-component of the electric field by the charges in the element on the ring between θ and $\theta + d\theta$ is

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda R d\theta}{R^2+z^2}\frac{z}{\sqrt{R^2+z^2}}.$$

To calculate the contributions from all charges on the ring we need to integrate this over the ring. This is done by the integration over the angular variable θ from 0 to 2π with the following result.

$$E_z = \frac{z}{4\pi\epsilon_0}\frac{2\pi R\lambda}{(R^2+z^2)^{3/2}}.$$

Therefore, the electric field at a point on the symmetry axis would be

$$\vec{E}_P = \frac{z}{4\pi\epsilon_0}\frac{2\pi R\lambda}{(R^2+z^2)^{3/2}}\hat{u}_z.$$

That is, the electric field at a point P(0,0,z) on the axis of the ring has magnitude $|\vec{E}|$ and pointed away from the center if the product $\lambda z > 0$ and towards the center if the product $\lambda z < 0$ as shown in Fig. 1.22.

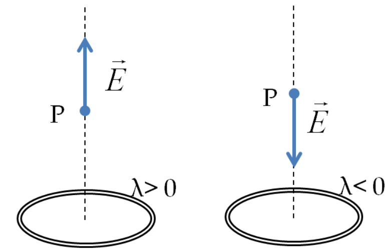

Figure 1.22: Electric field of a ring at a point on the axis.

Example 1.3.7. Electric Field of a Circular Disk Of Uniform Charge Density. A thin disk of radius R has a uniform surface charge density σ. Find the electric field at a point on the axis of the disk.

Solution. We divide the disk in ring-shaped cells and make use of the electric field of a ring worked out in Example 1.3.6. An infinitesimal-width cell between the cylindrical radial coordinates r and $r + dr$ is shown in the Fig. 1.23. The cell has the shape of a ring. We can write the electric field $d\vec{E}$ from the charged in the ring by using the result of the last example.

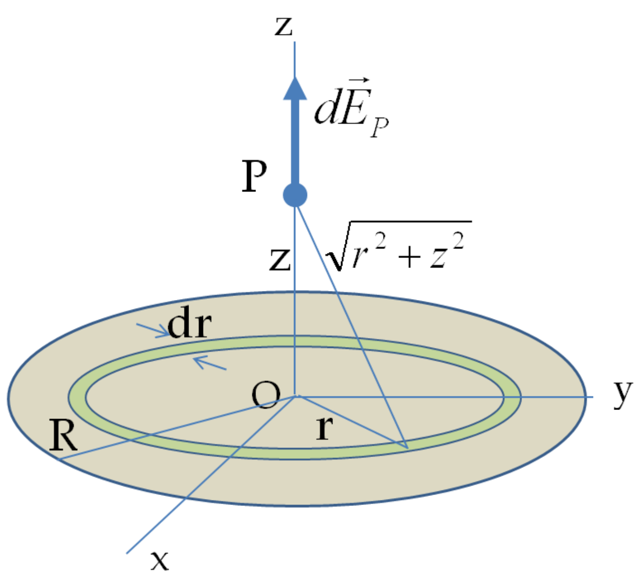

Figure 1.23: Example 1.3.7.

$$
\begin{aligned}
dE_x &= 0 \\
dE_y &= 0 \\
dE_z &= \frac{dq}{4\pi\epsilon_0} \frac{z}{\left(r^2 + z^2\right)^{3/2}} \quad [\text{P on the } z \text{ axis }]
\end{aligned}
$$

where dq is the charge in the ring-shaped cell, and r the radius of the ring. The charge in the ring will be equal to the product of the surface charge density and the area of the surface of the ring.

$$dq = \sigma \times 2\pi r dr.$$

Hence, the electric field on a point on the axis is all along the z-axis and the z-component is given by the following formula.

$$dE_z = \frac{\sigma z}{2\epsilon_0} \frac{rdr}{(r^2 + z^2)^{3/2}}$$

The integration variable is clearly the radius r of the ring-shaped cell. Adding up the contributions of all cells is accomplished by integrating over r from 0 to R, the radius of the disk.

$$E_z = \frac{\sigma z}{2\epsilon_0} \int_0^R \frac{rdr}{(r^2 + z^2)^{3/2}} = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right)$$

This problem illustrates how to use a known solution in a new problem rather than start from scratch. The reader is encouraged to take the $|z| >> R$ limit and confirm that, in this limit, the electric field of the charges on the disk become same as the electric field of a point charge when the field point is very far away compared to the radius of the disk. Why do you think we should expect this result? Note again that $\vec{E}$ at P has magnitude $|E_z|$ and the direction is away from the center of the disk if the product $\sigma z > 0$ and towards the center of the disk if the product $\sigma z < 0$ as shown in Fig. 1.24.

