Figure 3.13: How would you calculate direct distance between cities since the Earth is not flat? Map credits: www.cia.gov

3.5 VELOCITY AND SPEED

Recall that average velocity gives us only the overall rate of change of position for an entire interval of time. If the time interval is long, we miss the details of the motion, such as the change in velocity during the interval. Clearly, average velocities over smaller intervals will give us a more detailed picture of the motion. It turns out that, if you examine average velocity for successively smaller intervals around each instant of time, you discover a way to deduce another quantity which can be identified as an instantaneous rate of change of the position vector. To gain an insight into the process that gives rise to the definition of **velocity at each instant** it is helpful to examine a simple one dimensional motion.

3.5.1 Velocity in One-dimensional Motion

Suppose we drop a ball from a 20-meter tower. How fast will the ball be moving at the $t = 1$ *sec* mark? Recall that the average velocity is defined over an interval, not at an instant. So, how do we find how fast the ball is moving at a particular instant?

If we try to find the average velocity near the $t = 1$ sec mark for different time intervals near the 1 sec mark, what will we find?

We start by recording the location of the ball at various instants in time. The data for a hypothetical experiment is shown in Table 3.1, where we have displayed time to one microsecond (μs) precision and distance to 10 micrometer (μm) precision. From this table, we can find displacements for various size time intervals near $t = 1$ sec, and corresponding average velocities.

In Table 3.1 we have the data for the position of the ball as we approach the 1 sec mark as well as at instants immediately after

Table 3.1: Data for a ball dropped from 20 m.

Time (s)	Height (m)

that. We use the data in Table 3.1 to construct another table, Table 3.2, which contains average velocity for successively smaller intervals around the 1 sec instant. The details of constructing Table 3.2 is left as an important exercise for the student.

Table 3.2: Average velocity in successively smaller time intervals. As the interval nears $t = 1$ sec shrinks, the average velocity reaches closer and closer to a limit of { 4.9 m/s, down}.

Time interval (s)	Average velocity	
	Magnitude (m/s)	Direction
$1 \le\le 1.1$	5.414500	Down
$1 \le\le 1.01$	4.949245	Down
$1 \le\le 1.001$	4.904902	Down
$1 \le\le 1.0001$	4.900490	Down
$1 \le\le 1.00001$	4.900049	Down
$1 \le\le 1.000001$	4.900005	Down

Table 3.2 shows that, as we shorten the time interval, the average velocity gets closer and closer to { 4.9 m/s, down}. What would happen if we had a continuous record of the position in Table 3.1 instead of only the 10 μm resolution in distance and 1 μs resolution in time?

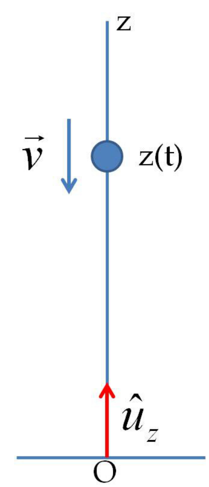

Figure 3.14: Coordinate system for analytic description of falling ball.

It is easier to discuss the continuous space and continuous time case in the analytic approach for vectors. To describe the motion analytically, we need to choose a coordinate system first. Here we will choose a coordinate system with z-axis pointed up with origin at the ground level as shown in Fig. 3.14. The choice of z is arbitrary, we could just as well have chosen x or y.

In this coordinate system, the z-coordinate of the ball equals its height from the ground. The negative z values are for points below the ground. The displacements for the vertically falling ball are all along the z-axis pointed towards negative z-axis.

The position is denoted by the position coordinate z, and the position vector is, of course the vector from the origin to the position of the ball, that is, $\vec{r} = z\ \hat{u}_z$, where the unit vector $\hat{u}_z$ is pointed up as shown in Fig. 3.14. The average velocity is given along the z-axis also, and can be written in terms of change in the z-coordinate over a time interval and the unit base vector.

$$\vec{v}_{ave} = v_z^{ave}\hat{u}_z = \left[\frac{z(t_2) - z(t_1)}{t_2 - t_1}\right]\ \hat{u}_z \tag{3.16}$$

Table 3.2 can now be interpreted as giving us the values of $v_z^{ave}\ \hat{u}_z$.

Since, the unit vector is pointed up, the down direction will mean $v_z^{ave} < 0$. According to Table 3.2, v_z^{ave} has different values depending upon the size of the interval: (-5.414500 m/s) when the interval has a size of 0.1 sec, (-4.949245 m/s) for an interval size of 0.01 sec, ..., (-4.900049 m/s) for an interval size of 0.00001 sec, and (-4.900005 m/s) for an interval size of 0.000001 sec, etc.

Important observation: The sequence of the values for v_z^{ave} for successively smaller time intervals near $t = 1$ *sec* appears to approach a unique limiting value of (-4.90000 m/s).

If you repeat the process described above for intervals around other instants in time, we obtained different values of v_z. For instance, v_z would be zero for $t = 0$ and -19.6 m/s at $t = 2$ sec, etc.

