3.5 DRIVEN OSCILLATOR

A damped oscillator will eventually come to rest if no additional energy is supplied to it. To supply energy to the oscillator you could apply any force that will do a net positive work on the oscillator. Among many forces that will do the job, a harmonically time varying force is of particular interest in physics for many applications. A harmonically varying force varies in time sinusoidally with a well-defined frequency ω_d.

$$\vec{F} = \vec{F}_0 \cos(\omega_d t), \tag{3.92}$$

where F_0 is the amplitude of the force. Note that $\omega_d = 0$ corresponds to just a constant force, such as gravity; a constant force will do a positive work in one half of the cycle and the same amount of a negative work in the second half of the cycle, therefore a constant force will not change the energy of the oscillator over one complete cycle.

To be concrete, consider one-dimensional motion of a mass m attached to a spring subject to a spring force $\vec{F}_s$, a viscous force $\vec{F}_{\text{visc}}$, and an applied force $\vec{F}$ as shown in Fig. 3.19. The direction of the velocity at the instant is also shown in the figure so that you can see that the damping force and the velocity are in the opposite direction to each other. A simple physical realization of a damped driven oscillator is shown in Fig. 3.20.

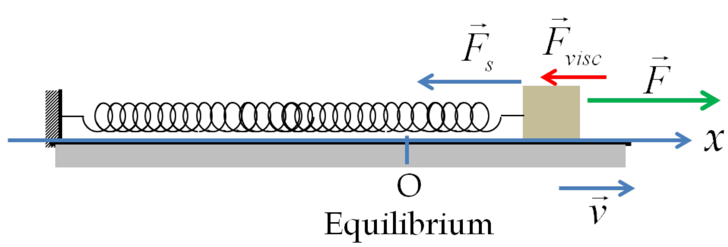

Figure 3.19: Three forces act on a damped driven oscillator: $\vec{F}_s$ is the force by the spring, $\vec{F}_{\text{visc}}$ is the damping force, and $\vec{F}$ is the sinusoidal driving force. The velocity vector is shown to indicate the relative opposite directions of the damping force and the velocity vector.

The block in Fig. 3.19 moves in a straight line about an equilibrium point. We choose the origin of our coordinate system at the location of the equilibrium for the block, and the motion occurs on the x-axis as before. Then x-component of Newton's second law of motion gives the following equation of motion for the x-coordinate of the block.

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \cos(\omega_d t). \tag{3.93}$$

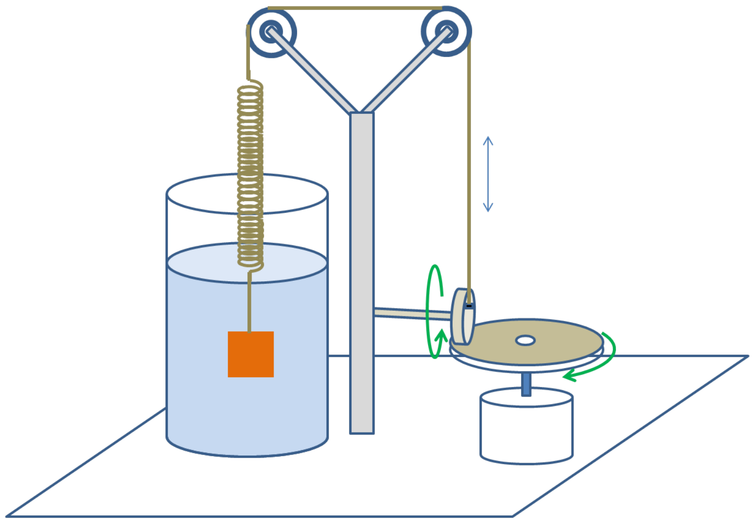

Figure 3.20: A physical realization of the damped driven oscillator, called the "Texas Tower" has been developed by J. G. King at the Education Research Center, Massachusetts Institute of Technology. A variable motor drives the up and down motion of the block in the fluid sinusoidally. The fluid provides the damping force and the spring the restoring force.