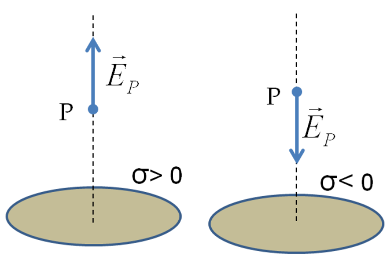

Figure 1.24: Electric field of a disk at a point on the axis.

Example 1.3.8. Electric Field of a Circular Cylinder Of Uniform Charge Density. A circular of radius R and length L has a uniform charge density ρ. Find the electric field at a point on the axis of the cylinder and outside the cylinder.

Solution. We can do this problem by making use of the solution for the disk. How? First we imagine the cylinder as being made of disks stacked on each other. The charges in each disk produce electric field at the field point whose formula is given in the last example. We can then add up these contributions from all the disks vectorially to obtain the final answer.

To implement this strategy let us place the cylinder such that the axis of the cylinder is along the z axis as shown in the figure. The distance from the disk at z' to the field point is $|z - z'|$. Consider one disk of thickness dz' at height z' from the origin. Let the field point be at $(0, 0, z)$ with $z > L/2$ so that the field point outside the cylinder. From the symmetry of the situation, the electric field at the field point P will have only the z-component non-zero. The surface charge density on the surface of the disk at z' will be $\rho dz'$ which is obtained by finding the charge in the volume of the disk and diviiding by the surface area of one surface since in the limit of infinitesimal dz' the thin disk will have only one surface. The the z component of

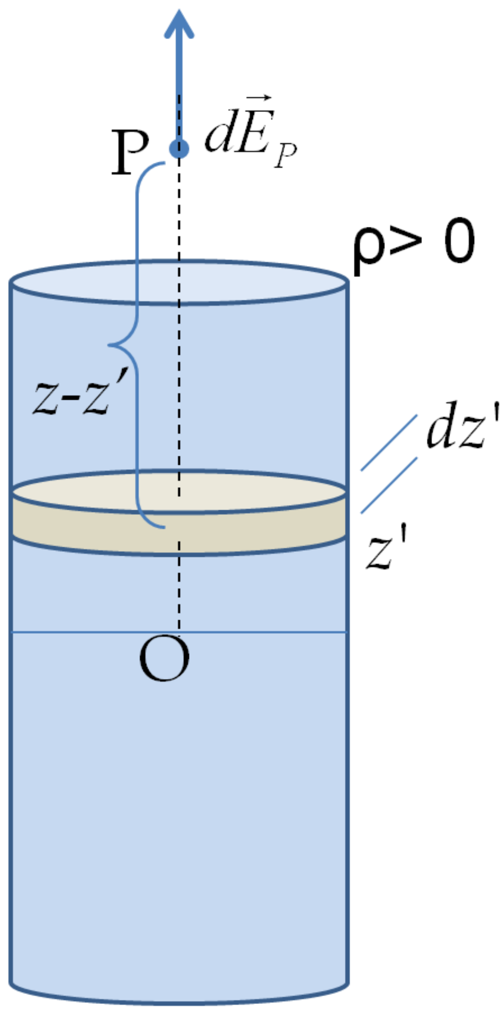

Figure 1.25: Example 1.3.8.

the electric field will be

$$dE_z = \frac{\rho dz'}{2\epsilon_0}\left[1 - \frac{|z-z'|}{\sqrt{R^2+(z-z')^2}}\right],$$

Now, we can integrate this from $z' = -L/2$ to $z' = L/2$ to obtain the contributions of all charges in the cylinder.

$$E_z = \frac{\rho}{2\epsilon_0}\int_{-L/2}^{L/2}\left[1 - \frac{|z-z'|}{\sqrt{R^2+(z-z')^2}}\right]\, dz'.$$

The electric field at the field point is $\vec{E}_P = E_z\hat{u}_z$. I leave the integral for the student as an exercise. Changing the integration variable from z' to $z - z'$ may be helpful.

