5.2 SOLVING STATICS PROBLEMS

Let us consider simple examples that help us understand the use of the static equilibrium in "ideal" undeformable objects. No object in reality is an undeformable object. But, if the applied forces are not large enough to overwhelm the internal forces, then, the "ideal" system analysis gives us most of the physics and is an important first step in any stability analysis. The method for solving static problems follow a few basic steps outlined below.

1. Pick the system. It can be either the entire structure, a part of the structure, or some small mass element inside the structure. The choice depends on the physical question under study.

2. Identify forces on the system by the external agents, making particular note of where they act and what their directions are. These usually fall into two categories:

   (a) Action-at-a-distance forces, usually gravity in our case, which can be taken to act at the center of mass of the system, and

   (b) Contact forces, usually a combination of normal and static frictional force but there may be other forces as well.

3. Draw a free-body diagram and choose x and y-axes judiciously for generating the F_x and F_y equations.

4. For the torque equation, you will need to go back to the original figure where you have both the location and the direction of forces. Pick a reference point O, so that torques are easy to calculate. A good location for O usually makes some of the torques zero for forces passing through O, especially for forces that are unknown. Use lever-arm method rather than the cross-product to calculate torques, and employ clock-wise and counter-clockwise sense of torques to assign plus and minus signs to the individual components of the torques. The algebraic sum of all torques is then set to zero.

5. The three equations given in Eq. 5.5-5.7 are then solved together. you may not need all three equations for every problem, so you would need to understand what you need to do for the particular problem at hand.

We will repeat the examples of this method presented in the chapter on forces. If you have already studied them, then this will be just a review. But, if you skipped those parts of the book, then here is your chance to study them before you proceed to more advanced topics in this chapter.

Example 5.2.1. Safety on a ladder. A ladder of length L and of mass M slides when the ladder leans against a frictionless wall at an angle greater than θ from the ground. What is the minimum static frictional coefficient between the ground and the ladder?

Solution. Let us first draw a picture of the situation and locate all the forces on the ladder. Then we will extract a free-body diagram from this diagram for writing out the equations corresponding to the balance of the external forces. To calculate the torque of the forces, the free-body diagram is not useful. For the torque calculation, we will go back to the original force diagram and choose a convenient pivot point. The choice of pivot point is arbitrary since the net force is zero.

The diagram of forces on the ladder shows various external forces on the ladder and the place those forces act on the ladder. This is given in Fig. 5.1. Note that there are two normal forces on the ladder, one from the floor and the other from the wall. Similarly, there are two frictional forces on the ladder, one from the floor and the other from the wall. The problem states that the frictional force from the wall is zero.

One more thing to note here is that I had make a decision about the direction of the static frictional force. Since, the ladder is not accelerating with respect to the ground, I anticipated that this frictional force must balance the normal force from the wall, and that is why I choose the frictional force to be in that direction.

In the free-body diagram of forces, we place the forces "coming out" from the same place even though the forces actually act at different points of the body. The free-body diagram of the forces and the direction of the axes are shown in Fig. 5.2 where we also show the directions of the positive x and y-axes. As you may already know that we do not need to pick the position of the origin for a free-body diagram since we use the free-body diagram to write the components of forces with respect to the axes.

Since the acceleration of the ladder is zero, the following equations are immediately found from the free-body diagram. **Note that the relations that include static frictional forces are inequalities, not equalities since the frictional force can take values up to a maximum value.** In the present problem, we will write an equality since the problem states that maximum frictional force should be

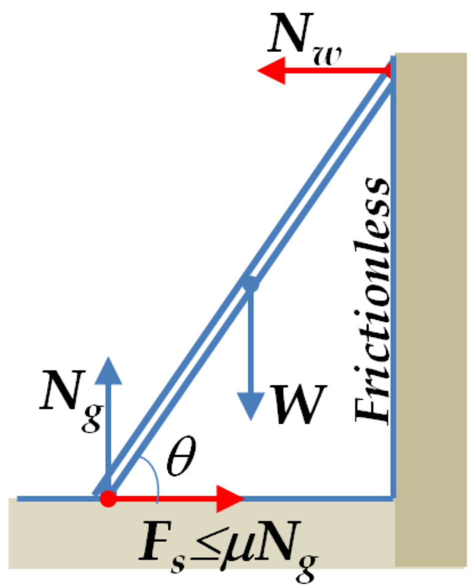

Figure 5.1: Example 5.2.1.

