2.3 ANALYTICAL VIEW OF VECTORS

The analytic view of vectors or the algebraic method for vectors is based on the decomposition of vectors along perpendicular axes. It is easily seen that an arbitrary vector can be constructed by adding appropriate vectors along x, y, and z-axes.

For simplicity of drawing let us look at an example of vectors in the xy-plane. In Fig. 2.20 you can see how an arbitrary vector $\vec{A}$ can be written as a sum of two vectors $\vec{A}_1$ and $\vec{A}_2$, one along each Cartesian axis. Similar arguments can be applied to a vector in the three-dimensional space. Therefore an arbitrary vector can be constructed from three vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$ along the x , y and z-axes respectively.

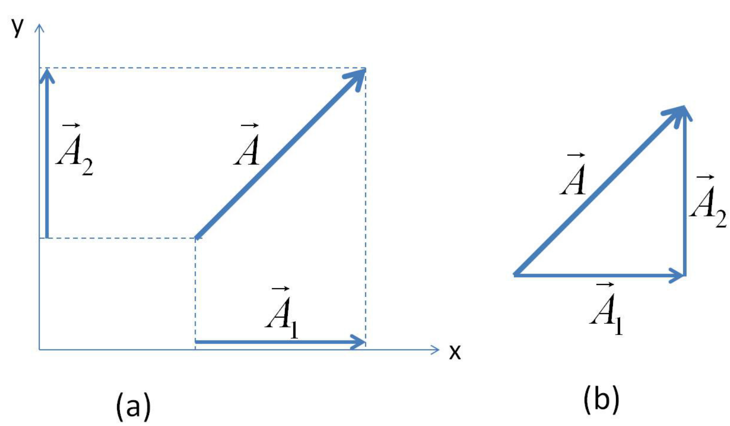

Figure 2.20: An arbitrary vector can be constructed by a sum of appropriate vectors parallel to the Cartesian axes. Here vector $\vec{A}$ is sum of vectors $\vec{A}_1$ and $\vec{A}_2$. (a) The vector $\vec{A}_1$ along the x-axis is formed by the projections of the two ends of the vector $\vec{A}$ on the x-axis. Similarly for $\vec{A}_2$ along the y-axis. (b) The triangle shows that $\vec{A} = \vec{A}_1 + \vec{A}_2$.

2.3.1 Base Vectors

Since we need vectors along the axes, either pointed towards the positive axes or in the opposite directions, we define unit vectors pointed towards the positive x, y and z-axes for convenience of writing them. Let $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ denote unit vectors in the direction of the positive x , y and z-axes respectively. These unit vectors are also called **base vectors**. Often these vectors are also written as $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively. We will use $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ notation to make the role of the coordinates explicit. If you are more comfortable with ($\hat{i}$, $\hat{j}$, $\hat{k}$) notation, you may continue to use them since they are less cumbersome to write and could save time.

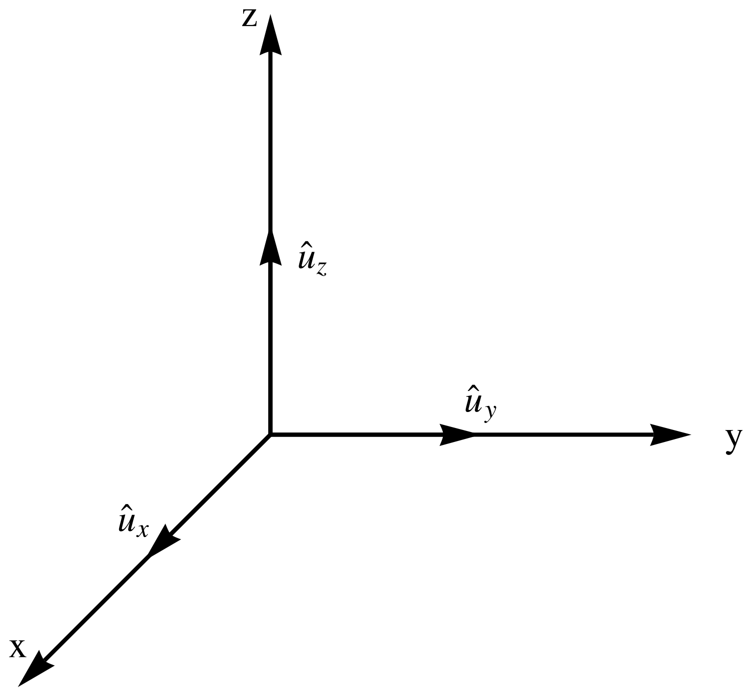

Figure 2.21: The unit vectors $\hat{u}_x$, $\hat{u}_y$, and $\hat{u}_z$ in the directions of the positive x , y and z-axes.

By definition, these vectors have unit magnitude. Since these vectors are perpendicular to each other, their dot products are also zero. Furthermore, a calculation of their cross-products will show that cross-product of two of them takes into plus or minus of the third.

$$\hat{u}_x \cdot \hat{u}_x = \hat{u}_y \cdot \hat{u}_y = \hat{u}_z \cdot \hat{u}_z = 1 \text{ (since unit vectors)} \tag{2.4}$$

$$\hat{u}_x \cdot \hat{u}_y = \hat{u}_x \cdot \hat{u}_z = \hat{u}_y \cdot \hat{u}_z = 0 \text{ (since perpendicular)} \tag{2.5}$$

$$\hat{u}_x \times \hat{u}_y = \hat{u}_z \tag{2.6}$$

$$\hat{u}_y \times \hat{u}_z = \hat{u}_x \tag{2.7}$$

$$\hat{u}_z \times \hat{u}_x = \hat{u}_y \tag{2.8}$$

Now, we can see that vector $\vec{A}_1$ along the x-axis in Fig. 2.20 can be written as a real number A_x times the unit vector $\hat{u}_x$. When the real number A_x is positive the vector $\vec{A}_1$ is pointed towards the positive x-axis and when A_x is negative, the vector $\vec{A}_1$ is pointed towards the negative x-axis.

