3.1 SIMPLE HARMONIC OSCILLATIONS

3.1.1 Oscillatory Motion

Consider a block attached to a spring and placed on a frictionless table as shown in Fig. 3.1. The other end of the spring is attached

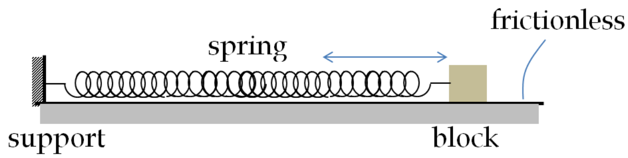


Figure 3.1: A spring between a block and a fixed support provides interaction between the block and the support. The spring force on the block causes the block to oscillate back and forth.

to a fixed support. You might say that the spring is the vehicle of interaction between the block and the support. The forces of gravity and normal on the block in the vertical direction are balanced.

Note that when the spring is in the relaxed state, there is no net force between the block and the support. This condition is called static equilibrium. The block is said to be in an **equilibrium** and the position of the block in this situation is called the **equilibrium position** of the block.

Restoring Force

Suppose, you pull the block a little distance from the equilibrium position away from the support, and release from rest, the block would begin to oscillate. The oscillations of the block will be accompanied by the successive contraction and expansion of the spring. The same thing happens if you push the block towards the support and let go of it. In either case, there would be a force on the block when it is away from the equilibrium position depending on the stretch or the contraction of the spring. The direction of this force is always towards the equilibrium position regardless of where the block is in its cycle. This type of force is called a **restoring force** since this force attempts to bring the object back to the equilibrium position.

Let us look at the direction of the force on the block at various points in one cycle of the motion of the block. Suppose the cycle of motion is $AOBOA$ in Fig. 3.2. When the block is at A, the force on the block is pointed to the left towards the equilibrium position O. When the block is at the equilibrium point O, there is no net force on the block. When the block is at B, which is on the other side of the equilibrium from point A, the force is pointed to the right towards O.

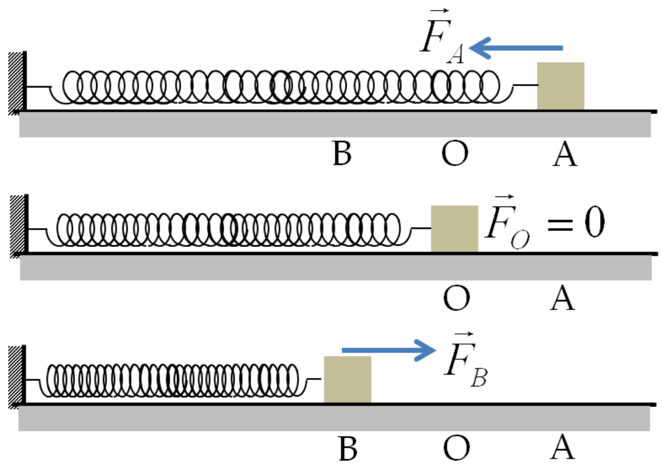


Figure 3.2: The force on the block changes direction as the block moves in one cycle about the equilibrium point O. The force on the block is always pointed towards the equilibrium point, and when the block is at the equilibrium point, the force is zero. The force on the block is labeled with the label of the point in the motion of the block.

Velocity Versus Acceleration

Since the direction of the acceleration is always towards the equilibrium point and the velocity is in the direction of motion, the acceleration and velocity will not be in the same direction at all points of the cycle of the oscillation. For instance, when the block is moving away from the equilibrium position, the velocity of the block would be pointed away from the equilibrium point but its acceleration will be pointed towards the equilibrium. Therefore, when the block is moving away from the equilibrium, the block will slow down and come to a stop and reverse its direction of motion as shown in Fig. 3.3.

At other points, when the block is moving towards the equilibrium, the velocity and acceleration of the block would be in the same direction. Therefore, when the block is moving towards the equilibrium, it will speed up. Suppose the block is moving towards the equilibrium from the right, say in the AOB direction in figure. As shown in the figure, when the block reaches the equilibrium point, its velocity will be pointed to the left and the acceleration will be zero at that point. The block will then continue to move to the left of the

