13.3 MAXWELL'S EQUATIONS IN POINT FORM

In our studies of the electricity and magnetism we have extensively studied two vector fields, namely the electric and magnetic fields $\vec{E}$ and $\vec{B}$. You may recall that we had introduced the concept of fields to get away from the action at a distance present in the non-field formulation, such as the Coulomb's force law and Newton's law of universal gravitation. However, we have found that the experimental observations are usually interpreted in terms of laws that integrate over these fields. Let us summarize the four laws of electric and magnetic field in the integral form as a reference.

$$\text{Gauss's Law for } \vec{E}\text{:} \quad \oint\!\!\!\int \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0} \tag{13.39}$$

$$\text{Gauss's Law for } \vec{B}\text{:} \quad \oint\!\!\!\int \vec{B}\cdot d\vec{A} = 0 \tag{13.40}$$

$$\text{Faraday's Law:} \quad \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{13.41}$$

$$\text{Ampere-Maxwell's Law:} \quad \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0\epsilon_0\frac{d\Phi_E}{dt} \tag{13.42}$$

These integral forms of the laws of electricity and magnetism are written in terms of the flux and circulations of the electric and magnetic fields. We have also learned that the flux and circulations of vector fields are related to the divergence and curl of the fields at space points. More specifically, the flux per unit volume through a closed surface about a point P is equal to the divergence of the field at that point, and the circulation per unit area of a loop about a point P is equal to the component of the curl of the vector field in the direction perpendicular to the loop. We can use these results from the last section to deduce equations that will relate properties of electric and magnetic fields at each point in space and at a particular instant in time.

For instance, let us apply Gauss's law given in Eq. 13.39 to a closed surface S around an arbitrary space point P which may or may not have any charge in the volume V enclosed by the surface S. Suppose we divide both sides by the volume V and take the infinitesimal volume limit.

$$\lim_{V\to 0}\left[\frac{1}{V}\oint\!\!\!\int_S \vec{E}\cdot d\vec{A}\right] = \frac{1}{\epsilon_0}\lim_{V\to 0}\left[\frac{Q_{\text{enc}}}{V}\right]$$

The limit on the left side will give the flux per unit volume at the point P which is equal to the divergence of the electric field at that point, and the limit on the right side will give charge per unit volume, which is the charge density ρ at point P.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This equation is the differential form of the Gauss's law. This equation is also called the point or local form of the Gauss's law. Often we say this is the Gauss's law. The Faraday's and the Ampere-Maxwell's laws are not as easily converted into the differential forms because they involve curls. I will just write the final answer for all the four laws for the electric and magnetic field here since they are so pretty in the differential form.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{13.43}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{13.44}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{13.45}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{13.46}$$

These equations together with the Lorentz force law gives a complete description of all electromagnetic phenomena. According to the Lorentz force law, the force on a point charge Q with velocity $\vec{v}$ is given by

$$\vec{F}_{em} = Q\vec{E} + Q\vec{v} \times \vec{B}. \tag{13.47}$$

We will now explore consequences of Maxwell's equations using the differential form.