8.1 WORK

Work has a different meaning in physics than it has in everyday life. In physics, we say that a force has done work on an object over some interval if the object is displaced in that time. A weightlifter lifting a barbell above his head, a lift pulling passengers up to higher floors, a tugboat pulling a ship, a student lifting a backpack full of books upon his shoulders, etc, in each case, a work is done when a force is exerted on an object to cause its displacement.

As we have seen in previous chapters, the displacement of a particle is caused by the net force on the particle, the direction of the displacement may or may not in the same direction as one of the forces that make up the net force. Therefore, only the component of a force in the direction of the displacement or in the opposite direction to the displacement can be thought of as being involved in the displacement.

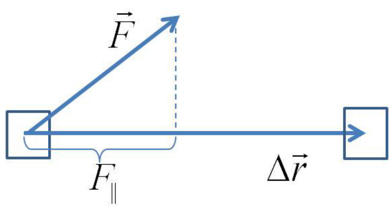

Figure 8.1: Work by a force is projection of force ($F_{\parallel}$) times the magnitude of displacement.

Mathematically, the projection of the force vector in the direction of displacement captures this physical reality. Therefore, quantitatively, we define the work of a force $\vec{F}$ during an interval Δt in which the particle has the displacement $\Delta\vec{r}$ by the product of projection of the force on the displacement and the magnitude of the displacement.

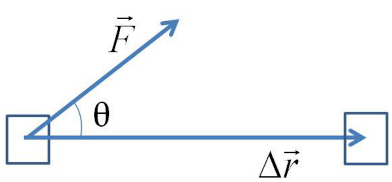

Figure 8.2: Work by a force is $W = \vec{F}\cdot\Delta\vec{r}$.

$$\begin{aligned} \text{Work } &= (\text{Projection of } \vec{F} \text{ on } \Delta\vec{r}) \times (\text{Magnitude of } \Delta\vec{r}) \\ &= F_{\parallel}\ |\Delta\vec{r}| \end{aligned} \tag{8.1}$$

Clearly, no work is done when there is no displacement. For instance, in the picture shown in Fig. 8.4, when the model holds the barbell in any one position, she applies a force on the barbell to balance the weight of the barbell, but no work is done by this force. Only during the moving of the barbell does she any work on the barbell.

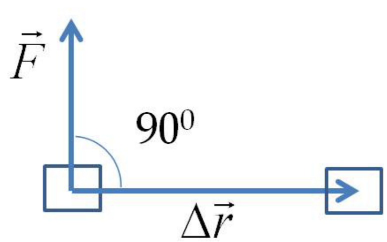

Figure 8.3: No work is done if $\vec{F}\perp\Delta\vec{r}$.

Recall that the scalar product between two vectors is also given by the product of the magnitude of one vector and the projection of the other vector. Therefore, the scalar product of the force vector and the displacement vector is also equal to the work by the force.

$$\boxed{W = \vec{F}\cdot\Delta\vec{r} = |\vec{F}||\Delta\vec{r}|\cos\ \theta,} \tag{8.2}$$

where θ is the angle between vectors $\vec{F}$ and $\Delta\vec{r}$.

The scalar product shows clearly that no work would be done by a force that is directed perpendicularly to the displacement since $\theta = 90^{\circ}$ in that case would give $\cos\ (90^{\circ}) = 0$ which would make

Figure 8.4: The girl in this picture does work on the barbell when she moves the weight from one position to another. When she is just holding the barbell in either position, she does not do work. Picture credits: Microsoft office clip-art

work equal to zero. Work is positive if force is in the same direction as the displacement, since $\theta = 0°$ and $\cos\ 0° = 1$, while work is negative if the force is oppositely directed to the displacement, since then $\theta = 180°$ and $\cos\ (180°) = -1$.

The analytic form of this definition with respect to a particular coordinate system can be written with the components of the force and the displacement.

$$W = F_x \Delta x + F_y \Delta y + F_z \Delta z. \tag{8.3}$$

From Eqs. 8.2 and 8.3, we note that the SI unit of work is N.m, which is also called Joule (J).

Example 8.1.1. Work done in pulling a cart. A cart of mass M is pulled a distance D on a flat horizontal surface by a constant force F that acts at an angle of θ with the horizontal direction. The other forces on the object during this time are gravity (F_w), Normal forces (F_{N1}) and (F_{N2}), and rolling frictions F_{r1} and F_{r2} as shown in Fig. 8.5. What is the work done by each force?

