2.2 GEOMETRICAL VIEW OF VECTORS

2.2.1 What is a Vector?

Vectors are physical quantities that have a direction and a magnitude, and add according to the parallelogram law of addition. Common examples of vectors in mechanics are: (1) the position vector - location of an object in relation to a reference point, (2) the displacement vector - change in position in the course of time, (3) the velocity vector - rate of change in the position vector, (4) the acceleration vector - rate of change of the velocity vector, (5) the force vector, (6) the momentum vector, and (7) the angular momentum vector.

From the list of vectors, it would appear that everything in mechanics is a vector, but that is not the case. Indeed, many important physical quantities are not vectors. For instance time, speed, energy and mass are not vectors. They are scalar quantities. Since vectors obey special rules of addition and multiplication, which are very different than the rules for scalars, you will need to keep track of what is a vector and what is not, so that you will use the appropriate rule. If you use scalar arithmetic rules for vectors, you will definitely get wrong results.

We will use an arrow over letters to denote a vector, for example $\vec{F}$ for a force vector, $\vec{v}$ for velocity vector, $\vec{a}$ for acceleration vector, etc. Occasionally, we will denote a vector by a bold-faced letters, e.g. in **A**. The magnitude of a vector will be denoted by either vertical bars around the vector symbol or by the same symbol without the arrow over it. For instance, the magnitude of the vector $\vec{F}$ will be denoted by either $|\vec{F}|$ or simply F.

Notation: Vectors will have an arrow over the symbol and their magnitude will be denoted by the same symbol without the arrow.

An arrow, whose length corresponds to the magnitude of the vector and whose direction refers to the direction of the vector in space, represents a vector geometrically. The length of the arrow requires a scale so that two arrows for the same physical quantity can be compared as shown in Fig. 2.3.

If two vectors have the same magnitude and the same direction, they are equal no matter where on the paper you happen to draw the two vectors. If two vectors have the same magnitude but different directions, they are not equal. For instance, a displacement of 200 m towards West and a displacement of 200 m East are not equal even though they have the same magnitude since they point in different

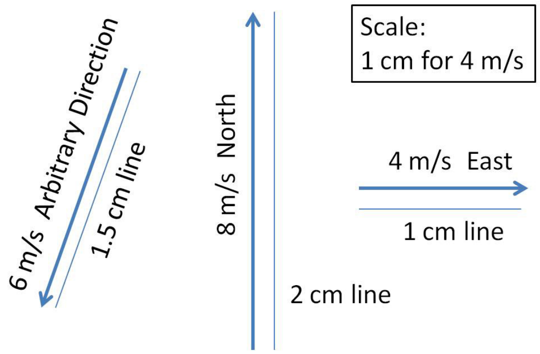


Figure 2.3: Vectors of different magnitudes are drawn using the same scale regardless of their directions.

directions.

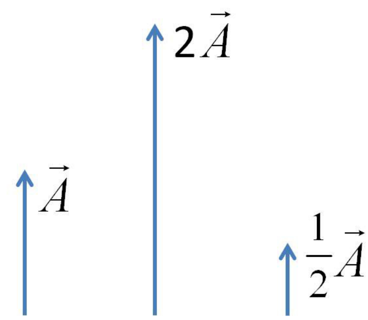


Figure 2.4: Multiplying a vector by positive real numbers gives vectors of different lengths and same direction as the original vector.

2.2.2 Multiplication of a Vector by a Scalar

A scalar is any real number, positive, negative, or zero. The effect of multiplying a vector by a scalar is very different for positive and negative scalars. Multiplication of a vector $\vec{A}$ by a scalar s gives a new vector $\vec{B} = s\vec{A}$. If s is a positive scalar, then $\vec{B}$ has the same direction as $\vec{A}$ but its magnitude is s times the magnitude of $\vec{A}$. That is, $|\vec{B}| = s|\vec{A}|$. If $s > 1$ then $|\vec{B}|$ is s times greater and if $s < 1$ then $|\vec{B}|$ is s times smaller. Note that dividing by the scalar s is equivalent to multiplying by $1/s$, so the division is not a different operation than multiplication. We can obtain a vector of any length in the same direction as the original vector by multiplying it by an appropriate positive real number.

Multiplication of a vector $\vec{A}$ by a negative number $(-s)$ [minus s] gives a vector $\vec{C}$ that is not only s times the original vector but also its direction is opposite to the direction of $\vec{A}$. For instance, when velocity vector of 10 m/s pointed North is multiplied by -2, you get another velocity vector that has a magnitude 20 m/s and pointed South.

