5.3 INTERNAL FORCES, ELASTICITY AND STRESS

In the last section we studied the equilibrium among external forces. If a system was perfectly rigid and not deformable, the story would end there. But, real materials respond to the external forces, and they are deformable to some extent, even to the point of breaking if the external force is sufficiently large. However, when the deformation is small compared to the physical dimensions of the object, many materials often recover the original shape when the external force is removed. This property is called **elasticity**.

The elastic behavior of a material can be seen by attempting to elongate a wire by hanging masses to the wire as shown in Fig. 5.8.

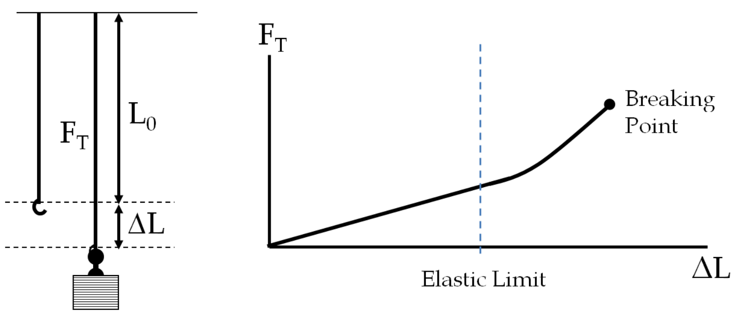

Figure 5.6: Elasticity of a string. The string is stretched as massed are added. The wire will go back to the origin length when you remove the weights if the stretching is within the elastic limit. If the wore is stretched to breaking point, the wire will break. Between the stretching limit and the breaking point, the wire becomes permanently stretched.

Initially as the weight is increased, the length of the string expands in proportion to the weight. This is the elastic region, since, if the weight is removed, the wire will go back to its original length. If you continue to increase the weight past a certain point, called the **elastic limit**, the elongated wire will not go back to the original length L_0. Instead, the wire will be permanently elongated. Now, if you continue to add more weight to the wire, eventually the wire will break at some point, called the **breaking point**. The breaking point gives us information about the ultimate strength of the material. The region between the elastic limit and the breaking point is called the **plastic region**. We will be concerned with the elastic range here since we want a predictable linear restoring force to help analyze the equilibrium behavior of structures so as to prevent permanent deformation or fracture.

In the linear elastic regime, the magnitude of the force $\vec{F}$ applied to the wire is proportional to the elongation ΔL.

$$F = k\Delta L, \tag{5.21}$$

where k is the proportionality constant. This observation was first made by Robert Hooke (1635-1703), a contemporary of Isaac Newton, and is often called Hooke's law, which states that "Ut tensio sic uis," in Latin, meaning "As the extension, so the force". We have already encountered Hooke's law in the context of our discussions on the spring and tension forces. The reason for a material "to act like" springs can be traced to the inter-atomic bonds in materials as illustrated in Fig. 5.7, which have a spring-like behavior for small displacement from equilibrium.