Solution. We can find the work done by each force by using either of the two ways of writing the dot product given in Eq. 8.2 and 8.3.

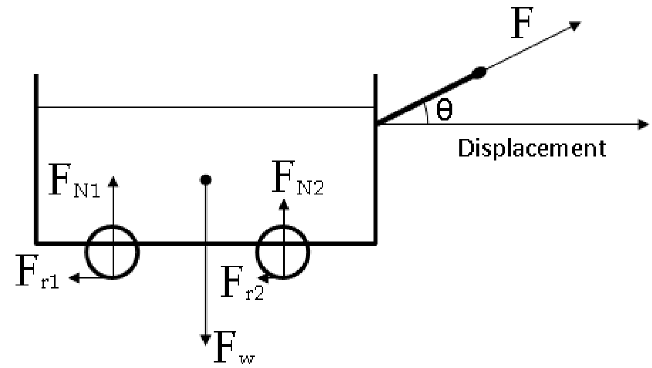

Figure 8.5: Example 8.1.1

Using Eq. 8.2 for the calculation of work

We can organize the information of magnitude and directions for each force with respect to the displacement vector in a table, and also present the answer for the work by the force.

Force	Magnitude	Angle with displacement	Work
$\vec{F}$	F	θ	$FD\cos\theta$
$\vec{F}_w$	Mg	90°	0
$\vec{F}_{N1}$	F_{N1}	90°	0
$\vec{F}_{N2}$	F_{N2}	90°	0
$\vec{F}_{r1}$	F_{r1}	180°	$-F_{r1}D$
$\vec{F}_{r2}$	F_{r2}	180°	$-F_{r2}D$

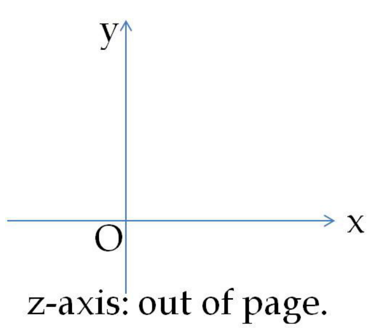

Figure 8.6: The coordinates for Example 8.1.1.

Using Eq. 8.3 for the calculation of work

We need directions of a Cartesian coordinate system to work out the components. We use the horizontal direction to the right in Fig. 8.5 to be positive x-axis and vertical direction up to be positive y-axis. The z-components of all forces and displacement will then be zero. Since, the displacement vector has only x-component non-zero, Eq. 8.3 simplifies to the following in this case.

$$W = F_x \Delta x = F_x D.$$

Therefore, if a force has a zero x-component, then the work will be zero. Once again, we can organize the information in a table.

Force	x-component	Work
$\vec{F}$	$F\cos\theta$	$FD\cos\theta$
$\vec{F}_w$	0	0
$\vec{F}_{N1}$	0	0
$\vec{F}_{N2}$	0	0
$\vec{F}_{r1}$	$-F_{r1}$	$-F_{r1}D$
$\vec{F}_{r2}$	$-F_{r2}$	$-F_{r2}D$

8.1.1 General Formula for Work

Work is a scalar number. This means that work over various displacements of an object will add as simple numbers. Fig. 8.7 shows work by a force on an arbitrary path of motion of an object can be obtained by "breaking up" the path into tiny straight displacements $\Delta\vec{r}_n$ and evaluating the work on each segment. The work from all infinitesimal displacements can be added to obtain the work over the whole path. Suppose the path of the object is divided up into N small straight displacements of length, $\Delta\vec{r}_1$, $\Delta\vec{r}_2$, $\cdots$, $\Delta\vec{r}_N$, then the net work is given by the following ordinary sum.

