3.2 ELECTRIC POTENTIAL

3.2.1 Definition

In chapter 1 we discussed that each electric charge changes the electrical properties of space and we were able to label each point in space by the net electric field by all charges. The electric field at a particular point told us the electric force a test charge will experience if placed at that point. In this chapter we learned that the test charge will also have an electrical potential energy when it is at that point. This potential energy is actually proportional to the charge.

$$U \propto \text{Charge}.$$

For instance, if you place a particle 1 of charge Q_1 at some point P and you find that the electrical potential energy is U_1, then keeping all else same, if you place another particle 2 of charge $Q_2 = 2Q_1$ at the same point P, then the electrical potential energy of B will be twice as much, that is $U_2 = 2U_1$. We will place a subscript P to U to indicate that the potential energy may depend on the location in question, and write the proportionality of the potential energy U_P of a particle of charge Q placed at point P as

$$\boxed{U_P(\text{of } Q) = QV_P,} \tag{3.9}$$

where V_P is called the electric potential at point P and represents the "energetic" influence of other charges on the charge Q when the later is at point P. If the potential energy of the particle varies in space, then we expect the electric potential will have different values at different points in space. That is, electric potential in a region is actually electric potential field. However, unlike the electric field, the electric potential field is a scalar field although it can take both positive and negative values.

Equation 3.9 says that the unit of electric potential in the SI system should be obtained from dividing the unit of energy by the unit of charge.

$$\text{Unit of Electric Potential} = \frac{\text{J}}{\text{C}}.$$

This unit is given its own name, Volt(V) after Alssandro Volta (1745-1827) who made numerous contributions to electricity, including the invention of the voltaic cell, the predecessor to the modern battery.

$$\text{V} = \frac{\text{J}}{\text{C}}.$$

Just as we did with the electric field, we can use the values of the electric potential at various points in space to map out the region and develop an electric potential map of the region. An electric potential map will give the electrical energetic influence in space due to the presence of the source charges. We will discuss some of these maps below, but first, we will work out an analytic expression for the electric potential of discrete charge distributions.

3.2.2 Calculations of Electric Potential

Faraday's picture says that each charge produces an electric field everywhere in space. We now say that each charge also produces an electric potential everywhere and Eq. 3.9 gives us an operational way of determining the electric potential at an arbitrary point. To determine the electric potential V_P at space point $\mathrm{P}(x, y, z)$, we would bring a test charge Q from the reference at infinity to that point and determine the change in the potential energy of the particle. Since the potential energy is taken to be zero at the reference, this will give us the potential energy of the particle when it is at point P. Once, we have found the potential energy we will divide it by the value of the charge and deduce the electric potential at P. During this procedure we make sure that the source charges are not disturbed. This procedure can be implemented for any system of charges. To illustrate with a simple case, we will use this procedure to find a formula for the electric potential of a point charge located at the origin of a coordinate system.

ELECTRIC POTENTIAL OF A POINT CHARGE AT THE ORIGIN Suppose a charge q is fixed to the origin of a Cartesian coordinate system. What is the electric potential at point P a distant r from the charge? To find the electric potential at P, let us "bring" a test charge charge Q from the reference at infinity to the point P, and deduce electric potential at P by dividing the change in the potential energy of Q by the charge Q.

The change in the electrical potential energy of Q is:

$$\Delta U_P = \frac{1}{4\pi\epsilon_0}\frac{Q\ q}{r} \tag{3.10}$$

Now, we divide this change in the electrical potential energy of Q by the charge Q to obtain the electric field at P due to the charge q at the origin.

$$\boxed{V_P = \frac{\Delta U_P}{Q} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.} \tag{3.11}$$

Note that the value of the electric potential at P depends on the sign of the charge. Thus, if the "source" charge q is positive, then the electric potential at P would be positive, and when q is negative the electric potential at P would negative. Let me emphasize again that the electric potential V_P exists at the space point P whether or not the test charge is placed there. The test charge Q only "discovers" an electric potential by having certain amount of electric potential energy compared to the energy at the reference when it is placed at that point.

SUPERPOSITION OF ELECTRIC POTENTIAL

Just as the electric field obeys a superposition principle, so does the electric potential. Consider a system consisting of N charges q_1, q_2, $\cdots$, q_N. What will be the net electric potential V_P at a space point P from these charges?

