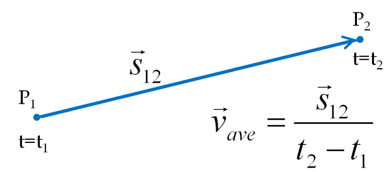
3.4 AVERAGE VELOCITY

The rate at which position changes with time gives us information about the flow of motion. A simple measure of the rapidity of a motion, both how fast the movement occurs and in which direction the movement is taking place, is obtained by dividing the displacement vector by the interval since the displacement vector has both the magnitude and direction information. This quantity is called the **average velocity** of the object in that interval. Let there be a displacement $\vec{s}_{12}$ between time t_1 and t_2 then the average velocity $\vec{v}_{ave}$ during this interval will be

$$\boxed{\vec{v}_{ave} = \frac{\vec{s}_{12}}{t_2 - t_1}.} \tag{3.6}$$



We saw above that the displacement vector can be written in a coordinate system using the position vectors for the starting and ending points of the interval. That is, if the position vectors at t_1 and t_2 are $\vec{r}_1$ and $\vec{r}_2$, respectively, then $\vec{s}_{12} = \vec{r}_2 - \vec{r}_1$. Therefore, we can write the average velocity in terms of the change in position vectors also.

$$\vec{v}_{ave} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}. \tag{3.7}$$

One often writes this relation in another notation. Let $\Delta t (= t_2 - t_1)$ denote the time interval and $\Delta\vec{r} (= \vec{r}_2 - \vec{r}_1)$ the displacement during that interval. Then, the average velocity is

$$\vec{v}_{ave} = \frac{\Delta\vec{r}}{\Delta t}, \tag{3.8}$$

which can be rearranged to get the change in position in the finite interval Δt

$$\Delta\vec{r} = \vec{v}_{ave}\Delta t. \tag{3.9}$$

Furthermore, using Eq. 3.3 we can express displacement in terms of the changes in coordinates, $(\Delta x, \Delta y, \Delta z)$ and write average velocity as

$$\vec{v}_{ave} = \left(\frac{\Delta x}{\Delta t}\right)\hat{u}_x + \left(\frac{\Delta y}{\Delta t}\right)\hat{u}_y + \left(\frac{\Delta z}{\Delta t}\right)\hat{u}_z. \tag{3.10}$$

The quantities multiplying the base vectors are called the x, y and z-**components of the average velocity**, and as you have studied in the last chapter, they are labeled with x, y and z subscripts similar to the labeling of components of other vectors.

Components of Average Velocity:

$$\boxed{v_x^{ave} = \frac{\Delta x}{\Delta t}} \quad (3.11)$$

$$\boxed{v_y^{ave} = \frac{\Delta y}{\Delta t}} \quad (3.12)$$

$$\boxed{v_z^{ave} = \frac{\Delta z}{\Delta t}} \quad (3.13)$$

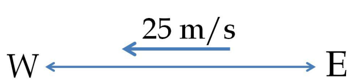
Note: The components v_x^{ave}, v_y^{ave}, and v_z^{ave} are not vectors!

Sometimes these components, v_x^{ave}, v_y^{ave}, and v_z^{ave} are called the average x, y and z-velocities respectively. Note that the names average x, y and z-velocities do not imply that they are vectors; they are the factors with which you must multiply the unit base vectors of the x, y, and z-axes and sum the resulting vectors along the axes to construct the actual vector, the average velocity. That is why it would be incorrect to put arrows over the symbols of the components since that would imply that they are vectors when they are not.

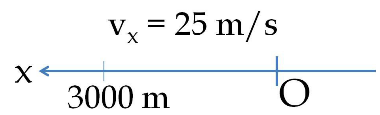
Example 3.4.1. Average Velocity of a Motion in a Straight Line. A train starts from Boston at 7 : 00 AM and goes towards New York City. In the first 2 minutes, the train moves on a straight track facing West and covers a direct distance of 3000 m from the starting point. What is the average velocity?