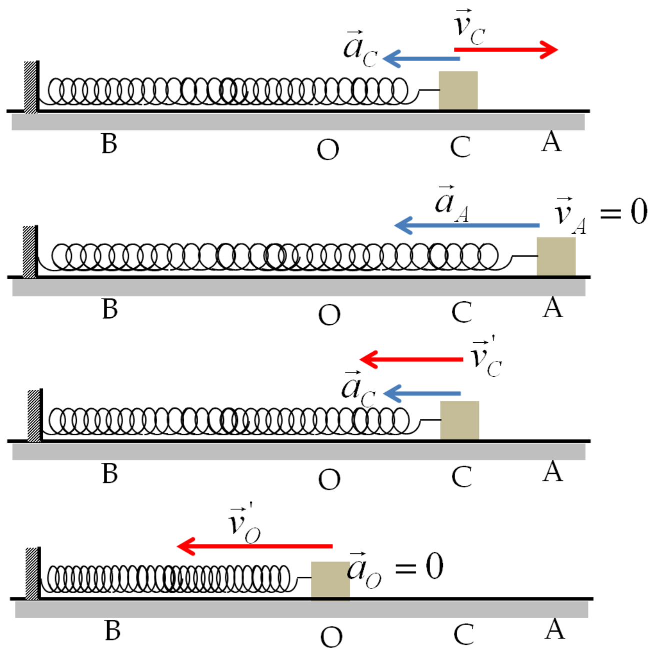


Figure 3.3: A part of the cycle of motion of a block attached to a spring is shown here. In the first figure at the top, the block is moving towards the right at point C and the acceleration is pointed to the left. This slows the block and brings the block to rest at point A. The acceleration of the block at A points in the direction of AO, i.e. towards the equilibrium O. When the block returns to point C, it is now moving with the velocity in the same direction as the acceleration. Therefore, the block continues to speed up and reaches the equilibrium point with the largest speed in the cycle. at the equilibrium the acceleration is zero. But, when the block goes on the other side of the equilibrium point, the velocity would be pointed to the left and the acceleration will be pointed towards the right. That would slow the block and would bring it to rest at point B. Now, due to the non-zero acceleration pointed to the right, the speed of the block will increase till it reaches back to the equilibrium point. Points A and B are the turning points of the motion of the block.

equilibrium. But, at those points, the acceleration would be pointed to the right which is in the opposite direction to the velocity at O at this instant. That means that, the block has the largest speed at the moment it passes through the equilibrium point O since after that the speed will start to decrease. The block would come to rest again at the other turning point B. After that, the block will pick up speed towards O, since during B to O, both the velocity and accelerations would be in the same direction.

The points where the velocity turns around are called **turning points**. In the present case, there will be two turning points, one to the right of O and one to the left of O. The block will move between these two turning points. The distance between the equilibrium point and the turning points is called the **amplitude** of the motion.

Thus, the block may start with zero velocity at a turning point, pick up speed, reach maximum speed at the equilibrium, then slow down to rest at the other turning point, then pick up speed again, reach maximum speed at the equilibrium, then slow down to rest at the first turning point. Therefore, a block attached to a spring is capable of back and forth motion indefinitely. This back and forth motion between the turning points is also called an **oscillatory motion**.

Further Remarks:

The oscillatory motion illustrated for a block attached to a spring is possible for many other systems, such as pendulum, bridge, buildings and towers. In all these physical situations, we notice that a back and forth oscillatory motion is possible due to the presence of **restoring force** in these systems. These systems are said to possess a **stable equilibrium** and the potential energy of these systems have the shape of a bowl as shown in Fig. 3.4.

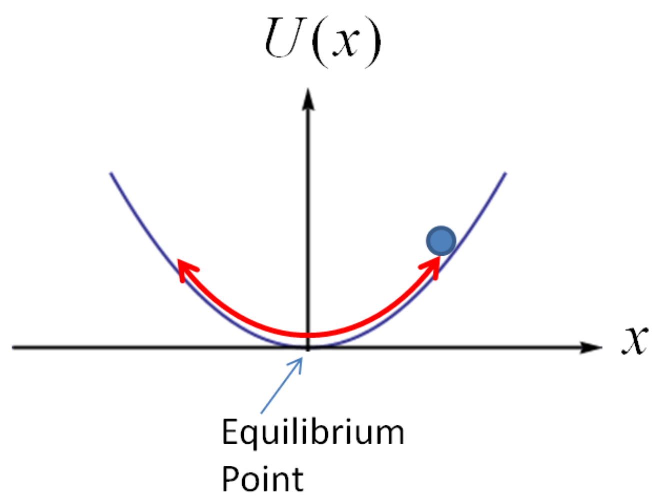


Figure 3.4: Stable equilibrium occurs near minimum of the potential energy. The force corresponding to this potential energy will be pointed towards the left for points on the right side of the equilibrium and towards the right for points on the left side of the equilibrium.

When the system is in an equilibrium state, there is no net force on the system, but when the system is displaced from the equilibrium, a restoring force develops that acts on the system. The restoring force acts to bring the system back to the equilibrium state. However, when the system returns to the equilibrium point, it overshoots and goes past the equilibrium point due to the motional inertia, causing the restoring force to act again, but this time in the opposite direction. In this way, a back and forth or oscillatory motion takes place when a system is released at a point that is away from a stable equilibrium point.

3.1.2 Analytic Study

To study the oscillatory motion of the block attached to a support through a spring analytically, we choose a coordinate system such that the origin is located at the equilibrium position and the positive x-axis is pointed to the right as shown in Fig. 3.5. Let m be the mass of the block and k be the spring constant of the spring. For simplicity, we will ignore the mass of the spring in our treatment here.

