7.4 CALCULATIONS OF CENTER OF MASS

7.4.1 Examples of Discrete Masses

The calculation of CM of discrete masses is based on applying the definition given in Eq. 7.24 to the given system as the following examples illustrate.

Example 7.4.1. CM of a two-mass system. Two blocks of masses 10 kg and 20 kg are attached at the ends of a 1-meter rod of negligible mass. Where is the CM located?

Solution. The calculation of the CM of a multiparticle system consisting of discrete masses is usually analytically by choosing a convenient set of Cartesian axes. In the present case, since we have only two masses, we place one of the masses at the origin and the other mass along the x-axis. Suppose we place the 20-kg block at the origin and the 10-kg block at $x = 1$ m.

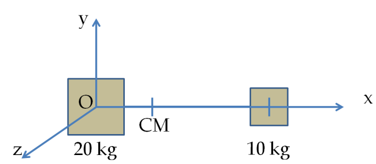


Figure 7.10: Example 7.4.1.

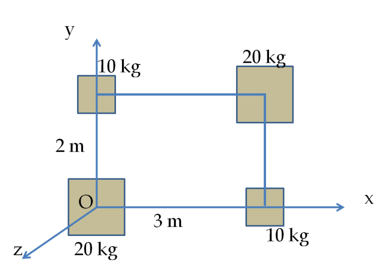
Now, we use the x-component of Eq. 7.24 to find the x-coordinate of the CM as

$$X_{\text{cm}} = \frac{(15\text{ kg})(1\text{ m}) + (20\text{ kg})(0)}{30\text{ kg}} = \frac{1}{3}\text{ m}.$$

Therefore, the CM is located $\frac{1}{3}$ m from the 20-kg block towards the other block. Note that the units of mass appear both in the numerator and the denominator, therefore it is not necessary to change the unit of mass in these calculation as long as you keep the same unit for all the masses.

Example 7.4.2. Masses at the corners of a rectangle. Find the location of the CM of the four masses given in Fig. 7.11.

Solution. Again, we perform the calculations analytically. Let us choose Cartesian coordinates so that the centers of the four masses fall in the xy-plane with one of the masses at the origin as indicated in the figure. In this coordinate system, the coordinates of the masses are organized in the following table.



Mass	x (m)	y (m)	z (m)
20 kg	0	0	0
10 kg	3	0	0
20 kg	3	2	0
10 kg	0	2	0

Here the z-coordinate is given for the sake of completeness since based on the choice of the coordinate system, it was already clear that the z-coordinates of all masses will be zero, and hence the z of CM will be zero also.

The x and y-coordinates of the CM are:

$$X_{\text{cm}} = \frac{10 \times 3 + 20 \times 3}{60} = 1.5 \text{ m}$$
$$Y_{\text{cm}} = \frac{10 \times 2 + 20 \times 2}{60} = 1.0 \text{ m}$$

This says that the CM of the system is actually at the center of the rectangle. This makes sense if you pair up the masses: the CM of the two 20-kg blocks will be at center and the CM of the two 10-kg blocks will also be at the center. The CM calculations can be done in steps: you can find CM of some parts of the total system and replace those parts by one point mass at the CM of those parts, successively simplifying the whole system.

In this example, we could have found the CM of the two 20-kg blocks and replace them by a 40-kg point mass at the center of the diagonal line joining them. Similarly, we could have replaced the two 10-kg blocks by one point mass of mass 20 kg at the center of the other diagonal. Since the centers of the two diagonals are at the same point we would obtain the center of mass at the center. This happened here because of the symmetry in the problem. Suppose the four masses were different, say 10 kg, 20 kg, 30 kg and 40 kg, would the CM be still at the center? The answer is no. Check it out.

Example 7.4.3. Masses at the corners of a triangle. Find the location of the CM of the three equal masses placed at the corner of a triangle.

