1.2 ROTATING FRAME

1.2.1 Kinematics in a Uniformly Rotating Frame

Since a rotating frame is not an inertial frame, the form of the second law, $\vec{F} = m\vec{a}$, is modified to $\vec{F} - m\vec{A} = m\vec{a}\,'$, where $\vec{A}$ is the acceleration of the non-inertial frame with respect to the inertial frame. In this section, we will work out the implications of the acceleration of the frame arising from the rotation of the frame with respect to an inertial frame.

Let us denote the kinematic variables in the accelerating frame with a prime as above. Let $\vec{r}\,'$, $\vec{v}\,'$, and $\vec{a}\,'$ be the position, velocity, and acceleration of the particle as observed in the rotating frame. Their definitions are the same in the rotating frame as in any other frame, viz., the velocity being the rate of change of the position and the acceleration the rate of change of the velocity.

To relate the position, velocity and acceleration vectors in the rotating frame to the corresponding quantities in the non-rotating inertial frame, it is helpful to examine these quantities for a point particle in two different frames that share a common origin but one rotating with respect to the other about the common z-axis as shown in Fig. 1.5.

For simplicity we pick the zero of time at the instant when the two frames have their axes lined up in the same directions so that all the coordinates of the same particle in the two frames have the same values at $t = 0$. In the following, we will consider two situations. In the first situation, we will choose a particle at rest in the rotating frame, and in the second situation, the particle will be allowed to move in a plane perpendicular to the rotation axis of the rotating frame.

A particle fixed at the x-axis of a uniformly rotating frame

Consider a particle fixed at $(x', 0, 0)$ as shown in Fig. 1.5. Although this particle's position in the rotating frame is fixed, its position in the fixed frame is changing. As a matter of fact, as far as the fixed frame is concerned, the particle moving in xy-plane in a circle of radius equal to x' with angular speed Ω.

The position of the particle in the two frames can be obtained by simple trigonometry of the situation in the xy-plane. We will omit

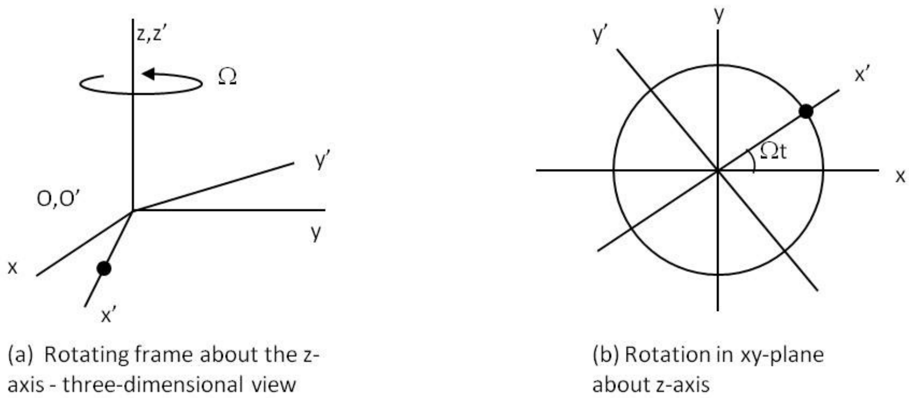

Figure 1.5: Looking at a particle at rest in a rotating frame.

the z-components since they are all zero in the two frames.

$$\begin{aligned} \text{Rotating frame:} \quad & x' = \text{fixed},\ y' = 0 \\ \text{Inertial frame:} \quad & x = x'\cos\Omega t,\ y = x'\sin\Omega t \end{aligned}$$

Therefore, the velocities of the particle in the two frames, obtained by taking derivatives of the positions, are as follows.

$$\begin{aligned} \text{Rotating frame:} \quad & v'_x = \frac{dx'}{dt} = 0,\ v'_y = \frac{dy'}{dt} = 0 \\ \text{Inertial frame:} \quad & v_x = \frac{dx}{dt} = -\Omega x'\sin\Omega t,\ v_y = \frac{dy}{dt} = \Omega x'\cos\Omega t \end{aligned}$$

