9.2 ANGULAR MOMENTUM

9.2.1 Angular Momentum of a Point Particle

We start with the simplest case of a motion of a point particle of mass m. Let $\vec{r}$ be the position of the particle with respect to an origin of a given coordinate system, and $\vec{p}$ the momentum. The angular momentum $\vec{L}$ with respect to the origin is defined by the cross product of $\vec{r}$ and $\vec{p}$.

$$\boxed{\vec{L} = \vec{r} \times \vec{p}.} \tag{9.30}$$

From this definition we see that the unit of angular momentum in meter-kg-sec system will be $kg.m^2/s$.

Example 9.2.1. Angular momentum of a particle moving in a circle. To get a feel for the angular momentum, we evaluate the angular momentum of particle of mass m that moves with a speed v in a circle of radius R shown in Fig. 9.9. Let the origin be at the

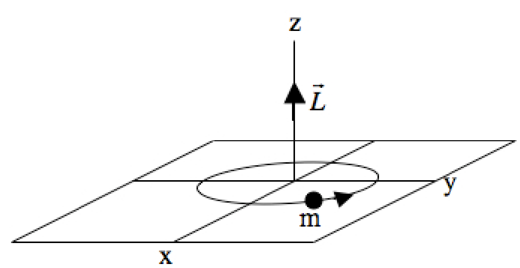


Figure 9.9: Example 9.2.1. The angular momentum of a particle moving in a circle is pointed perpendicular to the plane of the circle and has the magnitude $L = mvR$.

center of the circle and the circle be in the xy-plane of a Cartesian coordinate system as shown in Fig. 9.9. Since the particle moves in a circle, the position vector $\vec{r}$ is perpendicular to the momentum vector $\vec{p}$, hence the magnitude of the angular momentum is

$$\text{Magnitude, } L = (R)(mv)\,(\sin 90^\circ) = mvR$$

The direction is obtained by using the right-hand rule for the cross-product. This gives the direction of angular momentum along the positive z-axis in the figure.

The rules for the cross-product tell us about the magnitude and direction of the angular momentum. The direction is given by applying the right-hand rule of cross-product on the vectors $\vec{r}$ and $\vec{p}$ as shown in Fig. 9.10.

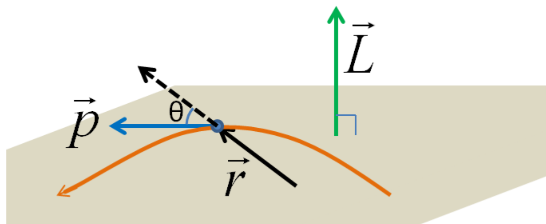


Figure 9.10: The direction of $\vec{L}$ is perpendicular to both $\vec{r}$ and $\vec{p}$ as given by the right-hand rule. If $\vec{r}$ and $\vec{p}$ are collinear, then the angular momentum is zero. When $\vec{r}$ and $\vec{p}$ are non-collinear, then they define a plane, and angular momentum is perpendicular to this plane.

Using the geometric definition of the cross product we can write angular momentum in terms of the magnitudes of the position and the momentum vectors and the angle θ between them.

$$\vec{L} = \vec{r} \times \vec{p} = \begin{cases} \text{Magnitude} = rp\sin\theta \\ \text{Direction: Use Right Hand rule} \end{cases} \tag{9.31}$$

The analytic method for evaluating a cross-product uses the decomposition of the two vectors into their Cartesian components and then computing the determinant.

$$\begin{aligned} \vec{L} = \vec{r} \times \vec{p} &= \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \\ &= \hat{u}_x \left(yp_z - zp_y\right) + \hat{u}_y \left(zp_x - xp_z\right) + \hat{u}_z \left(xp_y - yp_x\right). \end{aligned} \tag{9.32}$$

How would you obtain the angular momentum of a multiparticle system? Well, you just add up the angular momenta of all particles vectorially to obtain the angular momentum of the whole. For example, the angular momentum of a system consisting of N particles will be

$$\boxed{\vec{L} \equiv \vec{L}_{\text{net}} = \sum_{i=1}^{N} \vec{L}_i = \sum_{i=1}^{N} \left(\vec{r}_i \times \vec{p}_i\right).} \tag{9.33}$$

Note the role of the reference point and reference frame in the definition of the angular momentum. Since the position vector $\vec{r}$ is measured from a reference point O, the position vector will change if you pick another reference point, thereby changing $\vec{L}$. Similarly, since the momentum depends on the choice of the reference frame, the angular momentum will also depend on the choice of a reference frame.

Example 9.2.2. Angular momentum of a conical pendulum. A pendulum bob of mass m and length b can be rotated about the

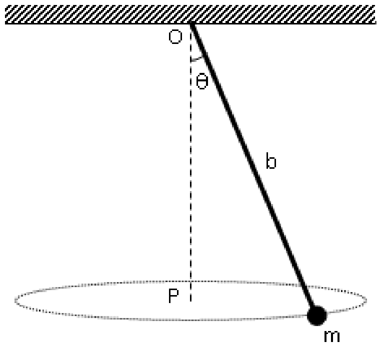


Figure 9.11: Example 9.2.2.

pivot so that the bob moves in a circular path with an angle of suspension θ as shown in Fig. 9.11. This arrangement is called a **conical pendulum**. Find the angular momentum of the pendulum (a) about the center P of the circle, and (b) about the pivot point O.