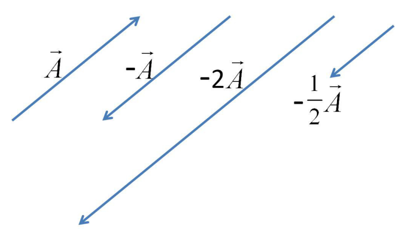


Figure 2.5: Multiplying a vector by a negative numbers gives vectors in the opposite direction.

2.2.3 Unit Vector

Evidently dividing a vector $\vec{A}$ by its magnitude $|\vec{A}|$ one should obtain a vector whose length is 1 with no units and the same direction as the direction of vector $\vec{A}$. This vector is called **unit vector** in the direction of $\vec{A}$. We will denote unit vectors obtained this way by placing a karat over the symbol of the original vector, $\hat{A}$. Thus, unit vector $\hat{A}$ is

$$\boxed{\hat{A} = \frac{\vec{A}}{|\vec{A}|}.} \tag{2.1}$$

Unit vectors are very useful for constructing vectors of arbitrary lengths. For instance, if we want a displacement vector of length 5 cm in the direction of some unit vector $\hat{u}$, then all we need to do is to multiply $\hat{u}$ by 5 cm, giving the desired vector (5 cm)$\hat{u}$. Now, a force vector of length 10 N in the same direction as $\hat{u}$ will be (10 N)$\hat{u}$. You can see that unit vectors have universal applicability regardless of how you construct them. They are holders of the information about directions in space. As you might have guessed it, there are infinite number of unit vectors, one for each direction in space.

2.2.4 Addition of Vectors

Vectors add differently than simple numbers as we have seen in the addition of two displacement vectors above. For instance, two vectors of magnitude 1 units each can add to yield a vector whose magnitude can be any number between zero and two units depending on the relative orientation of the two. That is, 1 + 1 for vectors can be anywhere between 0 and 2. That is why you need to be extra careful when dealing with vectors.

Let us take another example of addition of vectors. This time, I will use forces on a ring to illustrate the addition process of vectors. Suppose you pull a rigid ring by two forces, $\vec{F}_1$ and $\vec{F}_2$, in opposite directions and with equal strengths [represented in the figure with equal-length arrows] applied at the two opposite ends of the ring as shown in Fig 2.6. It would not come as surprise to you that the ring is not pulled in either direction. This is because the net force on the ring is zero because the vector sum also takes into account the directions of the vectors.

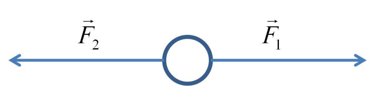


Figure 2.6: Two forces of equal magnitude but opposite directions acting on a rigid ring result in net zero force.

Now, suppose you apply three forces on the ring as in Fig. 2.7(a). This time, you will find that, if the three forces form sides of a triangle as shown in the Fig. 2.7(b), then the ring is not pulled in any direction. On the other hand, if the forces do not form a triangle, then there will be a net pull on the ring in some direction.

We interpret the experiment depicted in Fig. 2.7 as saying that the sum of two of the forces has exactly the same magnitude as the third force but acts in the opposite direction to it. Only then, the sum of the three will give a net zero force on the ring. The triangle diagram tells us about the addition rule of two vectors: if vectors $\vec{F}_1$

Check for yourself if the sum of vectors $\vec{F}_1$ and $\vec{F}_3$ cancels

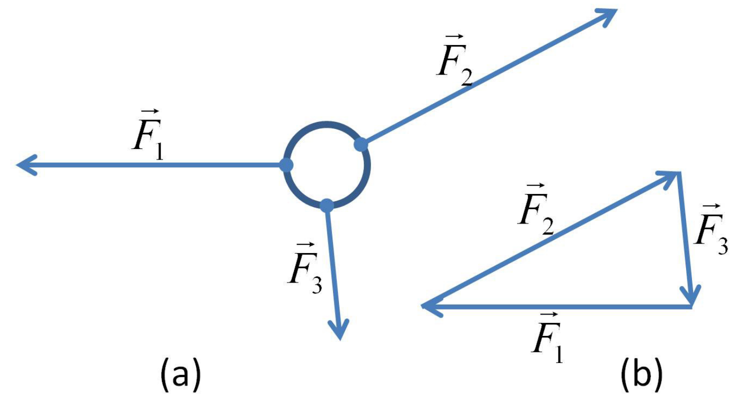


Figure 2.7: (a) Forces on a ring whose net force is zero. (b) When the three forces are balanced, they form sides of a triangle and vectors make a round trip in the triangle.

and $\vec{F}_2$ are non-collinear, they form adjacent sides of a parallelogram when the tail of one is placed at the tip of the other, and the sum of vectors $\vec{F}_1$ and $\vec{F}_2$ will be the vector along the diagonal of that parallelogram as shown in Fig 2.8.

