8.3 AMPERE'S LAW

8.3.1 Statement of Ampere's Law

What will happen if you evaluate the magnetic field given by Biot-Savart law? Recall that Biot-Savart law gives magnetic field by a current. Suppose a field point P is at coordinates (x, y, z). According to Biot-Savart law, the magnetic field $B(x, y, z)$ at (x, y, z) by a steady current I in an electric circuit is given by

$$\vec{B}(x, y, z) = \frac{\mu_0}{4\pi} \int_{\text{Circuit}} \frac{I d\vec{l} \times \hat{u}_s}{s^2}. \tag{8.22}$$

Note that the length element in this formula $d\vec{l}$ is an element of the wire carrying current I. The direction of $d\vec{l}$ is the direction of the current in the wire. The length element $d\vec{l}_A$ on the loop for circulation refers to a length vector at a different place.

Now, the circulation of the magnetic field give in Eq. 8.22 around a loop in space whose length vector elements are designated as $d\vec{l}_A$ can be formally written as

$$\oint_{\text{Space Loop}} \vec{B} \cdot d\vec{l}_A = \frac{\mu_0}{4\pi} \oint_{\text{Space Loop}} \left[\int_{\text{Circuit}} \frac{I d\vec{l} \times \hat{u}_s}{s^2} \right] \cdot d\vec{l}_A. \tag{8.23}$$

The double integral on the right side is difficult to do and you will learn how to do this integral in a more advanced course on electricity and magnetism. We will just quote the final result called Ampere's law.

$$\oint_{\text{Space Loop}} \vec{B} \cdot d\vec{l}_A = \mu_0 I_{enc} \tag{8.24}$$

$$I_{enc} = \text{Net current through the space loop.}$$

Fig. 8.14 shows three loops and a steady current segment. The enclosed currents through loops L_1 and L_2 are I. Since current does not go through loop L_3, the enclosed current is zero.

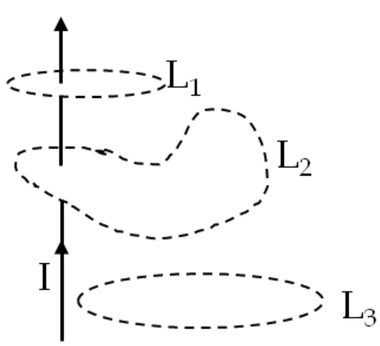

Figure 8.14: Circulation of magnetic field for various Amperian loops. Circulation through L_1 and L_2 are same, and circulation through L_3 is zero.

8.3.2 General Strategy for Using Ampere's Law

How can we use Ampere's law to find magnetic field? The statement of Ampere's law given in Eq. 8.24 shows that the unknown magnetic field is part of the integrand. Therefore, we can use Ampere's law to calculate magnetic field only in those situations when magnetic field can come out of the integrand. We find that if current distribution has one of the three symmetries then it is possible to choose closed paths, called the Amperian loops, where $\vec{B} \cdot d\vec{l}$ is either zero or a constant Bdl over segments of the closed paths.

$$\text{Choose Amperian loop such that} \\ \int \vec{B} \cdot d\vec{l} \text{ is either } 0 \text{ or } B_P L$$

where L is length of the segment contains point P where magnitude of the magnetic field is B_P. The three special symmetries that allow these special Amperian loops are

1. Cylindrical symmetry. This is the case when current is along a long straight wire.

2. Planar symmetry. This is the case if current is on a flat large surface.

3. Solenoidal symmetry. This is the case if current is along $\hat{u}_\phi$ of cylindrical coordinate. A cylinder that has an insulated wires wound on a cylinder has solenoidal symmetry if the cylinder is long and the area of cross-section is small.

The direction of the magnetic field is obtained as usual by the right-had rule of Biot-Savart law we have labeled RHR-II. In the following we discuss a six step process for systematically implementing a program for using Ampere's law where required symmetry exists.

Step 1: Obtain the direction of $\vec{B}_P$.

Step 2: Choose an Amperian loop.

Step 3: Deduce an expression for circulation.

Step 4: Deduce the amount of enclosed current.

Step 5: Solve for magnitude B_P.

Step 6: Write out the magnitude and direction of $\vec{B}_P$.

8.3.3 Examples

Example 8.3.1. Magnetic field of a steady current in a long wire

Solution. Let the wire be infinitely long so that the current has a cylindrical symmetry. We will find magnetic field $\vec{B}_P$ at a point P located at a distance r from the wire using Ampere's Law. We already know the answer to the current problem from our discussions of Biot-Savart Law, but we re-do this case to explicitly lay out the arguments common to all applications of Ampere's law.

Steps:

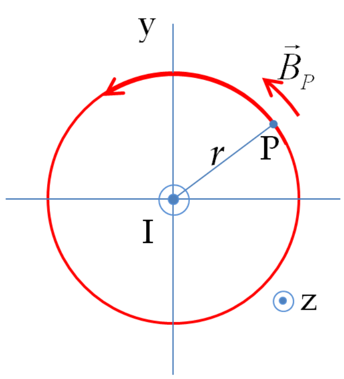

Figure 8.15: Direction of magnetic field at P.

Step 1: Obtain the direction of $\vec{B}_P$. This step is used to obtain the the direction of magnetic field $\vec{B}_P$ at the field point P and a general idea of the directions of $\vec{B}$ lines. By right-hand rule RHR-II used in Biot-Savart law for magnetic field, we can tell that if the current is towards the positive z-axis, then the magnetic field will circulate around the wire in the counter-clockwise sense when looked in from the positive z-axis. This will make the direction of the magnetic field in the polar unit vector $\hat{u}_\phi$ direction, which is tangent to circle about origin in the xy-plane.