Example 1.3.9. Electric Field of a Uniformly Charged Sheet. One side of a very large plastic sheet has a uniform charge density σ. Find the electric field at a point near the center and close to the sheet.

Solution. The electric field obtained on the axis of a uniformly charged disk in Example 1.3.7 will also be approximately correct at other points near the center of a large sheet. Furthermore, if the observation point is close to the sheet we will have $|z| << R$. Therefore, electric field near a large sheet of uniform charge density can be obtained from that of a finite disk by taking the $|z| << R$ limit.

$$E_z = \lim_{|z|<<R} \left[\frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right) \right] = \frac{\sigma}{2\epsilon_0} \tag{1.21}$$

The electric field is pointed away from the sheet if the sheet is positively charged and towards the sheet if negatively charged. The electric field of a large sheet near its center is seen to be a constant independent of the distance from the sheet.

This charge distribution provides a region of uniform electric field, which is very useful in experimental physics. The result obtained is also called electric field of an "infinite sheet".

The electric field lines of a uniformly charged large disk are shown in Fig. 1.26. In this figure, the formula given in Eq. 1.21 is applicable near the center of the disk only. Note that the electric field lines at the edges are different from the constant electric field given near the center by Eq. 1.21. The electric field lines shown as curved lines near the edges have to be worked out by a more exact calculation of the electric field at these points.

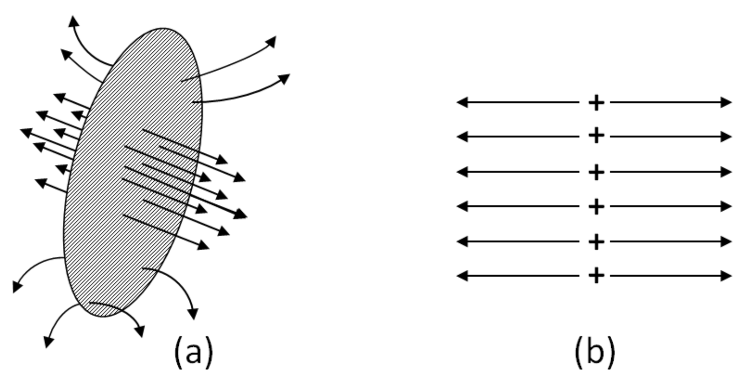

Figure 1.26: (a) The electric field lines of a large sheet of positive charge, (b) the uniform electric field line near the center (side view).

Example 1.3.10. Electric Field of Oppositely Charged Sheets. A most interesting and practically useful case is that of two oppositely charged large sheets of equal amount of charge but of opposite types facing each other with a separation d such that $d << R$, where R is linear dimension such as the radius of a disk-shaped sheet. Assume the surface charge density be $+\sigma$ on the positive plate, and $-\sigma$

on the negative plate. Find the electric field in a region near the center of the plates. Note, we must avoid the edges if are going to use the approximate formula derived above.

Solution. As long as we are near the center of the plates and the distance of the observation point from the plates is much less than the radius of either plates, we can use the approximate electric field formula obtain for a large plate for these regions as shown in Fig. 1.27. One can identify three regions, namely I, II and III, of which

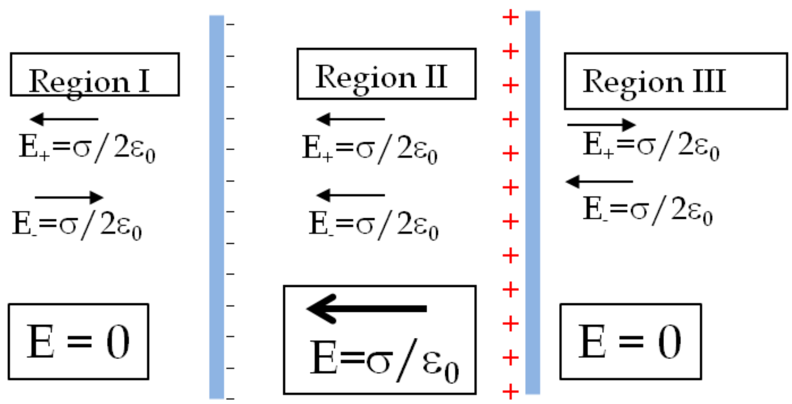

Figure 1.27: Electric field of two sheets of equal but opposite charges at points near the center and close to the plates.

