7.6 COLLISIONS

In a two-body collision, two objects collide with one another. In a collision, each body applies an impulse on the other body. Therefore, the momenta of the two bodies change in a collision.

However, since the collision event involves only internal forces between the two bodies, the total momentum of the colliding bodies after the collision will be equal to their total momentum before the collision, both in magnitude and direction.

$$\boxed{\left(\vec{P}_{\text{total}}\right)_{\text{after}} = \left(\vec{P}_{\text{total}}\right)_{\text{before}} \quad \text{(bodies isolated)}} \tag{7.55}$$

If two colliding objects 1 and 2 have momenta $\vec{p}_1$ and $\vec{p}_2$ immediately before the collision, and $\vec{p}_1{}'$ and $\vec{p}_2{}'$ immediately after the collision, the momentum conservation equation across the collision process produces the following vector equation.

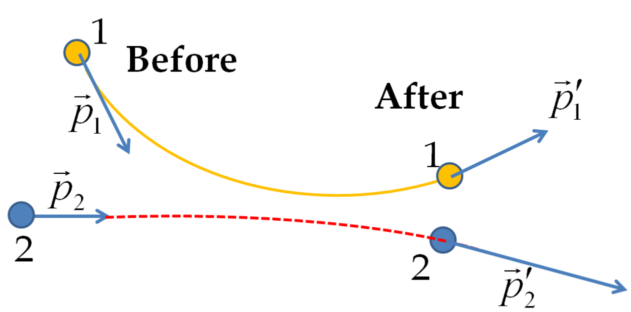

Figure 7.19: Collision process leads to transfer of momentum among bodies such that the total momentum before the collision is equal to the total momentum after the collision: $\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$.

$$\boxed{\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'.} \tag{7.56}$$

Therefore, during a collision, a transfer of momentum among the bodies takes place such that the total momentum of bodies before the collision is equal to the total momentum after the collision. Both the direction and the magnitude of the momenta of the two colliding objects may change, subject to the constraint that the total momentum remains fixed at the value immediately prior to the collision. Note that Eq. 7.56 is a vector equation. Often, it is helpful to work with the Cartesian components of this equation.

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x} \tag{7.57}$$

$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y} \tag{7.58}$$

$$p_{1z} + p_{2z} = p'_{1z} + p'_{2z} \tag{7.59}$$

If the incoming and outgoing momenta lie in one plane, then we usually take the plane to be the xy plane, and work with only the x and y-components.

The change in momentum of each object is due to the impulse by the other object. Suppose the collision lasts for a duration Δt when the two bodies are interacting and suppose that there is no interaction between the two bodies before or after the collision, then the change in momentum of each body during the collision gives us an idea of the average force between the bodies during the collision.

$$\boxed{\text{Change in momentum of 1: } \vec{p}_1{}' - \vec{p}_1 = \vec{F}_{\text{by 2 on 1}}\Delta t} \quad (7.60)$$

$$\boxed{\text{Change in momentum of 2: } \vec{p}_2{}' - \vec{p}_2 = \vec{F}_{\text{by 1 on 2}}\Delta t} \quad (7.61)$$

If we know the momentum change of one colliding body and have a good estimate of the duration Δt for which the two objects were in contact, these equations can be used to obtain an estimate of the average force $\vec{F}_{ave}$ between the two bodies.

Example 7.6.1. One dimensional collision. Two carts of masses 200 g and 300 g are moving towards each other on a straight track with speeds of 10 m/s and 15 m/s respectively. Immediately after the collision, the 200-g cart reverses direction and moves with a speed of 8 m/s. What is the velocity (i.e. speed and direction) of the 300-gram cart after collision?

Solution. To make the collision event clearer, we normally draw two figures, one for the situation **before the collision** and one for the situation **after the collision** as shown in Fig. 7.20. The direction of the 300-gram cart after the collision is not known yet, so we arbitrarily pick a direction. If we find that x-component of the momentum for 300-gram is positive then our chosen direction would be the correct one, otherwise the correct direction would be just the opposite of our choice. For component calculations, we do not need to know the position of the origin, we need only the directions of the axes. As shown in Fig. 7.20, we pick the x-axis to be the axis along which the collision takes place in this problem. This choice will give zero y and z-components and we will not worry about then any further.

Let us label the 200-gram cart as “1” and the 300-gram cart as “2”. Then we have the following quantities known from the statement of the problem when referred to the figure drawn. [To keep the expressions simple I will leave out the units from calculations, and put the units in at the end only. I will also drop the subscript x from the x-component of velocity in the notation.]