Step 2: Choose an Amperian loop. Symmetry allows us to make certain statements about the relation between the magnitudes at various points in space. In a cylindrically symmetric situation, we expect the magnitude of the magnetic field to depend only upon the radial distance r from the axis. That means magnetic field has same magnitude at all points of any circle about the axis.

Therefore, choosing a circle that passes through point P for the Amperian loop will give us a closed loop that has the same magnitude of magnetic field at all points as the magnitude at point P as shown in Fig. 8.16. Furthermore, we choose the direction of the Amperian loop to coincide with the direction of magnetic field $\vec{B}_P$ at field point P.

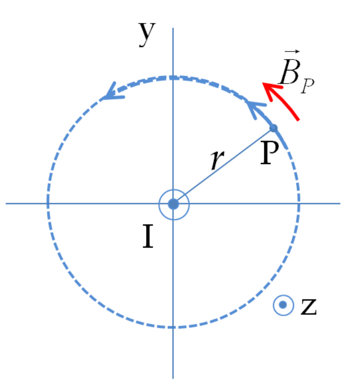

Figure 8.16: Amperian loop. Note that by symmetry argument, the magnitude of the magnetic field is same at all points of the Amperian loop.

Step 3: Deduce an expression for circulation. Since the direction of the elements of the loop and the magnetic field are same for points on the circle, the circulation of the magnetic field around the circle will simply be the product of the magnitude of magnetic field and the length of the circular path.

$$\text{Circulation: } \oint \vec{B} \cdot .d\vec{l} = B_P \times 2\pi r. \tag{8.25}$$

Step 4: Deduce the amount of enclosed current. Look at the loop used for the evaluation of circulation of the magnetic field. How much current passes through the space enclosed by the loop? The answer is enclosed current:

$$\text{Enclosed current: } I_{enc} = I. \tag{8.26}$$

Step 5: Solve for magnitude B_P. Ampere's law states that the circulation of magnetic field around a closed path will equal μ_0 times enclosed current. Therefore, we obtain the following relation between the magnitude of the magnetic field at point P and current in the wire.

$$B_P \times 2\pi r = \mu_0 I. \tag{8.27}$$

This can be solved for B_P.

$$B_P = \frac{\mu_0}{2\pi}\frac{I}{r}. \tag{8.28}$$

Step 6: Write out the magnitude and direction of $\vec{B}_P$. Now, we have obtained both the direction and magnitude of magnetic field at point P. We can summarize our finding by stating both direction and magnitude information as follows.

$$\text{Magnitude: } B_P = \frac{\mu_0}{2\pi}\frac{I}{r} \tag{8.29}$$

Direction: If current along the z-axis and point P on the x-axis, then $\vec{B}_P$ is pointed towards the y-axis. (8.30)

You could also say that the direction is given by the unit vector $\hat{u}_\phi$ of the cylindrical coordinate system if the current is along the z-axis. Other ways of describing the direction of the vector at point P would also be acceptable.

Example 8.3.2. Magnetic field of a non-uniform current density.

An infinitely long circular cylindrical wire of radius R carries current in the direction of its axis. The magnitude of the volume current density varies with the radial distant r from the axis of the wire according to the following function $J(r) = J_0(r/R)^n$, where J_0 and n are constants. Find magnetic field at an interior point of the wire by using Ampere's Law.

Solution. We apply the same steps as in the example above.

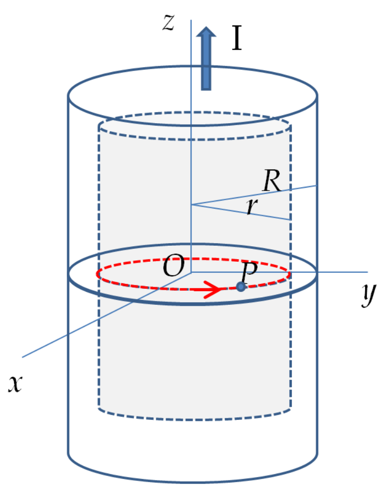

Figure 8.17: The Amperian loop is a circle of radius r and has the direction of the magnetic field $\vec{B}$. The magnitudes of the magnetic field at all points of the circle of radius r are equal to the magnitude B_P at point P. The enclosed current I_{enc} is equal to the current through the shaded wire. Note, the enclosed current is not equal to the full current carried by the wire.

Step 1: Obtain the direction of $\vec{B}_P$. By right-hand rule RHR-II used in Biot-Savart law for magnetic field, we can tell that, if the current is towards the positive z-axis, then the magnetic field will circulate in the counter-clockwise sense when looked from the positive z-axis. This will make the direction of the magnetic field in the polar unit vector $\hat{u}_\phi$ direction, which is tangent to circle about origin in the xy-plane.

Step 2: Choose an Amperian loop. Just as an infinite wire, symmetry again allows us to make certain statements about the relation between the magnitudes at various points in space. Here, we expect the magnitude of the magnetic field to depend only upon the radial distance r from the axis of the wire. That means, magnetic field would have the same magnitude at all points of any circle about the axis.

The Amperian loop is a closed loop in space. Choosing a circle about the axis that passes through point P will give us a closed loop that has the same magnitude of magnetic field as at point P. We choose the direction of the Amperian loop to coincide with the direction of magnetic field at the field point P as shown in Fig. 8.17.

