4.10 SUPERPOSITION OF WAVES

When you play two nearby notes on the piano you hear a low frequency beat that was not there before. How could a new note develop from two other notes? Similarly, in a pond when you drop two stones separated by a distance, then at the places the two circular waves meet you find a completely new wave pattern. Is there a way to find out the new wave pattern? Both of these phenomena and many like them result from the creation of the net wave from the sum of other waves overlapping in a region of space at the same time. The principle of the creation of a net wave from the addition of waves is called the **superposition principle**. Suppose there are waves ψ_1, ψ_2,$\cdots$, ψ_N in a region of space at some time t. Then, according to the superposition principle, the waves will superpose to create only one wave that is the sum of all.

$$\psi = \psi_1 + \psi_2 + \cdots + \psi_N. \tag{4.41}$$

The superposition of waves leads to some strange and unexpected consequences as we elaborate now with examples.

4.10.1 Beats

The beat phenomenon of sound is associated with the superposition of two sound waves of slightly different frequencies impinging on the ear drum simultaneously. For simplicity, consider two waves ψ_1 and ψ_2 of different angular frequencies ω_1 and ω_2 and with the same amplitude A meeting at the origin. Suppose, the waves at the origin are represented by the following functions of time.

$$\psi_1(0,t) = A\cos(\omega_1 t)$$
$$\psi_2(0,t) = A\cos(\omega_2 t)$$

In the beat phenomena we are usually interested in two nearby frequencies. Therefore, let us writeω_1 and ω_2 as

$$\omega_1 = \omega_0 - \frac{\Delta\omega}{2}$$
$$\omega_2 = \omega_0 + \frac{\Delta\omega}{2}$$

so that ω_0 is their average frequency and $\Delta\omega$ is the difference between the frequencies. By the superposition principle, the net wave at the origin will be

$$\psi = \psi_1 + \psi_2 = A\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right].$$

Now, the trigonometric identity

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

can be used to rewrite this relation as

$$\psi = 2A\cos\left[\left(\frac{\omega_1+\omega_2}{2}\right)t\right]\cos\left[\left(\frac{\omega_1-\omega_2}{2}\right)t\right]$$

Let us write this wave function in the terms of the average frequency ω_0 and the difference $\Delta\omega$ of the frequencies.

$$\psi = 2A\cos(\omega_0 t)\cos\left(\frac{\Delta\omega}{2}t\right) \tag{4.42}$$

The net vibration consists of fast oscillations at the average frequency ω_0 modulated by a slowly varying cosine as shown in Fig. 4.17. The

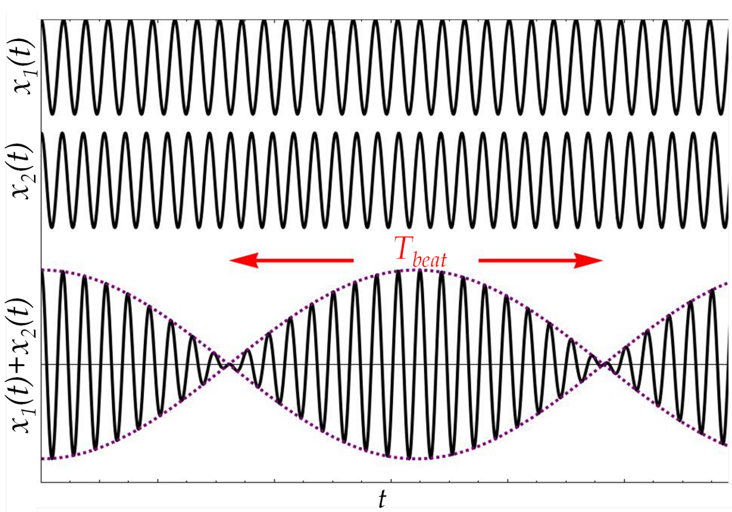

Figure 4.17: The beats as a result of superposition of two vibrations. The plot was made with $f_1 = 261.63$ Hz and $f_2 = 277.18$ Hz, corresponding to C_4 and $C_4^{\#}/D_4^{b}$ musical notes. The upper graph is the signal at frequency f_1, the middle at frequency f_2 and the bottom graph is the sum of the two. The peaks and troughs of the beats are separated by one beat period as shown.

amplitude of the fast oscillations is not constant in time but rather varies between $-2A$ and $2A$.