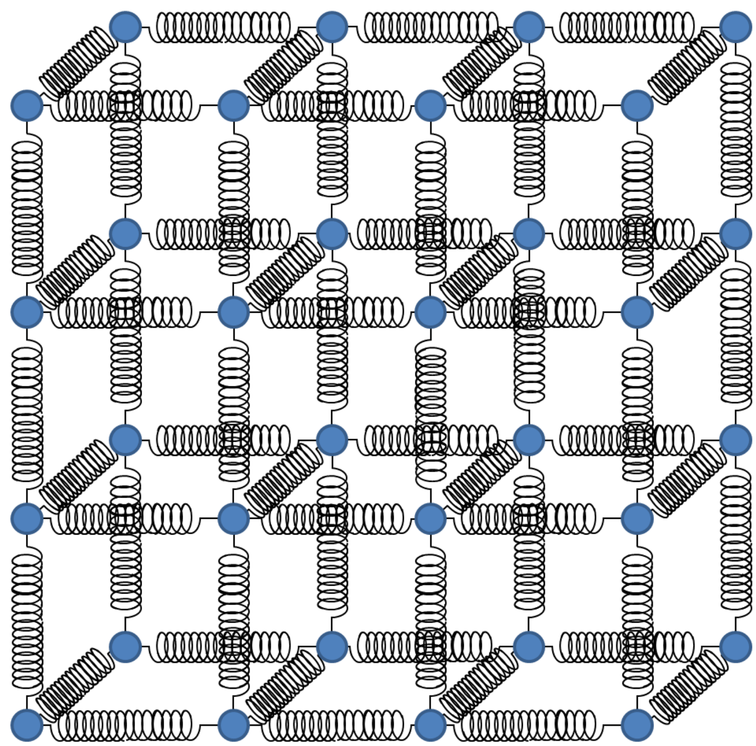

Figure 5.7: The elastic properties of materials emerge from spring-like bonding of molecules inside the material as illustrated here.

To measure the amount of deformation from the initial length as a fraction, it is customary to introduce a quantity, called the **strain** or **relative deformation**, by the following relation:

$$\text{Strain} = \frac{\text{Deformation}}{\text{Original Length}} = \frac{\Delta L}{L_0} \tag{5.22}$$

Clearly, strain is a dimensionless quantity since it is a ratio of two lengths. Experiments show that, for the same force, a wire stretches less if it is thicker. Therefore, the effectiveness of a force to cause strain in a material is contained in force per unit cross-sectional area of the wire rather than the force alone. To describe the cause of the strain, we define a quantity, called **stress**, which is equal to force per unit area. The magnitude of stress is given by

$$\text{Stress} = \frac{\text{Magnitude of Force}}{\text{Are of cross-section}} = \frac{F}{A}. \tag{5.23}$$

Note that stress will have direction since force has direction. Even the area of cross-section has an orientation. For instance, the six sides of a cube has different orientations in space. Therefore, in general, the "direction" of stress is dependent on the direction of the force and the orientation of the area. The stress is a tensor and has complicated rules for the direction. We will not delve into the general properties of stress here. We will be content to work with the stress given as the force on one area.

The dimension of stress can be found from the ratio of the dimensions of force and area.

$$[\text{stress}] = \frac{[F]}{[A]} = \frac{[M][L]/[T]^2}{[L]^2} = \frac{[M]}{[L][T]^2}. \tag{5.24}$$

This gives the unit of stress to be N/m^2, which is also called Pascal denoted by P. Now, Hooke's law given in Eq. 5.21 can be written using the stress and strain by dividing both sides of the equation by the area A and introducing the factor L_0/L_0 on the right side.

$$\frac{F}{A} = \frac{k}{A}\Delta L = \left(\frac{kL_0}{A}\right)\frac{\Delta L}{L_0} \tag{5.25}$$

Writing the constants given in parenthesis as one letter Y, we have the Hooke's law for a change in length of the wire as

$$\text{Stress} = Y \times \text{Strain} \tag{5.26}$$

The proportionality constant Y is called the Young's modulus. The unit of Young's modulus is the same as that of stress, i.e., N/m^2 or Pascal, since strain in unitless. Table 5.1 lists Young's modulus of a number of common materials.

Example 5.3.1. Stress inside a hanging wire. The stress at a point inside the wire is caused by hanging masses as well as the mass of the wire itself. In this example, I will work out the stress at a point P inside a wire of length L and mass m when a mass M is hung from the bottom as shown in Fig. 5.8.

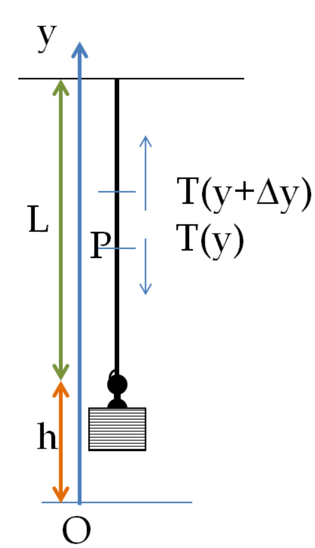

Figure 5.8: Example 5.3.1.

Let us use a coordinate system with the y-axis pointed up. Let y be the coordinate of the point P from the origin which is at the floor.

