4.13 EXERCISES

Coupled Vibrations

Ex 4.13.1. Two blocks of mass m each are connected by a spring of spring constant k, and then the blocks are placed on a frictionless track, where then can move freely on a straight track. Mark one point on the track as the origin and let the x-axis of a coordinate system be along the track. Let x_1 and x_2 be the positions of the two blocks at an arbitrary time. (a) Write the equations of motion of the two blocks. (b) Deduce the equations of motion for the center of mass and the relative coordinate. (c) Is the motion of the center of mass oscillatory? (d) Is the motion of the relative coordinate oscillatory? (e) Find the frequencies of the normal mode(s). (f) What are the expressions of x_1 and x_2 when the relative coordinate oscillates with a frequency ω? (g) What is the energy of the system of the two masses at an arbitrary time?

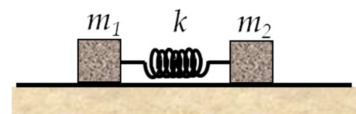

Figure 4.28: Exercise 4.13.1.

Ans: (g) $E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k\left(x_2 - x_1 - l_0\right)^2$.

Ex 4.13.2. Two balls of equal mass m are connected by a spring of spring constant k. The two-ball system is then hung from the ceiling by using another identical spring of spring constant k as shown in the figure. Let the point of suspension be the origin O and the y-axis pointed vertically down. Let y_1 and y_2 be the coordinates of the two balls at some time t. Let l be the stretched lengths of the two springs. (a) Write the equations of motion of the two balls. (b) Let $\bar{y}_1$ and $\bar{y}_2$ denote the y-coordinates of the balls when they are in equilibrium. Find $\bar{y}_1$ and $\bar{y}_2$. (c) Introduce new displacement variables for the two masses by $\eta_1(t) = y_1(t) - \bar{y}_1$ and $\eta_2(t) = y_2(t) - \bar{y}_2$. What are the equations of motions of $\eta_1(t)$ and $\eta_2(t)$? (d) Solving the equations of motion of $\eta_1(t)$ and $\eta_2(t)$ together will give you two normal mode frequencies. Show that the normal mode frequencies are given by $\omega^2 = \left(\frac{3\pm\sqrt{5}}{2}\right)\frac{k}{m}$.

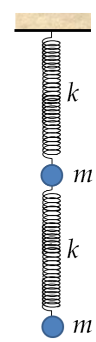

Figure 4.29: Exercise 4.13.2.

Ans: (a) $m\dfrac{d^2y_1}{dt^2} = -k(y_1 - l_0) + k(y_2 - y_1 - l_0) + mg$, $m\dfrac{d^2y_2}{dt^2} = F_{2y} = -k(y_2 - y_1 - l_0) + mg$.

Wave Function

Ex 4.13.3. A traveling wave along the x-axis is given by the following wave function

$$\psi(x,t) = 5\cos(2x - 10t + 0.4),$$

where x in meter, t in seconds, and ψ in meters. Read off the appropriate quantities for this wave function and find the following characteristics of this plane wave: (a) the amplitude, (b) the frequency, (c) the wavelength, (d) the wave Speed, and (e) the phase constant.

Ans: (a) 5 m; (b) 1.6 Hz; (c)3.14 m; (d) 5.1 m/s; (e) 0.4 rad.

Ex 4.13.4. (a) Plot the wave function given in Exercise 4.13.3 as a function of x at three different times, $t = 0$, $t = 0.1$ sec and $t = 0.3$ sec. These plots are called **wave profile**. (b) Which way is this wave traveling in time?

Ans: (b) The wave is traveling towards the positive x-axis.

Ex 4.13.5. (a) Plot the following two wave functions as a function of time at $x = 0$, $\psi_1(x,t) = 10\cos(2x - 10t)$ and $\psi_2(x,t) = 10\cos(2x - 10t + \pi/2)$, and (b) give an interpretation of phase constant. In the case of mechanical waves, these plots tell us the vibration of the particles at the origin.

