7.1 IMPULSE

7.1.1 Impulse of a Constant Force

When a constant force $\vec{F}$ acts for a duration Δt, it exerts an **impulse** $\vec{J}$ equal to $\vec{F}\Delta t$.

$$\boxed{Impulse,\ \vec{J} = \vec{F}\Delta t.} \tag{7.1}$$

Impulse is a vector and has the same direction as the force itself. Therefore, if you pull a cart with a constant force of magnitude 15 N for 2 sec, then you would impart an impulse of 30 N.s to the cart in the direction of the force. Same impulse of 30 N.s can be imparted by a smaller force, say 1 N in the same direction, that acts for 30 sec instead of 15 N for 2 sec. Note that the unit of impulse in the SI system of units is the same as the unit of momentum:

$$\text{Unit of impulse} = \text{N.s} = \frac{\text{kg.m}}{\text{s}^2}\text{s} = \frac{\text{kg.m}}{\text{s}} = \text{Unit of momentum.}$$

Impulses from different forces add vectorially. Suppose force $\vec{F}_1$ acts for a time Δt_1 and $\vec{F}_2$ acts for a time Δt_2, then the net impulse $\vec{J}_{net}$ will be the vector sum of the impulses $\vec{J}_1$ and $\vec{J}_2$ of the two forces.

$$\vec{J}_{net} = \vec{J}_1 + \vec{J}_2, \tag{7.2}$$

where $\vec{J}_1 = \vec{F}_1\Delta t_1$ and $\vec{J}_2 = \vec{F}_2\Delta t_2$.

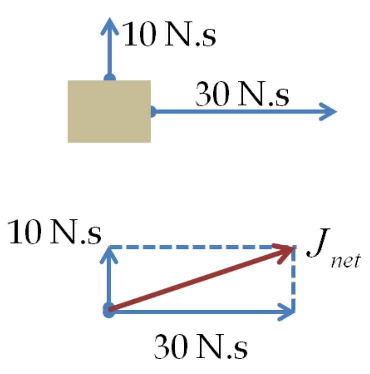

Figure 7.1: Adding impulse vectors, Example 7.1.1.

Example 7.1.1. Impulse as a vector. A box is on a level surface. A force of magnitude 3 N acts on the box in the direction towards the East for 10 sec and another force of magnitude 2 N acts on the box in the direction towards the North for 5 sec. What is the net impulse?

Solution. We will work this problem in the analytic picture for vectors. For that purpose, we will choose a coordinate system whose x-axis is pointed towards the East and the y-axis is pointed towards the North. We do not need the z-coordinate for this problem since the all the forces are in one plane, which is the xy-plane. The x and y-components of the two impulses and their sum are collected in the following table.

Impulse name	x-component (N.s)	y-component (N.s)
$\vec{J}_1$	30	0
$\vec{J}_2$	0	10
$\vec{J}_{net}$	30	10

The net impulse is now constructed from its components. We obtain the following magnitude and direction of the net impulse. Let us write J_x and J_y for the x and y-components of $\vec{J}_{net}$.

Magnitude:

$$J_{net} = \sqrt{J_x^2 + J_y^2} = 32 \text{ N.s},$$

Direction:

Since the vector is in the xy-plane, the angle of the vector with the positive x-axis can be used to indicate the direction.

$$\theta = \arctan\left(\frac{10}{30}\right) = 18^\circ,$$

Therefore, the direction of $\vec{J}_{net}$ is 18° North of East.

7.1.2 Impulse of a Time Dependent Force

Often, impressed forces on a body change with time. For instance, when you hit a base ball, the force between the bat and the ball varies with the time - having a zero magnitude before coming into contact, rising to a maximum magnitude after coming to contact, and then decreasing to zero at which time the two are no longer in contact. A representation of the magnitude of the force between the bat and the ball is shown in Fig. 7.2.

How can we calculate the impulse of the force of the bat on the ball? This is similar to how we extended the formulas for the constant acceleration for the case of varying acceleration. The general strategy is to think in terms of small steps of time over which a varying property can be replaced by an average value and then combine the effects from each subinterval.

Thus, we start by dividing the the total time, say from $t = 0$ to $t = T$, into smaller segments of time. We will then replace the force in each interval by the average force during that interval. This would give us a collection of constant forces acting on the body in each subinterval from which we can calculate the impulses in each subinterval. Finally, we can add the impulses in each subinterval to obtain the net impulse for the total time.

