10.5 EXAMPLES: CHANGING MAGNETIC FIELD

In this section I will illustrate the use of Faraday's flux rule in the situations of changing magnetic field. The other ways for changing the magnetic flux will be discussed in the next section. To begin the discussion we start with reviewing the calculation of flux of fields.

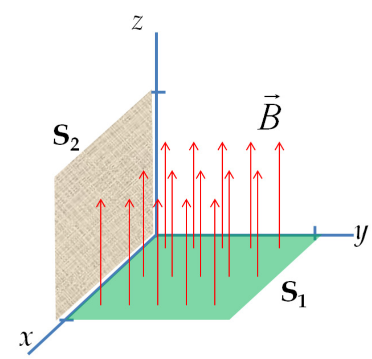

Figure 10.12: Example 10.5.1.

Example 10.5.1. Flux of Constant Magnetic Field # 1. A constant magnetic field $B_0\hat{u}_z$ is present in a region as shown in Fig. 10.12. Calculate the magnetic flux through two flat surfaces of area A (a) in the xy-plane (surface S_1) and (b) in the xz-plane (surface S_2).

Solution. (a) The given magnetic field is pointed towards the positive z-axis. The normal to an area in the xy-plane is along the z-axis. The unit vector normal to the xy-plane will be either $\hat{u}_z$ or $-\hat{u}_z$, which we can write as $\pm\hat{u}_z$. The magnetic flux is

$$\begin{aligned}\Phi_B &= (B_0\hat{u}_z)\cdot(\pm\hat{u}_z A)\\ &= \pm B_0 A\end{aligned}$$

(b) The normal to the area in the xz-plane is $\pm\hat{u}_y$. The dot product between the magnetic field and the normal now equals zero. Therefore, the flux of the magnetic field pointed towards the positive z-axis through an area in the xz-plane will be zero. A similar argument will show that the flux of the given magnetic field through an area in the yz-plane will also be zero.

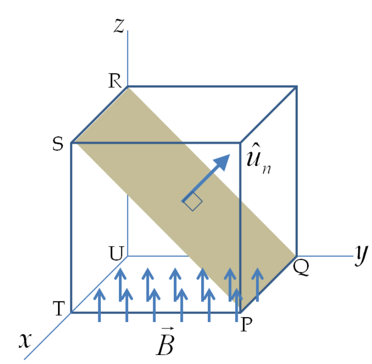

Figure 10.13: Example 10.5.2.

Example 10.5.2. Flux of Constant Magnetic Field # 2. Given a constant magnetic field of magnitude B_0 in the positive z-axis direction find the magnetic flux through the surface PQRS inside the cube as shown in Fig. 10.13.

Solution. Recall that a dot product also has a meaning in terms of the projection of vectors. For instance, $\vec{B}\cdot\vec{A}$ is equal to the product of the magnitude $|\vec{B}|$ and the projection of the vector $\vec{A}$ in the the direction of the vector $\vec{B}$. Here, the projection of the area denoted in the figure as PQRS in the direction of magnetic field is equal to the area at the base, viz. PQUT through which magnetic field lines pass perpendicularly. Therefore, the flux through PQRS is equal to the flux through PQUT.

$$\Phi_B = B_0 \times \text{Area of PQUT.}$$

Example 10.5.3. Flux of Inhomogeneous Magnetic Field. Magnetic field of a current in a wire is inhomogeneous and varies over

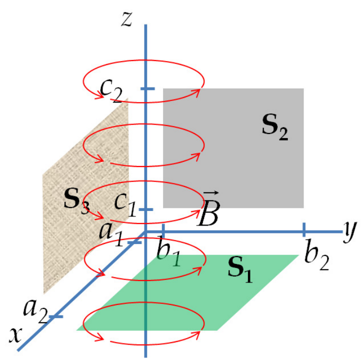

Figure 10.14: Example 10.5.3. The magnetic field lines are shown circulating about z axis.

space. In the Cartesian coordinate system shown in Fig. 10.14, the magnetic field of the current I towards the positive z-axis in a long straight wire is given by

$$\vec{B} = \frac{\alpha}{x^2+y^2}\left(-y\hat{u}_x + x\hat{u}_y\right),$$

where $\alpha = \mu_0 I/2\pi$. Find the flux through the surfaces (a) S_1, (b) S_2, and (c) S_3 shown in the figure.

Solution. Since the magnetic field varies over an area, we will think in terms of area element dA of each surface.