Solution. Suppose we place masses m at the corners of a triangle as shown in Fig. 7.12. Now, we replace the two masses on the base by a $2m$ point particle at the center of that side of the triangle. This leaves a two mass system of masses $2m$ and m along the dashed line in the figure. The CM of $2m$ and m system is at a distance $\frac{1}{3}^{rd}$

of the distance between the two masses as measured from the $2m$. Therefore, the CM of the three-mass system is at the centroid point of the triangle as shown in the figure. Again, if the masses were different, then the CM will not be at the **centroid**. For instance, if the mass at the top vertex was $2m$, then the CM will be half-way along the dashed line and not at the centroid.

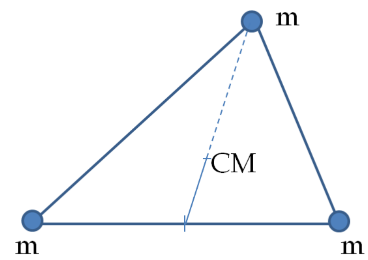


Figure 7.12: Example 7.4.3.

7.4.2 Examples of Continuous System

Most of the time we will be concerned with regular solids of uniform density whose CM can be easily found by inspection and an appeal to symmetry. We will show by explicit calculations that these intuitive feelings are indeed correct.

The general strategy of finding the center of mass of a continuous body is to first discretize the body by conceptually "breaking" it up into small cells. The shapes of the cells are arbitrary and are chosen to exploit the symmetry or other simplifying features of the body. Each cell is then replaced by its mass Δm at the centers of the cells. This process converts the original continuous body problem into an equivalent system of point masses, where we use the procedure of CM for point masses to obtain the CM of the original body. With infinitesimal cells, the sum becomes integral over the body. In the following examples we will demonstrate the use of this general strategy.

Example 7.4.4. CM of a rod of uniform density. The simplest system of continuous mass is a uniform rod of length L and area of cross-section A. Where is the CM of the rod located?

Solution. Let M be the mass of the rod. The density of the rod ρ is then given by

$$\rho = \frac{M}{AL}$$

In calculations concerning a uniform rod, the area of cross-section usually plays a passive role and it is useful to define another density, called **linear density**, which is mass per unit length, denoted by μ.

$$\mu = \frac{M}{L},$$

which is related to the mass per unit volume by

$$\mu = \rho A.$$

To implement this strategy for a uniform rod, let us place the rod along x-axis with one end at the origin at the other end at $x = L$ as shown in Fig. 7.13.

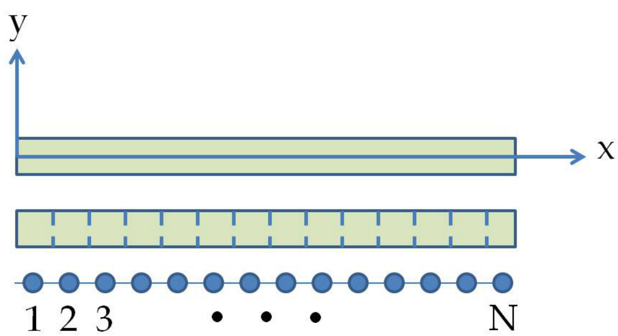


Figure 7.13: Calculating CM of a rod by partitioning the rod into smaller cells.

If we divide the rod into N cells, each of length $\Delta x = L/N$, then the procedure gives us a discrete system of masses m_1, m_2, $\cdots$, m_N at the following places.

$$\begin{aligned} &m_1 \text{ at } x_1 = \Delta x/2 \\ &m_2 \text{ at } x_2 = (3/2)\Delta x \\ &\quad\vdots \\ &m_N \text{ at } x_N = (L - 1/2\Delta x)\Delta x \end{aligned}$$

Therefore an approximate formula for X_{cm} will be given by applying definition of CM for discrete masses to m_1, m_2, $\cdots$, m_N.

$$X_{\text{cm}} = \frac{1}{M}\left[m_1\frac{1}{2}\Delta x + m_2\frac{3}{2}\Delta x + \cdots + m_N\left(L - \frac{1}{2}\Delta x\right)\right]. \tag{7.33}$$

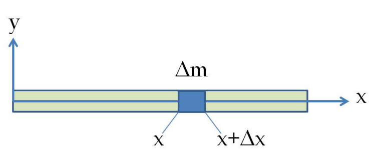


Figure 7.14: Calculating CM of a rod - infintesimals.