Similarly, we can write vector $\vec{A}_2$ along the y-axis as a real number A_y times the unit vector $\hat{u}_y$. If we had a vector $\vec{A}_3$ along z-axis, then we would write the vector as a real number A_z times the unit vector

$\hat{u}_z$.

$$\vec{A}_1 = A_x \hat{u}_x \text{ (vector along } x \text{ axis)} \tag{2.9}$$
$$\vec{A}_2 = A_y \hat{u}_y \text{ (vector along } y \text{ axis)} \tag{2.10}$$
$$\vec{A}_3 = A_z \hat{u}_z \text{ (vector along } z \text{ axis)} \tag{2.11}$$

The real numbers A_x, A_y and A_y are called the components of vector $\vec{A}$, which can be written as the sum of the vectors $\vec{A}_1$, $\vec{A}_2$, and $\vec{A}_3$ along the axes. Fig. 2.22 shows how vector $\vec{A}$ can be constructed by adding the vectors along the Cartesian axes. Therefore, we can write any vector in terms of the base vectors.

$$\boxed{\vec{A} = A_x \hat{u}_x + A_y \hat{u}_y + A_z \hat{u}_z,} \tag{2.12}$$

where the sum of $A_x \hat{u}_x$, $A_y \hat{u}_y$ and $A_z \hat{u}_z$ is a vector sum. A simpler notation for the representation of a vector is to just list the components in order, as in $\vec{A} = (A_x, A_y, A_z)$. We will refrain from this notation since it ignores the explicit role of unit base vectors. However, if you are comfortable with this notation, it can save you some time since you do not have to write down the unit vectors all the time, especially when you are doing long calculations.

Vectors in terms of components: $\vec{A} = A_x \hat{u}_x + A_y \hat{u}_y + A_z \hat{u}_z = (A_x, A_y, A_z)$

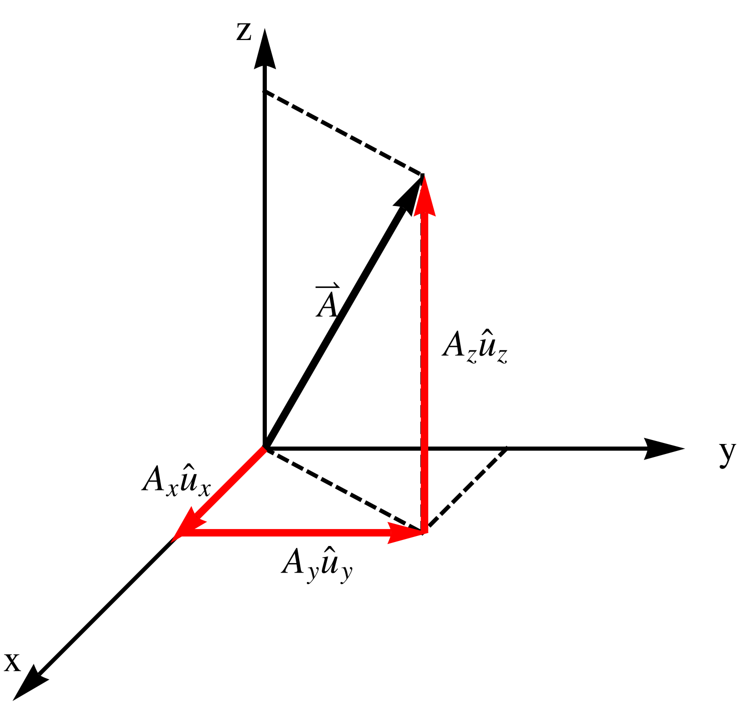

Figure 2.22: Placing the tail of vector $A_y \hat{u}_y$ at the tip of vector $A_x \hat{u}_x$ and then placing the tail of vector $A_z \hat{u}_z$ at the tip of $A_y \hat{u}_y$ give the vector $\vec{A}$ as the sum of the three: $\vec{A} = A_x \hat{u}_x + A_y \hat{u}_y + A_z \hat{u}_z$.

2.3.2 Magnitude of Vectors

You can find the magnitude of a vector from its components by using Pythagoras theorem in Fig. 2.22. It is immediately seen that the magnitude of a vector is equal to the square root of the sum of the squares the components.

$$\boxed{|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.} \tag{2.13}$$

2.3.3 Directions of Vectors

The directions of vectors is automatic in the graphical or geometric picture of vectors. You always draw the complete vectors when working geometrically. In the analytic or algebraic picture we work with components when performing algebraic manipulations. In the end, we must put together the information contained in the components into the magnitudes and directions of vectors. The specification of a physical direction in space very much depends upon whether the vector has only one non-zero component, or two non-zero components, or three non-zero components. These will be called one-dimensional, two-dimensional and three-dimensional situations respectively.

1. Direction for 1-D situations

If a vector $\vec{A}$ has only one non-zero component, say the x-component, the vector will be $A_x\hat{u}_x$ analytically. Depending upon the sign of the x-component A_x, the vector $\vec{A}$ will be pointed either towards the positive x-axis if $A_x > 0$ or towards the negative x-axis if $A_x < 0$. Therefore, if only one component of a vector is non-zero, the sign of the value of the component will have sufficient information to deduce the direction of the vector with respect to the coordinate axis. Due to this reason, sometimes the sign itself is mistakenly taken to be the direction of the vector. Note that direction of a vector is the physical direction in space and cannot be plus or minus something.