Solution. From the given description of the motion we need to deduce two parts of the average velocity vector: (1) magnitude and (2) direction. Since, there are two viewpoints of vectors, often there are two ways to address these problems: (a) Geometrically, and (b) Analytically.

(a) Geometric approach: For a geometric answer, we find the magnitude of average velocity vector by dividing the magnitude of displacement vector by the interval. The magnitude of the displacement is equal to the direct distance between the points of space at the end points of the interval, which is given to be 3000 m. The interval has a duration of 2 min. Therefore, the magnitude of the average velocity is 3000 m/2 min = 1500 m/min. It is usually customary to write numerical values in standard units, which would be kg, m, and (sec or s). Therefore, we rewrite the answer in m/s, which gives the magnitude of the velocity to be 25 m/s. The direction is towards West and can be written as such. Or, you could draw an arrow and name the direction "West".



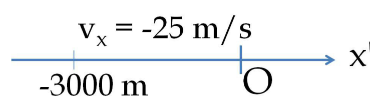
(b) Analytic approach: Analytically, we would proceed by choosing a coordinate system and describe the direction using that coordinate system. Suppose, we choose a coordinate system such that its x-axis is pointed West with its origin at the starting point on the track. Then, the unit base vector $\hat{u}_x$ will be pointed towards the West direction. The coordinates of the beginning and end points are $(0, 0, 0)$ and $(3000\ \text{m}, 0, 0)$ respectively. Therefore, the change in x-coordinates, Δx, in the two-minute interval is 3000m. Dividing Δx by Δt we find the x-component of the average velocity vector to be 25 m/s. Therefore, using the chosen coordinate system, we will write the velocity vector as $\vec{v}_{ave} = (25\ \text{m/s})\hat{u}_x$. The answer makes sense when you refer to what the direction $\hat{u}_x$ means for the chosen coordinate system. Here, it means the direction towards West. Therefore, $(25\ \text{m/s})\hat{u}_x$ means that the velocity is { 25 m/s, West }.



Further Observations

Note that, in the analytic method, the form of the final answer has a reference to the coordinate system employed for calculations. Now, suppose, we had chosen the positive x-axis to point towards East, instead of West. What, then, will be our answer? Clearly, in the new coordinate system, the coordinates of the two points under study will be $(0, 0, 0)$ and $(-3000\ \text{m}, 0, 0)$.

Note the x-coordinate of the final point will be negative in the new system. The x-component of the average velocity, obtained by dividing the final x minus the initial x by the interval, now gives -25 m/s. We need to multiply the unit base vector by this number to obtain the average velocity vector. The unit base vector along the x-axis now points towards East, which is just the opposite of the direction the unit base vector was pointing in the previous choice of coordinate system. Therefore, let us use a prime to denote the unit base vectors in this system, which gives the average velocity vector in this system to be $\vec{v}_{ave} = (-25\ m/s)\hat{u}'_x$. [You should confirm that this is the right result.]



On the surface, the answer now does not appear to be same as the answer using the other coordinate system: We have a minus sign in the answer here and we didn't have the minus sign before. But, is the average velocity different? No, the average velocity is same, as an interpretation of the meaning of these numerical answers immediately reveals. Since $\hat{u}'_x$ points towards East, its negative will point towards West. Therefore, $(-25\ \text{m/s})\hat{u}'_x$ is a velocity of 25 m/s pointed towards West as before.

You do not have to use x-axis for the track. Try using the y-axis to point along the track. What do you get for the answer?

Example 3.4.2. Average Velocity for a Motion in a Straight

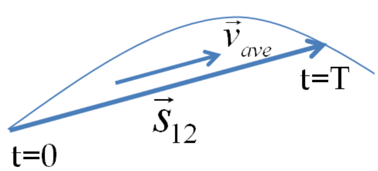
Line.