The acceleration of the particle in the two frames can be obtained by taking derivatives of the corresponding velocities.

$$\begin{aligned} \text{Rotating frame:} \quad & a'_x = \frac{dv'_x}{dt} = 0, \quad a'_y = \frac{dv'_y}{dt} = 0 \\ \text{Inertial frame:} \quad & a_x = \frac{dv_x}{dt} = -\Omega^2 x'\cos\Omega t, \quad a_y = \frac{dv_y}{dt} = -\Omega^2 x'\sin\Omega t \end{aligned}$$

The x and y-components of acceleration in the inertial frame are equivalent to the radially inward acceleration, i.e. the centripetal acceleration.

Inertial frame:

$$\vec{a} = \begin{cases} \text{Magnitude} = \Omega^2 x' \\ \text{Direction: Towards the origin.} \end{cases} \tag{1.10}$$

This makes sense, because from the perspective of the fixed frame, the particle moves in a uniform circular motion of radius equal to x' at constant angular speed Ω. Therefore, the acceleration in the fixed frame must be pointed radially inwards.

Vector Notation

It is instructive to write the kinematic quantities in the fixed and rotating frames given above in a vector notation. The example particle does not move with respect to the rotating frame. This particle will appear to rotate in a circle about the z-axis when observed from the fixed frame. Therefore, in the fixed frame the angular velocity vector will be pointed along the z-axis.

$$\vec{\Omega} = \Omega \hat{u}_z. \tag{1.11}$$

The velocity of the particle in the fixed frame is tangential to the circle in the xy-plane and is given by the cross product of the angular velocity vector and the position vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r}. \tag{1.12}$$

Since acceleration is equal to the time-derivative of velocity, we obtain the following for the acceleration in the fixed frame.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} == \vec{\Omega} \times \frac{d\vec{r}}{dt} \\ \Longrightarrow \quad \vec{a} &= \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r} \right) \quad (\text{Const } \vec{\Omega}) \end{aligned}$$

We find that, although the particle is not moving in the rotating frame, the particle has an acceleration in the fixed frame.

A particle moving in the xy-plane of a uniformly rotating frame

Let us consider a particle that is not fixed but moves in the xy-plane of a uniformly rotating frame $Ox'y'z'$ which is rotating with respect to a fixed frame $Oxyz$ about their common z-axis. Since the frame is rotating about the z-axis, the particles moves in the xy-plane of the fixed frame as well. The situation between time t and $t + \Delta t$ is shown in Fig. 1.6. For simplicity, let the axes of the two frames be coincident at time t.

Let (x, y) and (x', y') be the coordinates of the particle at some time t in the $Oxyz$ and $Ox'y'z'$ frames respectively. The change of the coordinates over the interval from t to $t + \Delta t$ are $(\Delta x, \Delta y)$ and $(\Delta x', \Delta y')$ respectively. What are their relations? The relation between the displacements in the two frames is more conveniently written in the vector notation. Therefore, we will work with vector notation here.

At time t, let the particle be at point P along momentarily coincident x and x'-axes (Fig. 1.7). The non-rotating frame marks

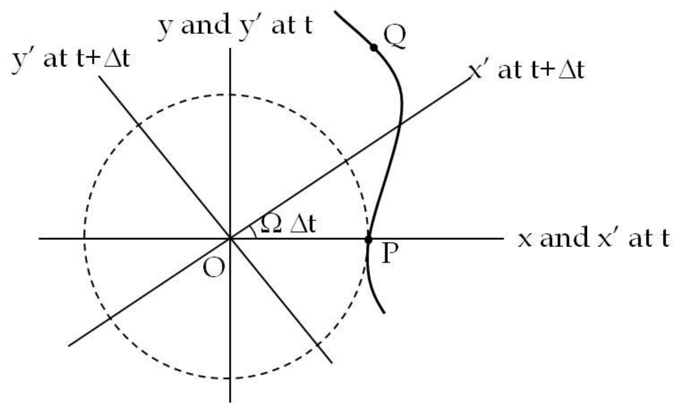

Figure 1.6: The physical situations at times t and $t + \Delta t$. In duration t to $t + \Delta t$, the particle moves from P to Q, and the rotating frame rotates by an angle Ωt. The common z-axis pointed out-of-page.