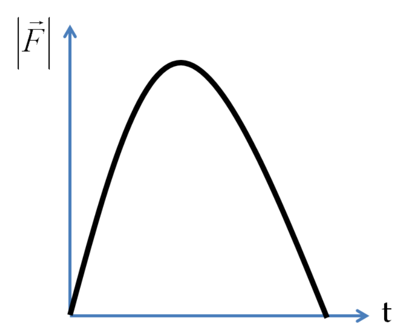

Figure 7.2: Magnitude of the force on a base ball by the bat changes with time.

Let $\vec{F}_1$, $\vec{F}_2$, $\cdots$, $\vec{F}_N$ be the average forces in the N subintervals, Δt_1, Δt_2, $\cdots$, Δt_N, respectively. Each subinterval of time will generate an impulse vector. Let $\vec{J}_1$, $\vec{J}_2$, $\cdots$, $\vec{J}_N$ be the corresponding impulse vectors.

$$\begin{aligned}\vec{J}_1 &= \vec{F}_1 \Delta t_1 \\ \vec{J}_2 &= \vec{F}_2 \Delta t_2 \\ &\vdots \\ \vec{J}_N &= \vec{F}_N \Delta t_N,\end{aligned} \tag{7.3}$$

with the total time T:

$$T = \Delta t_1 + \Delta t_2 + \cdots + \Delta t_N.$$

The impulse in different subintervals must now be added to obtain the net impulse.

$$\vec{J}_{net} = \vec{J}_1 + \vec{J}_2 + \cdots + \vec{J}_N. \tag{7.4}$$

The sum of vectors in Eq. 7.4 can be done either geometrically or analytically. Typically, an addition of vectors is more easily done in the analytic picture for vectors. For this purpose, we choose a coordinate system with respect to which we can determine the components of the forces, obtain the components of each impulse, and then add the components separately to obtain the components of the net impulse. This exercise results in the following set of equations.

$$\begin{aligned}J_x^{net} &= F_{1x}\Delta t_1 + F_{2x}\Delta t_2 + \cdots + F_{Nx}\Delta t_N \\ J_y^{net} &= F_{1y}\Delta t_1 + F_{2y}\Delta t_2 + \cdots + F_{Ny}\Delta t_N \\ J_z^{net} &= F_{1z}\Delta t_1 + F_{2z}\Delta t_2 + \cdots + F_{Nz}\Delta t_N.\end{aligned}$$

As long as the subintervals $\{\Delta t_i,\ i = 1, 2, \cdots, N\}$ are finite intervals, no matter how small, we will only get an approximation of the exact value of the impulse of a time-dependent force. However, the difference between the exact answer and the approximate answer can be minimized to an arbitrary precision by making the sizes of the subintervals arbitrarily small, and consequently, making the number of subintervals infinitely large. The result is then written using the integral sign.

$$J_x^{net} = \int_0^T F_x(t)dt \tag{7.5}$$

$$J_y^{net} = \int_0^T F_y(t)dt \tag{7.6}$$

$$J_z^{net} = \int_0^T F_z(t)dt. \tag{7.7}$$

A definite integral such as in Eqs. 7.5 - 7.7 have also the meaning of the "area under the curve" with area being positive if function is positive and negative if function is negative as we have encountered before. Therefore, we can use area under the curve for each of the components of forces if the forces are given as plots rather than as analytic functions. Formally, we can write the three equations for the components in vector notation.

$$\boxed{\vec{J}_{net} = \int_0^T \vec{F}(t)dt.} \tag{7.8}$$

Example 7.1.2. Impulse of a sinusoidally varying force. The time dependence of a sinusoidally varying force can be written as a sine or cosine of time. For instance, when an electron is illuminated by a monochromatic light, the force on the electron can be represented by a cosine or sine function. Suppose the force on an electron at some time t has the following form with respect to a coordinate system, $\vec{F} = A\cos(t)\hat{u}_x + B\sin(t)\hat{u}_y$, where $\hat{u}_x$ and $\hat{u}_y$ are the unit base vectors along the x and y-axes respectively, and A and B are some constants. What will be the impulse on the electron during an interval $t = 0$ to $t = T$?