The limiting value obtained by the procedure outlined above is called the instantaneous rate of change of the z-coordinate of the object, which is the z-component, v_z, of the instantaneous velocity vector $\vec{v}$. It is customary to write the process in the following notation:

$$\text{Symmetric difference: } v_z = \lim_{\Delta t \to 0} \left[\frac{z(t + \Delta t/2) - z(t - \Delta t/2)}{\Delta t} \right], \tag{3.17}$$

It can be shown that, instead of taking a symmetric interval of time for the change in z-coordinates, one may take forward or backward coordinate differences without affecting the limit of the sequence.

$$\text{Forward difference: } v_z = \lim_{\Delta t \to 0} \left[\frac{z(t + \Delta t) - z(t)}{\Delta t} \right] \tag{3.18}$$

$$\text{Backward difference: } v_z = \lim_{\Delta t \to 0} \left[\frac{z(t) - z(t - \Delta t)}{\Delta t} \right] \tag{3.19}$$

It is instructive to write the change in coordinates as Δz for the interval of time Δt. Then, the symmetric, forward, and backward difference formulas can be combined into one form.

$$v_z = \lim_{\Delta t \to 0} \left[\frac{\Delta z}{\Delta t} \right], \tag{3.20}$$

The procedure described above for obtaining v_z defines v_z as the time derivative of the z-coordinate.

$$\boxed{v_z(t) = \frac{dz}{dt}.} \tag{3.21}$$

This says that the z-component of velocity is equal to the rate of change of the z-coordinate of the particle. The velocity at time t for the particle that moves only along the z-axis is given by multiplying the unit vector for the z-axis by v_z.

$$\vec{v}(t) = v_z(t)\ \hat{u}_z. \tag{3.22}$$

The change in z-coordinate of the particle can be obtained from the z-component of the velocity by integrating. Let the particle be at z_1 at time t_1 and at z_2 at t_2, then

$$\boxed{z_2 - z_1 = \int_{t_1}^{t_2} v_z(t)dt.} \tag{3.23}$$

Further Observations

What would happen if we choose the x or y-axis to point up in place of the z-axis used above? Had we chosen the x-axis to point in the vertical direction, our analysis would have been in terms of the x-coordinate of the ball. This would have led us to the velocity as $\vec{v} = v_x\hat{u}_x$, where v_x would be the rate of change of the x-coordinate, called the x-component of the velocity or the x-velocity.

$$\boxed{v_x = \frac{dx}{dt}.} \tag{3.24}$$

Similarly, if the y-axis was chosen in the vertical direction, we would have written the velocity as $\vec{v} = v_y\hat{u}_y$, where v_y is the rate of change of the y-coordinate, i.e. the y velocity.

$$\boxed{v_y = \frac{dy}{dt}.} \tag{3.25}$$

For a general motion in three dimensions, all three coordinates of the object will change with time at their respective rates, and we will write velocity as the vector sum of velocities along the axes.

Components of velocity:

$$v_x = \frac{dx}{dt}$$
$$v_y = \frac{dy}{dt}$$
$$v_z = \frac{dz}{dt}$$

$$\boxed{\vec{v} = v_x\hat{u}_x + v_y\hat{u}_y + v_z\hat{u}_z = \left(\frac{dx}{dt}\right)\hat{u}_x + \left(\frac{dy}{dt}\right)\hat{u}_y + \left(\frac{dz}{dt}\right)\hat{u}_z.} \tag{3.26}$$

We will have more to say about the three-dimensional case below. Here we continue the discussion of one-dimensional motion.

Example 3.5.1. Instantaneous velocity as derivative of position You are driving a race car towards East on a straight East-West road. Your position from an intersection is given in a particular reference in which origin is at the intersection and the x-axis is along the road with positive x-axis in the direction of East from the intersection. The x-coordinate of your car with time is given by the function, $x(t) = 20 + 15t + 4t^2$, where x is in meters and t in seconds. Find the velocity of the car at (a) $t = 0$, (b) $t = 5$ sec, and (c) $t = 10$ sec?

Solution. This problem demonstrates a direct calculation of the x-component of velocity, v_x from the calculation of derivative of $x(t)$. Since the car is moving only on the x-axis, the velocity of the car at an arbitrary instant t will be given by

$$\vec{v}(t) = v_x(t)\ \hat{u}_x.$$

First, let us calculate the derivative of $x(t)$ and then evaluate the resulting x-component of the instantaneous velocity at different instants.

$$\begin{aligned} v_x &= \frac{dx}{dt} \\ &= \frac{d}{dt}\left(20 + 15t + 4t^2\right) \\ &= 15 + 8t. \end{aligned} \tag{3.27}$$

(a) Now, we put $t = 0$ in Eq. 3.27 to find the x-component of velocity at $t = 0$.

