4.5 STANDING WAVES ON A TAUT STRING

Each normal mode of a stretched string actually consists of two traveling waves, one moving to the right and the other moving to the left as we will see below. We have determined above that the transverse vibrations of a stretched string fixed at both ends are given by the following displacement function.

$$y_n(x,t) = A_n \sin\left(k_n x\right) \cos\left(\omega_n t - \delta_n\right), \quad 0 \le x \le L, \tag{4.23}$$

where the wavenumber and frequencies of the modes are

$$k_n = n\pi/L \;\text{ and }\; \omega_n = n\omega_1,$$

with

$$\omega_1 = \sqrt{\frac{T}{\mu}}\frac{\pi}{L},$$

and $n = 1, 2, 3, \cdots$. Here A_n is an amplitude of the n^{th} mode and δ_n the corresponding phase of the mode. The amplitude and phase depend on the initial conditions on the string. We will set $\delta_n = 0$ to focus on other properties that are insensitive to the initial conditions. The following trig identity helps one write the product of a sine and a cosine as a sum of two sine functions.

$$2 \sin A \cos B = \sin(A - B) + \sin(A + B).$$

Using this identity we write Eq. 4.23 as a sum of two sine functions

$$\begin{aligned} y_n(x,t) &= \frac{A_n}{2}\left[\sin\left(k_n x - \omega_n t\right)\right. \\ &\qquad \left. + \sin\left(k_n x + \omega_n t\right)\right], \quad 0 \le x \le L, \end{aligned} \tag{4.24}$$

The first term in Eq. 4.24 is a wave traveling towards the positive x-axis and the second term is a wave traveling towards the negative x-axis. Therefore, the normal modes of vibration of a string $y_n(x,t)$ turns out to be a sum of two waves, $\sin(k_n x - \omega_n t)$, which moves to the right towards the positive x-axis, and $\sin(k_n x + \omega_n t)$, which moves to the left towards the negative x-axis. Since the two waves have the same wavelength and same frequency, they travel at the same speed v.

$$v = \frac{\omega_n}{|k_n|} = \frac{n\sqrt{\frac{T}{\mu}}\frac{\pi}{L}}{n\pi/L} = \sqrt{\frac{T}{\mu}}.$$

We can imagine the way a standing wave is set-up: the right-moving wave is reflected off from the right boundary, and the left-moving wave is similarly reflected off from the left boundary. The waves then interfere with each other and create a standing pattern, called the standing wave, which shows only the up and down vibrations of each element and no traveling wave.

4.5.1 Forced Oscillations of a String

One way of vibrating a stretched string in one of the normal modes is to pull the string in the shape of the normal mode profile, and then releasing the deformed string from the rest. The subsequent motion of the string will be the oscillatory motion of the mode. Although this method of exciting a string to oscillate in a normal mode is possible, but it is very difficult to achieve.

Oscillating the string at one end with a frequency of the normal mode can also set the string into the mode as illustrated in Fig. 4.10 and 4.11. An oscillating force at one end near the node of a normal mode generates a wave that travels towards the fixed end at the wave speed $v = \sqrt{T/\mu}$, where T is the tension in the string and μ mass per unit length. When the wave arrives at the fixed end, a reflected wave is generated that has the phase opposite to that of the incident wave. The two waves travel in the opposite directions. If the oscillations correspond to one of the normal modes, then the right-traveling and left-traveling waves add such that the sum is a standing wave. A student will benefit by playing with simulation programs of the standing wave generation available online at various sites. One good site is *http://phet.colorado.edu/sims/wave-on-a-string/wave-on-a-string_en.html*.