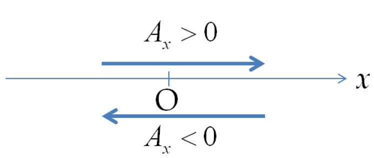

Figure 2.23: One-dimensional vector $\vec{A} = A_x\hat{u}_x$. The direction of the vector is towards the positive x-axis if $A_x > 0$ and towards the negative x-axis if $A_x < 0$.

2. Direction for 2-D situations

If a vector has two non-zero components, the vector will fall in a plane. For instance, if only the x and y-components of a vector are non-zero, the vector will fall in the xy-plane. Similarly, if only the x and z-components are non-zero, the vector will be in the xz plane, etc. In these cases, the direction in the plane is specified by an angle with one of the axes: the vector is drawn, or re-drawn if needed, so that the tail of the vector is at the origin, then the angle that the arrow makes with one of the axis is sufficient to tell the direction of the vector in the plane.

To be concrete, consider a vector $\vec{A}$ that falls entirely in the xy-plane such that we have the following analytic representations.

$$\vec{A} = A_x \hat{u}_x + A_y \hat{u}_y.$$

Depending upon the signs of A_x and A_y, the vector arrow may point in the first, second, third, or fourth quadrant as shown in Fig. 2.24. Beware of the common formula based on the trigonometry of the

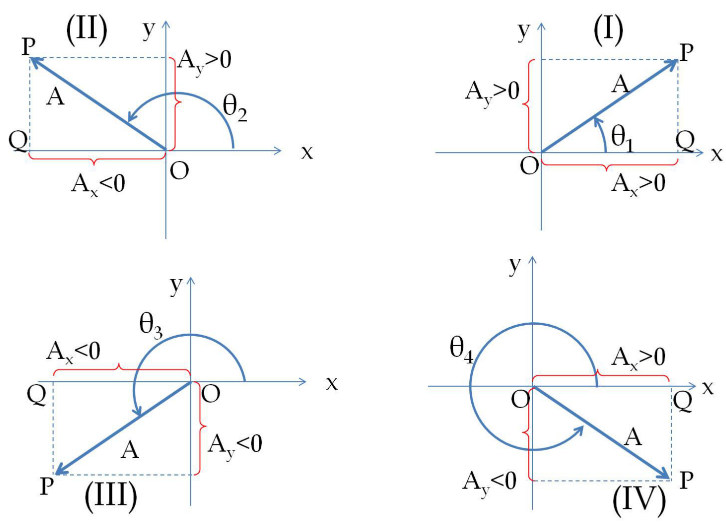

Figure 2.24: Vector in a plane. The direction of a vector in a plane can be in one of the four quadrants and can be assigned with respect to the positive x-axis direction. The common formula $\theta = \arctan(A_y/A_x)$ gives all four angles if correction is made for the particular quadrant: $\theta_1 = \theta$, $\theta_2 = \theta + 180^\circ$, $\theta_3 = \theta + 180^\circ$, and $\theta_4 = \theta + 360^\circ$.

vector in the first quadrant in Fig. 2.24. The direction of the vector in the first quadrant with respect to the positive x-axis direction is given by angle θ_1. From the right-angled triangle $\triangle OPQ$ in Fig. 2.24(I) it is seen that the tangent of this angle is related to the components A_x and A_y as follows.

$$\tan(\theta_1) = A_y/A_x \quad \Longrightarrow \quad \theta_1 = \arctan(A_y/A_x). \tag{2.14}$$

When the vector falls in the second quadrant, the right-angled triangle $\triangle OPQ$ in Fig. 2.24(II) gives the supplementary angle to the angle θ_2, that is the tangent of angle $\angle QOP$ is equal to A_y/A_x. Therefore, we must add 180° to the value obtained by arc-tangent of the ratio A_y/A_x.

$$\theta_2 = 180^\circ + \arctan(A_y/A_x). \tag{2.15}$$

If a vector falls in the third quadrant, the direction is opposite to

Warning! Beware of

the direction in the first quadrant with the signs of both A_x and A_y reversed. Therefore, the angle from the positive x-axis is 180° more than θ_1, which is equal to $\arctan(A_y/A_x)$.

$$\theta_3 = 180^\circ + \arctan(A_y/A_x). \tag{2.16}$$

Finally, the direction in the fourth quadrant is exactly opposite to that of the direction in the second quadrant when we reverse the signs of both A_x and A_y. Therefore,

$$\theta_4 = 360^\circ + \arctan(A_y/A_x). \tag{2.17}$$

Example 2.3.1. Direction of a vector in a plane A velocity vector of a projectile is given in the xy-plane of a Cartesian coordinate system with the right horizontal direction for the positive x-axis and the direction up for the positive y-axis. At a particular instant the velocity has the following value: $\vec{v} = 3\hat{u}_x - \sqrt{3}\hat{u}_y$ in units of $[m/s]$, where I have kept the unit separate for convenience. What is the direction of the velocity vector?

Solution. Geometrically, we could draw the vector and read off the angle with respect to either of the two axes. That would be sufficient for specifying the direction.

