2.4 THE THIN LENS EQUATION

2.4.1 Derivation of the Thin Lens Equation

In the last section we discussed refraction at a single spherical surfaces. Once a light ray enters the second medium, it will travel in a straight line till it encounters the other edge and refracts at the second surface. A refracting material that has at least one side spherical is called a lens. The cross-sections of various types of lenses commonly in use are shown in Fig. 2.19.

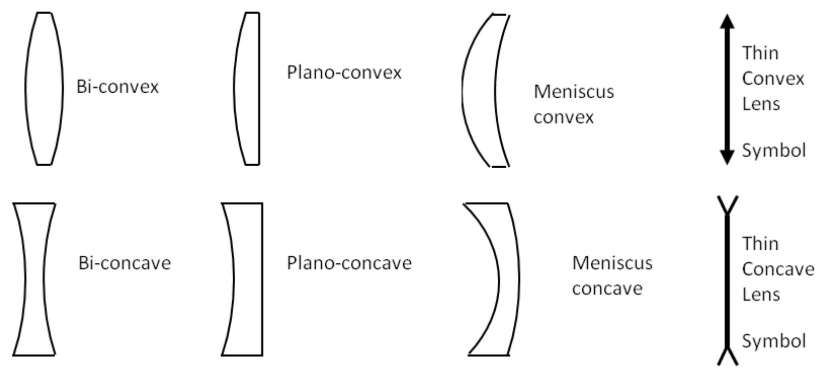


Figure 2.19: Various types of lenses.

We will now study refraction of light at the two interfaces of a lens as shown in Fig. 2.20. For illustrative purposes, we choose a point object P on the symmetry axis of a bi-convex lens. Let the refractive index of the surrounding media be n_1 and that of the lens be n_2. Let the radii of curvatures of the two sides be R_1 and R_2 as shown in the figure. We wish to find a relation between the object distance, the image distance and the parameters of the lens.

Locating the image graphically as a result of refraction through a lens is straightforward. As usual we draw two rays of light PV_1V_2Q and PXYQ tracing their paths through the lens making sure to take into account the refractions at the interfaces according to the Snell's law. We find that the rays intersect at point Q for the case shown in Fig. 2.20. The crossing of real rays at point Q generate a **real image** there.

We now examine the path PXYQ in Fig. 2.20 more closely. The incident ray PX bends towards normal and goes in the direction XY

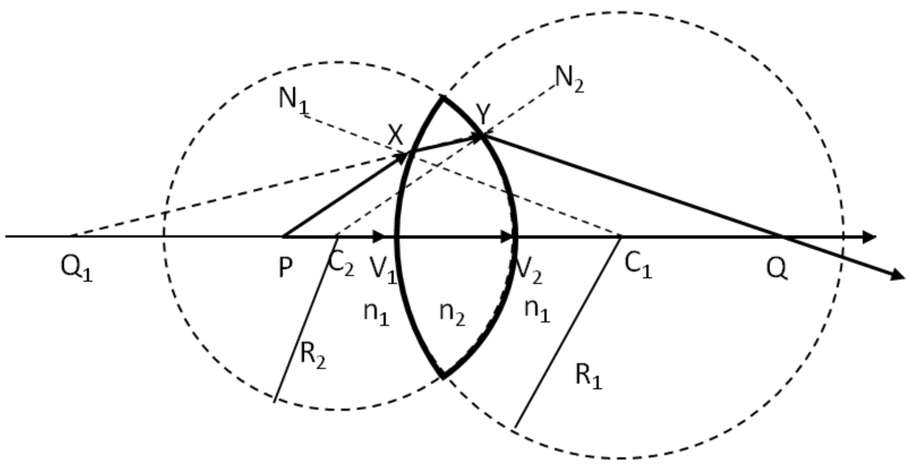


Figure 2.20: Image formation in a lens. Here d is the thickness of lens. We shall take d-¿ 0 limit to obtain formula for thin lens.

inside the glass, but it is diverging away from the inside path V_1V_2 on the axis, therefore we extend XY backward towards the axis meeting the axis at Q_1 which is a virtual image of P created by the front surface of the lens. The image Q_1 is the image due to refraction at the front surface.