$$v_x(0) = 15 + 8(0) = 15 \text{ m/s}.$$

Therefore, the velocity at $t = 0$ mark is

$$\vec{v}(0) = (15 \text{ m/s})\ \hat{u}_x.$$

(b) The instant now is $t = 5$ *sec*. Therefore, we put $t = 5$ sec in Eq. 3.27 to find the instantaneous velocity at $t = 5$ sec.

$$v_x(5 \text{ sec}) = 15 + 8(5) = 55 \text{ m/s}.$$

Therefore, the velocity at 5 sec mark is

$$\vec{v}(5\ sec) = (55 \text{ m/s})\ \hat{u}_x.$$

(c) Same process gives $v_x(10 \text{ sec}) = 95$ m/s, and $\vec{v}(t) = (95 \text{ m/s})\ \hat{u}_x$.

Example 3.5.2. Change in x-coordinate from constant v_x. A ball is rolling on the floor such that its x-coordinate in a particular Cartesian coordinate system changes at a constant rate u_0. If the x-coordinate is x_0 at $t = t_0$ what will be the x-coordinate at $t = T$? [NOTE: This example and the next use integration. Integrations appear naturally when dealing with varying rates as we will see in the next chapter.]

To the student: If you can-

Solution. From our discussion we know that the rate of change in a coordinate is given by the derivative of the coordinate with time.

$$\frac{dx}{dt} = \text{Rate of change of } x \text{ coordinate.}$$

Here the rate is given to be u_0. Therefore, we have the following equation for the derivative of $x(t)$.

$\Longrightarrow$

$$\frac{dx}{dt} = u_0.$$

$v_x(t)dt$.

This equation can also be written for differential elements, dx and dt by multiplying both sides by dt.

$$dx = u_0 dt,$$

which we can integrate on both sides. On the left side, the limit of integration is x_0 to $x(T)$ and the limit on the right side is the times for those x values, viz. from t_0 to T.

$$\int_{x_0}^{x(T)} dx = \int_{t_0}^{T} u_0 dt,$$

which gives the following result.

$$x(T) - x_0 = u_0(T - t_0).$$

Further Observations

An integration is actually not necessary here since the rate of change is constant in time. The constant rate implies that the average rate is the same as instantaneous rate of change. Therefore,

$$\frac{\Delta x}{\Delta t} = u_0,$$

which can be multiplied both sided by Δt to solve for Δx. This immediately gives the change over the interval from t_0 to T as

$$x(T) - x_0 = u_0(T - t_0).$$

Example 3.5.3. Change in x-coordinate from varying v_x. A particle moves in the xy-plane of a coordinate system. The rate of change of its x-coordinate varies linearly with time such that the rate at any particular time is $v_x = u_0 + bt^2$, where u_0 and b are constants. If the x-coordinate is x_0 at $t = t_0$ what will be the x-coordinate at $t = T$?

Solution. The discussion of the last example takes us to the following equation for the differentials

$$dx = \left(u_0 + bt^2\right) dt,$$

which can be integrated to give

$$x(T) - x_0 = u_0(T - t_0) + \frac{b}{3}\left(T^3 - t_0^3\right). \tag{3.28}$$

Further Observations - Computing the average x-velocity from changing v_x.

Here we must resort to integration since the rate is changing with time. The short-cut method applicable for constant rate does not give the correct answer. The average rate is also not equal to the average of the x-component of velocity at the end and the x-component of velocity at the beginning.

$$v_x^{ave} \neq \frac{v_x(T) + v_x(t_0)}{2}.$$

The x-component of the average velocity is equal to the change in the x-coordinate divided by the interval. Using Eq. 3.28 we find v_{ave} to be

$$v_x^{ave} = \frac{x(T) - x_0}{T - t_0} = u_0 + \frac{b}{3}\left(T^2 + t_0 T + t_0^2\right),$$

which can also be obtained by integrating the x-component of the instantaneous velocity over time and dividing by the interval.

$$v_x^{ave} = \frac{\int_{t_0}^{T} v_x(t)dt}{T - t_0} = \frac{\text{Integration of } x\text{-instantaneous velocity}}{\text{Duration}}.$$

This gives the same result.

Computing Derivatives from the Slopes of Tangents

In experiments we measure position at a finite number of instants and deduce the instantaneous velocity by making a plot of position versus time. You will now see that the components of velocity, i.e. $v_x(t)$, $v_y(t)$ and $v_z(t)$, can be determined from the plots of x vs t, y vs t and z vs t respectively, since the derivative of a function is also equal to the slope of the tangent of the curve when the function is plotted along the ordinate (the vertical axis) and time along the abscissa (the horizontal axis).

