9.1 MAGNETIC DIPOLE IN MAGNETIC FIELD

9.1.1 Force And Torque In A Uniform Magnetic Field

We have seen before that a magnetic dipole can be thought of as a loop of current. To **represent a magnetic dipole** $\vec{\mu}$ by a loop of current we envision a loop of current such that the product of the current and the area of the loop is equal to the magnitude of the magnetic dipole moment as illustrated in Fig. 9.1. The direction of the current is such that when you look at the loop from the direction of the tip of the dipole moment the current goes in counterclockwise sense. The equivalent current loop is also called "bound current" because it cannot flow like a current in a conducting wire - these current loops are bound to specific atoms in the material.

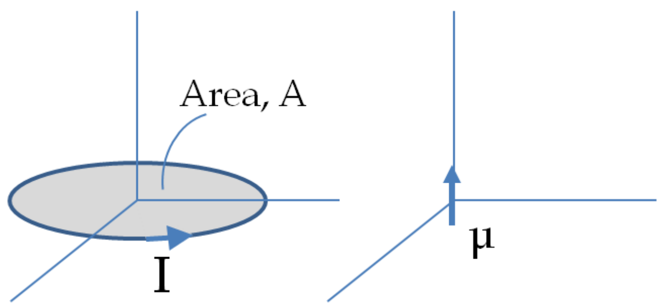

Figure 9.1: Two equivalent pictures of an infinitesimal current loop: either think of the current loop as current loop or a magnetic dipole of magnetic dipole moment $\mu = IA$ in the direction as displayed here.

This tells us that we can find the force and torque on a dipole in a magnetic field by the corresponding expressions for the force and torque on the equivalent loop of current. Therefore, if we place a magnetic dipole $\vec{\mu}$ in an external magnetic field $\vec{B}$, the force on the dipole will be

$$\vec{F}_{\text{on dipole}} = \oint I d\vec{l} \times \vec{B}, \tag{9.1}$$

where uniform current I runs in a loop of area A is such that

$$\boxed{IA = |\vec{\mu}|.}$$

In a uniform magnetic field $\vec{B}$ is constant. Therefore, in Eq. 9.4, $\vec{B}$ and I are constants and we can take them outside the integral.

$$\vec{F}_{\text{on dipole}} = -I\vec{B} \times \oint d\vec{l}, \tag{9.2}$$

where $\oint d\vec{l}$ is the sum of infinitesimal displacement vectors around the closed circuit of the loop of current, and therefore must be 0. See Figure 9.2.

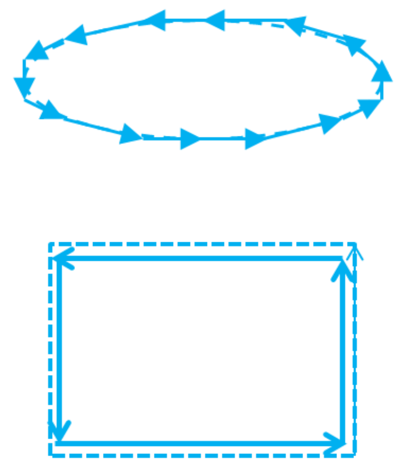

Figure 9.2: Loop vector elements around a close loop gives a net zero vector for any shape loop.

$$\oint d\vec{l} = 0 \quad \text{(by vector sum)} \tag{9.3}$$

Therefore the net force on a dipole in a uniform magnetic field must be zero.

$$\vec{F}_{\text{on dipole}} = 0 \quad (\text{uniform } \vec{B}) \tag{9.4}$$

Even though the net force on a magnetic dipole in a uniform magnetic field is zero, the torque on the dipole is not zero if magnetic dipole moment is pointed at an angle to the magnetic field. The torque on an equivalent current loop whose normal makes an angle with the external magnetic field was shown to be

$$\boxed{\vec{\tau} = I\vec{A} \times \vec{B},} \tag{9.5}$$

which gave the torque on a magnetic dipole to be

$$\boxed{\vec{\tau}_{\text{on dipole}} = \vec{\mu} \times \vec{B}.} \tag{9.6}$$

The cross product says that the torque on a magnetic dipole is perpendicular to both the magnetic dipole moment vector and the magnetic field vector. Therefore, this torque will give angular acceleration to the particle which will tend to align the particle's magnetic dipole moment to the external magnetic field.