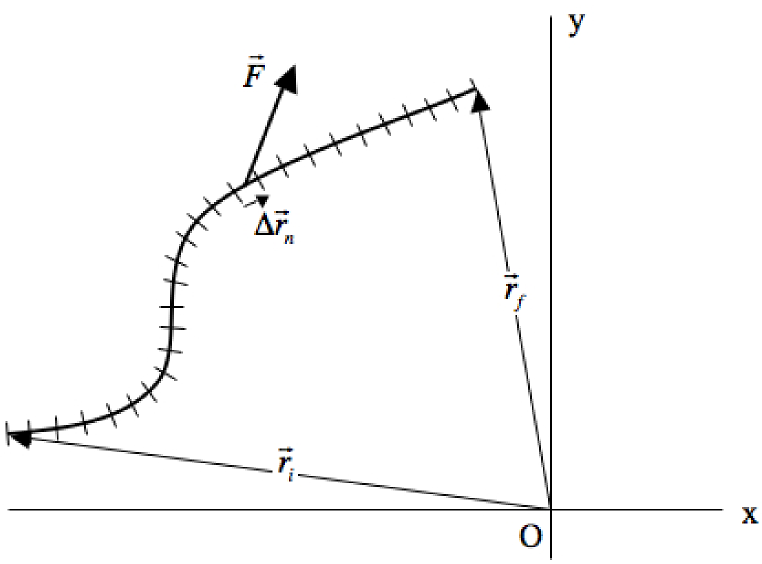

Figure 8.7: Work done by a force is the sum of scalar product of force with displacements on the segments of the path between initial and final positions.

$$W_{if} = \sum_{n=1}^{N} \vec{F}\cdot\Delta\vec{r}_n. \tag{8.4}$$

When the segments of the displacement are made smaller indefinitely, the result is written as an integral.

$$\boxed{W_{if} = \int_{i,path}^{f} \vec{F} \cdot d\vec{r},} \tag{8.5}$$

where subscript "path" has been added to the symbol of the integral to indicate the role of path in the definition, and i stands for the initial point of the path with coordinates (x_i, y_i, z_i) and f for the final point with coordinates (x_f, y_f, z_f). Often, the analytic form of Eq. 8.5 is useful for calculations. Using the analytic form of the infinitesimal displacement $d\vec{r}$ in Cartesian coordinates,

$$d\vec{r} = dx\ \hat{u}_x + dy\ \hat{u}_y + dz\ \hat{u}_z, \tag{8.6}$$

where $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ are the unit base vectors towards the positive axes, and the components of the force vector we can rewrite Eq. 8.5 as:

$$W_{if} = \int_{i,path}^{f} \left(F_x dx + F_y dy + F_z dz\right), \tag{8.7}$$

which is a collection of three one-variable integrals but the integrations have to be done on the path of motion of the object, which constrains the independent variables x, y and z to the coordinates on the path only. To perform these integrals, we need to specify a path and the end points on the path in addition to the integrands, which are specified as functions of position, i.e, $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$. This type of integration is variously called a line or path integral.

We will now have some practice of doing these types of integrals below. Note that the subscripts x, y, and z refer to the components and the same symbols in the argument (x, y, z) refer to the coordinates of the points on the path over which we seek to compute the work integral.

Further Remarks

For situations where the force is constant, i.e. if the force does not depend on the position of the object, Eqs. 8.2 and 8.3 are more suitable for a calculation of work. However, if the force depends on the position, such as the spring force, then we must use Eq. 8.5 or 8.7 for a calculation of the work.

Example 8.1.2. Work done by a constant force - path along an axis. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1 = 3\ \text{N}\ \hat{u}_x + 4\ \text{N}\ \hat{u}_y$. As a result, the particle moves along the x-axis from $x = 0$ to $x = 5$ m position in some time interval. What is the work done by $\vec{F}_1$?

Solution. Note that, since force is constant, it is easier to use Eq. 8.2 or 8.3 to calculate the work. Here the displacement is along the x-axis, therefore, the work done will only use the x-component of the force. Therefore, the work will be equal to $W = 3\text{ N} \times 5\text{ m} = 15$ N.m.

Example 8.1.3. Using path integral. Do the example above by Eq. 8.7.

Solution. The path here is along the x-axis as shown in Fig. 8.8.

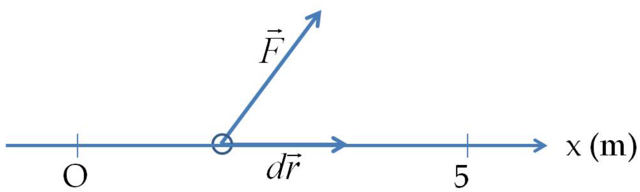

Figure 8.8: Example 8.1.3.

Therefore, $d\vec{r} = dx\hat{u}_x$, which means that the integral will be

$$W_{if} = \int_0^{5\text{ m}} (3\text{ N})dx = 15\text{N.m}.$$

Example 8.1.4. Work done by a constant force - path in a plane.

Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1 = 3\text{ N } \hat{u}_x + 4\text{ N } \hat{u}_y$. As a result, the particle moves in the xy-plane of a Cartesian coordinate system, first on the x-axis from A$(0, 0)$ to B$(5\text{ m}, 0)$ and then parallel to the y-axis from B$(5\text{ m}, 0)$ to C$(5\text{ m}, 6\text{ m})$ as shown in Fig. 8.9. What is the work done by $\vec{F}_1$?

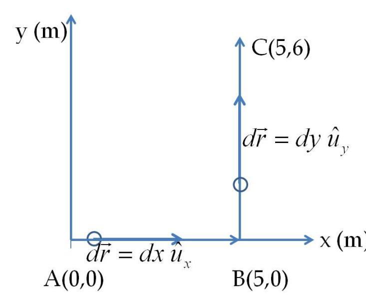

Figure 8.9: Example 8.1.4.

Solution. Since the force is constant, it is easier to work out the work done by the force using Eq. 8.2 or 8.3 on each path. As work is a scalar number, the total work on the full path is equal to the scalar sum of the parts. Let W_{AB}, W_{BC} and W_{ABC} denote work from A to B, B to C, and for the full path from A to B to C on the given path. Following the calculation presented in Example 8.1.2 it is readily seen that

$$W_{AB} = 15\text{ N.m}$$
$$W_{BC} = 24\text{ N.m}.$$

Therefore, the net work is

$$W_{ABC} = W_{AB} + W_{BC} = 39\text{ N.m}.$$

This result can also be obtained by setting up the path integral over the full path with different expressions for the infinitesimal displacement $d\vec{r}$ on the two segments, $d\vec{r} = dx\ \hat{u}_x$ for the path parallel to the x-axis, and $d\vec{r} = dy\ \hat{u}_y$ for the path parallel to the y-axis. Performing the resulting integral on the x and y variables on each segment is simple also, and we leave them for the student to work out.

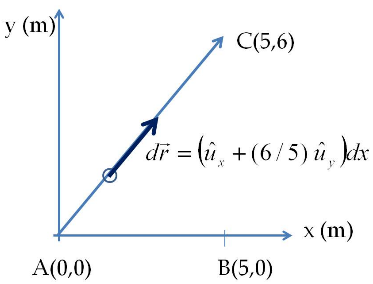

Figure 8.10: Example 8.1.5.

Example 8.1.5. Work done by a constant force - path in a plane. The same force as the last example, but now the path is the straight path from A$(0,0)$ to C$(5\text{ m}, 6\text{ m})$. What is the work done by $\vec{F}_1$ now?

Solution. Again, since the force is constant, it is easier to work out the work done by the force using Eq. 8.2 or 8.3. We have components of the force and the displacements, viz., $\vec{F}_1 = (3\text{ N}, 4\text{ N})$ and $\Delta\vec{r} = (5\text{ m}, 6\text{ m})$. Therefore work done is

$$W_{\text{AC-direct}} = F_x \Delta x + F_y \Delta y = (3\text{ N}) \times (5\text{ m}) + (4\text{ N}) \times (6\text{ m}) = 39\text{ N.m}.$$

How would we do the path integral (Eq. 8.7) for the direct path AC? Now, the path is not along the axes, but in the xy-plane. Furthermore, we have a relation between the x and y on the points on the path given by

$$y = \frac{6}{5}x.$$

Therefore, the infinitesimals dx and dy are related on the path by

$$dy = \frac{6}{5}dx.$$

We use this to simplify the infinitesimal displacement vector

$$d\vec{r} = \left(\hat{u}_x + \frac{6}{5}\hat{u}_y\right) dx.$$

Therefore the dot product $\vec{F} \cdot d\vec{r}$ becomes

$$\vec{F} \cdot d\vec{r} = \left((3\text{ N}) \times 1 + (4\text{ N}) \times \frac{6}{5}\right) dx,$$

which can be integrate from $x = 0$ to $x = 5$ m to give the result, $W = 39$ N.m.

Further Remarks

Work done by a constant force $\vec{F}_1$ in going from A to C directly is same as going by way of A-B-C worked out in Example 8.1.4. We got the same result because the force was constant. For every constant force, such as gravity near Earth, this will be the case.

Many forces, some even non-constant forces, have this feature that the work done by them is independent of path; when that is the case, we say that the force is a conservative force.