Table 5.1: **Young's, shear and bulk moduli**

Source: Kaye and Laby, Tables of physical & chemical constants, 16th edition (1995). Young's Modulus Y, Shear Modulus G, Bulk Modulus B. (Temp= $20°C$)

Material	Y (GPa)	G (GPa)	B (GPa)
Metals and alloys			
Aluminum	70.3	26.1	75.5
Copper	129.8	48.3	137.8
Lead	16.1	5.59	45.8
Nickel (unmagnetized., hard)	219	83.9	187.6
Silver	82.7	30.3	103.6
Titanium	115	43.8	107.7
Tungsten	411.0	160.6	311.0
Brass (70 Zn, 30 Cu)	100.7	37.3	111.8
Steel, Stainless	215.3	83.9	166.0
Tungsten Carbide	$\sim$ 534.4	$\sim$ 219	$\sim$ 319
Glass and plastics			
Glass, Heavy Flint	80.1	31.5	57.6
Glass, Crown	71.3	29.2	41.2
Quartz, fused	73.1	31.2	36.9
Epoxy	$\sim$ 3.2		
Polycarbonate	2.4		
Polyethylene	0.4-1.3		
Polyvinylchloride (PVC)	2.4-4.1		
Liquids Glycerin		20.5	
Mercury		26.2	
Olive oil		1.60	
Paraffin oil		1.62	
Turpentine		1.28	
Water (150)		2.05	
Water, sea		2.32	

Let h be the y-coordinate of the lowest part of the wire when the length of the wire is L. Let us examine the balancing of forces on the element of the wire between y and $y+\Delta y$. The forces on this element of wire are tension pointed up above the element, which has the magnitude $T(y+\Delta y)$, the tension below the element pointed down, which has the magnitude $T(y)$, and the weight of the element pointed down, which has the magnitude $m\Delta y/L$. Since the acceleration of the element is zero, the forces are balanced. This gives us the following equation.

$$T(y+\Delta y) - T(y) - \frac{m\Delta y}{L} g = 0, \quad h \le y \le h+L. \tag{5.27}$$

Rearranging and dividing by Δy, and taking the infinitesimal Δy limit we obtain the following equation for $T(y)$.

$$\frac{dT}{dy} = \frac{mg}{L}, \quad h \le y \le h+L. \tag{5.28}$$

We can solve this equation for the tension in the wire at point P by rewriting this equation as $dT = (mg/L)dy$ and integrating from $y = h$ to $y = y$.

To specify the values of tension at the two limits in this integration, we need the value of tension when $y = h$, i.e. at the bottom point of the string. To obtain this value, we will apply Newton's second law to the element of the wire that is at that point. This element of length Δy has a tension pulling it up with magnitude T_h (which we wish to determine), weight of the element $mg\Delta y/L$, and the weight of the hanging mass of magnitude Mg pointed down. Therefore,

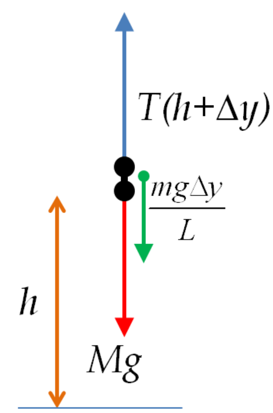

Figure 5.9: Free-body diagram for an element at the tip of the wire.

$$T(h+\Delta y) - \left[M + \frac{m}{L}\Delta y\right] g = 0. \tag{5.29}$$

We are interested in the infinitesimal limit of Δy so that we can get the information at the tip of the wire. This will leave only M in the bracket and $T(h+\Delta y)$ will go into $T(h)$, which we will write as T_0 to indicate that T_0 is the tension at the bottom point of the wire. This gives

$$T_0 = Mg. \tag{5.30}$$

Now, the integration of Eq. 5.28 goes as

$$\int_{T_0}^{T(y)} dT = \int_h^y \frac{mg}{L} dy, \quad h \le y \le h+L. \tag{5.31}$$

This equation along with Eq. 5.30 gives the result we seek.

$$T(y) = Mg + \frac{mg}{L}(y-h). \tag{5.32}$$

This result shows that if the mass of the wire cannot be neglected compared to the mass hung from the wire, then, the tension in the wire will vary along the wire.

