12.2 PASSIVE CIRCUIT ELEMENTS

12.2.1 Voltage Across an Inductor

The self-inductance of a circuit L tells us how the rate of change of magnetic flux Φ_B through the area of the circuit depends on the rate at which the current in the circuit changes.

$$\frac{d\Phi_B}{dt} = L\frac{dI}{dt}. \tag{12.9}$$

Faraday's law relates this change in magnetic flux to the loop line integral of electric field in the circuit. With the direction of the loop for the line integral in the direction of the electric field in the loop, we obtain the following equation.

$$\oint \vec{E}\cdot d\vec{l} = -L\frac{dI}{dt}. \quad \text{(at each instant)} \tag{12.10}$$

Since the electric field in the circuit with a solenoid is induced by the changing magnetic field in the coils of a solenoid, the electric field is a non-conservative field and it is not possible to write the electric field as the gradient of the electric potential. In the chapter on Faraday's law we came across the circuits that involved inductance and we dealt with them in terms of the induced EMF in the Faraday's flux rule rather than voltages. We can continue to proceed with the same approach here.

However, an alternative somewhat wrong method of dealing with the dynamic circuits is widely practised which can be justified by making "reasonable" assumptions. We will like to be sure about these assumptions so that we can go ahead and use these simpler methods when applicable and resort to the more fundamental method based on Farday's law when the simpler methods are not applicable. These assumptions make it possible to think in terms of voltages even in dynamic circuits and one can make use of Kirchhoff's rules. We have already seen the types of assumptions we need when we discussed the concept of the voltage across an AC generator. Similar assumptions are required here as well. We will repeat the arguments for an inductor here.

Assumptions

1. The wires in the inductors are perfect conductors.
2. The magnetic field outside the inductor is zero.

These assumptions in the circuit shown in Fig. 12.6 allow us to split the line integral for electric field in the flux rule to one part inside the inductor and the other outside.

$$\oint \vec{E}\cdot d\vec{l} = \int_{a\ \text{[inside]}}^{b} \vec{E}_{\text{in}}\cdot d\vec{l} + \int_{b\ \text{[outside]}}^{a} \vec{E}_{\text{out}}\cdot d\vec{l}$$

The line integral inside the inductor gives zero since the electric field in a perfect inductor is required to be zero.

$$\int_{a\ \text{[inside]}}^{b} \vec{E}_{\text{in}}\cdot d\vec{l} = 0 \quad \text{(since perfect conductor)}$$

Therefore, only the contribution from the loop that is outside the inductor can be non-zero. With the assumption that the magnetic field outside is zero, the curl of the electric field outside is zero, which means that the electric field outside the inductor would be a conservative field. Therefore, the line integral over the electric field outside the inductor will be independent of the path and will depend only on the end points which are the two terminals of the inductor.

$$\int_{b\ \text{[outside]}}^{a} \vec{E}_{\text{out}}\cdot d\vec{l} = -(V_a - V_b) \quad \text{(since conservative } \vec{E} \text{ field).}$$

Now, equating the line integral of the electric field around the entire loop to the induced EMF due to the change in magnetic flux we obtain the voltage across the inductor.

$$\boxed{V_L \equiv V_a - V_b = -\int_b^a \vec{E}_{\text{out}}\cdot d\vec{l} = -\oint \vec{E}\cdot d\vec{l} = L\frac{dI}{dt}.} \tag{12.11}$$

Note that the current direction inside the solenoid is assumed to be in the a to b direction. Thus, if the current is increasing, then $V_a > V_b$. That is voltage will drop as you go in the direction of the current when the current is increasing. This decrease corresponds to the back EMF in the circuit, which mostly comes from the inductor part of the circuit.

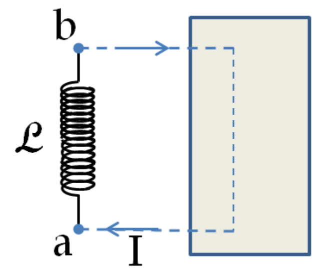

Figure 12.6: Voltage across an inductor becomes equal to the EMF induced in the circuit with perfect conductor and localized magnetic field assumptions.

Is there an Ohm's law for an inductor?

What is the relation between the voltage across an inductor and the current through the inductor? Let us say, there is a current $I(t)$ through the inductor given by

$$I(t) = I_0 \cos(\omega\, t). \tag{12.12}$$

The voltage across the inductor at that instant will be

$$V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega\, t). \tag{12.13}$$

This says that when the current varies as cosine, then voltage varies as negative sine. We can write both of these functions as sines or cosines. The usual practice is to write everything as cosine and use the phasor diagram to compare the phases of sinusoidal functions. Therefore, we write the voltage as cosine also, which is

$$V_L = \omega L I_0 \cos\left(\omega\, t + \frac{\pi}{2}\right). \tag{12.14}$$

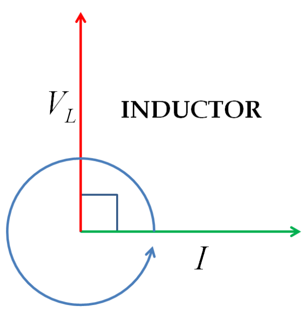

Figure 12.7: The phasor for the voltage across an inductor leads the current into the inductor by 90°.

Remember that the factor multiplying the cosine has to be positive. We find that there is no simple relation we can call "Ohm's Law". Actually, the phasors for I and V_L are perpendicular to each other. The phase shows that when the phasors rotate the phasor for V_L leads the phasor for I being always in front of it by 90° as shown in Fig. 12.8.