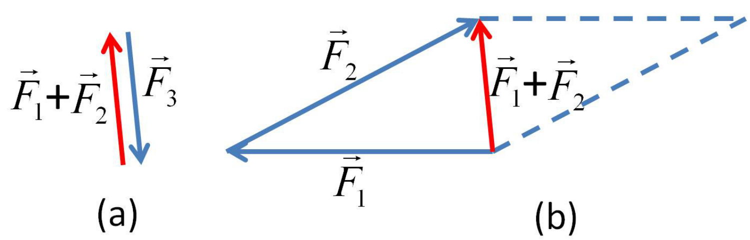


Figure 2.8: Balanced forces on the ring in Fig. 2.7. (a) To cancel force $\vec{F}_3$, the sum of forces $\vec{F}_1$ and $\vec{F}_2$ must be in the opposite direction to $\vec{F}_3$ and have the same magnitude. (b) The sum of two forces according to the parallelogram law of addition - the tail of vector $\vec{F}_2$ is placed at the tip of vector $\vec{F}_1$ and the sum is read from the tail of vector $\vec{F}_1$ to the tip of $\vec{F}_2$.

Our procedure of addition of vectors parallels the procedure Isaac Newton gave in his masterpiece Principia Mathematica regarding the addition of forces:

"A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately."

The same rule of addition applies to all vectors. Geometrically it amounts to drawing the second vector at the tip of the first vector and the method is known as **tip-to-tail method** of adding vectors. For instance, to add vector $\vec{B}$ to vector $\vec{A}$, i.e. to obtain $\vec{A}+\vec{B}$, draw vector $\vec{A}$ and then, draw vector $\vec{B}$ by placing the tail of vector $\vec{B}$ at the tip of the arrow for vector $\vec{A}$. The arrow from the tail of $\vec{A}$ to the tip of $\vec{B}$ is the sum. This rule is called the parallelogram law of addition because the two vectors being added form the sides of a parallelogram and the sum is one of the diagonals of the parallelogram.

You will find that, vector $\vec{B}+\vec{A}$ obtained by placing the tail of vector $\vec{A}$ at the tip of vector $\vec{B}$ is equal to the vector $\vec{A}+\vec{B}$ obtained by placing the tail of $\vec{B}$ to the tip of $\vec{A}$ as shown in Fig. 2.9. Thus, adding two vectors can be done in any order, i.e., vector addition is commutative.

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}.$$

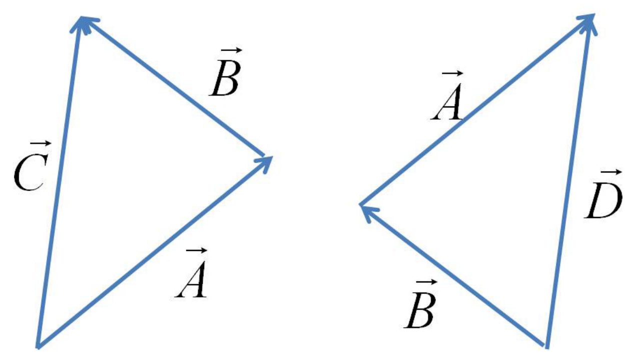


Figure 2.9: $\vec{C}=\vec{A}+\vec{B}$ and $\vec{D}=\vec{B}+\vec{A}$ are equal.

The parallelogram law of addition can be extended to the addition of more than two vectors by simply placing each additional vector starting from the tip of the last vector added as illustrated in Fig. 2.10.

