1.8 Doppler Effect

1.8.1 Non-relativistic Doppler Effect

Due to the Doppler effect one hears a higher pitch of a car horn when the car is approaching and a lower pitch when the car is receding. Any relative motion between an emitter and a detector changes the pitch. In the case of sound we have three systems: the source, the observer, and the medium. Let us denote the speed of the sound wave in the stationary medium by v_s and the source (emitter) and the observer (detector) be placed along the x-axis with source to the left of the observer.

Suppose the source emits a sound of frequency f_0 when it is not moving. That is, if you were in the frame in which the source is at rest with you, you would hear a sound of frequency f_0. Let the source be at the origin of the S' frame which moves towards the positive x-axis with speed V with respect to an observer at the origin of the S frame.

In the rest (S') frame, the crests of the waves are a distance $\lambda_0 = v_s/f_0$ apart and all wave fronts are spreading out of the same point in stationary space, which is the origin of the S' frame. In the lab (S) frame, the wave has been generated at different points on the x-axis since the source has been moving. Let us denote the period of the vibrations at the source by τ_0 which is equal to $1/f_0$. The waves are generated every t_0 interval, during which the source has moved a distance $V\tau_0$ with respect to the observer at the origin of the S frame. Therefore, the distance between crests in the S frame, which is equal to the wavelength λ of the wave in this frame, will be

$$\lambda = \lambda_0 + V\tau_0 = \lambda_0 + \frac{V}{f_0}. \tag{1.74}$$

Writing this in terms of frequency of the waves by using

$$\lambda_0 = v_s/f_0, \quad \lambda = v_s/f, \tag{1.75}$$

we find

$$\boxed{f = f_0\left(\frac{v_s}{v_s + V}\right) \quad \begin{cases} \text{Observer receding: } V > 0 \\ \text{Observer approaching: } V < 0 \end{cases}} \tag{1.76}$$

In the case when the source is fixed but the observer moves we place the source at the origin of the S frame and the detector moves with the origin of the S' frame. Let τ_0 be the time period of the source in the S frame. In the S frame the waves will be spreading from the origin with crests a distance $v_s\tau_0$ apart. The detector at the origin of S' frame will encounter crests at less distance than $v_s\tau_0$ if the detector is moving towards the source. Let us consider a situation in which the detector (the origin of S' frame) is on the positive x-axis and moving towards the source (the origin of S frame). That is, let $V < 0$ in our equations.

Let the detector encounters crests every τ second. The observer will note that he/she encounters wave crests at periodic distances of period $v_s\tau$ since the speed of the wave with respect to the stationary media is v_s. However, during this time the observer has moved a distance $V\tau$ in space. Therefore, we must have the following relation.

$$v_s\tau_0 = v_s\tau - V\tau. \tag{1.77}$$

Writing this equation in terms of frequencies, $f_0 = 1/\tau_0$ at the emitter and $f = 1/\tau$ at the moving detector we get

$$\boxed{f = f_0\left(1 - \frac{V}{v_s}\right) \quad \begin{cases} \text{Observer receding: } V > 0 \\ \text{Observer approaching: } V < 0 \end{cases}} \tag{1.78}$$

1.8.2 Relativistic Doppler Effect

The non-relativistic Doppler effect shows the effect is different when the source is moving and when the observer is moving. Thus, using the non-relativistic Doppler effect it is possible to tell which one of the two frames, the source frame or the observer frame, is moving even though they may be moving at uniform velocity with respect to the medium. This would not happen if you do not have the medium which takes part in the motion of sound waves.

In the case of light waves no medium is necessary. When we work out the frequency of light observed by a detector the shift in frequency is same whether the source is moving or the observer is moving - the effect depends only on the relative motion of the source and the observer. Once again, let f_0 be the frequency of the emitted light in the rest frame of the emitter and f be the frequency of the light detected regardless of the state of motion of the detector. Then, we will find below that the Doppler effect, now called relativistic Doppler effect, give the following relation.

$$\boxed{f = f_0\sqrt{\frac{1 - V/c}{1 + V/c}} \quad \begin{cases} \text{Observer receding: } V > 0 \\ \text{Observer approaching: } V < 0 \end{cases}} \tag{1.79}$$

To derive this result we will look at a plane wave source traveling along the x-axis. To study a moving source we will place the source at the origin of the S' frame which moves relative to the S frame towards the positive x-axis with speed V as before as shown in Fig. 1.5.