Solution. Since the angular momentum depends on the reference point we will organize the answer in the two parts as follows.

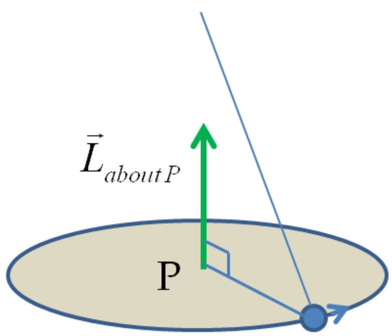


Figure 9.12: Example 9.2.2. The direction of the angular momentum about P.

(a) $\vec{L}$ about P

To find the angular momentum about point P we need the position and momentum vectors with respect to P. The position vector of m from point P has the magnitude of the radius of the circle and a direction radially outward in the plane of the circle. The momentum of m has the magnitude mv and the direction tangent to the circle. Therefore, the position vector is perpendicular to the momentum vector. That means that the magnitude of the angular momentum will be equal to product of the radius R of the circle and the momentum mv. Since the radius of the circle about P is $R = b\sin\theta$, we have

$$\text{The magnitude of } \vec{L} \text{ about P} = mvR = mvb\sin\theta.$$

The direction of the angular momentum is obtained by using the right-hand rule on vectors $\vec{r}$ and $\vec{p}$. This gives the direction of $\vec{L}$ to be perpendicular to the plane of the circle, pointed up as shown in Fig. 9.12.

(b) $\vec{L}$ about O

The position vector now has a magnitude b and pointed in the direction from O to m. Since the direction of the momentum is perpendicular to the plane containing points O, P, and m in the figure, the angle between the position vector and the momentum vector is 90°. The magnitude of angular momentum will therefore be simply the product of the magnitudes of position and momentum vectors. The position of the particle with respect to the reference point O has magnitude b now. Therefore, the angular momentum about O has the following magnitude.

$$\text{The magnitude of } \vec{L} \text{ about O} = mvb.$$

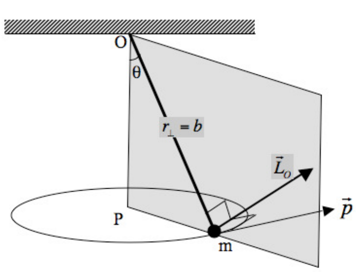


Figure 9.13: Example 9.2.2.

The direction of the angular momentum vector is again obtained by applying the right-hand rule to the position and the momentum vectors. The resulting angular momentum vector would be in the plane containing the points O, P, and m, in the direction shown in Fig. 9.13. Make sure that the angular momentum vector is perpendicular to both the position vector and the momentum vector.

9.2.2 Angular Momentum of an Extended Body and the Moment of Inertia

The angular momentum of an extended body can be constructed from the sum of angular momenta of particles that make up the body. Suppose an extended body is made up of N masses, m_1, m_2, $\cdots$, m_N, whose positions with respect to a reference point O be $\vec{r}_1$, $\vec{r}_2$, $\cdots$, $\vec{r}_N$, respectively, and the spatial velocity be $\vec{v}_1$, $\vec{v}_2$, $\cdots$, $\vec{v}_N$, respectively. The angular momentum of i^{th} mass $\vec{L}_i$ about point O

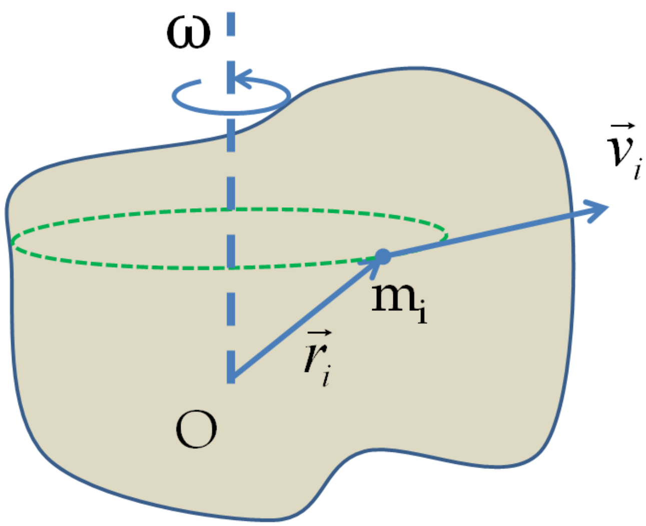


Figure 9.14: Angular momentum of an extended body is obtained by a sum of angular momenta of its parts. For mass m_i, the angular momentum is $\vec{L}_i = \vec{r}_i \times m\vec{v}_i$.

can be written using the point mass formula given above.