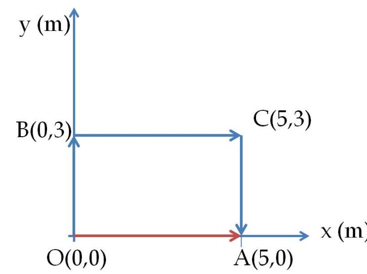

Figure 8.11: Example 8.1.6.

Example 8.1.6. Work done by a position-dependent force. Consider a particle on which several forces act, one of which is known to be a force whose magnitude and direction depends on the position of the particle. In a particular coordinate system, the force is given by $\vec{F}_2(x, y) = 2y\hat{u}_x + 3x\hat{u}_y$, where x and y are the coordinates of the position of the particle.

(a) Find the work done by this force when the particle moves from the origin to a point A$(5\ \text{m}, 0)$ on the x-axis. (b) Find the work done by this force, if the particle first moves on the y-axis from origin to a point B$(0, 3\ \text{m})$, turns 90°, moves parallel to the x-axis from point B to point C$(5\ \text{m}, 3\ \text{m})$, moves parallel to the y-axis from C to A, and finally ends up at A.

Solution. This example illustrates additional feature present in the evaluation of path integral given in Eqs. 8.5 and 8.7. So far, we have studied constant forces and shown that the infinitesimal displacement on paths assume different forms when get restricted to the path. A position-dependent force is usually given for an arbitrary position (x, y, z) which must be also restricted to the path. For instance, here $F_{2x}(x, y, z) = 2y$, which says that $F_{2x} = 0$ for a path on x-axis since on that path $y = 0$. Let us work out the work on each segment.

(a) On path OA the displacement of the particle is on the x-axis. Therefore, the displacement vector element will be $d\vec{r} = dx\ \hat{u}_x$, with $y = 0$. We now take the dot product of the force $\vec{F}_2$ with $d\vec{r}$, and set $y = 0$ in the result to restrict the integration to this path.

$$\vec{F}_2 \cdot d\vec{r} = 2ydx\Big|_{y=0} = 0.$$

The work done by $\vec{F}_2$ when the particle moves from origin to $x = 5\ m$ on the x-axis is zero.

(b) There are three path segments: OB, BC, CA.

SEGMENT OB:

Work done on OB, $W_{OB} = 0$ since $F_{2y}(x = 0) = 0$.

SEGMENT BC:

Displacement vector infinitesimal element, $d\vec{r} = \hat{u}_x\ dx$.

Other condition on the path: $y = 3$ m.

Dot product of the displacement vector element with force: $\vec{F}_2 \cdot d\vec{r} = 2ydx$.

Evaluating this for $y = 3$ m we obtain $\vec{F}_2 \cdot d\vec{r} = 6dx$.

Now, we integrate $\vec{F}_2 \cdot d\vec{r}$ for the segment. The integration variable is x, which changes from 0 to 5 m over the segment. Therefore, the work on BC segment is

$$W_{BC} = \int_0^{5\text{ m}} 6dx = 30 \text{ N.m.}$$

$\cdots$ continued

SEGMENT CA:

We repeat the process for this segment.

Displacement vector infinitesimal element, $d\vec{r} = dy\ \hat{u}_y$.

Other condition on the path: $x = 5$ m.

Dot product of displacement vector element with force: $\vec{F}_2 \cdot d\vec{r} = 3xdy$.

Evaluating this for $x = 5$ m we obtain $\vec{F}_2 \cdot d\vec{r} = 15\ dy$.

Now, we integrate this for the segment. The integration variable is y, which changes from 3 m to 0 over the segment. Therefore, work on BC segment is

$$W_{CA} = \int_{3\text{m}}^{0} 15dy = -45 \text{ N.m.}$$

The net work done on the path O-B-C-A path is

$$W_{OBCA} = W_{OB} + W_{BC} + W_{CA} = -15 \text{ N.m.}$$

We find that work done on path OA is different from the work on path $OBCA$. If the work of a force depends on the path we say that the force is non-conservative. Therefore, the force $\vec{F}_2$ given here is an example of a non-conservative force.

Example 8.1.7. Work against a spring force

A block is attached to a spring of spring constant k and placed on a horizontal table. The other end of the spring is attached to a support. A force $\vec{F}$ is applied to compress the spring by a distance A. What is the work done?

Solution. To compress the spring we need to apply a force that overcomes the spring force by an infinitesimal amount. We will assume that the applied force has magnitude equal to the spring force and has an opposite direction. Fig. 8.12 shows the forces on the block at an arbitrary instant during the compression when the spring is compressed by a distance $|x|$.

