1.2 COULOMB'S LAW

1.2.1 The Electrostatic Force Between Static Charges

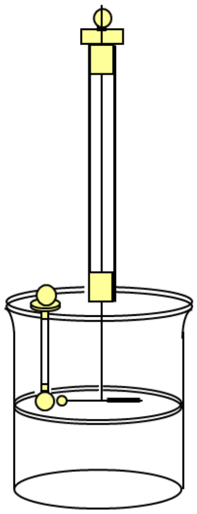

Figure 1.9: Coulomb's torsion balance apparatus. A pith ball with charge is hung from a fine wire which can rotate when electric force from the charge on the fixed pith ball.

The interaction between charges were extensively studied in 1780s by the French physicist **Charles Coulomb** using a torsion balance apparatus first used by Henry Cavendish in 1798 to study the gravitational force (Fig. 1.9). Coulomb's torsion balance worked by charging two pith balls, one of which was fixed and the other attached to the end of a needle which was hung using a silver wire or hair to a torsion micrometer. The apparatus allowed him to determine the force by way of the torque and permitted him to accurately determine the dependence of force on distance and charges on the pith balls.

Coulomb found that electric force by a charged body on another is proportional to the magnitudes of each charge, and inversely proportional to the square of the distance between them. Coulomb also found that the force between two charges was independent of the presence of other charges. The electric force is also called **Coulomb force**. Coulomb force is repulsive if two charges are of same type, i.e., either both positive or both negative, and attractive if two charges are of opposite types. The direction of the force is along the line joining the two particles.

Henry Cavendish also discovered the inverse square dependence of the electrostatic forces prior to Coulomb, but he did not publish his findings and his discovery was not known until James Clerk Maxwell publicized it in 1879. Other people who had known or suspected similarities with the Gravitational force included Benjamin Franklin and Joseph Priestly, whose work was the basis for Coulomb's experiments.

Coulomb's law

Electric force $\vec{F}_e$ between two charges q_1 and q_2 separated by a distance r has the following magnitude and direction.

Magnitude: $F_e \propto \dfrac{|q_1||q_2|}{r^2}$.

Direction: use the rule, "like charges repel, opposite charges attract."

The direction of the force on any one of the charges depends on the type of the two charges and the charge on which the force acts as shown in Fig. 1.10.

Note that the magnitude of the force is always positive regardless of the charges involved. That is why we have placed the absolute sign around the symbols for charges. The magnitude of electric force $\vec{F}_e$ can be written as an equality by introducing a proportionality constant k independent of q_1, q_2, or r.

$$\boxed{F_e = k\frac{|q_1||q_2|}{r^2}.} \tag{1.3}$$

The value of constant k depends on the units in which F_e, q_1, q_2, and r are expressed. For instance, in the Systéme Internationale (SI)

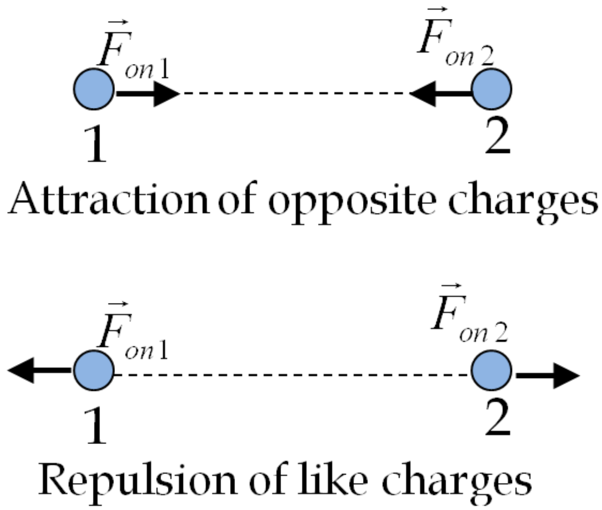

Figure 1.10: Direction of Coulomb forces between charges.

family of units, where force is expressed in Newton (N), charge in Coulomb (C) and distance in meter (m), the approximate value of k is:

$$k \approx 9 \times 10^9 \frac{\mathrm{N.m}^2}{\mathrm{C}^2}$$

Often one expresses k in terms of another constant, ϵ_0, called the **permittivity of vacuum** (or free space) by the following relation.

$$k = \frac{1}{4\pi\epsilon_0}.$$

The value of ϵ_0 is then approximately given by the following.

$$\epsilon_0 \approx 8.843 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N.m}^2}.$$

We can write Coulomb's law in a vector form by introducing a unit vector to represent the direction, and then multiplying the magnitude of the force by the unit vector. For instance, the directions of the Coulomb forces on q_1 by q_2 and q_2 by q_1 depend on the relative nature of the two charges as shown in the Fig. 1.11, although the magnitudes of the forces do not depend on the nature of the charges, only their amounts.

