12.3 ELEMENTARY AC CIRCUITS

We will now work out several examples of AC circuits. You should work through these examples. Just reading them will not be enough.

12.3.1 Series RL in AC Circuit

For our first example, let us consider a circuit containing a resistance and an inductance driven by a sinusoidal EMF as shown in Fig. 12.11. Using the voltage drops across the resistor and the inductor we find

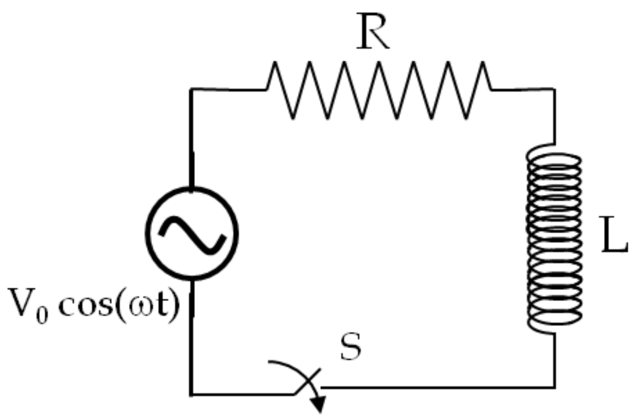


Figure 12.11: A circuit with a resistance and an inductance driven by a sinusoidal EMF.

the equation governing the current $I(t)$ at any instant t.

$$L\frac{dI}{dt} + RI = V_0\,\cos(\omega t). \tag{12.24}$$

When the switch is closed, there is a transient behavior of the circuit, where the current in the circuit does not behave with the same frequency as the frequency of the driving EMF. But, after some time a steady state is reached and the current in the circuit oscillates with the same frequency as the frequency of the driving EMF. Therefore, we can write the steady state solution of Eq. 12.24 in the following form that has two unknowns the amplitude I_0 and phase constant ϕ.

$$I(t) = I_0\,\cos(\omega t + \phi). \quad \text{(Steady state)} \tag{12.25}$$

To find I_0 and ϕ, we substitute this solution in Eq. 12.24 to obtain

$$-L\,\omega\,I_0\,\sin(\omega t + \phi) + RI_0\,\cos(\omega t + \phi) = V_0\,\cos(\omega t). \tag{12.26}$$

Now, we expand the trigonometric functions and rearrange terms to obtain the following.

$$\begin{aligned}[V_0 + L\,\omega\, I_0\ \sin\phi - R\, I_0\ \cos\phi]\cos(\omega t)\\ + [L\,\omega\, I_0\ \cos\phi - R\, I_0\ \sin\phi]\sin(\omega t) = 0.\end{aligned} \tag{12.27}$$

In this equation, the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ must be independently zero. This can be shown by multiplying the equation by $\cos(\omega t)$ and integrating over one period. You should do this step. you will find that this integration leads to the conclusion that the coefficient of the cosine term must be zero. Similarly, when you multiply Eq. 12.27 by $\sin(\omega t)$ and integrate over one period, you find that the coefficient of the sine term must be zero.

$$V_0 + L\,\omega\, I_0\ \sin\phi - R\, I_0\ \cos\phi = 0. \tag{12.28}$$

$$L\,\omega\, I_0\ \cos\phi - R\, I_0\ \sin\phi = 0. \tag{12.29}$$

From Eq. 12.29, we obtain

$$\tan\phi = -\frac{\omega L}{R}. \tag{12.30}$$

We can use this equation to write expressions for $\cos\phi$ and $\sin\phi$ to be used in Eq. 12.28, which can be solved for I_0.

$$\cos\phi = \frac{R}{\sqrt{R^2 + (\omega L)^2}}. \tag{12.31}$$

$$\sin\phi = -\frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}. \tag{12.32}$$

Putting these expressions for $\sin\phi$ and $\cos\phi$ in Eq. 12.28, we find I_0.

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}} \tag{12.33}$$

The ratio of the amplitude of the voltage of the source to that of the current through the source has units of resistance, it consists of contributions from the inductance and the frequency. This quantity "acts" like an overall resistance in the circuit and is called the **amplitude of impedance** or simply **impedance** of the circuit, denoted usually by the letter $|Z|$. The amplitude of the impedance does not have the information of the phase difference between the voltage and the current. The complete impedance is defined to contain both of these informations into one complex quantity we will study in the next section. Here, the amplitude of the impedance of the RL-circuit is found to be

$$\boxed{|Z| = \frac{\text{Amplitude of Voltage}}{\text{Amplitude of Current}} = \sqrt{R^2 + (\omega L)^2}.} \tag{12.34}$$

The expressions for the phase and amplitude of the AC current in the RL-circuit, Eqs. 12.30 and 12.33, respectively, can be used to obtain useful rules for inductive circuits.