Step 3: Deduce an expression for circulation. As explained above, since the magnetic field is parallel to the direction of the loop, the dot product $\vec{B}\cdot d\vec{l}$ is equal to product of the magnitudes, that is, Bdl. Since, B at any point of the circle is equal to to the magnitude at point P, that is $B = B_P$, the circulation around the loop is given by the product of the magnitude of the magnetic field at point P and the length of the loop, which is same as circumference of the circle.

$$\text{Circulation: } \oint \vec{B}\cdot .d\vec{l} = B_P \times 2\pi r. \tag{8.31}$$

Step 4: Deduce the amount of enclosed current. As shown in Fig. 8.17 the current through the loop is only the current passing through the area of the loop. A cross-sectional view of the current shown in Fig. 8.18 helps with this calculation.

The current dI through a ring of infinitesimal thickness ds with inner radius s and outer radius $s + ds$ is given by the product of the current density and the area of the infinitesimal ring since current density is perpendicular to the ring. The current density at ring is $J(s) = J_0(s/R)^n$ and the area of the element

$2\pi s ds$. Therefore,

$$dI = \frac{J_0}{R^n} s^n \times 2\pi s ds \quad \text{(since } \vec{J} \text{ perpendicular to area.)} \tag{8.32}$$

Integrating this from $s = 0$ to $s = r$ (not $s = R$) will give all the current for I_{enc}.

$$\begin{aligned} I_{enc} &= \frac{2\pi J_0}{R^n} \int_0^r s^{n+1} ds \\ &= \left(\frac{2\pi J_0}{n+2}\right) \frac{r^{n+2}}{R^n}. \end{aligned} \tag{8.33}$$

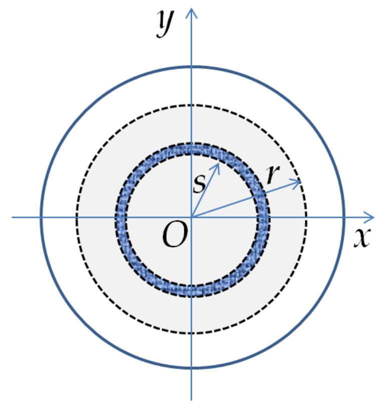

Figure 8.18: The enclosed current is current through the circle of radius r. To calculate the enclosed current we find the current through an infinitesimal ring of thickness ds with inner radius s and outer radius $s + ds$. Summing over contributions of all rings from $s = 0$ to $s = r$ gives the net enclosed current.

Step 5: Solve for magnitude B_P. Now, we equate the expression of circulation around the Amperian loop to $\mu_0 I_{enc}$

$$B_P \times 2\pi r = \left(\frac{2\pi J_0}{n+2}\right) \frac{r^{n+2}}{R^n} \tag{8.34}$$

and solve for B_P, to obtain

$$B_P = \left(\frac{J_0}{n+2}\right) \frac{r^{n+1}}{R^n} \tag{8.35}$$

Step 6: Write out the magnitude and direction of $\vec{B}_P$. The magnitude and direction of magnetic field at point P can be written in a compact notation by using the unit vector $\hat{u}_\phi$ of the cylindrical coordinate system or the $\hat{u}_\phi$ of the polar coordinate system in the xy-plane.

$$\vec{B}_P = \left(\frac{J_0}{n+2}\right) \frac{r^{n+1}}{R^n} \hat{u}_\phi \quad (r < R) \tag{8.36}$$

Further Remarks:

1. To find the magnetic field at a point outside the wire, we will set up the problem the same way. The only difference will come from the enclosed current expression. The Amperian loop in this case would be a circle of radius $r > R$ which will include full current in the wire. The integral in Eq. 8.33 will be from $s = 0$ to $s = R$ giving the following result.

$$I_{enc} = \frac{2\pi J_0}{n+2} R^2.$$

Relating this enclosed current to circulation gives the following magnetic field at a point P with $r > R$ is

$$\vec{B}_P = \left(\frac{J_0}{n+2}\right) \frac{R^2}{r} \hat{u}_\phi \quad (r > R) \tag{8.37}$$

2. The results obtained above can be used to answer magnetic field of a uniform current over the cross-section of the wire. Suppose the current density is uniform over the cross-section of the wire and has magnitude J_0, then all we need to do is set $n = 0$ to obtain the following results for magnetic field at points inside and outside the wire. Sometimes, the field at points inside are labeled "in" and fields outside "out".

$$\vec{B}_P^{in} = \frac{1}{2} J_0 r \quad (r < R) \tag{8.38}$$

$$\vec{B}_P^{out} = \frac{1}{2} J_0 \frac{R^2}{r} \quad (r > R) \tag{8.39}$$

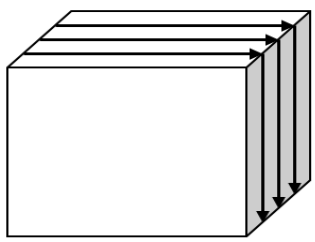

Figure 8.19: Making planar surface current by winding insulated wires tightly around a rectangular box.

Example 8.3.3. Magnetic field of a planar current Take a large rectangular box and wind a wire with plastic coatings around the box so that on each side we get one layer of wire, with two sides with no wire as shown in the Fig. 8.19. What is the magnetic field at a point very close to one of the sides carrying current?

Solution. If the point of interest is very close to the surface, we can assume the surface to be infinite and ignore the magnetic field from other surfaces. This gives us a situation of a planar current.