regions I and III are outside the space bounded by the plates, and II in between the plates. From the previous example, we know the electric field of each of the sheets has a magnitude $\sigma/2\epsilon_0$, and the direction as shown in the figure, where E_+ is the magnitude of the electric field of the positive plate and E_- that of the negative plate. We find that the electric fields from the two plates are in the same direction in region II, and therefore they add giving a total of $\sigma/2\epsilon_0 + \sigma/2\epsilon_0 = \sigma/\epsilon_0$. In regions I and III, the electric fields from the two plates are in the opposite directions and therefore they subtract, canceling each other.

Once again, a reminder that the electric field of two oppositely charged parallel plates is not constant near the edges. We have only addressed the electric field near the center and close to the plates. The electric field at the edges, called **fringing fields**, is non-uniform as shown in the Fig. 1.28 for one plate.

1.3.4 Graphical Representation of Electric Field

The graphs of the electric field in a region provide a powerful pictorial view and give an alternative way for understanding the electrical phenomena. Since the electric field is a vector, we need both its

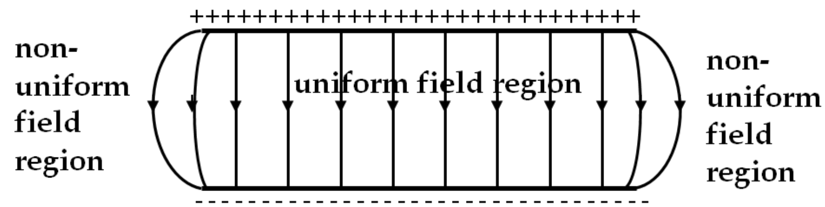

Figure 1.28: Electric field lines of two oppositely charged large plates.

magnitude and direction in space. There are two different ways of presenting the electric field maps pictorially: Electric field Vector maps and Electric field lines map. Each method has its advantages and disadvantages, and we will study examples of each separately in the following.

Electric field vector maps A vector is represented by an arrow of the appropriate size consistent with a chosen scale for the drawing. The size of the arrow corresponds to the magnitude of the vector and the arrowhead points in the direction of the field. For drawing a vector field we need to draw arrows at many points to give an understanding of the strength and direction of the field, and their variation in space. Since electric field drops off as the inverse square of the distance from a charge, it is often difficult to draw the electric field vectors at space points that are far apart in the same diagram.

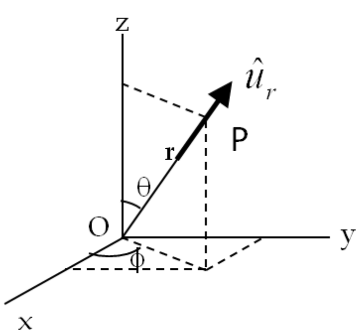

Figure 1.29: Spherical Coordinates.

As an example of the electric field vector map, consider a point charge $+q$ located at the origin. Since the electric field of a point charge is spherically symmetric, it is best written in the spherical coordinates shown in Fig. 1.29. The electric field at a space point P at a distance r from the charge will be

$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{u}_r$$

where $\hat{u}_r$ is the unit vector in the direction from the charge at the origin to the field point P. Using this expression I evaluated the electric field vectors at a number of points near the origin. These vectors are shown to-scale in Fig. 1.30.

