6.4 DIFFRACTION THROUGH A DOUBLE-SLIT

In the last chapter when we were studying interference in the Young's double-slit experiment we ignored the diffraction effect in each slit. We assumed that the slits were so narrow that on the screen you only saw the interference of light from just two point sources. If the slit is smaller than the wavelength then Fig. 6.9 shows that there is just a spreading of light and no peaks or troughs on the screen. Therefore, it was reasonable to leave out the diffraction effect in the last chapter. But, if you make the slit wider, Fig. 6.9 shows that you cannot ignore diffraction. In this section we will study the complications to the double-slit experiment when you also need to take into account the diffraction effect of each slit.

Fig. 6.11 shows the basic arrangement of a double-slit experiment. The intensity on a point P on the screen is usually written as a function of the angle θ that the point makes at the symmetry point O as shown in the figure.

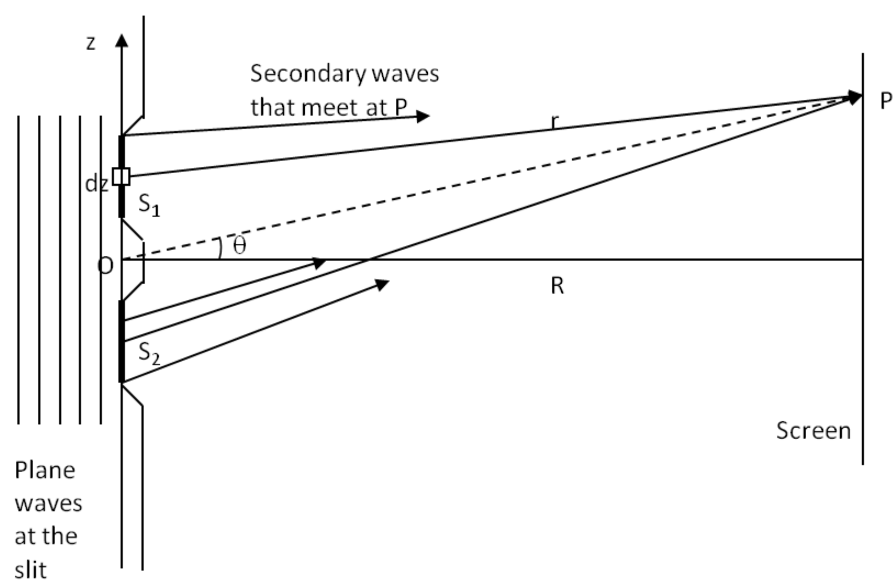

Figure 6.11: Fraunhofer diffraction through a double-slit.

The secondary spherical wavelets from each slit would interfere with other wavelets from the same slit as well the wavelets from the other slit. The result would be a combined diffraction and interference effects resulting in the following formula for the intensity in the direction θ.

$$I(\theta) = 4I_{\text{ref}} \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha. \tag{6.21}$$

Here I_{ref} is the intensity of light from each slit treated as a point source. The constants β and α are defined by the following relations as we have done before.

$$\alpha \equiv \frac{\pi a \sin\theta}{\lambda} = \frac{ka\sin\theta}{2} \tag{6.22}$$

$$\beta \equiv \frac{\pi b \sin\theta}{\lambda} = \frac{kb\sin\theta}{2} \tag{6.23}$$

The factor $(\sin\beta/\beta)^2$ in Eq. 6.21 comes from the diffraction of the waves originating from the same slit and the factor $\cos^2\alpha$ arises from the interference of the waves from two different slits. The diffraction pattern has a minimum whenever $I(\theta)$ becomes a minimum.

Minima:

$$\alpha = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \cdots \tag{6.24}$$

$$\beta = \pm\pi, \pm 2\pi, \pm 3\pi, \cdots \tag{6.25}$$

The minima due to α are called the **interference minima** and the ones due to β are called the **diffraction minima**. It is also useful to rewrite the minima conditions in terms of wavelength λ and the slit dimensions a and b.