Solution. Since the force is already given in an analytic form, we will express the impulse in the same coordinate system. Using Eq. 7.8 and writing out the integrals for each component, we find

$$\begin{aligned} \vec{J}_{net} &= A\left[\int_0^T \cos(t)dt\right]\hat{u}_x + B\left[\int_0^T \sin(t)dt\right]\hat{u}_y \\ &= A\sin(T)\hat{u}_x + B\left[1-\cos(T)\right]\hat{u}_y. \end{aligned}$$

Further Remarks:

Is the magnitude of impulse equal to the magnitude of a force times the duration? The answer depends on whether or not the force is changing with time.

1. When a single constant force acts on the system, then the impulse will have the same direction as the direction of the force, and the magnitude of the impulse will be simply the product of the amplitude of the force and the duration over which the force has acted on the system.

2. When multiple constant forces act on the system, then the net force will also be constant. This situation is same as the one force case with one force now being the net force. Therefore, the impulse will have the same direction as the direction of the net force, and the magnitude of the impulse will be the product of the amplitude of the net force and the duration over which the force has acted on the system.

3. When a time varying force acts on the system, the situation is more complicated. As a concrete example of this situation, let us examine in detail the case of a force of constant magnitude but time-dependent direction. Let $F_x(t)$ and $F_y(t)$ be the non-zero components of a force whose magnitude is constant in time.

   $F = \sqrt{F_x(t)^2 + F_y(t)^2}$ = constant, even if the components are not.

   For instance, if $F_x(t) = F_0 \cos(bt)$ and $F_y(t) = F_0 \sin(bt)$, then the magnitude will be constant, even when the components are time-dependent. The components of impulse will be

   $$J_x^{net} = \int_0^T F_x(t)dt.$$
   $$J_y^{net} = \int_0^T F_y(t)dt.$$

   This gives the magnitude of the impulse

   $$J_{net} = \sqrt{\left(J_x^{net}\right)^2 + \left(J_y^{net}\right)^2}.$$

   The question now becomes: Can this J be written as F times T? The answer is no, since F_x and F_y are time-dependent.

   $$J_{net} \neq FT.$$

   Therefore, when the force varies with time, you cannot simply take the magnitude of the force and multiply it by the duration to obtain the impulse. You will have to calculate the magnitude and direction of the impulse vector from its components, which would be separately obtained from the time-varying force components.

7.1.3 Area Under the Curve Method

Often the data for force with respect to time is in terms of a plot of force versus time as shown in Fig. 7.2. If you can write the data as an analytic function of time, then you could do the integrals of Eqs. 7.5 - 7.7 to obtain the components of the impulse vector.

Alternately, we can utilize another interpretation of the definite integral of one variable: a definite integral of a function $f(t)$ of one independent variable t is equal to the area under the curve of $f(t)$ versus t plot. The area for a curve above the abscissa gives a positive area and the area for a curve below the abscissa has a negative area.

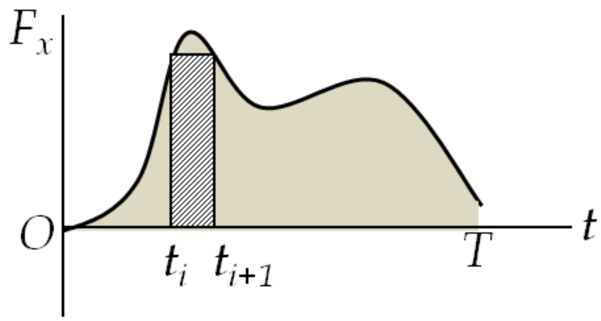

Figure 7.3: Area under curve shown shaded is evaluated from the sum of the areas of rectangles of the area. In the limit of infinitesimal t subintervals the area obtained from summing the areas of the rectangles approaches the exact area under the curve.

The integral value for integration from $t = t_1$ to $t = t_2$ is the sum of all areas under the curve between the limits, $t_1 \leq t \leq t_2$.

Let us look at one of the Cartesian components, say the x-component. A plot of x-component of a force is shown in Fig. 7.3.