When the rays XY and V_1V_2 reach the second surface of the lens, they refract at that surface. This situation is similar to the rays originating in the medium n_2 which extends to infinity on the left of the second interface and refracting into the medium n_1. [By the way, the medium to the right does not have to have the refractive index n_1. We are using the same refractive index on the two sides of the lens to keep our formulas simpler.] The rays XY and V_1V_2 can be considered to come from the same point Q_1 in medium of refractive index n_2. Actually, all rays starting from P and having refracted at the front surface would appear to be coming from Q_1 as far as the second interface of the lens is concerned.

The ray XY refracts in the direction YQ and meets the ray $Q_1V_1V_2Q$ on the axis. To find the location of the image point Q graphically we need to draw the figure as precisely as possible. However, there is also an algebraic method of analysis that gives algebraic relations which is often easier to use that to draw rays. We still draw rays to get an overall picture but do the calculations using the algebraic equations to obtain quantitative values.

Various distances in the figure will be denoted by the following symbols.

$$p_1 = \mathrm{V_1P};\quad R_1 = \mathrm{V_1C_1};\quad q_1 = \mathrm{V_1Q_1};\quad d = \mathrm{V_1V_2};$$
$$p_2 = \mathrm{V_2Q_1};\quad R_2 = \mathrm{V_2C_2};\quad q_2 = \mathrm{V_2Q}.$$

The algebraic method relies on the refraction formulas for the spherical surfaces we have obtained in the last section. First we locate the image $\mathrm{Q_1}$ due to the refraction at the first interface, and then use $\mathrm{Q_1}$ as an object for the second interface to find the final image at Q.

Front face (refraction at the convex surface):

$$\frac{n_1}{p_1} + \frac{n_2}{q_1} = \frac{n_2 - n_1}{R_1}, \quad (q_1 < 0 \text{ and } R_1 > 0). \tag{2.26}$$

Back face (refraction at the concave surface):

$$\frac{n_2}{p_2} + \frac{n_1}{q_2} = \frac{n_1 - n_2}{R_2}, \quad (p_2 > 0, q_2 > 0, R_2 < 0)\,. \tag{2.27}$$

We have the following additional relation. The object distance for the second interface is from $\mathrm{V_2}$ to $\mathrm{Q_1}$, and since $q_1 < 0$, the distance p_2 will depend on q_1 and the width d of the lens.

$$p_2 = |q_1| + d = -q_1 + d. \tag{2.28}$$

Note that q_1 is a negative number here, therefore $-q_1$ is positive. Put p_2 from Eq. 2.28 into Eq. 2.27 and add Eq. 2.26 and Eq. 2.27 to obtain the following.

$$\frac{1}{p_1} + \frac{1}{q_2} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{n_2}{n_1}\,\frac{d}{q_1(q_1 - d)}. \tag{2.29}$$

In the $d \to 0$ limit, called the **thin lens approximation**, the last term goes to zero, and $\mathrm{V_1}$ and $\mathrm{V_2}$ become one point. All distances are then measured from the center of the lens rather than either vertex. We can drop the subscripts on p and q since the distances are measured from the center.

$$\boxed{\frac{1}{p} + \frac{1}{q} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).} \tag{2.30}$$

This equation is called the **lens maker formula**. Lenses are usually used in air whose refractive index is very close to 1.0. In that case n_1 = 1.0 and n_2 is the refractive index of the material of the lens, which will now be denoted by the letter n without a subscript.

$$\frac{1}{p} + \frac{1}{q} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \quad (lensinair) \tag{2.31}$$

