7.7 COLLISIONS IN THE CM FRAME

If the colliding bodies together form an isolated system, then, a coordinate system in which the CM of the colliding bodies is fixed at the origin is very helpful in the analysis of the collision. This frame of reference, calles the **CM frame**, is also an inertial frame since the CM would not have an acceleration if the system is an isolated system. In this section we examine the collision process from the CM frame and discuss the relation to the collision as observed in the laboratory frame.

Recall that the total momentum of a system in any frame is equal to the total mass of the system times the velocity of CM in that frame. In the CM frame, the velocity of the CM is zero, by definition. Therefore, the total momentum of the system would be zero in the CM frame.

$$\vec{P}_{\text{System}} = 0 \quad \text{(CM frame)} \tag{7.62}$$

Let us choose a notation for the symbols for quantities with respect to the CM frame. We will attach a subscript or superscript "CM" to the symbol to indicate the quantity with respect to the CM frame. The standard frame in the laboratory is also called **LAB frame**, and sometimes we will attach a subscript or superscript "LAB" to the symbol to emphasize the reference frame.

Let $\vec{P}_1^{\text{CM}}$ and $\vec{P}_2^{\text{CM}}$ be the momenta of the two masses in the CM frame. Then, as stated above, their vector sum will be zero.

$$\boxed{\vec{P}_{\text{System}}^{\text{CM}} = \vec{P}_1^{\text{CM}} + \vec{P}_2^{\text{CM}} = 0.} \tag{7.63}$$

Therefore, in the CM frame, the magnitudes of the momenta of two colliding particles must be equal and directions must be opposite.

$$\vec{P}_1^{\text{CM}} = -\vec{P}_2^{\text{CM}}. \tag{7.64}$$

As shown in Fig. 7.24, before the collision, the colliding bodies approach each other toward the CM, where the origin of the CM frame is located.

Let $\vec{P}_1{}'^{\text{CM}}$ and $\vec{P}_2{}'^{\text{CM}}$ be the momenta of objects 1 and 2 after the collision. Since, the conservation of momentum applies to all inertial frames, and since the CM-frame of an isolated system is also an inertial frame, the sum of the momenta of the two objects would also zero after the collision since their sum was zero before the collision in this frame.

$$\vec{P}_1^{\text{CM}} + \vec{P}_2^{\text{CM}} = 0 = \vec{P}_1{}'^{\text{CM}} + \vec{P}_2{}'^{\text{CM}}. \tag{7.65}$$

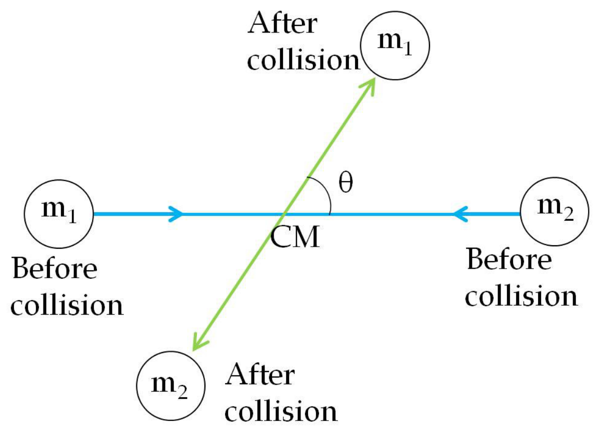

Figure 7.24: Collision in CM frame.

Therefore, the momenta of the two objects after the collision are also collinear. The colliding objects after the collision move away from each other with equal magnitude of momentum! In the CM frame, the line joining the colliding objects rotates by an angle θ, called the **scattering angle** (Fig. 7.24). The scattering angle can be obtained by taking a dot product of $\vec{P}_1^{CM}$ and $\vec{P}_1{}'^{\text{CM}}$.

$$\cos\theta = \frac{\vec{P}_1^{\text{CM}} \cdot \vec{P}_1{}'^{\text{CM}}}{\left|\vec{P}_1^{\text{CM}}\right| \left|\vec{P}_1{}'^{\text{CM}}\right|}. \tag{7.66}$$

Relation between the CM frame and the LAB frame

It is easy to relate the momentum an object in the CM-frame to the momentum of the same object the LAB-frame. Let m_1 and m_2 be the masses of the two bodies, and their velocities be $\vec{v}_1^{\text{LAB}}$ and $\vec{v}_2^{\text{LAB}}$ in the LAB-frame. From the definition of the CM we know the velocity of the CM of the two masses will be

$$\vec{V}_{\text{CM}}^{\text{LAB}} = \frac{m_1\vec{v}_1^{\text{LAB}} + m_2\vec{v}_2^{\text{LAB}}}{m_1 + m_2}$$