RULES FOR INDUCTIVE CIRCUITS

1. **High frequency ($\omega \rightarrow \infty$ Limit)**

   As frequency of the driving EMF is raised, we note that the inductive reactance increases without bound, and hence the resistor becomes less and less important in the RL-circuit. At very high frequencies, an RL-circuit acts purely inductive with phase constant of current reaching close to $-\pi/2$ radians and the amplitude I_0 given by simply ignoring the resistance in the circuit, $I_0 \approx V_0/\omega L$.

2. **Low frequency ($\omega \rightarrow 0$ Limit)**

   At low frequency the circuit becomes more like a DC circuit. Hence, the inductance becomes less important since a back EMF requires a changing magnetic flux. The current in the circuit will be in-phase with the driving EMF, i.e., $\phi \rightarrow 0$, and the amplitude I_0 is obtained by ignoring the inductor altogether, i.e., $I_0 \approx V_0/R$.

Example 12.3.1. EMF and Current in an RL Circuit

(a) Find the current in a series RL-circuit with the following values: $V_0 = 150$ V, $f = 60$ Hz, $R = 1\ \Omega$; $L = 0.01$ H. (b) Plot the current in the circuit and the EMF of the source as functions of time on the same graph. This type of plot can give you a visual understanding of the timings of the two functions. (c) Draw a phasor diagram to show the phase relations of the EMF of the source and the current in the circuit.

Solution. (a) Putting the numerical values we obtain I_0 = 38 A and ϕ = -1.3 radians.

(b) I have used Mathematica® to plot the functions $V(t) = V_0 \cos(\omega t)$ and $I(t) = I_0 \cos(\omega t + \phi)$ using the parameters f = 60 Hz, R = 1 Ω; L = 0.01 H. The plot is displayed in Fig. 12.12. The current in the circuit is behind the voltage of the source by approx. 75 degrees or 1.3 rad in the phase.

(c) Phasor Diagram For RL-Circuit

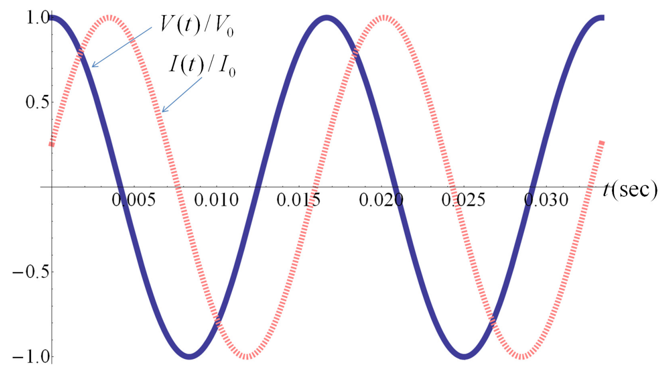


Figure 12.12: The EMF if the source (solid line) and the current in the circuit (dashed) in the RL-circuit. The voltage is plotted in units of V_0 = 150 V and the current in units of I_0 = 38 A. Other parameters of the circuit are f = 60 Hz, R = 1 Ω; L = 0.01 H.

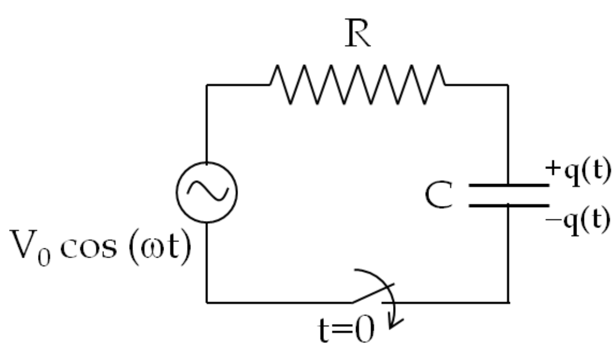
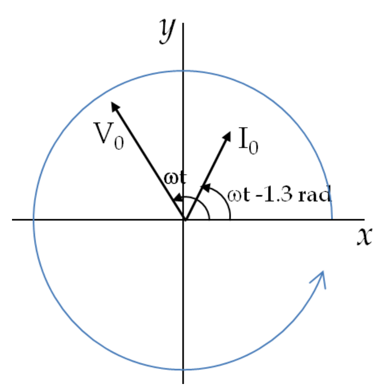


Figure 12.13: An RC-circuit driven by an alternating EMF.

For a phasor diagram, we are going to represent the source voltage and the current in the circuit as vectors in a two-dimensional space as discussed before for phasors. The phasor diagram for the present RL-circuit is shown on the right.



