1.10 Relativistic Energy

Everybody has heard of Einstein's famous equation, $E = mc^2$. We will now see where it comes from. Einstein came to this equation from a consideration of energy in electromagnetic waves. Although Einstein's derivation is quite pretty, we will do the derivation in a different way in keeping with the development of dynamics here.

To obtain the energy picture from Newton's equation, we had simply studied the work done by a force and had found that it gives the change in the kinetic energy. From Eq. 1.130 we had obtained

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 \frac{d\vec{p}}{dt}\cdot d\vec{r}, \tag{1.132}$$

which gave the work energy theorem,

$$W_{12} = \frac{1}{2}mu_2^2 - \frac{1}{2}mu_1^2. \tag{1.133}$$

From the right side we had defined the concept of energy of motion or kinetic energy K for a particle of mass m and speed u.

$$\text{Newtonian kinetic energy:} \qquad K = \frac{1}{2}mu^2. \tag{1.134}$$

What happens to this expression of the kinetic energy when we use the relativistic momentum as in Eq. 1.131 in Eq. 1.132? For simplicity, let us say, the initial speed is $u_1 = 0$ and the final speed $u_2 = u$.

$$\begin{aligned}\int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 \frac{d\vec{p}}{dt}\cdot d\vec{r} &= \int_0^u d\vec{p}\cdot\vec{u} \\ &= \int_0^u d\left(m\vec{u}\right)\cdot\vec{u} = \int_0^u \left(\vec{u}\,dm + m\,d\vec{u}\right)\cdot\vec{u} \\ &= \int_0^u \left(u^2\,dm + m\,u\,du\right) \quad \left[\because\ d\vec{u}\cdot\vec{u} = \frac{1}{2}d(\vec{u}\cdot\vec{u} = u\,du.)\right]\end{aligned} \tag{1.135}$$

Now from the definition of speed-dependent mass we have

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}} \quad\Longrightarrow\quad m^2c^2 - m^2u^2 = m_0^2c^2. \tag{1.136}$$

Working out the differential of this equation gives

$$d(m^2)c^2 - \left[d(m^2)u^2 + m^2 d(u^2)\right] = 0. \tag{1.137}$$

Simplifying this expression yields

$$u^2\,dm + m\,u\,du = c^2 dm. \tag{1.138}$$

Putting this back in Eq. 1.135 gives rise to a simple integral for the relativistic kinetic energy. We use the limits of integration as m_0 when $u = 0$ and $m = m$

when $u = u$. Therefore the energy of the particle due to motion, or kinetic energy when it has a speed u will be

$$K = c^2 \int_{m_0}^{m} dm = m(u)c^2 - m_0 c^2. \tag{1.139}$$

Therefore, the relativistic kinetic energy has the following definition.

$$\boxed{K = m(u)c^2 - m_0 c^2 = m_0 c^2 \left[\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right].} \tag{1.140}$$

where the quantity $m_0 c^2$ is called the **rest energy** of the particle. The subtraction of $m_0 c^2$ in this expression gives the correct limit for $u \to 0$ corresponding to the particle at rest and hence $K = 0$. Thus, this expression gives us the indication that even when a free particle is at rest, it has an innate energy.

$$\boxed{\text{Rest Energy } = m_0 c^2.} \tag{1.141}$$

Adding the rest energy to the kinetic energy should give us the total energy E of a free particle moving at speed u.

$$\boxed{E = K + m_0 c^2 = mc^2 = \frac{m_0}{\sqrt{1 - u^2/c^2}} c^2 = \gamma m_0 c^2.} \tag{1.142}$$

We now verify that the expression of relativistic kinetic energy in Eq. 1.140 gives rise to $K = \frac{1}{2} m_0 u^2$ when $u << c$, i.e. in the non-relativistic regime when we expect Newton's formulas to be applicable. To start the calculation we first replace m by its relativistic expression in this equation.

$$K = \frac{m_0}{\sqrt{1 - u^2/c^2}} c^2 - m_0 c^2 \approx m_0 c^2 \left[1 + \frac{1}{2} \frac{u^2}{c^2} \right] - m_0 c^2 = \frac{1}{2} m_0 u^2. \tag{1.143}$$

Thus, $K = \frac{1}{2} m_0 c^2$ when $u << c$, which is the kinetic energy in Newtonian mechanics.