The largest amplitude oscillations occur at the peak and trough of the slowly varying modulation $cos\left(\frac{\Delta\omega}{2}t\right)$. If the working frequencies are in the audible range, then the peaks and troughs of the slowly varying modulation corresponds to the loudest sound while the sound is faintest when the modulation function is smallest. Thus, sound from two sources of nearby frequencies appear to go loud and soft periodically - the loud sounds are called beats.

Since both the troughs and peaks of $\cos\left(\frac{\Delta\omega}{2}t\right)$ correspond to the loudest sound, there are two beats in one period of this function. Therefore, the beat period is the half of the period of the modulating function $\cos\left(\frac{\Delta\omega}{2}t\right)$.

$$T_{beat} = \frac{1}{2}\left(\frac{2\pi}{\Delta\omega/2}\right) = \frac{2\pi}{\Delta\omega}. \tag{4.43}$$

Therefore, the beat frequency is the difference in frequencies of the two component vibrations:

$$\omega_{beat} = \Delta\omega = \left|\omega_1 - \omega_2\right|. \tag{4.44}$$

By setting $\omega = 2\pi f$, we can write this relation for the beat frequency.

$$\text{Beat frequency:} \quad f_{\text{beat}} = \left|f_1 - f_2\right|. \tag{4.45}$$

4.10.2 Interference of Waves

In the last section we described how waves add to create new waves that may not look anything like the original waves. This is clearly visible when two circular water waves on the surface of a pond overlap. The term **interference** refers to the phenomenon observed in the resulting intensity when the two waves overlap as shown below.

Let ψ_1 and ψ_2 be the wave functions of two waves overlapping in a particular region of space at time t. The resultant wave function would be the sum of ψ_1 and ψ_2.

$$\psi = \psi_1 + \psi_2 \tag{4.46}$$

The definition of intensity of waves says that the intensity is proportional to the time average of the square of the wave function. This gives the following for the resultant intensity where the two waves overlap.

$$I = \alpha < (\psi_1 + \psi_2)^2 >= I_1 + I_2 + I_{12} \tag{4.47}$$

where α is a constant prefactor that relates amplitudes to intensities in the medium, the symbol $< \cdots >$ stands for the time averaging, I_1 and I_2 are the individual intensities of the two waves when the other wave is not present there, and the cross-term I_{12} is called the interference term. Hence, intensity of the sum of the two waves is not equal to the sum of the intensities.

$$I \neq I_1 + I_2 \tag{4.48}$$

While the intensities of the two waves I_1 and I_2 are positive numbers, the interference term I_{12} can be positive or negative. Therefore the actual intensity observed can be different than simply the sum of the intensities of the separate waves. As a matter of fact, if the two waves have equal intensities to begin with, then the net wave can have intensity anywhere between zero and four times the intensity of one of the waves, not just twice.

Mixing two waves of equal amplitude:

For a simple example, we consider two waves of equal frequency and equal amplitude. Let the two waves start out at S_1 and S_2 in synchronization with each other, but wave 1 travels a distance L_1 to reach a point P at the detector while wave 2 travels a distance L_2 as shown in the Fig. 4.18.

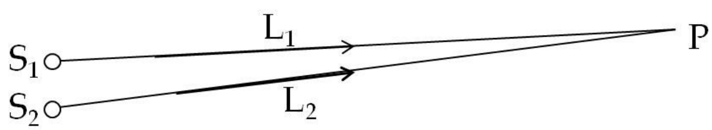

Figure 4.18: Interference of two waves.

This experiment can be set up, for example, by having two slits in a plate through which waves must exit on the other side and move to P where they mix. For instance, we can shine light from a laser on an opaque plate with two slits at S_1 and S_2. The experimental arrangement with two slits is also called **Young's double slit experiment** who studied interference of light waves by this method.

Even if the two waves start out in sync at S_1 and S_2, they might not be in sync at point P on the screen, depending upon relative distances from the source points to P. If the difference between L_1 and L_2 is a multiple of a wavelength, then we expect them to end up in sync again at P and a maximum intensity should result there. When this happens, we say that the two waves have a constructive interference at point P. This is shown in Fig. 4.19 where two waves which start out at points S_1 and S_2 in sync end up in sync at point P.