Now we divide both sides of the equation by m and define the following composite parameters.

$$\omega_0 = \sqrt{k/m}, \text{ the natural frequency,} \tag{3.94}$$
$$\beta = b/2m, \text{ the damping constant,} \tag{3.95}$$
$$D = F_0/m. \tag{3.96}$$

The quantity D is the acceleration of the block if only the driving force were to act on the system at zero driving frequency. Using the new parameters the equation of motion can be rewritten as follows,

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = D\cos(\omega_d t), \tag{3.97}$$

whose solution describes the motion of the block. The differential equation obtained here is considerably more difficult to solve than the ones we had encountered for the undamped and the viscously damped oscillators.

Since the block is being forced to oscillate at the driving frequency, ω_d, we expect that in the long run, the block should oscillate at this frequency. The solution for a harmonically driven damped oscillator does exhibit this behavior when $t >> 1/\beta$. While in the short time of turning on the driving force, the x-coordinate of the block is a complicated function of time, the long-term behavior is a simple harmonic motion at the driving frequency. The complete solution of Eq. 3.97 that describes the short-term solution accurately is called the

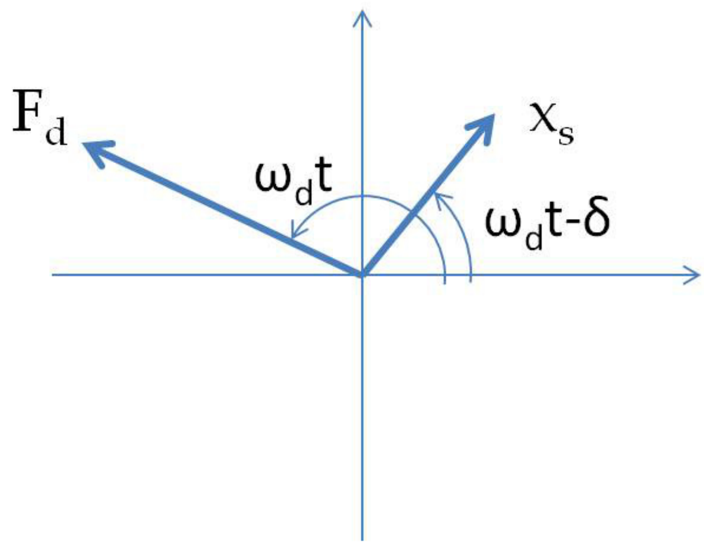

Figure 3.21: Phase lag of displacement with respect to the driving force shown here using the circular motion analogy for phase given earlier in the chapter.

transient solution. The long-term solution, where the transients have died out and the block executes a simple harmonic motion, is called the **steady state solution**.

When the block reaches a steady state, it just oscillates at the frequency of the driving force with an amplitude and phase that depend upon the frequency of the driving force, among other parameters. We will state here the steady state solution without actually solving the equation of motion. Even though we do not solve the equation, various aspects of the solution are important for us to discuss here. The steady state solution of Eq. 3.97 can be written either as a cosine or a cosine with a phase constant or a mixture of sine and cosine functions. To be specific, let us write the solution as a cosine with a phase constant.

$$x_s(t) = A\cos(\omega_d t - \delta). \tag{3.98}$$

Other ways of writing the same answer are: $A'\sin(\omega_d t - \delta')$ and $C_1\cos(\omega_d t) + C_2\sin(\omega_d t)$. Recall that the driving force is $F_x = F_0\cos(\omega_d t)$. Therefore, the phase constant δ of the displacement x_s represents the phase lag of the displacement with respect to that of the driving force as shown by using the circular motion analogy for the phase constant in Fig. 3.21.

The value of the phase constant says the relative position of the displacement and the driving force in their own cycles. For instance, if $\delta = 0$, then the displacement of the block and the driving force are synchronized in the sense that the when displacement is at the peak then so is the driving force and when the displacement is at the trough so is the driving force. If $\delta = \pi$, the displacement of the block is pointed in opposite direction to that of the driving force, and if $\delta = \pi/2$, then the displacement is quarter cycle behind the force.