It is often desirable to write the total energy of a particle in terms of the momentum. To obtain the relation, we note that the square of the magnitude of momentum p will be given by

$$p^2 = \vec{p} \cdot \vec{p} = \frac{m_0^2 u^2}{1 - u^2/c^2} = \frac{m_0^2 c^2 u^2}{c^2 - u^2}. \tag{1.144}$$

Square of total energy from Eq. 1.142 gives

$$E^2 = \frac{m_0^2 c^2}{1 - u^2/c^2} = \frac{m_0^2 c^4}{c^2 - u^2} \tag{1.145}$$

From Eqs. 1.144 and 1.145 yields

$$\boxed{E^2 = p^2 c^2 + m_0^2 c^4.} \tag{1.146}$$

Example 1.11. Rest energy of a proton. Evaluate the rest energy of a proton.

Solution. The mass of a proton is $m_0 = 1.67 \times 10^{-27}$ kg. Therefore, its rest energy will be

$$E = 1.67 \times 10^{-27} \text{ kg} \times \left(3 \times 10^8 \text{ m/s}\right)^2 = 1.5 \ \times 10^{-10} \text{ J}.$$

We write this energy usually in eV or GeV. Let us convert this to eV.

$$E = 1.5 \ \times 10^{-10} \text{ J} = \frac{1.5 \ \times 10^{-10} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 9.39 \times 10^8 \text{ eV}.$$

This is 939 MeV, or alternately, 0.939 GeV.

Example 1.12. Rest energy of an electron. Evaluate the rest energy of an electron.

Solution. The mass of a proton is $m_0 = 9.1 \times 10^{-31}$ kg. Therefore, its rest energy will be

$$E = 9.1 \times 10^{-31} \text{ kg} \times \left(3 \times 10^8 \text{ m/s}\right)^2 = 8.2 \times 10^{-14} \text{ J}.$$

We write this energy usually in eV or MeV. Let us convert this to eV.

$$E = 8.2 \times 10^{-14} \text{ J} = \frac{8.2 \times 10^{-14} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 0.511 \times 10^6 \text{ eV}.$$

This is 0.511 MeV.

Example 1.13. Energy of a fast proton. Evaluate (a) the total energy, (b) the kinetic energy, and (c) the momentum of a proton in an accelerator beam in which the proton is moving at a speed of 0.99 c.

Solution. Let us calculate γ first since it will go into other calculations.

$$\gamma = \frac{1}{\sqrt{1 - 0.99^2}} = 7.1.$$

(a) Recall that the total energy of a moving particle is $E = \gamma m_0 c^2 = \gamma \times$(rest energy). We have calculated the rest energy of proton above.

$$E_{\text{rest}} = m_0 c^2 = 939 \, \text{MeV}.$$

Therefore the total energy of the proton will be

$$E = mc^2 = \gamma m_0 c^2 = 7.1 \times 939 \, \text{MeV} = 6.667 \, \text{GeV}.$$

(b) The kinetic energy K is the difference between the total energy and the rest energy.

$$K = E - E_{\text{rest}} = 6.667 \, \text{GeV} - 0.939 \, \text{GeV} = 5.728 \, \text{GeV}.$$

Here $K >> m_0 c^2$. This type of particle is also called ultra-relativistic particle. (c) The momentum will be

$$p = \gamma m_0 u = 7.1 \times 1.67 \times 10^{-27} \text{ kg} \times 0.99 \times 3 \times 10^8 \text{ m/s} = 3.52 \times 10^{-18} \text{ kg.m/s}.$$