6.4 PROBLEMS CONCERNING MOTION OF PARTICLES

Example 6.4.1. Change in momentum due to a constant force. A particle of mass m interacts with another object A such that the force on the particle is constant with magnitude F and direction towards A. What is the change in momentum of the particle over the interval t_1 to t_2?

Solution. This problem is just an application of Newton's second law. Since, the rate of change of momentum is constant, we can write Newton's second law for finite interval of time rather than the instantaneous form.

$$\frac{\Delta\vec{p}}{\Delta t} = \vec{F},$$

which gives the change in momentum over the interval t_1 to t_2 as

$$\Delta\vec{p} = \vec{F}\Delta t = \vec{F}\,(t_2 - t_1)\,.$$

Therefore, the change in momentum has a magnitude of $F \times (t_2 - t_1)$ and has the same direction as the direction of the force.

Example 6.4.2. Instantaneous acceleration due to a constant force. A particle of mass m interacts with another object A in the time interval $t_1 \leq t \leq t_2$ such that the force on the particle is constant with magnitude F and direction towards A. What is the instantaneous acceleration at an arbitrary time t?

Solution. This problem is an application of Newton's second law for constant mass case. At any instant, the net force is equal to mass times acceleration. Therefore, acceleration is obtained by dividing the force on the particle by the mass of the particle.

$$\vec{a} = \frac{\vec{F}}{m}.$$

This says that the acceleration at as arbitrary instant t during t_1 to t_2 has a magnitude of F/m and has the same direction as the direction of the force.

Example 6.4.3. Change in velocity due to a constant force. A particle of mass m interacts with another object A such that the force on the particle is constant with magnitude F and direction towards A. (a) What is the change in velocity of the particle over the interval t_1 to t_2? (b) If the velocity of the particle at $t = 0$ was $\vec{v}_0$, what is its velocity $\vec{v}$ at an arbitrary instant t?

Solution. (a) We can make use of either Example 6.4.1 or 6.4.2. From Example 6.4.1, we know the change in momentum from t_1 to t_2 for the same situation, and since mass is constant here, we find the change in velocity by simply dividing by m.

$$\Delta\vec{v} = \frac{\vec{F}}{m}\left(t_2 - t_1\right).$$

(b) Now, setting $t_1 = 0$, $t_2 = t$, and $\Delta\vec{v} = \vec{v} - \vec{v}_0$ in the result of part (a) we find the velocity at an arbitrary time.

$$\vec{v} = \vec{v}_0 + \frac{\vec{F}}{m}t. \tag{6.17}$$

Example 6.4.4. Change in velocity due to a constant force. A particle of mass m falls near Earth under its weight. (a) What is the change in velocity of the particle over the interval t_1 to t_2? (b) If the velocity of the particle at $t = 0$ was $\vec{v}_0$, what is its velocity $\vec{v}$ at an arbitrary instant t?

Solution. (a) The constant force is equal to mg pointed down. Therefore, we find the change in velocity

$$\Delta\vec{v} = \{g\left(t_2 - t_1\right), \text{pointed down}\}.$$

(b) Now, setting $t_1 = 0$, $t_2 = t$, and $\Delta\vec{v} = \vec{v} - \vec{v}_0$ in the result of part (a) we find the velocity at an arbitrary time.

$$\vec{v} = \vec{v}_0 + \{gt, \text{pointed down}\}. \tag{6.18}$$

We can write these results analytically in a coordinate system. Let y-axis be pointed up, and the horizontal plane have x and z-axes. Then, we will get the following components for Eq. 6.18.

$$\begin{aligned} v_x &= v_{0x}, \\ v_y &= v_{0y} - gt, \\ v_z &= v_{0z}, \end{aligned}$$

which says that the vertical motion is a constant acceleration motion and the horizontal motion a constant velocity motion, as we expect in a situation of an object that is falling freely. This is the projectile motion problem we have studied before.

Example 6.4.5. Change in position due to a constant force. A particle of mass m interacts with another object A such that the force on the particle is constant with magnitude F and direction towards A. (a) What is the displacement of the particle over the interval t_1 to t_2? (b) If the position of the particle at $t = 0$ was $\vec{r}_0$, what is its position at an arbitrary instant t?

Solution. (a) Newton's second law relates force with the rate of change of momentum or acceleration if mass is constant. So, from the given information of a constant force on the particle, we obtain the constant acceleration. A constant acceleration gives velocity that changes linearly with time as given in Eq. 6.17. Let us write the velocity at an arbitrary time as $\vec{v}(t)$ to display the time-dependence of the velocity vector. We wish to determine the displacement vector, which can be obtained by making use of the definition of velocity.

$$\frac{d\vec{r}}{dt} = \vec{v}(t),$$

which can be written as a differential relation:

$$d\vec{r} = \vec{v}(t)dt.$$

Now we can integrate over the time interval. On the left side the limits are $\vec{r}_1$ and $\vec{r}_2$ corresponding to the limits on the right of t_1 and t_2.

$$\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} = \int_{t_1}^{t_2} \vec{v}(t)dt.$$

This integral is usually done by expressing the displacement vector in Cartesian components giving us three integrations, one each for x, y and z.