(a) The normal $\hat{u}_n$ to surface S_1 is along the z-axis. Therefore, the dot product of $\vec{B}$ with $\hat{u}_n$ will vanish, which makes the flux through S_1 zero.

(b) The normal $\hat{u}_n$ to surface S_2 along the x-axis, giving $\hat{u}_n = \pm\hat{u}_x$. The area element on S_2 in the yz-plane is

$$d\vec{A} = \pm dydz\hat{u}_x.$$

The surface S_2 has $x = 0$. The magnetic flux through S_2 will be given by the following integral based on the general formula.

$$\begin{aligned}\Phi_B &= \iint \vec{B}\cdot d\vec{A} \\ &= \mp\alpha \int_{y=b_1}^{y=b_2}\int_{z=c_1}^{z=c_2}\left[\frac{y}{x^2+y^2}\right]_{x=0} dydz \\ &= \mp\alpha\left(c_2-c_1\right)\ln\left(\frac{b_2}{b_1}\right)\end{aligned}$$

(c) A similar calculation as part (b) will give the following for the magnetic flux through S_3.

$$\Phi_B = \pm\alpha\left(c_2-c_1\right)\ln\left(\frac{a_2}{a_1}\right)$$

Example 10.5.4. Induced current in a varying magnetic field. A circular loop of a copper wire of radius a and resistance R is placed in a uniform magnetic field whose direction is normal to the plane of the loop. While the direction of the magnetic field does not change with time, its magnitude changes with time according to $|\vec{B}(t)| = B_0 \exp\left(-\alpha t\right)$, where B_0 is the magnitude at $t = 0$. Find the magnitude and direction of the induced current in the copper loop.

Solution. From Ohm's law we know that the induced current will be equal to the induced EMF in the loop divided by the equivalent resistance of the loop of wire. To find the induced EMF, we need to evaluate the rate of change of the magnetic flux through the loop. Here the flux is changing due to the changing magnetic field whose analytic form has been provided. Therefore, we find the rate of change of the magnetic flux by taking the the time derivative of the magnetic flux.

Since the magnetic field is perpendicular to the area, the magnetic flux will be simply a product of the magnitude of the magnetic field and the area up to a sign. We will ignore the sign in our calculations of the magnitude of the induced current. The sign is related to the direction of the induced current, which we will obtain by applying Lenz's law.

We will calculate the absolute value of the magnetic flux through the loop since we are gong to work with magnitudes first and we will find the direction of the induced current from Lenz's law. The magnetic flux has the following magnitude here.

$$|\Phi_B| = \pi a^2 B_0 \exp{(-\alpha t)}.$$

The induced EMF in the loop will be

$$\begin{aligned}\mathcal{E} &= \left|\frac{d\Phi_B}{dt}\right| \\ &= \pi a^2 \alpha B_0 \exp{(-\alpha t)}\end{aligned}$$

Dividing the magnitude of the induced EMF by the equivalent resistance of the loop will give the magnitude of the induced current.

$$I = \frac{\mathcal{E}}{R} = \frac{\pi a^2 \alpha B_0}{R} \exp{(-\alpha t)}.$$

We now utilize Lenz's law to determine the direction of the induced current. Let the direction of magnetic field be pointed up. The operational steps for the application of the Lenz's law for this problem are shown in Fig 10.15. We start by noting that, since the magnitude of the magnetic field decreasing with time, the flux of the magnetic field of the type pointed-up would be decreasing with time. Therefore, according to the Lenz's law, the magnetic flux of the magnetic field of the induced current must add to the magnetic flux of the pointed-up type field. That is, the magnetic field of the induced current should be pointed up at points inside the loop. This gives the direction of counter-clockwise for the induced current as looked from above as shown in Fig. 10.15.