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times m\vec{v}_i = m\vec{r}_i \times \vec{v}_i \quad \text{[About O]} \tag{9.34}$$

Now, we sum up vectors $\vec{L}_i$ for each particle to get the angular momentum $\vec{L}$ of the body as a whole about point O.

$$\vec{L} = \sum_{i=1}^{N} \vec{L}_i = \sum_{i=1}^{N} (m_i\vec{r}_i \times \vec{v}_i) \quad \text{[About O]} \tag{9.35}$$

Each particle's spatial velocity can also be expressed in terms of the angular velocity of the body using Eq. 9.29.

$$\boxed{\vec{L} = \sum_{i=1}^{N} [m_i\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] \quad \text{[About O]}} \tag{9.36}$$

This formula for the angular momentum of a body is very complicated for an arbitrarily rotating body [no kidding!]. If the axis of rotation is towards the z-axis, the formula simplifies considerably. In this case, we can perform the cross products rather easily with the result that the angular momentum is also pointed along the z-axis. In the

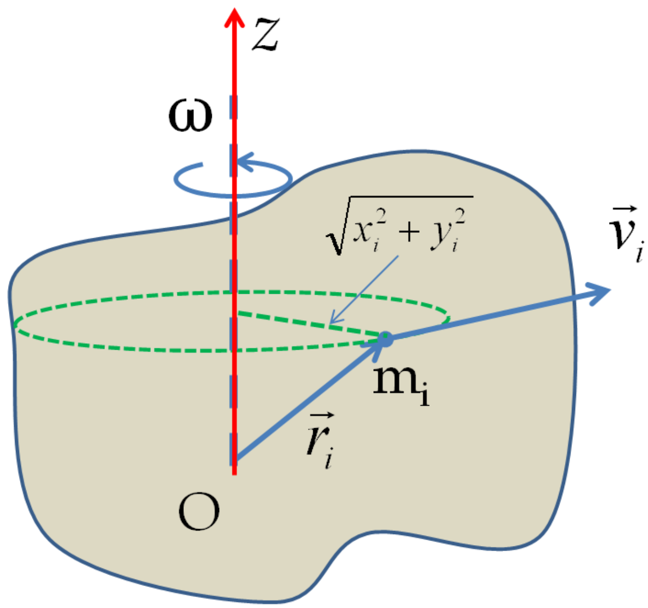


Figure 9.15: Choosing a Cartesian axis along the axis of rotation simplifies calculations for fixed axis rotations.

component form the result is:

$$\left\{\begin{array}{l} L_x = 0 \\ L_y = 0 \\ L_z = \left[\sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2\right)\right] \, \omega_z \end{array}\right\} \quad \text{[Rotating about } z\text{-axis.]} \tag{9.37}$$

Proof: Let us write the vectors in component form so that we can perform the cross products in Eq. 9.36.

$$\vec{r}_i = x_i \hat{u}_x + y_i \hat{u}_y + z_i \hat{u}_z$$
$$\vec{\omega} = \omega_z \hat{u}_z \quad \text{[since rotating about the } z\text{-axis]}$$

The spatial velocity vector $\vec{\omega} \times \vec{r}_i$ is then obtained by a direct calculation.

$$\vec{\omega} \times \vec{r}_i = \omega_z x_i \hat{u}_y - \omega_z y_i \hat{u}_x.$$

Now we can take the cross product of $\vec{r}_i$ with this expression giving the following result.

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \omega_z \left(x_i^2 + y_i^2\right) \hat{u}_z.$$

Since ω_z and $\hat{u}_z$ are same for every particle, they come outside the sum giving the relation in Eq. 9.37.

The quantity in brackets in Eq. 9.37 is the coefficient by which we must multiply the z-component of angular velocity to obtain the z-component of the angular momentum. It is the zz component of a quantity called the moment of inertia, which is usually denoted by the letter I. We will denote the zz component of moment of inertial by I_{zz}.

$$\boxed{I_{zz} = \sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2\right),} \tag{9.38}$$

which can be used to write the z-component of the angular momentum more compactly as

$$\boxed{L_z = I_{zz}\omega_z.} \tag{9.39}$$

This equation shows that, for the fixed-axis rotation, the formula for the z-component of angular momentum is analogous to the definition of the z-component of the linear momentum $p_z = mv_z$. The moment of inertia reflects the geometrical distribution of masses in the body, and plays the role of inertia in rotation analogous to the role played by mass in the translational motion. Therefore, <u>the fixed-axis rotation is analogous to translational motion in a straight line</u>.

For a continuous body, the sum in Eq. 9.38 will turn into an integral. Suppose dm is a mass element located in a small volume at the coordinates (x, y, z), then the zz component of the moment of inertia cam be calculated by performing the following "conceptual" integral.

$$\boxed{I_{zz} = \int_{\text{body}} \left(x^2 + y^2\right) dm.} \tag{9.40}$$

This integral is usually performed by first dividing the body into elements and then writing the mass of a representative element as a product of the density ρ and infinitesimal volume $dxdydz$ at the point (x, y, z).

$$dm = \rho\; dx\; dy\; dz. \tag{9.41}$$

The choice of the shape of the volume element depends on the symmetry of the situation as you will see in the examples below. For many cases of interest, the integral becomes an integration over only one variable if appropriately shaped elements are chosen for the calculation.

Further Remarks

If the axis of rotation is not along the z-axis, then, the angular velocity may have non-zero x, y and z-components and can be written as:

$$\text{Rotation about arbitrary axis: } \vec{\omega} = \omega_x \hat{u}_x + \omega_y \hat{u}_y + \omega_z \hat{u}_z. \tag{9.42}$$

Using Eq. 9.42 in Eq. 9.36 gives rise to the following components of angular momentum in terms of the components of the angular velocity that have nine coefficients, denoted by $\{I_{ij},\ (i = x, y, z), (j = (x, y, z)\}$. The values of the components I_{ij} tell us the way masses of the system are distributed with regard to the axis of rotation as captured by the Cartesian coordinates of the masses. For instance, if the body rotates about the z-axis, you need only I_{zz} which tells us about the distribution of masses of the system with regard to the z-axis.

