7.1 MASS CONSERVATION AND STEADY FLUID FLOW

The rate of flow of a fluid is described by the current or mass rate of flow, which is the rate at which mass of the fluid moves past a point.

$$\boxed{\text{Mass rate of flow, Current } = \frac{dm}{dt}} \quad (7.1)$$

To be concrete, consider a uniform flow through a pipe of cross-section area A with velocity $\vec{v}$. In duration Δt, you will find that the fluid within a distance $v\Delta t$ would pass through the cross-section of area A (Fig. 7.2). If the density of the fluid is ρ, then the mass of fluid flowing through A per unit time would be equal to $[\rho \times (Av\Delta t)]$. Therefore, the **mass rate of flow**, also called **current**, is

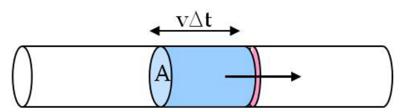

Figure 7.2: Volume of the liquid that will cross the area of cross-section in unit time.

$$\text{Mass rate of flow, Current } = \rho A v. \quad (7.2)$$

The volume rate of flow is obtained by dividing both sides by the density.

$$\boxed{\text{Volume rate of flow } = Av.} \quad (7.3)$$

In a steadily flowing fluid, fluid does not accumulate at any point. Therefore the density does not change in time. We can use this observation about a steadily flowing fluid to relate the flow properties at two different points. Towards that end, let us consider a steadily flowing fluid in a pipe that varies in the cross-section as shown in Fig. 7.3. Let A_1 and A_2 be the cross-sectional areas of the entrance and exit respectively, and v_1 and v_2 be the corresponding speeds. Let the density of fluid at the entrance and exit be ρ_1 and ρ_2 respectively.

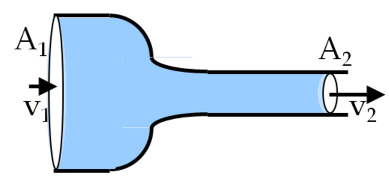

Figure 7.3: Geometry for derivation of the equation of continuity. The amount of liquid passing 1 must equal the amount leaving 2 if the liquid is incompressible.

Equating the mass of the material entering the space between A_1 and A_2 in an arbitrary time interval Δt with the amount exiting.

$$A_1 v_1 \rho_1 \Delta t = A_2 v_2 \rho_2 \Delta t.$$

Therefore in a steady flow

$$\boxed{A_1 v_1 \rho_1 = A_2 v_2 \rho_2.} \quad (7.4)$$

Equation 7.4 is called the equation of continuity. The continuity equation, as you have seen above, is just a statement about the conservation of mass of the fluid. For a uniform density fluid, the constant density cancels out from the two sides and we obtain a simpler equation.

$$\boxed{A_1 v_1 = A_2 v_2 \ \text{ (uniform density).}} \quad (7.5)$$

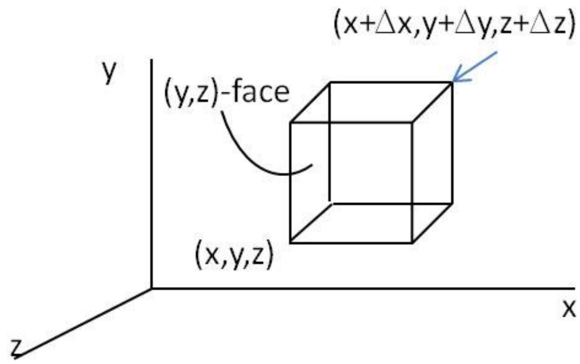

Figure 7.4: Set-up for calculation of general continuity relation.

The area of cross-section times the velocity of flow is called volume rate of flow. Thus we find that in a steady flow of uniform density fluid, the volume flow rate is constant.

The equation of continuity, either Eq. 7.4 or 7.5, together with the Bernoulli's equation, to be derived in the next section, is sufficient to understand the steady non-viscous flow of fluids.

Further Remarks: The Continuity Equation - General

Although we used the entire volume between entrance and exit of a pipe carrying a fluid, the equation of continuity holds for any fluid volume element whatsoever. To derive a more general equation, consider a small rectangular volume between (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$ inside the fluid.

Let $\vec{v}(x, y, z)$ be the velocity at the (y, z)-face at (x, y, z) and $\vec{v}(x+\Delta x, y, z)$ be the velocity at the (y, z) face at $(x+\Delta x, y, z)$. Then the amount of fluid moving in interval Δt in the volume element through (y, z) face of the cube at (x, y, z) is $\rho v_x(x, y, z)\Delta y\Delta z\Delta t$, while the amount leaving the (y, z) face at $(x+\Delta x, y, z)$ is $\rho v_x(x+\Delta x, y, z)\Delta y\Delta z\Delta t$. Similarly for the (x, y) and (z, x) faces of the cube shown in Fig. 7.4. Hence, the net change of material inside the volume $\Delta x\Delta y\Delta z$ is:

$$\Delta m = \Delta t \begin{bmatrix} \{\rho v_x(x, y, z) - \rho v_x(x+\Delta x, y, z)\}\,\Delta y\Delta z + \\ \{\rho v_y(x, y, z) - \rho v_y(x, y+\Delta y, z)\}\,\Delta z\Delta x + \\ \{\rho v_z(x, y, z) - \rho v_z(x, y, z+\Delta z)\}\,\Delta x\Delta y \end{bmatrix}$$

Now, we divide both sides by the volume $\Delta x\Delta y\Delta z$ of the volume element at (x, y, z) and by the time interval Δt, and we take the $\Delta t \to 0$ and $\Delta x\Delta y\Delta z \to 0$ limits to obtain the general statement of continuity equation.

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right], \tag{7.6}$$

where I have replaced $\Delta m/\Delta x \Delta y \Delta z$ by $\Delta\rho$ and written $\Delta\rho/\Delta t$ as $\partial\rho/\partial t$. The continuity equation will hold at every point in the fluid as long as matter is neither created nor lost at that point. By introducing a vector current density $\vec{J}$ as

$$\vec{J} = \rho\vec{v}, \tag{7.7}$$

we can write the continuity equation more compactly as

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}, \tag{7.8}$$

where

$$\vec{\nabla} = \hat{u}_x \frac{\partial}{\partial x} + \hat{u}_y \frac{\partial}{\partial y} + \hat{u}_z \frac{\partial}{\partial z}.$$