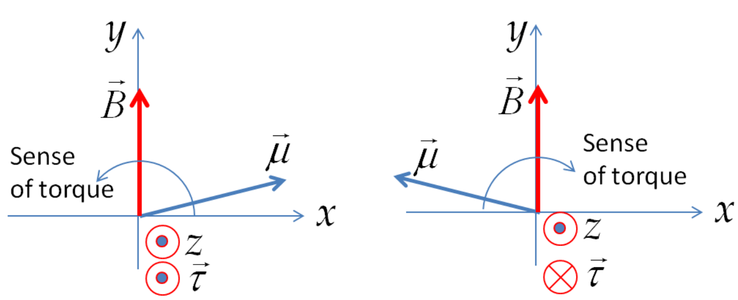

Figure 9.3: The torque on magnetic dipole is pointed such that the angular acceleration of the particle as a result of the torque tends to align the dipole moment to the external field.

9.1.2 Force On A Dipole In A Non-Uniform Magnetic Field

We saw above that in a uniform magnetic field the force on a magnetic dipole is zero. But we know from our experience that a magnetic dipole, such as a magnetic compass needle, experiences a net force near a bar magnet. How does that happen? The difference has to do with a nonuniform magnetic field versus a uniform magnetic field. While it is true that there will be no net force on a dipole in a uniform magnetic field, we will see below that there would be a net force if a dipole is placed in a non-uniform field.

In the case of similar situation of force on an electric dipole we had looked at the Coulomb force $q\vec{E}$ on the individual positive and negative charges that made up the dipole. But, since we do not have a corresponding force law on the individual poles of a magnet, we cannot proceed in the same way. However, we dop have an equivalent current-loop representation of a magnetic dipole and we do have force law for force on current, therefore, to find a formula for the force on a magnetic dipole we will replace the dipole with a small ring of current as we have done above. For the non-uniform field we will place the ring of current above a magnetic pole as shown in Fig. 9.4.

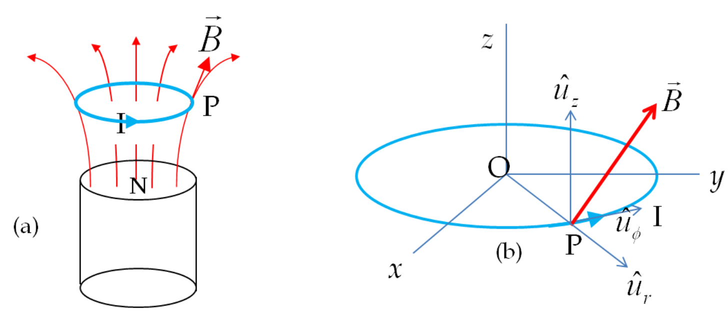

Figure 9.4: Magnetic force in a non-uniform magnetic field. (a) Position the ring of current in the nonuniform fringing field above a solenoid. (b) Force on the ring is from the z and r components of $\vec{B}$. The right-hand rule of magnetic force on a current shows that the force from the z component of $\vec{B}$ are radially outward and their sum over the ring vanishes. Only the force from the r component of $\vec{B}$ survives and is pointed down in the figure.

Since the current in the loop is best thought of as azimuthal having the direction of $\hat{u}_\phi$ as shown in Fig. 9.4, it is helpful to resolve the magnetic field into z, r and ϕ components of a cylindrical coordinate system. These components of the magnetic field would be $B_z = \vec{B}\cdot\hat{u}_z$, $B_r = \vec{B}\cdot\hat{u}_r$, and $B_\phi = \vec{B}\cdot\hat{u}_\phi$ for the unit vectors shown in Fig. 9.4.