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

These relations show that the x-component of the angular momentum depends not only on the x-component of the angular velocity but also on the y and z-components of the angular velocity. Therefore, in general, the angular momentum and angular velocity may be in different directions. This is very different than the relation between the momentum and velocity for translational motion, where the two are in the same direction.

In the case of a fixed-axis rotation, angular momentum and angular velocity are in the same direction since only one component of angular velocity is non-zero which can always be taken to be the z-axis. In this case $L_z = I_{zz}\omega_z$.

9.2.3 Calculations of Moments of Inertia

Moment of inertia of systems with point masses

The general formula for the moment of inertia component I_{zz} for discrete masses is given in Eq. 9.38.

$$I_{zz} = \sum_{i=1}^{N} m_i \left(x_i^2 + y_i^2\right)$$

where x_i and y_i are x and y-coordinates of the mass m_i. The quantity $x^2 + y^2$ can be replaced by the square of the distance r from z-axis.

$$\boxed{I_{zz} = \sum_{i=1}^{N} m_i r_i^2, \quad r_i = \sqrt{x_i^2 + y_i^2}.}$$

Now, we work out a few simple examples.

One particle of mass m

Consider a particle of mass m located at point P with Cartesian coordinates (x, y, z). This gives the three principal moments as follows. Note: we don't need any information about the rotation axis to calculate moments of inertia components; we only need the mass of the particle and its Cartesian coordinates.

$$\begin{aligned} I_{xx} &= m\left(y^2 + z^2\right) \\ I_{yy} &= m\left(z^2 + x^2\right) \\ I_{zz} &= m\left(x^2 + y^2\right) \end{aligned}$$

Two particles of masses m_1 and m_2

Let two point masses located at $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. We can immediately write down the three principal moments.

$$\begin{aligned} I_{xx} &= m_1\left(y_1^2 + z_1^2\right) + m_2\left(y_2^2 + z_2^2\right) \\ I_{yy} &= m_1\left(z_1^2 + x_1^2\right) + m_2\left(z_2^2 + x_2^2\right) \\ I_{zz} &= m_1\left(x_1^2 + y_1^2\right) + m_2\left(x_2^2 + y_2^2\right) \end{aligned}$$

Example 9.2.3. Moment of inertia of the methane molecule. To illustrate an application to molecular rotations consider determining the moment of inertia of the methane molecule for a coordinate system with the origin at the carbon atom and the z-axis through one of the C-H bonds.

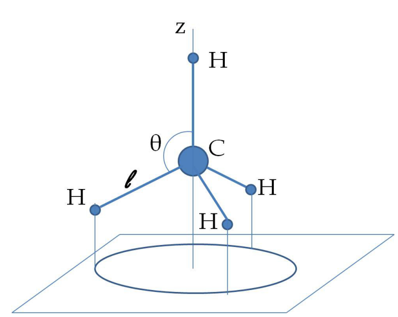


Figure 9.16: Methane molecule.

The methane molecule has four hydrogen atoms at the corners of a tetrahedron and one carbon atom at the center. Let l be the bond length between C and H. By symmetry all four bonds are of equal length. Let the angle between any two C-H bonds be θ.

We will calculate I_{zz} in the given coordinate system. For I_{zz} we need only the x and y-coordinates of the three H atoms that are not on the z-axis. By symmetry the three H atoms will fall on a circle of radius $l \sin\theta$ around z-axis separated by 120°. Therefore,

$$I_{zz} = 3ml^2 \sin^2\theta.$$

A uniform thin rod with the axis through center and perpendicular to the rod

Consider a uniform thin rod of mass M and length L. We assume that the area of cross-section of the rod is small and the rod can be thought of as a string of masses in one straight line. Suppose the

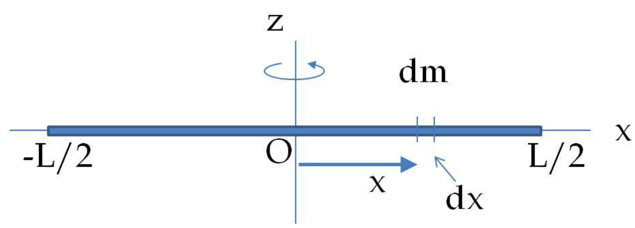


Figure 9.17: Calculation of I_{zz} of a thin rod about an axis through the center.

axis of rotation is perpendicular to the rod and passes through the mid-point. We wish to find the moment of inertia about this axis. As before, we will orient the axes so that the z-axis is the axis of rotation. Since, we are dealing with a rod, and the integral is to go over the volume of the rod, it will be helpful to orient the x or y-axis in the direction of the rod. For definiteness, let the x-axis pass through the length of the rod as shown in fig. 9.17.