The wave emitted by the source will be traveling towards the negative x-axis to reach the detector at the origin of the S frame which is to the left of the origin of the source at the origin of S' frame for $t > 0$. Therefore, the wave at the detector has the following wave function at points on the x'-axis in the S' frame.

$$\psi'(x', t') = \psi_0' \cos\left(\frac{2\pi}{\lambda_0}x' + 2\pi f_0 t'\right). \tag{1.80}$$

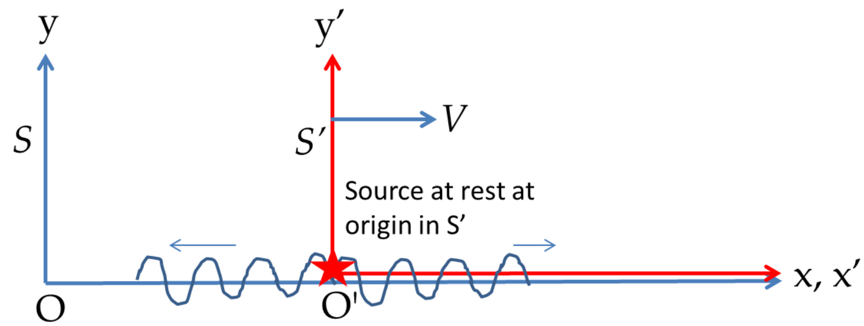

Figure 1.5: To calculate the Doppler effect we place the source of sinusoidal wave at rest at the origin of the S' frame.

The Lorentz transformation gives the following relations among the coordinates of S and S' frame.

$$t' = \gamma\left(t - \frac{V}{c^2}\, x\right), x' = \gamma\,(x - Vt)\,, y' = y, z' = z. \tag{1.81}$$

Now, we express x' and t' in Eq. 1.80 in terms of x and t and collect terms in the argument of cosine as a term multiplying x and another term multiplying t.

$$\psi'(x,t) = \psi'_0 \cos\left[2\pi\gamma\left(\frac{1}{\lambda_0} - \frac{Vf_0}{c^2}\right)\, x + 2\pi\gamma\left(f_0 - \frac{V}{\lambda_0}\right)\, t\right]. \tag{1.82}$$

In the S frame the wave will be represented by

$$\psi(x,t) = \psi_0 \cos\left(\frac{2\pi}{\lambda_0} x + 2\pi f_0 t\right). \tag{1.83}$$

We can ignore the amplitudes and look at the phase to determine the frequency of the wave in the S frame in relation to the frequency of the same wave in the S' frame. Equating the factor that multiplies $2\pi t$ in the argument of the cosine function in Eqs. 1.82 and 1.83 gives the desired relation of the frequency detected in the lab (S) frame in terms of the frequency in the rest (S') frame.

$$f = \gamma\left(f_0 - \frac{V}{\lambda_0}\right). \tag{1.84}$$

We can write the right-side of this equation in terms of f_0 by noting that $f_0\lambda_0 = c$, where c is the speed of light. Thus,

$$\boxed{f = f_0\sqrt{\frac{1 - V/c}{1 + V/c}} \quad \begin{cases} \text{Observer receding: } V > 0 \\ \text{Observer approaching: } V < 0 \end{cases}} \tag{1.85}$$

Example 1.7. Doppler shift in Astronomy. The spectrum of light from radio galaxy 8C1435+635 is shifted in wavelength considerably from what you see when emitting atoms are at rest or moving slowly. For instance, the Lyman α line from hydrogen atom spectrum is observed at 639.2 nm instead of 121.6 nm observed in lab. Since the wavelength is increased, which is same as frequency decreased, the galaxy must be receding with respect to us. Suppose, the galaxy is receding directly away from us, what will be the speed of recession?

Solution. We use the Doppler formula in the wavelength form.

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}\sqrt{\frac{1-V/c}{1+V/c}}$$

Rearranging we find

$$\frac{1+V/c}{1-V/c} = (\lambda/\lambda_0)^2 = (639.2/121.6)^2 = 27.6.$$

Therefore,

$$V = \left(\frac{27.6-1}{27.6+1}\right) c = 0.93\, c.$$

This says that the radial velocity of the galaxy is 93% of the speed of light. In Astronomy, one defines a related quiantity call the redshift or z of recession by

$$z = \frac{\Delta\lambda}{\lambda_0}.$$

This galxy has z of

$$z = \frac{638.2\ \text{nm} - 121.6\ \text{nm}}{121.6\ \text{nm}} = 4.25.$$

1.8.3 Doppler Effect and Aberration

In the last subsection we studied the relativistic Doppler effect for the relative motion along the line joining the source and the observer. Now, we suppose the source of the wave is in the xy-plane and is at rest in the S' frame at an angle θ' in the first quadrant from the origin and counterclockwise from the x'-axis.