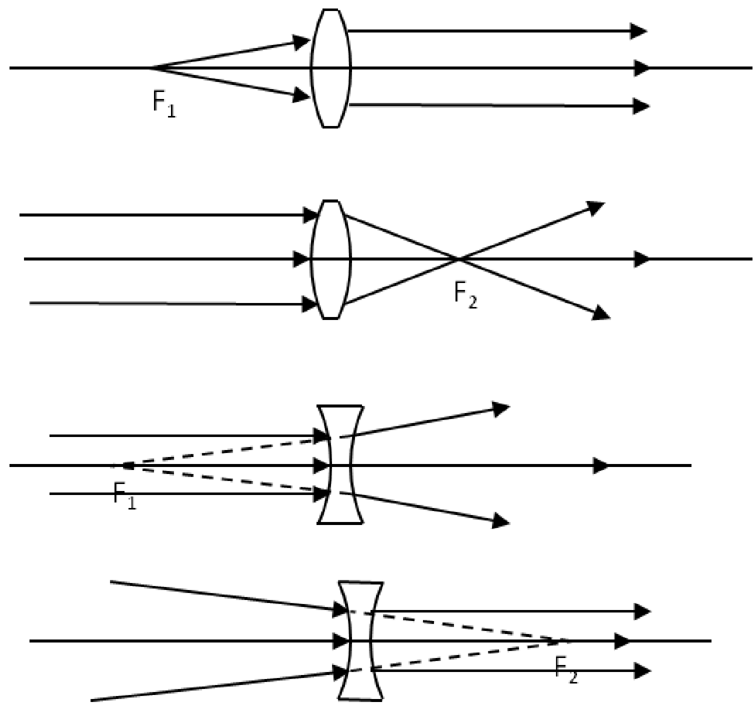


Figure 2.21: First and second foci of lenses.

The right hand side of this equation is a constant depending only on the optical property and geometrical parameters of the lens. We can also define foci F_1 and F_2 here. Recall that F_1 is the place where if the object is placed then the image is formed at infinity, and F_2 is the place where image is formed when the object is at infinity. But, since Eq. 2.30 has a symmetry of p and q switch, the distance of F_1 on the left side of the lens will be equal to distance of F_2 on the right side of the lens. Hence for a lens we have a single focal length denoted by f.

Object focal point F_1 and image focal point F_2 are symmetrically placed for thin lenses, i.e., they are at an equal distance from the lens as illustrated in Fig. 2.21.

To obtain an expression for focal length of a lens of refractive index n_2 placed in a medium of refractive index n_1, we set q to ∞ and p to f in Eq. 2.30 and solve for f.

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{2.32}$$

Writing the right side of Eq. 2.30 as $1/f$ we obtain the famous **thin**

lens equation.

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \quad \text{(sign convention also needed)} \tag{2.33}$$

To properly use the thin lens equation, the following sign conventions must be obeyed.

Sign Conventions for lenses:

1. p positive if left of the lens otherwise negative.
2. q positive if image to the right of the lens otherwise negative.
3. f positive for convex lens and negative for concave lens.

By using a finite size object on the axis you can prove that the magnification of image defined as the ratio of the image height to the object height is equal to the negative of the image distance to object distance $-q/p$.

Magnification m: By placing the object with bottom at the axis and the tip at an off-axis point you can study the relative size and orientation of the image from the location of the image point of the tip of of the object. The magnification m of an image is defined by the ratio of image height (h_i) to the object height (h_o).

$$m \equiv \frac{h_i}{h_o}. \tag{2.34}$$

If magnification is positive ($m > 0$) the image has the same orientation in space as the object as far as the vertical orientation is concerned. If magnification is negative ($m < 0$) then image is inverted with respect to the object. By using the geometry of the rays you can show that the ratio of the heights are related to the ratio of the image and object distances in the following way

$$m = \frac{h_i}{h_o} = -\frac{q}{p}. \tag{2.35}$$

2.4.2 Using the Thin Lens Equation

Example 2.4.1. Using Lens Equation # 1. Suppose you have been asked to make a bi-concave lens of focal length 20 cm. Find the radius of curvature of a symmetrically ground biconcave lens from a glass of refractive index 1.55 so that the its focal length in air is 20 cm.

Solution. We use the following,

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

where $R_1 < 0$ and $R_2 > 0$. Since we are using symmetric bi-concave lens, we have $|R_1| = |R_2|$. Let us write the radius of curvature as R.

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(-\frac{2}{R}\right).$$

Putting $f = -20$ cm, $n_2 = 1.55$ and $n_1 = 1.0$ we can solve for R.

$$R = 2 \times 0.55 \times 20 \text{ cm} = 22 \text{ cm}.$$

Example 2.4.2. Using Lens Equation # 2.

Find the location, orientation and magnification factor of the image in each of the following positions of an object of height 3 cm in front of a convex lens of focal length 10 cm. (a) $p = 50$ cm, (b) $p = 5$ cm, (c) $p = 20$ cm.