Let the plane of current of interest be the xy-plane with current towards the positive x-axis (Fig. 8.20). We will use Ampere's law to find the magnetic field at a point P on the z-axis at $z = a$ by following the steps given above.

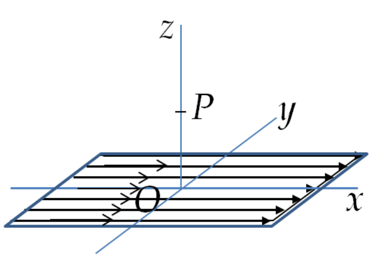

Figure 8.20: The planar current in the xy-plane. The magnetic field is sought at point P.

Step 1: Obtain the direction of $\vec{B}_P$. The right-hand rule RHR-II of Biot-Savart law tells us that the magnetic field at point P from currents in different wires on the surface will be in different directions as shown in Fig. 8.21.

Since we have assumed that the planar current is spread over the entire xy-plane we look at magnetic field of two currents I_1 and I_2 that are symmetrically placed on the positive y- and negative y-axes as shown in Fig. 8.21.

First, not that the magnitudes of the magnetic field by the two currents at point P will be equal since point P is same distance from the two wires. Next we notice that the magnetic fields of the two currents I_1 and I_2 cancel each other's vertical components and only horizontal component is left. This is shown graphically in Fig. 8.21.

This cancellation at point P will occur for every pair of current that is symmetrically placed about the point underneath P. Therefore, the direction of the magnetic field at point P will

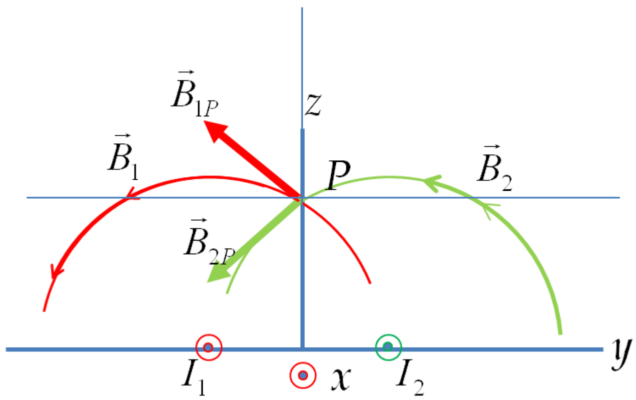

Figure 8.21: Magnetic field by two currents placed symmetrically about the observation point P. The vector addition of the two magnetic fields will result in a net magnetic field that is pointed along the negative y-axis. Since every wire has another wire placed symmetrically about point P, the direction of the magnetic field of the planar current is pointed along the negative y-axis.

be horizontal towards the negative y-axis as displayed in Fig. 8.21. This argument can be made for any point above the plane since the plane is infinite in extent and placement of the z-axis is arbitrary. The same argument for points below $z = 0$ would show that the magnetic field there is pointed towards the positive z-axis.

Step 2: Choose an Amperian loop. Since, the current is spread over the entire xy-plane, the magnetic field at arbitrary space point cannot depend on the x and y coordinates of the point. Therefore, we expect magnetic field will have the same magnitude at all points in planes $z = \pm a$.

We want to choose an Amperian loop that contains point P and makes use of the constant amplitude planes. A rectangular loop in the yz-plane, shown as abcda in Fig. 8.22, one side of which passes through point P and the other side at $z = -a$, will give circulation that will contain the magnitude of magnetic field at point P.

Step 3: Deduce an expression for circulation. Let w denote the lengths of the parallel sides ab and cd. We work out the integral over $\vec{B} \cdot d\vec{l}_A$ on the four straight segments to obtain the circulation around the loop abcda to be

$$\text{Circulation: } \oint \vec{B} \cdot d\vec{l} = B_P w + 0 + B_P w + 0 = 2B_P w, \quad (8.40)$$

where 0's are for the sides where magnetic field is perpendicular

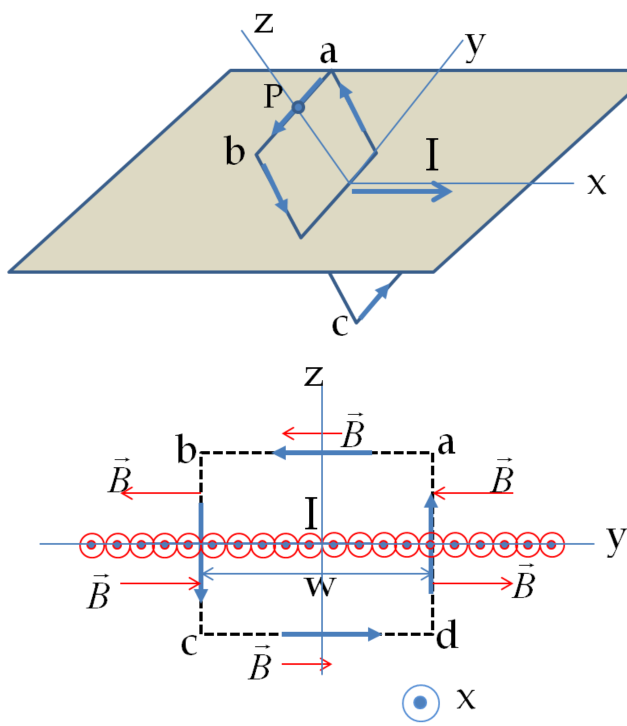

Figure 8.22: The Amperian loop for a planar current.

to the loop direction, hence resulting in zero due to the scalar product.