Writing this equation in terms of the momenta of the two bodies in the LAB frame we have

$$\vec{V}_{\text{CM}}^{\text{LAB}} = \frac{\vec{p}_1^{\text{LAB}} + \vec{p}_2^{\text{LAB}}}{m_1 + m_2}.$$

The momentum of body 1 in the CM frame will be obtained by multiplying the velocity of this body with respect to the CM and its mass.

$$\vec{P}_1^{\text{CM}} = m_1\left(v_1^{\text{LAB}} - V_{\text{CM}}^{\text{LAB}}\right) = \vec{p}_1^{\text{LAB}} - m_1\left(\frac{\vec{p}_1^{\text{LAB}} + \vec{p}_2^{\text{LAB}}}{m_1 + m_2}\right).$$

Simplifying, we obtain,

$$\vec{P}_1^{\text{CM}} = \frac{m_2\vec{p}_1^{\text{LAB}} - m_1\vec{p}_2^{\text{LAB}}}{m_1 + m_2}.$$

Similarly, you can show the following for the momentum of body 2 in the CM frame.

$$\vec{P}_2^{\text{CM}} = -\left(\frac{m_2\vec{p}_1^{\text{LAB}} - m_1\vec{p}_2^{\text{LAB}}}{m_1 + m_2}\right).$$

Example 7.7.1. Scattering of alpha particles from gold nucleus. Alpha particles of mass 4 amu are incident on gold nucleus of mass 197 amu [atomic mass unit]. Before the collision, an alpha particle is moving with a speed of 2×10^5 m/s in the LAB-frame. After the collision, the alpha particle comes out with a speed of 1.5×10^5 m/s at an angle of 10° from the original direction in the LAB frame. Find the angle between the incoming and outgoing directions in the CM-frame. (1 amu $= 1.66053\times 10^{-27}$ kg)

Solution. Let us orient our axes of the LAB and CM frames so that their axes are parallel, and initially the motion is along the x-axis. The velocity of the origin of the CM-frame with respect to the LAB-frame is along the x-axis.

$$\vec{V}_{\text{CM}}^{\text{LAB}} = V_x^{\text{LAB}}\hat{u}_x = \frac{4\times 2\times 10^5 + 0}{4 + 197}\hat{u}_x = 4\times 10^3\hat{u}_x \text{ m/s}.$$

Using the velocity of CM, we can obtain the velocity of the colliding particles as seen from the CM-frame. Let $\vec{v}_1^{\text{CM}}$ and $\vec{v}_1{}'^{\text{CM}}$ be velocities of the alpha particle in the CM-frame before and after the collision, respectively. We use the same letters for the velocities in the LAB frame.

$$\begin{aligned}
\vec{v}_1^{\text{CM}} &= \vec{v}_1^{\text{LAB}} - \vec{V}_{\text{CM}}^{\text{LAB}} = \left[2\times 10^5 - 4\times 10^3\right]\hat{u}_x \text{ m/s}.\\
\vec{v}_1{}'^{\text{CM}} &= \vec{v}_1{}'^{\text{LAB}} - \vec{V}_{\text{CM}}^{\text{LAB}}\\
&= \left[1.5\times 10^5\cos(10^\circ)\hat{u}_x + 1.5\times 10^5\sin(10^\circ)\hat{u}_y\right] \text{ m/s}\\
&\quad -4\times 10^3\hat{u}_x \text{ m/s}.
\end{aligned}$$

Therefore, the momenta of the alpha particle before and after collision are:

$$\begin{aligned}
\vec{p}_1^{\text{CM}} &= m_1\vec{v}_1^{\text{CM}}.\\
\vec{p}_1{}'^{\text{CM}} &= m_1\vec{v}_1{}'^{\text{CM}}.
\end{aligned}$$

Therefore the angle between the incoming direction and outgoing direction of alpha particles in the CM-frame.

$$\cos\theta = \frac{\vec{P}_1^{\text{CM}}\cdot\vec{P}_1{}'^{\text{CM}}}{\left|\vec{P}_1^{\text{CM}}\right|\left|\vec{P}_1{}'^{\text{CM}}\right|} = 0.986 \quad\Longrightarrow\quad \theta = 9.6^\circ.$$

Note that mass and common constant factors cancels out in the calculation of the angle.