Let us look at an example of the x-component of velocity from the x vs t plot. Consider the graphs of a function $x(t)$ given in Fig. 3.15. Suppose we are interested in the x-component of velocity at $t = 1$ sec. The slopes of the secants joining a point of the curve before $t = 1$ sec and a point after $t = 1$ sec give the x-component of the average velocity for various time intervals. The figure shows that, as we examine shorter and shorter time intervals, the secants tend to become parallel to the tangent to the curve at $t = 1$ sec, the instant of interest. Therefore, the slope of the tangent will equal the instantaneous rate of change. This gives us another way of computing derivatives, which is particularly important for relating to experimental results.

$$\frac{dx}{dt} = \text{Slope of the tangent to the } x \text{ vs } t \text{ curve.} \tag{3.29}$$

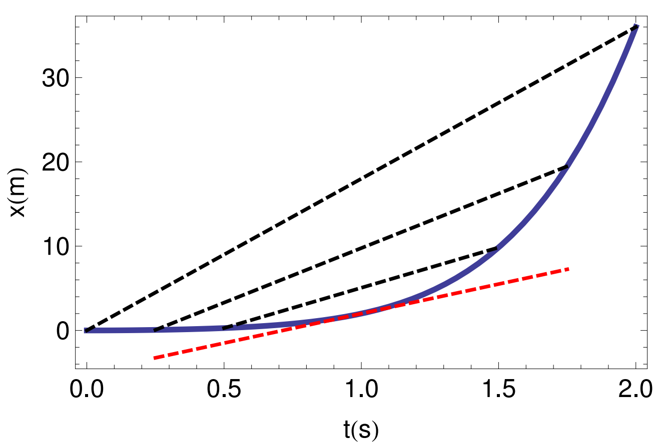

Figure 3.15: The x-component of the instantaneous velocity at the instant $t = 1$ sec is equal to the slope of x vs t curve at $t = 1$ sec. The slopes of the secants for various time intervals around $t = 1$ sec give average velocities in those time intervals. This figure shows visually that the slopes of secants change as we zero in near the instant of interest, eventually the secant becomes parallel to the tangent to the curve.

Example 3.5.4. Instantaneous velocity from slope. Figure 3.16 shows an example of calculation of the x-component of instantaneous velocity from a plot of x vs t. To obtain the x-velocity at an instant, say at $t = t_1$, we draw a tangent to the curve at the time of interest, i.e., $t = t_1$ as shown in the figure. The rise of the tangent line gives the change in position Δx and the run of the tangent gives the duration Δt for that change. Therefore, dividing Δx by Δt is equal to the rate of change of the x-coordinate of the object at the instant $t = t_1$, which is the x-component of instantaneous velocity at time $t = t_1$.

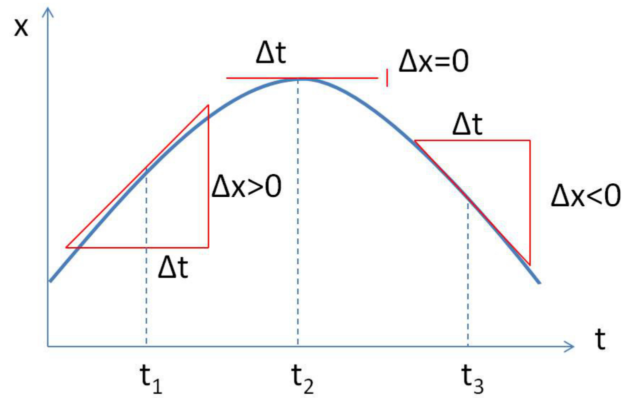

Figure 3.16: Example 3.5.4. The slope of tangents to the curve $x(t)$ gives the x-component of the instantaneous velocity. The slopes of the tangents obtained by dividing the rise Δx by the run Δt at instants $t = t_1$, $t = t_2$, and $t = t_3$ give the values of the x-component of velocity at those instants. Here, $v_x > 0$ at $t = t_1$, $v_x = 0$ at $t = t_2$ and $v_x < 0$ at t_2.

At time $t = t_1$, the slope of the tangent, i.e., v_x, is positive, meaning the velocity is pointed in the positive x direction for a one-dimensional motion on the x-axis. Just to remind you that velocity for one-dimensional motion on the x-axis is $\vec{v} = v_x \hat{u}_x$, where $\hat{u}_x$ is the unit vector pointed towards the positive x-axis. Note that at time $t = t_2$ in Fig. 3.16, the slope is zero, meaning that the x-velocity is zero there. At time $t = t_3$, the slope is negative, meaning the velocity will be pointed in the negative x-axis direction for a one-dimensional motion.

We wish to emphasize here that an x vs t plot gives you only v_x. In a three-dimensional motion, you will need x vs t, y vs t, and z vs t so that you can deduce x, y, and z-components of velocity, viz. v_x, v_y, and v_z, and construct the velocity vector from them.

Example 3.5.5. Velocity from slope. The position of a box in a one-dimensional motion is recorded by placing the y-axis on the line of motion. The data is plotted as a y vs t plot and shown in Fig. 3.17. Find the y-component of velocity at $t = 1$ sec, 3 sec, 5 sec, and 7 sec.