You can solve for the two unknowns A and δ in the solution given in Eq. 3.98 by putting the solution into Eq. 3.97. The algebra is tedious but do-able. You should attempt to show that one would get the following expressions for the amplitude and phase constant.

$$A = \frac{D}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + \left(2\beta\omega_d\right)^2}} \tag{3.99}$$

$$\tan\delta = \frac{2\beta\omega_d}{\omega_0^2 - \omega_d^2} \tag{3.100}$$

3.5.1 Steady state and resonance

In the steady state, the oscillator oscillates between $x = -A$ and $x = A$ at the angular frequency ω_d. The solution for A given above shows that the amplitude A of the motion varies with the strength of the force through $D = F_0/m$, as you would expect, but also, more importantly, with the frequency ω_d of the harmonic driving force. That is, you can get different amplitudes of oscillation for the same magnitude of the force F_0 if you apply the fore at a different frequency.

Fig. 3.22 shows the variation of A with respect to the driving frequency. The figure illustrates visually that when you vary the frequency ω_d of the driving force, the steady state amplitude of the oscillations of the block would change. We see that the amplitude takes its maximum value at a particular value of ω_d. This special frequency is called the **resonance frequency**, which we will denote by ω_R. To find the expression of the resonance frequency in terms of other quantities, we can apply Calculus to the function $A(\omega_d)$ and set its first derivative to zero and solve for ω_d. This ω_d will equal the resonance frequency ω_R.

$$\left.\frac{dA}{d\omega_d}\right|_{\omega_d=\omega_R} = 0. \tag{3.101}$$

Rather than do the calculation indicated in this equation, we note that the extrema of A with respect to ω_d will occur at the same place as the extrema of the quantity inside the radical in the denominator of A given in Eq. 3.99. This provides a much simpler calculation for the same result.

$$\left.\frac{d[(\omega_0^2 - \omega_d^2)^2 + (2\beta\omega_d)^2]}{d\omega_d}\right|_{\omega_d=\omega_R} = 0. \tag{3.102}$$

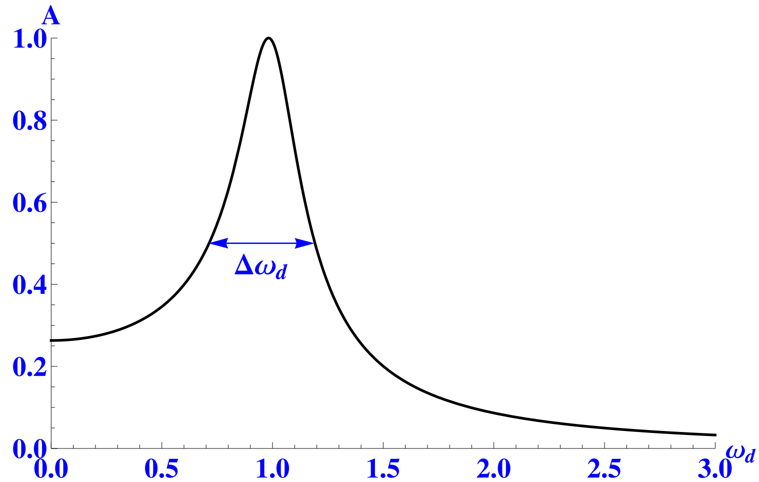

Figure 3.22: The amplitude of the steady state oscillations as a function of driving frequency. Here the amplitude A divided by the amplitude at the resonance A_R is plotted along y and driving frequency in units of ω_0 is plotted along x-axis for an oscillator of damping constant $\beta = 0.1\omega_0$. The width at half-height $\Delta\omega_d$ characterizes the sharpness of the peak.

We can solve this equation for ω_R. Keeping only the positive root, we find the following expression for the resonance frequency.

$$\boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}.} \tag{3.103}$$

The frequency $\omega_d = \omega_R$ is not only an extremum of $A(\omega_d)$ but also its maximum as you can verify by finding the sign of the second derivative of A with respect to ω_d for $\omega_d = \omega_R$. Therefore, the oscillator will vibrate with the largest amplitude if it driven at this frequency.