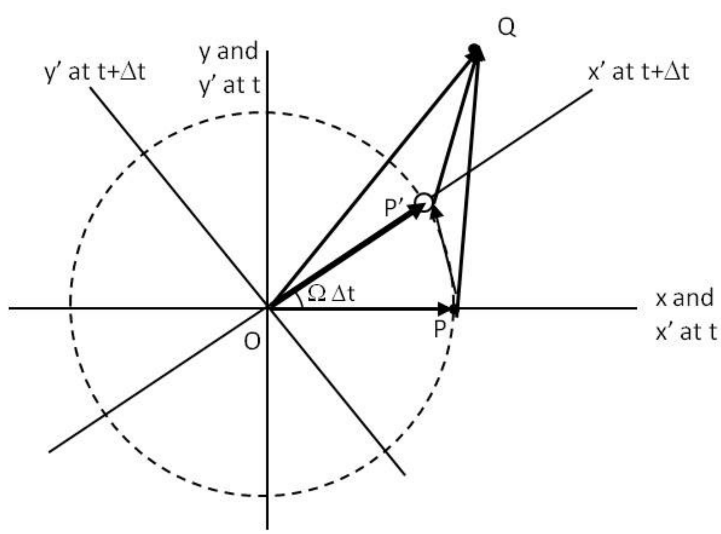

Figure 1.7: Position of a point particle P from rotating frame.

the location with a marker at P at its x-axis and the rotating frame marks the same location in its frame as P' at its x-axis. As time passes, the particle moves to another location shown as Q at time $t + \Delta t$. Meanwhile the rotating frame has also moved.

Therefore, from the perspective of the rotating frame, the displacement of the particle between t and $t + \Delta t$ is the vector from P' on its x-axis where the particle was at t to the space point Q. The same displacement of the particle from the perspective of the non-rotating frame is PQ vector in space. From $\triangle PP'Q$ we obtain the following relation in space.

$$\overrightarrow{PQ} = \overrightarrow{PP'} + \overrightarrow{P'Q}. \tag{1.13}$$

Now, $\overrightarrow{PP'}$ is on the arc of a circle of radius $|\vec{r}|$ with the arc angle

$\Omega\Delta t$, therefore

$$\overrightarrow{PP'} = \left(\vec{\Omega}\Delta t\right) \times \vec{r}.$$

The vector $\overrightarrow{PQ}$ is the displacement $\Delta\vec{r}$ of the particle in the fixed frame and $\overrightarrow{P'Q}$ is the displacement $\Delta\vec{r}\,'$ of the particle in the rotating frame. Putting these quantities in Eq. 1.13 we have the following relation among displacements in the time interval Δt.

$$\Delta\vec{r} = \left(\vec{\Omega}\Delta t\right) \times \vec{r} + \Delta\vec{r}\,'.$$

Dividing by Δt and taking the $\Delta t \to 0$ limit, we find that the velocity in the inertial frame is equal to the sum of the velocity of the particle in the rotating frame and an additional term resulting from the rotation of the frame.

$$\vec{v} = \vec{v}\,' + \vec{\Omega} \times \vec{r}. \tag{1.14}$$

Rather than use prime for the quantities in the rotation frame, sometimes a different notation is use in which we attach a subscript "in" and "rot" for quantities in the fixed frame and the rotating frame respectively.

$$\vec{v}_{in} = \vec{v}_{rot} + \vec{\Omega} \times \vec{r}. \tag{1.15}$$

There is no need to put a subscript to $\vec{r}$ since $\vec{r}$ is a vector from the origin to the position of the particle and the origins of the two frames are at the same point. The result of relation between the velocity vectors is also often written as

$$\left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{r}. \tag{1.16}$$

where the subscript "in" denotes the quantity in the inertial or non-rotating frame and the subscript "rot" for a quantity with respect to the rotating frame.