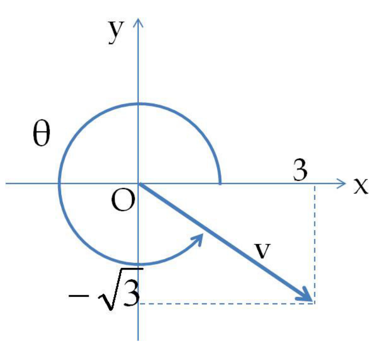

Figure 2.25: Example 2.3.1

The discussion above has shown that, alternately, we could obtain the angle from the given components without drawing the vector in the xy-plane. Here the components of the vector tell us that the vector is pointed in the fourth quadrant since $v_x > 0$ and $v_y < 0$. Therefore, the larger angle of the vector going counter-clockwise from the positive x-axis is given by the angle θ_4, which we will write simply as θ (see Fig. 2.25).

$$\theta = 360^\circ + \arctan\left(-\sqrt{3}/3\right) = 360^\circ - 30^\circ = 330^\circ.$$

The calculation reveals that using analytic method can save a lot of work.

Other Choices for Specifying Directions in a Plane

There is no reason why you should have to refer the direction of a vector in the xy-plane with respect to the direction of the positive x-axis. Often it is more convenient to provide the direction of a vector with respect to the closest axis. The negative axis directions are usually labeled with a "bar" over the axis symbol. For instance, while the positive x-axis is Ox, the negative x-axis is written as $O\bar{x}$, and similarly for other axes. Then, we have a number of convenient choices for giving directions of a vector $\overrightarrow{OP}$ in a plane drawn from the tail at the origin: $\angle xOP$, $\angle\bar{x}OP$, $\angle yOP$, and $\angle\bar{y}OP$.

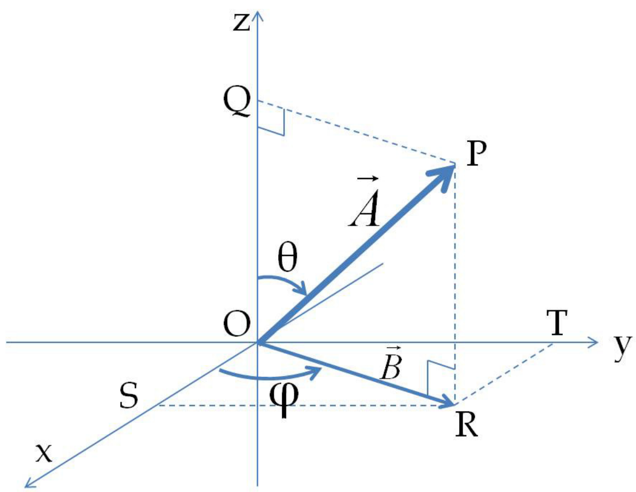

Figure 2.26: Vector in space can be given by using spherical coordinates given by the polar angle θ and the azimuthal angle ϕ. The polar angle is the angle between the vector and the positive z-axis, and the azimuthal angle is the angle between the positive x-axis and the direction of the projection of the vector on the xy-plane.

For instance, the direction of a vector in the third quadrant does not have to be specified by the large angle θ_3 but rather by the small angle $\angle QOP$, which is $\angle \bar{x}OP$ the vector makes with the negative x-axis.

$$\text{Third quadrant: } \angle QOP = \angle \bar{x}OP = \arctan\left(A_y/A_x\right). \qquad (2.18)$$

Similarly, the direction of the vector in the fourth quadrant is more conveniently specified with the smaller angle $\angle QOP$ with respect to the x-axis, that is, going in the clockwise direction from the x-axis rather than going counterclockwise direction. This is easily seen to equal $\arctan\left(A_y/A_x\right)$.

$$\text{Fourth quadrant: } \angle QOP \text{ in } \triangle QOP = \angle xOP = \arctan\left(A_y/A_x\right). \qquad (2.19)$$

3. Direction for 3-D situations

The directions for vectors with all three of its components non-zero require two angles. Typically, we use the polar and azimuthal angles of a spherical coordinate system. The commonly used symbols in physics are θ for the polar angle and ϕ for the azimuthal angle, which is exactly opposite of the convention in math books for these angles. Since we are studying physics here we will stick to the physics notation.

The polar angle θ for a vector $\vec{A}$ is the angle the vector makes with the z-axis when the tail of the vector is placed at the origin as shown in Fig. 2.26. The range of polar angle is between 0 and 180° or π radians. The zero degree direction is towards positive z-axis and the 180° direction is towards the negative z-axis.

Azimuthal angle is somewhat complicated: we first draw the vector in space with its tail at the origin, and then project the vector on the xy-plane by drawing a normal from the tip of the vector to the xy-plane as shown in the figure. The projection on the xy-plane defines another vector $\vec{B}$, which may be called projected vector, which has only x and y-components non-zero. The x and y-components of the projected vector $\vec{B}$ are equal to the x and y-components of the original vector $\vec{A}$ as you can verify from the given figure: $B_x = A_x$ and $B_y = A_y$. The direction of the projected vector $\vec{B}$ in the xy-plane with respect to the positive x-axis is given by the angle the projected vector makes with the positive x-axis direction. This angle is called the azimuthal angle of the original vector $\vec{A}$ as shown in the figure.

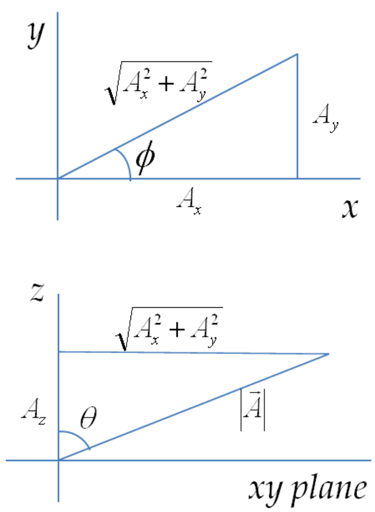

Figure 2.27: Angles ϕ and θ.