Let dm be a small element of mass located between $(x, 0, 0)$ and $(x + dx, 0, 0)$. The element has mass equal to ρdx, where ρ is mass per unit length, $\rho = M/L$.

$$dm = \frac{M}{L} dx, \tag{9.43}$$

which says that, to take into account the contribution from the entire rod, we need to integrate over the x variable. Now, using the coordinates $(x = x, y = 0)$ in Eq. 9.40 for the element and replacing dm by the expression for the rod given in Eq. 9.43, we obtain

$$I_{zz} = \int_{-L/2}^{L/2} \left(x^2 + 0\right) \frac{M}{L} dx.$$

This integral can be readily performed to yield the zz component of the moment of inertia of a thin rod about an axis through the center and perpendicular to the length of the rod.

$$\boxed{I_{zz} = \frac{1}{12} M L^2 \quad \text{[for rod about an axis through center]}}$$

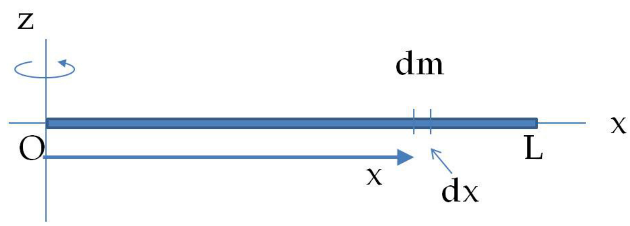


Figure 9.18: Calculation of I_{zz} of a thin rod about an axis at the edge.

A uniform thin rod with axis at the end and perpendicular to the length

The origin of the coordinate system will be at one end as shown in Fig 9.18. This changes only the range of integration. Now, the integration will be from $x = 0$ to $x = L$, giving a different formula for I_{zz} of a rod.

$$I_{zz} = \int_0^L \left(x^2 + 0\right) \frac{M}{L} dx = \frac{1}{3} M L^2.$$

Performing the integration we find

$$\boxed{I_{zz} = \frac{1}{3} M L^2 \quad \text{[for rod about an axis through on end]}}$$

This shows that the moment of inertia of a system depends on the axis of rotation. We have found that the I_{zz} for the same rod about an axis placed at the end of the rod is four times the moment of inertia when the axis passes through the center.

Why are the two I_{zz} for the same rod different? When the axis passes through the center, all mass elements of the rod rotate within a distance of $L/2$ from the axis, but when the axis passes through a point at the end, half of the masses of the rod rotate in larger circles, having distances between $L/2$ and L from the axis. Clearly, having masses more distant from the axis makes a big difference in rotation.

A uniform thin ring about an axis through the center and perpendicular to the ring

Consider a thin ring of mass M and radius R. We assume that the ring is so thin that we can place all masses on the ring at the same distance R from the center. To find the component I_{zz} of the moment of inertia for the rotation about the z-axis perpendicular to the ring and passing through the center, we place the ring in the xy-plane with the origin at the center as shown in Fig. 9.19.

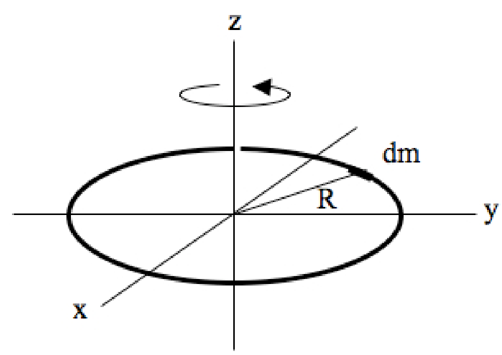


Figure 9.19: Calculation of I_{zz} of a ring.

Let dm be an element of the ring located at point $(x, y, 0)$ on the ring. Since $(x, y, 0)$ is located at the ring and the origin is at the center of the ring, the coordinates x and y are related by the radius R of the ring.

$$x^2 + y^2 = R^2.$$

Therefore, the formula for the conceptual integral for I_{zz} given in Eq. 9.40 simplifies and the integral can be done without any effort.

$$I_{zz} = \int_{\text{body}} \left(x^2 + y^2\right) dm = \int_{\text{body}} R^2 dm = R^2 \int_{\text{body}} dm.$$

The integration over dm is simply a sum of mass of all elements of the ring, which will give the total mass of the ring for the integral. Therefore, the result is

$\boxed{I_{zz} = MR^2 \ \ \text{[Ring; Axis through the center and perpendicular to the ring]}}$

A uniform thin ring about an axis through the center and in the plane of the ring

Consider again a thin ring of mass M and radius R. Let us place the ring in the xy-plane so that z-axis passes through its center and is perpendicular to the ring (Fig. 9.20).

From the symmetry, it is clear that rotation about x-axis is equivalent to a rotation about the y-axis. Therefore, the components I_{xx} and I_{yy} of the moment of inertia will be equal.

$$I_{xx} = I_{yy}.$$

The component I_{xx} has a similar expression as I_{zz} given above, specifically,

$$I_{xx} = \int_{\text{body}} \left(y^2 + z^2\right) dm.$$

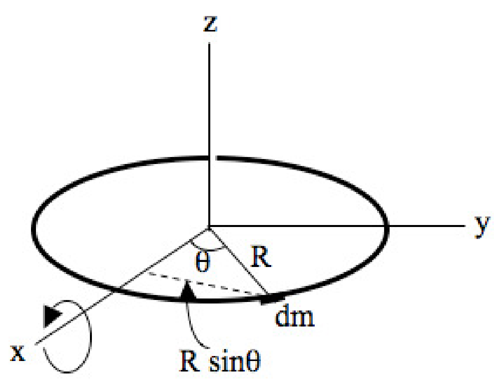


Figure 9.20: Calculation of I_{xx} of a ring.

