3.4 ELECTRIC FIELD AND ELECTRIC POTENTIAL

3.4.1 Electric Potential from Electric Field

Consider a distribution of charges that produce an electric field $\vec{E}(x, y, z)$ and electric potential $V(x, y, z)$ at space point P(x, y, z). What is the relation between them? The potential difference $V - V_R$ is the work done against the electric force when a unit charge is moved from some reference point R to the space point P. Therefore,

$$V(x, y, z) - V_R = -\int_{\text{Reference point}}^{P(x,y,z)} \vec{E} \cdot d\vec{r}. \tag{3.46}$$

This equation is a very useful relation for finding V if you can use Gauss's law to find electric field $\vec{E}(x, y, z)$ first. The reference point should be chosen carefully when setting up the integral in Eq. 3.46 although reference is an arbitrary point. First of all, the reference cannot be at a point where a charge is already present. Second, for charges that are distributed over a finite space you can use infinity as the reference point, but for charge distributions that go on to infinity, such as an infinitely long wire or an infinite sheet, you cannot use infinity as a reference, and you must place the reference at some point that is a finite distance from the charges in the system. Below we will work out examples illustrating uses of Eq. 3.46.

Example 3.4.1. Infinitely Long Charged Wire. Find the electric potential of a an infinitely long wire charged uniformly with λ (C/m).

Solution. Since we have already worked out the potential of a finite wire of length L in Example 3.3.1 we might wonder if taking $L \rightarrow \infty$ in Eq. 3.39 might give us the desired result.

$$V_P = \lim_{L\rightarrow\infty} \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{L + \sqrt{L^2 + 4x^2}}{-L + \sqrt{L^2 + 4x^2}}\right] \quad ?$$

Since this limit does not exist due to the argument of logarithm becoming [1/0] as $L \rightarrow \infty$, this way of finding V of an infinite wire will not work. The reason for this problem may be traced to the fact that the charges are not localized in some space but contiue to infinity in the direction of the wire. The fundamental formula used in deriving Eq. 3.39 assumed reference zero potential at infinity in all directions.

$$V_{\text{point particle}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \text{(Reference zero at } r = \infty)$$

For the systems where charges go on to infinity in any direction we need to pick a reference at a finite space point. Therefore, we find potentials in these cases starting from scratch and performing the integral in Eq. 3.46 from an arbitrary reference point whose choice is made after doing the integral. This requires a knowlege of the formula for electric field at space between the reference point and field point P.

The infinite wire with uniform charge density provides an excellent example for working out potential from electric field since the electric field can be obtained easily by using Gauss's law as given in Example 2.3.3. To be concrete we place the wire with charge on the z-axis and determine potential at a point P in the xy-plane. Since choice of direction of axes is arbitrary, we can always choose the x-axis to pass through P to reduce the calculational complexity. Let us also pick a reference point R on the x-axis at some other place than the field point P or origin. The electric field for an infinitely long wire has been worked out by Gauss's law and can be written for the points on the x-axis between R and P as follows.

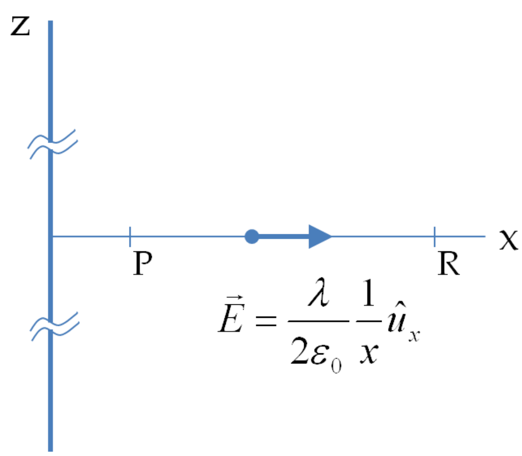

Figure 3.22: Electric field at points on the x-axis.

$$\vec{E}(x) = \frac{\lambda}{2\epsilon_0}\frac{1}{x}\hat{u}_x \quad \text{(on x-axis)}$$

Therefore, after taking dot product with $d\vec{r} = dx\ \hat{u}_x$, the potential difference between the reference point with $x = x_R$ and field point P with $x = x_P$ is given by Eq. 3.46 is

$$V_P - V_R = -\int_{x_R}^{x_P} \frac{\lambda}{2\epsilon_0}\frac{1}{x}dx. \tag{3.47}$$

The integral is easy to work out with the following result.

