13.6 SOLUTIONS WITH CHARGES AND CURRENTS

In the last section we saw that Maxwell's equation provides an explanation of the electromagnetic waves in vacuum. Vacuum is a very special condition and we would like to know what Maxwell's equation says about the electric and magnetic fields of charges and currents. In the last several chapters we have found solutions in particular situations. Let us list some of the particular solutions we have studied so far.

1. **Charge q fixed at the origin:**

$$\vec{E}(x,y,z,t) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\hat{u}_r \quad \text{(spherical coordinates)} \tag{13.100}$$
$$\vec{B}(x,y,z,t) = 0. \tag{13.101}$$

2. **Charged sphere of radius R with uniform charge density ρ:**

$$\vec{E}(x,y,z,t) = \begin{cases} \frac{1}{4\pi\epsilon_0}\,\frac{q_{\text{tot}}}{r^2}\,\hat{u}_r & r > R \\ \frac{1}{4\pi\epsilon_0}\,\frac{q_{\text{enc}(r<R)}}{r^2}\,\hat{u}_r & r \le R \end{cases} \tag{13.102}$$
$$\vec{B}(x,y,z,t) = 0, \tag{13.103}$$

where

$$q_{\text{tot}} = \frac{4}{3}\pi R^3 \rho \tag{13.104}$$
$$q_{\text{enc}(r<R)} = \frac{4}{3}\pi r^3 \rho. \tag{13.105}$$

3. **Steady current I in a long wire along the z-axis:**

$$\vec{E}(x,y,z,t) = 0 \tag{13.106}$$
$$\vec{B}(x,y,z,t) = \frac{\mu_0}{2\pi}\,\frac{I}{s}\,\hat{u}_\phi \quad \text{(cylindrical coordinates)} \tag{13.107}$$

These solutions give electric and magnetic fields for static charges and steady currents. We would like to know how to find electric and magnetic fields under non-steady situations such as a moving point charge or the current in an antenna that varies with time.

The seemingly simple problem of electric field of an arbitrarily moving charge turns out to be very complicated. You need much more powerful mathematical techniques to solve these problems than an introductory course in Calculus. To get you started on that path

I will now discuss the formulation of Maxwell's equations in terms of potentials. Although I will not solve the resulting equations here, the process of deducing the working equations is instructive for you and will give you a peek into the more advanced concepts to come in your future studies of this subject.

13.6.1 Maxwell's Equations for Potentials - Advanced Topics

The general approach to solving Maxwell's equations relies on first simplifying the Maxwell's equations through new fields, called scalar and vector potentials. In this section we will now look at how that is done.

We start with the Gauss's law of the magnetic field $\vec{\nabla}\cdot\vec{B} = 0$. Now, if you look at the identities for the del operator, you will find that the divergence of a curl of a vector field is also zero. Therefore, $\vec{\nabla}\cdot\vec{B} = 0$ means that we should be able to write $\vec{B}$ as the curl of another vector field, usually denoted by $\vec{A}$.

$$\vec{\nabla}\cdot\vec{B} = 0 \quad\Longrightarrow\quad \boxed{\vec{B} = \vec{\nabla}\times\vec{A}.} \tag{13.108}$$

The vector field $\vec{A}$ is called the **vector potential**. It appears that instead of simplifying we are complicating things. We had one field $\vec{B}$, now we have the curl of another field $\vec{A}$. But, hold on. Things do improve. Let us see what happens to the Faraday's law when we replace $\vec{B}$ there.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) \quad\Longrightarrow\quad \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0. \tag{13.109}$$

We find a combination of the electric field and vector field whose curl is zero. The identities of the del operator says that curl of a gradient is zero. That means, the combination of the electric field and vector potential can be set equal to the gradient of a scalar field, called scalar potential. We put a minus sign for convention.

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi. \tag{13.110}$$

In the case of the static electricity, this relation says that ϕ will be same as the electric potential V.

$$\text{Static:}\quad \vec{E} = -\vec{\nabla}\phi \;\leftrightarrow\; \vec{E} = -\vec{\nabla}V. \tag{13.111}$$

Presently, the electric field is dynamic and requires both the scalar potential ϕ and the vector potential $\vec{A}$, not just ϕ or V.

$$\text{Dynamic:} \quad \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.} \tag{13.112}$$

Let us summarize our findings so far. We learned that based on the Gauss's law of the magnetic field and the Faraday's law we can replace two vector fields $\vec{E}$ and $\vec{B}$ by one vector field $\vec{A}$ and one scalar field ϕ.

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{13.113}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{13.114}$$

You must agree that that there is simplification of sorts: given any set of charges and currents we need to find only four functions, $\{\phi, \vec{A}\}$, rather than six functions, $\{\vec{E}, \vec{B}\}$. Once we find $\{\phi, \vec{A}\}$, we can calculate $\{\vec{E}, \vec{B}\}$ by using Eqs. 13.113 and 13.114.