Azimuthal angle has a range of 0 to 360° or 2π radians, with the direction taken from the positive x-axis towards the positive y-axis, also called counter-clockwise as seen from the positive z-axis, as shown in Fig. 2.26. Sometimes, the range of azimuthal angle is given between −180° and 180° or $-\pi$ radians to π radians where the negative angle is for clockwise direction from the positive x-axis when seen from the perspective of the positive z-axis.

The right-angled triangles $\triangle OPQ$ and $\triangle OSR$ in Fig. 2.26 can be used to obtain the following relations for the polar and azimuthal angles in terms of the Cartesian components (A_x, A_y, A_z) of the vector.

$$\boxed{\tan\theta = \frac{\sqrt{A_x^2 + A_y^2}}{A_z} \quad \text{(Add 180° if } A_z < 0.\text{)}} \tag{2.20}$$

$$\boxed{\tan\phi = \frac{A_y}{A_x} \quad \text{(Be mindful of the quadrant.)}} \tag{2.21}$$

The triangles in Fig. 2.26 can also be used to obtain the components of a vector from its magnitude and direction given by polar and azimuth angles. I will give the answer and leave the exercise of proving them to the student as an exercise. Let the magnitude of a vector be A and the direction in three-dimensional space be given by polar angles θ and ϕ. The Cartesian components of the vector will be:

$$\boxed{A_x = A\sin\theta\cos\phi} \tag{2.22}$$

$$\boxed{A_y = A\sin\theta\sin\phi} \tag{2.23}$$

$$\boxed{A_z = A\cos\theta} \tag{2.24}$$

Example 2.3.2. Vector in space The displacement vector from the center of the Earth to St. Louis, Missouri has a length of $6.4 \times 10^6\ m$. A Cartesian axis is chosen with its origin at the center of the Earth, z-axis pointed towards polar North and x and y-axes in the equatorial plane. With a particular choice of x and y-axes, the Cartesian components of the vector are (-14780 m, -4234000 m, 4800000 m). What are the values of polar and azimuthal angles for the direction of the vector?

Solution. This is a straightforward application of formulas given above. We only need to be careful in using arctangent and make appropriate choice for the quadrant. The data shows that the angle will be in the third quadrant in the xy-plane.

$$\tan\theta = \frac{\sqrt{A_x^2 + A_y^2}}{A_z} = \frac{4234025}{4800000} = 0.882 \quad \Longrightarrow \quad \theta = 41.4^\circ.$$

$$\tan\phi = \frac{A_y}{A_x} = 286.5$$

$$\Longrightarrow \quad \text{Since, third quadrant, } \phi = 180^\circ + \arctan(286.5) = 269.8^\circ.$$

2.3.4 Adding and Subtracting Vectors Analytically

Adding and subtracting vectors is a rather simple task if done analytically. Each vector is first written as a sum of vectors along the three Cartesian axes using their components. Then, the component vectors along each Cartesian axis are summed to obtain the components of the resultant vector. Let $\vec{A} = A_x\hat{u}_x + A_y\hat{u}_y + A_z\hat{u}_z$ and $\vec{B} = B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z$ be two vectors we want to add. The Cartesian components of their sum $\vec{C} = \vec{A} + \vec{B}$ is

$$\begin{aligned}
\vec{C} &= \vec{A} + \vec{B} \\
&= (A_x\hat{u}_x + A_y\hat{u}_y + A_z\hat{u}_z) + (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \\
&= (A_x + B_x)\,\hat{u}_x + (A_y + B_y)\,\hat{u}_y + (A_z + B_z)\,\hat{u}_z
\end{aligned}$$

Therefore, the components of the sum are

$$\boxed{C_x = A_x + B_x}$$
$$\boxed{C_y = A_y + B_y}$$
$$\boxed{C_z = A_z + B_z}$$

Similarly, subtracting vector $\vec{B}$ from vector $\vec{A}$ gives the following result.

$$\begin{aligned}
\vec{D} &= \vec{A} - \vec{B} \\
&= (A_x - B_x)\,\hat{u}_x + (A_y - B_y)\,\hat{u}_y + (A_z - B_z)\,\hat{u}_z
\end{aligned}$$

Example 2.3.3. Calculation of a net force A pendulum bob has two forces on it - gravity of Earth and tension in the string. At a particular instant the pendulum is at an angle 30° from the vertical. At that instant it has a tension of 5 N (unit N called Newton will be explained later) pointed towards the point of suspension, and a force of gravity equal to 10 N pointed down. Find the magnitude and direction of the net force, defined as the sum of all forces.

Solution. It is helpful to make a drawing of the physical situation as shown in Fig. 2.28. The figure also shows a choice of axes for ana-

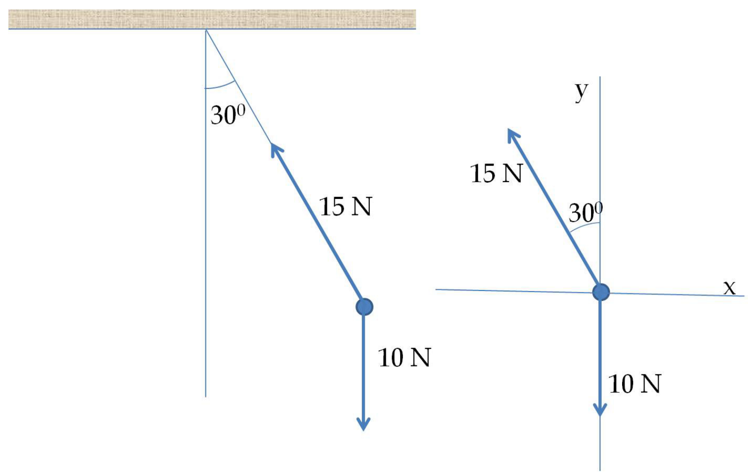

Figure 2.28: Example 2.3.3. Normally, we draw two figures. The figure on the left shows the physical setting of the problem and the figure on the right collects all the forces emanating from the same point where the origin of coordinate system is placed. The choice of axes are shown in the diagram. The components of the vectors are obtained from their projections on the axes.