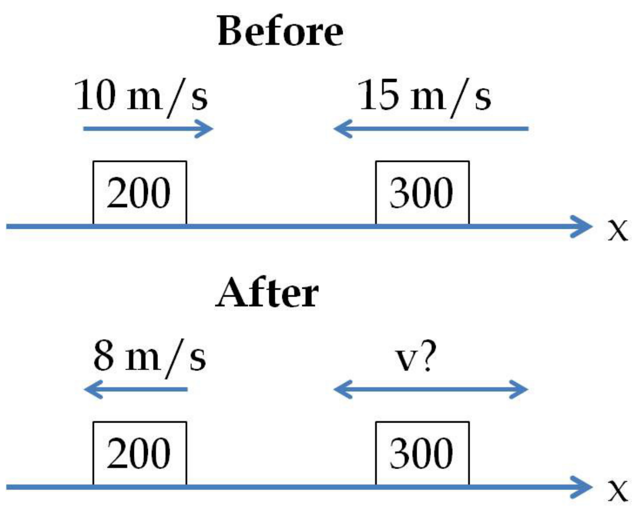

Figure 7.20: Example 7.6.1. Normally two figures are drawn to understand the collision process, one for the physical situation before the collision and the other after the collision.

Before collision:

$$p_{1x} = +(0.2)(10) = +2;$$
$$p_{2x} = (0.3)(-15) = -4.5.$$

After collision:

$$p'_{1x} = (0.2)(-8) = -1.6;$$
$$p'_{2x} = +(0.3)(v) = \ 0.3v.$$

The conservation of the x-component of the total momentum across the collision yields an equation for the unknown v.

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}$$
$$+2 - 4.5 = -1.6 + 0.3\ v, \quad \Longrightarrow \quad v = -3\ m/s.$$

Therefore, we conclude that the 300-gram cart continues to move towards the negative x-axis with a speed of 3 m/s, the same direction as the direction before the collision.

Example 7.6.2. Two-dimensional collision. Two balls A and B of masses 0.4 kg and 0.45 kg are moving towards each other with speeds 5 m/s and 8 m/s respectively and collide at an angle of 60°, which is the angle between their velocities before the collision. After the collision, ball A continues in the direction that is 15° to its original direction at a speed of 10 m/s. What is the velocity of ball B after the collision?

Solution. We usually draw four figures when the collision takes place in two or more dimensions. These consist of a schematic diagram for the situation before, another for the situation after the collision, and two diagrams with coordinate choices to help with the components of vectors as shown in Fig. 7.21.

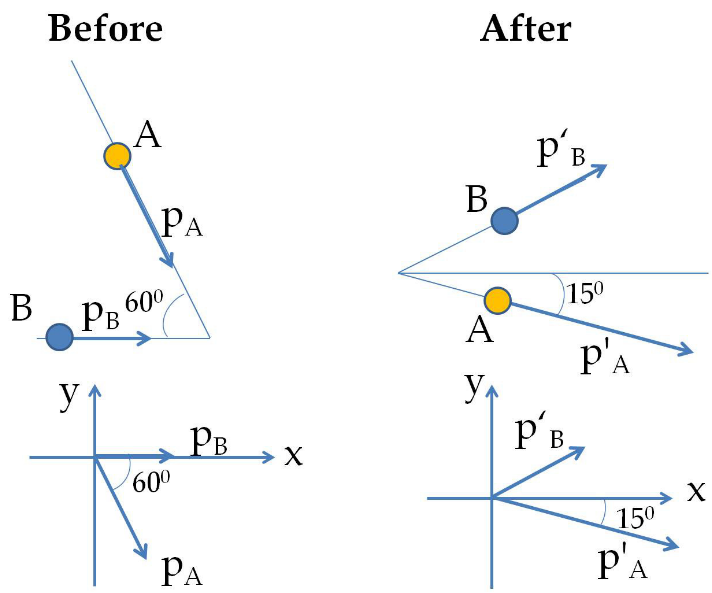

Figure 7.21: Example 7.6.2.

Just as before, when dealing with the addition or subtraction of vectors, it is helpful to organize the information about the components in a table format. Using the magnitudes from the given data and the direction with respect to the axes shown in the figure, we work out the components of various momenta. Since we do not know either the magnitude or the direction of the momentum of B, we will leave the momentum in terms of the x and y-components. Furthermore, to simplify the writing, we will denote the components of the unknown momentum by p_x and p_y in place of p'_{Bx} and p'_{By}, respectively, since the later notation is too cumbersome. We put the units in the column heads in the table.