Minima:

$$\text{Interference:} \quad a\sin\theta = \pm\frac{\lambda}{2}, \pm 3\frac{\lambda}{2}, \cdots \tag{6.26}$$

$$\text{Diffraction:} \quad b\sin\theta = \pm\lambda, \pm 2\lambda, \pm 3\lambda, \cdots \tag{6.27}$$

The maxima of the factor $\cos^2\alpha$ are called the maxima due to interference between waves from two slits, i.e. one wave from slit S1 and the other from slit S2. These maxima occur in directions θ which correspond to the interference condition

$$a\ \sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \cdots \tag{6.28}$$

The minima due to the interference between the two slits occur at

$$a\ \sin\theta = m'\lambda/2, \quad m' = \pm 1, \pm 3, \cdots \tag{6.29}$$

The maxima and minima due to the factor $(\sin\beta/\beta)^2$ are called diffraction maxima and minima due to the interference of waves from points from the same slit. As we have seen that these minima occur at

$$b\ \sin\theta = m\lambda, \quad m = \pm 1, \pm 2, \cdots \tag{6.30}$$

The two effects may produce minima at different angles. This gives rise to a complicated pattern on the screen - some of the maxima of

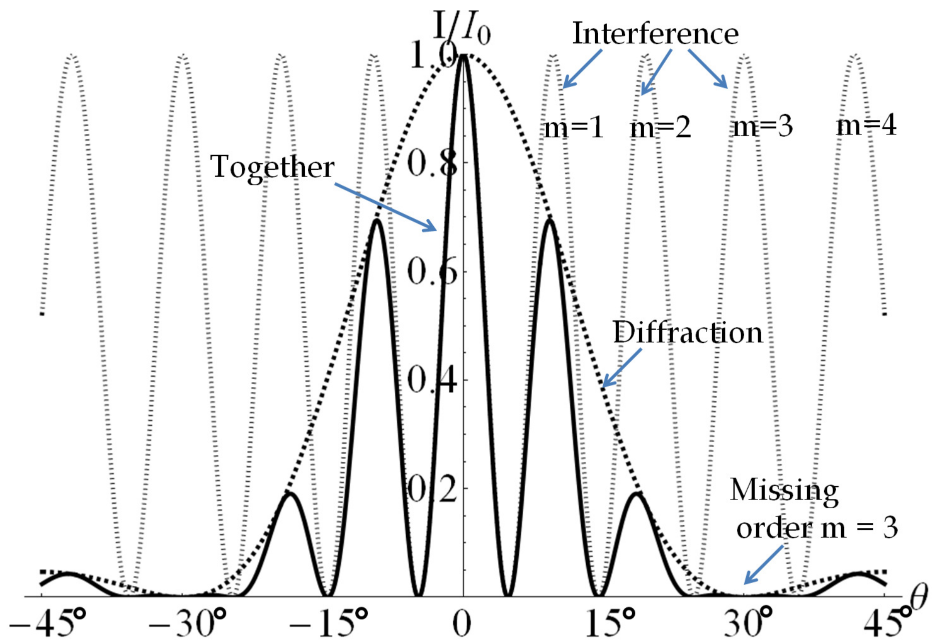

Figure 6.12: Diffraction from a double-slit. The dashed line with same height peaks are from the interference of the waves from two slits, the dashed line with one big hump in the middle is the diffraction of waves from within one slit and the solid line is the product of the two, which is the pattern observed on the screen. The plot shows the expected result for a slit width $b = 2\lambda$ and slit separation $a = 6\lambda$. The maximum of $m = \pm 3$ order for the interference is missing because the minumum of the diffraction occurs in the same direction.

interference from the two slits would be missing if the maximum of the interference is in the same direction as the minimum of the diffraction. We refer to these missing peaks as **missing order**. One example diffraction pattern on the screen is presented in Fig. 6.12. The solid line with multiple peaks of various heights is the intensity observed on the screen. It is a product of the interference pattern form waves from separate slits and the diffraction of waves from within one slit.