$$V_P - V_R = -\frac{\lambda}{2\epsilon_0}\ln\frac{x_P}{x_R}. \tag{3.48}$$

If we choose $x_R = 1\ m$ and $V_R = 0$, that is we choose potential to be zero at 1 m on the x-axis, we obtain potential at any point on the x-axis as

$$V_P = -\frac{\lambda}{2\epsilon_0} \ln x_P, \tag{3.49}$$

where x_P is in meters. By symmetry we write the potential at a distance s from the wire with zero potential at $s = 1$ meter.

$$V_P = -\frac{\lambda}{2\epsilon_0} \ln s. \tag{3.50}$$

Note the potential blows up as $s \to \infty$, and that is why we could not use infinity as the reference here. Of course, all physical systems are finite in extent, so we need the mathematical device used in this section only for idealized systems. The idealization that results from extending a charge distribution to infinity allows us to take advantage of mathematical symmetries without any significant loss of physics information. The arbitrariness of choice of a reference for potential does not bother us because the physical information is contained in the difference of potential energy between two points in space which is unaffected by the choice.

Example 3.4.2. Potential of a Uniformly Charge Sphere The electric field of a uniformly charged sphere of radius R and charge density ρ_0 (C/m^3) is given in spherical coordinates (r, θ, ϕ) as

$$\vec{E}(r,\theta,\phi) = \begin{cases} \hat{u}_r \left(\frac{\rho_0}{3\epsilon_0}\right) r & (r < R) \\ \hat{u}_r \left(\frac{\rho_0}{3\epsilon_0}\right) \frac{R^3}{r^2} & (r > R) \end{cases} \tag{3.51}$$

Find electric potential t points inside and outside the sphere.

Solution. We can find potential by integrating electric field between an arbitrary reference point R and the field point P(r, θ, ϕ). Since the charge distribution is in a finite space, the common practice is to choose the zero reference at $r = \infty$. We need to perform integration from reference point to the field point. We will get two different expressions for points inside the sphere and points outside the sphere, which we will label as V_{in} and V_{out} respectively.

With $r = \infty$ as reference it is better to work out potential at a point outside the sphere first.

$$
\begin{aligned}
V_{out} &= -\int_{\infty}^{r>R} \vec{E}\cdot d\vec{r} \\
&= -\int_{\infty}^{r} \vec{E}_{out}\cdot d\vec{r} \\
&= -\left(\frac{\rho_0}{3\epsilon_0}\right)\int_{\infty}^{r} \frac{R^3}{r^2} dr \\
&= \left(\frac{\rho_0}{3\epsilon_0}\right)\frac{R^3}{r}
\end{aligned}
$$

Now, to find the potential at a point inside the sphere, we need integral from the reference at $r = \infty$ to a point inside the sphere. The integral will break up into $r = \infty$ to $r = R$ and $r = R$ to $r < R$. The first integral gives V_{out} evaluated at $r = R$, which we will label $V_{out}(R)$, and we need to do the second integral using the electric field for $r < R$.

$$
\begin{aligned}
V_{in} &= -\int_{\infty}^{r<R} \vec{E}\cdot d\vec{r} \\
&= -\int_{\infty}^{R} \vec{E}_{out}\cdot d\vec{r} - \int_{R}^{r} \vec{E}_{in}\cdot d\vec{r} \\
&= V_{out}(R) - \left(\frac{\rho_0}{3\epsilon_0}\right)\int_{R}^{r} r dr \\
&= \left(\frac{\rho_0}{3\epsilon_0}\right)\frac{R^2}{2}\left[3 - \left(\frac{r}{R}\right)^2\right]
\end{aligned}
\tag{3.52}
$$

3.4.2 Electric Field from Potential

(** NOTE TO STUDENTS: This section uses somewhat advanced mathematics than most other parts of the book. You will encounter these concepts in Calculus III course. The material in this section is presented here to show you that you can go back and forth between electric field and electric potential in both directions. Skipping this section is OK if you are not not comfortable with multivariate calculus. **) Electric field can be obtained from electric potential by the use of fundamental theorems of calculus. We do not go in detail here, but give the final result. The relation between electric field and electric potential for electrostatic situations is written in a compact notation using the nabla notation for the gradient.

$$\vec{E} = -\vec{\nabla} V \tag{3.53}$$

where $\vec{\nabla}$ is a vector operator and takes the following form in Cartesian coordinates.