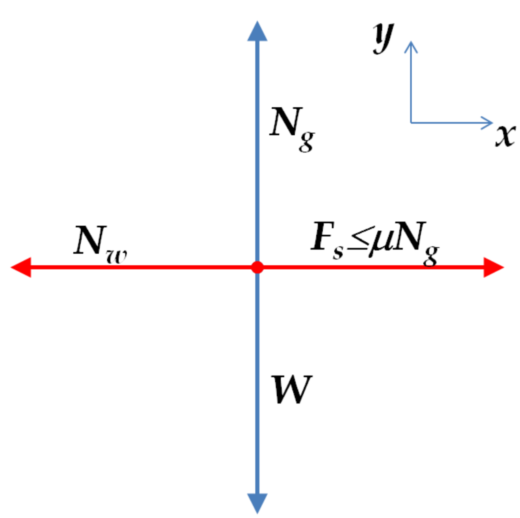

Figure 5.2: Example 5.2.1.

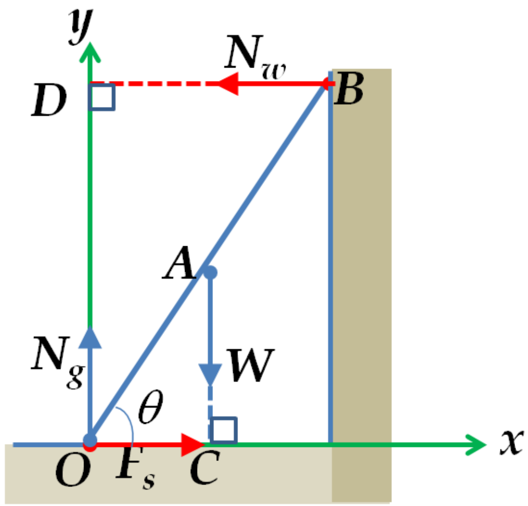

Figure 5.3: Example 5.2.1.

acting since we are looking for the minimum value of μ.

$$x \text{ equation:} \quad N_w - \mu N_g = 0 \tag{5.8}$$

$$y \text{ equation:} \quad N_g - W = 0 \tag{5.9}$$

Note that we have three unknowns, N_g, N_w and μ, and we have only two relations so-far. The third relation is obtained when we apply the vanishing of torque. The torque of a force depends on the force and the lever arm of that force, which is the distance from the pivot point to the line of action of the force. The information about where the force is acting is lost in the free-body diagram. Therefore, we need to go back to the original diagram given in Fig. 5.1 to examine the situation regarding the torques.

Since the net force is zero, the choice of pivot point would be arbitrary. In the static problems, we choose a pivot point so that the lever arm for some of the unknown forces would be zero for that choice. In the present case, we notice that if we choose the pivot point at the point where the ladder touches the ground, and then the torques from two of the forces, viz, N_g and $F_s = F_s^{\max} = \mu N_g$ will be zero. The digram for the torque calculation is displayed in Fig. 5.3, where the lever arms OC for the force $\vec{W}$ and OD for the force $\vec{N}_w$ are also shown. Let us denote these lever arms by L_w and L_{nw} respectively. The z-axis is coming out of the page in this figure. We find the z-components of the torques and equate their sum to zero for the equilibrium condition to obtain the third relation we seek.

$$N_w \; L_{nw} - W \; L_w = 0. \tag{5.10}$$

The lever arms can be written in terms of the length L of the ladder and the angle θ the ladder makes with the horizontal.

$$L_w = \tfrac{L}{2}\cos\theta \tag{5.11}$$
$$L_{nw} = L\sin\theta \tag{5.12}$$

Substituting in the torque equation, Eq. 5.10, we find the following.

$$N_w\ \sin\theta - \frac{W}{2}\cos\theta = 0. \tag{5.13}$$

Now, we can solve Eq. 5.8, 5.9, and 5.13 together to find the result.

$$\mu = \frac{1}{2}\cot\theta. \tag{5.14}$$

Example 5.2.2. Cantilever. A beam that extends beyond its sup-

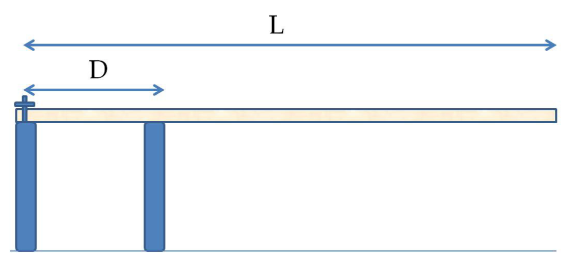

Figure 5.4: Example 5.2.2.

port is called a **cantilever**. Consider a cantilever of length L and mass M resting and bolted to two supports, one at the end and the other is at a distance D from it. (a) Find the conditions on the forces such that the cantilever is in static equilibrium. (b) For L=30 m, M=1000 kg, and D=12 m, find the magnitude and directions of the forces on the beam from the supports.