Equation 3.103 shows that, normally, the resonance frequency is below the natural frequency of the oscillator. However, for a lightly-damped oscillator ($\omega_0 >> \beta$) the resonance frequency will be very near its natural frequency, $\omega_0 \equiv \sqrt{k/m}$.

$$\text{Lightly damped oscillator:}\ \ \omega_R \approx \omega_0. \tag{3.104}$$

Therefore, in a lightly-damped oscillator, an experimental determination of the resonance frequency gives a good indicator of the natural frequency of the system. What happens if the oscillator is driven at the resonance frequency? We can find the answer by directly putting $\omega_d = \omega_R$ in Eqs. 3.99 and 3.100. Let us denote the values of the amplitude and phase lag at the resonance by A_R and δ_R respectively.

$$A_R \equiv A(\omega_d = \omega_R) = \frac{D/2\beta}{\sqrt{\omega_0^2 - \beta^2}} \tag{3.105}$$

$$\tan\delta_R \equiv \delta(\omega_d = \omega_R) = \frac{2\beta\omega_R}{\omega_0^2 - \omega_R^2} = \frac{\pi}{2}. \tag{3.106}$$

Therefore, at the resonance, the position of mass and the driving force are 90-degrees out of phase with each other. This says that at the resonance, while the driving force varies as $\cos(\omega_R t)$, the position of the block varies as $\sin(\omega_R t)$.

$$x_s\big|_{\omega_d=\omega_R} = A_R\ \cos\left(\omega_R t - \frac{\pi}{2}\right) = A_R\ \sin\left(\omega_R t\right). \qquad (3.107)$$

The occurrence of the largest amplitude for the oscillator at the resonance implies that the external agent is able to transfer energy to the oscillator most effectively under these conditions. This is seen quite readily in a swing when you try to make the person in the swing to oscillate at larger amplitudes. You must time the push on the swing properly in order to make it oscillate with a large amplitude.

The resonance phenomenon can also cause serious problems in physical structures. Gusty winds can drive buildings and other structures to swing, and if the gusts have a frequency component that matches with the resonance frequency of the structure, the resonance can then drive the structure to larger amplitude oscillations. A famous disaster as a result of gusty winds happened in 1940 when Tacoma Narrows Bridge in Washington state broke under large amplitude oscillations, which may have been caused by resonance. To prevent the resonance to take place, engineers use dampers to critically damp large buildings.

The resonance phenomenon is also important in the microscopic world. For instance, we use the resonance of nuclear moments of protons in the nuclear magnetic resonance (NMR) to investigate the structure of matter and for medical diagnostic applications. Similarly, in lasers we use resonance to amplify the electromagnetic field inside the cavity.

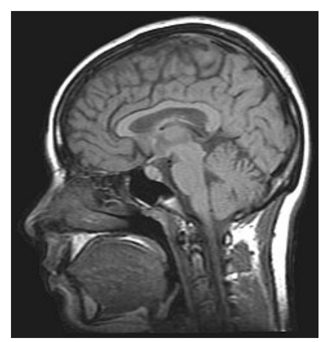

Figure 3.23: Nuclear magnetic resonance is used in the Magnetic Resonance Imaging to look inside human body. The inside picture of a brain tells activities in various parts of the brain. Picture credits: Wikicommons.

3.5.2 Role of Damping in Resonance

We have introduced the Quality factor $Q = \omega_0/2\beta$ to characterize the effect of damping on harmonic oscillations. Therefore, let us rewrite our formulas for the amplitude A and the phase lag δ in terms of the quality factor Q by substituting β in terms of Q using $\beta = \omega_0/2Q$ in the Eqs. 3.99 and 3.100.