Vector	x-component (kg.m/s)	y-component (kg.m/s)
$\vec{p}_A$	$0.4 \times 5\cos(60°) = 1$	$-0.4 \times 5\sin(60°) = -1.732$
$\vec{p}_B$	$0.45 \times 8 = 3.6$	0
$\vec{p}'_A$	$0.4 \times 10\cos(15°) = 3.86$	$-0.4 \times 10\sin(15°) = -1.04$
$\vec{p}'_B$	p_x	p_y

The conservation of the x and y-components of momenta give the following values for p_x and p_y.

$$p_x = 0.74 \text{ kg.m/s}$$
$$p_y = -0.70 \text{ kg.m/s}$$

The magnitude of momentum of ball B after collision is

$$p_B = \sqrt{p_x^2 + p_y^2} = 1.0 \text{ kg.m/s},$$

The direction of the momentum of ball B is in the fourth quadrant of the Cartesian coordinates in Fig. 7.21. The clockwise angle from the positive x-axis would suffice to determine the direction of the momentum vector. This angle is given by

$$\theta = \arctan\left(\frac{-0.70}{0.74}\right) = -43^\circ,$$

where the negative value indicates that the direction of the vector is at an angle of 43° clockwise from the positive axis towards the negative y-axis. The velocity of ball B is in the same direction as the momentum and has the magnitude

$$\text{Magnitude of velocity, } v = \frac{1 \text{ kg.m/s}}{0.45 \text{ kg}} = 2.2 \text{ m/s}.$$

Example 7.6.3. Colliding bodies stuck together. Two balls A and B of mass 0.4 kg and 0.45 kg are moving towards each other with speeds 5 m/s and 8 m/s respectively, and then they collide at an angle of 60° as in the last example. Upon collision, they stick together, and then move as one object after the collision. Find the speed and direction in which the pair moves.

Solution. See Fig. 7.21 for the physical situation except that after the collision, now, the two bodies stick together and move in some direction to be determined. Let the components of the two bodies together after the collision be P_x and P_y. The values for components of momenta before and after the collision are

Vector	x-component (kg.m/s)	y-component (kg.m/s)
$\vec{p}_A$	1	−1.732
$\vec{p}_B$	3.6	0
$\vec{P}$	P_x	P_y

The conservation of momentum gives the values of P_x and P_y.

$$P_x = 4.6 \text{ kg.m/s}; \quad P_y = -1.732 \text{ kg.m/s}.$$

The magnitude of velocity is equal to the magnitude of momentum divided by the mass of the two bodies moving together.

$$\text{Magnitude of velocity: } v = \frac{\sqrt{4.6^2 + 1.732^2}\ \text{kg.m/s}}{0.85\ \text{kg}} = 5.8\ \text{m/s}.$$

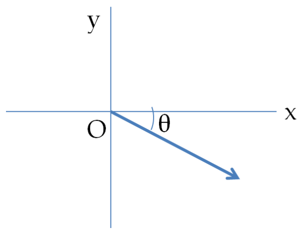

Figure 7.22: Example 7.6.3.

The direction of the velocity is in the fourth quadrant, and can again be given by the clockwise angle θ from the positive x-axis.

$$\theta = \arctan\left(\frac{-1.732}{4.6}\right) = -21^\circ.$$

Example 7.6.4. A projectile breaking up in mid air. A projectile of mass 2 kg moving at 50 m/s breaks up in mid air into two pieces. The larger piece comes out at 60 m/s in the forward direction, and the smaller piece at 100 m/s in the backward direction. What are the masses of the two pieces?

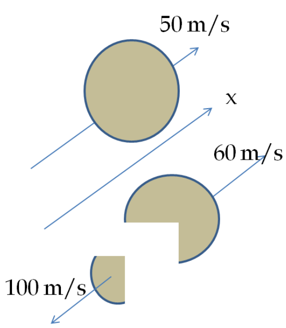

Figure 7.23: Example 7.6.4.

Solution. Since an explosion is the opposite of a collision, only internal forces are involved in explosion. Therefore, the total momentum of the multiparticle system after explosion must equal the total momentum of the exploding body before the explosion.

The given situation has a one-dimensional scenario. Hence, we will use only one of the Cartesian axes for the analysis. Let the positive x-axis be pointed in the forward direction. Let m be the mass of the piece in the forward direction. Then, the x-component of momentum equation conservation for the problem yields the following condition.

$$2\ \text{kg} \times 50\ \text{m/s} = m 60\ \text{m/s} - (2 - m) \times 100 \text{m/s}.$$

Solving this equation for m we find that $m = 1.875\ kg$. Therefore, the larger piece has a mass of 1.875 kg and the smaller piece 0.125 kg.