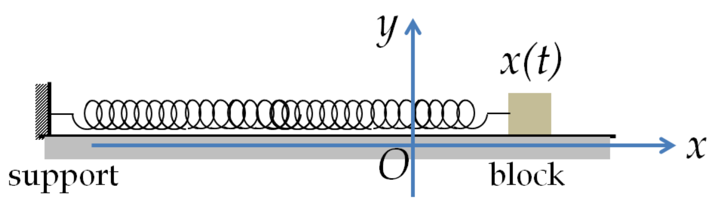


Figure 3.5: The coordinate system for the study of oscillations of a mass attached to a spring with origin at a point on the table where the equilibrium point is located. The position of the block at an arbitrary time t is given by its x-coordinate at that time. The table is frictionless so that the only force on the block is the spring force. The y and z-components of net force are zero.

As mentioned above, the forces on the block in the vertical direction cancel out, and we are left with only the spring force on the block. At an arbitrary time t, the x-component of the net force on the mass is from the change in the length of the spring. The difference between the x-coordinates of the block and the support is the length of the spring. Let x_s be the x-coordinate of the spring. Then, the length l_0 of the spring when at equilibrium state and the length l of the spring at an arbitrary time are

$$\begin{aligned}&\text{Relaxed state: } l_0 = 0 - x_s = -x_s\\&\text{Arbitrary state: } l = x - x_s\end{aligned}$$

Therefore, the change in the length of the spring at arbitrary time is

$$\Delta l = l - l_0 = x.$$

Therefore, the magnitude of the spring force $\vec{F}_s$ will be

$$\boxed{\text{Magnitude of spring force } \ |\vec{F}_s| = k\Delta l.}$$

The change in length of the spring can be written in terms of the x-coordinate of the block. Since, $x = 0$ corresponds to no change in the length of the spring, the absolute value of the x-coordinate of the block is equal to the change in the length of the spring.

$$\Delta l = |x|.$$

To write the x-component of the spring force in terms of the x-coordinate of the block, we notice that, when $x > 0$ the spring is stretched and the spring force is pointed towards the negative x-axis. When $x < 0$, the spring is compressed and the spring force is pointed towards the positive x-axis. Therefore, the x-component of the spring force will be given by multiplying the x-coordinate of the block by minus one.

$$\boxed{F_x = -k\ x.} \tag{3.1}$$

Newton's second law then gives us the following x-component of the equation of motion.

$$\boxed{m\ a_x = -k\ x,} \tag{3.2}$$

where a_x is the x-component of the acceleration of the block. We see that the acceleration of the block is proportional to the displacement of the block and points in the opposite direction to the position vector. Therefore, the acceleration of the block is not constant but depends on where the block is at the time.

A word of warning for students in order here. Since the acceleration of the block is not constant, you cannot use the formulas for the constant acceleration motion for a block attached to a spring. In particular,

$$\boxed{v_x \neq v_{0x} + a_x t}$$

$$\boxed{x - x_0 \neq v_{0x}t + \frac{1}{2}a_x t^2}$$

To understand and solve for the position of the block as a function of time, we find it helpful to write the acceleration in Eq. 3.2 as second derivative of position with respect to time. This makes the dependence of x with time more explicit.

$$\boxed{m\ \frac{d^2x}{dt^2} = -k\ x.} \tag{3.3}$$

Thus, the equation of motion of the block is a second-order ordinary differential equation. The solution of this equation will give us the position of the block at a particular time. When we specify the initial position and velocity we can obtain a unique solution of Eq. 3.3. The solution, written as position as a function of time, $x(t)$, gives us the position of the block at all times. You will learn to solve this equation systematically in a more advanced course in mathematics, but in the next subsection we will solve this equation by guessing the answer and some reasonable arguments.

3.1.3 Solving Equation of Motion

To solve Eq. 3.2 or 3.3 means finding $x(t)$ and $v_x(t)$ when the position and velocity of the block are given at another time, say at $t = 0$. To carry out the calculations, it is helpful to combine the parameters m and k into one parameter.

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2\ x,} \tag{3.4}$$

where we have introduced

$$\boxed{\omega^2 = k/m.} \tag{3.5}$$

The square for ω makes sure that k/m correspond to a positive number since both k and m are positive. We will see below that $|\omega|/2\pi$ is the frequency of the oscillator. We will often refer to ω itself as frequency, although it should be more appropriately called the **angular frequency**, since for each cycle ω changes by 2π radians. The frequency itself corresponds to the number of cycles per unit time and not to any radians per unit time. We will frequently ignore the problem with the naming and probably call ω frequency and hope that, from the context, you would understand that we are talking about the angular frequency.