The shaded part has an "area under the curve" of $F_x(t_i)\Delta t$, which is the contribution to the x-component of the impulse for the time segment t_i to t_{i+1} of duration Δt. You can easily see that the area $F_x(t_i)\Delta t$ is not really equal to the "true" area under the curve - the rectangle misses the curve by some amount depending upon the size of Δt and how curved the curve is at that point in the plot. However, as you make Δt progressively smaller, the difference between the exact area and the approximate area evaluated by rectangles becomes progressively smaller, and the two become more and more equal to each other. By choosing small enough Δt we can make the error less than any desired value.

To obtain an exact value of the integral of F_x from $t = 0$ to $t = T$ from a plot of F_x versus t, we divide the total interval, $0 \leq t \leq T$, into subintervals of size Δt, and sum over the areas of the rectangles. If Δt is small enough, then the value obtained by summing the areas gives us the exact value of the integral.

$$\begin{aligned} \int_0^T F_x dt &= \lim_{\Delta t \to 0} [F_x(0)\Delta t + F_x(\Delta t)\Delta t + F_x(2\Delta t)\Delta t + \cdots] \quad (7.9) \\ &= \text{"Area under } F_x \text{ vs } t \text{ from } t = 0 \text{ to } t = T\text{".} \end{aligned}$$

The same procedure will work for the y and z-components also. The "area under the curve" method provides an alternate way of evaluating components of impulse vector given in Eqs. 7.5 - 7.7.

$$x\text{-component:}\quad J_x^{net} = \int_0^T F_x(t)dt = \text{“Area under } F_x \text{ vs } t\text{”} \tag{7.10}$$

$$y\text{-component:}\quad J_y^{net} = \int_0^T F_y(t)dt = \text{“Area under } F_y \text{ vs } t\text{”} \tag{7.11}$$

$$z\text{-component:}\quad J_z^{net} = \int_0^T F_z(t)dt = \text{“Area under } F_z \text{ vs } t\text{”} \tag{7.12}$$

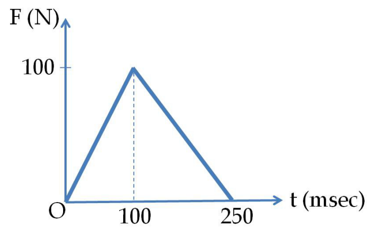

Figure 7.4: Magnitude of the force on a base ball by the bat changes with time.

Example 7.1.3. Impulse of a time-varying force The force between a baseball and a bat is an example of a time-varying force. The magnitude of the force varies with time in a complicated way as shown in Fig. 7.2. A simpler picture of the magnitude of the force is shown in Fig. 7.4. What is the impulse of the force given in Fig. 7.4?

Solution. Figure 7.4 gives us the magnitude of the force on the ball. Although, the variation of the magnitude of the force is an important information for impulse, but we also need information about the direction of the force to compute the impulse vector. Here, the force on the ball has the same direction during the entire time the bat is in contact with the ball. Therefore, if we choose the x-axis to point in the direction from the bat to the ball, then Fig. 7.4 is same as the x-component of the force as shown in Fig. 7.5.

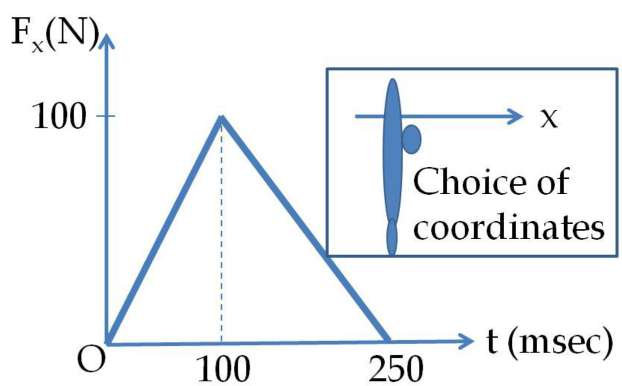

Figure 7.5: x-component of force on the baseball.

The area under the curve in Fig. 7.5 is the area of the triangle under the curve. The formula for the area of a triangle is base times height divided by 2. We convert the unit of time from millisecond to sec before putting the numbers in the area formula. This gives the x-component of impulse to be

$$J_x = \frac{1}{2} \times 100 \text{ N} \times 0.25 \text{ s} = 12.5 \text{ N.s},$$

which is equal to the magnitude of the impulse since the y and z-components are zero, and the direction of the impulse is from the bat towards the ball.