$$\vec{\nabla} = \hat{u}_x\frac{\partial}{\partial x} + \hat{u}_y\frac{\partial}{\partial y} + \hat{u}_z\frac{\partial}{\partial z} \tag{3.54}$$

Therefore electric field in Cartesian coordinates is

$$\vec{E} = -\hat{u}_x \frac{\partial V}{\partial x} - \hat{u}_y \frac{\partial V}{\partial y} - \hat{u}_z \frac{\partial V}{\partial z} \tag{3.55}$$

The expression for the gradient is different in different coordinate systems. for instance, in the cylindrical and spherical coordinates the gradient of the potential looks as follows.

$$\text{Cylindrical:} \quad \vec{\nabla} = \hat{u}_s \frac{\partial}{\partial s} + \hat{u}_\phi \frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}_z \frac{\partial}{\partial z} \tag{3.56}$$

$$\text{Spherical:} \quad \vec{\nabla} = \hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.57}$$

We will now illustrate calculation of electric field from known potentials.

Example 3.4.3. Electric Field of a Dipole Find the electric field of a dipole consisting of two charges $+q$ and $-q$ separated by a distance d.

Solution. For concreteness let us place the two charges on the z-axis at $z = \pm d/2$ with positive charge above the origin and the negative charge below the origin. The electric potential at an arbitrary space point P(x, y, z) is given as

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d/2)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d/2)^2}}\right]$$

We drop the subscript P from symbols to keep them less cluttered. The Cartesian components of the electric field at space point P(x, y, z) are then given by taking the corresponding partial derivatives.

$$E_x = \frac{q}{4\pi\epsilon_0}\left[\frac{x}{(x^2+y^2+(z-d/2)^2)^{3/2}} - \frac{x}{(x^2+y^2+(z+d/2)^2)^{3/2}}\right]$$

$$E_y = \frac{q}{4\pi\epsilon_0}\left[\frac{y}{(x^2+y^2+(z-d/2)^2)^{3/2}} - \frac{y}{(x^2+y^2+(z+d/2)^2)^{3/2}}\right]$$

$$E_z = \frac{q}{4\pi\epsilon_0}\left[\frac{(z-d/2)}{(x^2+y^2+(z-d/2)^2)^{3/2}} - \frac{(z+d/2)}{(x^2+y^2+(z+d/2)^2)^{3/2}}\right]$$

Example 3.4.4. A Calculation in Cylindrical Coordinates

In Cylindrical coordinates, (s, ϕ, z), the potential of uniformly charged wire with charge density λ_0 (C/m) is given as

$$V(s,\phi,z) = -\frac{\lambda_0}{2\pi\epsilon_0} ln(s) \ \ \text{(independent of } z \text{ and } \phi\text{)}.$$

Note, we use letter s for radial distance perpendicular to the z-axis. Find the electric field in cylindrical coordinates.

Solution. We have given the representation of the nabla vector operator in Cartesian coordinates. The vector operator nabla, $\vec{\nabla}$ has the following representation in cylindrical coordinates.

$$\vec{\nabla} = \hat{u}_s \frac{\partial}{\partial s} + \hat{u}_\phi \frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}_z \frac{\partial}{\partial z} \tag{3.58}$$

Therefore, the cylindrical components of electric field are

$$\begin{aligned}
E_s &= -\frac{\partial V}{\partial s} = \frac{\lambda_0}{2\pi\epsilon_0} \\
E_\phi &= -\frac{1}{s}\frac{\partial V}{\partial \phi} = 0 \\
E_z &= -\frac{\partial V}{\partial z} = 0
\end{aligned}$$

Thus electric field is radial. If the wire is positively charged, the electric field lines are directed away from the wire and if the wire is negatively charged, the filed lines are pointed towards the wire.

Example 3.4.5. An Example Using Spherical Coordinates

In spherical coordinates, (r, θ, ϕ), the potential of a point charge at origin is given as

$$V(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \text{(independent of } \theta \text{ and } \phi\text{).}$$

Find the electric field in spherical coordinates.

Solution. We have given the representation of nabla vector operator in Cartesian and cylindrical coordinates. The vector operator nabla, $\vec{\nabla}$ has the following representation in spherical coordinates.

$$\vec{\nabla} = \hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{u}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.59}$$

Therefore, the spherical components of electric field are

$$\begin{aligned}
E_r &= -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\
E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} = 0 \\
E_\phi &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0
\end{aligned}$$

Thus electric field is radial, as we know from previous studies of a point charge.