5.3.1 Types of Stress

Tensile versus Compressive Stress

A stress that causes strain in the same direction as the applied force can lead to either the extension of the body or a compression of the body as illustrated in Fig. 5.10. The stress that leads to an extension of the body is called a **tensile stress** and the stress that compresses the body is called a **compressive stress**.

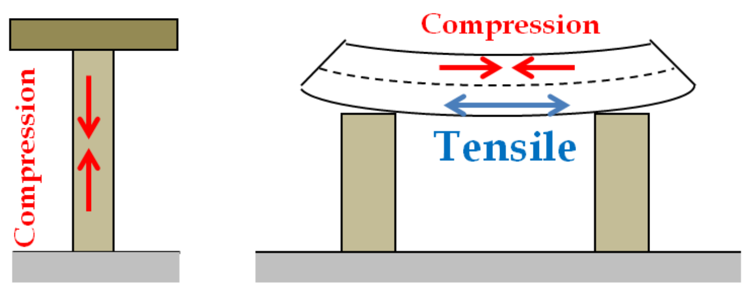

Figure 5.10: If the strain in the material is negative, i.e. the material is compressed as a result of the stress, then the stress is called a compressive stress. If the strain in the material is positive, i.e. the material stretches as a result of the stress, then the stress is called a tensile stress.

Usually Young's moduli for the two stresses are equal, but they do not have to be equal. For instance, the Young's modulus for concrete is 2 MPa for the tensile stress and 20 MPa for the compressive stress. Thus, a column made of concrete will break more easily when pulled than when compressed; a concrete column is able to support considerably more weight on the column than it can hold up weight hanging from it.

Shear Stress

When a force is applied parallel to the surface of an object, instead of stretching or compressing the object, the force may deform the object if some other part of the object is held fixed. For instance, suppose you fix the bottom of a box by gluing the bottom part and apply a side-ways force on the top part as illustrated Fig. 5.11, you will deform the box. This deformation is used to define an **angular or shear strain**.

The shear strain is defined by the ratio $\Delta L/L_{\perp}$, where ΔL is the displacement of the shifted face compared to the fixed face and $l_{\perp}$ is the distance between the shifted face and the fixed face. For small deviations, the angle of the deviation $\Delta\theta$ is equal to $\Delta L/L_{\perp}$, and is

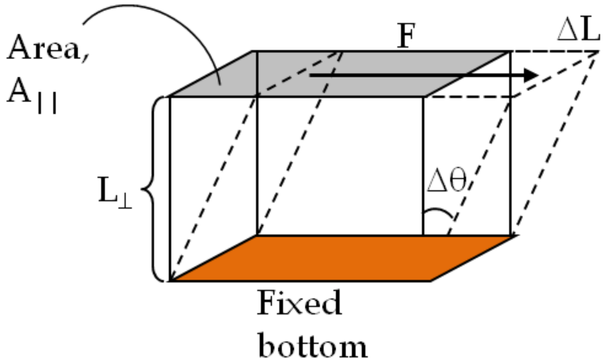

Figure 5.11: Defining shear strain and shear stress. The bottom part of the box is fixed to a table and a force parallel to the top surface is applied. The top face shifts by a distance ΔL compared to the bottom face. The shear strain is defined by the ratio $\Delta L/L_{\perp}$, where $L_{\perp}$ is the distance between the shifted face and the fixed face. Shear stress is defined as F/A, where A is the area of the shifted face.

a measure of the shear strain.

$$\text{Angular or shear strain} = \frac{\Delta L}{L_{\perp}} \approx \Delta\theta. \tag{5.33}$$

The resulting stress in the body is called the **shear stress**. This time, the applied force is spread over the area of the surface parallel to the force and not perpendicular to the force, as was the case for the tensile and compressive stresses. Therefore, the stress corresponding to a shear strain is defined as the magnitude of the force $\vec{F}$ divided by parallel area, $A_{\parallel}$.

$$\text{Shear stress} = \frac{F}{A_{\parallel}}. \tag{5.34}$$

Hooke's law is also applicable for this situation if the angular deviation is small. That is, we expect the shear stress to be proportional to the shear strain.

$$\text{Shear stress} = G \times \text{Shear strain} \tag{5.35}$$

where the proportionality constant G is called the shear modulus of the material. The shear modulus G has the units of N/m^2 or Pa. The shear moduli of some common materials are listed in Table 5.1. As a general rule, the shear modulus is less than the Young's modulus for the same material, which means that it is usually easier to deform an object sideways than to compress or elongate it.