For simplicity, consider an object moving in a straight line such as a train on a straight track. When we place a coordinate system such that its x-axis falls on the straight line of motion, the displacement will be a vector that can point either towards the positive x-axis or towards the negative x-axis only. Therefore, displacement in this coordinate system can be written entirely as a scalar multiple of the unit vector $\hat{u}_x$, which will mean that the average velocity has only the x-component v_x^{ave}

$$\vec{v}_{ave} = v_x^{ave}\hat{u}_x. \tag{3.14}$$

If $v_x^{ave} > 0$, then the average velocity will be in the same direction as the vector $\hat{u}_x$, i.e. towards the positive infinity of the x-axis, and if the $v_x^{ave} < 0$, then the average velocity will be in the direction opposite to that of the vector $\hat{u}_x$, i.e. towards the negative infinity of the x-axis. The direction towards the positive infinity of the x-axis is also referred to as moving to the right or moving toward the positive x-axis. Similarly we say moving to the left or towards the negative x-axis for the direction towards negative infinity of the axis.

We will use similar descriptions of motions on a straight line, if we happen to use a coordinate system oriented such that its y or z-axis coincides with the straight line where the motion takes place.



Example 3.4.3. Average velocity of a motion in a plane. A projectile is flying through the air in a plane. At $t = 0$, the projectile is at the origin of a coordinate system, and at $t = T$, the coordinates of the projectile are $(X, Y, 0)$. What is the average velocity of the projectile over the time interval 0 to T?

Solution. The analytic viewpoint of average velocity tells us that we can construct the average velocity vector from its components, which we can readily find here.

$$\begin{aligned} v_x^{ave} &= \frac{X-0}{T-0} = \frac{X}{T} \\ v_y^{ave} &= \frac{Y-0}{T-0} = \frac{Y}{T} \\ v_z^{ave} &= \frac{0-0}{T-0} = 0 \end{aligned}$$

Therefore, the average velocity has the following representation in the given coordinate system.

$$\vec{v}_{ave} = \left(\frac{X}{T}\right)\hat{u}_x + \left(\frac{Y}{T}\right)\hat{u}_y.$$

The magnitude and direction of the average velocity can be determined as usual from the components. The magnitude of the average velocity is

$$v_{ave} = \frac{\sqrt{X^2 + Y^2}}{T}.$$

Since, the motion occurs in a plane, one angle will be sufficient to specify the direction of the vector. The angle θ from the positive x-axis in the xy-plane is given by

$$\theta = \arctan\left(\frac{Y}{X}\right).$$

This is the angle we usually specify when we want to give direction of a vector in the xy-plane.

3.4.1 Average Speed

Often we are interested in the total distance traveled over some time interval and not necessarily in the direction of the motion. For instance, if you travel by car between two cities, the road will not always point in the same direction. The distance you must travel will be more than the direct distance between the cities. Let D be the actual distance moved in total time T, then the ratio D/T, called the **average speed** is a measure of how fast the distance was covered. We will denote average speed by v_s.

$$\boxed{v_s = \frac{D}{T}.} \tag{3.15}$$

Average speed and magnitude of average velocity may be very different, especially in a motion where there are changes in direction. An extreme example will be a motion where an object returns to the original place after some time - the magnitude of average velocity in this case will be zero but average speed will not be zero.

Example 3.4.4. Average speed of a road trip. Google maps shows that a road trip from Boston to Los Angeles will cover 4800 km and take two days and zero hour. What is the average speed for the trip in km/h? From the longitudinal and latitude of the two cities the direct distance can be calculated assuming the Earth to be a sphere. The direct distance between the two cities is approximately 4100 km. What is the average velocity for the trip?

Solution. We divide the distance by time to obtain the average speed:

$$v_s = \frac{4800 \text{ km}}{48 \text{ hr}} = 100 \text{ km/hr}.$$

The magnitude of the average velocity will be equal to the direct distance divided by time, which gives $v_{ave} = 4100/48 = 84\ km/hr$. The direction is pointed from Boston towards Los Angeles.

Question to student: Will