We driving a standing wave by attaching the string to an oscillator near a node of the corresponding normal mode as shown in Fig. **??**. This experimental situation presents conditions on the ends of the string, called **boundary conditions**. We can write the boundary conditions on the wave equation analytically as:

$$y(x,t)|_{x=-\epsilon} = A_0\ cos(\omega t), \tag{4.25}$$

$$y(x,t)|_{x=L} = 0, \tag{4.26}$$

where $\epsilon << \lambda$ and $x = \epsilon$ is near the node at the origin and A_0 is the the amplitude of the oscillation at the oscillator. When the frequency ω is near the frequency of a normal mode, the amplitude of the wave A is much larger than A_0, and the place of the vibrator can be considered to be at a node. The solution of the wave problem with conditions give in Eqs. 4.25 and 4.26 can be written as

$$y(x,t) = Asin(kx)cos(\omega\ t). \tag{4.27}$$

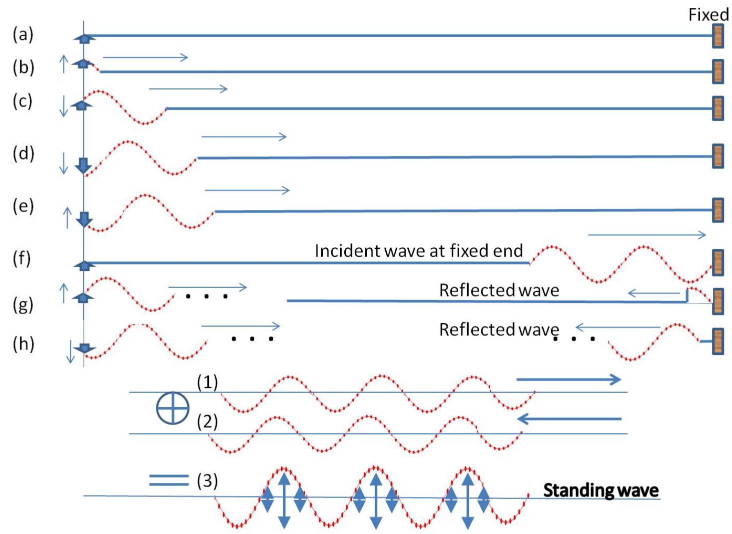

Figure 4.10: Generation of standing wave by driving vibration at a node of a normal mode. (a) The stretched string at equilibrium. (b)-(e): the right-moving traveling wave generated as the vibrator moves up and down. (f) the incident wave at the fixed support. (g) the generation of the reflected wave with the phase flipped with respect to the incident wave. (h) the reflected wave traveling back. (1)-(3) shows the addition of two traveling waves generates the standing wave in which the particles vibrate in phase with a frequency equal to the frequency of the normal mode.

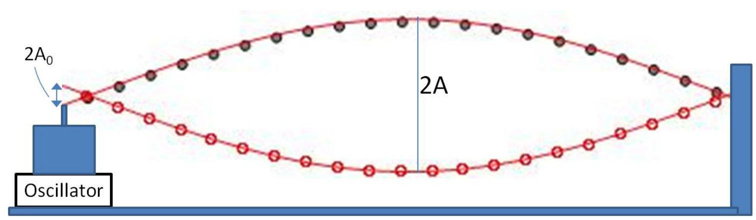

Figure 4.11: Generation of standing wave by driving vibrations by an oscillator near the node of a normal mode.

This will satisfy Eqs. 4.25 and 4.26 if

$$A\ sin(k\epsilon) = -A_0, \tag{4.28}$$
$$sin(kL) = 0. \tag{4.29}$$

Note that Eq. 4.29 is satisfied automatically for the mode of the string with nodes at $x = 0$ and $x = L$, and Eq. 4.28 gives the relation between the amplitude of the oscillator near the origin and the amplitude of the wave riding on the string. The wave function given in Eq. 4.27 is equal to the sum of two waves - one traveling to the right and the other traveling to the left.

$$y(x,t) = \frac{A}{2}\left[sin\left(kx - \omega\ t\right) + sin\left(kx + \omega\ t\right)\right]. \tag{4.30}$$

Fig. 4.11 shows the lowest normal mode driven by an oscillator. A string under tension is tied to the plunger of an oscillator. If the oscillator vibrates at the mode frequency of the lowest frequency mode, we find the string oscillate at a magnified amplitude since the oscillator is near a node confirming the relation between the distance A_0 traveled by the plunger of the oscillator and the amplitude A of the wave.