6.4.1 Derivation of the Intensity Formula

In case you are wondering how one can obtain the formula for intensity cited above, here it is. Consider plane waves incident on an opaque shield with two long horizontal slits each with width a and center-to-center distance b as shown in Fig. 6.11. Then the superposition of secondary wavelets from the plane waves at the two slits will proceed in a similar way as for a single slit, except now the integration over z will have two parts, one over each slit.

$$E_P = \text{Contribution from the wave through slit 1} \\ + \text{Contribution from the wave through slit 2.}$$

We have already worked out the math for one slit in detail when we studied the diffraction through a single slit. The same math is applied here with the range for z for the slits to be: $\frac{a}{2} - \frac{b}{2}$ to $\frac{a}{2} + \frac{b}{2}$ for the slit S_1 and $-\frac{a}{2} - \frac{b}{2}$ to $-\frac{a}{2} + \frac{b}{2}$ for the slit S_2.

$$E_P = \frac{A}{R}\left[\int_{a/2-b/2}^{a/2+b/2} \sin(kR - kz\sin\theta - \omega t)dz\right]_{\text{slit 1}} \\ + \frac{A}{R}\left[\int_{-a/2-b/2}^{-a/2+b/2} \sin(kR - kz\sin\theta - \omega t)dz\right]_{\text{slit 2}}$$

where A is the amplitude at the slits. For simplicity we have assumed the same amplitude at the points of the two slits. The integrals are easy to do with the following result.

$$E_P = 2\frac{bA}{R}\left(\frac{\sin\beta}{\beta}\right)\cos\alpha\ \sin(kR - \omega t), \tag{6.31}$$

where α and β are same as defined above. The intensity at a point P on the screen is obtained from the fluctuating electric field of the wave by time-averaging the square of the field and multiplying by the permittivity ϵ_0 and the speed of light c. The Intensity at point P of screen is then found to be

$$I(\theta) = \left[2\epsilon_0 c\left(\frac{bA}{R}\right)^2\right]\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha, \tag{6.32}$$

where the quantity in the square bracket has units of intensity. As we are mostly interested in how the intensity changes on the screen and not in the absolute value, it will be nice to find a reference intensity I_{ref} to write $I(\theta)$ in terms of. The intensity of light from one slit on the screen at the horizontal spot from the slit when the other slit is covered provides a nice reference intensity. This intensity is easily obtained from noting that the electric field has amplitude A at any point of the slit and accounting for the spreading out in a sphere of radius R when traveling to the screen.

$$I_{\text{ref}} = \frac{1}{2}\epsilon_0 c\left(\frac{bA}{R}\right)^2. \tag{6.33}$$

Therefore intensity $I(\theta)$ from two slits is written as follows.

$$I(\theta) = 4I_{\text{ref}}\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha. \tag{6.34}$$

Note that when both slits are open, there is 4 times intensity of one slit rather than 2 times. This is because the amplitudes add giving 2 $\times$ the amplitude of one slit, but the intensity is proportional to the square of amplitudes, and therefore you get 4 $\times$ the intensity of one slit.

Example 6.4.1. Interference and diffraction in a two-slit diffraction

Find the angular positions of the interference maxima within the central peak of a double-slit diffraction for a monochromatic light of wavelength 628 nm on slits of width of 1.5 μm separated by 4 μm.

Solution. First let us find the range of angle included within the central maximum of the diffraction by locating the first diffraction minima.

$$b \sin\theta = \pm\lambda.$$

Therefore, the central peak is between $-\sin^{-1}(\lambda/b)$ to $+\sin^{-1}(\lambda/b)$, which gives $-23.3°$ to $+23.3°$. We need to find the interference maxima within this range. The condition for interference maxima are

$$a \sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \cdots .$$

The value of m for which the edge of the central peak of diffraction is reached is obtained by setting angle to 23.3°.

$$m = \frac{a \sin\theta}{\lambda} = \frac{4\ \mu\text{m}\ \ \sin(23.36\circ)}{0.628\ \mu\text{m}} = 2.5.$$

Hence, the central peak will have five interference peaks, corresponding to $m = 0$, ±1, ±2.