On the other hand, if the difference of distances from the sources at S_1 and S_2 to the point P is an odd multiple of half a wavelength, then the crest of one wave will fall on the trough of the other, making the net amplitude zero there. The waves are said to interfere destructively at point P. If the amplitudes of the waves from S_1 and S_2 are equal, then, the net amplitude at the point of destructive interference will be zero since the two waves will be completely out of sync there. This is shown in Fig. 4.20 where two waves which start out in sync at points S_1 and S_2 end up out of sync at point P and

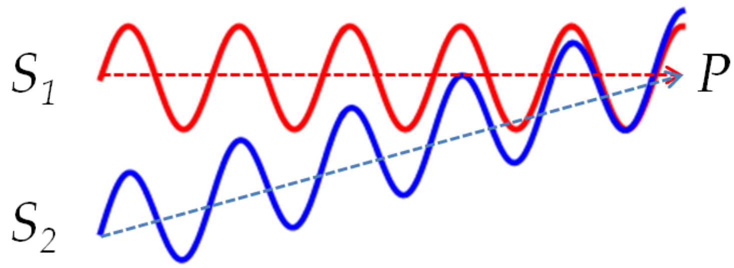

Figure 4.19: Constructive interference of two waves.

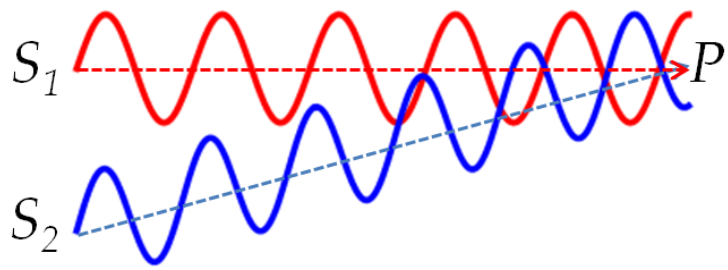

Figure 4.20: Destructive interference of two waves.

interfere destructively there.

Let us now see how the constructive and destructive interferences show up analytically as a result of the addition of the amplitudes of the two waves. For this, we need to work out the interference term more fully.

$$I_{12} = 2\alpha < \psi_1 \psi_2 >$$

Since waves 1 and 2 travel distances L_1 and L_2 their wave functions at point P will take on the following values at an arbitrary time t.

$$\psi_1(t) = A\cos(kL_1 - \omega t) \tag{4.49}$$

$$\psi_2(t) = A\cos(kL_2 - \omega t) \tag{4.50}$$

Note the waves have the same frequency, wavenumber and amplitude for our exercise here. Note also that we have dropped the phase constant since we have assumed that the waves start out in sync at S_1 and S_2. Multiplying the two wave functions we find

$$\begin{aligned} \psi_1(t)\psi_2(t) &= A^2\cos(kL_1)\cos(kL_2)\cos^2\omega t + A^2\sin(kL_1)\sin(kL_2)\sin^2\omega t \\ &\quad -A^2\left[\cos(kL_1)\sin(kL_2) + \sin(kL_1)\cos(kL_2)\right]\cos\omega t\sin\omega t \end{aligned}$$

Since the wave functions are periodic in time, the time average of the product $< \psi_1(t)\psi_2(t) >$ can be done by integrating the product with respect to time over one cycle and then dividing the result by the period.

$$\text{Time average of a periodic function: } < f(t) >= \frac{1}{T}\int_0^T f(t)dt.$$

Here the period of the wave function is $T = 2\pi/\omega$. The integration involves separate integrals over $\cos^2(\omega t)$, $\sin^2(\omega t)$, and $\cos(\omega t)\sin(\omega t)$ with the following results.

$$< \cos^2(\omega t) >= \frac{1}{T}\int_0^T \cos^2(\omega t) dt = \frac{1}{2} \tag{4.51}$$

$$< \sin^2(\omega t) >= \frac{1}{T}\int_0^T \sin^2(\omega t) dt = \frac{1}{2} \tag{4.52}$$

$$< \cos(\omega t)\sin(\omega t) >= \frac{1}{T}\int_0^T \cos(\omega t)\sin(\omega t) dt = 0. \tag{4.53}$$

Therefore, the interference term evaluates to

$$I_{12} = \alpha A^2 \cos\left[k\left(L_1 - L_2\right)\right]$$

Note the intensity of each wave is

$$I_1 = \alpha < \psi_1^2 >= \frac{1}{2}\alpha A^2 = I_2$$

Let us write the intensity of the individual waves as I_0.

$$I_0 \equiv I_1 = I_2.$$

Now, writing the intensity observed at point P, all in terms of I_0, the intensity of separate waves, is

$$I = 2I_0\left\{1 + \cos\left[k\left(L_1 - L_2\right)\right]\right\} \tag{4.54}$$