Note that you must pay attention to the particular area and length in the definition of shear stress given in Eq. 5.34. Unlike the tensile stress, the area here is of the surface that is parallel to the applied force. Further more, when you find the angle of deviation to use in the formula for the the angular deviation, you need to make sure the displacement ΔL and the length $L_\perp$ are perpendicular to each other. Also, while ΔL is parallel to the applied force, $L_\perp$ is perpendicular to the applied force.

Ultimate strength

I have mentioned above that if the magnitude of the applied force exceed a certain value, the material will ultimately break up. This was illustrated in Fig. 5.8 for stretching of a wire. We expect the same physics should happen for the compressive, tensile and shear stresses.

The compressive, tensile and shear stresses lead to three kinds of fractures in materials, called tensile fracture, compressive fracture and shear fracture, as displayed in Fig. 5.12.

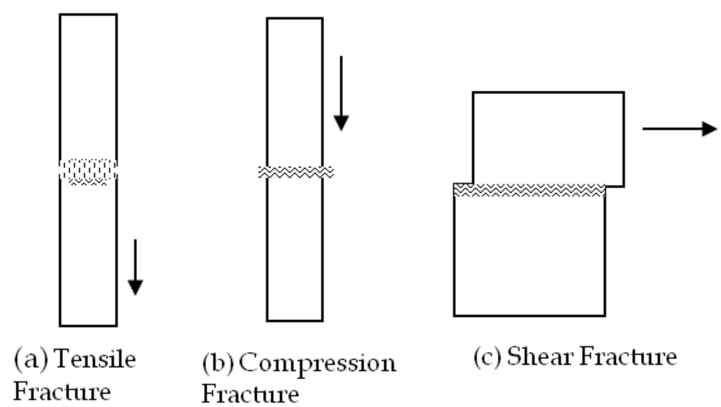

Figure 5.12: Compressive, tensile and shear fractures.

Note than the shear stress rupture is in the perpendicular dimension to the direction of the shear force while the tensile or compression related ruptures take place in the same direction as the corresponding force. The ultimate stress of each type that a material can withstand before giving in is called the **ultimate strength** of the material.

Whenever you design a mechanical structure, you must take into account the ultimate strength of the material to be used and the type of stress to be expected under extreme conditions of use. This is especially important when designing the structures that are subject to large stresses such as buildings and bridges. The ultimate strength for the shear stress is usually less than that for compressive and tensile stresses as we see in Table 5.2 and therefore, a building is more susceptible to a shear stress than to compressive or tensile stress.

Pressure

Table 5.2: Ultimate strength of materials (MPa) Source: Various

Material	Tensile strength (MPa)	Compressive strength (MPa)	Shear strength (MPa)
Aluminum	90-100		
Brass, rolled	230-270	~ 250	~ 200
Concrete	~ 3	20-30	~ 2
Iron, cast	100-230	~ 550	~ 170
Iron, wrought	290-450		
Steel	400-1500	~ 500	~ 250
Wood, pine (parallel to grain)	20-50	~ 35	~ 5

We have looked at stresses caused in a material by forces applied perpendicular to the opposite faces of the material and those applied parallel to the surfaces of the material. Now, we consider forces applied on the system from all directions, as would be the case, for instance, when an object in submerged in a fluid, where fluid will apply the force from all directions.

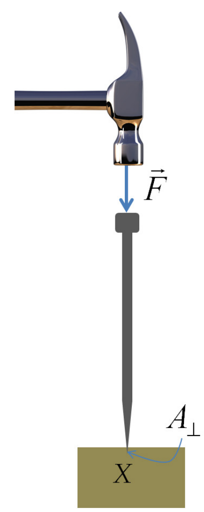

Figure 5.13: The pressure at the point marked X is the magnitude F of the force divided by the area $A_\perp$ of the tip of the nail.