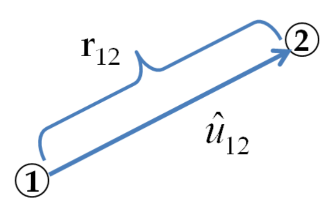

Figure 1.12: Unit vector and direct distance notation.

We will like to write one formula that captures all four cases: both q_1 and q_2 positive, both q_1 and q_2 negative, q_1 positive but q_2 negative, and q_1 negative but q_2 positive. To illustrate the analytic way of writing these forces, I will illustrate here the force on q_2 which I will denote by $\vec{F}_2$ for brevity. To do that we define a unit vector in the direction from q_1 to q_2 by $\hat{u}_{12}$ and the direct distance between the two charges by r_{12} as in Fig. 1.12. With this notation, all four cases of force on charge q_2 can be stated by a single equation:

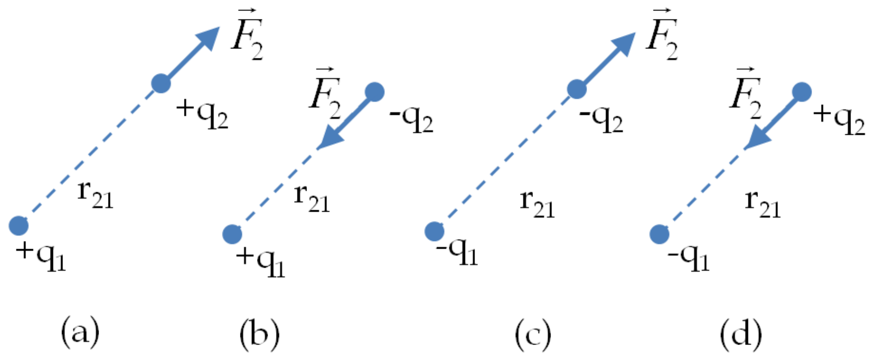

Figure 1.11: Coulomb force on charge q_2 by charge q_1 are shown for four different cases depending upon the type of charges q_1 and q_2. Note the direction of the force depends on the relative type of each charge: (a) both positive, (b) q_1 positive, q_2 negative, (c) both negative and (d) q_1 negative, q_2 positive. For concreteness the figure shows only force on q_2; there is also an equal magnitude force in opposite direction on q_1 for each case.

$$\vec{F}_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{u}_{12}, \tag{1.4}$$

where we use a positive number for a positive charge and a negative number for a negative charge in order for the direction of the force to come out right. To write the force on q_1 by q_2 all you have to do in replace the unit vector by $\hat{u}_{21}$, the vector from 2-to-1, which is negative of the unit vector $\hat{u}_{12}$.

$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{u}_{21}, \tag{1.5}$$

Note that there is no distance in the unit vectors $\hat{u}_{12}$ or $\hat{u}_{21}$. All the distance-dependence of the force between the charges is in the direct distance r_{12}, and the force goes as inverse square of the direct distance.

Example 1.2.1. The Static Equilibrium. Two Styrofoam balls of equal diameter are painted with a silver paint. An unknown amount of charge is put on one of them and then the other ball is brought in contact with it. As a result, an equal amount of charge is distributed on each ball. When the balls are hung by using a 1-meter long cotton thread each from the same point, they repel each other and come to rest with 60 degrees angle between them. If the masses of the balls are 10 grams each, what is the amount of charge on each ball?

Solution. Since the acceleration of each ball is zero, the net force on each ball would also be zero. To obtain relations among the forces on the balls we will identify all the forces on one of the balls, and then set the sum of their Cartesian components zero independently. Let F_e,

T, and W denote the magnitudes of the Coulomb force, the tension in the string, and the weight of the ball, respectively. Then, from the free-body diagram shown in Fig. 1.13, we obtain the following components of the equation $\vec{F}_{net} = 0$.