lytical calculations. The magnitude of the forces and their directions are also shown on the figure for convenience. Let us denote force of gravity by $\vec{F}_g$ and the force of tension by $\vec{F}_T$. From the figure, we find the following decomposition of the vectors, where we keep track of units on the right side of the equation. The force of gravity has only the y-component and the force of tension has a negative x-component and a positive y-component with respect to the axes chosen in the figure. To work out the components I have redrawn the vector in Fig. 2.28 with tails at the origin of the axes so that simple projections onto the axes gives the corresponding components. I will also place the units separately so that you can focus on the other aspects of the calculations.

$$\vec{F}_g = -10\hat{u}_y \quad [\text{N}]$$
$$\vec{F}_T = -15\sin\ 30°\hat{u}_x + 15\cos\ 30°\hat{u}_y \quad [\text{N}]$$

Now, the sum of these two forces will have the following components.

$$\begin{aligned}\vec{F}_{net} &= \vec{F}_g + \vec{F}_T \\ &= -7.5\hat{u}_x + 3.0\hat{u}_y \quad \text{[N]}\end{aligned}$$

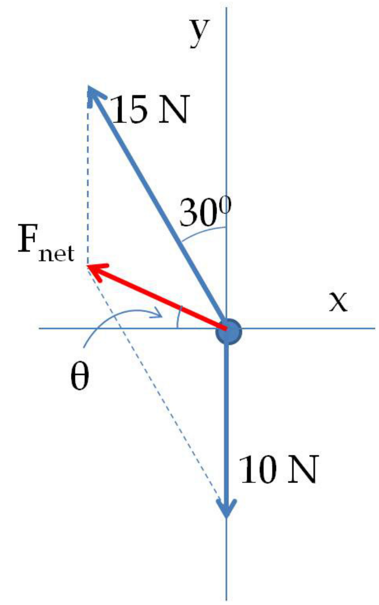

Figure 2.29: Example 2.3.3. The direction of the net force.

We are not done yet. We have found the representation of the net force in the chosen coordinate system. The magnitude and direction of the net force can now be obtained from the components.

Magnitude of the net force: $\left|\vec{F}_{net}\right| = \sqrt{(-7.5)^2 + (3.0)^2} = 22.5$ N.

The direction is in the second quadrant given by the angle θ with respect to the negative x-axis, $O\bar{x}$.

$$\theta = \arctan\left(3/(-7.5)\right) = -22^\circ.$$

The negative sign says that the angle is clockwise from the negative x-axis into the second quadrant. We could, of course, specify the counterclockwise angle with respect to the positive x-axis, which will be 158°.

2.3.5 Scalar Products Analytically

Using the dot products between the base vectors one can write the scalar product between two vectors $\vec{A}$ and $\vec{B}$ in terms of their components. The calculation is left as an exercise for the student.

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z,} \tag{2.25}$$

which must be identical to the geometric definition of the dot product. Therefore,

$$|\vec{A}||\vec{B}|\cos\theta = A_xB_x + A_yB_y + A_zB_z. \tag{2.26}$$

Applying the result in Eq. 2.25 to the dot product of a vector $\vec{A}$ with itself gives

$$|\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2. \tag{2.27}$$

Since the amplitude of a vector is a positive quantity, only the positive root is physical. Taking the positive root of both sides, we see that the amplitude of a vector can be computed from its components.

$$\boxed{|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}.} \tag{2.28}$$

Projections

From our geometrical discussions we know that the scalar product between two vectors also gives information about the projection of one vector onto another.

$$\vec{A}\cdot\vec{B} = AB\cos\theta, \tag{2.29}$$

which gives the projection of A on B, namely $A\cos\theta$, as:

$$A\cos\theta = \frac{\vec{A}\cdot\vec{B}}{B}. \tag{2.30}$$

If vector $\vec{B}$ is a unit vector then we can use Eq. 2.30 to find the component of the vector along the direction of the unit vector. For instance if $\vec{B}$ is the unit vector along the x-axis, we obtain the x-component of $\vec{A}$ as the following calculation demonstrates.

$$\begin{aligned}
\vec{A}\cdot\hat{u}_x &= (A_x\hat{u}_x + A_y\hat{u}_y + A_z\hat{u}_z)\cdot\hat{u}_x \\
&= A_x\hat{u}_x\cdot\hat{u}_x + A_y\hat{u}_y\cdot\hat{u}_x + A_z\hat{u}_z\cdot\hat{u}_x \\
&= A_x\times 1 + A_y\times 0 + A_z\times 0 \\
&= A_x
\end{aligned}$$

This shows that to obtain any component of a vector we just need a dot product of the vector with the corresponding base vector.

$$\vec{A}\cdot\hat{u}_x = A_x \tag{2.31}$$
$$\vec{A}\cdot\hat{u}_y = A_y \tag{2.32}$$
$$\vec{A}\cdot\hat{u}_z = A_z \tag{2.33}$$