Ans: (b) ψ_1 vibrations are ahead of the ψ_2 vibrations by a quarter cycle.

Sound wave

Ex 4.13.6. The sound produced in the air by a speaker is given by the following pressure differential wave function.

$$\Delta p = (10\ Pa)\cos(kx + 6000t + \pi)$$

where t is in seconds, x in meters and pressure in Pascals. Assume the speed of sound in air to be 343 m/s., and the density of air to be 1.2 kg/m^3. Find the following quantities.

(a) Angular frequency, (b) Frequency, (c) Wavelength, (d) Amplitude of the wave, (e) The direction the wave is traveling, (f) Bulk modulus of air, (g) Pressure differential at $x = 0$ at $t = 0$, (h) Pressure at $x = 0$ at $t = 0$, (i) Pressure differential at $x = 0$, $t = 1/5000$ sec, (j) Plot the pressure differential at $x = 0$ versus t, (k) Plot the pressure differential at $t = 0$ versus x.

Ans: (a) 6000 rad/s. (b) 955 Hz. (c) 0.359 m. (d) 10 Pa. (e) negative x-axis. (f) 0.14 MPa. (g) -10 Pa. (h) 1 atm - 10 Pa. (i) -3.63 Pa. (j)Plot $(10\ \text{Pa})\cos(6000\ t + \pi)$ versus t. (k) Plot $(10\ \text{Pa})\cos((343/6000)x + \pi)$ versus x.

Ex 4.13.7. The sound produced in air by a speaker is given by the following displacement wave function.

$$\psi(x,t) = (100\ \mu m)\cos(120x + \omega t + \pi/2),$$

where t is in seconds, x in meters and ψ in micrometers. Assume the speed of sound in air to be 343 m/s and the density of air to be 1.2 kg/m^3. Find the following.

(a) Wavelength, (b) Frequency, (c) Angular frequency, (d) Amplitude of the displacement wave, (e) The direction the wave is traveling in, (f) Displacement of particles at $x = 0$ at $t = 0$, (g) Displacement of particles at $t = 0$, $x = 5$ mm, (h) Displacement of particles at $x = 0$ vs t, (i) Plot pressure differential at $t = 0$ versus x, (j) The pressure differential wave function.

Energy in three-dimensional waves

Ex 4.13.8. A planar sound wave in the air of density 1.1 kg/m^3 is given by the following function.

$$\psi(x, y, z, t) = 4\cos\left(z - 35,000t + \frac{\pi}{2}\right)$$

where z is in cm, t in sec and ψ in nN/m^2.

(a) What are the wavelength, frequency, and speed of the wave?

(b) Find the amplitude of the wave at $t = 0$ at the points whose coordinates (x, y, z) are given as follows. (i) (0,0,0), (ii) (0,1,0), (iii) (1,0,0), (iv) (1,1,0), (v) (1,1,1), (vi) (-1,1,1), (vii)(1,-1,1), (viii) (-1,-1,1).

(c) What are the intensities of the wave at the following points (i) (0,0,0) and (ii) (1,1,1)?

(d) How much energy will pass through an area 3 cm^2 in 2 sec if the area is in the (i) xy-plane, (ii) yz-plane and (iii) xz-plane?

Ans: (a) 6.28 cm, 5,571 Hz, 350 m/s, (b) (i)-(iv): 0, (v)-(viii): -3.37 nN/m^2; ; (c) (i) and (ii): 0.47 μW/m^2; (d) (i) 282 pJ, (ii) 0, (iii) 0.

Ex 4.13.9. A spherical wave is given by the following function.

$$\psi(r, \theta, \phi, t) = \frac{4}{r}\cos(2r - 18,000t)$$

where r is in cm, t in seconds and ψ in μN/m^2. Here r, θ, and ϕ are spherical coordinates of a space point (x, y, z) with $r = \sqrt{x^2 + y^2 + z^2}$.