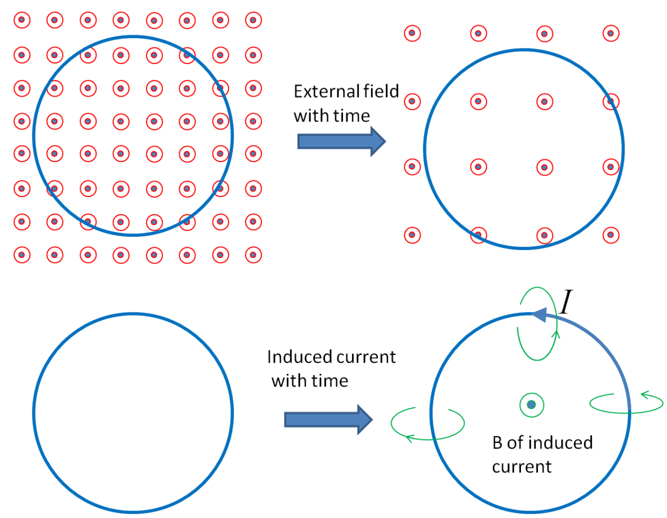

Figure 10.15: Direction of the induced current by Lenz's law. We use the fact that the magnetic field of the induced current opposes the change in the magnetic flux through the space of the loop. The top figures show that the magnetic flux for magnetic field pointed out-of-page is decreasing with time. The bottom figures show the induced current tending to compensate the loss of the magnetic flux by producing magnetic field in the direction in which the new magnetic field can mitigate the loss.

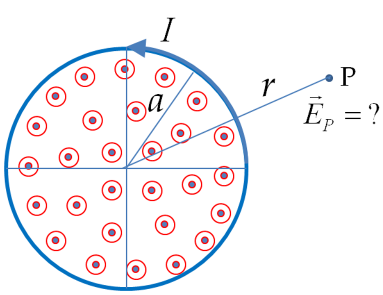

Figure 10.16: Example 10.5.5. For clarity only a cross-section of the solenoid is shown here.

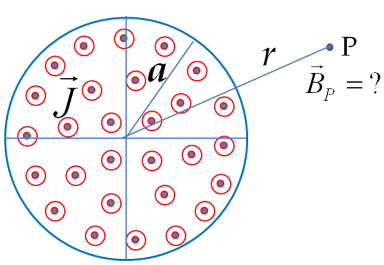

Figure 10.17: Example 10.5.5. The magnetic field from a uniform current in a wire.

Example 10.5.5. Electric field from a varying magnetic field. A long cylindrical solenoid of radius a having n turns per unit length carries a time-dependent current $I(t)$. Find electric field at a point P outside the solenoid as shown in Fig. 10.16.

Solution. Recall that the current in a solenoid produces a uniform magnetic field inside the solenoid of the magnitude $\mu_0 n I$ and zero outside the solenoid. The direction of the magnetic field depends on the current direction given by the right-hand rule of Biot-Savart's law: if you look down the axis, then the direction of magnetic field will be towards you if the current loops around in a counter-clockwise direction as shown in Fig. 10.16.

From the analogy of Ampere's law and Faraday's law discussed above, we note that the problem here is analogous to the Ampere's law problem of finding magnetic field outside a straight wire that carries a uniform volume current density $\vec{J}$ as shown in Fig. 10.17.

Therefore, in the present situation we can claim that the electric field will circulate in circles around the magnetic field just as the magnetic field of a straight wire circulates around the current. Following the Ampere's law procedure for a straight wire, we choose an "Amperian loop" in the shape of a circle that passes through the field point P as shown in Fig. 10.18. Now, we equate the circulation of electric field around the Amperian loop with the "enclosed" rate of change of magnetic flux through the Amperian loop.

Circulation of $\vec{E} = -$ Enclosed rate of change of magnetic flux.

Since the electric field has the same amplitude at all points of the circle the circulation is just a product of the amplitude of the electric field and circumference of the circle of radius r, which is the distance of field point P from the axis of solenoid. The enclosed magnetic flux is also easy to calculate here since the magnetic field is zero outside the solenoid.