Each of these charges are "source" charges that would produce their own electric potential at point P, independent of whatever other changes may be doing. Let $V_1, V_2, \cdots, V_N$ be the electric potentials at P produced by the charges $q_1, q_2, \cdots, q_N$, respectively. Then, the net electric potential V_P at that point will be equal to the sum of these individual electric potentials as you can easily show by calculating the potential energy of a test charge when you bring the test charge from the reference point at infinity.

$$V_P = V_1 + V_2 + \cdots + V_N. \tag{3.12}$$

Further Remarks

We can show that the superposition of electric potential occurs by explicitly working out the potential energy of a test charge when the test charge is at point P. Suppose there are N-charges, q_1, q_2, $\cdots$, q_N, fixed in space. Let the direct distances of a space point P from these charges be r_1, r_2, $\cdots$, r_N, respectively. The expression for the potential energy of a test charge Q was worked out in Eq. 3.7, where each of the term was proportional to the test charge Q, and therefore we can factor out Q. We write the potential energy as U_P now.

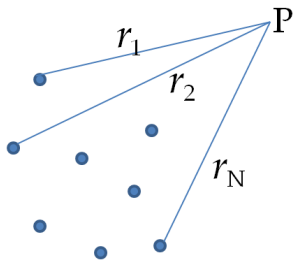

Figure 3.8: Notation for direct distances from charges to space point P.

$$U_P = Q\left[\frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_N}{r_N}\right] \tag{3.13}$$

The quantity in bracket $[\cdots]$ does not depend on charge Q, therefore must be the electric potential at P produced by the source charges q_1, q_2, $\cdots$, q_N.

$$V_P = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2} + \cdots + \frac{1}{4\pi\epsilon_0}\frac{q_N}{r_N} \tag{3.14}$$

Each one of the terms is the potential at P produced by one of the charges. Therefore, the superposition principle follows from this equation.

$$V_P = V_1 + V_2 + \cdots + V_N.$$

3.2.3 Electric Potential Maps

We can use Eq. 3.9 to determine the potential map of a space by measuring changes in the potential energy of a test particle of charge Q as we move the test particle over space points in the region of interest. Although we have chosen infinity as the reference point in the calculations above, in experiments we choose a reference point at some finite place, and move the test charge Q from the reference point to the point of interest and note the amount of work done on the test charge Q by an applied force that balances the electric force on the test charge. Work done would equal to the potential energy of the test charge with respect to the reference. Then, we divide the potential energy of the test charge by its charge Q to obtain the electric potential at that space point. Do this for every space point in the region of interest, and you would have an electric potential map.

Since electric potential is a scalar quantity, each space point in a potential map is labeled with a real number, which can be positive or negative. The usual practice for presenting a potential map is to connect all the points of equal potential values.

In a three-dimensional map of the electric potential, the points of the equal potential vlaues fall on one surface, which are called the **equipotential surfaces**. The points of equal potential values in a plane are connected curves or contours, which are called **equipotential lines**. These contours are labeled by the corresponding electric potential values and look similar to the topographical maps of the Earth. The contours of the electric potential maps are akin to the height above the sea level in a topographical map of the surface of the Earth.

Electric potential map of a point particle

Consider a point particle with charge q. What electric potential map would we discover near the charge? First we will fix the particle at some place so that we do not disturb the "source" charge q when working with the test charge to find the electric potential map of the region near q. Since the force is spherically symmetric, we expect the potential to be spherically symmetric. Th equipotential points fall on spherical surfaces around the point particle as shown in Fig.

3.9. If we look at the plane containing the charge, we obtain a two-dimensional map which has circles around the charge. Rotating the two-dimensional map about an axis through the charge produces the three-dimensional map.

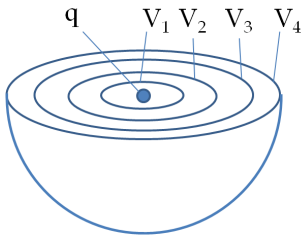

Figure 3.9: Potential map of a point charge. Equipotential points fall on spherical surfaces around the charge.

Electric potential map of two point particles

The potential map of two charges $+q$ and $-q$ is more complicated. Figure 3.10 displays a three-dimensional map of electric potential where lines on the map are for equipotential points. The hill is at the positive charge and the trough at the negative charge. The potential is zero far away from the charges.