Note that to derive this relation, we only used the geometric property of the position vector. A similar argument can be made about the rate of change of any arbitrary vector $\vec{w}$ in the two frames.

$$\boxed{\left(\frac{d\vec{w}}{dt}\right)_{in} = \left(\frac{d\vec{w}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{w}.} \tag{1.17}$$

For instance, we can obtain the time-derivative of the velocity vector $\vec{v}_{in}$ in the inertial frame by setting $\vec{w} = \vec{v}_{in}$ in this equation.

$$\boxed{\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{in}}{dt}\right)_{rot} + \vec{\Omega} \times \vec{v}_{in}.} \tag{1.18}$$

The left hand side of this equation is the acceleration of the particle in the inertial frame since it is the rate of change of the corresponding velocity in the same frame. The first term on the right side of this equation is not acceleration of anything since this term mixes information from the two frames - this term is the rate of change of the velocity in the inertial frame as observed from the rotating frame.

Finally, by substituting $\vec{v}_{in}$ in the right-side of Eq.rot-inert-eq-2 in terms of $\vec{v}_{rot}$ as given in Eq. 1.15 we can work out the relation between accelerations in the two frames.

$$\begin{aligned}\left(\frac{d\vec{v}_{in}}{dt}\right)_{in} &= \left[\frac{d}{dt}\left(\vec{v}_{rot} + \vec{\Omega}\times\vec{r}_{in}\right)\right]_{rot} + \vec{\Omega}\times\left(\vec{v}_{rot}\times\vec{\Omega}\times\vec{r}_{in}\right)\\ &= \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\end{aligned}$$

Therefore, the accelerations of the particle in the two frames are related as follows.

$$\boxed{\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right) \quad \text{(Constant } \vec{\Omega}\text{).}} \tag{1.19}$$

1.2.2 Newton's Second Law in Uniformly Rotating Frame

Newton's second law for a particle of fixed mass m is written in inertial frame as $m\vec{a}_{in} = \vec{F}$, where $\vec{F}$ is the net real force. Substituting $\vec{a}_{in}$ from Eq. 1.19 into the second law we find that mass times acceleration in a rotating frame has the following form.

$$\boxed{m\vec{a}_{rot} = \vec{F} - m\left[2\vec{\Omega}\times\vec{v}_{rot} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right] \quad \text{(Constant } \vec{\Omega}\text{)}} \tag{1.20}$$

Therefore, in a rotating frame, mass times acceleration is not equal to the net real force on the particle. In addition to the net real force $\vec{F}$, there are terms that have units of force but are dependent on the rotation of the frame. These are the "**fictitious forces**" or "**inertial forces**". In a rotating frame, the "real" and "fictitious" forces are indistinguishable in their effect on the acceleration except that there appears to be no agent(s) for the fictitious forces as far as the rotating frame is concerned.

The first term $\left(-2m\vec{\Omega}\times\vec{v}_{rot}\right)$ is called the **Coriolis force**, and the second term $\left[-m\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right]$ the **centrifugal force**. While the direction of the Coriolis force is perpendicular to the axis of rotation and the velocity of the particle, the centrifugal force is always pointed away from the axis of rotation and remains perpendicular to the axis.

1.2.3 Newton's Second Law in Earth's Frame

An Earth-based frame is a rotating frame. Therefore, the equation of motion of a particle in an Earth-based frame will have Coriolis force and centrifugal force. The centrifugal force is included in mg, therefore, the equation of motion in the rotating frame of the Earth has Coriolis force only.

$$\boxed{m\vec{a}_{rot} = m\vec{g} + \vec{F}_{\text{other}} - 2m\vec{\Omega} \times \vec{v}_{rot},} \tag{1.21}$$

where $\vec{F}_{\text{other}}$ are forces other than gravity and $\vec{g}$ is the acceleration due to gravity written as a vector to include the direction information in the equation. We will use Eq. 1.21 to study the motion the motion of particles near the surface of the Earth in an Earth-based frame.

