8.1 MAGNETIC FIELD OF A STEADY CURRENT

8.1.1 Biot-Savart Law

A magnetic field is associated with all moving charges in addition to the electric field of the charges, which is always present regardless of the state of motion of the charge. Unlike the electric field, however, the magnetic field of a moving point charge is quite difficult to compute. However, if a collection of charges move such that they form a steady current, such as the steady current in a wire connected to the terminals of a battery, then the magnetic field is given by a much simpler law called Biot-Savart law.

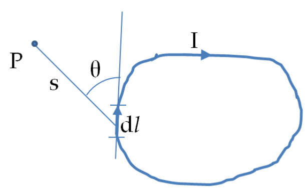

Figure 8.1: Biot-Savart law gives magnetic field at a space point P from a steady current I in a circuit.

Biot-Savart law says that electric currents are accompanied by magnetic fields whose magnitude and direction at a space point P depends on the magnitude and direction of the current and the magnitude and direction of the displacement vector from current to P.

It is more convenient for application to write Biot-Savart law from current in an infinitesimal element of a circuit (see Fig. 8.1). The magnitude of the magnetic field dB_P at a point P in space by a steady current I through an element dl of a circuit varies directly with the strength of the current I, length dl of the element and sine of the angle θ between current direction and the direction from the element to P, and varies inversely to the square of the distance s of the space point from the current. The direction of the magnetic field at P is given by a right-hand rule as illustrated in Fig. 8.2.

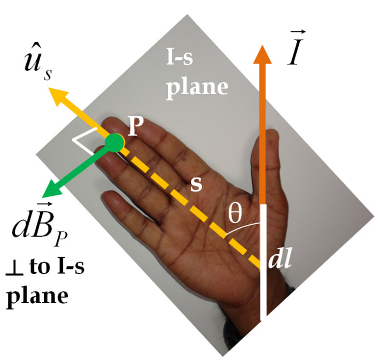

Figure 8.2: Biot-Savart Law and Right hand rule (RHR-II) for magnetic field of a current. To find the direction of magnetic field at space point P, place current direction and the line from current element to point P in one plane, labeled I-s plane. Magnetic field at point P is perpendicular to the I-s plane as shown with thumb along the current and any other finger along line s to point P. The magnitude of the magnetic field $dB_P = (\mu_0/4\pi) I \sin\theta dl/s^2$.

Biot Savart Law: (8.1)

$$\text{Magnitude: } dB_P = \text{Constant} \times \frac{I\ \sin\theta\ dl}{s^2}$$

Direction: Use Right-Hand Rule

where the constant is often written as

$$\text{Constant} = \frac{\mu_0}{4\pi},$$

where μ_0 is called the **magnetic permeability** of free space. When magnetic field is expressed in the SI unit Tesla (T), the current in A and the distance in m, then the constant has an exact value 10^{-7} T.m/A.

$$\boxed{\mu_0 = 4\pi \times 10^{-7}\ \frac{\text{T.m}}{\text{A}}}$$

We will refer the right-hand rule of Biot-Savart law as RHR-II to distinguish it from other uses of right-hand rule arising from cross-products among vectors elsewhere, such as in torque, angular momentum, magnetic force on moving charge, etc. One way of memorizing RHR-II is shown in Fig. 8.2. Place your right thumb along the current, and point any other finger of right-hand towards the space point P of interest. The magnetic field at point P will be coming out of the palm perpendicular to the plane of the current and the line from the current element to the space point P.

Biot-Savart law can be compactly written in a vector form in terms of the unit vector $\hat{u}_s$ in the direction from the current element dl to the space point P and the current vector $\vec{I}$.

$$d\vec{B}_P = \frac{\mu_0}{4\pi}\frac{dl\ \vec{I}\times\hat{u}_s}{s^2}. \tag{8.2}$$

Frequently, we transfer the vector information of the current to the current element and write the latter as vector $d\vec{l}$, which has magnitude dl and direction of the current.

$$\boxed{d\vec{B}_P = \frac{\mu_0}{4\pi}\frac{I\ d\vec{l}\times\hat{u}_s}{s^2}.} \tag{8.3}$$

The contributions from all elements of the circuit add vectorially to yield the magnetic field at point P. We can write this instruction for adding up the contributions from all elements of the circuit as a conceptual integral:

$$\boxed{\vec{B}_P = \frac{\mu_0}{4\pi}\int_{\text{Circuit}}\frac{I\ d\vec{l}\times\hat{u}_s}{s^2},} \tag{8.4}$$

where quantities s, $d\vec{l}$, and $\hat{u}_s$ depend on the circuit and the field point P. To make use of this formula, we normally employ a coordinate system and convert these quantities for an arbitrary element on the circuit to appropriate variables as demonstrated in examples below.