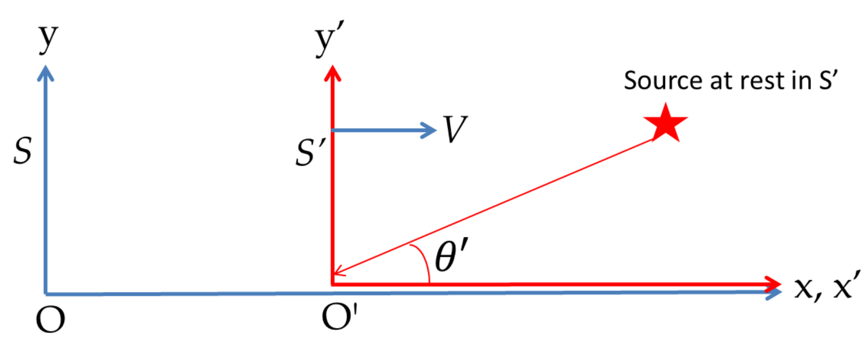

Figure 1.6: To calculate the aberration we place the source of sinusoidal wave at rest in the S' frame in the direction θ'.

The propagation vector for the ray from the source will be

$$\vec{k} = \frac{2\pi}{\lambda_0}\left(-\cos\theta'\,\hat{\imath} - \sin\theta'\,\hat{\jmath}\right), \tag{1.86}$$

where k is the wavenumber, and $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors towards the positive x' and y' axes respectively. The wavenumber k is related to the wavelength λ by $k = 2\pi/\lambda$. Using a general form for oscillating part of plane waves, $\sim \cos(\vec{k}\cdot\vec{r} - 2\pi f t)$, the wave function for this wave will be

$$\psi'(x', t') = \psi_0' \cos\left[\frac{2\pi}{\lambda_0}(-x'\cos\theta' - y'\sin\theta') - 2\pi f_0 t'\right]. \tag{1.87}$$

Now, we express x', y', and t' in Eq. 1.80 in terms of x, y, and t using the Lorentz transformations, and collect terms in the argument of cosine as a term multiplying x, a term multiplying y, and another term multiplying t. We also use $\lambda_0 f_0 = c$ to simplify.

$$\begin{aligned}\psi'(x, t) = \psi_0' \cos\Bigg[&2\pi\gamma\frac{1}{\lambda_0}\left(-\cos\theta' + \frac{V}{c}\right) x \\ &- 2\pi\frac{\sin\theta'}{\lambda_0}\, y - 2\pi\gamma f_0\left(1 - \frac{V}{c}\cos\theta'\right) t\Bigg].\end{aligned} \tag{1.88}$$

The detector/observer at the origin of the S frame will find this wave coming from the direction θ and assign the following wave function.

$$\psi(x, t) = \psi_0 \cos\left[\frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta) - 2\pi f t\right]. \tag{1.89}$$

Now, we equate the coefficients of x, y, and t in the argument of cosine in the last two equations to obtain

$$\frac{\cos\theta}{\lambda} = \gamma\frac{1}{\lambda_0}\left(-\cos\theta' + \frac{V}{c}\right) \tag{1.90}$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda_0} \tag{1.91}$$

$$f = \gamma\left(1 - \frac{V}{c}\cos\theta'\right) f_0 \tag{1.92}$$

We can eliminate $\cos\theta'$ in Eq. 1.92 in favor of $\cos\theta$. This gives the Doppler shift from a source whose direction is θ in the frame of the observer while the source is moving at velocity in the direction of $\theta = 0$, i.e., the positive x-axis.

$$f = \left[\frac{1 - (V/c)\cos\theta}{\sqrt{1 - V^2/c^2}}\right] f_0. \tag{1.93}$$

There is a Doppler effect even when the source is moving in the perpendicular direction. To see that we set $\theta = 90°$ in this equation gives the Doppler shift of an object that is towards the y-axis and moving towards the positive x-axis. This effect is called the **Transverse Doppler Effect**.

$$f = f_0 \sqrt{1 - V^2/c^2}. \tag{1.94}$$

The transverse Doppler effect is much weaker than the longitudinal Doppler effect obtained in Eq. 1.85. Let us expand Eqs. 1.85 and 1.94 in powers of V/c. Let us designate the longitudinal Doppler by $f_{\|}$ and the transverse Doppler by $f_{\perp}$.

$$f_{\|} = f_0 \left[\left(1 - \frac{V}{c}\right) \frac{1}{\sqrt{1 - V^2/c^2}} \right] = f_0 \left(1 - V/c + V^2/2c^2 + \cdots\right) \tag{1.95}$$

$$f_{\perp} = f_0 \sqrt{1 - V^2/c^2} = f_0 \left(1 - V^2/2c^2 + \cdots\right) \tag{1.96}$$

Dividing Eqs. 1.90 and 1.90 gives us the following relation between the direction of the source in the two systems.

$$\tan\theta = \frac{\sin\theta' \sqrt{1 - V^2/c^2}}{\cos\theta' - V/c}. \tag{1.97}$$