As the cell size is made progressively smaller, the limit of the sum gives the exact value of X_{cm}.

The path to infinitesimal cell sizes is better described by the following procedure, we will call **the method of infinitesimals**. Consider a representative cell between x and $x+\Delta x$ is shown in Fig. 7.14. The mass Δm in this cell can be written using the linear density μ as

$$\Delta m = \mu \Delta x. \tag{7.34}$$

Then, the sum over cells given in Eq. 7.33 can be formally written as

$$X_{\text{cm}} = \frac{1}{M}\sum_{cells} x\Delta m,$$

In the limit of infinitesimal cells, this sum becomes an integral.

$$\boxed{X_{\text{cm}} = \frac{1}{M}\int_{rod} x\ dm.} \tag{7.35}$$

The integral in Eq. 7.35 is often called a "conceptual" integral since we have not actually specified the limits for the integration variable. Integration over rod is better thought of in terms of the length of the rod. If we can convert the conceptual integral over dm into an integration over the x-coordinate of the cells, then we can specify the limits of integration of the x variable more easily, as from $x = 0$ to $x = L$. We change from the "conceptual integration variable" m to the operational variable x by replacing dm using the infinitesimal form of Eq. 7.34.

$$dm = \mu \, dx, \tag{7.36}$$

which gives the following integral to evaluate for the X_{cm}.

$$X_{\text{cm}} = \frac{1}{M} \int_0^L x\mu \, dx. \tag{7.37}$$

If mass density is constant throughout, as is the case in this example, the density $\mu = M/L$ will come out of the integral and the integral would be easily evaluated and simplified to give the X_{cm}.

$$X_{\text{cm}} = \frac{L}{2} \quad \text{(Uniform rod)} \tag{7.38}$$

Of course, $Y_{\text{cm}} = 0$ and $Z_{\text{cm}} = 0$ here if the x-axis goes right through the middle of the rod. Therefore, the CM of the uniform rod is at the mid-point in the rod as expected.

Example 7.4.5. CM of a triangle of uniform density

Consider a plate of uniform density and uniform thickness cut in the shape of right-angled triangle with base b and height h. Where is the center of mass?

Solution. Let thickness of the plate be t. Along the thickness of the plate, the CM will be in the plane half-way between the two faces of the plate. So, we only need to find the center of mass coordinates on the triangular shape surface.

Let us place the triangular face in the xy-plane as shown in Fig. 7.15. We will proceed with the method of infinitesimals and will try to find the definite integral for the X_{cm} first. Once we complete our calculation for X_{cm} we will guess the answer for Y_{cm}. For the calculation of X_{cm}, the "conceptual integral" is

$$X_{cm} = \frac{1}{M} \int_{plate} x \, dm. \tag{7.39}$$

Here dm is the mass of an element in a cube-shaped element of thickness t and area $dxdy$ located at the point (x, y) in the xy plane. The

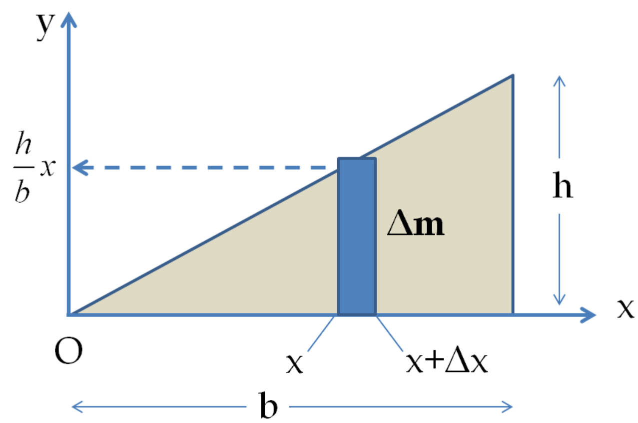


Figure 7.15: Calculating CM of right-angle triangle.