Step 4: Deduce the amount of enclosed current. The enclosed current is the current through the area of the Amperian loop. Suppose there are n wires per unit perpendicular distance along the y-axis. Then, through a distance w the number of wires will be nw, each wire carrying current I. Therefore, enclosed current is

$$I_{enc} = nwI. \tag{8.41}$$

Step 5: Solve for magnitude B_P. Now, we equate the expression of circulation around the Amperian loop to $\mu_0 I_{enc}$. This gives

$$2B_P w = \mu_0 nwI. \tag{8.42}$$

We now solve this equation for the magnitude of magnetic field at point P.

$$B_P = \frac{\mu_0 nI}{2}. \tag{8.43}$$

Step 6: Write out the magnitude and direction of $\vec{B}_P$. Finally, we write the magnitude and direction of magnetic field at point P can in a compact notation by using the unit vector for the

y-axis, $\hat{u}_y$, since the magnetic field is pointed towards the negative y-axis. We will need to multiply $\hat{u}_y$ by minus 1 to get the direction correspond to the negative y-axis.

$$\vec{B}_P = -\frac{\mu_0 n I}{2}\hat{u}_y. \tag{8.44}$$

Further Remarks:

1. This result can also be written using surface current density $\vec{K}$ and a unit normal vector $\hat{u}_n$ perpendicular to the plane of the current, defined for the direction from the plane of the current towards point P. The surface current density is given by current per unit cross-sectional length. In the present situation since n is number of wires per unit cross-sectional length and each wire carries a current I in the direction of the positive x-axis, the vector surface current density would be

$$\vec{K} = nI\hat{u}_x. \tag{8.45}$$

Define the normal direction from plane current towards the field point P. This gives unit normal vector $\hat{u}_n$ same as unit vector towards the positive z-axis.

$$\hat{u}_n = \hat{u}_z. \tag{8.46}$$

Equation 8.44 can now be written in a notation that is independent of Cartesian axes as

$$\vec{B}_P = \frac{\mu_0}{2}\vec{K}\times\hat{u}_n. \tag{8.47}$$

This form for the magnetic field is free of reference to a coordinate system.

2. The magnetic field given by Eq. 8.44 or 8.47 is independent of the distance to the point P from the plane! The physical situation, however, is somewhat fictitious, since we have ignored the contributions of the wires on other faces that was necessary for the currents to form a closed circuit in the first place. Therefore, in practice, we will not have a constant magnetic field in any realistic arrangement of currents on a cube of finite size. However, our approximation works better as we look at points that are nearer to the center of any one face.

Example 8.3.4. Magnetic field of an ideal solenoid

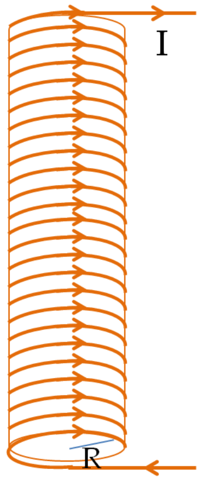

Figure 8.23: Solenoid.

Solution. A solenoid is constructed by tightly winding a wire about a cylinder. Let there be n turns of the wire per unit length with current I flowing in each wire. Note that the windings are insulated from each other by a thin plastic coating on the wire so that they are not shorted. To be specific, let the cross-section of the solenoid be circular of radius R, although any other shape of the cross-section will work also.

For analytic treatment of the problem, we choose coordinate axes so that the z-axis is along the axis of the cylinder such that when you look from the positive z-axis the current in each loop flows counter-clockwise. current in each loop flows in a circle of radius R parallel to the xy-plane.

We will use Ampere's law to find the magnetic field at points outside the solenoid and at points inside the solenoid. We will present the solution in the steps outlined above.

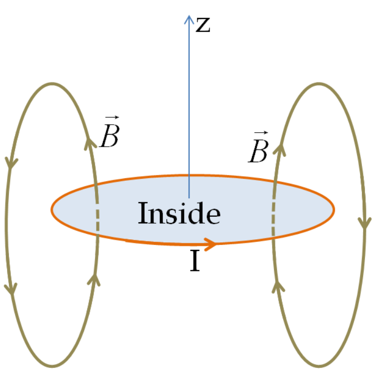

Figure 8.24: Direction of magnetic field inside and outside the solenoid from current in one winding.

Step 1: Obtain the direction of $\vec{B}_P$. A solenoid carries current in each of its windings. As we have learned from Biot-Savart law, each element of current produces its own magnetic field. Figure 8.24 shows two magnetic field lines from one winding. It shows that magnetic field lines circulate around the current with direction going up inside the solenoid and going down outside the solenoid.

We can deduce the direction of the magnetic field at any point P from all the windings of a solenoid by considering two rings of current located symmetrically with respect to point P. Figure 8.25 shows that net magnetic field at an inside point P_1 is pointed up for the counterclockwise current (seen from the positive z-axis) occurring as a result of cancellation of the horizontal components of the magnetic field. Figure 8.25 also shows that the net magnetic field at an outside point P_2 is pointed down for the current direction assumed in the figure.

Step 2: Choose an Amperian loop. Since the ideal solenoid is of infinite length along the z-axis, we expect that the magnitude of the magnetic field cannot depend on the z-coordinate of the field point P as there is no way to distinguish $z = 0$ and $z = a$ for the system. Since the current density is independent of the direction in the xy-plane, the magnitude of the magnetic field will also be independent of the polar angle in the xy-plane. The magnitude of magnetic field is therefore expected to depend only upon the distance r from the axis of the solenoid.