Example 1.2.1. Freely Falling Particle In An Earth-Based Frame. The rotation of earth has observable effects on a freely falling object. If a particle of mass m is released at rest from a height h above the surface of the Earth, it will fall towards the center of the Earth if the Earth were not rotating. But, because the Earth is rotating, the path of the particle will deviate from this direction in an Earth-based frame. In this example we wish to determine the deviation at the equator.

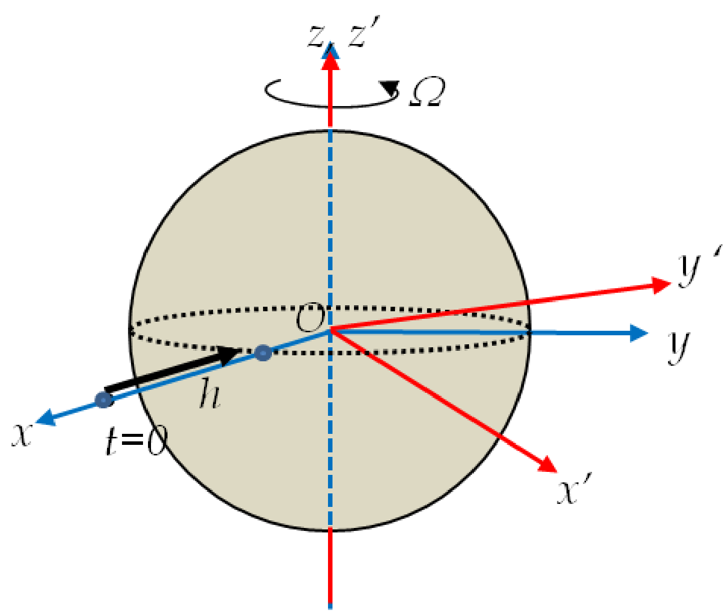

Figure 1.8: Example 1.2.1. The rotating frame $O'x'y'z'$ and the fixed frame $Oxyz$ coincide at $t = 0$. That is, Ox' and Ox are in the same direction at $t = 0$ when the particle is released on the x-axis. The particle falls down the x-axis of the fixed coordinate system.

Solution. Consider two coordinate frames $Oxyz$ and $Ox'y'z'$ with their origins at the center of Earth that have the z-axis pointed in the direction of the axis of rotation of the Earth. Let the frame $Ox'y'z'$ rotate with respect to the fixed inertial frame $Oxyz$ at angular speed Ω, and let the axes of the two systems be coincident at $t = 0$. We will use M for the mass of the Earth and assume Earth to be a sphere of radius R.

Suppose now a particle of mass m is released at $t = 0$ from rest at $x = x' = h + R$. In the inertial frame the particle moves straight down the x-axis and reaches $x = R$ at some time Δt. During Δt the x' and y' axes of the rotating frame move out to another direction. Since the rotation is about the z-axis, the particle will fall in the $x'y'$ plane of the rotating frame. We wish to determine the y' component of the displacement when the particle has dropped a distance of h in the inertial frame. We start by writing the components of the equation in the rotating frame.

$$\frac{dv'_x}{dt} = -g - \left(2\vec{\Omega} \times \vec{v}_{rot}\right)_{x'} \tag{1.22}$$
$$\frac{dv'_y}{dt} = 0 - \left(2\vec{\Omega} \times \vec{v}_{rot}\right)_{y'} \tag{1.23}$$
$$\frac{dv'_z}{dt} = 0 - \left(2\vec{\Omega} \times \vec{v}_{rot}\right)_{z'} \tag{1.24}$$

The x' and y' components of the Coriolis and centrifugal terms are as follows.

$$\vec{\Omega} \times \vec{v}_{rot} = \begin{vmatrix} \hat{u}_{x'} & \hat{u}_{y'} & \hat{u}_{z'} \\ 0 & 0 & \Omega \\ v_{x'} & v_{y'} & v_{z'} \end{vmatrix} = -\Omega v_{x'} \hat{u}_{y'} + \Omega v_{x'} \hat{u}_{z'}.$$