Pressure at a point is defined as the magnitude of the force per unit area. The area is the area perpendicular to the force and we can indicate this in the formula for pressure. For instance, if you apply a force on a nail, as illustrated in Fig. 5.13, then the area will be the area of cross-section of the tip of the nail. Suppose a force $\vec{F}$ acts on an area $A_\perp$. Then the pressure corresponding to the force will be defined as

$$\text{Pressure, P} = \frac{F}{A_\perp}. \tag{5.36}$$

Formally, this equation looks identical to the equation for the definition of compressive and tensile stresses. However, when we speak of pressure, we are usually interested in a different strain than the change in linear dimension. In the case of pressure as a stress, we are interested in the change in volume. The strain of interest for the pressure is the relative change in the volume, which is also called the **bulk strain**, which is defined as the change in the volume ΔV divided by the original volume.

$$\text{Bulk strain} = -\frac{\Delta V}{V_0}. \tag{5.37}$$

Hooke's law also holds for the pressure and the bulk strain

$$\boxed{\text{Pressure} = B \times \text{Bulk strain},} \tag{5.38}$$

where the proportionality constant B is called the **bulk modulus**. When you apply a force from all sides of an object, the volume of the object will change and a pressure will develop inside the object.

Example 5.3.2. Stress in a steel beam. A steel beam of length L and area of cross-section A is posted vertically on a concrete slab. Find the tensile stress in the beam as a function of height h.

Solution. We have done a similar problem for a wire under stress. Same technique will give the stress at any point equal to the weight of the steel beam above that point. Suppose the origin of the y-coordinate is at the top point of the beam and the direction of the y-axis is pointed down. Then, the weight of the beam above the point with y-coordinate equal to y will be My/L. Therefore, the stress at that point will be

$$\text{Stress at a distance } y \text{ from the top} = \frac{M}{L}y.$$

We can state the answer for the height from the bottom as.

$$\text{Stress at a height h from the bottom} = \frac{M}{L}(L-h).$$

Example 5.3.3. Compression of a support beam. Two cylindrical concrete posts are to support a heavy machine of mass 20,000-kg set in the middle of a 2000-kg beam resting on the posts. For safety reasons, it is required that the stress in the posts does not exceed 5 times the ultimate strength of concrete, what must be the minimum radii of the two posts.

Solution. The forces on the beam are given in Fig. **??**. In the figure M and m are the masses of the machinery and the beam respectively. The force $\vec{F}_1$ and $\vec{F}_2$ are the forces on the beam from the two posts respectively. By symmetry in the situation, the two forces must be equal. You can prove this also by demanding that the net torque be zero. Let us represent both of them by $\vec{F}$.

$$\vec{F}_1 = \vec{F}_2 \equiv \vec{F}.$$

From the static condition for the beam we immediately obtain the magnitude of the force $\vec{F}$.

$$F = \frac{1}{2}(M+m)\,g.$$

This gives the compressive force in each column, just from the weight of the machinery and the beam. There is also compressive stress in each support from the weight of the supports above the point of consideration.

Suppose the stress in the support is largely from the machinery and the steel beam. Then, the stress in each support will be F/A, where A is the area of cross-section of the support. Now, if we equate the compressive stress to safety factor c times the ultimate strength for compressive stress.

$$\frac{F}{A} = c \times \text{Ultimate strength}$$

This gives the following for the area of cross-section of the support we need.

$$A = \frac{F}{c \times \text{Ultimate strength}}.$$

We look up the ultimate strength of concrete for compression mode in the table. This says that the ultimate strength $= 20 \times 10^6\ \text{N/m}^2$. Therefore, to be safe, the area of cross-section that meets the safety requirement is $A = 0.027\ \text{m}^2$. Therefore minimum radius of the posts must be $9.25\ \text{cm}^2$.

Example 5.3.4. Tear under shear stress. A steel bolt is used to connect two steel plates. It is required that the bolt must withstand shear forces up to 4,000 N. Assuming a safety factor of 5, what must be the minimum diameter of the bolt?

Solution. Here, the shear force divided by the area of the bolt must be one-fifth of the breaking shear stress. We look up the breaking shear stress for steel in the table and use the value to find

$$\frac{4000\ \text{N}}{\pi d^2/4} = \frac{250 \times 10^6\ \text{Pa}}{5}.$$

Therefore, $d = 10.1$ cm.