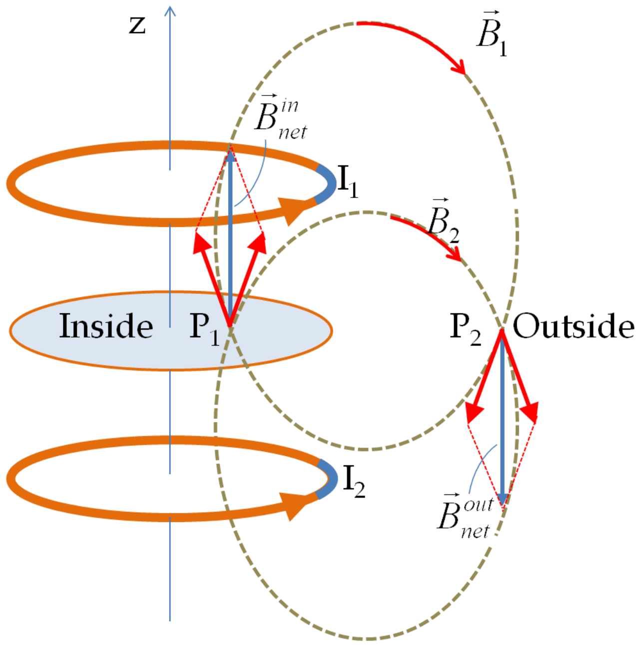

Figure 8.25: Directions of magnetic field inside and outside the solenoid are parallel and anti-parallel to the axis of the solenoid.

Since the magnitude depends only upon the distance r from the z-axis and the direction of the magnetic field is parallel to the z-axis, we choose a rectangular loop shown in Fig. 8.26. The Amperian loop abcda has two sides ab and cd parallel to the z-axis and two other sides bc and da perpendicular to the z-axis. One these sides parallel to the z-axis goes through the field point P. The circulation of magnetic field around this Amperian loop will be in terms of the magnitude of magnetic field at distances of the sides ab and cd from the z-axis.

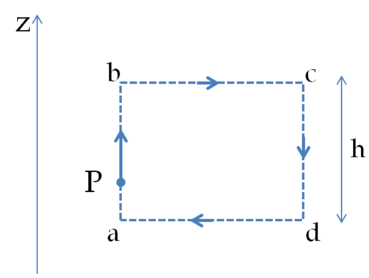

Figure 8.26: Choice of Amperian loop.

Step 3: Deduce an expression for circulation. The expression of the circulation of magnetic field around a rectangular loop abcda described above depends on the location of the loop in space. Let us denote the magnetic field at various points as $\vec{B}^{\rm out}_1$, $\vec{B}^{\rm out}_2$, $\vec{B}^{\rm in}_3$, and $\vec{B}^{\rm in}_4$ for magnetic field at distance r_1, r_2 , r_3, and r_4 from the axis respectively. The distance r_1 and r_2 are greater than the radius of the solenoid and r_3 and r_4 are less than the radius of the solenoid.

Amperian loop completely outside the solenoid

For a question about magnetic field at a point P outside the solenoid we will place the Amperian loop such that side ab goes through the field point P. Now, since vector $\vec{B}^{\rm out}_1$ is opposite to the loop direction at ab, the contribution to circulation from side ab will be negative. If the length of the side ab is denoted by h, the contribution of the side ab to the circulation around abcda will be

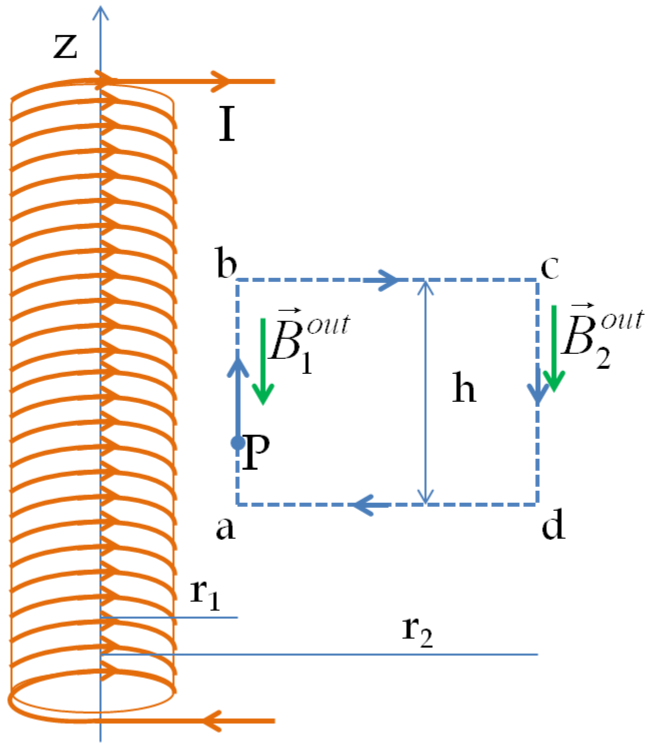

Figure 8.27: Amperian loop outside the solenoid.

$$\text{Contribution of side ab to circulation } = -B_1^{\text{out}} h.$$

On the other hand, since $\vec{B}_2^{\text{out}}$ is in the same direction as the loop direction at side cd, the contribution to circulation from side cd will be positive.

$$\text{Contribution of side cd to circulation } = B_2^{\text{out}} h.$$

Since, the magnetic fields at the other two sides are perpendicular to the loop directions for those sides, their contributions to the circulation will be zero. Therefore, the circulation around the loop

$$\text{Circulation(loop placed outside)} = \left(B_2^{\text{out}} - B_1^{\text{out}}\right) h. \quad (8.48)$$

Amperian loop completely inside the solenoid

For a question about magnetic field at a point P inside the solenoid we will need to place the Amperian loop such that side ab goes through the field point P. There are two choice for side cd: either cd is inside the solenoid or cd is outside the solenoid.