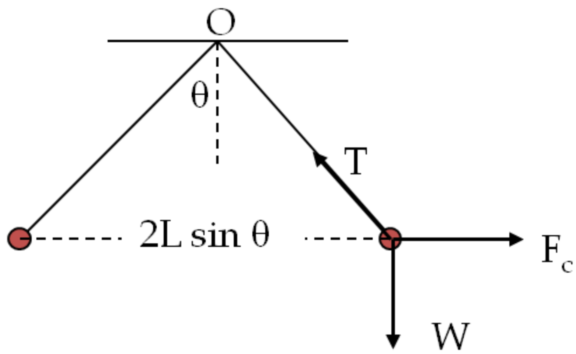

Figure 1.13: Forces on one of the charged balls are: tension T, weight W, Coulomb F_e.

$$\text{Horizontal:} \quad F_e - T\sin\theta = 0 \tag{1.6}$$

$$\text{Vertical:} \quad T\cos\theta - W = 0 \tag{1.7}$$

Hence,

$$F_e = W\tan\theta. \tag{1.8}$$

The Coulomb force F_e on this ball is by the charges on the other ball. Since the two balls have equal charges and the direct distance between them is $2L\sin\theta$, the magnitude of the coulomb force can also be written using the Coulomb's law as

$$F_e = kq^2/(2L\sin\theta)^2. \tag{1.9}$$

Eqs. 1.8 and 1.9 are two expressions for the magnitude of the same Coulomb force. Equating them gives us an equation in q which we can solve readily to find the following.

$$q = \pm\,(2L\sin\theta)\,\sqrt{(W/k)\tan\theta}$$

Now, we put in the numbers $L = 1$ m, $\theta = 30°$, $W = 0.01 \times 9.8$ N, $k = 9.0 \times 10^9 \text{N.m}^2/\text{C}^2$ to obtain the numerical value of the charge q.

$$q = \pm 2.5 \times 10^{-6}\ \text{C} = \pm 2.5\ \mu\text{C}.$$

Since the two balls repel each other, the charges on the two are of the same type, either both positive or both negative. They are both either $+2.5\ \mu$C or $-2.5\ \mu$C. However, based on the data given, we cannot determine which is the case here. If we knew the type for one of the charges, we would have immediately known the type for the other.

Example 1.2.2. The Direct Distance Vector

Two point charges q_1 and q_2 are located at (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Using the unit vectors $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ along the three Cartesian axes, find the unit vector pointed from q_1 to q_2?

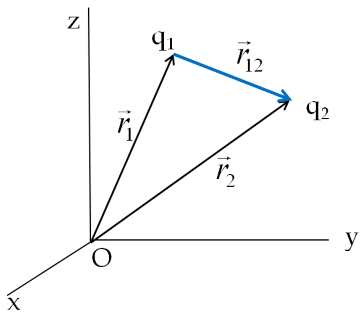

Figure 1.14: Positions vectors of charges and direct vector from q_1 to q_2.

Solution. First we will construct the direct distance vector $\vec{r}_{12}$ from (x_1, y_1, z_1) to (x_2, y_2, z_2), and then divide $\vec{r}_{12}$ by its magnitude to find the unit vector. To construct the direct distance vector $\vec{r}_{12}$ from (x_1, y_1, z_1) to (x_2, y_2, z_2), we make use of the diagram in Fig. 1.14, where $\vec{r}_1$ is the position vector from the origin to point (x_1, y_1, z_1), and $\vec{r}_2$ from the origin to point (x_2, y_2, z_2).

The direct distance vector from q_1 to q_2 is readily obtained from the following vector addition rule in triangle Oq_1q_2.

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1.$$

Since $\vec{r}_1 = x_1\hat{u}_x + y_1\hat{u}_y + z_1\hat{u}_z$ and $\vec{r}_2 = x_2\hat{u}_x + y_2\hat{u}_y + z_2\hat{u}_z$, the direct distance vector $\vec{r}_{12}$ is given as

$$\vec{r}_{12} = (x_2 - x_1)\,\hat{u}_x + (y_2 - y_1)\,\hat{u}_y + (z_2 - z_1)\,\hat{u}_z.$$

The magnitude of $\vec{r}_{12}$ is obtained by using the Pythagoras theorem.

$$r_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Thus the unit vector pointing from 1 to 2 is

$$\hat{u}_{12} = \frac{\vec{r}_{12}}{r_{12}} = \frac{(x_2 - x_1)\,\hat{u}_x + (y_2 - y_1)\,\hat{u}_y + (z_2 - z_1)\,\hat{u}_z}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Further observations:

Clearly, $\hat{u}_{12}$ is dimensionless as you can easily verify by dividing the units of the numerator and denominator. Therefore, $\hat{u}_{12}$ does not have any information about the distance. It only has the information about the direction from q_1 to q_2. Although this formula looks quite complicated, it is quite simple conceptually. You should learn this formula well because it will be very useful throughout this textbook.