Note that Eq. 3.4 shows that the displacement $x(t)$ from equilibrium is a function of time t, whose second derivative gives us back the same function multiplied by a negative constant. You may know from Calculus that there are two functions with this property: sine and cosine.

$$\frac{d^2\cos(\omega t)}{dt^2} = -\omega^2\cos(\omega t), \tag{3.6}$$

$$\frac{d^2\sin(\omega t)}{dt^2} = -\omega^2\sin(\omega t). \tag{3.7}$$

Any linear combination of sine and cosine with the same arguments will also work, as you can easily verify for the following function $f(t)$.

$$f(t) = A\ \cos(\omega t) + B\ \sin(\omega t)$$
$$\frac{d^2}{dt^2} f(t) = -\omega^2\ f(t). \tag{3.8}$$

Thus, a general solution of Eq. 3.4 can be written as

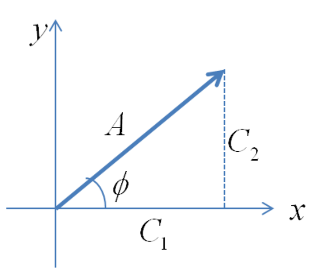
$$\boxed{x(t) = C_1\ \cos(\omega t) + C_2\ \sin(\omega t),} \tag{3.9}$$

where C_1 and C_2 are arbitrary constants, which are determined from the initial position and initial velocity of the block. Different initial condition are different ways the oscillator can get started. The solution given in Eq. 3.9 can also be written in the following form.

$$\boxed{x(t) = A\ \cos(\omega t + \phi).} \tag{3.10}$$

where A and ϕ are constants. When the solution is written in the form of Eq. 3.10 or as $x = A\sin(\omega t + \phi)$, then the constant A is called the **amplitude** and ϕ the **phase constant**. The amplitude and phase constant depend on the initial conditions on the block just as C_1 and c_2. As a matter of fact by demanding that the two solutions give the same function $x(t)$, you can find relation between the two ways of writing the solution given in Eqs. 3.9 and 3.10 by deducing the relations between (C_1, C_2) pair and (A, ϕ) pair.

$$A = \sqrt{C_1^2 + C_2^2}, \quad \tan\phi = \frac{C_2}{C_1}. \tag{3.11}$$



You might say that C_1 and C_2 are "Cartesian" and A and ϕ are "polar" if you think of C_1 as some "x", C_2 as some "y", A as "r" and ϕ as "θ". We say that the pair (A, ϕ) is a "vector" of magnitude A and angle ϕ with respect to x-axis and C_1 and C_2 are the x and y-components of this abstract vector. The (A, ϕ) representation is also called a phasor representation.

Definition of Simple Harmonic Motion

Equation 3.4 defines Simple Harmonic Oscillator. Any dynamical variable whose dynamics follows a differential equation that can me mapped on Eq. 3.4 is called a **Simple Harmonic Oscillator (SHO)**. As we have seen here, the dynamical variable can be written as a linear combination of sines and cosines of the same argument. The solution Eq. 3.9 or 3.10 is also said to describe a **Simple Harmonic Motion (SMH)** because the solution implies a dynamics governed by Eq. 3.4.

3.1.4 Specifying Initial Position and Velocity

The initial position and velocity of the block determine the amplitude and phase constant of the motion as we show in this section. Let the initial position and velocity be

$$x(0) = x_0;\ v_x(0) = v_{0x}. \tag{3.12}$$

Working with solution Eq. 3.9: If we use the solution in the form given in Eq. 3.9 the initial condition can be used to find the constants C_1 and C_2 for the given oscillator. When we set $t = 0$, the sine term becomes zero.

$$x_0 = C_1. \tag{3.13}$$

Taking the time derivative turns cosine into sine and vice versa. Therefore the velocity at initial time is related to C_2.

$$v_{0x} = \omega C_2. \tag{3.14}$$

Finally, the solution incorporating initial conditions on position and velocity is

$$\boxed{x(t) = x_0\ \cos(\omega t) + \frac{v_{0x}}{\omega}\ \sin(\omega t),} \tag{3.15}$$

Working with solution Eq. 3.10: Using the initial condition on position, we find the following from Eq. 3.10

$$x_0 = A\ \cos\phi. \tag{3.16}$$

To use the initial condition on velocity, we first take the first derivative of x with respect to time and then set $t = 0$.

$$v_{0x} = v_x(0) = \left.\frac{dx}{dt}\right|_{t=0} = -A\ \omega\ \sin\phi. \tag{3.17}$$

From Eqs. 3.16 and 3.17 we find the following for A and ϕ.

$$A = \sqrt{x_0^2 + (v_{0x}/\omega)^2}\ ;\ \tan\phi = -\frac{v_{0x}}{\omega x_0}. \tag{3.18}$$

3.1.5 Physical Meaning of Amplitude, A

The amplitude represents the maximum displacement of the oscillator from the equilibrium on either side of the equilibrium as a plot of the solution in Fig. 3.6 shows. The block turns around at $x = \pm A$, which are the **turning points** of motion. The velocities of the block at the turning points are zero.