(a) What are the wavelength, frequency, and speed of the wave?

(b) Find the amplitude at following points (x, y, z) at $t = 0$. (i) (0,0,0), (ii) (0,1,0), (iii) (1,0,0), (iv) (1,1,0), (v) (1,1,1), (vi) (-1,1,1), (v)(1,-1,1), (vi) (-1,-1,1).

(c) What are the intensities of wave at (i) (0,0,0), (ii) (0,0,1), (iii) (0,0,2), and (iv) (0,0,3)?

(d) How much energies will pass through an area 3 cm^2 perpendicular to the radial direction in 2 seconds if the area is in spherical shells at following distances from the origin (i) 2 cm, (ii) 4 cm and (iii) 8 cm?

Ex 4.13.10. A planar sound wave of wavelength 3 cm and speed 350 m/s travels through a medium of density 1.2 kg/m^3 towards the positive y-axis of a Cartesian coordinate system with an intensity of 49 W/m^2. Write the wave function for this wave.

Ans: $(6.6 \times 10^{-6}\ Pa) \cos(209y - 73,300t + \phi)$.

Ex 4.13.11. A spherical wave of wavelength 3 cm and speed 350 m/s travels through a medium of density 1.2 kg/m^3. It is found to have an intensity of 49 W/m^2 at $r = 2$ cm from the source. Write the wave function for this wave.

ans: $\psi(r,t) = \frac{\left(1.2\times10^{-5}\ \text{Pa}\right)}{r} \cos(209\ r - 73,300\ t + \phi)$.

Ex 4.13.12. A planar ultrasound wave of wavelength 0.1 mm and speed 6,500 m/s travels through a medium of density 8,000 kg/m^3 towards the positive x-axis of a Cartesian coordinate system with an intensity of 4,900 W/m^2. Write the wave function for this wave.

Ans: $(33.6\ \text{pN/m}^2) \cos(62,800\ x - 4.08 \times 10^8\ t)$.

Decibels

Ex 4.13.13. The intensity at a point 10 m from a source of spherical sound waves is measured to be 20 W/m^2. (a) Find the intensity level in decibels if the reference intensity is $1.0 \times 10^{-12}\ \text{W/m}^2$. (b) What is the total power emitted by the source?

Ans: (a) 133 dB, (b) 25,000 W.

Ex 4.13.14. A microphone detects that the sound level in the room is 3 dB with respect to the standard level of sound $I_0 = 10^{-12}\ \text{W/m}^2$. What is the intensity of sound in W/m^2?

Ans: $2.0 \times 10^{-12}\text{W/m}^2$.

Ex 4.13.15. The intensity of sound drops by half. What is the change in dB?

Ans: -3dB

Ex 4.13.16. The intensity of sound drops to 5% of its original intensity. What is the change in dB?

Ans: -13.0 dB.

Ex 4.13.17. A person hears a sound level of 40 dB from an air plane engine. (a) If the eardrum has the area of cross-section area of 0.4 cm^2, how much energy does the eardrum receive is one minute? (b) If the sound intensity level goes up to 80 dB, by what factor has the pressure on the eardrum gone up?

Ans: (a) 2.4×10^{-11} J; (b) 100.

Wave properties

Ex 4.13.18. Find the speed of a mechanical wave in a rope of tension 100 N and mass density 20 g/cm.

Ans: 7.1 m/s

Ex 4.13.19. Find the speed of a mechanical wave in air of density 1.2 kg/m^3 and bulk modulus 10^5 N/m^2.

Ans: 290 m/s.

Ex 4.13.20. Find the speed of a mechanical wave in water of density 1 g/cc and bulk modulus 2×10^9 N/m^2.

Ans: 1414 m/s.