1.2.2 Electric Force Compared with Gravitational Force

The electric force F_e between two objects depends on the excess charges on them. The gravitational force F_g, on the other hand, acts between all objects with mass whether they are charged or not.

Therefore, one is likely to draw no definite conclusions by comparing the two forces between arbitrary objects - you can just as easily set up situations where the gravitational force is greater than the electric force as the opposite case. To compare their strengths fairly, we will examine the magnitudes of the gravitational and electric forces between charged elementary particles. There we find without exception that $F_e >> F_g$ if an elementary particle is electrically charged. As a concrete example, consider two electrons located at a separation r from each other. The ratio of F_e to F_g is found to be of the order of 10^{42}.

$$\left(\frac{F_e}{F_g}\right)_{\text{electron}} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} \times \frac{r^2}{G_N m^2} \approx 10^{42}.$$

This shows that the electric force between two electrons at any distance is 10^{42} times stronger than the gravitational force between them. For two protons, the ratio is approximately 2.5×10^{35}, which is still a very large number. For composite objects, such as atoms and molecules and larger systems, gravitational force may be greater or less than the electric force depending on the excess charge on the bodies or an asymmetry in the charge distribution of the charges on the bodies.

Example 1.2.3. Balancing Electric and Gravitational Forces.

How much excess charge per kilogram must two objects have if the electric force between them is equal to the gravitational force?

Solution. To answer this question, consider two objects of mass m and charge q at a distance r. Setting the magnitudes of the electric force between them equal to the gravitational force between them (note: not mg), and then solving for the required charge to mass ratio we find

$$\frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = \frac{G_N m^2}{r^2} \quad \Longrightarrow \quad \frac{q}{m} = \sqrt{4\pi\epsilon_0 G_N} = 8.6 \times 10^{-11}\text{C/kg}.$$

The value for $q/m = 8.6 \times 10^{-11}$ C/kg needed for a balance between electric and gravitational forces corresponds to approximately 5.4×10^8 excess electrons per kilogram of a material. What percent of atoms in the material are charged then? We can estimate the number of atoms in 1 kilogram of a material although it will depend on the material itself. For instance, there are 6.022×10^{26} atoms in 1 kilogram of hydrogen, 2.6×10^{25} atoms in 1 kilogram of sodium, and about 10^{25} atoms in 1 kilogram of iron. No matter which one of these materials you examine, you find that only a small fraction (1 in 10^{16}) of the material needs to acquire charge to overcome the gravitational attraction between them. For elementary particles such as electrons and protons, discussed above, $q/m >> 8.61 \times 10^{-11}$ C/kg;

the q/m of an electron is 1.76×10^{11} C/kg, and that of the proton is 9.6×10^{7} C/kg.

1.2.3 The Coulomb Force and The Superposition Principle

The Coulomb's law gives the electric force between two charges. What happens if you have more than two charges? Experiments show that the force between any two charges is independent of the presence of any other charges. This means that, the force on a charge q_1 by two charges q_2 and q_3 will just be the vector sum of the forces by q_2 and q_3 given by Coulomb's law for each. This property of the electric force, where the Coulomb's forces on a charge by different charges can be added up one by one individually, is called the **Superposition Principle** of Coulomb force. While adding forces we must be cognizant of the fact that forces are vectors, and therefore, should be added according to the rules of vector addition. A good notation for writing direct distances and direction unit vectors for Coulomb force is very helpful in this regard.

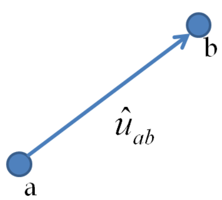

Figure 1.15: The unit vector from a to b denoted by $\hat{u}_{ab}$. Note that in this notation the second subscript indicates the particle towards which the vector is pointed.

NOTATION: We will use $\hat{u}_{ab}$ for the unit vector from particle a to particle b as shown in Fig. 1.15 and r_{ab} for the direct distance between them. The second subscript b will refer to the particle towards which the vector is pointed while the first subscript a refers to the other particle. Thus, subscript ab will stand for the direction from a to b.

Consider three charges q_1, q_2 and q_3 as shown in Fig. 1.16. What will be the force on q_1 by the other two charges? Using the notation developed above, it is rather easy to write down the vector sum of two forces on q_1. In Fig. 1.16, we also show the direct distances r_{21} and r_{31}, and the unit vectors $\hat{u}_{21}$ and $\hat{u}_{31}$. Therefore, the net force on q_1, denoted as $\vec{F}_1$, is calculated as follows.