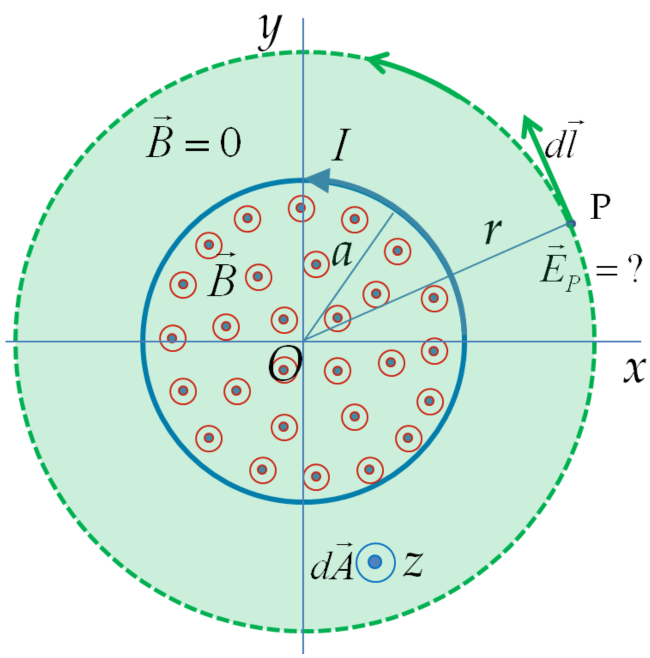

Figure 10.18: Example 10.5.5. The Amperian loop passes through the field point and the magnetic flux is calculated over the area enclosed by the Amperian loop shown shaded here. The direction of the Amperian loop and area vector are related by the right-hand rule given in Fig. 10.11. Although there is no magnetic field at P, there is an electric field there due to the changing magnetic field inside the solenoid.

To include the direction information, we cast the circulation and magnetic flux in terms of components. The right-hand rule for Faraday's law shows that, if the direction of the loop for the circulation is taken along $\hat{u}_\phi$, then the direction of the area element will be towards the positive z-axis as shown in Fig. 10.18. This will give the following for the ϕ-component of electric field.

$$(2\pi r)\, E_\phi = -\left(\mu_0 \frac{dI}{dt} n\right)\left(\pi a^2\right),$$

where the minus sign on the right side comes from the minus sign in the Faraday's law. Solving for E_ϕ we find

$$E_\phi = -\frac{\mu_0 n a^2}{2r}\frac{dI}{dt}.$$

Therefore, the electric field induced at a point P outside the solenoid is

$$\vec{E}_P = -\frac{\mu_0 n a^2}{2r}\frac{dI}{dt}\hat{u}_\phi.$$

We find that the induced electric field at the space point P circulates clockwise for an increasing current ($dI/dt > 0$) and counter-clockwise for a decreasing current ($dI/dt < 0$). If you place a circular metal wire of radius r which coincides with the Amperian loop for the circulation of $\vec{E}$ calculation, you will notice that there is an induced current in the direction of the induced electric field. Also, if you place a charge anywhere outside the solenoid, the charge will experience an electric force in the axial direction when the current in the solenoid is changing, but no such force when the current in the solenoid is steady.

The result obtained here implies a rather strange conclusion: the electric field at a point outside of a solenoid with a time-varying current is not zero, although the magnetic field there is zero! Note that this takes place only when the current in the solenoid is changing; once the current has become steady, there would be no electric field outside the solenoid.

Electric field inside solenoid

To find electric field induced at a point inside the solenoid we will choose Amperian loop with radius $r < a$ as shown in Fig. 10.19. The enclosed magnetic flux will be less than the flux through the entire cross-section of the solenoid - instead, the magnetic flux through only an area πr^2 will be enclosed within the Amperian loop. This gives the electric field that increases with the distance from the axis inside the solenoid.

$$\vec{E}_P = -\frac{\mu_0 n r}{2}\frac{dI}{dt}\hat{u}_\phi \quad \text{(at a point P inside).}$$

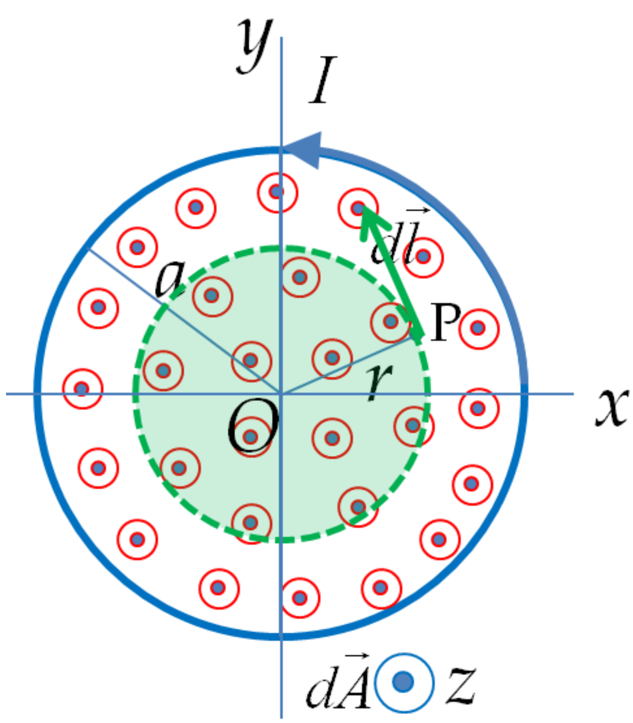

Figure 10.19: The Amperian loop and are enclosed (shaded) for a point inside the solenoid.