Solution. The plot of y vs t is a segment-wise straight line, which makes finding slopes easier. We do not need to draw any tangents since the tangent to a straight line is the straight line itself. The slopes at $t = 1$ sec and $t = 3$ sec are equal, $v_y = 1$ m/s. The slope at $t = 5$ sec is zero since the line is flat and rise is zero. Therefore at $t = 5$ sec, $v_y = 0$. Finally, the slope is negative at $t = 7$ sec. The y-coordinate changes by -2 m in 3 sec steadily. This gives $v_y = -2/3$ m/s.

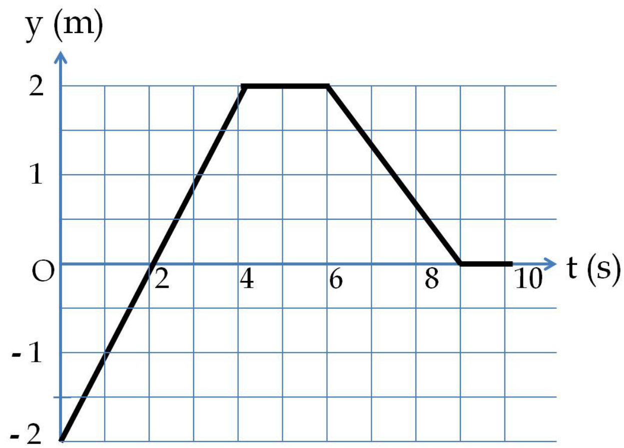

Figure 3.17: Example 3.5.5. Assume corners to be smooth.

Further remarks: Note that in a plot of x, y, or z vs t, there cannot be sharp corners since the velocity of an object cannot change abruptly. That is, the velocity at an instant before the corner must be arbitrarily close to the velocity an instant after the corner. In the limit of infinitesimal interval around the corner, the velocity obtained at an instant before a corner will not equal the one for an instant after the corner. Therefore, the plots of coordinates versus time must be smooth at all points, meaning that $x(t)$, $y(t)$ and $z(t)$ are smooth functions of t, not just continuous functions.

3.5.2 Velocity - General

We now generalize the discussions of the one-dimensional cases presented in the last subsection. As usual for vectors, it is helpful to look at the velocity vector from both geometric and analytic viewpoints. The geometric viewpoint gives a pictorial view of vectors and the analytic viewpoint based on a particular choice of coordinates is more suited for calculations. Note that, while in the geometric viewpoint one always works with the complete vector, in the analytic approach we work with components of the vector and in the end we must recover the magnitude and direction of the vectorial physical quantity from the components.

Geometric Viewpoint

Consider the trajectory of an arbitrary motion given in Fig. 3.18. What would be the velocity vector at time t when the object is at the point labeled P? We define the velocity vector in the same way

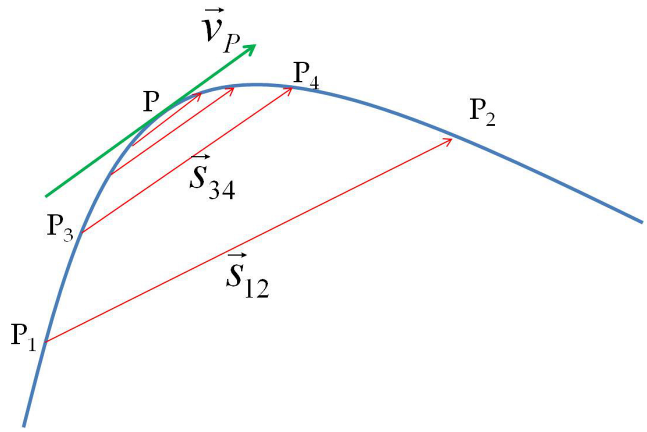

Figure 3.18: The instantaneous velocity $\vec{v}_P$ at the instant t when the object is at point P is pointed in the direction of tangent to the trajectory at point P. The object moves on the trajectory in the direction of P_1, P_3, P, P_4, P_2, $\cdots$. The limit of the sequence of average velocity vectors: { $\vec{s}_{12}/(t_2 - t_1)$, $\vec{s}_{34}/(t_4 - t_3)$, $\cdots$ }, as time segments become increasingly smaller around point P defines the velocity at point P. Here t_1, t_2, t_3, t_4 correspond to time when the object is at P_1, P_2, P_3, P_4, respectively.

here as we did for the one-dimensional motion: the displacements for increasingly smaller time intervals around point P are divided by their time intervals to evaluate average velocity vectors for those intervals. The process is shown schematically in Fig. 3.18. The limit of the sequence of average velocity vectors { $\vec{s}_{12}/(t_2 - t_1)$, $\vec{s}_{34}/(t_4 - t_3)$, $\cdots$ } for increasingly smaller time intervals is the instantaneous velocity at time t. Informally, we will write velocity at time t as

$$\vec{v} = \lim_{\Delta t \to 0} \left[\frac{\Delta \vec{r}}{\Delta t} \right], \tag{3.30}$$

where $\Delta \vec{r}$ is the displacement in the interval from $(t - \Delta t/2)$ to $(t + \Delta t/2)$. Here, $\Delta r = \vec{r}(t + \Delta t/2) - \vec{r}(t - \Delta t/2)$ is the change in the position vector over the interval. As the interval near t is made increasingly smaller, we obtain a limiting vector which represents the instantaneous rate of change of the position vector. The limit is called the **instantaneous velocity** and can be formally written as