Recall that the force on a current element is given by a cross product, viz., $d\vec{F} = I d\vec{l}\times\vec{B}$. Therefore, the force on the current loop with the current is along $\hat{u}_\phi$ comes from only the z and r components of the magnetic field. The force from the z-component of the magnetic field is radially outward at all points of the loop and therefore cancels out. Therefore, the force on the "current loop" is from the radial component only. This force is in the negative z-direction in Fig. 9.4 and has the magnitude

$$\left|\vec{F}\right| = 2\pi RI\,|B_r|\,, \tag{9.7}$$

where B_r is the radial component of magnetic field $\vec{B}$. This expression can be written in terms of the magnetic dipole moment as

$$\left|\vec{F}\right| = 2\mu\frac{|B_r|}{R}. \tag{9.8}$$

Note that the radius of the current loop that represents a magnetic dipole moment is arbitrary, the only requirement being that the loop is small in size and the product of the area of the loop and the current in the loop be equal to μ. It is therefore important to write this equation such that R does not appear in the formula. We now show that this can be done by using Gauss's law for the magnetic field, which says that

$$\oiint \vec{B}\cdot d\vec{A} = 0. \tag{9.9}$$

That is, the magnetic flux through a closed surface is zero. To obtain the relation between the components of the magnetic field as a consequence of this law, we will examine the magnetic flux through a small cylindrical box enclosing the ring of current as shown in Fig. 9.5.

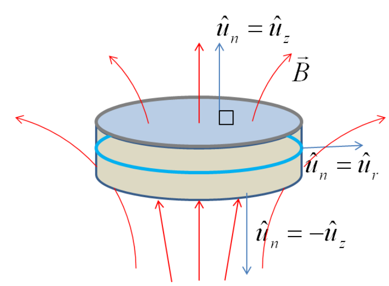

Figure 9.5: Closed surface for the application of Gauss's law for magnetic field. The current ring is at z, the top of the box is at $z+\Delta z$ and bottom at $z-\Delta z$. the outward normals at top end, bottom end and the sides are shown.

Let the top of the box be at $z+\Delta z$ and the bottom at $z-\Delta z$ and the radius of the box be R, the same as the radius of the current loop. The outwards flux through the side is $B_r 2\pi R \times 2\Delta z$. The net flux outward from the top and bottom ends are in the box $B_z(z+\Delta z)\pi R^2$ and $-B_z(z-\Delta z)\pi R^2$, respectively. Therefore, equating the net flux out of the closed box to zero gives us the following relation.

$$B_r 2\pi R \times 2\Delta z + B_z(z+\Delta z)\pi R^2 - B_z(z-\Delta z)\pi R^2 = 0. \tag{9.10}$$

Dividing this equation be $2\pi R^2\Delta z$ and taking the limit $\Delta z \to 0$, we obtain B_r/R in terms of the derivative of B_z.

$$\frac{2B_r}{R} = -\frac{\partial B_z}{\partial z}. \tag{9.11}$$

We have written the derivative as a partial derivative because we expect the z-component B_z of the magnetic field to vary in space and depend on the x, y and z coordinates of the space point. Now, we can write the force on the magnetic dipole moment all in terms of the dipole moment and the magnetic field by using Eq. 9.11 into Eq. 9.8.

$$\boxed{|\vec{F}| = \mu \left| \frac{\partial B_z}{\partial z} \right|.} \tag{9.12}$$

which is pointed downward in the figure towards the negative z. It would not come as a surprise that if we reverse the direction of the dipole, the force on the dipole will be upward towards the positive z-axis.

Equation 9.12 says that if a magnetic field has a positive component along a magnetic dipole, then the dipole will be attracted towards increasing field. If a magnetic field has a negative component then the force on the dipole will be towards decreasing field. Hence, a paramagnetic material, whose dipoles tend to line up with the external magnetic field, will be attracted towards a magnet, while a diamagnetic material, whose dipoles tend to be anti-parallel to the external magnetic field, will be repelled by a magnet.