If the Amperian loop is completely inside the solenoid, we use arguments similar to what we have used for the loop that was completely outside to obtain:

$$\text{Circulation(loop placed inside)} = \left(B_3^{\text{in}} - B_4^{\text{in}}\right) h, \qquad (8.49)$$

where notation "in" is attached for fields at points inside the solenoid.

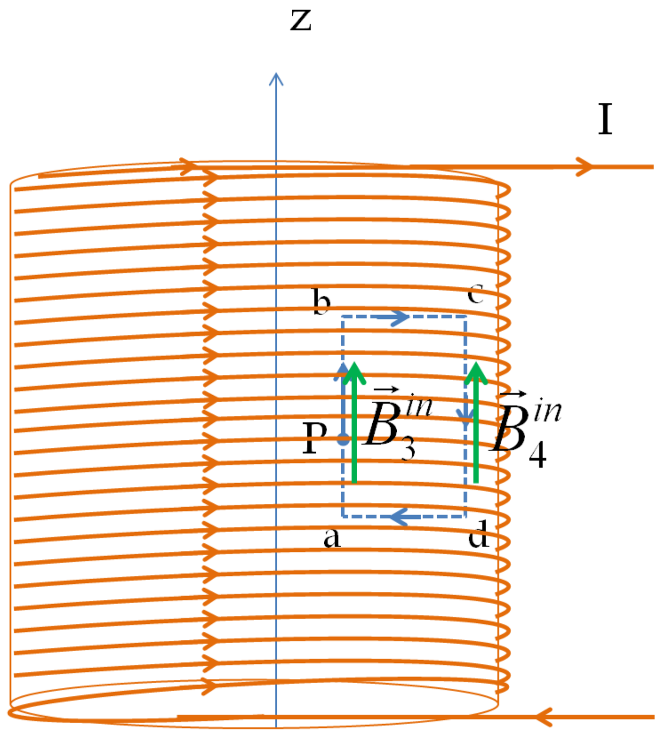

Figure 8.28: Amperian loop inside the solenoid.

Amperian loop straddling the solenoid

An Amperian loop that has one leg inside the solenoid and another leg outside will relate the magnitudes of magnetic field at inside and outside points. If leg ab of the loop is inside the solenoid and leg cd outside the solenoid, then, we will get

$$\text{Circulation(loop straddling)} = \left(B_3^{\text{in}} + B_2^{\text{out}}\right) h, \qquad (8.50)$$

where we have placed field point P on leg ab that is inside and side cd is a distance where the magnetic field is $\vec{B}_2^{\text{out}}$. Note that, for this loop the two magnetic fields $\vec{B}_2^{\text{out}}$ and $\vec{B}_3^{\text{in}}$ are along the direction of the loop.

Step 4: Deduce the amount of enclosed current. The amount of enclosed current depends on the location of the Amperian

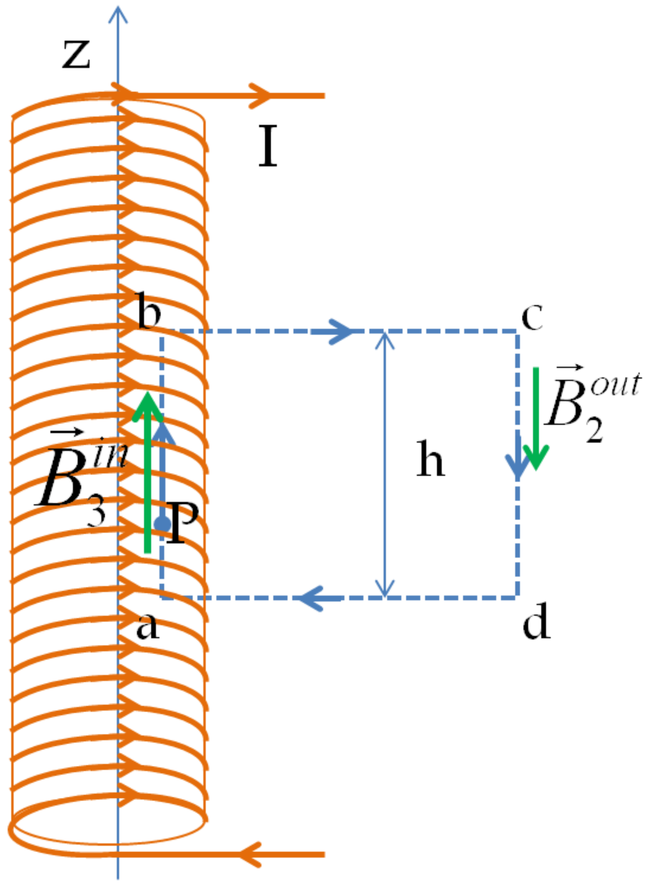

Figure 8.29: Amperian loop straddling the solenoid.

loop in space. The three cases of the loops given above have the following enclosed currents.

$$I_{enc}(\text{loop placed outside}) = 0 \tag{8.51}$$
$$I_{enc}(\text{loop placed inside}) = 0 \tag{8.52}$$
$$I_{enc}(\text{loop straddling}) = nhI \quad (\text{see Fig. 8.30.}) \tag{8.53}$$

Note nh gives the number of current windings that pass through the loop when the loop is straddling the solenoid since n is the number of windings per unit length.