8.1.2 Calculations of Magnetic Field Using Biot-Savart Law

Example 8.1.1. Magnetic Field of Current in Straight Wire. One part of an electric circuit that carries a steady current I is a straight wire of length L. What is the magnetic field at a point in the plane that divides the wire in half from the current in the straight segment only?

Solution. Recall that Biot-Savart law is written as a conceptual integral and require setting up a coordinate system to make the integral explicit and ready for calculations. The type of coordinate system suitable for a problem depends in the geometry of the current carrying structure. Since, the current in this problem is carried on a straight segment, we will choose Cartesian coordinates as shown in Fig. 8.3, where we have placed the wire along the z-axis symmetrically about the origin. The field point P will then be in the xy-plane.

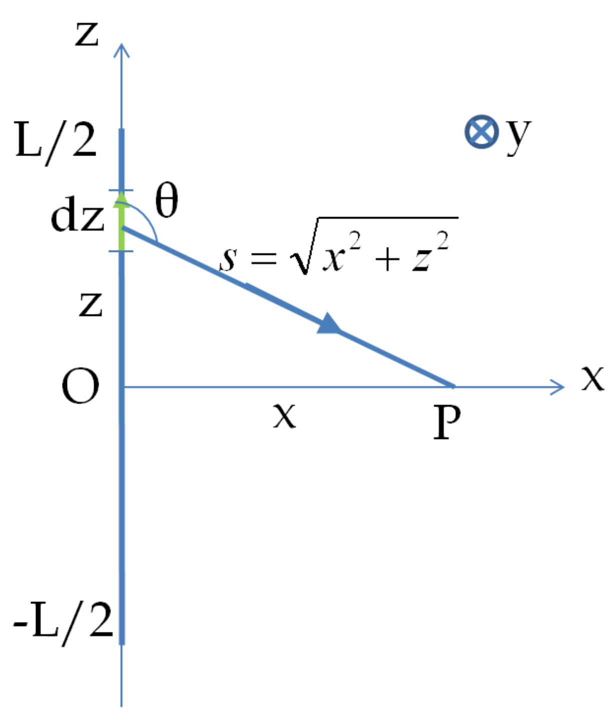

Figure 8.3: Geometry for calculation of magnetic field at a point in the symmetry plane of a straight wire. The wire is along the z-axis and the symmetry plane is the xy-plane. Only xz plane is shown since all points on the symmetry plane that are same distance from the wire are equivalent. The y-axis is pointed into the page here.

To be concrete we choose the x-axis direction to pass through the field point P. These choices give us the following quantities to be used for writing an expression for the magnetic field at P.

$$s = \sqrt{z^2 + x^2}$$
$$dl = dy$$
$$\sin\theta = \sin(\pi - \theta) = \frac{x}{s} = \frac{x}{\sqrt{x^2 + z^2}}$$

Now, we write the expression for magnetic field by a current element between z and $z + dz$.

$$\text{Magnitude: } dB_P = \frac{\mu_0}{4\pi}\frac{Ixdz}{(x^2+z^2)^{3/2}}$$

Direction: towards the positive y-axis.