For calculations, it is convenient to define a dimensionless frequency by dividing the frequency ω_d of th driving force by the natural frequency ω_0 of the oscillator. We introduce a dimensionless parameter $\Omega = \omega_d/\omega_0$ that will vary with ω_d.

$$\Omega \equiv \frac{\omega_d}{\omega_0}$$

It is helpful in the calculations to replace ω_d by $\Omega\omega_0$. We will see that ω_0 will cancel out from the expressions. After some algebra, which is left as an exercise for the student and is highly recommended to the student, we find that

$$A = \frac{D}{\omega_0^2}\frac{1}{\sqrt{(1-\Omega^2)^2+\Omega^2/Q^2}}, \tag{3.108}$$

$$\tan\delta = \frac{\Omega/Q}{1-\Omega^2}, \tag{3.109}$$

The resonance frequency in the units of ω_0 takes the following form.

$$\Omega_R \equiv \frac{\omega_R}{\omega_0} = \sqrt{1-\frac{1}{2Q^2}}. \tag{3.110}$$

We plot Eqs. 3.108 and 3.110 in Fig. 3.24, and Eqn. 3.109 in Fig. 3.25. The resonance peaks of the amplitude versus frequency shows that the resonance becomes taller and sharper with the lowering of damping which is the same as increasing of the Quality factor Q. The resonance peak does not coincide with the natural frequency ω_0 as seen in Fig. 3.24(b) but approaches ω_0 as seem from $\Omega_R \to 1$ when $Q \to \infty$. For good oscillators with low damping we can usually take the resonance frequency to be equal to the natural frequency.

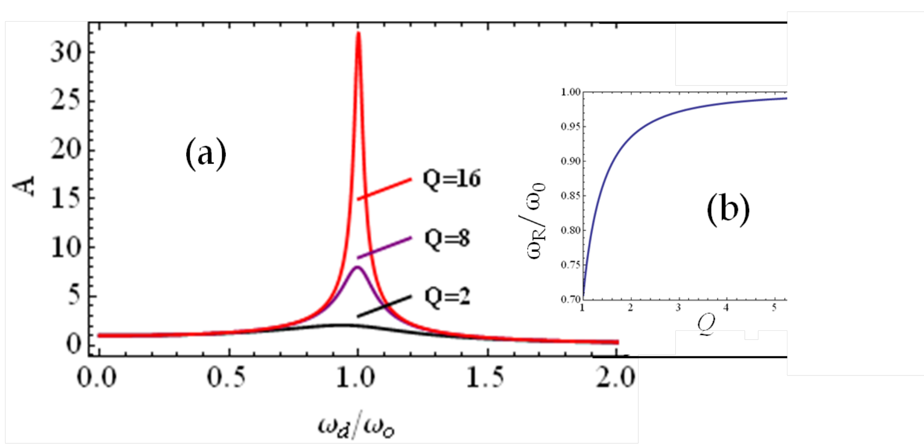

Figure 3.24: The Resonance for different Q. (a) Plot of amplitude A as a function of driving frequency for Q = 2, 4, and 32. (b) The resonance frequency ω_R, here the ratio $\Omega_R = \omega_R/\omega_0$ as a function of Q shows that resonance frequency tends to the natural frequency ω_0 as Quality Q of the oscillator rises.

Figure 3.25 shows the variation of phase lag for different Quality factor. The phase lag goes from zero to π when the driving frequency goes from low frequencies to high frequencies with a transition at the resonance frequency. The transition at resonance frequency becomes sharper for higher Q oscillators. For an undamped oscillator, the transition is a step function.

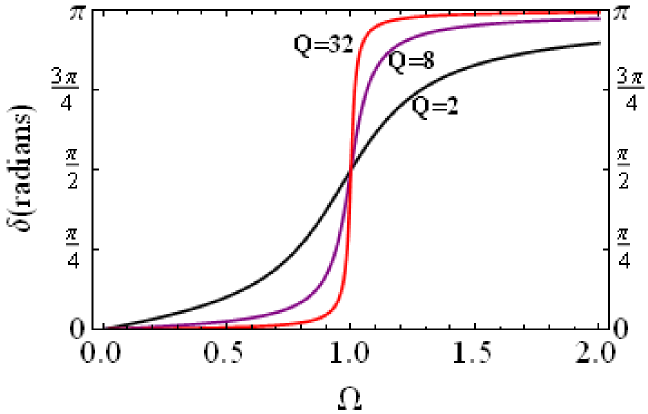

Figure 3.25: The phase lag as a function of driving frequency for different Q.