The ratio of the magnitude of the voltage to the magnitude of current will have units of resistance, but it is not resistance. It is just a resistance-like quantity, called the reactance of the inductor, and denoted by X_L.

$$\boxed{X_L = \frac{\text{Amplitude of } V_L}{\text{Amplitide of } I} = \omega L} \tag{12.15}$$

12.2.2 Voltage Across a Capacitor

The voltage across a capacitor C in a dynamic circuit also requires certain assumptions to make it a valid concept. The following assumptions will be made so that the line-integral of the electric field in a loop will end up leaving only that part of the integral nonzero where the electric field is conservative by having its curl vanish there. This requires us to have the following assumptions.

1. Plates and all wires are perfect conductors. This will make electric field zero everywhere except between the plates.

2. All electric field lines from one plate land on the other plate. This will make sure the two plates have equal charges of opposite types at all times.

3. No magnetic fields near the capacitor. This will make the electric field in the space near the capacitor conservative. We can close a loop connecting the two terminals of the capacitor and obtain a loop in which the line integral of $\vec{E}$ will be zero.

$$\oint \vec{E}\cdot d\vec{l} = 0. \tag{12.16}$$

Armed with these assumptions, we can carry out the calculation in Eq. 12.16 for the loop shown in Fig. ??. Let the positive side of the capacitor be connected to the terminal a. Then, the loop in the direction of the electric field will be a-P_1-P_2-b-a. Here b-a is the part of the loop outside the capacitor.

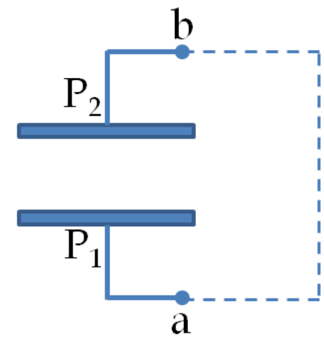

Figure 12.8: The loop for the calculation of voltage across a capacitor.

$$\oint \vec{E}\cdot d\vec{l} = \int_a^{P_1} \vec{E}\cdot d\vec{l} + \int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} + \int_{P_2}^{b} \vec{E}\cdot d\vec{l} + \int_b^{a} \vec{E}\cdot d\vec{l} = 0.$$

The integration between the plates would yield Q/C as shown for capacitors before. The integral from a to P_1 and from P_2 to b will be zero since the wires are perfect conductors. The integral between the plates will give $q(t)/C$ where charge on the plate P_1 is assumed to be positive at the instant. The integral from b to a will give the potential difference, $V_b - V_a$. Therefore, we obtain the following for the voltage across the capacitor

$$\boxed{V_C(t) \equiv V_a - V_b = \frac{q(t)}{C}.} \tag{12.17}$$

Is there an Ohm's Law for a capacitor?

What is the relation between the voltage across a capacitor and the current through the capacitor? The relation between the voltage and current for a capacitor is obtained by taking the derivative of $V_C(t)$ and using the definition of current as

$$I(t) = \frac{dq}{dt}. \tag{12.18}$$

We obtain the following for the relation between voltage across the capacitor $V_C(t)$ and the current $I(t)$ into the positive plate of the capacitor.

$$\frac{dV_C}{dt} = \frac{1}{C}\, I(t) \tag{12.19}$$

Now, we can work out the relation between sinusoidal $V_C(t)$ and $I(t)$. Suppose at some instant the voltage across the capacitor is

$$V_C(t) = V_0 \cos(\omega\, t). \tag{12.20}$$

What will be the current at that instant? Using Eq. .12.19 we find

$$I(t) = -\omega C V_0 \sin(\omega\, t) = \omega C V_0 \cos\left(\omega\, t + \frac{\pi}{2}\right). \tag{12.21}$$

This says that when the voltage across a capacitor varies as cosine the current varies as the negative sine. Writing both as cosine, we find that in a capacitor, the phasor situation for the current and voltage

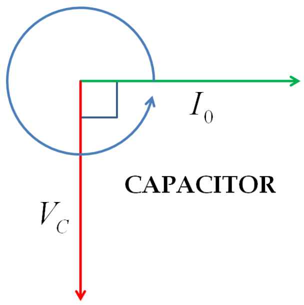

Figure 12.9: The phasor for the voltage across an inductor leads the current into the inductor by 90°.

phasors is completely opposite of the case for the inductor. Just like the inductor, there is no simple relation we can call the "Ohm's Law" for capacitors.

Actually, the phasors for I and V_C are perpendicular to each other just like the inductor, but here, when the phasors rotate the phasor for V_C lags the phasor for I being always behind by 90° as shown in Fig. 12.9.

Once again, the ratio of the magnitude of the voltage to the magnitude of current will have the unit of resistance, but it is not resistance. It is just a resistance-like quantity, called the reactance of the capacitor, and denoted by X_C.

$$\boxed{X_C = \frac{\text{Amplitude of } V_C}{\text{Amplitude of } I} = \frac{1}{\omega C}} \tag{12.22}$$

12.2.3 Voltage Across a Resistor

The resistors act similarly in the AC circuits as they do in the DC circuits as long as the frequency is not too high. The voltage drop V_R across a resistor follows the Ohm's law and given by the usual Ohm's law equation.

$$V_R = IR. \tag{12.23}$$

Since there is no difference in the phases of the current through a resistor and the voltage across the resistor, their phasors are pointed in the same direction and rotate at the same rate.

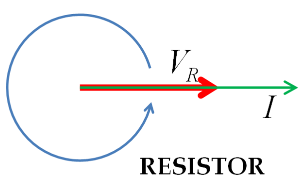

Figure 12.10: The phasors for the voltage across a resistor and the current through the resistor are in the same direction and rotate together at the same rate.