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}.} \tag{3.31}$$

Analytic Viewpoint

For the analytic viewpoint we make use of a Cartesian coordinate system and follow the changes in the Cartesian coordinates of the points on the trajectory of the motion. The x, y, and z-coordinates of the points on the trajectory change with time. The trajectory is said to be described by a triplet of functions $\{\ x(t), y(t), z(t)\ \}$ such that the position vector of the object at time t is given by the following vector:

$$\vec{r}(t) = x(t)\ \hat{u}_x + y(t)\ \hat{u}_y + z(t)\ \hat{u}_z, \tag{3.32}$$

where $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ are the usual unit vectors pointed in the directions of positive x, y, and z-axes respectively. The process of taking the limit of the sequence of average velocity vectors $\{\ \vec{s}_{12}/(t_2 - t_1)$, $\vec{s}_{34}/(t_4 - t_3),\ \cdots\ \}$ for increasingly smaller time intervals separates into similar processes for the three coordinate functions since the unit vectors have the same orientations for every point on the trajectory. Therefore, a separation of motion into motions along different axes occurs as illustrated in Fig. 3.19 for a two-dimensional case.

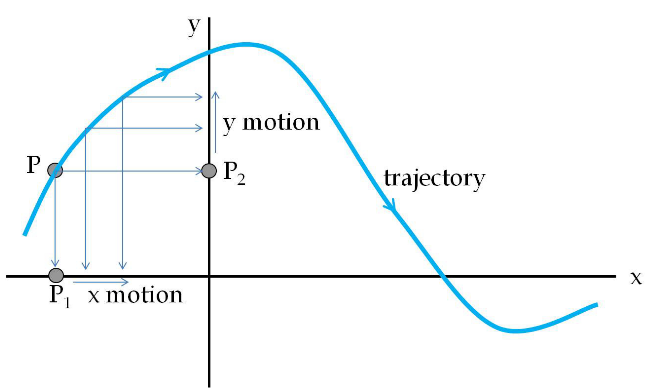

Figure 3.19: The coordinates of a particle P moving in the xy-plane change independently giving an equivalent picture of the motion. One particle P along the actual trajectory is analyzed as the motion of two particles P_1 and P_2, one moving along the x-axis and the other moving along the y-axis. The motions of P_1 and P_2 reproduce the changes in the x and y-coordinates of the original particle. The separate motion of the coordinates is a consequence of the vector nature of kinematic quantities.

By looking at the way the x-coordinates change with time, we determine the x-component of the velocity vector.

$$v_x = \frac{dx}{dt}.$$

Similarly for the y and z-components of the velocity vector. Therefore, the analytic expression for the velocity at time t is given by adding the velocities along the axes. Formally,

$$\boxed{\vec{v}(t) = \frac{dx}{dt}\ \hat{u}_x + \frac{dy}{dt}\ \hat{u}_y + \frac{dz}{dt}\ \hat{u}_z,} \tag{3.33}$$

which can be written more compactly as the expression in Eq. 3.31, viz.

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt}.}$$

The magnitude of the velocity vector can be obtained from its components as

$$\boxed{\text{Magnitude: } |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},} \tag{3.34}$$

and the information about the direction is obtained from the components by drawing a vector from $(0, 0, 0)$ to (v_x, v_y, v_z) in a three-dimensional coordinate system whose axes are marked with velocity component values. For a three-dimensional situation, you will need two angles for the specification of the direction of the vector, which are usually θ and ϕ of the spherical coordinate system. And, for a two dimensional situation, the direction can be specified by one angle only, which is usually the angle going counter-clockwise from the x-axis in the xy-plane. The way to figure out the directions of vectors have been treated earlier in this chapter and in the chapter on vectors (see Chapter 2).

Direction of $\vec{v}$:

One-dimensional along (x) axis: $x > 0$ or $x < 0$.

Two dimensional in (xy) plane: θ counter-clockwise from $+x$ axis.

Three dimensional: θ and ϕ of spherical coordinate system

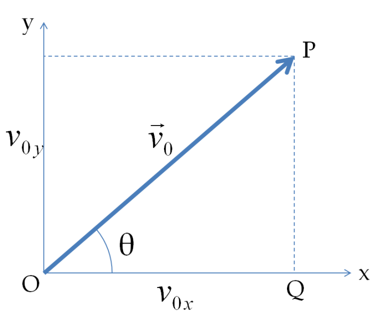

Figure 3.20: Example 3.5.6.