This says that the integration that sums up the contributions from all current elements will be from $z = -L/2$ to $z = L/2$. Since the integrand is an even function of z, we can double the result for integration from $z = 0$ to $z = L/2$.

$$\begin{aligned}\text{Magnitude: } B_P &= \frac{\mu_0}{4\pi}2Ix\int_0^{L/2}\frac{dz}{(x^2+z^2)^{3/2}}\\ &= \frac{\mu_0}{2\pi}\frac{IL}{x\sqrt{L^2+4x^2}}.\end{aligned}$$

We can write the answer in a vector form by using the unit vector $\hat{u}_y$ for the direction along the positive y-axis. Therefore, the magnetic field at a point P on the x-axis in the coordinate system used above is given by

$$\vec{B}_P = \frac{\mu_0}{2\pi}\frac{IL}{x\sqrt{L^2+4x^2}}\hat{u}_y. \tag{8.5}$$

Since the magnitude of the magnetic field depends only on the distance of the field point P from the wire and not on the polar angle in the xy-plane, we conclude that the magnetic field in the xy-plane will have the same magnitude at the same distance from the origin in all directions. Only the direction of the magnetic field will be different at different points in the xy-plane. Using the polar coordinates (r, ϕ) for points in the xy-plane, point P will be located with polar coordinates (r, ϕ) instead of $(x, y, 0)$, and the magnetic field at point P will be given by

$$\vec{B}_P = \frac{\mu_0}{2\pi}\frac{IL}{r\sqrt{L^2+4r^2}}\hat{u}_\phi, \tag{8.6}$$

where $r = \sqrt{x^2+y^2}$ and $\hat{u}_\phi$ is a unit vector tangential to the circle of radius r at point having polar coordinates (r, ϕ). In Fig. 8.4 we draw the $\vec{B}$ field lines in the symmetry plane to get a visual picture of the magnetic field near the wire. The field is stronger near the wire and weaker away from the wire which is shown by the thickness of the lines.

A very useful formula for the magnetic field of a long wire can be obtained from this result by taking $L \to \infty$ limit while keeping r finite.

$$\text{Infinitely long wire: } \vec{B}_P = \frac{\mu_0 I}{2\pi r}\hat{u}_\phi, \tag{8.7}$$

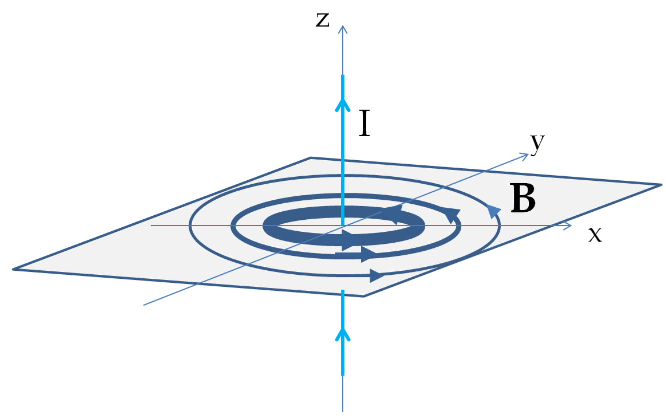

Figure 8.4: Magnetic field of current in a straight wire in the symmetry plane. Note that magnetic field is stronger near the wire shown here with thicker field lines near the wire. Right-hand rule II gives the direction of the magnetic field lines as shown.

Example 8.1.2. Comparing to Magnetic Field of Earth

Evaluate magnetic field of a 5-A current in a very long straight wire at the following distance from the wire: (a) 1 cm , (b) 10 cm, and (c) 100 cm. Compare the strengths of magnetic fields to that of magnetic field of Earth, whose magnitude is approximately 1/2 Gauss.

Solution. The formula for current in infinitely lone wire can be used to find the magnetic field values here. The calculations for the magnitudes give the following results.

(a) $B = \frac{\mu_0}{2\pi}\frac{I}{r} = 2\frac{\text{T.m}}{\text{A}} \times \frac{5\ \text{A}}{0.01\ \text{m}} = 10^{-4}\text{T} = 1\ \text{G}.$

(b) $B = 0.1$ G.

(c) $B = 0.01$ G.

At 1 cm distance, the magnetic field by the 5 Amp current in the wire is twice the Earth's magnetic field. At 10 cm , B of the current is 1/5 th of the Earth's magnetic field, and at 100 cm from the wire it is 1/50 th of the Earth's magnetic field.

Example 8.1.3. Numerical Example of Magnetic Field.