3.5.3 Power of the Driving Force

The driving force must do work to overcome the dissipation of energy due to the friction of the damping force in order to drive a steady motion of the oscillator. The power expended by the driving force is equal to the driving force times the velocity of the oscillator. We use the situation in the steady state and the steady state velocity to obtain the following instantaneous power $P(t)$ by the driving force.

$$\begin{aligned}
P(t) &= \vec{F}_d \cdot \vec{v} \\
&= F_0 \cos(\omega_d t) \frac{dx_p}{dt} \\
&= F_0 \cos(\omega_d t) \frac{d}{dt} \left[A \cos(\omega_d t - \delta)\right] \\
&= -F_0 \omega_d A \cos(\omega_d t) \sin(\omega_d t - \delta) \\
&= \frac{F_0 \omega_d A}{2} [\sin\delta + \sin\delta \cos(2\cos\omega_d t) \\
&\qquad - \cos\delta \sin(2\cos\omega_d t)] \qquad (3.111)
\end{aligned}$$

What is important is power expended by the force over a full cycle - this is the energy that the driving force must supply per cycle to maintain the steady motion of the oscillator. The average power P_{av} is obtained by integrating Eq. 3.111 over one cycle. The integration over the cosine and sine in Eq. 3.111 will give zero leaving only the contribution from the constant term.

$$\begin{aligned}
P_{av} &= \frac{\omega_d}{2\pi} \int_0^{2\pi/\omega_d} P(t) dt \\
&= \frac{1}{2} F_0 \omega_d A \sin\delta, \qquad (3.112)
\end{aligned}$$

The average power is also a function of the driving frequency. Let us write the full expression for the average power using explicit expressions for $A(\omega_d)$ and $\delta(\omega_d)$. After much algebra, it can be shown that

$$P_{av} = \left(\frac{\beta F_0^2}{m}\right) \frac{\omega_d^2}{\left(\omega_0^2 - \omega_d^2\right)^2 + (2\beta\omega_d)^2}. \tag{3.113}$$

Unlike the amplitude A of the displacement of the oscillator, the average power of the driving force is maximum when $\omega_d = \omega_0$ regardless of the value of the damping constant as shown in Fig. 3.26. Often the resonance of power is taken to mean as the resonance phenomenon rather than the resonance of the amplitude A. Note that there are differences in the resonance of the average power of the driving force and the resonance of the amplitude of the displacement of the oscillator from the equilibrium. The resonance of the power is important experimentally, since in many experimental settings, it is easier to measure power than the displacement.

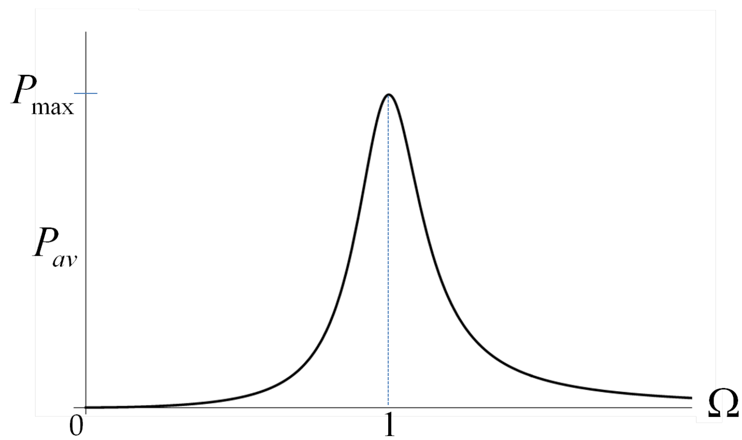

Figure 3.26: The power curve of driven oscillator. The resonance of power occurs at $\Omega = \omega_d/\omega_0 = 1$. The maximum power at the resonance has the value $P_{\max} = F_0^2/4\beta m$. The width of the curve at half height of the peak, denoted as $\Delta\Omega_{1/2}$ is approximately 2β. and the width for a lightly damped oscillator is determined by the damping constant $2\beta/\omega_0$.