Ex 4.13.21. A sonar is an ultrasound device that is used to map the surface of the ocean. At a particular place the echo is heard 4 seconds after the sonar sends an ultrasound. How deep is the ocean there? Use 1,500 m/s for the speed of sound in salt water.

Ans: 3,000 m.

Superposition of waves

Beats

Ex 4.13.22. The beat phenomenon is very sensitive for detecting the deviation of sound from a pitch. This helps piano tuners as well as members of orchestra to adjust their musical instruments. When an instrument key and a standard reference for that key do not produce any beats, then the two are said to be in tune. Suppose that a particular key of a piano vibrates at 256 Hz when tuned. However, it is out of tune. When a tuning fork of 256 Hz is sounded at the same time as the key, one hears a beat of 30 Hz, instead of no beat. By how much is the piano key out of tune?

Ans: Piano key is at either 246 Hz or 266 Hz.

Ex 4.13.23. Two police cars are producing sirens primarily at 2000 Hz when they are not moving. You hear a beat of 200 Hz when one of the cars is standing still while the other car is moving towards you due to the Doppler effect. The frequency of the car moving towards you is shifted up relative to the frequency when the car was not moving. Determine the frequency of the siren from the moving car.

Ans: 2,200 Hz.

Ex 4.13.24. A piano key of 512 Hz is producing 500 Hz sound. (a) What will be the beat frequency if it is checked against a violin producing 520 Hz? (b) If the wire corresponding to the key is 1.2 meters long, and has a mass density of 0.1 kg/m, by what percentage should the tension of the wire be adjusted so that the pitch is adjusted to 512 Hz? Assume insignificant change in mass per unit length while tightening.

Ans. (a) 20 Hz, (b) 5%.

Interference

Ex 4.13.25. Two coherent microwave sources of wavelength 3 cm are separated by 6 cm. Their electromagnetic waves interfere at a detector 200 cm away. The detector can be moved about in a line parallel to the line joining the two sources. The arrangement is similar to Young's double-slit experiment. Find the locations of the detector the constructive interference for order 0, ± 1 and ± 2.

Ans: $y = 0, \quad y = \frac{200}{\sqrt{3}}$ cm, and $y = -\frac{200}{\sqrt{3}}$ cm.

Ex 4.13.26. A light of wavelength 632.8 nm wave is passed through a lens. The beam spreads out and is incident perpendicularly on an opaque board that has two very narrow slits which are separated by 3 μm. The emerging waves from the two slits will interfere constructively and destructively. (a) Find the directions in which the destructive interferences occur. (b) How many constructive interference fringes are allowed by the small angle formula if the formula is used for all angles between $\theta = -90°$ and $\theta = +90°$? What are their directions?

Ans: (a) Eight destructive places. Four lowest orders, $\theta = \pm 6.1°$, $\pm 18.4°$. (b) Nine constructive places: $\theta = 0, \pm 12.2°, \pm 24.9°, \pm 39.2°$, $\pm 57, 5°$.

Ex 4.13.27. Two coherent 3-mm wavelength sources separated by 5-mm shine towards a screen as shown. If the intensity of individual source at the screen is 3 W/m^2, what is the intensity at the points P and Q marked on the screen?

Ans: $I_P = 3\ W/m^2$;$I_Q = 0.3\ W/m^2$.

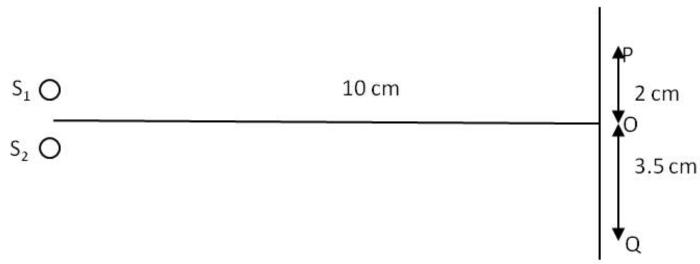

Figure 4.30: Exercise 4.13.27.