Solution. (a) $p = 50$ cm, $f = +10$ cm, $q = ?$

$$\begin{aligned} &\frac{1}{50 \text{ cm}} + \frac{1}{q} = \frac{1}{10 \text{ cm}} \\ &\frac{1}{q} = \frac{1}{10 \text{ cm}} - \frac{1}{50 \text{ cm}} = \frac{4}{50 \text{ cm}}. \end{aligned}$$

Therefore, $q = 12.5$ cm, which is positive, meaning the image is on the right side of the lens.

$$\text{Magnification,} \quad m = -\frac{q}{p} = -\frac{12.5 \text{ cm}}{50 \text{ cm}} = -\frac{1}{4}.$$

The negative magnification means that the image would be inverted. Since $|m| < 1$, the image would be smaller than the object. The size of the image is given by

$$|h_i| = |m| \times h_o = 0.25 \times 3 \text{ cm} = 0.75 \text{ cm}.$$

(b) $p = 5$ cm, $f = +10$ cm, $q = ?$

$$\begin{aligned} &\frac{1}{5 \text{ cm}} + \frac{1}{q} = \frac{1}{10 \text{ cm}} \\ &\frac{1}{q} = \frac{1}{10 \text{ cm}} - \frac{1}{5 \text{ cm}} = -\frac{1}{10 \text{ cm}}. \end{aligned}$$

Therefore, $q = -10$ cm, which is negative, meaning the image is on the left side of the lens, the same side as the object.

$$\text{Magnification,} \quad m = -\frac{q}{p} = -\frac{-10 \text{ cm}}{5 \text{ cm}} = +2.$$

The positive magnification means that the image would be upright, the same orientation as the object. Since $|m| > 1$, the image would be larger than the object, i.e. the image will be magnified. The size of the image is given by

$$|h_i| = |m| \times h_o = 2 \times 3 \text{ cm} = 6 \text{ cm}.$$

(c) $p = 20$ cm, $f = +10$ cm, $q = ?$

$$\frac{1}{20 \text{ cm}} + \frac{1}{q} = \frac{1}{10 \text{ cm}}$$
$$\frac{1}{q} = \frac{1}{10 \text{ cm}} - \frac{1}{20 \text{ cm}} = \frac{1}{20 \text{ cm}}.$$

Therefore, $q = 20$ cm, which is positive, meaning the image is on the right side of the lens.

$$\text{Magnification}, \quad m = -\frac{q}{p} = -\frac{20 \text{ cm}}{20 \text{ cm}} = -1.$$

The negative magnification means that the image would be inverted. Since $|m| = 1$, the image would be of the same size as the object. The size of the image would be given by

$$|h_i| = |m| \times h_o = 1 \times 3 \text{ cm} = 3 \text{ cm}.$$

Example 2.4.3. Using Lens Equation # 3. Find the location, orientation and magnification factor of the image in each of the following positions of a 3 cm tall object in front of a concave lens of focal length 10 cm. (a) p = 50 cm, (b) p = 5 cm, (c) p = 20 cm.

Solution. In this example we have a concave lens. This means $f < 0$ in the lens equation.

(a) $p = 50$ cm, $f = -10$ cm, $q = ?$

$$\frac{1}{50 \text{ cm}} + \frac{1}{q} = \frac{1}{-10 \text{ cm}}$$
$$\frac{1}{q} = -\frac{1}{10 \text{ cm}} - \frac{1}{50 \text{ cm}} = -\frac{6}{50 \text{ cm}}.$$

Therefore, $q = -8.33$ cm. The negative image distance means the image is on the left side of the lens, i.e. on the same side as the object.

$$\text{Magnification}, \quad m = -\frac{q}{p} = -\frac{-[50 \text{ cm}/6]}{50 \text{ cm}} = +\frac{1}{6}.$$

The positive magnification means that the image would have the same orientation as the object. Since $|m| < 1$, the image would be smaller than the object. The size of the image would be given by

$$|h_i| = |m| \times h_o = \frac{1}{6} \times 3 \text{ cm} = \frac{1}{2} \text{ cm}.$$

(b) $p = 5$ cm, $f = -10$ cm, $q = ?$

$$\frac{1}{5\text{ cm}} + \frac{1}{q} = \frac{1}{-10\text{ cm}}$$
$$\frac{1}{q} = -\frac{1}{10\text{ cm}} - \frac{1}{5\text{ cm}} = -\frac{3}{10\text{ cm}}.$$

Therefore, $q = -3.33$ cm. The negative image distance means the image is on the left side of the lens, i.e. on the same side as the object.

$$\text{Magnification,}\ \ m = -\frac{q}{p} = -\frac{-[10\text{ cm}/3]}{5\text{ cm}} = +\frac{2}{3}.$$