Equations 2.31 to 2.33 are very useful, not only for finding components of a vector, but also the angles a vector makes with the Cartesian axes as we see by explicitly writing out the scalar product of the left side of these equations. Let α, β, and γ be the angles that vector $\vec{A}$ makes with the positive x, y and z-axes respectively as shown in Fig. 2.30 . That is, the angles between the vector $\vec{A}$ and the unit vectors $\hat{u}_x$, $\hat{u}_y$ and $\hat{u}_z$ are α, β, and γ, respectively. We find

$$\begin{aligned}
|\vec{A}||\hat{u}_x|\cos\alpha &= A_x \\
|\vec{A}||\hat{u}_y|\cos\beta &= A_y \\
|\vec{A}||\hat{u}_z|\cos\gamma &= A_z
\end{aligned}$$

Since the magnitudes of the unit vectors are all 1, we obtain the following formulas for the cosines of the angles. These cosines are

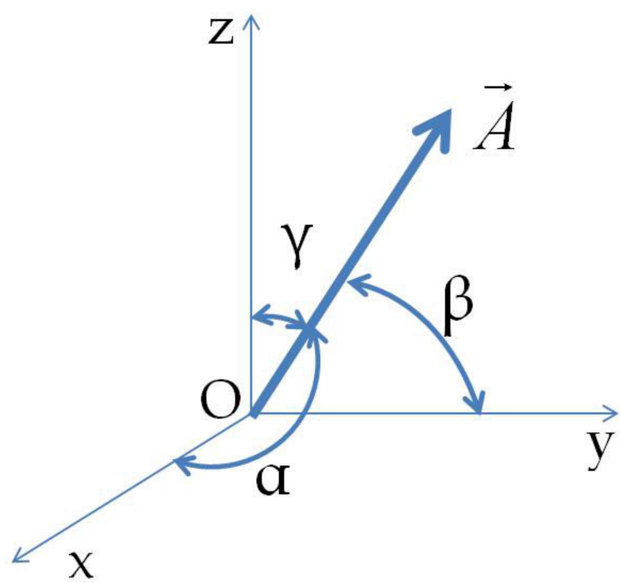

Figure 2.30: Angles for Direction Cosines.

called direction cosines of the vector.

$$\boxed{\cos\alpha = \frac{A_x}{|\vec{A}|} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}} \tag{2.34}$$

$$\boxed{\cos\beta = \frac{A_y}{|\vec{A}|} = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}} \tag{2.35}$$

$$\boxed{\cos\gamma = \frac{A_z}{|\vec{A}|} = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}} \tag{2.36}$$

Example 2.3.4. Angle between two vectors. Two vectors have the following Cartesian representations: $\vec{A} = 3\hat{u}_x + 4\hat{u}_y + 5\hat{u}_z$ and $\vec{B} = -6\hat{u}_x + 7\hat{u}_y + 8\hat{u}_z$. What is the angle between the the vectors when they are drawn so that their tails are at the same point?

Solution. This example is an application of Eq. 2.26, which is a very useful formula for finding angles between any two vectors. We need the magnitudes of the vectors and their components to compute the angle between them. From the given components, the magnitudes of the two vectors are:

$$A = \sqrt{3^2 + 4^2 + 5^2} = 7.07.$$
$$B = \sqrt{(-6)^2 + 7^2 + 8^2} = 12.2.$$

Therefore, the cosine of the angle between these vectors is

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} = \frac{-3\times 6 + 4\times 7 + 5\times 8}{7.07\times 12.2} = 0.58,$$

which gives the following angle between the two vectors.

$$\theta = \arccos(0.58) = 55^\circ.$$

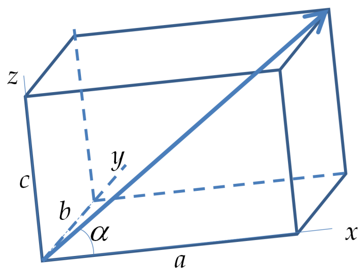

Figure 2.31: Body diagonal of a parallelepiped.

Example 2.3.5. Use of direction cosines. Find the angle between the body diagonal of a parallelepiped of dimensions $a \times b \times c$ and the edges. Use the result to find the angle the body diagonal of a cube makes with the edges.

Solution. Let us place a Cartesian coordinate system so that the body diagonal from one corner to farthest corner be given by the following vector $\vec{A} = a\hat{u}_x + b\hat{u}_y + c\hat{u}_z$. Therefore, the direction cosines with the edges are

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

If the parallelepiped is a cube, then all the sides are equal and the three angles will also be equal. Setting $a = b = c$ in these equations results in

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}.$$

Therefore, the angle the body diagonal makes with the edges is 54.7°.

Example 2.3.6. Law of cosines using the scalar product. Some standard trigonometric results can be obtained rather easily with the use of vectors. One such result is the law of cosines which can be derived using the scalar product since the scalar product involves cosine of the angle between two vectors. Let $\vec{A}$ and $\vec{B}$ be vectors along two adjacent sides of a triangle drawn with their tails at the vertex. Let the angle between $\vec{A}$ and $\vec{B}$ be denoted by θ. We now place a third vector from the tip of $\vec{B}$ to the tip $\vec{A}$ on the third side of the triangle. Then, it is seen that vector $\vec{C}$ is equal to $\vec{A} - \vec{B}$.

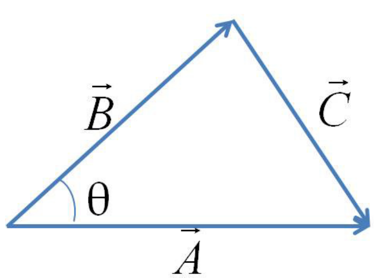

Figure 2.32: Diagram for law of cosines.

$$\vec{C} = \vec{A} - \vec{B}.$$

We now take the scalar product of each side with itself to obtain.

$$\vec{C}\cdot\vec{C} = \left(\vec{A} - \vec{B}\right)\cdot\left(\vec{A} - \vec{B}\right).$$

After expansion on the right side we find

$$C^2 = A^2 + B^2 - 2\vec{A}\cdot\vec{B},$$

where we have used the fact that the order of the vectors in a dot product does not matter, i.e. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$. We can now use the dot product between $\vec{A}$ and $\vec{B}$ to get the law of cosines for a triangle.