Hence the equations of motion of the particle are:

$$\frac{dv_{x'}}{dt} = -g - 2\Omega v_{y'} \tag{1.25}$$
$$\frac{dv_{y'}}{dt} = 2\Omega v_{x'} \tag{1.26}$$
$$\frac{dv_{z'}}{dt} = 0 \tag{1.27}$$

Solving these equations with the initial condition $x' = h + R$, $y' = 0$, $z' = 0$, and $v_{0x'} = v_{0y'} = v_{0z'} = 0$ will give us the trajectory of the particle from the perspective of the rotating frame, i.e., from someone observing the particle from Earth-based frame. Since there is no motion along the z-axis, we will work out the solution for x' and y' components only.

Approximate solution: Observe that the particle will pick up velocity more along the vertical direction than along the horizontal direction. Therefore, we can assert that in time t, $|v'_x| \approx gt$. Using this in v'_x equation, we find the following for v'_y.

$$\frac{dv_{y'}}{dt} \approx -2\Omega gt \quad \Longrightarrow \quad v_{y'} = -\Omega gt^2,$$

where we have used the initial condition on $v_{y'}$. Integrating $v_{x'}$ and $v_{y'}$ we obtain the following for x' and y' coordinates after the particle is released at $(x' = h + R,\, y' = 0)$ at $t = 0$.

$$x' = h + R - \frac{1}{2}gt^2 \tag{1.28}$$

$$y' = -\frac{1}{3}\Omega gt^3 \tag{1.29}$$

From the x' equation we can determine the time T for the particle to the surface of Earth, which has $x' = R$. Using this time in the y' equation we find the horizontal deviation y'.

$$\boxed{\Delta y' = -\frac{1}{3}\Omega g \left(\frac{2h}{g}\right)^{3/2}}$$

For a 100 meter drop we will find the deviation to be

$$\begin{aligned} \Delta y' &= -\frac{1}{3}\frac{2\pi}{24 \times 3600\ s} \times 9.81\ \mathrm{m/s^2} \times \left(\frac{2 \times 100\ \mathrm{m}}{9.81\ \mathrm{m/s^2}}\right)^{3/2} \\ &= -2.2 \times 10^{-2}\ \mathrm{m}. \end{aligned}$$

Since the rotation axis is towards the North, the y' axis is towards the East at the surface of the Earth. Therefore, a particle dropped from 100 m above the ground will land approximately 2.2 *cm* to the West of the line connecting the original position to the center of the Earth.

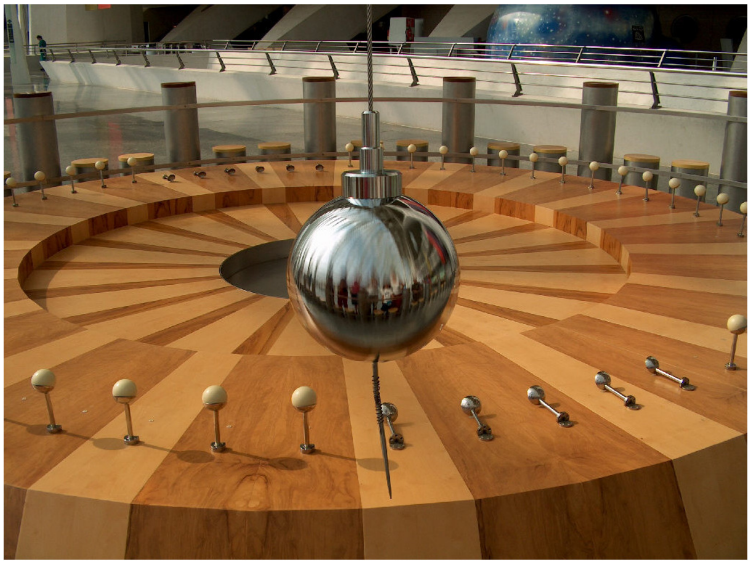

Figure 1.9: Foucault pendulum. The rotation of the Earth causes the plane of oscillation of the pendulum to change over time. With changing plane of oscillation, the pins at different positions are knocked down at different times. Photo credit: Ciudad de las Artes y de las Ciencias de Valencia by Daniel Sancho, Wikicommons.