The positive magnification means that the image would have the same orientation as the object. Since $|m| < 1$, the image would be smaller than the object. The size of the image would be given by

$$|h_i| = |m| \times h_o = \frac{2}{3} \times 3\text{ cm} = 2\text{ cm}.$$

(c) $p = 20$ cm, $f = -10$ cm, $q = ?$

$$\frac{1}{20\text{ cm}} + \frac{1}{q} = \frac{1}{-10\text{ cm}}$$
$$\frac{1}{q} = -\frac{1}{10\text{ cm}} - \frac{1}{20\text{ cm}} = -\frac{3}{20\text{ cm}}.$$

Therefore, $q = -6.67$ cm. The negative image distance means the image is on the left side of the lens, i.e. on the same side as the object.

$$\text{Magnification,}\ \ m = -\frac{q}{p} = -\frac{-[20\text{ cm}/3]}{20\text{ cm}} = +\frac{1}{3}.$$

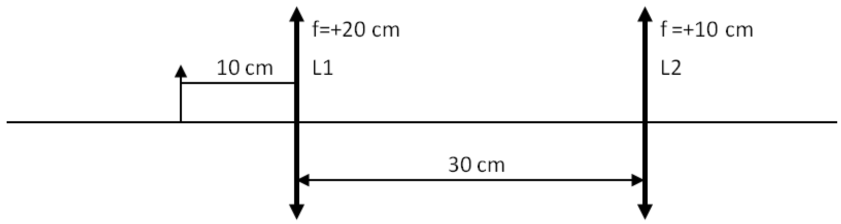
The positive magnification means that the image would have the same orientation as the object. Since $|m| < 1$, the image would be smaller than the object. The size of the image would be given by

$$|h_i| = |m| \times h_o = \frac{1}{3} \times 3\text{ cm} = 1\text{ cm}.$$

Example 2.4.4. Using Lens Equation # 4

Two convex lenses of focal lengths 20 cm and 10 cm are placed 30 cm apart. An object of height 2 cm is placed 10 cm in front of the lens of the focal length 20 cm. Find the location, orientation and magnification factor of the final image.

Solution. Let us first draw a schematic picture just to get the sense of the problem and not to do any exact ray tracing.



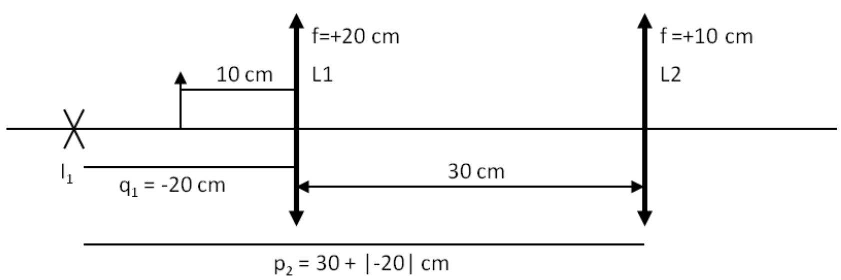
We will work one lens at a time. First we find the image from lens L1 and use this image as object for lens L2.

L1: $p = +10$ cm, $f = +20$ cm.

Therefore,

$$\frac{1}{10\text{ cm}} + \frac{1}{q} = \frac{1}{20\text{ cm}} \implies q = -20\text{ cm}.$$

The image distance is negative. This means image I_1 form lens L1 is formed on the left of L1. We usually mark the point I_1 in the figure now.



Magnification:

$$m_1 = -\frac{q}{p} = -\frac{-20\text{ cm}}{10\text{ cm}} = +2.$$

Now we use image I_1 as the "object" for lens L2.

$$p = +50\text{ cm}; \quad f = +10\text{ cm}.$$

Hence,

$$\frac{1}{50\text{ cm}} + \frac{1}{q} = \frac{1}{10\text{ cm}} \implies q = 12.5\text{ cm}.$$

The image distance is positive. This means image I_2 forms on the right of lens L2.

$$\text{Magnification,}\ \ m_2 = -\frac{q}{p} = -\frac{12.5\text{ cm}}{50\text{ cm}} = -1/4.$$

Hence the final image forms to the right of the two lens system at a distance 12.5 cm from the second lens. It is a real image since only the real rays from P will cross there. The net magnification of the image is

$$m = m_1 m_2 = 2 \times \left(-\frac{1}{4}\right) = -\frac{1}{2}.$$

Hence, the final image it is inverted and half the size of the object. That is, the final image is 1 cm tall and upside down.