$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

2.3.6 Vector Product Analytically

A vector product is also relatively easy to work out by making use of the results of the vector products among the base vectors given in Eqs. 2.6-2.8. Let us rewrite them here also:

$$\boxed{\hat{u}_x \times \hat{u}_y = \hat{u}_z}$$
$$\boxed{\hat{u}_y \times \hat{u}_z = \hat{u}_x}$$
$$\boxed{\hat{u}_z \times \hat{u}_x = \hat{u}_y}$$
$$\boxed{\hat{u}_x \times \hat{u}_x = \hat{u}_y \times \hat{u}_y = \hat{u}_z \times \hat{u}_z = 0.}$$

Now, to find the cross product of two vectors we will first express each vector in terms of the base vectors and then expand the product out by using distributive, associative, and linearity properties of the vector product operation.

$$\vec{A} \times \vec{B} = (A_x\hat{u}_x + A_y\hat{u}_y + A_z\hat{u}_z) \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \quad (2.37)$$
$$\begin{aligned} &= A_x\hat{u}_x \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \\ &\quad + A_y\hat{u}_y \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \\ &\quad + A_z\hat{u}_z \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \end{aligned} \quad (2.38)$$

The calculations for the three lines in Eq. 2.38 are similar. Let us work out the first line in detail.

$$\begin{aligned} A_x\hat{u}_x \quad & \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) \\ &= A_xB_x(\hat{u}_x \times \hat{u}_x) + A_xB_y(\hat{u}_x \times \hat{u}_y) + A_xB_z(\hat{u}_x \times \hat{u}_z) \\ &= A_xB_y\hat{u}_z - A_xB_z\hat{u}_y. \end{aligned}$$

The second and third lines in Eq.2.38 work out similarly with the following results.

$$A_y\hat{u}_y \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) = -A_yB_x\hat{u}_z + A_yB_z\hat{u}_x$$
$$A_z\hat{u}_z \times (B_x\hat{u}_x + B_y\hat{u}_y + B_z\hat{u}_z) = A_zB_x\hat{u}_y - A_zB_y\hat{u}_x.$$

Summarizing, the vector product in terms of components gives the following.

$$\boxed{\vec{A} \times \vec{B} = \hat{u}_x(A_yB_z - A_zB_y) + \hat{u}_y(A_zB_x - A_xB_z) + \hat{u}_z(A_xB_y - A_yB_x).} \quad (2.39)$$

The result of the vector product can also be obtained by evaluating the following determinant, which serves as a good way of organizing terms in the cross product.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (2.40)$$
$$= \hat{u}_x(A_yB_z - A_zB_y) + \hat{u}_y(A_zB_x - A_xB_z) + \hat{u}_z(A_xB_y - A_yB_x).$$

Once cross product has been calculated, the magnitude and direction of the resulting vector can be determined in the usual way described above.

Example 2.3.7. Numerical example of vector product. Evaluate the vector product of the following vectors given in a particular Cartesian coordinate system: $\vec{A} = 2\hat{u}_y + 3\hat{u}_z$ and $\vec{B} = 4\hat{u}_x + 5\hat{u}_y + 6\hat{u}_z$.

Solution. The determinant notation of a vector product helps organize the information well. Note that in $\vec{A}$ there is no component vector along the x-axis, which will cause an entry of zero in the corresponding place in the determinant.

$$\begin{aligned} \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \\ &= \hat{u}_x (2 \times 6 - 3 \times 5) + \hat{u}_y (3 \times 4 - 0) + \hat{u}_z (0 - 2 \times 4) \\ &= -3\hat{u}_x + 12\hat{u}_y - 8\hat{u}_z. \end{aligned}$$

Example 2.3.8. Unit normal to the plane of two vectors. Two non-collinear vectors $\vec{A}$ and $\vec{B}$ define a plane in space since they can always be drawn from the same point. In a particular coordinate system suppose the two vectors have the following representations $\vec{A} = 2\hat{u}_y + 3\hat{u}_z$ and $\vec{B} = 4\hat{u}_x + 5\hat{u}_y + 6\hat{u}_z$. Find a unit vector normal to the plane.

Solution. We know that a vector product of two vectors is normal to the plane of the two vectors. Therefore, we can determine the unit normal vector from the cross product of the two vectors. We have already worked out the cross product of these vectors in Example 2.3.7.

$$\vec{A} \times \vec{B} = -3\hat{u}_x + 12\hat{u}_y - 8\hat{u}_z.$$

Now, we must divide this vector by its magnitude to obtain the unit normal, $\hat{n}$.

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{-3\hat{u}_x + 12\hat{u}_y - 8\hat{u}_z}{\sqrt{3^2 + 12^2 + 8^2}} = -0.20\hat{u}_x + 0.82\hat{u}_y - 0.54\hat{u}_z.$$

Note that the magnitude of $\hat{n}$ is equal to 1.00 for two decimal places. Beyond that the rounding error in the calculation makes the magnitude deviate from 1. Note also that there is another unit vector in addition to $\hat{n}$ that is normal to the plane of vectors $\vec{A}$ and $\vec{B}$: the other vector is $-\hat{n}$ which is pointed in exactly the opposite direction to $\hat{n}$.