13.8 PROBLEMS

Problem 13.8.1. The divergence of electric field is equal to volume charge density divided by the permittivity of free space:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

The electric field of a uniformly charged non-conducting sphere of radius R with charge density given as ρ_0 for $r \leq R$ and zero for $r > R$, where r is the distance from the center of the sphere, is given by the following.

$$\vec{E} = \begin{cases} \dfrac{\rho_0}{3\epsilon_0}\, r\, \hat{u}_r & r \leq R \\ \\ \dfrac{Q_{\text{tot}}}{4\pi\epsilon_0} \frac{1}{r^2}\, \hat{u}_r & r > R \end{cases}$$

Here Q_{tot} is the total charge in the sphere. (a) Calculate the divergence of electric field at a point outside the sphere. (b) Calculate the divergence of electric field at a point inside the sphere.

Ans: (a) 0,(b) ρ_0/ϵ_0.

Problem 13.8.2. The magnetic field of a steady current volume density of magnitude J_0 pointed in positive z-axis in a very long wire of radius R is given in polar coordinates as follows.

$$\vec{B} = \begin{cases} \dfrac{\mu_0 J_0}{2}\, r\, \hat{u}_\phi & r \leq R \\ \\ \dfrac{\mu_0 J_0}{2}\, \dfrac{R^2}{r}\, \hat{u}_\phi & r > R \end{cases}$$

Here r is the distance from the wire and $\hat{u}_\phi$ is a unit vector tangent to the circle centered at the wire of radius r passing through the field point. (a) Find the curl of the given magnetic field. (b) Prove that for a steady current $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$. Hint: You can write the given magnetic field in Cartesian coordinates and then calculate the curl. Ans: (a) The curl is 0 outside and $\mu_0 \vec{J}$ inside.

Problem 13.8.3. According to the Maxwell's equations, a dynamic electric field that is uniform in y and z coordinates but allowed to vary along the x-axis obeys the following equations in free space, i.e. at points in space where there are no charges or currents.

$$E_x = 0$$
$$\frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}$$
$$\frac{\partial^2 E_z}{\partial t^2} = c^2 \frac{\partial^2 E_z}{\partial x^2}$$

where $c^2 = 1/\mu_0\epsilon_0$. Consider the following wave solution of these equations.

$$\vec{E} = \vec{E}_0 \cos(kx - \omega t),$$

where $\vec{E}_0$ is a constant vector in space whose x-component is zero.

(a) Show that the components of the assumed solution satisfy the equations above.

(b) Find the direction of motion of the two components of the electric field.

(c) Write the associated magnetic field.

Problem 13.8.4. Repeat (a), (b) and (c) of the previous problem with the following form for electric field.

$$\vec{E} = \vec{E}_0 \cos(kx + \omega t),$$

Problem 13.8.5. The following represents an electromagnetic wave traveling in the direction of the positive x-axis.

$$E_x = 0;\ E_y = E_0 \cos(kx - \omega t);\ E_z = 0.$$

$$B_x = 0;\ B_y = 0; B_z = B_0 \cos(kx - \omega t).$$

The wave is passing through a wide tube of circular cross-section of radius R whose axis is along the x-axis. Find the expression for the displacement current through the tube.

Problem 13.8.6. Prove the following identities for the del vector operator.

(a) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$, where ϕ is a scalar field.

(b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$, where $\vec{A}$ is a vector field.

(c) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$, where $\vec{A}$ is a vector field.

Hint: Prove by actually calculating the given expressions using Cartesian components. For (a) and (c) you may show the relation to be true for the x-component of the equation since the same will be true for the other components as well.

Problem 13.8.7. The electric field of an electromagnetic wave is given by the superposition of two waves.

$$\vec{E}(x, y, z, t) = \hat{u}_y E_0 \cos(kx - \omega t) + \hat{u}_y E_0 \cos(kx + \omega t).$$

You can see that the first wave is moving towards the positive x-axis and the second moving towards the negative x-axis.

(a) What is the associated magnetic field wave?

(b) Write the Poynting vector and calculate the average energy per unit area per unit time transported by this wave.

(c) If the wave is confined between two perfectly reflecting walls, what would be the radiation pressure on the walls?

Ans: (a) $\hat{u}_z \dfrac{E_0}{v} \cos(kx - \omega t) - \hat{u}_z \dfrac{E_0}{v} \cos(kx + \omega t)$.

Problem 13.8.8. A microscopic spherical dust particle of radius 2 μm and mass 10 μg is moving in outer space at a constant speed of 30 cm/sec. A wave of light strikes it from the opposite direction of its motion, and gets absorbed. Assuming the particle decelerates uniformly to zero speed in one second, what is the average electric field amplitude in the light? Ans: 164 N/C.

Problem 13.8.9. A Styrofoam spherical ball of radius 2 mm and mass 20 μg is to be suspended by the radiation pressure in a vacuum tube in a lab. How much intensity will be required if the light is completely absorbed the ball? Ans: 4.7×10^6 W/m^2 .

Problem 13.8.10. A conservative force is given by the negative of the gradient of a potential energy function U. Based on the following given potential energy when the masses are separated by a distance r deduce the expression of the gravitational force on a mass m by another mass M.

$$U = -\frac{G_N M m}{r} \quad \text{(Reference at } r = \infty.)$$

Ans: $-\left(\frac{G_N M m}{r^2}\right) \hat{u}_r$ with M at the origin.