12.3.2 Series RC Circuit

As our second example of the AC circuit, let us replace L by C in the RL circuit to obtain an RC circuit driven by an alternating EMF (Fig. 12.13). We have studied this circuit before when the source was a DC-voltage. We know that this circuit would be controlled by the charging and discharging of the capacitor. We expect the same behavior here, except that, now, due to the changing source voltage, the capacitor may or may not be able to fully charge or discharge in one cycle of the source voltage depending upon the time constant of the circuit in relation to the frequency of the driving source.

The voltage loop equation gives:

$$RI(t) + \frac{1}{C}\, q(t) = V_0\ \cos(\omega t), \tag{12.35}$$

with current at instant t given as

$$I(t) = \frac{dq}{dt},$$

where the convention of direction of current into the positive plate was used. Let us convert this equation to an equation for the current only. We can achieve that by taking a derivative with respect to time and using the relation of current to the charge on the capacitor. The voltage loop equation gives:

$$R\frac{dI}{dt} + \frac{1}{C}\, I(t) = -\omega\, V_0\ \sin(\omega t). \tag{12.36}$$

The steady state solution of this equation, written as

$$I(t) = I_0\ \cos(\omega t + \phi) \tag{12.37}$$

proceeds in a similar way as we have done for the RL circuit above. The calculation is left as an exercise for the student to complete. The student will find the following expressions for I_0 and ϕ here.

$$I_0 = \frac{V_0}{\sqrt{R^2 + (1/\omega C)^2}}. \tag{12.38}$$

$$\tan\phi = \frac{1}{\omega RC}. \tag{12.39}$$

The ratio of the amplitude of the voltage of the source to that of the current through the source has units of resistance. However, this quantity here consists of the contributions from the capacitance and the frequency. This quantity "acts" like the overall resistance in the circuit. Just as the similar quantity in the RL circuit was called the amplitude of the impedance of the circuit, here too it is called the **amplitude of the impedance** or simply the impedance of the circuit, denoted usually by the letter $|Z|$. Here, the amplitude of the impedance of the RC-circuit is found to be

$$\boxed{|Z| = \frac{\text{Amplitude of Voltage}}{\text{Amplitude of Current}} = \sqrt{R^2 + (1/\omega C)^2}.} \tag{12.40}$$

The expressions for AC current can be used to obtain useful rules for capacitative circuits.

RULES OF CAPACITATIVE CIRCUITS

1. **High frequency ($\omega \to \infty$ Limit)**

   As the frequency of the driving EMF is increased, the capacitative reactance becomes smaller. Therefore an RC-circuit becomes more and more like a purely resistive circuit as capacitance becomes less important. Basically, at high frequency the capacitor never gets a chance to charge to any significant extent. The current in the circuit will be in-phase with the EMF, i.e., $\phi \to 0$, and the amplitude I_0 is obtained by ignoring the presence of capacitor, i.e., $I_0 \approx V_0/R$.

2. **Low frequency ($\omega \to 0$ Limit)**

   At low frequencies of the driving EMF, the capacitative reactance becomes large and the resistor becomes less and less important in an RC-circuit. The phase constant of the current reaches $+\pi/2$ and the amplitude I_0 is simply given by ignoring the resistance R in the circuit, $I_0 \approx \omega C V_0$.

Example 12.3.2. EMF and current in an RC circuit

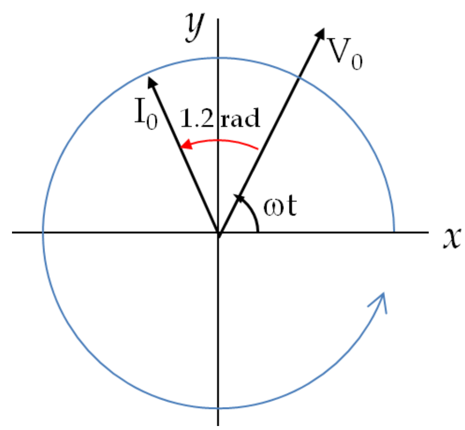
(a) Find the current in a series RC-circuit with the following values: $V_0 = 150$ V, $f = 600$ Hz, $R = 1\ \Omega$, $C = 1$ mF. (b) Plot the current and EMF as functions of time. (c) Draw a phasor diagram to show the phase relations of the EMF of the source and the current in the circuit.

Solution. (a) On putting the given numerical values in the equations above, we find the peak current $I_0 = 53$ A and its phase with respect to the driving voltage $\phi = 1.2$ rad.

(b) I have used Mathematica to plot the functions $V(t) = V_0 \cos(\omega t)$ and $I(t) = I_0 \cos(\omega t + \phi)$ using the parameters $f = 60$ Hz, $R = 1\ \Omega$; $C = 0.001$ F. The plot is displayed in Fig. 12.14. The current in the circuit is ahead of the voltage of the source by approx. 69 degrees or 1.2 rad in the phase, or a little less than a quarter cycle.