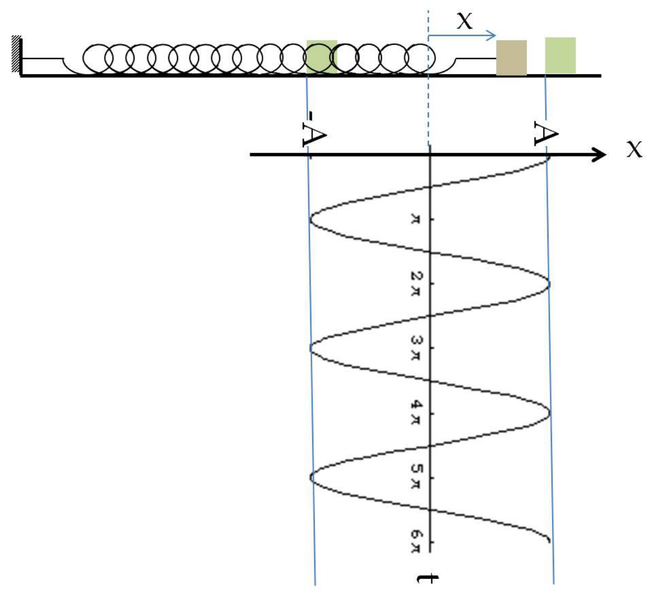


Figure 3.6: Plot of x vs t for $\omega = 1$, and $\phi = 0$. A one-dimensional harmonic oscillator moves within a displacement A from the equilibrium on either side.

When the block approaches the turning point on the right its velocity is positive and decreasing since the force on the block would be pointed to the left and increasing. At the turning point, the force, and hence the acceleration of the block, is the largest and the velocity is zero. After the block's motion turns around at the turning point $x = A$, the velocity and acceleration both point to the left, which accelerates the block to the equilibrium point at $x = 0$. There is a similar turning of the motion near the other turning point at $x = -A$.

3.1.6 Physical Meaning of ω

The solution of simple harmonic motion is periodic in time since it describes the periodic motion of the mass attached to the spring. Let T be the time for one complete cycle of motion. The time T is called the **time period** of the oscillator. We expect both the position and the velocity to be periodic functions of time with the same time period T.

$$x(t+T) = x(t) \quad \text{and} \quad v_x(t+T) = v_x(t). \tag{3.19}$$

Demanding the periodicity in $x(t)$ given in Eq. 3.10 we find that ω is related to the period:

$$\begin{aligned} & A\cos(\omega t + \omega T + \phi) = A\cos(\omega t + \phi). \\ \Longrightarrow \quad & \omega T = 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots , \end{aligned} \tag{3.20}$$

Clearly $n = 0$ is not the solution since that would mean T be zero, which is not the case here. The $n = 1$ case refers to a situation after one time period, and $n = -1$, to a situation one time period earlier. The constant ω is therefore 2π over the time period.

The inverse of time period gives the number of cycles of oscillations completed by the oscillator in a unit time. This is called the **frequency** of the oscillator. We will denote the frequency by the letter f.

$$f = \frac{1}{T}. \tag{3.21}$$

The SI unit of frequency is 1/sec, which is given its own name, **the Hertz (Hz)**. Therefore, ω can be written in terms of frequency f as

$$\omega = 2\pi f. \tag{3.22}$$

This says that ω "counts" radians in unit time instead of cycles in unit time. Therefore, we call ω an angular frequency and f regular frequency. Because f is number of cycles per unit time, f is a positive number. Since ω is a positive number times f, ω will also be positive. Therefore when we take square-root of ω^2, we keep only the positive root.

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}. \tag{3.23}$$

The angular frequency ω depends only on the mass and the spring constant. That is why ω is also called the **natural frequency** of the oscillator.

3.1.7 Analogy to Circular Motion

Some aspects of a particle moving in a circle provide useful tools for visualizing the Simple Harmonic Motion in another way. If a particle is moving in a circle of radius r about the origin and be at an angle θ at time t, then the x and y-coordinates of the position would be given by $r\cos(\theta)$ and $r\sin(\theta)$ respectively.

We found above that the displacement of a Simple Harmonic Motion can be given by a cosine function of time. Therefore, we can represent a simple harmonic motion by the x-component of the motion of a fictitious particle moving uniformly in a circle as shown in Fig. 3.7. In this picture of a Simple Harmonic Motion, ω refers to the angular speed of the fictitious particle. Since the fictitious particle moves uniformly, the magnitude of its angular displacement $\Delta\theta$ of the fictitious particle in time interval Δt would be given by