When you rotate the ring about the x-axis, the mass elements move in circles in planes parallel to the yz plane. To calculate the moment of inertia component I_{xx} you can take the configuration at any instant. Here, we choose to work at the instant the ring is in $z = 0$ plane. With the ring in the $z = 0$ plane, I_{xx} simplifies to

$$I_{xx} = \int_{\text{body}} y^2 dm. \tag{9.44}$$

The shape of the ring suggests that this integral will be easier to do if we formulate the problem in the polar coordinate. In the polar coordinate the radial variable is fixed to the radius of the ring.

$$r = R.$$

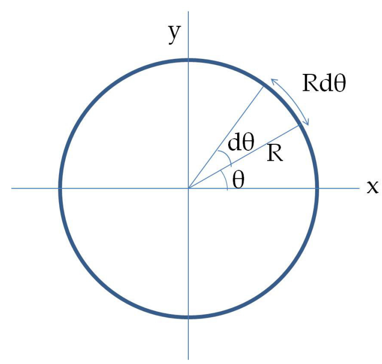


Figure 9.21: Calculation of dm.

The mass element has a shape of arc length $Rd\theta$, where $d\theta$ is the angle the arc element subtends at the center. The mass dm in an arc element is

$$dm = \text{ Mass per unit length } \times \text{ Arc length } = \frac{M}{2\pi R} \times Rd\theta.$$

The y-coordinate written in terms of polar coordinate has the following form.

$$y = R \sin\theta.$$

Therefore, the integral in Eq. 9.44 transforms into an integral in one variable, the polar angle from 0 to 2π.

$$I_{xx} = \frac{MR^2}{2\pi} \int_0^{2\pi} \sin^2\theta d\theta = \frac{1}{2} MR^2.$$

Performing this integral we find

$$\boxed{I_{xx} = \frac{1}{2} MR^2 \ \text{[Ring; Axis through the center and in plane of the ring]}}$$

A uniform thin disk about an axis through the center and perpendicular to the disk

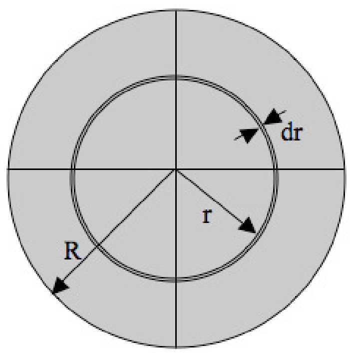


Figure 9.22: Calculation of I_{zz} of a disk.

Consider a thin disk of mass M and radius R. Since the disk is thin, its thickness can be ignored. For calculations, we place the disk in the xy-plane so that the z-axis is perpendicular to the disk and passes through its center. Now, we divide the disk into thin rings of various radii r and infinitesimal thickness dr.

One ring whose radius is less than R is shown in Fig. 9.22. The disk can be considered to be made up of rings like these from $r = 0$ to $r = R$. The thin ring shown in Fig. 9.22 contains an infinitesimal mass given by

$$dm = \frac{M}{\pi R^2} \times 2\pi r dr.$$

All masses in the infinitesimally thin ring are on circles of radius between r and $r + dr$. since dr is infinitesimal, it has a value that can be as small as you like, except zero. Therefore, we can say that all the masses in the infinitesimal ring between r and $r + dr$ are on a circle of radius r.

$$x^2 + y^2 = r^2. \text{ (Note: it is not } R^2.\text{)}$$

Using this expression in the definition of I_{zz} given in Eq. 9.44, we find that the conceptual integral for a disk becomes an integral over the radial coordinate. The range $r = 0$ to $r = R$ includes all the infinitesimal rings that make up the disk.

$$I_{zz} = \int_0^R \left(r^2\right) \left(\frac{M}{\pi R^2} 2\pi r dr\right) = \frac{2M}{R^2} \int_0^R r^3 dr = \frac{1}{2} M R^2.$$

After performing the integration, we find

$$\boxed{I_{zz} = \frac{1}{2} M R^2 \quad \text{[Disk; Axis through the center and perpendicular to the disk]}}$$

A uniform thin disk about an axis through the center and in the plane of the disk

Once again consider a thin disk of mass M and radius R. From symmetry we note that rotation about x and y-axes are equivalent. Therefore, components I_{xx} and I_{yy} of the moment of inertia for rotations about the x and y-axes respectively, must be equal. We will perform the calculation for I_{xx} as we did for the ring above.

To make use of the cylindrical symmetry in the disk, we will do calculation in the polar coordinates. We focus on writing dm, and

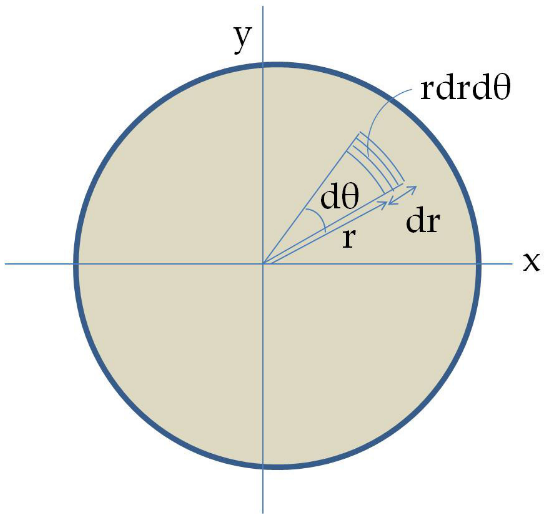


Figure 9.23: Calculation of I_{xx} of a disk.

the x and y-coordinates of an element of the disk so that we can write the conceptual integral in Eq. 9.44 into a definite integral.