volume of the element is $tdxdy$. Let ρ be the density of the material, then we have

$$dm = \rho\ tdxdy \tag{7.40}$$

Therefore Eq. 7.39 can be transformed in operational form, where we need the limits of x and y variables that go over the plate.

$$X_{cm} = \frac{\rho\ t}{M}\int_{plate} x\ dxdy. \tag{7.41}$$

The integral in Eq. 7.41 is a double integral: there is an integration over x and another one over y, but the two integrations are linked since we must restrict the points to the points on the surface of the triangular plate. Suppose we do the integration over y first, keeping fixed at some value x, then we see from Fig. 7.15 that the limit will be from $y = 0$ to $y = y_{\max}$, where $y_{\max}$ is the value of y at the hypotenuse. Writing the equation of the line for the hypotenuse we find that

$$y_{\max} = \frac{h}{b}x.$$

After we have done the integration over y, the value of x goes over the entire base to include all the elements of the triangle. Therefore, Eq. 7.41 becomes

$$X_{\rm cm} = \frac{\rho\ t}{M}\int_0^b dx\left[x\int_0^{y_{\max}} dy\right]. \tag{7.42}$$

Doing the integration over y leaves x integral to do in the following

$$X_{\rm cm} = \frac{\rho\ t}{M}\frac{h}{b}\int_0^b x^2 dx. \tag{7.43}$$

Finally, we obtain the following answer for X_{cm} which can be simplified.

$$X_{\rm cm} = \frac{\rho\ t}{M}\frac{h}{b}\frac{b^3}{3}. \tag{7.44}$$

Note that the volume of the plate is $t\times$ area of the surface, or $tbh/2$, which gives $M = \rho \times \text{Volume} = \rho \times tbh/2$. Therefore,

$$X_{\text{cm}} = \frac{2}{3}b. \tag{7.45}$$

This says that X_{cm} is two-third of the way along the base from the tip, or one-third of the distance from the edge opposite to the tip.

To obtain the Y_{cm} we do not need to do another calculation. We can make use of the result for the X_{cm} calculation, which says that that the CM in this direction will be located $\frac{2}{3}^{rd}$ from the tip or $\frac{1}{3}^{rd}$ from the side. In the choice of coordinates in Fig. 7.15, the Y_{cm} is from the edge. Therefore,

$$Y_{\text{cm}} = \frac{1}{3}h. \tag{7.46}$$

Example 7.4.6. CM of a composite system. CM of composite systems are found by using the CM of individual parts. Where is the CM of a composite system of a rectangle of mass M_1 and a right angle triangle of mass M_2, both of uniform mass density of dimensions and separation shown in Fig. 7.16.

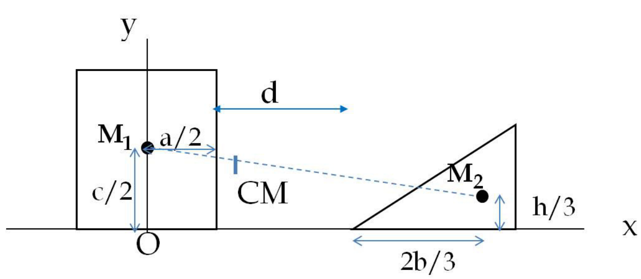


Figure 7.16: CM of a composite system.

Solution. To find the CM of a composite system the best strategy is to replace each object with a point masses at its center of mass. This process turns a complex problem into a problem of point mass. In the present case, the problem turns into a mass M_1 at $(0, c/2)$ and M_2 at $(a/2 + c + 2b/3, h/3)$. Therefore, the coordinates of the CM are

$$X_{cm} = \frac{M_2}{M_1 + M_2}\left(\frac{1}{2}a + d + \frac{2}{3}b\right)$$
$$Y_{cm} = \frac{1}{M_1 + M_2}\left(\frac{1}{2}M_1 c + \frac{1}{3}M_2 h\right)$$