$$\Delta\theta = \omega\Delta t. \tag{3.24}$$

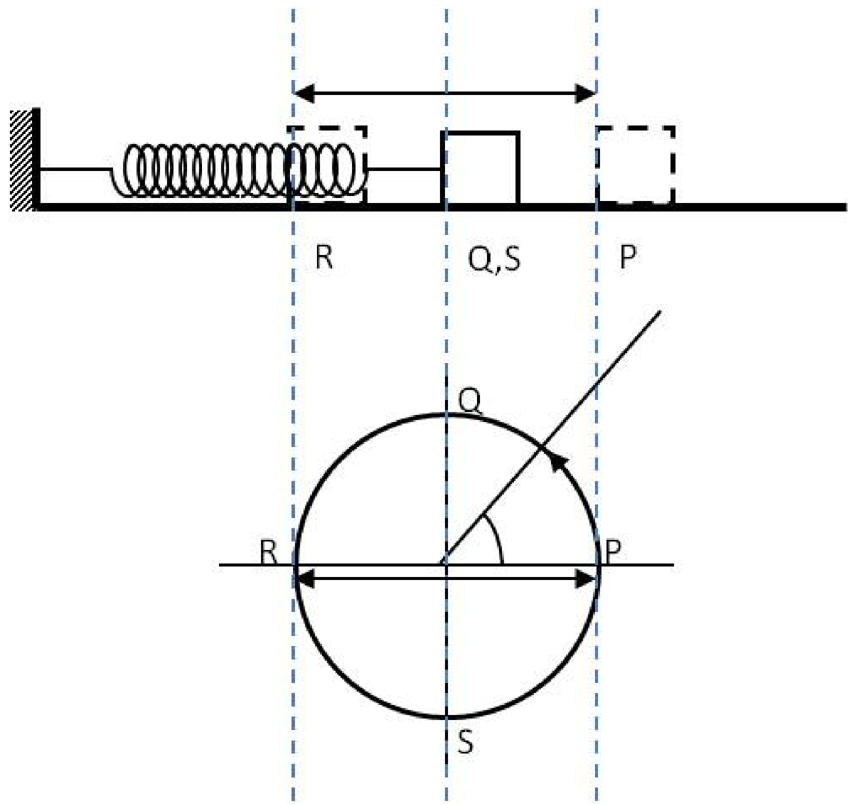
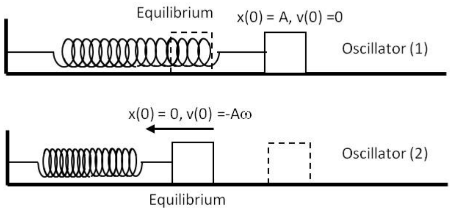


Figure 3.7: The cyclic process of oscillatory motion of the block represented as a point moving on a circle. The angular frequency ω corresponds to the angle in radians covered by the point on the circle per unit time, so that angle covered in time Δt is $\omega\Delta t$. Here P, Q, R, S are successive positions of the block as it executes simple harmonic motion.

3.1.8 Physical Meaning of Phase Constant, ϕ

The phase constant ϕ is related to the relative position of the block in its cycle if the cycle is represented by a rotation of 2π radians of a fictitious particle in the circular motion as explained above. To get a feel for the physical information contained in the phase constant, we examine two oscillators with $\phi_1 = 0$ and $\phi_2 = \frac{\pi}{2}$ radians respectively as shown in Fig. 3.8. Notice that the second oscillator is always ahead of the first by quarter of a cycle: for instance, $x = A$ is reached by oscillator 2 before oscillator 1 in any cycle. In terms of 2π radians in one cycle, this corresponds to a phase difference of $\frac{\pi}{2}$ radians.

Thus, the phase constant ϕ represents a measure of time in the cycle of the oscillator as measured in terms of angle of the fictitious particle moving in a circle. You may practice drawing displacements



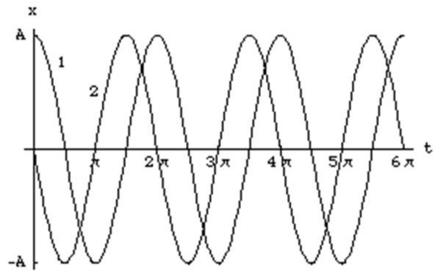


Figure 3.8: Oscillators started in two different ways such that they have the same amplitude but have different phase constants. We plot their position in time for $\omega = 1$ rad/sec and the following initial conditions: (1) $x = A, v = 0$, and (2) $x = 0, v = -A$. The relative phase different is $\frac{\pi}{2}$ radians or $\frac{1}{4}$ cycle.

of two oscillators of same frequency and amplitude but differing in phase by π radians or $\frac{\pi}{3}$ radians and see for yourself the relation between oscillators.

3.1.9 Energy of a Simple Harmonic Oscillator

The energy of the block is the sum of the kinetic and potential energies of the block. The kinetic energy of the block is given as

$$KE = \frac{1}{2}mv^2, \tag{3.25}$$

and the potential energy of the block for the spring force is given as

$$PE = \frac{1}{2}kx^2, \tag{3.26}$$

with the reference for the potential energy set at zero when the spring is relaxed. Now, since the block oscillates sinusoidally, meaning as a sine or cosine function, the kinetic and potential energies oscillate sinusoidally also. However, the total energy E of the oscillator is constant since the spring force is a conservative force.

$$E = KE + PE = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}. \tag{3.27}$$

You can verify that $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is constant for a harmonic oscillator by using either Eq. 3.9 or 3.10 for $x(t)$.