Example 3.5.6. The Components of Velocity in a Plane. Suppose a cannon ball is fired with a speed v_0 at an angle θ with the horizontal direction. A coordinate system is chosen such that the x-axis points horizontally and the y-axis points vertically up. What are the x and y-components to the velocity vector?

Solution. It is helpful to draw the vector and the axes as shown in Fig. 3.20. The hypotenuse of the right-angled triangle $\triangle OPQ$ has length equal to the magnitude of the velocity vector, and the sides OQ and PQ are the x and y-components, to be denoted as v_{0x} and v_{0y} respectively. Trigonometry yields the following expressions for the components immediately.

$$v_{0x} = v_0 \cos\theta$$
$$v_{0y} = v_0 \sin\theta$$

Example 3.5.7. Velocity of a Projectile at Different Points. The trajectory of a projectile is shown in Fig. 3.21 with the assumption that the initial velocity was 80 m/s at the angle of 72° from the horizontal direction. The position of the projectile at successive 1 sec instants are shown in the figure. You can see that the

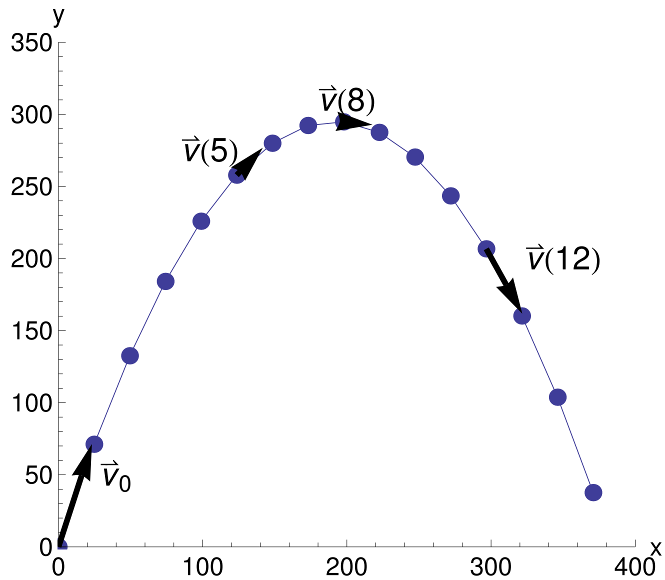

Figure 3.21: Example 3.5.7. The velocity vectors at $t = 0$, 5 sec, 8 sec, 12 sec are shown to demonstrate the changing magnitude and direction of velocity.

direction as well as the magnitude of the velocity of the projectile changes with time. As the projectile rises, the distance covered in each subsequent second decreases as a result of slowing down. At the top of the trajectory, the projectile only has horizontal velocity since the vertical component of the velocity is zero there. The velocity is never pointed vertically straight down since the projectile always has a non-zero horizontal component of the velocity.

Example 3.5.8. Instantaneous Velocity of a Projectile. The velocity of the projectile whose trajectory is shown in Fig. 3.21 at $t = 12$ sec has the following components: $v_x = 24.7$ m/s, $v_y = -41.6$ m/s, $v_z = 0$. Find the magnitude and direction of the velocity at that instant.

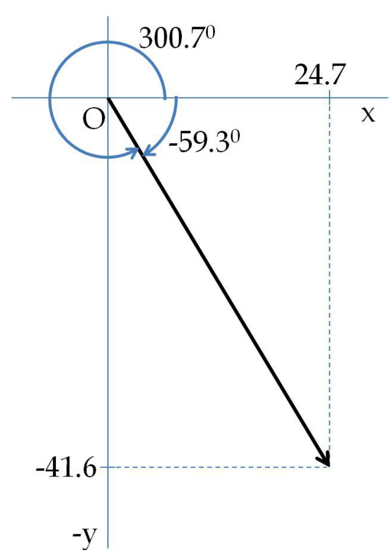

Figure 3.22: Example 3.5.8.

Solution. The magnitude of the velocity vector is easy to work out from its components.

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = 48.4 \text{ m/s}.$$

Since velocity vector is contained in a plane, here the xy-plane, we need only one angle. Normally we give angle from the positive x-axis. The x-component of the velocity is positive and y-component negative, which places the direction in the fourth quadrant. Therefore, the angle we use is either the clockwise angle from the positive

x-axis which will be negative or the counterclockwise angle from the positive x-axis as shown in Fig. 3.22.

$$\text{Clockwise angle from positive } x \text{ axis} = \arctan\left(\frac{-41.6}{24.7}\right) = -59.3^\circ.$$
$$\text{Counter-clockwise angle from positive } x \text{ axis} = 360^\circ - 59.3^\circ = 300.7^\circ.$$

Example 3.5.9. Analyzing Two-dimensional Data. A ball is rolling on the floor. The position of the ball is recorded as (x, y) of a coordinate system. The z-coordinate of the ball is always zero in this coordinate system and therefore ignored. The plots of x vs t and y vs t are displayed here. Find the velocity of the ball at (a) $t = 3$ sec, (b) $t = 5$ sec, and (c) $t = 7$ sec.