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v_x(t)dt.$$
$$\int_{y_1}^{y_2} dy = \int_{t_1}^{t_2} v_y(t)dt.$$
$$\int_{z_1}^{z_2} dz = \int_{t_1}^{t_2} v_z(t)dt.$$

The velocity for this problem given in Eq. 6.17 can be integrated to find the change in position. After performing the integrations separately, we combine the results and write it in the vector form.

$$\vec{r}_2 - \vec{r}_1 = \vec{v}_0 (t_2 - t_1) + \frac{1}{2}\frac{\vec{F}}{m} (t_2 - t_1)^2 . \tag{6.19}$$

(b) We put $t_1 = 0$, $t_2 = t$, $\vec{r}_1 = \vec{r}_0$ and $\vec{r}_2 = \vec{r}$ in Eq. 6.19 to obtain the standard constant acceleration kinematics equation in the vector form.

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\frac{\vec{F}}{m} t^2. \tag{6.20}$$

Note that this equation is a vector equation, which will give rise to three equations for components.

$$x = x_0 + v_{0x}t + \frac{1}{2}\frac{F_x}{m}t^2.$$
$$y = y_0 + v_{0y}t + \frac{1}{2}\frac{F_y}{m}t^2.$$
$$z = z_0 + v_{0z}t + \frac{1}{2}\frac{F_z}{m}t^2.$$

Typically, if the force is constant, you would orient one of the Cartesian axes along the direction of the constant force vector. Suppose, you orient the x-axis in the direction of the force then y and z motions will be constant velocity the motions.

$$x = x_0 + v_{0x}t + \frac{1}{2}\frac{F_x}{m}t^2.$$
$$y = y_0 + v_{0y}t.$$
$$z = z_0 + v_{0z}t.$$

Example 6.4.6. Uniform circular motion. A stone of mass m is tied with a string and swung in a horizontal circle of radius R with a constant speed v. (a) What is the acceleration of the stone? (b) Which force is responsible for the acceleration? Find the magnitude and direction.

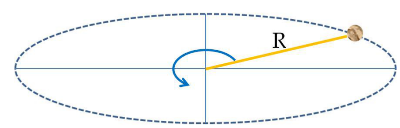


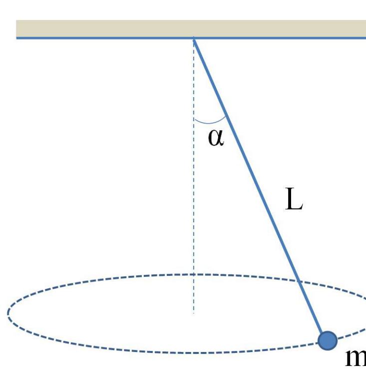
Figure 6.6: Example 6.4.6.

Solution. (a) A particle in a uniform circular motion, defined as a circular motion with constant speed, has the acceleration pointed towards the center of the circle with magnitude v^2/R.

(b) The force is the force of tension between the stone and the hand linked by the string. The tension force acting on the stone is directed towards the hand which is at the center of the circle. The magnitude of the tension force on the stone is $ma = mv^2/R$.

$$\vec{T} = -\left(\frac{mv^2}{R}\right)\hat{u}_r.$$

Note that this force is not a constant force! Why? Although the magnitude is constant, the direction is changing, since, in order to be always pointed towards the center of the circle, the direction in space would have to change as the position of the particle changes around the circle.



Example 6.4.7. Conical pendulum. A ball of mass m is tied to a string of length L and suspended from the ceiling. The ball is then swung smoothly with a constant speed in a horizontal circle so that the point of suspension and the horizontal circle form a cone of apex angle α (Fig. 6.7). Find the tension in the string and the speed of the ball in terms of m, L, α, and g.

Solution. Let v be the constant speed of the ball. Since speed of the ball is constant, its acceleration is all centripetal. That is, the acceleration of the ball is pointed towards the center of the circle of motion and has the magnitude given by v^2/R, where $R(= L\ \sin\alpha)$ is the radius of the circle as seen from Fig. 6.8.

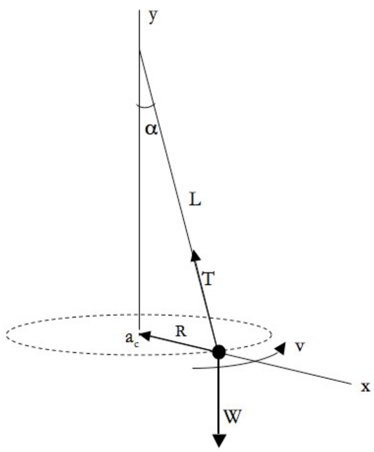


Figure 6.8: Example 6.4.7. The forces on a conical pendulum and choice of axes. The figure is drawn at an instant when the ball is crossing the x-axis so that the centripetal acceleration is pointed towards the negative x-axis.