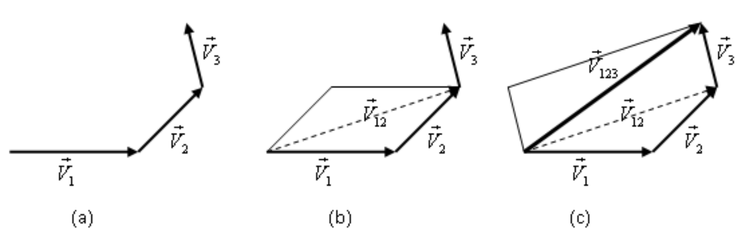


Figure 2.10: Addition of more than one vector according to the parallelogram law of addition id done by placing each successive vector at the tip of the last vector added. (a) Vectors to be added, (b) Sum $\vec{V}_{12}$ of $\vec{V}_1$ and $\vec{V}_2$, and (c) Sum $\vec{V}_{123}$ of $\vec{V}_{12}$ and the third vector V_3.

2.2.5 Subtraction of Vectors

Subtracting a vector $\vec{B}$ from another vector $\vec{A}$ can be turned into adding the negative of vector $\vec{B}$ to $\vec{A}$.

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right).$$

Therefore, the geometrical procedure to obtain $(\vec{A} - \vec{B})$ will be to draw vector $\vec{A}$ and at the tip of vector $\vec{A}$ draw the vector $(-\vec{B})$, which is obtained from vector $\vec{B}$ by reversing the direction of the later. From the tail of $\vec{A}$ to the tip of $(-\vec{B})$ is the vector $(\vec{A} - \vec{B})$. You would obtain the same result for $(\vec{A} - \vec{B})$ by drawing $\vec{B}$ such that its tip meets the tip of $\vec{A}$. Then $(\vec{A} - \vec{B})$ is from the tail of $\vec{A}$ to the tail of $\vec{B}$.

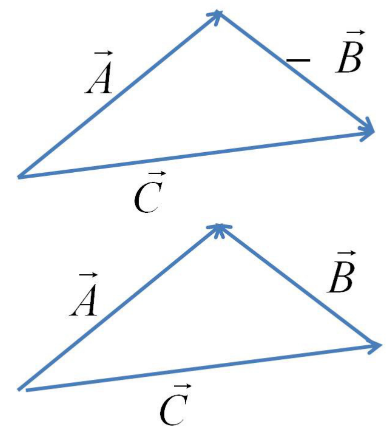


Figure 2.11: Two ways of obtaining $(\vec{A} - \vec{B})$: (a) From tail of $\vec{A}$ to the tip of $(-\vec{B})$, and (b) From tail of $\vec{A}$ to the tail of $\vec{B}$.

2.2.6 Vector Equations and Polygons

Recall that when you walk around a polygon and end up at the starting place, then your direct distance from the starting place to ending place is zero. This means that the net displacement must be a zero vector, also called the null vector. Thus, if you add vectors by placing the tail of one vector at the tip of another vector and the result is a closed polygon, the net sum of all vectors will be zero vector as illustrated in Fig. 2.12.

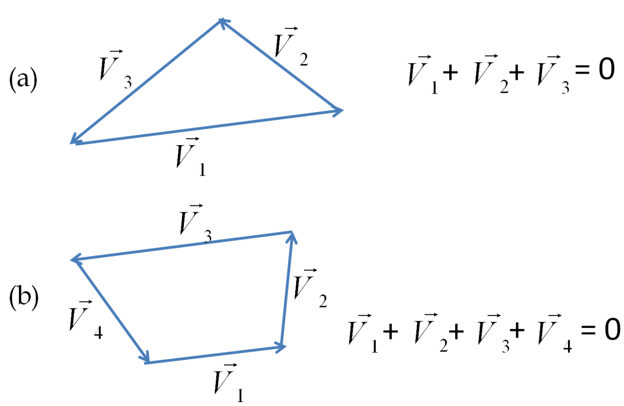


Figure 2.12: Polygon of vectors arranged tip-to-tail add up to zero or null vector.

Therefore, for any polygon of vectors we can write a vector equation. In Fig. 2.12(a), the three vectors give rise to the vector equation

$$\vec{V}_1 + \vec{V}_2 + \vec{V}_3 = 0,$$

which is the same as

$$\vec{V}_1 + \vec{V}_2 = (-\vec{V}_3),$$

which shows that the sum of the two vectors $\vec{V}_1$ and $\vec{V}_2$ is equal to the vector $(-\vec{V}_3)$, i.e. another vector that has the same magnitude as vector $\vec{V}_3$ but has an opposite direction to that of vector $\vec{V}_3$. Similarly, Fig. 2.12(b) corresponds to the vector equation

$$\vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4 = 0.$$

You will find many vector equations in this textbook. For example, the second law of motion is a vector equation $\vec{F} = m\vec{a}$, which says that the force vector obtained by vector addition of all the forces on the object of mass m is equal to the vector obtained by multiplying the acceleration vector with the mass.

