6.7 DISPERSION AND RESOLVING POWER

A diffraction grating produces sharp lines in its diffraction patterns of a monochromatic light. The angular positions of the maxima depend upon the slit separation and wavelength of light used. Therefore a common use of the diffraction gratings is in the separation of lights of different colors and measurements of their wavelengths. However if the wavelengths of light to be separated are too close then their diffraction patterns may overlap considerably and not be distinguishable. Once again the Raleigh criterion is used to deduce the minimum separation for resolvability. The Raleigh criterion is based on the fact that the peaks of the diffraction pattern have width, even though they are very narrow and when two peaks overlap they may mask each other. First we discuss a measure of the width of peaks, called the half-width, and then discuss the criterion of resolvability below.

6.7.1 Half-Width

A peak is defined by the bright spot surrounded by the minima around it. The width of a peak is therefore equal to the angle subtended at the center of the diffraction grating by the minima around the peak. For instance, the width of the central bright spot has a width given by the angle subtended by the dark edges of the bright spot. From the formula for the intensity in diffraction for an N-slit grating we know that the minima around the peaks are from the interference part.

$$I(\theta) = \left[\frac{\sin(N\alpha)}{N \sin\alpha} \right]^2. \tag{6.40}$$

The α for the minimum around the central peak will be given by

$$N\alpha = \pm\pi. \tag{6.41}$$

This condition in terms of the angle θ is

$$\sin\theta = \pm\frac{\lambda}{Na}.$$

The half-width is defined as the angle from the center to one of these minima as shown in Fig. 6.19.

Hence, half-width $\theta_{\rm hw}$ of the central peak will be

$$\sin\theta_{\rm hw} = \frac{\lambda}{Na}.$$

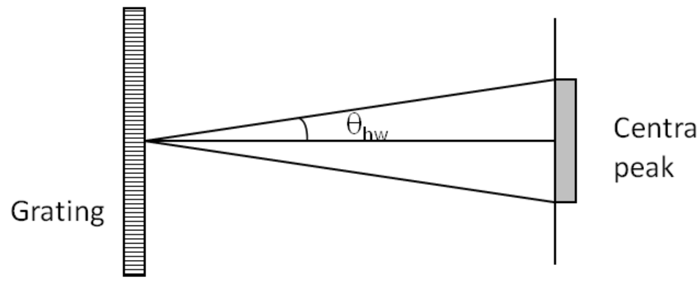

Figure 6.19: Half-width for central peak

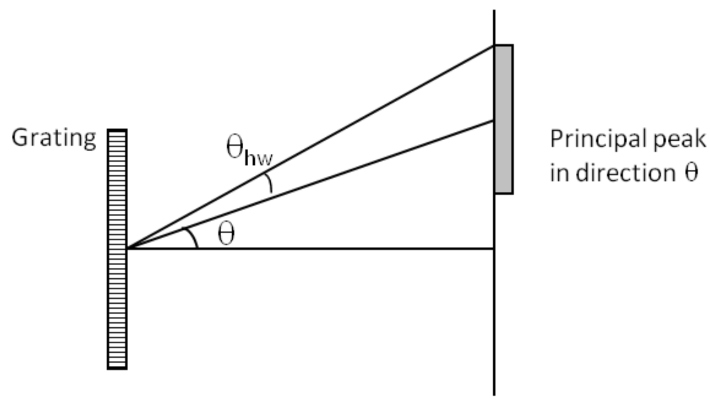

Figure 6.20: Half-width for at an angle.

Since the angle is small, we can replace $\sin\theta_{\rm hw}$ by $\theta_{\rm hw}$ and obtain a simpler relation for half-width of central maximum.

$$\theta_{\rm hw} = \frac{\lambda}{Na}. \tag{6.42}$$

The width of other principal maxima depend upon the angle θ they make with the horizontal line (Fig. 6.20).