There are only two forces on the ball: the weight W of the ball and the tension T in the string, ignoring minor effects of air resistance. At any instant, the forces fall in one plane. We examine the equation of motion at an arbitrary instant when the ball is crossing x-axis such that the two forces have only x and y-components non-zero, and the acceleration has only x-component non-zero. Organizing the information about forces and their components in a table is quite helpful.

Force name	x-component	y-component
$\vec{W}$	0	$-mg$
$\vec{T}$	$-T\ \sin\alpha$	$T\ \cos\alpha$
$\vec{F}_{\text{net}}$	$-T\ \sin\alpha$	$T\ \cos\alpha - mg$

The corresponding components of the acceleration are:

$$a_x = -\frac{v^2}{R} = -\frac{v^2}{L\ \sin\alpha},\quad a_y = 0.$$

Here a_x is negative since the acceleration at the instant is pointed towards the negative x-axis. Now equating the x-component of force to mass times x-component of acceleration, and similarly for the y-components, we find

$$-T\ \sin\alpha = m\left(-\frac{v^2}{L\ \sin\alpha}\right) \tag{6.21}$$

$$T\ \cos\alpha - mg = 0. \tag{6.22}$$

From Eq. 6.22 we find that the magnitude of tension in the string is

$$T = \frac{mg}{\cos\alpha}. \tag{6.23}$$

Now, putting T from Eq. 6.23 into Eq. 6.21 and solving for speed v we find

$$v = \sqrt{g\ L\ \sin\alpha\ \tan\alpha}.$$

It is interesting to note that the speed of the ball is independent of its mass. Therefore, all conical pendula of the same length L will have an identical period ($2\pi R/v = 2\pi L\sin\alpha/v$) if swung with the same angle α regardless of their masses.

Example 6.4.8. Plane pendulum - An example of a motion in a vertical circle.

A ball of mass m tied to a string of length L is suspended from the ceiling. The ball is pulled to one side so that the string makes an angle θ_0 with the vertical and let go from rest. The ball swings in an arc of total angle $2\theta_0$ in a circle of radius L such that its speed is varying with time. Find relations between the tension in the string, the speed of the ball, and the rate at which the speed of the ball is changing with time at an arbitrary instant when the string makes an angle θ.

Solution. Let v be the speed of the ball when the string makes an angle θ with the vertical. The acceleration of the ball has both a radial component v^2/L pointed towards the suspension point, and a tangential component dv/dt pointed towards increasing speed tangential to the circle of motion. The two forces on the ball are its weight $\vec{W}$ and the tension $\vec{T}$ in the string which varies with time and depends on the instantaneous angle θ of suspension.

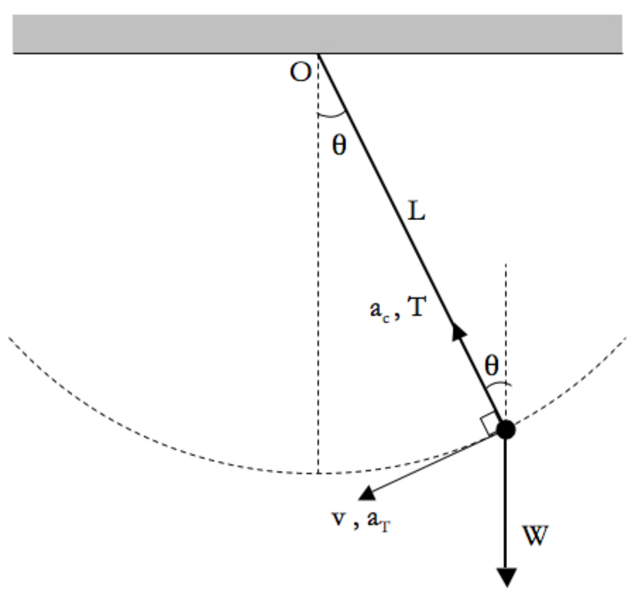


Figure 6.9: The free-body diagram of forces on a plane pendulum. The acceleration has both radial and tangential components. Weight vector $\vec{W}$ also has both radial and tangential components, while tension vector $\vec{T}$ has only a radial component.

Again, we use a table to organize the information about forces and their components. Here we use radial and tangential directions for components.

Force name	radial	tangential
$\vec{W}$	$-mg\ \cos\theta$	$mg\ \sin\theta$
$\vec{T}$	T	0
$\vec{F}_{net}$	$T - mg\ \cos\theta$	$mg\ \sin\theta$

The corresponding components of the acceleration are:

$$\text{Radial inward: } a_c = \frac{v^2}{L}, \quad \text{Tangential: } \quad a_T = \frac{dv}{dt}.$$

Equating the radial and tangential components of the net force to mass times the corresponding components of acceleration we find the required relations.

$$\begin{aligned} &T - mg\ \cos\theta = m\frac{v^2}{L}, \\ &mg\ \sin\theta = m\frac{dv}{dt}. \end{aligned}$$