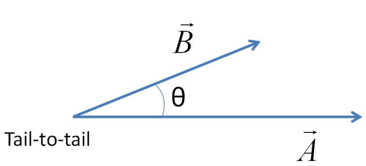
2.2.7 Multiplication of a Vector with Another Vector

You have already learned what happens when you multiply a vector by a real number (also called a scalar). How do you multiply a vector by another vector? A vector has a numerical part in the magnitude and a non-numerical part in the direction. For instance, a velocity vector can be 3 m/s East and another velocity can be 4 m/s Up. Granted you can think of a way to multiply 3 and 4, but how would you multiply East and Up?

It turns out that two types of multiplication are found in physics. (1) In one type of multiplication, called the scalar product, the product of two vectors is actually a scalar number that depends upon the magnitudes of the two and the angle between them. We call this type of multiplication a scalar product. Examples of this type of multiplication are work and energy. (2) In the other type of multiplication between two vectors, called the vector product, the product is another vector. Examples of this vector products are torque and angular momentum.

2.2.8 Scalar Product Or Dot Product

The **scalar product** of two vectors $\vec{A}$ and $\vec{B}$, often called the **dot product**, is defined as the product of three quantities: (1) magnitude of the vector $\vec{A}$, (2) magnitude of the vector $\vec{B}$ and (3) cosine of the angle θ between the two vectors when they are drawn tail-to-tail. The angle θ is the smaller of the two angles that the two vectors make with each other. We denote the scalar product by placing a dot between the two vectors as in $\vec{A} \cdot \vec{B}$. The result of multiplying the three numbers can be positive, zero or negative real number.



$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta.} \tag{2.2}$$

Geometrically, you can see that $|\vec{B}| \cos \theta$ is the length of the projection of vector $\vec{B}$ on vector $\vec{A}$ as shown in Fig. 2.14. The projection of $\vec{B}$ over $\vec{A}$ is obtained by drawing the two vectors so that their tails are at the same point, and then drawing a perpendicular line from the tip of $\vec{B}$ onto the line of $\vec{A}$. From the right-angled triangle so-formed, you can immediately see that the projection of $\vec{B}$ on $\vec{A}$ will have the value $|\vec{B}| \cos \theta$.

Therefore, you could state the scalar product in another way: the scalar product is equal to the product of $|\vec{A}|$ and the projection of vector $\vec{B}$ on vector $\vec{A}$. To draw a projection, sometimes, you may need to extend the vector in one or the other direction. If the projection of vector $\vec{B}$ falls on the opposite side of the direction of $\vec{A}$, then we say that the projection has a negative value, otherwise projection will have a positive value. If vector $\vec{B}$ is perpendicular to vector $\vec{A}$, then the projection will be zero.

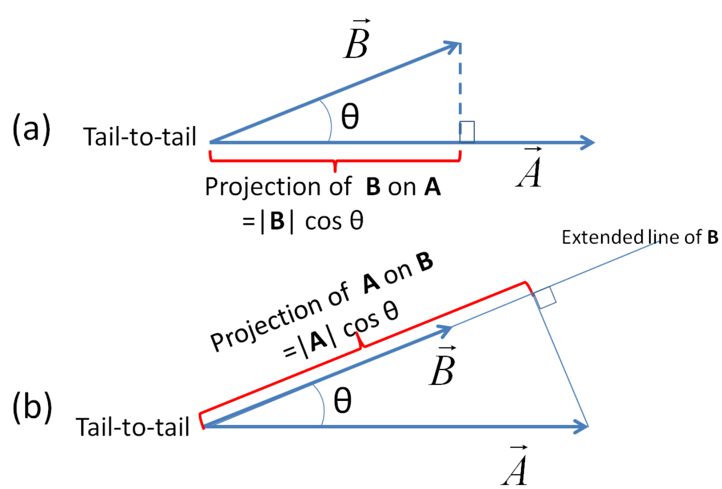


Figure 2.14: Projection of one vector over another. For two vectors, you have two projections: (a) the projection of $\vec{B}$ over $\vec{A}$ and (b) the projection of $\vec{A}$ over $\vec{B}$. Sometimes, you need to extend the line of one vector to draw a projection as shown in (b). If the projection from one vector falls on the extension on the opposite side, the projection has a negative value as shown in Fig. 2.15.