This relation says light emitted at a direction θ' in one frame will coming from the direction θ in the other frame. The phenomenon is called the **aberration of light**. For instance, if the source is on the y' axis, the angle $\theta' = 90°$. But the angle θ in the S frame will be

$$\theta = -\tan^{-1}\left(\frac{\sqrt{1 - V^2/c^2}}{V/c} \right) \approx -\tan^{-1}\left(\frac{c}{V}\right). \tag{1.98}$$

1.8.4 Ives-Stilwell Experiment

In 1938 Ives and Stilwell conducted an experiment to verify the longitudinal Doppler effect. Due to technical reasons they could not perform the transverse Doppler effect which would require examining the light emitted perpendicular to the emitting particle. In Ives-Stilwell experiment canal rays (which are rays of positive ions) from hydrogen gas discharge were accelerated to high speeds as shown in Fig. 1.7. The ions then recombined with electrons to produce light.

Ives and Stilwell detected the hydrogen emission in the blue, the Hβ line, for both both forward and backward emitted light with the arrangement of a mirror in the tube. By varying the accelerating voltage they could control the speed of the emitted ion. They found the emitted light contained light of original wavelength 4849.32 Åand two symmetrically displaced wavelengths, one at a higher wavelength and the other at a lower wavelength. The magnitude of displacement of the wavelength depended upon the voltage of the accelerating voltage. From theory of Doppler effect, if we ignore v^2/c^2 compared to v/c, then from Eq. 1.93 we expect

$$f \approx f_0 \left[1 - (V/c)\cos\theta\right],$$

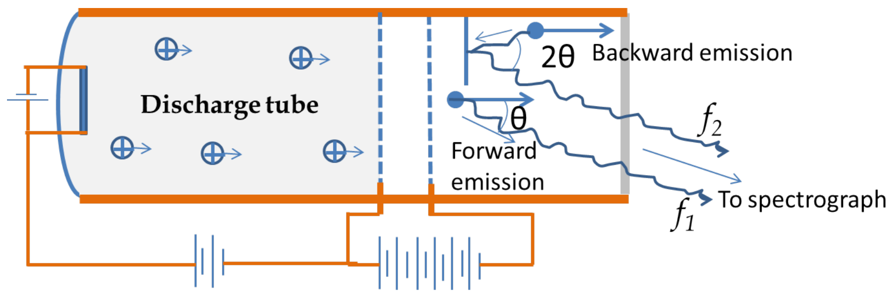

Figure 1.7: A schematic picture of the apparatus used by Ives and Stilwell to investigate Doppler shift.

which can be written for the wavekength as

$$\frac{c}{\lambda} \approx \frac{c}{\lambda_0}\left[1 - (V/c)\cos\theta\right],$$

which can be written as

$$\left|\frac{\Delta\lambda}{\lambda_0 \cos\theta}\right| = V/c. \tag{1.99}$$

Now, the speed of the light emitting particle can be related to the accelerating voltage $\mathcal{E}$.

$$e\mathcal{E} = \frac{1}{2}MV^2, \tag{1.100}$$

where e is the electronic charge and M is the mass of the emitting particle, which was either H_2 or H_3 in their experiment. From Eqs. 1.99 and 1.101 we see that

$$\left|\frac{\Delta\lambda}{\cos\theta}\right| = \text{const} \times \sqrt{\mathcal{E}}. \tag{1.101}$$

In the Ives-Stilwell experiment, they examined rays at 7°. A plot of $\Delta\lambda$ adjusted by dividing by $\cos 7°$ versus $\mathcal{E}$ from the paper of Ives and Stilwell is shown in Fig. 1.8. The data shows a good fit to the prediction up to the linear approximation.

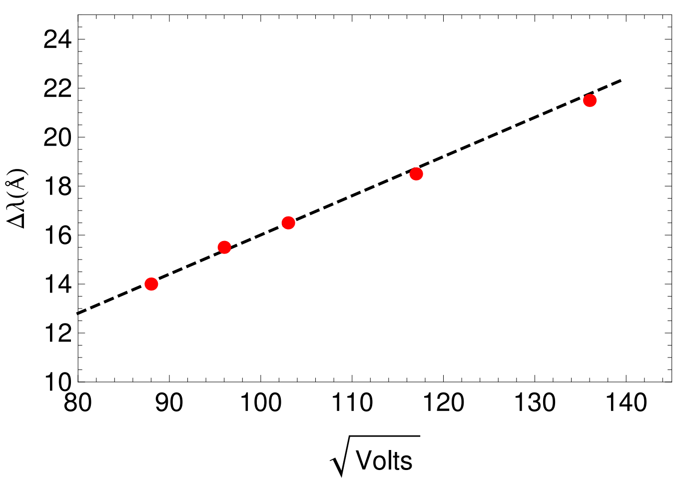

Figure 1.8: Doppler shift is plotted versus square-root of voltage, which is proportional to the speed of the light-emitting particle. The dashed line is the theoretical prediction in the linear approximation.