Equation 3.27 asserts that the energy of a Simple Harmonic Oscillator contains two terms, an inertial term that is quadratic in velocity and an elastic restoring term that is quadratic in displacement. Any system whose conserved energy can be written this way can be treated as a Simple Harmonic Oscillator (SHO). Suppose the dynamical variable of an abstract system is $f(t)$ and its time derivative is

$\dot{f}$, then, this system will be an SHO if its energy takes the following mathematical form for some constant a and b.

$$E = a\dot{f}^2 + bf^2. \tag{3.28}$$

From the expression for the energy of oscillator given in Eq. 3.27, it is clear that the expression for the energy can also be used to deduce the frequency of the oscillator. The ratio of the constant $(k/2)$ multiplying the displacement squared and the constant $(m/2)$ multiplying the velocity squared is equal to the angular frequency squared.

$$\omega^2 = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } v^2} = \frac{k}{m}. \tag{3.29}$$

As the block gets closer to the extreme points of the motion, the magnitude $|x|$ of the displacement from equilibrium becomes closer to the maximum. At the turning points, the potential energy has the largest value, since at these points, there is no kinetic energy and all the energy must be in the potential energy. When the block returns to the point which corresponds to a point when the spring is relaxed, then the potential energy of the block will be zero and the energy would all be in the kinetic energy.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2, \tag{3.30}$$

where v_{max} is the speed when the block passes the equilibrium point. This energy conservation is for a block attached to a spring and placed on a frictionless table and the other end of the spring is attached to a rigid support.

Note that if there are other forces on the block in the block/spring system, then the zero potential energy and zero force may not be the same place. We will analyze the hanging mass attached to a spring later in the chapter where there will be two forces along the axis of the spring, the spring force and gravity.

Example 3.1.1. Conservation of Energy of a Harmonic Oscillator. A block of mass 0.6 kg is attached to a spring of spring constant 180 N/m and negligible mass compared to the mass of the block. The block is placed on a frictionless horizontal table, pulled horizontally 3 cm from its equilibrium position, and released with zero speed. Evaluate the speed of the block at the time (a) the block crosses the equilibrium position, and (b) when the block is 1.5 cm from the equilibrium point.

Solution. Since no friction acts on the block, the energy is conserved. Let us denote the original position as point A, the equilibrium as point B and 1.5 cm from the equilibrium as point C.

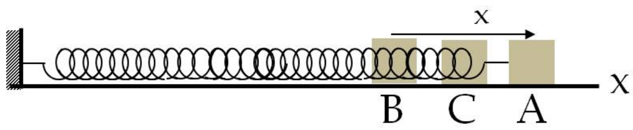


Figure 3.9: Example 3.1.1

(a) When the block is at point A, it is not moving, therefore all the energy is in the potential energy stored in the spring.

$$E_A = \frac{1}{2}kx_A^2 = \frac{1}{2}(180 \text{ N/m})(0.03 \text{ m})^2 = 0.081 \text{ J}.$$

When the block is at point B, the spring is neither stretched not compressed, therefore there is no potential energy and the entire energy is contained in the Kinetic energy.

$$E_B = \frac{1}{2}mv_B^2 = \frac{1}{2}(0.6 \text{ kg})v_B^2.$$

Equating the energy at B to the energy at A we find the speed of the block when it is moving past the equilibrium point B.

$$0.3\ v_B^2 = 0.081 \Rightarrow v_B = 0.52 \text{ m/s}.$$

(b) When the block is at point C, which could be on either side of the equilibrium a distance of 1.5 cm away from the equilibrium, the spring is either compressed or stretched. Therefore, there will be potential energy stored in the spring. But, the stored potential energy is less than the starting energy, hence, the rest of the energy will be in the form of kinetic energy of the block.

$$E_C = \frac{1}{2}mv_C^2 + \frac{1}{2}kx_C^2 = 0.3\ v_C^2 + 0.02\text{J}.$$

Equating E_C to E_A, and solving for v_C we find the speed when the block is 1.5 cm from the equilibrium to be

$$0.3\ v_C^2 + 0.02 = 0.081 \Rightarrow v_C = 0.45 \text{ m/s}.$$

3.1.10 Vertically Oscillating Block

We now examine a block attached to a spring but hung vertically by fixing the other end to a support in the ceiling. There are two forces on block in the vertical direction and none in the horizontal direction. Let m be the mass of the block and k the spring constant of the spring. We also assume that the mass of the spring is negligible.

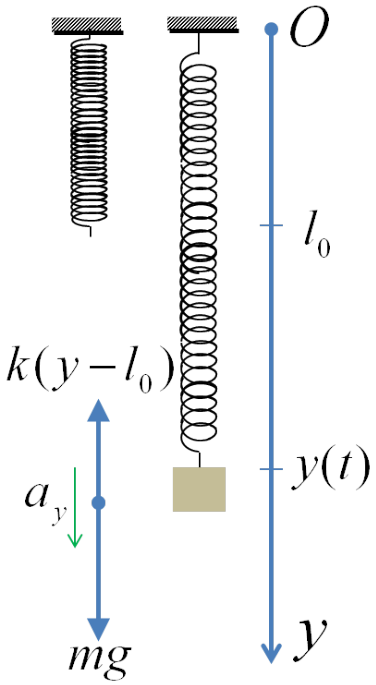


Figure 3.10: Choice of origin and axis for the calculations regarding a vertically oscillating block attached to a spring.