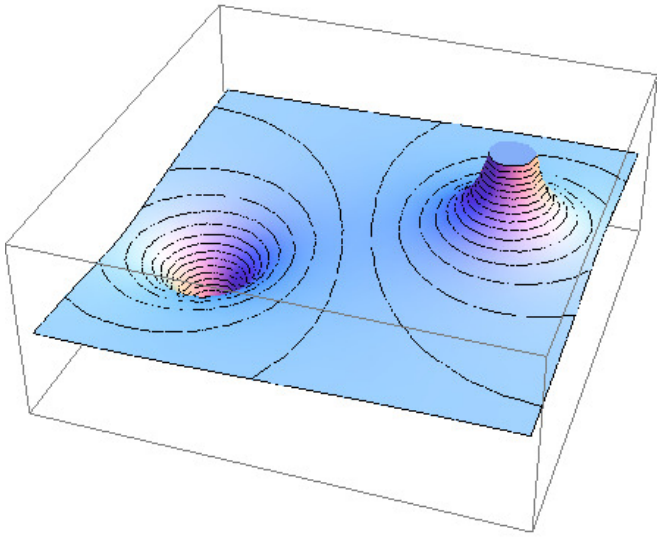

Figure 3.10: Electric potential map of two opposite charges of equal magnitude. The potential is negative near the negative charge and positive near the positive charge.

A map of the cross-sectional plane that contains both charges is shown in Fig. 3.11. The line that is equidistant from the two opposite charges corresponds to zero potential since, at the points of the line, the positive potentials of the positive charge cancel the negative potentials of the negative charge. Equipotential lines in the cross-section plane are closed loops, which are not necessarily circles since at each point the net potential is the sum of potential from each charge.

Example 3.2.1. Calculating Electric Potential.

Three charges (+2 μC, -4 μC, +6 μC) are fixed at the corners of a square of side 3 cm as shown in the figure. Find the potential by

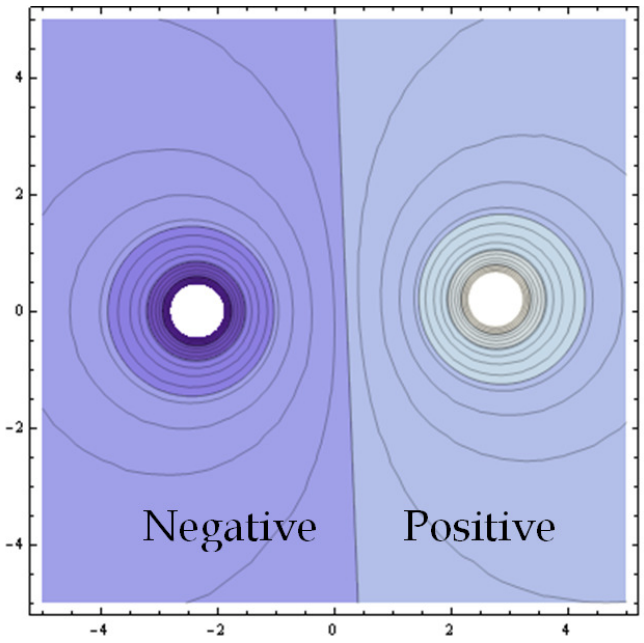

Figure 3.11: A cross-section of the electric potential map of two opposite charges of equal magnitude. The potential is negative near the negative charge and positive near the positive charge.

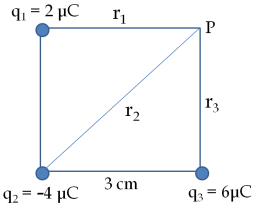

Figure 3.12: Example 3.2.1.

at the fourth corner.

Solution. Each charge has its own potential at point P and the net potential at point P is equal to the algebraic sum of these potentials. We calculate the direct distances from the charges to point P to plug into the formula for the potentials. Referring to figure we find the distances to point P as

$$r_1 = 3 \times 10^{-2} \text{ m}; \quad r_2 = 3\sqrt{2} \times 10^{-2} \text{ m}; \quad r_3 = 3 \times 10^{-2} \text{ m}.$$

Therefore the potential at P is given as:

$$\begin{aligned} V_P &= V_1 + V_2 + V_3 \\ &= 9 \times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \times \left[\frac{2 \times 10^{-6}\text{C}}{3 \times 10^{-2} \text{ m}} + \frac{-4 \times 10^{-6}\text{C}}{3 \times \sqrt{2} \times 10^{-2} \text{ m}} + \frac{6 \times 10^{-6}\text{C}}{3 \times 10^{-2} \text{ m}} \right] \\ &= 1.55 \times 10^4 \text{ V}. \end{aligned} \tag{3.15}$$