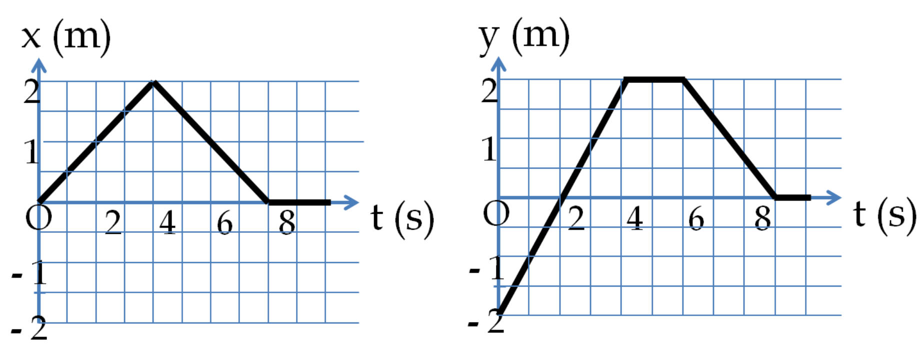

Figure 3.23: Example 3.5.9.

Solution. From the given plots, we can read off slopes at the indicated times to obtain the x and y-components of the velocity. They are:

$$\begin{aligned}
t = 3 \text{ sec}: &\quad v_x = 0.5 \text{ m/s},\ v_y = 1.0 \text{ m/s}.\\
t = 5 \text{ sec}: &\quad v_x = -0.5 \text{ m/s},\ v_y = 0.0 \text{ m/s}.\\
t = 7 \text{ sec}: &\quad v_x = -0.5 \text{ m/s},\ v_y = -1.0 \text{ m/s}.
\end{aligned}$$

From the components we determine the magnitude and directions of the velocity the indicated instants as usual with the following results.

$$\begin{aligned}
t = 3 \text{ sec}: &\quad \text{Magnitude: } v = \sqrt{v_x^2 + v_y^2} = 1.1 \text{ m/s};\\
&\quad \text{Direction: } 63^\circ \text{ counterclockwise from } +x\text{-axis}.\\
t = 5 \text{ sec}: &\quad \text{Magnitude: } v = \sqrt{v_x^2 + v_y^2} = 0.5 \text{ m/s};\\
&\quad \text{Direction: towards negative } x\text{-axis}.\\
t = 7 \text{ sec}: &\quad \text{Magnitude: } v = \sqrt{v_x^2 + v_y^2} = 1.1 \text{ m/s};\\
&\quad \text{Direction: } 63^\circ \text{ clockwise from the } +x\text{-axis}.
\end{aligned}$$

3.5.3 Speed

The rate at which an object covers physical distance on the trajectory of its motion is called its **instantaneous speed** or simply speed. At any instant the infinitesimal distance ds covered on the actual trajectory, whether curved or not, will be identical to the magnitude of the infinitesimal displacement $d\vec{r}$in that interval.

$$\boxed{ds = |d\vec{r}|}$$

Therefore, speed at an instant is equal to the magnitude of the velocity vector at that instant.

Instantaneous speed = Magnitude of instantaneous velocity.

Therefore, we denote speed by the same symbol as the velocity except for the arrow over the symbol.

$$\boxed{\text{Instantaneous speed, } v = |\vec{v}|\,.} \tag{3.35}$$

Since instantaneous speed is only the magnitude of instantaneous velocity vector, it does not have the information about the direction of motion. For example, a missile moving at 10 m/s towards East and another one at 10 m/s towards North, both have the same speed of 10 m/s, but their velocities are different, the first one being { 10 m/s, East} and the second one { 10 m/s, North }. While the velocity vector is represented by an arrow, speed is simply a non-negative real number.

If the velocity vector is examined analytically in a coordinate system, we will have its x, y and z-components v_x, v_y and v_z. In that case, we can write the speed in terms of the components of the velocity vector.

$$\boxed{\text{Instantaneous speed, } v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.} \tag{3.36}$$

Note that, although average speed and magnitude of average velocity were not related since they correspond to any arbitrary size interval, the instantaneous speed is actually equal to the magnitude of the instantaneous velocity since the interval size is infinitesimal. In an infinitesimal interval the change in position is equal to the distance covered since there is not change in direction.

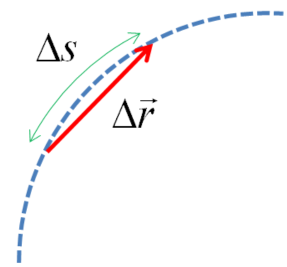

Figure 3.24: For finite interval of time Δt, the actual distance Δs traveled on the trajectory may differ from the magnitude of the displacement, $|\Delta\vec{r}$. When the time interval becomes infinitesimal, the difference between Δs and $|\Delta\vec{r}$ disappears for all motions.