Example 1.2.2. Surface of a Rotating Fluid in a Bucket.

A bucket of water is rotated with a uniform rotational speed Ω. It is found the surface assumes a steady shape. Determine the shape. Ignore the rotation of earth.

Solution. Due to the symmetry in the problem, it is sufficient to work in a plane containing the axis of rotation and a horizontal direction as shown in Fig. 1.10. We will call the axis of rotation the

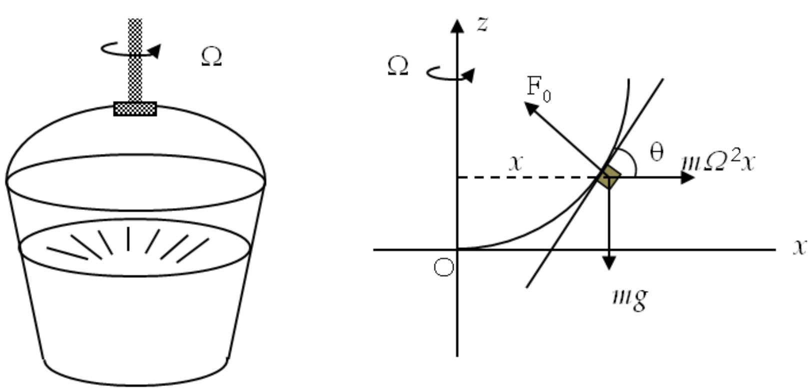

Figure 1.10: Example 1.2.2.

z-axis and the horizontal direction will be taken to be the x-axis. Therefore, to find the equation of the surface, we need to work out the function $z(x)$. We will make use of steady condition on a mass element at the surface.

Here, notice that it is easier to work in the rotating frame of the bucket since in this frame, liquid in the bucket will not be moving and will have zero acceleration. That is, in this frame, real forces will be balanced by inertial force(s). Let us figure out the real and inertial forces on a mass element at the surface.

In the rotating frame once the steady state has reached, water would not be moving any more. Therefore the Coriolis force will be zero. Thus, the only inertial force on particles of water will be the centrifugal force. The only other force is the weight.

Consider a mass element of mass m at the surface of the steady fluid. The weight has magnitude mg and acts straight down. The centrifugal forces on various water particles and the weight of the water molecules will press on the layers of water in contact. The water molecules on the surface will press on the molecules just below the surface. The reaction force from the layer just below the surface will be in the normal direction of the surface as shown by F_0 in the figure.

Therefore, the x and z-components of the equations of motion of an element at the surface of water in the rotating frame for a mass at the surface are:

$$\begin{aligned} x\text{-component:} &\quad m\Omega^2 x - F_0 \sin\theta = 0 \\ z\text{-component:} &\quad F_0 \cos\theta - mg = 0 \end{aligned}$$

Therefore,

$$\tan\theta = \frac{\Omega^2}{g} x$$

But this tangent must equal the slope of the tangent to the curve $z(x)$ at the point.

$$\frac{dz}{dx} = \tan\theta.$$

Therefore, we obtain the following equation for $z(x)$.

$$\frac{dz}{dx} = \frac{\Omega^2}{g} x,$$

which can be immediately integrated to yield

$$z = \frac{\Omega^2}{2g} x^2 + C,$$

where C is the constant of integration. From the figure, $x = 0$ corresponds to $z = 0$ on the surface, therefore $C = 0$. Hence, the equation for the surface is

$$z = \frac{\Omega^2}{2g} x^2.$$

The situation is symmetric in the xy-plane and there is nothing special about x-axis. To obtain the equation for the entire surface and not just a slice of the surface, all we need to do is to replace x by the radial distance r of polar coordinates.

$$z = \frac{\Omega^2}{2g} r^2. \tag{1.30}$$

Hence, the surface is a paraboloid of revolution about the axis of rotation.