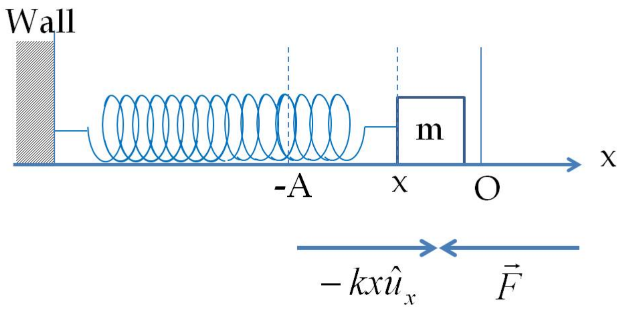

Figure 8.12: Example 8.1.7.

When dealing with spring force, you need to be careful with the direction of the vectors and signs of components. The spring force is always pointed towards the point when the spring is unstretched. Since the coordinate x of the point in Fig. 8.12 is negative, the spring force on the block will be equal to -$kx\hat{u}_x$. With the minus sign, the spring force would be pointed towards origin, which is towards the positive x-axis. The external force will be pointed such that the forces will add to give zero result for the x-components of the forces on the block.

$$\vec{F} - kx\ \hat{u}_x = 0,$$

and the infinitesimal displacement vector (setting $y = z = 0$ in the general formula for $d\vec{r}$)

$$d\vec{r} = dx\ \hat{u}_x.$$

Therefore, the work integral for x to go from $x = 0$ to $x = -A$ is given by

$$W = \int_i^f \vec{F} \cdot d\vec{r} = \int_0^{-A} kxdx = \frac{1}{2}kA^2.$$

Example 8.1.8. Work by a varying direction between displacement and force. In the motion of a pendulum the direction between the weight and the displacement of pendulum changes along the path of the pendulum. Consider a pendulum of mass m and length L. Find the work done by the force of gravity (i.e. weight) of the pendulum for a displacement of the pendulum from a position where the suspending thread makes an angle θ_0 with respect to the vertical axis to the point where the pendulum is hanging vertically.

Solution. Although weight mg of the pendulum is constant, it does different amount of work on different segments $d\vec{r}$ along the path of the pendulum since the angle between weight vector and $d\vec{r}$ changes along the path. Therefore, we must use the fundamental definition of work given in Eq. 8.5.

To set up the work integral, we look at an infinitesimal displacement at an arbitrary displacement in its motion, say between the suspension angles θ and $\theta + d\theta$, where $d\theta$ is negative for the motion under study. The displacement vector has the magnitude of the corresponding arc, $-Ld\theta$, which is positive since $d\theta < 0$. Since the direction of the displacement is tangent to the circle of motion, the angle between the weight and the displacement vector is equal to $(90° - \theta)$ as shown in Fig. 8.13.

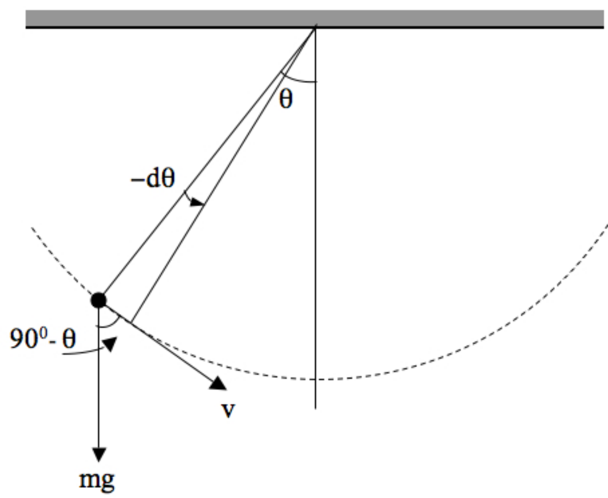

Figure 8.13: Example 8.1.8

Therefore, the work done by the weight during the infinitesimal displacement under consideration is

$$dW = (mg)(-Ld\theta)\cos(90° - \theta) = -mgL\sin\theta \; d\theta.$$

Now, we integrate from $\theta = \theta_0$ to $\theta = 0$ to obtain the required work.

$$W = \int_{\theta_0}^{0} [-mgL\sin\theta] \; d\theta = mgL(1 - \cos\theta_0).$$