3.2.4 Dipole Potential And Dipole Moment

Two equal magnitude charges of opposite types displaced from each other make an **electric dipole**. The electric potential map of a dipole was presented above. We would now like to study the electric potential of a dipole analytically. Dipoles appear naturally at the molecular level. However, their effect can be felt even at macroscopic

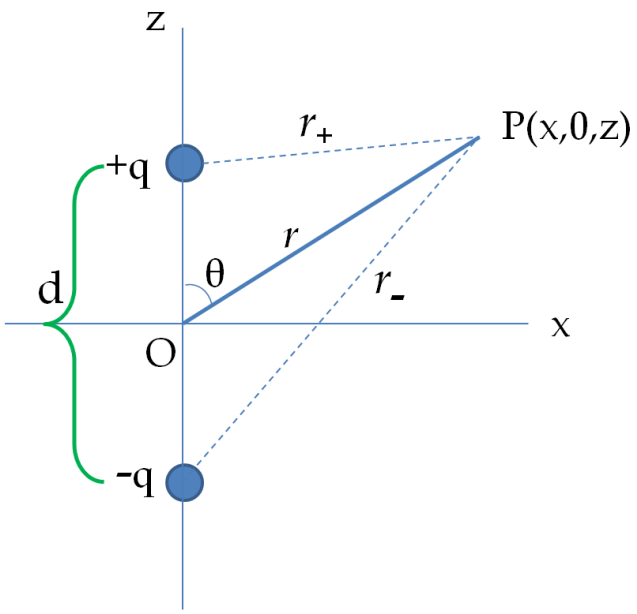

Figure 3.13: Electric Dipole.

distances. Therefore, we will also derive an approximate formula, called the dipole potential, for the electric potential of a dipole at a far away point.

Figure 3.13 shows a dipole with charges $+q$ and $-q$ separated by a distance d. The charges of the dipole in this figure are placed on the z-axis of a Cartesian coordinate system symmetrically about the origin. From the symmetry of the situation, it is clear that we need only work out the electric potential at a point in any one plane containing the z-axis. Therefore, without any loss of generality, we can choose to work in the xz-plane. In the following we will deduce the potential at a point P whose Cartesian coordinates are $(x, 0, z)$.

The potential V_P at P will be given by the superposition of potentials V_+ and V_- from the charges $+q$ and $-q$ respectively.

$$V_P = V_+ + V_- = \frac{1}{4\pi\epsilon_0}\frac{q}{r_+} + \frac{1}{4\pi\epsilon_0}\frac{-q}{r_-}, \tag{3.16}$$

where r_+ and r_- are the distances from the charges $+q$ and $-q$ to the field point P respectively. We can express these distances in terms of the Cartesian coordinates of $(x, 0, z)$ for the field point P and those of the charges, $(0, 0, d/2)$ for $+q$ and $(0, 0, -d/2)$ for $-q$.

$$r_+ = \sqrt{x^2 + \left(z - \frac{d}{2}\right)^2} \tag{3.17}$$

$$r_- = \sqrt{x^2 + \left(z + \frac{d}{2}\right)^2} \tag{3.18}$$

Often we write both of these expressions in one formula using $\pm$ notation.

$$r_{\pm} = \sqrt{x^2 + \left(z \mp \frac{d}{2}\right)^2}, \tag{3.19}$$

where the upper symbols correspond to one formula and the lower symbols to another formula.

The potential given by Eq. 3.16 is exact. As mentioned above, we are often interested in the potential far away from the dipole. By "far away" we mean that the space point P for which the distance r from the dipole is much larger than the distance d between the charges, $d << r$. The exact formula can be simplified for these cases as follows.

To find the approximation of the exact potential given in Eq. 3.16 for $d << r$, we first express the distances r_+ and r_- in terms of the distance r and the angle θ in place of the coordinates x and z of the field point P and take $d << r$ limit. Note that the polar coordinates r and θ given in Fig. 3.13 are related to the Cartesian coordinates x and z as follows.

$$x = r\sin\theta \tag{3.20}$$

$$z = r\ cos\theta \tag{3.21}$$

This gives the following for the distances from the charges to the field point.

$$r_{\pm} = \sqrt{r^2\sin^2\theta + \left(r\cos\theta \mp \frac{d}{2}\right)^2} \tag{3.22}$$

Since we are interested in the limit $d << r$, we introduce a dimensionless small parameter ϵ to facilitate the limit-taking procedure.