Ex 4.13.28. In Exercise 4.13.27, compute the intensities at points P and Q if the sources are replaced by those of wavelength 2-mm.

Ex 4.13.29. In Exercise 4.13.27, let the sources of wavelength 700 nm be separated by 3 micrometer. Find the intensities at the same P and Q as in Exercise 4.13.27.

Ans: $I_P = 0$.

Standing Wave

Ex 4.13.30. A guitar string of length 78 cm and mass per unit length of 30 grams per meter is under a tension of 200 N. (a) Find the frequency of the fundamental mode of vibration. (b) Find the frequencies of the first, second and third overtones. (c) Find the distance between the nodes of the second overtone. (d) Find the distance between nodes of the third overtone.

Ans: (a) 52.3 Hz. (b) 104.6 Hz, 156.9 Hz and 209.2 Hz. (c) 26 cm. (d) 19.5 cm.

Ex 4.13.31. A 2-m long string is under a tension of 300 N. It is found that the frequency of its fundamental tone of vibration is 250 Hz. Find the mass per unit length of the string.

Ans: 3.0×10^{-4} kg/m.

Ex 4.13.32. A 82 cm string of mass density 20 grams per m is to be tightened to produce a fundamental note of 510 Hz. What must be the tension in the string?

Ans: 14,000 N

Ex 4.13.33. A string of length 120 cm, density 20 g/m, and tension 20 N, is vibrating perpendicularly to the length of the string in its first harmonic with maximum displacement of 0.5 cm from equilibrium. (a) Where is/are the antinode(s), and what is the value of amplitude at an antinode? (b) What is the value of the amplitude of oscillations at the following points from one end of the string: (i) 10 cm, (ii) 20 cm, (iii) 40 cm, and (iv) 60 cm? (c) How much time do the particles in (b) take to go from maximum amplitude to zero amplitude?

Ans: (a) 30 cm from either end. (b) (i) 0.25 cm, (ii) 0.433 cm, (iii) 0.433 cm, (iv) 0. (c) 9.47 msec.

Doppler effect

Ex 4.13.34. A car horn at rest sounds a frequency of 400 Hz. (a) You are running towards the horn at a speed of 10 m/s, what frequency will you hear? (b) What frequency will you hear if you were running away from the horn? Assume the speed of sound in air to be approximately 343 m/s.

Ans: (a) 412 Hz; (b) 388 Hz.

Ex 4.13.35. A car horn at rest sounds a frequency of 400 Hz. (a) If the car is moving towards you at speed 10 m/s, what frequency will you hear? (b) If the car is moving away from you at speed 10 m/s, what frequency will you hear? Assume the speed of sound in air to be approximately 343 m/s.

Ex 4.13.36. While sitting on your porch you hear a fire engine siren at a frequency 1620 Hz when approaching towards you, and a frequency 1590 Hz when receding from you. From this data, determine the speed of the fire engine assuming it was constant. Use 343 m/s for the speed of sound.

Ans: 3.2 m/s or 11.5 km/hr.

Ex 4.13.37. A humming bird is flying towards you with speed 10 m/s while you are running towards it at 20 m/s, both speeds with respect to the ground. You hear sound of frequency 450 Hz. What is the frequency of the sound the Humming bird is making? Use 343 m/s for speed of sound.

Ans: 413 Hz.

Ex 4.13.38. Two police cars are producing sirens primarily at 2000 Hz when they are not moving. You hear a beat of 100 Hz when one of the cars is standing still while the other car is moving towards you due to the Doppler effect. The frequency of the car moving towards you is shifted up relative to the frequency when the car was not moving. (a) Determine the frequency of the siren from the moving car. (b) How fast is the police car moving with respect to you? Use the speed of sound in air to be 340 m/s.

Ans: (a) 2,100 Hz, (b) 16.2 m/s.