Consider a small element on the disk between r and $r + dr$ and θ and $\theta + d\theta$. The element has a length equal to the arc length $rd\theta$ and width dr. Therefore, the area of this element is $rdrd\theta$. The total mass M of the disk is spread over the total area πR^2 of the disk. Therefore, the mass in the mass element is

$$dm = \frac{M}{\pi R^2} rdrd\theta.$$

This says that we have to perform a two-dimensional integral, one for r and the other for θ. For I_{xx} calculation, the conceptual integral has the distance of the element from x-axis, which is $y^2 + z^2$. Here, the disk is placed in $z = 0$ plane, therefore, we have only y^2. Thus, we need y-coordinate of the element to go in the integral. Since the element is at the polar coordinates (r, θ), its y-coordinate is

$$y = r \sin\theta.$$

Therefore we find I_{xx} to be,

$$\begin{aligned} I_{xx} &= \int y^2 dm = \int\int (r\sin\theta)^2 \left(\frac{M}{\pi R^2} rdrd\theta\right) \\ &= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} \sin^2\theta d\theta \end{aligned}$$

This gives the following After performing the integration, we find

$$\boxed{I_{zz} = \frac{1}{4} MR^2. \text{ [Disk; Axis through the center and in plane of the disk]}}$$

A uniform sphere about an axis through the center

Consider a sphere of mass M and radius R. Since all axes through the center of a sphere are equivalent, it suffices to work out only the component I_{zz} of moment of inertia for rotation about z-axis. It

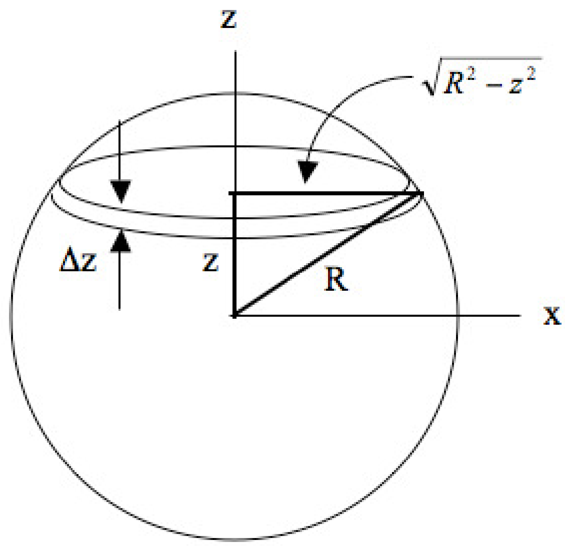


Figure 9.24: Calculation of I_{zz} of a sphere.

turns out that we can make use of the results of the disk if we think of the sphere as thin disks of thickness dz stacked on top each other. The disk between z and z+ z has a radius of $\sqrt{R^2 - z^2}$. The moment of inertia of the thin disk between z and $z + dz$ is found as follows.

$$\begin{aligned}
&\text{Radius of the disk} = \sqrt{R^2 - z^2} \\
&\text{Volume of the disk} = \pi \left(R^2 - z^2\right) dz \\
&\text{Mass of the disk} = \text{density} \times \text{volume} = \rho\pi \left(R^2 - z^2\right) dz \\
&I_{zz} \text{ of the disk } = \frac{1}{2} \left[\rho\pi \left(R^2 - z^2\right) dz\right] \left(R^2 - z^2\right).
\end{aligned}$$

The moment of inertia of the sphere will be obtained by summing up moments of inertia of all the disks, which is obtained by integrating over z from $-R$ to R.

$$I_{zz} = \int_{-R}^{R} \frac{1}{2} \left[\rho\pi \left(R^2 - z^2\right) dz\right] \left(R^2 - z^2\right) = \frac{8}{15}\pi\rho R^5.$$

This formula can be written by eliminating the density ρ by mass over volume.

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}.$$

Hence, I_{zz} component of the moment of inertia of the sphere written in terms of its mass and radius is

$$\boxed{I_{zz} = \frac{2}{5} M R^2 \quad \text{[Sphere; Axis through the center}}$$

THE PARALLEL AXIS THEOREM

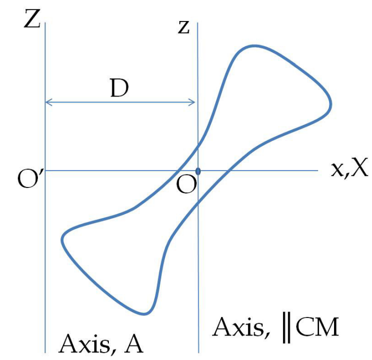


Figure 9.25: Parallel axis theorem. The origin O is at the CM.

In calculating the standard formulas for the components of the moments of inertia for regular shapes, we usually place origin at the center of mass (CM) of the body. Often, the axis of rotation of interest in a particular problem does not go through the CM. The parallel axis theorem relates the moment of inertia corresponding to two parallel axes, one of which passes through the center of mass and the other through an arbitrary point in space.