Step 5: Solve for magnitude B_P. We obtain the following equations for the three loops.

$$(\text{loop placed outside}):\ B_2^{\text{out}} - B_1^{\text{out}} = 0 \tag{8.54}$$
$$(\text{loop placed inside}):\ B_3^{\text{in}} - B_4^{\text{in}} = 0 \tag{8.55}$$
$$(\text{loop straddling}):\ B_3^{\text{in}} + B_2^{\text{out}} = \mu_0 nI \tag{8.56}$$

These equations tell us that magnetic field is same at all inside points, which we can write simply as B_{in},

$$B_3^{\text{in}} = B_4^{\text{in}} \equiv B_{in}, \tag{8.57}$$

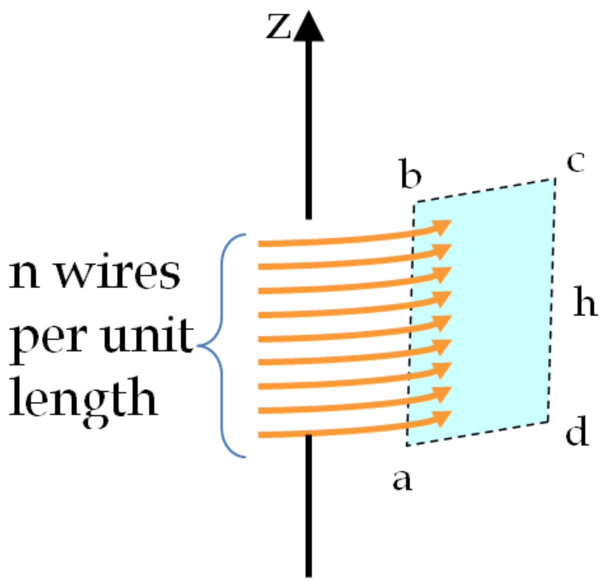

Figure 8.30: Current enclosed for a loop that straddles the solenoid. For n wires per unit length each carrying a current I, the enclosed current is $I_{enc} = nhI$ since the height of the loop is h.

and same at all points outside, which we can write simply as B_{out}.

$$B_1^{\text{out}} = B_2^{\text{out}} \equiv B_{out}. \tag{8.58}$$

According to Eq. 8.56, we can find the value of B_{in} in terms of the current if we can somehow figure out the value of B_{out}.

A trick for figuring out B_{out}

Let us look at the magnetic field by solenoid in terms of magnetic field lines. Recall that magnetic field lines of any current make loops around the current. Therefore, every field line generated by current in any winding of the solenoid must loop through inside space of the solenoid, but they return outside over much greater space.

As you move further away from the solenoid, the density of lines per unit cross-section area, which is a measure of the strength of magnetic field, decreases. This says that if you go out to infinity in the xy-plane, the magnetic field will go to zero. Since we have shown that magnetic field outside the solenoid has the same magnitude at all points, this argument proves that B_{out} is zero everywhere.

$$B_{\text{out}} = 0. \tag{8.59}$$

Now, using this in Eq. 8.56 gives the following for magnitude

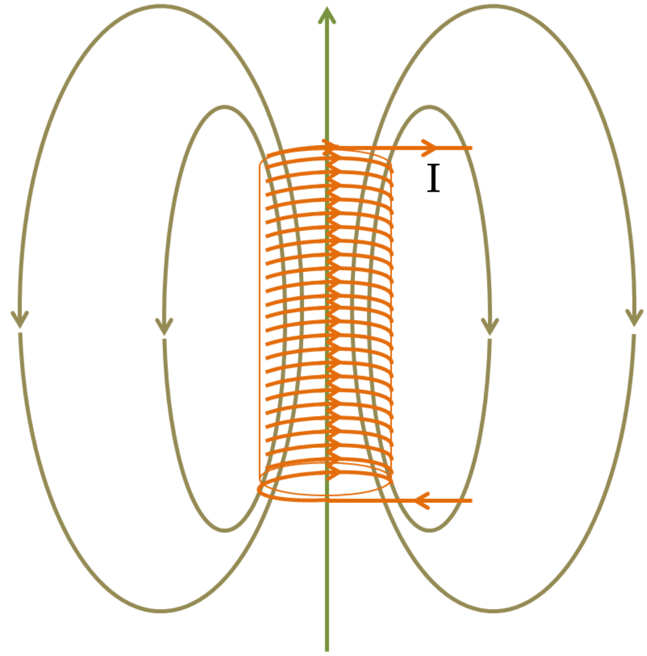

Figure 8.31: All magnetic field lines of the solenoid pass through the interior but return over much larger space outside. The density of field lines decreases with distance from the solenoid, showing that magnetic field outside the solenoid goes to zero as distance from the axis of solenoid goes to infinity.

of the magnetic field inside the solenoid.

$$B_{\text{in}} = \mu_0 n I \tag{8.60}$$

Step 6: Write out the magnitude and direction of $\vec{B}_P$. Now, we summarize our results for the magnetic field of an ideal solenoid where current I in the windings of the solenoid flows counter-clockwise when looked from the positive z-axis and there are n windings per unit length.

$$\vec{B}_P = \begin{cases} \mu_0 n I \hat{u}_z, & \text{if P is inside the solenoid} \\ 0, & \text{if P is inside the solenoid} \end{cases} \tag{8.61}$$