A calculation based on the intensity formula shows that half-width of the principal maxima in the direction θ depends on the angle θ with respect to the horizon as given by the following formula.

$$\theta_{\rm hw} = \frac{\lambda}{Na\cos\theta}. \tag{6.43}$$

6.7.2 Dispersion

The diffraction pattern of a light consisting of multiple wavelengths consists of separated colors, similar to the light coming out of a prism or in a rainbow. The phenomenon is called dispersion. But unlike

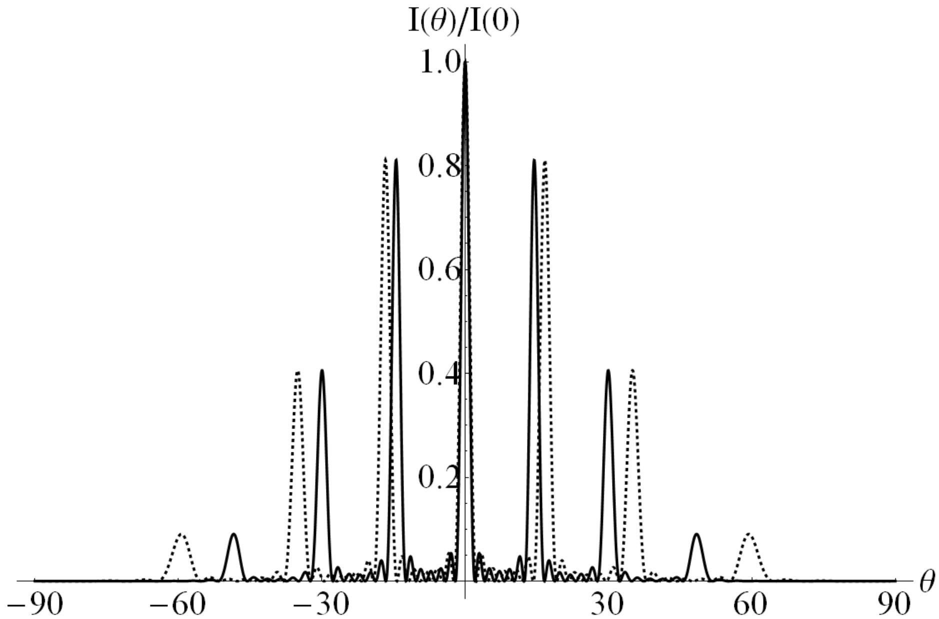

Figure 6.21: Diffraction pattern for the light consisting of two wavelengths, $\lambda_1 = 1.0\mu$m and $\lambda_1 = 1.15\ \mu$m, incident on a grating with a separation of 4 μm, slit width of 1 μm over $N = 7$ slits. The solid curve is for λ_1 and the dashed curve for λ_2. The peaks for $m = 0$ are unresolved. The peaks for $m = 1$ order are resolvable based on the Raleigh criterion. The principal peaks of the two waves are seen to separate out more for the $m = 2$ order than for the $m = 1$ order.

separation of colors in prisms, the diffraction pattern shows separation of colors in each order. Thus, many "rainbows" come out of a diffraction grating when white light is incident on it (Fig. 6.21).

The magnitude of separation of two colors $\Delta\theta$ depends upon the difference of the wavelength $\Delta\lambda$. Hence, we define a quantity called dispersion as

$$D = \frac{\Delta\theta}{\Delta\lambda}. \tag{6.44}$$

Since the angular separation of two colors depends on the parameters of the diffraction grating and the order m of the principal maximum, we expect the dispersion D to depend upon these quantities as well. Consider a grating consisting of N slits with inter-slit separation a. Then the principal maxima occur at angles consistent with the following condition.

$$\text{Principal maxima:}\quad a\sin\theta = m\lambda,\quad m = 0, \pm1, \pm2, \cdots . \tag{6.45}$$

Taking the differential of both sides, we find

$$a\cos\theta\Delta\theta = m\Delta\lambda. \tag{6.46}$$

Therefore, the dispersion of a diffraction grating corresponding the the m^{th} order diffraction is

$$D = \frac{\Delta\theta}{\Delta\lambda} = \frac{m}{a\cos\theta}. \tag{6.47}$$

The result clearly shows that to achieve a greater angular separation we must have a smaller spacing a between the slits and look at higher order m. With increasing m, the numerator increases as well as the denominator $\cos\theta$ decreases, both helping to increase the dispersion D.