$$\text{Let} \quad \epsilon \equiv \frac{d}{r}.$$

Now we write the distances $r_{\pm}$ in terms of the small parameter as

$$r_{\pm} = r\sqrt{\sin^2\theta + \left(\cos\theta \mp \frac{\epsilon}{2}\right)^2} \tag{3.23}$$

We can expand the inverse of $r_{\pm}$ in Maclaurin series about $\epsilon = 0$ for use in the formula for the potential at P. Suppose we write $1/r_{\pm}$ as a function $f_{\pm}$ of ϵ,

$$f(\epsilon) \equiv \frac{1}{r_{\pm}} = \frac{1}{r}\frac{1}{\sqrt{1 \mp \epsilon\cos\theta + \frac{\epsilon^2}{4}}}$$

Then, the **Maclaurin series** of the two functions will be [Note: I have dropped $\pm$ from the notation for simplification. To the student:

if you have not encountered this series or the more general series called the Taylor series yet, you would have to take the series given here on faith now and work out the calculational steps as indicated. The series will make much more sense when you have studies the chapter on Series in your Calculus book.]

$$f(\epsilon) = f(0) + f'(0)\epsilon + f''(0)\frac{\epsilon^2}{2} + f'''(0)\frac{\epsilon^3}{3!} + \cdots , \tag{3.24}$$

where $f(0)$ is the value of f for $\epsilon = 0$, $f'(0)$ is the value of first derivative of f with respect to ϵ and then setting $\epsilon = 0$, $f''(0)$ the second derivative of f with respect to ϵ and then setting $\epsilon = 0$, etc. The main question here is how many terms of Eq. 3.24 should we keep fora good approximation of the exact answer. The answer is that we need only keep the most dominant term in the potential V_P. Therefore, we will work with one term of Eq. 3.24 at a time, put the approximation in V_P, and evaluate the result. [You might be wondering why bother about the approximate answer when we have the exact answer. A good question. It turns out that the approximate answer is quite simple and quite good far away from dipoles as would be the case when dealing with the effect of the microscopic dipoles at everyday distances.]

Step 1: Keep the first term only

Is $f(\epsilon) = f(0)$ enough? Depends on whether the resulting potential has any desired information about the charges and the field point. Let us see what happens by actually working out the potential with this approximation.

$$\begin{aligned} f_{\pm}(\epsilon) &= \frac{1}{r} \\ V_P &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{r}\right] = 0. \end{aligned} \tag{3.25}$$

We get a zero for the potential if we stop at the first term. Although $1/r$ is the most leading behavior of f, it is not the leading behavior of V_P. The physical reason for the cancellation is that at a far away place, the distances to point P from $+q$ and $-q$ are almost identical up to d. Hence the potentials from $+q$ and $-q$ cancel each other out in this approximation.

Step 2: Keep up to the second term

We already know that the first term is zero. Let us evaluate up

to the second term and see what we get.

$$f_{\pm}(\epsilon) = \frac{1}{r} \mp \frac{\cos\theta}{2r^2}\epsilon$$
$$V_P = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} + \frac{d\cos\theta}{2r^2} - \frac{1}{r} + \frac{d\cos\theta}{2r^2}\right]$$
$$= \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2} \tag{3.26}$$

Now, we have a non-zero distance-dependence result for the potential, and therefore we will stop here as we seek only the leading behavior. The potential function given in Eq. 3.26 is also called the **dipole potential** V_{dip}.

$$\boxed{V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.} \tag{3.27}$$

The product qd is called the **electric dipole moment** or simply the **dipole moment** p of the dipole of charges, $+q$ and $-q$ separated by a distance d as in Fig. 3.13.

$$p = qd. \tag{3.28}$$

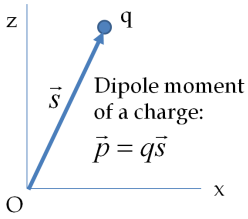

Figure 3.14: Electric dipole moment of a charge located away from origin.

