4.3 CONSTANT ACCELERATION

Motion with a constant acceleration is our most important example in this chapter. Throughout physics you will find many applications of the material presented in this section. An object moves with **constant acceleration** with respect to a reference point provided that both the magnitude and the direction of the acceleration vector do not change with time.

The analysis of the motion with constant acceleration is normally done in the analytic approach of vectors. The selection of axes for constant acceleration motion is aided by the constant direction of the acceleration. We point one of the Cartesian axes in the direction of the acceleration.

Note that direction of acceleration may or may not be same as direction of velocity: while velocity is in the direction of motion, acceleration is in the direction of force as we will learn in the next chapter.

Traditionally, different axes have served this role in different types of problems. For instance, for an object sliding down an incline, where the acceleration is pointed either down the incline or up the incline, one chooses the x-axis along the incline, but for a projectile motion whose acceleration is pointed vertically down, one chooses the y-axis to point vertically.

As an example, suppose, we choose the positive x-axis to be the direction of the constant acceleration of magnitude a. Then, the acceleration vector will have the following components.

$$\begin{aligned} a_x &= a \\ a_y &= 0 \\ a_z &= 0 \end{aligned}$$

The acceleration vector will have the following representation in this coordinate system.

$$\vec{a} = a\hat{u}_x,$$

where $\hat{u}_x$ is the unit vector pointed towards the positive x-axis. The velocity and position vectors may have all three components nonzero.

$$\begin{aligned} \vec{r} &= x(t)\hat{u}_x + y(t)\hat{u}_y + z(t)\hat{u}_z \\ \vec{v} &= v_x(t)\hat{u}_x + v_y(t)\hat{u}_y + v_z(t)\hat{u}_z \end{aligned}$$

Since the velocity and position vectors can be in any direction, a motion with constant acceleration does not have to be a one-dimensional motion unlike the motion with a constant velocity. In the following we will see that by choosing our axes appropriately, one of the coordinates can always be made to be zero. This will prove that the most general type of motion of an object with a constant acceleration is a planar motion, i.e. the motion will occur in a flat plane. We call such motions **planar motions**.

4.3.1 Planar Motion

When an object has constant acceleration, the components of velocity in the plane perpendicular to the direction of the acceleration cannot change. Therefore, the motion is confined to a plane. This plane contains the vectors of the net velocity and the acceleration.

For instance, suppose the acceleration is pointed towards the positive x-axis. Then, the y and z-components of the velocity cannot change. That means $v_y\hat{u}_y + v_z\hat{u}_z$ will be a constant vector, which allows us to choose new y- or z-axis in the direction of $v_y\hat{u}_y + v_z\hat{u}_z$ vector. Therefore, the original motion can be described completely in terms of the x-axis and the new y-axis.

Hence, in the case of a constant acceleration motion we can always choose axes such that the motion is completely confined to the xy-plane. That would simplify the position, velocity and acceleration vectors to the following:

$$\begin{aligned}\vec{r} &= x(t)\hat{u}_x + y(t)\hat{u}_y\\ \vec{v} &= v_x(t)\hat{u}_x + v_y(t)\hat{u}_y\\ \vec{a} &= a_x\hat{u}_x \quad (a_x \text{ constant})\end{aligned}$$

Therefore, we will have the following equations for the variation of the components of displacement and velocity.

$$\begin{aligned}x\text{-component:}\quad &(a)\ dv_x/dt = a_x;\\ &(b)\ dx/dt = v_x(t) \qquad (4.5)\\ y\text{-component:}\quad &(a)\ dv_y/dt = 0;\\ &(b)\ dy/dt = v_y(t) \qquad (4.6)\end{aligned}$$

Now, we ask: what do these equations tell us about the velocity and position at an arbitrary time if they are given at some time $t = t_0$?

Change in velocity

Let us work out the change in velocity first. From Eqs. 4.5(a) and 4.6(a) we immediately see that if the velocity at time $t = t_0$ has components (v_{0x}, v_{0y}), then the components (v_x, v_y) at an arbitrary time t will be given by the following equations. (We will write a_x instead of a for the acceleration to emphasize the choice of x-axis).

$$x\text{-component:} \quad v_x = v_{0x} + a_x(t - t_0), \tag{4.7}$$

$$y\text{-component:} \quad v_y = v_{0y}, \tag{4.8}$$

Change in position

We use the expressions for velocity components in Eqs. 4.5(b) and 4.6(b) to obtain the following equations for the rate of change in coordinates.

$$x\text{-component:} \quad dx/dt = v_{0x} + a_x(t - t_0)$$

$$y\text{-component:} \quad dy/dt = v_{0y}$$

Multiplying these equations with dt and integrating with appropriate limits we obtain

$$x\text{-component:} \quad \int_{x_0}^{x(t)} dx = \int_{t_0}^{t} \left[v_{0x} + a_x(t - t_0)\right] dt$$

$$y\text{-component:} \quad \int_{y_0}^{y(t)} dy = \int_{t_0}^{t} v_{0y} dt$$

Integrating these equations and rearranging terms you can show that the result is

$$x\text{-component:} \quad x(t) - x_0 = v_{0x}(t - t_0) + \frac{1}{2}a_x(t - t_0)^2 \tag{4.9}$$

$$y\text{-component:} \quad y(t) - y_0 = v_{0y}(t - t_0) \tag{4.10}$$

Equations 4.7, 4.8, 4.9, and 4.10 give us the change in components of velocity and position for a motion which has a constant acceleration pointed along x-axis. We can simplify these equations further by choosing initial time to be $t_0 = 0$ and initial position to be at the origin.

$$\text{Choose}: \; t_0 = 0 \quad x_0 = 0; \quad y_0 = 0; \quad z_0 = 0.$$

With these choices, the x and y-components of the position and velocity vectors at an arbitrary time when the constant acceleration is pointed towards the positive x-axis are

$$x\text{-component:} \quad \boxed{v_x(t) = v_{0x} + at} \tag{4.11}$$

$$\boxed{x(t) = v_{0x}t + \frac{1}{2}a_x t^2} \tag{4.12}$$

$$y\text{-component:} \quad \boxed{v_y(t) = v_{0y}} \tag{4.13}$$

$$\boxed{y(t) = v_{0y}t} \tag{4.14}$$

Table 4.1: Constant Accel. Relations

$(x_0 = y_0 = z_0 = 0)$
$(t_0 = 0; a_y = 0; v_z = 0)$
1. $v_x(t) = v_{0x} + a_x t$
2. $x(t) = v_{0x}t + \frac{1}{2}a_x t^2$
3. $v_y(t) = v_{0y}$
4. $y(t) = v_{0y}t$

The time variable t can be eliminated from the two equations of the x-components of the position and velocity vectors to obtain the following useful formula.

$$\boxed{x\text{-component:} \quad v_x^2 - v_{0x}^2 = 2a_x x} \tag{4.15}$$

One can also eliminate t from x and y- equations to obtain the equation of the trajectory in the xy-plane.

$$x = \left(\frac{v_{0x}}{v_{0y}}\right) y + \left(\frac{a_x}{2v_{0y}^2}\right) y^2. \tag{4.16}$$

Equations 4.11 to 4.16 are the basic equations that describe the motion of an object that has a constant acceleration pointed towards x-axis. We see that the direction of the initial velocity with respect to the direction of acceleration vector decides if both the x and y-components are needed.