Will the block have the same frequency of oscillation and same energy formulas as the block on the horizontal frictionless table discussed above?

To analyze the motion of the block we need to choose the origin and one axis. We will choose a coordinate axes so that the positive y-axis is pointed vertically down as shown in Fig. 3.10. Pointing y-axis vertically down will be helpful in writing out the expression for the spring force as you will see below.

Where should we place the origin? Note the force by the spring is proportional to the change in the length of the spring with respect to the length the spring has when it is relaxed. Suppose we place the origin at the ceiling where the spring is held. This is a convenient place for out initial calculations as you will see below. We may end up choosing another place for the origin after we have understood the calculations a little bit more.

Let l_0 be the relaxed length of the spring when it is hanging from the ceiling and the block is not attached. When the block is attached, the spring stretches. Let y be the position of the block at an arbitrary time t. Then, the change in the length of the spring at this time is given by $y - l_0$. The y-component F_y of the spring force is now readily written down.

$$F_y = -k(y - l_0).$$

Let us make sure this gives correct directions for the force. When $y - l_0 > 0$, then the spring is stretched and so the force should be pointed up. That means towards negative y-axis. That works out with the negative sign in this formula. You should do a similar analysis to make sure the formula works for $y - l_0 < 0$.

The free-body diagram of the block with acceleration given by two time derivatives of the y-coordinate gives the equation of motion along y-axis as

$$m\frac{d^2y}{dt^2} = -k(y - l_0) + mg. \tag{3.31}$$

Note the y-component of weight is positive since y-axis is pointed down. This equation is more complicated than the equation for the block on the horizontal table. This equation is simpler if written with respect to origin at the equilibrium point of the block. We can find the y-coordinate of the equilibrium point by setting the acceleration in Eq. 3.31 to zero. Let the equilibrium be at $y = y_e$. Then we find that

$$-k(y_e - l_0) + mg = 0,$$

which gives

$$y_e = l_0 + \frac{mg}{k}. \tag{3.32}$$

To move the origin to the new place, all we have to do is change to new y-variable, say y' which is related to the old y by

$$y' = y - y_e. \tag{3.33}$$

Therefore, y' will be the y-coordinate from the point in space where the equilibrium is located. The acceleration in the y' coordinate is equal to the acceleration in the y-coordinate as you can see by taking two time derivatives in this equation. Now, let us substitute Eq. 3.33 into Eq. 3.31. We find

$$m\frac{d^2y'}{dt^2} = -ky' \tag{3.34}$$

The equation with respect to the equilibrium is same as the Simple Harmonic Oscillator (SHO) equation. Therefore, we will have the same solution for the y' coordinate as for any other SHO.

$$y'(t) = A\cos(\omega t + \phi) \quad \text{or} \quad C_1\cos(\omega t) + C_2\sin(\omega t), \tag{3.35}$$

with angular frequency, $\omega = \sqrt{k/m}$ as obtained from Eq. 3.34.

Energy Considerations in Vertical Oscillations: The energy of the block will again be equal to the sum of the kinetic and potential energies of the block. Here, there are two sources of potential energy. Therefore, the energy can be written as

$$E = \frac{1}{2}mv(t)^2 + \frac{1}{2}k\left[\Delta l(t)\right]^2 + mgh(t) \tag{3.36}$$

where $\Delta l(t)$ is the change in the length of the spring at time t, $h(t)$ is the height of the block at that instant from a reference height for gravitational potential energy and $v(t)$ is the speed of the block at that time. We can write more explicit expressions of the potential energy with respect to the origin at the ceiling as the reference for the zero of the gravitational energy. With this choice, the gravitational potential energy is $-mgy$, where y is the y-coordinate of the block. The change in the length of the spring can be written as $y - l_0$ in this coordinate system. Therefore, the energy of the block in the coordinate system with origin at the ceiling and y-axis pointed down is

$$E = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}k\left[y(t) - l_0\right]^2 - mgy(t). \tag{3.37}$$

This can be written in terms of y' coordinate of the block as

$$E = \frac{1}{2}m\left(\frac{dy'}{dt}\right)^2 + \frac{1}{2}ky'^2 - \frac{mg}{2k}\left(mg + 2kl_0\right), \tag{3.38}$$

which shows that the potential energy in the y' does not the simple $1/2kx^2$ type formula in the case of vertical oscillations. We can make another change in variable to "discover" the SHO nature of this system in the energy picture as well. Let

$$u(t) = y' - \frac{1}{k}\sqrt{mg(mg + 2kl_0)}.$$

In the u variable which shifts the origin to yet another place on the y-axis, the energy becomes

$$E = \frac{1}{2}m\left(\frac{du}{dt}\right)^2 + \frac{1}{2}ku^2.$$