(c)

The phasor diagram of the RC-circuit shown on the right gives the phase relation between the source EMF and current more clearly. The figure shows that, in a capacitative circuit, as the phasors for the source EMF and the current in the circuit rotate counterclockwise, the current I leads the source EMF V.



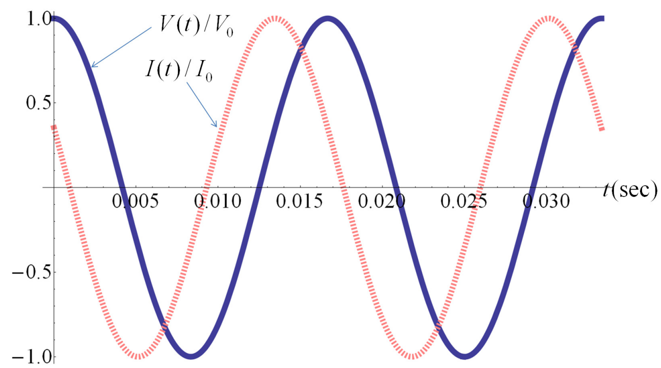


Figure 12.14: The EMF if the source (solid line) and the current in the circuit (dashed) in the RC-circuit. The voltage is plotted in units of V_0 = 150 V and the current in units of I_0 = 53 A. Other parameters of the circuit are f = 60 Hz, R = 1 Ω; C = 0.001 F.

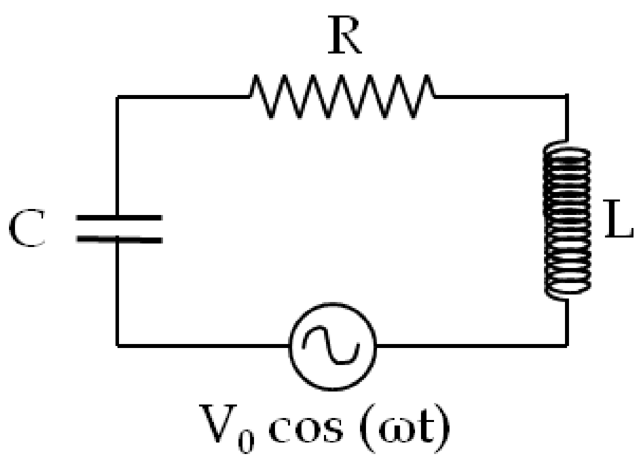


Figure 12.15: A series RLC-circuit driven by an alternating EMF.

12.3.3 Series RLC Circuit

Figure 12.15 shows a circuit that has all three basic circuit elements, namely a resistor R, an inductor L and a capacitor C connected in series with an alternating EMF source with voltage $V(t) = V_0 \cos(\omega t)$. We have already studied this circuit in the last chapter. An AC circuit analyis of this circuit refers to only the steady state solution, which we recall here. The current in the circuit is given by

$$I(t) = I_0 \, \cos(\omega t + \phi),$$

with I_0 and ϕ can be given as

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \tag{12.41}$$

$$\tan(\phi) = \frac{1}{\omega RC} - \frac{\omega L}{R}. \tag{12.42}$$

By taking the ratio V_0 to I_0 we determine the amplitude of the impedance $|Z|$ of the series RLC circuit to be

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \tag{12.43}$$

Due to the presence of L and C in the circuit, we expect electromagnetic oscillations in the circuit and due to R we expect damping of the oscillations. We had found in the last chapter that the series RLC circuit connected to a sinusoidal source is a driven damped oscillator.

From the solution it is clear that the peak current I_0 will change with the driving frequency ω. As the frequency ω is varied, while keeping V_0, L, R and C fixed, the peak current changes. That is, even when the voltage of the source is fixed, the current in the circuit changes when we change the frequency with which we drive the circuit. We had found that the largest peak current occurs when the frequency of the EMF source is equal to the natural frequency ω_0 of the circuit.

$$I_0 \text{ max when: } \omega = \frac{1}{\sqrt{LC}} \equiv \omega_0.$$

The phenomenon is called the resonance of the circuit, and the frequency for which the maximum peak current occurs is called the resonance frequency of the current, which we had denoted as ω_I, which is also the resonanace frequency of the power, denoted by ω_R.

$$\omega_R = \frac{1}{\sqrt{LC}}.$$

It is also observed that at the resonant frequency, the phase difference between the driving EMF and the current disappears.

$$\phi|_{\omega=\omega_R} = 0.$$

Therefore, at resonance the driving EMF and the current are in synchronization with each other, and the current in the circuit at resonance is given by

$$I_R(t) = \frac{V_0}{R}\cos(\omega_R t),$$

which would have been the current if the circuit contained only the resistor and no capacitor or inductor. At resonance, the circuit "forgets" that the capacitor and inductor are there!