For a magnetic dipole moment pointed in an arbitrary direction, the magnetic force is a generalization of the above result.

$$\begin{aligned} \vec{F} &= \hat{u}_x \frac{\partial}{\partial x}\left(\vec{\mu}\cdot\vec{B}\right) + \hat{u}_y \frac{\partial}{\partial y}\left(\vec{\mu}\cdot\vec{B}\right) + \hat{u}_z \frac{\partial}{\partial z}\left(\vec{\mu}\cdot\vec{B}\right), \\ &= \vec{\nabla}\left(\vec{\mu}\cdot\vec{B}\right), \end{aligned} \tag{9.13}$$

where $\vec{\nabla} = \hat{u}_x \frac{\partial}{\partial x} + \hat{u}_y \frac{\partial}{\partial y} + \hat{u}_z \frac{\partial}{\partial z}$ is the nabla vector operator. This expression for the force shows that the **potential energy** U of a magnetic dipole $\vec{\mu}$ in a magnetic field $\vec{B}$ is given by

$$\boxed{U = -\mu \cdot \vec{B},} \tag{9.14}$$

since a conservative force and potential energy due to that force are related $\vec{F} = -\vec{\nabla} U$. We will deduce this expression of the potential energy in another way below.

Example 9.1.1. Magnetic force on a copper atom. Find the magnitude of force on a magnetic moment of an atomic modeled as arising from an electron circulating in orbit of radius $R = 0.05$ nm around a nucleus with speed 1×10^5 m/s in a magnetic field that varies with a gradient of 20 T/m in the direction parallel to the magnetic dipole.

Solution. We start with a calculation of the magnetic dipole moment of the current loop from area of the orbit and the current of electron in the orbit. Since, electron moves past the same point in its orbit once every period, we can find the current of the electron by dividing the charge of the electron by the time period. Note that in our notation charge of en electron is written as $-e$, where $e = 1.6 \times 10^{-19}\ C$.

$$I = \frac{e}{\text{Period}} = \frac{ev}{2\pi R}.$$

Therefore, the magnetic dipole moment of an electron due to its orbital motion, to be denoted by μ_l is

$$\mu_l = I\pi R^2 = \frac{evR}{2}.$$

Now, putting the numerical values we find

$$\mu_l = \frac{1}{2} 1.6 \times 10^{-19}\text{C} \times 1 \times 10^5 \text{m/s} \times 0.05 \times 10^{-9}\text{m} = 4.0 \times 10^{-25}\text{A.m}^2.$$

Now, we can calculate the magnitude of force on the dipole.

$$|\vec{F}| = \mu_l \left(\frac{\partial B_z}{\partial z} \right) = 4.0 \times 10^{-25}\text{A.m}^2 \times 20\ \text{T/m} = 8.0 \times 10^{-24}\text{N},$$

where we have replaced A.m.T by N. You should check if A.m.T is actually equal to N. How does this force compare with the force of gravity on the atom? Suppose the atom is an atom of the element copper. Then, $m = 63$ g/$6.022 \times 10^{23} = 1.05 \times 10^{-25}$ kg. Therefore, the weight of one atom would be

$$mg = 1.05 \times 10^{-25}\text{kg} \times 9.81\ \text{m/s}^2 = 1.0 \times 10^{-24}\text{N}.$$

The magnetic force in the present example is seen to be 8 times the force of gravity.

9.1.3 Energy Of A Magnetic Dipole In An External Field

As mentioned above, we can deduce the energy from $\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B})$ as $U = -\vec{\mu} \cdot \vec{B}$. Here, we will deduce this result from the rotational work as we did for the electric dipole. It takes work to rotate the direction of magnetic dipole in a magnetic field since there is torque on the dipole. Consider a magnetic dipole $\vec{\mu}$ pointed at an angle θ with an external magnetic field $\vec{B}$ as in Fig. 9.6.