Equivalently, you can draw a projection of vector $\vec{A}$ on vector $\vec{B}$. The length of this projection will be $|\vec{A}| \cos \theta$, which we can multiply with the length of vector $\vec{B}$ to get the value of the scalar product. Therefore, we can state the scalar product in yet another way: the scalar product is equal to the product of $|\vec{B}|$ and the projection of vector $\vec{A}$ on vector $\vec{B}$.

When the vector upon which a projection is being sought needs to be extended backwards as shown in Fig. 2.15, you must be careful with signs. It is easier to work with the supplementary angle ϕ in the triangle formed by the projection of $\vec{B}$ onto the backward extended line of $\vec{A}$, but in the end, we are interested in the cosine of the other angle, the angle between the vectors, which is here θ. Therefore, we need to place a negative in front of the value $|\vec{B}|\cos\phi$.

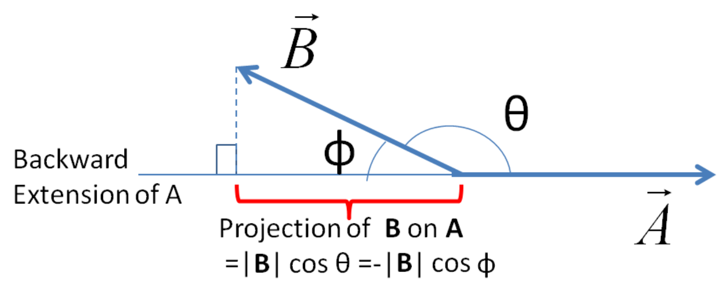


Figure 2.15: Projection of one vector over another sometimes requires extension of the line of action of the vector. If the projection from one vector falls on the extension on the opposite side, the projection has a negative value.

Note that if the dot product between two vectors is zero, then, either one or both of the vectors has a zero length, or the angle between them in ninety degrees since $\cos 90° = 0$.

$$\boxed{\text{If } \vec{A}\cdot\vec{B} = 0,\ \text{either } |\vec{A}| = 0,\ \text{or } |\vec{B}| = 0,\ \text{or } \theta = 90°.}$$

Another useful property of the scalar product occurs when you take the scalar product of a vector with itself. Since $\cos\theta = 1$ here, the scalar product is equal to the square of the magnitude of the vector.

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2.$$

In particular, the scalar product of a unit vector $\hat{u}$ is equal to 1 since the magnitude of a unit vector is 1 by definition.

$$\hat{u}\cdot\hat{u} = 1^2 = 1.$$

Based on the definition of the scalar product given above, you can prove the following algebraic properties of the scalar product.

1. Linearity: $\vec{A}\cdot(s\vec{B}) = s\vec{A}\cdot\vec{B}$, where s is scalar.
2. Distributive: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$.
3. Commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

2.2.9 Vector or Cross Product

The other product of interest to us is the vector product, which is often called the cross product. Unlike the result of the scalar product, the result of a vector product is another vector, whose magnitude depends upon magnitudes of the two vectors and the angle between them, and whose direction depends on the orientations of the two vectors being multiplied. So, to define the vector product, we need rules for the magnitude as well as the direction for the product.

The **vector product** or the **cross product** between two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$, which we will represent by the symbol $\vec{C}$.

$$\vec{C} \equiv \vec{A}\times\vec{B}.$$

Rule for the magnitude of $\vec{C}$:

Let θ be the angle between the two vectors $\vec{A}$ and $\vec{B}$ when they are drawn with their tails at the same point, just as we have done when we defined the scalar product.

$$\boxed{|\vec{C}| = |\vec{A}||\vec{B}|\sin\theta.} \quad (2.3)$$

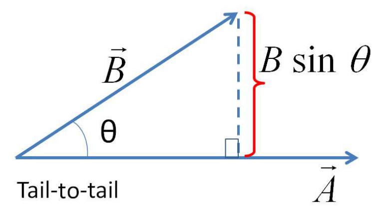


Figure 2.16

Note that $|\vec{B}|\sin\theta$ is the length of the perpendicular when you draw the projection of vector $\vec{B}$ onto vector $\vec{A}$ as shown in Fig. 2.16. Therefore, we see that $|\vec{A}||\vec{B}|\sin\theta$ is equal to the area of the parallelogram formed by the two vectors drawn so that they come out of the same point as illustrated in Fig. 2.17.