Solution. We will consider the beam of the cantilever as the system. This makes everything else external to the system. The beam has the forces from the support and the bolt at one end, an upward normal force from the support in the middle, and the force of gravity from the Earth. As shown in Fig. 5.5, we will combine the forces of the bolt and the support at the end into a horizontal force $\vec{F}_{11}$ and a vertical force $\vec{F}_{12}$, the sum of the two forces at the end give us the net force $\vec{F}_1$ at that point.

$$\vec{F}_1 = \vec{F}_{11} + \vec{F}_{12}.$$

Normally, one writes the force $\vec{F}_{11}$ as $\vec{F}_{1x}$ and $\vec{F}_{12}$ as $\vec{F}_{1y}$. This notation is confusing since symbols F_{1x} and F_{1y} stand for x and y-components of the vector $\vec{F}_1$ and are not vectors themselves. Therefore, we will use $\vec{F}_{11}$ and $\vec{F}_{12}$ for the the horizontally and vertically pointed vectors whose sum is the force at the end.

Based on physical grounds we expect that all four forces $\vec{F}_{11}$, $\vec{F}_{12}$, $\vec{F}_2$ and the weight $\vec{W}$ will be in one plane, which we take to be the xy-plane of the coordinate system shown.

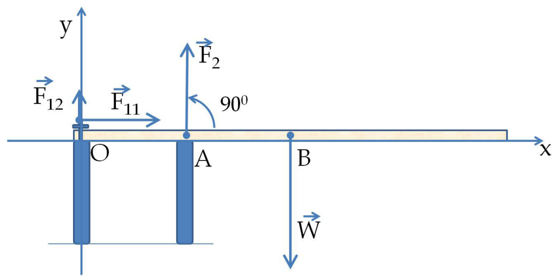

Figure 5.5: Example 5.2.2. Forces on the cantilever beam.

Balancing the forces immediately gives the following equations.

$$F_{11} = 0 \tag{5.15}$$

$$F_12 + f_2 - W = 0 \tag{5.16}$$

Although you can do the torque part graphically, I will use the analytic approach just to illustrate the kind of considerations that goes into this method. For the torque equation, we need to choose a suitable point P about which we will work out the torques. As the net force is zero, the choice of a pivot point is arbitrary. Since we do not know either $\vec{F}_1$ (or equivalently $\vec{F}_{11}$ and $\vec{F}_{12}$) or $\vec{F}_2$, it does not save us much work whether we place the pivot point at O or A. Let us us pick O as the pivot point because it is nicely on one end so that we can measure displacements of the forces from the end point. We will also name the displacements as $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_w$ for the four forces. Let us organize the information regarding forces and displacements in a table.

Force	Force in components	Displacement	Torque
$\vec{F}_{11}$	$F_{11}\hat{u}_x$	$\vec{r}_1 = 0$	0
$\vec{F}_{12}$	$F_{12}\hat{u}_y$	$\vec{r}_1 = 0$	0
$\vec{F}_2$	$F_2\hat{u}_y$	$\vec{r}_2 = D\hat{u}_x$	$DF_2\hat{u}_z$
$\vec{W}$	$-Mg\hat{u}_y$	$\vec{r}_2 = (L/2)\hat{u}_x$	$-(L/2)Mg\hat{u}_z$

Balancing the torques gives the following equation.

$$DF_2 = \frac{1}{2}LMg. \tag{5.17}$$

Equations 5.15, 5.16, and 5.17 can be solved to yield the following for the magnitudes of the forces.

$$F_{11} = 0 \tag{5.18}$$

$$F_{12} = Mg\left(1 - \frac{L}{2D}\right) \tag{5.19}$$

$$F_2 = \left(\frac{L}{2D}\right) Mg \tag{5.20}$$

Equation 5.18 says that there is no horizontal component of force $\vec{F}_1$ at the end, and Eq. 5.19 says that if the second support is less that $L/2$ from the end, then the force in the vertical direction at the end support is downward, i.e. the force there is dominated by the push back from the bolt at the top of the beam.

(b) Plugging in the numerical values we find the following.

$$F_{11} = 0,$$
$$F_{12} = -2500\ N,$$
$$F_2 = 12000\ N,$$

where I have rounded off the magnitudes to the two significant digits.