For fixed source points S_1 and S_2, the intensity depends upon the location of the observation point P since depending upon the value of $k(L_1 - L_2)$ the cosine function fluctuates between -1 and 1. When cosine is equal to -1, the intensity at P will be zero, and when cosine is $+1$, the intensity is four times as much as the intensity in each wave. Hence, we have the following conditions for constructive and destructive interference.

$$k\left(L_1 - L_2\right) = \begin{cases} 2n\pi & n = 0, \pm1, \pm2, \cdots \quad \text{Constructive} \\ (2n+1)\pi & n = 0, \pm1, \pm2, \cdots \quad \text{Destructive} \end{cases} \tag{4.55}$$

Let us write the wave number k in terms of the wavelength to relate to the qualitative pictures we obtained above.

$$\left(L_1 - L_2\right) = \begin{cases} n\lambda, & n = 0, \pm1, \pm2, \cdots \quad \text{Constructive} \\ \left(n + \frac{1}{2}\right)\lambda, & n = 0, \pm1, \pm2, \cdots \quad \text{Destructive} \end{cases} \tag{4.56}$$

Therefore, the waves from two synchronized sources of the same frequency interfere constructively when the distances to an observation point P differ by multiples of a wavelength and interfere destructively if the distances differ by half a wavelength, or odd multiples of half a wave length.

SIMPLIFICATION FOR LARGE L

Often we are interested in interference of two waves far away from the sources. In this case, we can use trigonometry to simplify the interference conditions when the difference in the path lengths of the two sources to the detector is small compared to the distance to the detector.

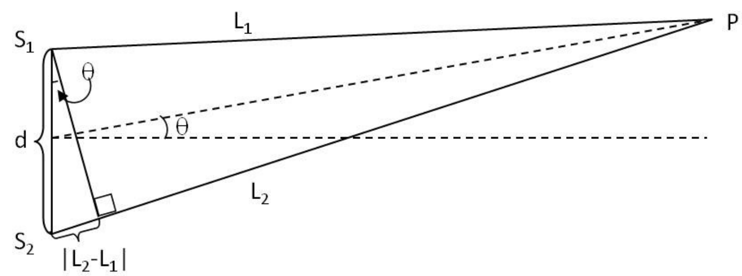

Figure 4.21: Geometry for approximating the difference in the path length from the source points and a point on the screen.

From the geometry in the figure, the difference in paths is related to the angle and the separation of the sources.

$$|L_1 - L_2| = d \sin\theta \tag{4.57}$$

Therefore, the conditions for interference are:

$$d \sin\theta = \begin{cases} n\lambda, & n = 0, \pm 1, \pm 2, \cdots \quad \text{Constructive} \\ \left(n + \frac{1}{2}\right)\lambda, & n = 0, \pm 1, \pm 2, \cdots \quad \text{Destructive} \end{cases} \tag{4.58}$$

Example 4.10.1. Constructive and Destructive Interference. Waves of wavelength 50 cm are emitted from two point sources separated by 70 cm. In what directions will there be constructive interferences at a far away detector?

Solution. This example illustrates the use of the formulas for constructive and destructive interferences in a double-slit experiment. We have been given the wavelength and the distance between the slits. Since the detector is far away, we can use the approximate formula for the interference conditions which gives the directions in which constructive and destructive interferences will occur in terms of the wavelength λ and the distance d between the slits. The constructive interferences are in the directions given by:

$$\sin\theta = \frac{n\lambda}{d}, \quad n = 0, \pm 1, \pm 2, \cdots$$

Clearly, the conditions fails for n for which $\sin\theta > 1$ since sine of an angle must be between -1 and 1, i.e. $-1 \leq \sin\theta \leq 1$. Also, the allowed angles are symmetric about $\theta = 0$ since if $n = 1$ is allowed, then $n = -1$ is also allowed. Now, we systematically work out the allowed angle starting with $n = 0$. We find the following allowed the directions of constructive interference:

$$
\begin{aligned}
&n = 0: \quad \theta = 0 \\
&n = \pm 1: \quad \theta = \arcsin\left(\frac{\pm 1 \times 50\text{cm}}{70\text{cm}}\right) = \pm 45.6^\circ \\
&n = \pm 2: \quad \theta = \arcsin\left(\frac{\pm 2 \times 50\text{cm}}{70\text{cm}}\right) \quad \Longrightarrow \text{no solution}
\end{aligned}
$$

Hence, only three constructive places are possible for the given conditions.