6.7.3 Resolving Power

The ability of a diffraction grating to separate lights of different wavelength is often designated by its resolving power. A grating with higher resolving power helps resolve two wavelengths by producing narrower peaks. We have already encountered dispersion of a grating that gives us the separation of waves by a diffraction grating. The resolving power is a related quantity defined by making use of the smallest wavelength interval that can be barely resolved. Let $\Delta\lambda \equiv |\lambda_1 - \lambda_2|$ be the smallest wavelength interval resolvable so that λ_1 and λ_2 can be barely resolved by the given diffraction grating. Then the resolving power R is defined as

$$R = \frac{\lambda_{\text{ave}}}{\Delta\lambda}, \tag{6.48}$$

where λ_{ave} is the average of the two wavelengths,

$$\lambda_{\text{ave}} = \frac{\lambda_1 + \lambda_2}{2}.$$

Using the Raleigh criterion that two peaks are barely resolvable if the peak of one is at the zero of the other peak. That means for the wavelengths λ_1 and λ_2 to be resolvable by the Raleigh criterion their peaks will be separated by half-width.

$$\theta_{\text{hw}} = \frac{\lambda_{\text{ave}}}{Na\cos\theta} = \text{separation between peaks of } \lambda_1 \text{ and } \lambda_2 = \Delta\theta. \tag{6.49}$$

Now, using the condition for the maxima of the principal peaks we find

$$(a\cos\theta)\Delta\theta = m\Delta\lambda \implies \Delta\theta = \frac{m\Delta\lambda}{a\cos\theta}. \tag{6.50}$$

Therefore,

$$\frac{\lambda_{\text{ave}}}{Na\cos\theta} = \frac{m\Delta\lambda}{a\cos\theta}. \tag{6.51}$$

Hence, the resolving power R of a diffraction grating with N slits spread over a distance Na is given by

$$R = \frac{\lambda_{\text{ave}}}{\Delta\lambda} = Nm. \tag{6.52}$$

Example 6.7.1. Resolving the Sodium doublet.

The D-line of sodium consists of two different wavelengths, 589.0 nm and 589.6 nm. A beam of light from a sodium lamp forms a beam of width 10 mm. The beam is incident perpendicularly on a diffraction grating that has 8000 rulings over a width of 10 mm. (a) Find the directions of the first-order principal peaks for the two wavelengths. (b) Decide if the two peaks resolvable in the first order. (c) Find the number of lines of a grating so that peaks in the first order are barely resolved?

Solution. (a) We need separation a between the slits to figure out the direction of the first-order principal peak.

$$a = \frac{10 \text{ mm}}{8000 \text{ lines}} = 1250 \text{ nm}.$$

For $m = 1$, the condition for the principal peak is $a \sin\theta = \lambda$. Let us denote the angle for the wavelength 589.00 nm by θ and the angle for the wavelength 589.59 nm by ϕ. Hence the directions of $m = 1$ peaks for the the two wavelengths are

$$\theta_1 = \sin^{-1}\left(\frac{589.00 \text{ nm}}{1250 \text{ nm}}\right) = 0.49065 \text{ rad}$$
$$\phi_1 = \sin^{-1}\left(\frac{589.59 \text{ nm}}{1250 \text{ nm}}\right) = 0.49119 \text{ rad}$$

(b) We can compare the resolution required to the resolving power of the grating to see if the peaks will be resolved. The resolving power required for resolving the the two given wavelengths is

$$R_{\text{req}} = \frac{\lambda_{\text{ave}}}{\Delta\lambda} = \frac{589.295 \text{ nm}}{0.59 \text{ nm}} = 999.$$

The resolving power of the given grating for $m = 1$ peaks is

$$R_{\text{grating}} = Nm = 8000.$$

Since $R_{\text{grating}} > R_{\text{req}}$ the peaks will be resolved.

(c) Using the required resolving power, we can work backwards and deduce the number of lines required in 10 mm, the beam width, so that the grating can resolve the two given waves.

$$Nm = R_{\text{req}} \implies N \times 1 = 999 \implies N = 999.$$