A more fundamental treatment is based on the **dipole moment vector** of a charge. You can define the dipole moment vector of a charge with respect to an origin as follows. Let the position of a charge q with respect to the origin be given by the position vector $\vec{s}$. We define the dipole moment vector $\vec{p}$ of the charge by

$$\vec{p} = q\vec{s}. \tag{3.29}$$

The dipole moment vector of a single does not make physical sense, it is just a mathematical quantity, but when you use this definition on the electric dipole you obtain the dipole moment vector of the dipole whose magnitude is qd as above.

$$\vec{p}_+ = \frac{qd}{2}\hat{u}_z \tag{3.30}$$
$$\vec{p}_- = -\frac{qd}{2}\hat{u}_z \tag{3.31}$$

Therefore, the total dipole moment vector of a dipole is

$$\vec{p} = q\vec{d}, \tag{3.32}$$

where $\vec{d}$ is displacement vector from the negative charge to the positive charge.

The dipole potential of an electric dipole can be written in a coordinate-independent form by making use of the dipole moment

vector as follows. Let $\hat{u}_r$ be the unit vector from the origin, which is at the midpoint between the charges, to the field point P. Then, we find that the dipole potential in Eq. 3.27 is given by the following compact formula

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{u}_r}{r^2} \tag{3.33}$$

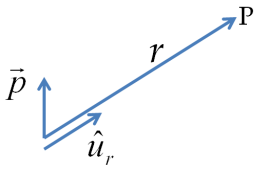

Figure 3.15: Geometry for dipole potential formula.

We see that while the potential of a single charge drops as $1/r$ from the charge, the potential of a dipole drops much faster as $1/r^2$. The $1/r$ behavior of a potential is called the **monopole potential**, the $1/r^2$ the **dipole potential**, $1/r^3$ the **quadrupole potential**, $1/r^4$ **the octupole potential**, etc. The higher-pole potentials become important only when lower order potentials are somehow zero. For instance, in the case of the dipole we have discussed, the total charge is zero, and far away from the dipole, the charges appear to add and act as a net zero charge entity, but since dipole moment is not zero, we still have the dipole effect.

By using the general relation between the electric field and the electric potential, which will be discussed in a section below, we can deduce the electric field for an electric dipole in the dipole potential approximation of this section. We write the electric field of a dipole in the spherical coordinates of the field point $\text{P}(r,\theta,\phi)$ and the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$.

$$\vec{E}_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta\hat{u}_r + \sin\theta\hat{u}_\theta}{r^3} \tag{3.34}$$

Notice that $\vec{E}_{\text{dip}}$ is independent of the polar angle ϕ, reflecting the symmetry of rotation about the z-axis in which the dipole points. Therefore, we only need study one plane containing the axis of the dipole. Fig. 3.16 shows the electric field lines in the xz-plane for a dipole at the origin that is pointed in the z-axis.

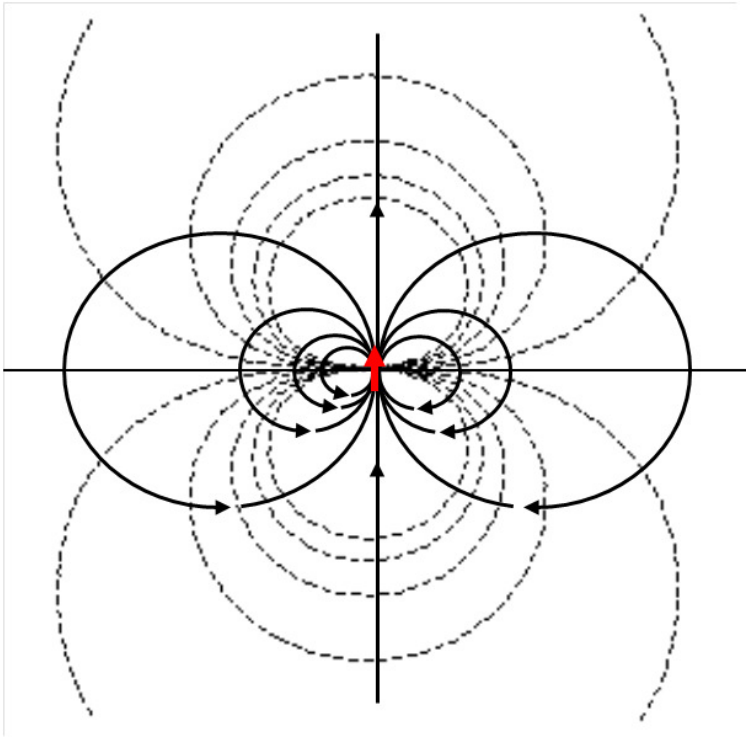

Figure 3.16: The electric field lines (solid lines) and equipotential lines (dashed) of a microscopic dipole. Rotate the figure about the axis of the dipole to obtain the three-dimensional picture.