Since the curl of a gradient of an arbitrary function is zero, the vector potential $\vec{A}$ given in Eqn. 13.113 is not unique. Two vector potentials $\vec{A}$ and $\vec{A}\,'$ that differ by the gradient of an arbitrary function f will give the same magnetic field.

$$\vec{B} \text{ same for } \vec{A} \text{ and } \vec{A}' = \vec{A} + \vec{\nabla} f. \tag{13.115}$$

That is,

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}\,' = \vec{B}.$$

But if you replaced $\vec{A}$ by $\vec{A}\,'$ you wouldn't have the same electric field unless you change ϕ also. You can convince yourself that by simultaneous replacements of $\vec{A}$ by $\vec{A}\,'$ and ϕ by ϕ' related as follows does not change the physical fields $\vec{E}$ and $\vec{B}$.

$$\vec{A} \text{ by } \vec{A}\,' = \vec{A} + \vec{\nabla} f \quad \text{and} \quad \phi \text{ by } \phi' = \phi - \frac{\partial f}{\partial t}, \tag{13.116}$$

This freedom, called **gauge degree of freedom**, can be used to simplify equations obeyed by the potentials as we will see below.

You may have noticed that we have used up two of the Maxwell's equations to arrive at the representations of the fields in terms of the potentials. Now, let us write out the remaining two Maxwell's equations in terms of the potentials and see how to use the freedom.

When we substitute $\vec{E}$ in $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ in terms of the potentials we find

$$\vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}, \tag{13.117}$$

which can be rewritten as

$$\boxed{\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0}.} \tag{13.118}$$

We have one more Maxwell's equation to work out - the Ampere-Maxwell's law. Substituting for $\vec{E}$ and $\vec{B}$ there and moving fields on the left side of the equation and the source on the other side we obtain

$$\vec{\nabla}\times(\vec{\nabla}\times A) + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{\partial\vec{A}}{\partial t}\right) = \mu_0\,\vec{J}. \tag{13.119}$$

The equations 13.118 and 13.119 for the potentials are much more complicated than the equations for $\vec{E}$ and $\vec{B}$ in the Maxwell's equation. At first sight the whole math exercise appears to make matters worse. But, wait, we have not used the gauge freedom in $\{\phi,\vec{A}\}$ yet. Let us expand Eq. 13.119 further to identify a strategy for the simplification using the gauge freedom.

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right] = -\mu_0\vec{J}. \tag{13.120}$$

Let us summarize again, the final equations for the potentials from Maxwell's equations. Note that the function f is an arbitrary function with defined time derivative and gradient.

$$\begin{cases} \nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\epsilon_0} \\ \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t}\right] = -\mu_0\vec{J} \\ \text{Gauge freedom: } \vec{A}\,' = \vec{A} + \vec{\nabla}f; \quad \phi' = \phi - \frac{\partial f}{\partial t} \end{cases} \tag{13.121}$$

We notice that, if we could set the quantity in the bracket in the second equation to zero, then the equations for ϕ and $\vec{A}$ will separate out. Actually, this is possible since f is an arbitrary function. Therefore, we choose

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial\phi}{\partial t} = 0, \tag{13.122}$$

and make sure that this is also true for $\{\phi',\vec{A}\,'\}$ pair.

$$\vec{\nabla}\cdot\vec{A}\,' + \mu_0\epsilon_0\frac{\partial\phi'}{\partial t} = 0, \tag{13.123}$$

This requires that f be a solution of the equation

$$\nabla^2 f - \mu_0\epsilon_0\frac{\partial^2 f}{\partial t^2} = 0. \tag{13.124}$$

The existence of any solution of this equation would be sufficient to guarantee that the gauge choice given in Eq. 13.122 can be used. The choice of a condition such as the one in Eq. 13.122 is called a **gauge condition** or simply **gauge**. With this choice the equations for the potentials start to look familiar.

$$\boxed{\nabla^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}.} \tag{13.125}$$

$$\boxed{\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}.} \tag{13.126}$$

These equations look like the wave equation for the electric and magnetic fields in vacuum, except that on the right side we have charges and currents. It turns out that these equations are easier to solve than the equations for $\vec{E}$ and $\vec{B}$. The solutions of Eqs. 13.125 and 13.126 are obtained by integrating over the position of the charges and currents. Let the field point P be at the position vector $\vec{r}$ and a point in the source, charge or current be denoted by $\vec{r}\,'$. Let $\mathcal{R}$ denote the direct distance between the source and field points.

$$\mathcal{R} = |\vec{r} - \vec{r}\,'\,| \tag{13.127}$$

Then, a formal solution can be written as

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\,(\vec{r}\,',t-\mathcal{R}/c)}{\mathcal{R}}\; d^3\vec{r}\,' \tag{13.128}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,(\vec{r}\,',t-\mathcal{R}/c)}{\mathcal{R}}\; d^3\vec{r}\,' \tag{13.129}$$

The time in the source is replaced by $t - \mathcal{R}/c$, which is called the **retarded time**. Since we need to use the retarded time for locating the sources, the potential at instant t depends on where the charge or current was at an earlier time. We say that the "news" of the position of the charge moves as electromagnetic wave at the speed of light. This section completes the theory of electricity and magnetism for us. The mathematics necessary for an application of these formulas is quite a bit beyond the scope of this textbook. In the next two chapters we will apply Maxwell's equations to study electromagnetic oscillations and AC circuits.