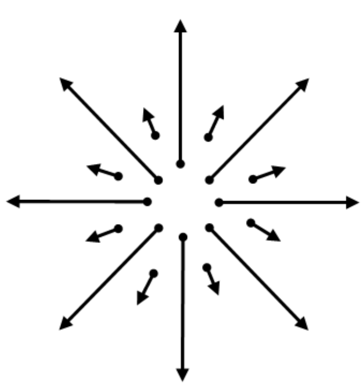

Figure 1.30: Electric Field of a point charge q. The arrows are drawn to scale. As the dis-

The electric field of a single negative charge located at the origin is similar to a positive charge there, except that all the arrows would be reversed in the direction, pointing towards the negative charge. A more complicated picture results in the case of two or more charges.

Electric Field Line Maps The electric field line maps are constructed by drawing the electric field lines that originate either at a positive charge or at infinity and land either on a negative charge or continue to infinity. The tangent to an electric field line at a point corresponds to the direction of the electric field there. The magnitude of the electric field at a space point is represented by the number of lines per unit area passing through an imaginary surface normal to the direction of the lines. Thus, you draw lines closer together in a region of a stronger electric field and further apart in a region of a weaker field. In Fig. 1.31, we illustrate the field lines maps of two and three charge configurations.

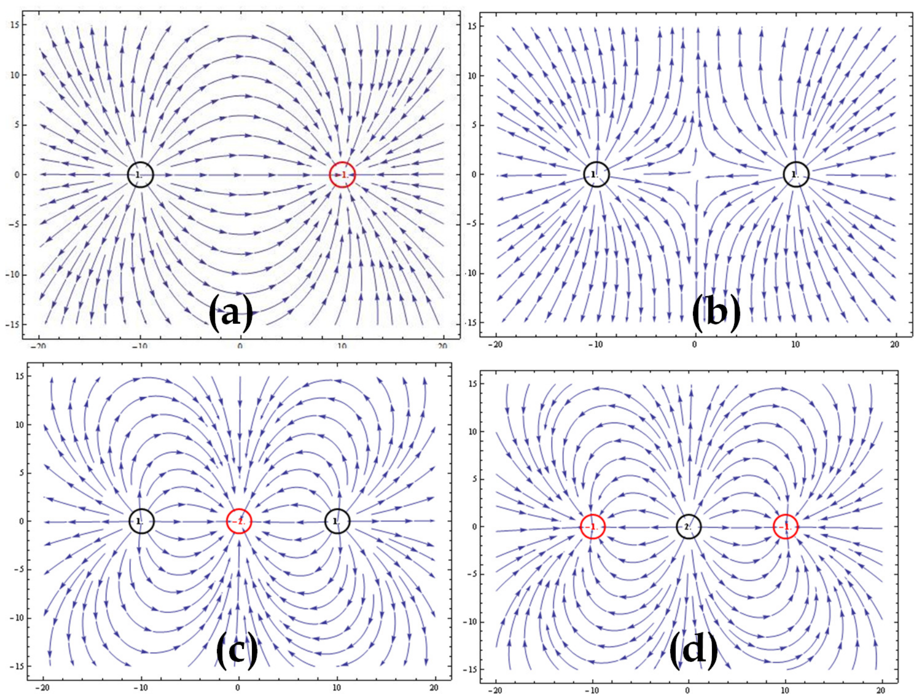

Figure 1.31: Electric field lines: (a) Two oppositely charged particles of equal amount, (b) Two positive charges of equal amount, (c) Three charges, +1C, -2C, +1C, (d) Three charges, -1C, +2C, -1C. Reference: "Electric Fields for Three Point Charges" from the Wolfram Demonstrations Project http://demonstrations.wolfram.com/ElectricFieldsForThreePointCharges/

From the definition of electric field the direction of the electric field at a point in space is the direction of the force on a positive test charge placed there, and since force is a physical quantity with a unique direction determinable by experiment, the direction of electric field must also be unique. Consequently, the electric field lines cannot cross each other, for if they did, that would imply two different directions for the same force which is not physically possible.

Graphical Definition of Electric Field

Using the idea of electric field lines it is possible to introduce a qualitative and more pictorial definition of electric field. According to this approach, we envision space filled by electric field lines from charges according to the following rules.

1. The tangent to an electric field line at a point gives the direction of the electric field at that point.
2. The number of field lines per unit are passing through an imaginary surface normal to the lines corresponds to the magnitude of the electric field.

This graphical definition of electric field is useful in visualizing electric field.