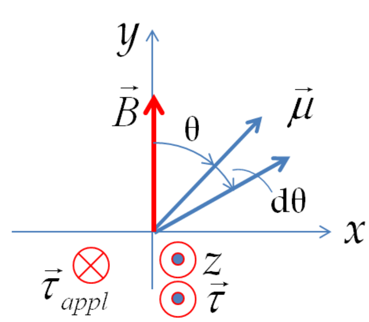

Figure 9.6: Magnetic dipole $\vec{\mu}$ is rotated in the xy plane by applying a torque $\vec{\tau}_{appl}$ which balances torque $\vec{\tau}$ by magnetic field $\vec{B}$ along y axis.

To find the potential energy of magnetic dipole when the dipole moment is pointed at an angle with respect to the magnetic field, we find work done by an applied force which applies a torque that balances the torque by the magnetic field.

$$\vec{\tau}_{\text{appl}} = -\vec{\tau}, \tag{9.15}$$

which is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B}. \tag{9.16}$$

Therefore, the applied torque is

$$\vec{\tau}_{\text{appl}} = -\vec{\mu} \times \vec{B}. \tag{9.17}$$

This torque will do the following rotational work when it rotates the magnetic dipole by an angle $d\theta$ in the direction

$$dW = -\vec{\mu} \times \vec{B} \cdot \left(-d\theta \hat{u}_z\right). \tag{9.18}$$

where the direction of the angular displacement is obtained by using the right-hand rule as was discussed in the chapter on rotation. The work done on the dipole in rotating the dipole from $\theta = \theta_i$ to another angle $\theta = \theta_f$ is obtained by integrating.

$$\begin{aligned} W &= \mu B \int_{\theta_i}^{\theta_f} \sin\theta d\theta \\ &= \mu B \left(\cos\theta_i - \cos\theta_f\right) \end{aligned} \tag{9.19}$$

Work done by $\vec{\tau}_{\text{appl}}$ against the torque by $\vec{B}$ is stored as the potential energy U of the dipole. Choosing $\theta_i = \pi/2$ as reference simplifies the formula and we arrive at the following result for the potential energy of a magnetic dipole in an external magnetic field when the angle between them is θ.

$$U = -\mu B \cos\theta, \tag{9.20}$$

which can be written using vector properties as follows.

$$U = -\vec{\mu} \cdot \vec{B}. \tag{9.21}$$

Note that the potential energy of a dipole in an external magnetic field is lowest when the magnetic dipole moment is pointed in the same direction as the magnetic field, and highest when the magnetic dipole moment is in the opposite direction.

$$U = \begin{cases} -\mu B & (\vec{\mu} \text{ parallel to } \vec{B}) \\ +\mu B & (\vec{\mu} \text{ anti-parallel to } \vec{B}) \end{cases} \tag{9.22}$$

Hence, the natural tendency of a dipole is to align with the external magnetic field when placed in a magnetic field.

Example 9.1.2. Rotating a magnetic dipole In a nuclear magnetic resonance experiment, a nuclear magnetic dipole is first aligned with a static external magnetic field. Then a dynamic magnetic field of an electromagnetic radiation is applied on the sample to flip the orientation of the dipole to face opposite to the direction of the static field. Suppose it takes 4.35×10^{-7} eV of energy to flip the magnetic dipole of a proton when aligned to a magnetic field of magnitude 2.5 T. Find the magnetic dipole moment of a proton.

Solution. Note that the change in energy when a magnetic dipole moment is flipped from the anti-aligned to the aligned direction is given by $\Delta U = 2\mu B$. This gives $\mu = \Delta U/2B$. Putting in the numbers we find

$$\begin{aligned}\mu &= 4.35 \times 10^{-7}\text{eV} \times \frac{1.6 \times 10^{-19}\text{J}}{1 \text{ eV}} \times \frac{1}{2 \times 2.5 \text{ T}} \\ &= 1.4 \times 10^{-26}\text{J/T} = 1.4 \times 10^{-26}\text{A.m}^2.\end{aligned}$$