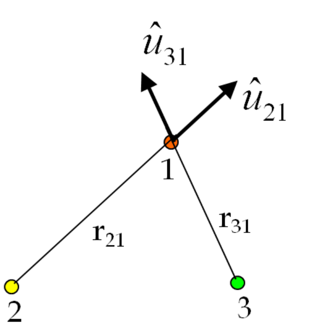

Figure 1.16: Direct distances and unit vectors.

$$\begin{aligned}\vec{F}_1 &= \vec{F}_{\text{by 2 on 1}} + \vec{F}_{\text{by 3 on 1}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{u}_{21} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{r_{31}^2}\hat{u}_{31} \qquad (1.10)\end{aligned}$$

It takes almost no extra effort to write down the net Coulomb force on a charge Q_0 by N charges q_1, q_2, $\cdots$, q_N.

$$\begin{aligned}\vec{F}_0 &= \vec{F}_{\text{by 1}} + \vec{F}_{\text{by 2}} + \cdots + \vec{F}_{\text{by N}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q_0 q_1}{r_{10}^2}\hat{u}_{10} + \frac{1}{4\pi\epsilon_0}\frac{Q_0 q_2}{r_{20}^2}\hat{u}_{20} + \cdots + \frac{1}{4\pi\epsilon_0}\frac{Q_0 q_N}{r_{N0}^2}\hat{u}_{N0}. \qquad (1.11)\end{aligned}$$

Example 1.2.4. Superposition of Coulomb Force. Consider three charges $+2\ \mu$C, $-3\ \mu$C, and $+4\ \mu$C held in the xy-plane at

$(0,0,0)$, $(3\text{ cm}, 1\text{ cm}, 0)$, and $(-2\text{ cm}, 4\text{ cm}, 0)$ respectively. Find the net Coulomb force on the $+2\ \mu$C charge.

Solution. First we should work out the direct distances and unit vectors that we need here.

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = (-3\text{ cm})\hat{u}_x + (-1\text{ cm})\hat{u}_y$$
$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = (2\text{ cm})\hat{u}_x + (-4\text{ cm})\hat{u}_y$$

Therefore,

$$r_{21} = \sqrt{(-3\text{ cm})^2 + (-1\text{ cm})^2} = \sqrt{10}\text{ cm}$$
$$\hat{u}_{21} = -\frac{3}{\sqrt{10}}\hat{u}_x - \frac{1}{\sqrt{10}}\hat{u}_y$$
$$r_{31} = \sqrt{20}\text{ cm}$$
$$\hat{u}_{31} = \frac{1}{\sqrt{5}}\hat{u}_x - \frac{2}{\sqrt{5}}\hat{u}_y$$

Note that when calculating the magnitude of the force, the sign of charges are dropped so that the magnitude is a positive number. The magnitudes of the forces by $-3\ \mu C$, and $+4\ \mu C$ charges are

$$F_{21} = 9 \times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \frac{2 \times 10^{-6}\text{C} \times (3 \times 10^{-6})\text{ C}}{\left(\sqrt{10} \times 10^{-2}\text{ m}\right)^2} = 54\text{ N}.$$
$$F_{31} = 9 \times 10^9 \frac{\text{N.m}^2}{\text{C}^2} \frac{2 \times 10^{-6}\text{C} \times (4 \times 10^{-6})\text{ C}}{\left(\sqrt{20} \times 10^{-2}\text{ m}\right)^2} = 36\text{ N}.$$

The force $\vec{F}_{21}$ is attractive and $\vec{F}_{31}$ is repulsive. Therefore the two forces on $+2\ \mu C$ charge are

$$\vec{F}_{21} = -F_{21}\hat{u}_{21}$$
$$\vec{F}_{31} = F_{31}\hat{u}_{31}$$

Adding these froces gives the following net force in terms of Cartesian components.

$$\vec{F}_{net} = -F_{21}\hat{u}_{21} + F_{31}\hat{u}_{31} = (67\text{ N})\hat{u}_x - (15\text{ N})\hat{u}_y.$$

The components of the net force give the magnitude of the net force on $+2$ C as 69 N and the direction by the angle $12.6°$ clockwise from the positive x-axis direction.

Magnitude: $F_{net} = \sqrt{(67\text{ N})^2 + (15\text{ N})^2} = 69\text{ N}.$

Direction: $\theta = \arctan(-15\text{ N}/67\text{ N}) = 12.6°$, clockwise from x-axis.