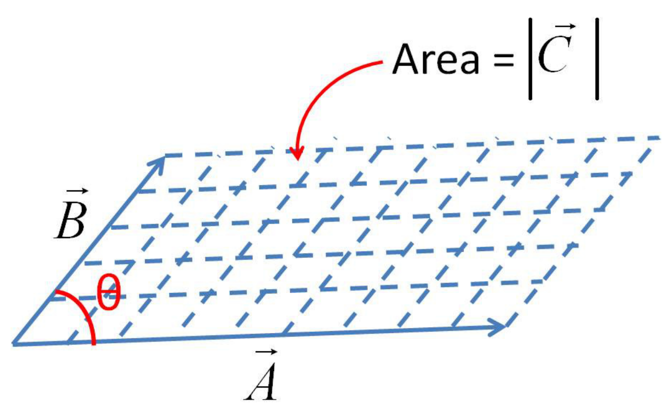


Figure 2.17: The magnitude of a vector product is equal to the area of the parallelogram formed by placing the two vectors on the sides of the parallelogram.

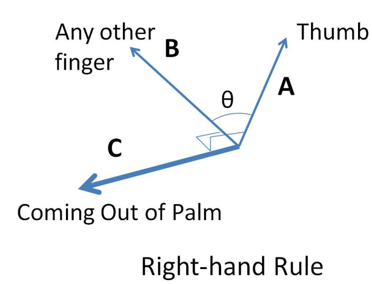


Figure 2.18: Directions of vectors in the Right Hand Rule.

Rule for the direction of $\vec{C}$:

When vectors $\vec{A}$ and $\vec{B}$ are drawn so that their tails are at the same point, they define a plane if they are non-collinear. [If the

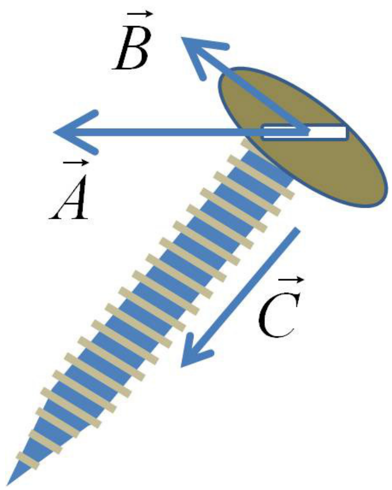


Figure 2.19: Right Hand Rule illustrated by the advancing right-handed screw.

vectors are collinear, their vector product will be zero since θ is either zero or 180° making $\sin\theta = 0$, so you do not need a rule for direction.] The direction of vector $\vec{C}$ is perpendicular to this plane. Now, there are two directions for perpendicular to a plane. The direction of $\vec{C}$ is obtained by a **Right Hand Rule**: if the thumb of your right hand points towards vector $\vec{A}$ and any of the other fingers point in the direction of vector $\vec{B}$, then the vector product $\vec{C}$ points in the direction coming out of palm as shown in Fig. 2.18.

Another useful way to determine the direction of $\vec{C}$ is that of a right-handed screw (Fig. 2.19). Place vectors $\vec{A}$ and $\vec{B}$ in the plane of the head of the screw so that when you turn the screw so that the screw advances forward when $\vec{A}$ rotates towards $\vec{B}$. Then the direction of the screw advancing is the direction of the vector product $\vec{C}$.

From the definition of a vector product you can conclude that the vector product of a vector with itself will be zero since the angle with itself is zero and $\sin\theta = \sin 0 = 0$. Another useful result of cross product is the cross product of two vectors that are perpendicular to each other. In that case $\sin\theta = \sin\ 90° = 1$. Therefore, the magnitude of a vector product of two mutually perpendicular vectors is simply the product of the magnitudes of the two vectors.

$$\vec{A}\times\vec{A} = 0$$
$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|,\ \text{if } \vec{A} \text{ is perpendicular to } \vec{B}.$$

By using the definition for the vector product you can prove the following algebraic properties of the vector product.

1. Linearity: $\vec{A}\times(s\vec{B}) = s\vec{A}\times\vec{B}$, where s is scalar.
2. Distributive: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$.
3. Anti-commutative: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

Of particular importance is the anti-commutative property: it shows that $\vec{A}\times\vec{B}$ is not equal to $\vec{B}\times\vec{A}$. That is, the order matters.