To illustrate the relation we seek, we calculate I_{zz} about the two z-axes, one for a coordinate system with origin at the CM and the other with origin at some point O' on the x-axis as shown in Fig. 9.25. Let the the two z-axis be separated by a distance D. Let us denote the two moments of inertial by I_{zz} and I_{ZZ} respectively. Then we will prove that,

$$I_{ZZ} = I_{zz} + M D^2. \tag{9.45}$$

Proof

We will prove the result by writing out I_{ZZ} from the definition of this component of moment of inertia and then use the following relation between the coordinates of any point in the two systems.

$$X = x + D \tag{9.46}$$
$$Y = y \tag{9.47}$$
$$Z = z \tag{9.48}$$

Hence,

$$\begin{aligned} I_{ZZ} &= \int \left(X^2 + Y^2\right) dm \quad \text{(Definition)} \\ &= \int \left[(x+D)^2 + y^2\right] dm \quad \text{(Eqs. 9.46 - 9.48)} \\ &= \int \left(x^2 + y^2\right) dm + \int D^2 dm + \int 2Dx dm \end{aligned}$$

The first term is I_{zz}, the second term evaluates to MD^2, and the last integral is zero since the origin of $Oxyz$ is the CM of the body.

$$\int x dm = 0 \quad \text{(using definition of } X_{cm}\text{)}$$

Therefore,

$$I_{ZZ} = I_{zz} + MD^2.$$

Often, we state this result in more "practical" language. The quantity I_{ZZ} is called the moment of inertia about axis A that does not pass through the CM. The quantity I_{zz} is called the moment of inertia about an axis parallel to axis A that passes through the CM. Using this nomenclature, we can write our result as

$$\boxed{I_{\text{Axis}} = I_{\|\text{CM}} + MD^2}$$

Example 9.2.4. Moment of inertia about an axis through the end of a rod.

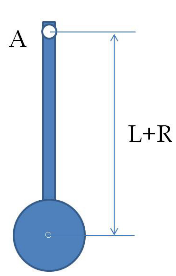
As an application of the parallel axis theorem let us work out the I_{zz} of a rod about an axis through its end. We have already calculated this I_{zz} above. Here, we show that from the answer for I_{zz} about the axis through the center, which was equal to $\frac{1}{12}ML^2$, we can find I_{zz} about the axis at the end.

To find I_{zz} about the axis at the end, we add MD^2 to the I_{zz} about the axis through the center, where D is the distance between the two parallel axes. In the present case, the distance between the axes is $\frac{1}{2}L$. Therefore from the parallel axis theorem, I_{zz} about the axis at the end of the rod is

$$I_{\text{Axis}} = I_{\|\text{CM}} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2,$$

which is the answer we had found by a direct calculation.

Example 9.2.5. Moment of inertia of a spherical ball. We have found the moment of inertia of a spherical ball about an axis through the center of the ball. The answer was $I_{zz} = \frac{2}{5}MR^2$ for a ball of mass M and radius R. Suppose the ball is hung by attaching it to a light rod of length L and suspending the rod from the other end. Find the moment of inertia about an axis perpendicular to the plane shown in Fig. 9.26 with origin at the suspension point A.



Solution. Using the parallel axis theorem, the moment of inertia about an axis through A can be written in terms of the moment of inertia about another axis through the center of mass (CM) of the ball as long as the two axes are parallel to each other. Here, the distance between the two axes are $D = L + R$, therefore

$$I_{\text{Axis}} = I_{\|\text{CM}} + MD^2 = \frac{2}{5}MR^2 + M(L+R)^2.$$

RADIUS OF GYRATION

The moment of inertia has dimensions of mass times distance squared. If you have a single mass M at a distance b from the axis of rotation, then its moment of inertia about that axis would be simply Mb^2. Note: By the distance of a point from a line we mean the length of a line that is perpendicular from the point to the line (see Fig. 9.27).

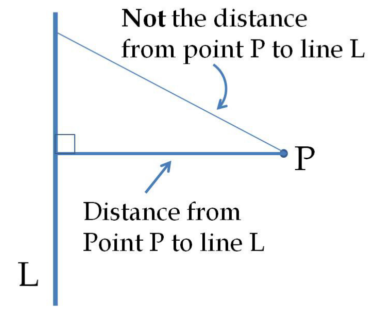


Figure 9.27: Meaning of distance between a point and a line.

For more complicated objects, different points of the object will be at different distances from the axis, and as a result the formula for I_{zz} or any other component of the moment of inertia that will contain a numerical pre-factor multiplying the mass and a length squared.

The length in the moment of inertia formulas contains the information about the location of the axis relative to the body and the dimensions of the body. For instance, I_{zz} of a sphere with the origin at the center of the sphere is equal to $\frac{2}{5}MR^2$, or that of a sphere with the origin at the edge of the sphere is $\frac{7}{5}MR^2$.

We now ask: suppose you replace the extended object by a point mass, what distance will you have to place this point mass so that the moment of inertia of the point mass will be equal to the moment of inertia of the extended body? This distance is called the radius of gyration R_G of the body.

For instance, the radius of gyration of a sphere of radius R for the rotation about an axis through the center of the sphere can be obtained by equating I_{zz} of the point mass at a distance R_G from the axis to the I_{zz} of the sphere with respect to the same axis.

$$MR_G^2 = \frac{2}{5}MR^2,$$

which gives the following for the radius of gyration of sphere

$$R_G(\text{sphere}) = \sqrt{\frac{2}{5}}\, R.$$

The same body may have different radii of gyration depending on the axis. For instance, $R_G = \frac{1}{\sqrt{2}}R$ for a disk of radius R corresponding to I_{zz} and $R_G = \frac{1}{2}R$ corresponding to I_{xx}.