A copper wire of radius 1 mm and length 3 m is connected across a 3 V battery with 1 mΩ internal resistance. The wire is straightened so that there is a one meter long straight segment. Find the magnetic field at a distance of 2 cm from the straight segment near the midpoint of the segment and compare the magnetic field to the magnetic field of the Earth which can be taken to be $\sim \frac{1}{2}G$, where G stands for Gauss. Use $1.7 \times 10^{-8}\ \Omega.m$ for the resistivity of copper.

Solution. First we find the resistance in the copper wire.

$$R = \rho \frac{L}{A} = 16 \times 10^{-3}\Omega.$$

Since copper wire and the internal resistance of the battery are in series, the current in the circuit is

$$I = \frac{3\text{V}}{(16+5)\times 10^{-3}\Omega} = 143 \text{ A}.$$

The distance of 2 cm from the wire is much smaller than the length of the wire. Therefore, we use the very long wire approximate formula to obtain the magnitude of the magnetic field at a distance of 2 cm from the wire as

$$\text{Magnitude: } B = \frac{\mu_0}{2\pi} \times \frac{I}{r} = 0.00143 \text{ T} = 14.3 \text{ G}.$$

This magnetic field is approx. 28.6 times stronger than the magnetic field of Earth.

Example 8.1.4. Force Between Oppositely Directed Currents.

Solution. As an application of the formula for magnetic field of a straight wire derived above, we find the force per unit length between two infinitely long parallel wires separated by a distance d and carrying currents I_1 and I_2 in the opposite directions as shown in Fig. 8.5. Two wires configured this way are also called anti-parallel.

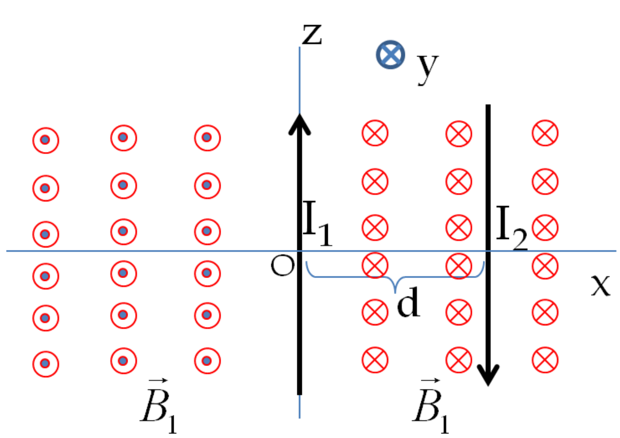

Figure 8.5: Geometry of two wires carrying current in opposite direction. Wire 2 is in the magnetic field $\vec{B}_1$ of current $\vec{I}_1$ in wire 1. Therefore, effective force on wire 2 per unit length of wire 2 is obtained by $\vec{I}_2 \times \vec{B}_1$. The coordinate system shown in this diagram is used for analytic calculations in the text.

To find the effective force of wire 1 on wire 2, we will first find magnetic field of wire 1 at the location of the wire 2. The conduction electrons of wire 2 then experience a magnetic force as a result of the magnetic field of wire 1. The electrical response of the atoms of the material of the wire results in an effective force on wire 2 from wire 1. This effective force is also called magnetic force between two wires. Since the effective force between the wires can do work while magnetic force on charges cannot do any work, we will call these forces effective forces.

From the formula for the magnetic field of a very long wire given above and applied to the present situation using the coordinates in Fig. 8.5, the magnetic field of current I_1 at the location of wire 2 would be

$$\vec{B}_1 = \hat{u}_y \frac{\mu_0 I_1}{2\pi d} \tag{8.8}$$

Therefore, force $\vec{F}_2$ on a length L_2 of wire 2 will be given by

$$\vec{F}_2 = L_2 \vec{I}_2 \times \vec{B}_1 = I_2 L_2 \frac{\mu_0 I_1}{2\pi d} \left(-\hat{u}_z \times \hat{u}_y\right) = \frac{\mu_0 I_1 I_2 L_2}{2\pi d} \hat{u}_x$$

We see that the effective force on wire 2 is away from wire 1. Hence, the effective force between the wires is repulsive with magnitude $\mu_0 I_1 I_2 / 2\pi d$ per unit length. Since the magnitude of the force between the wires is symmetric in $I_1 \leftrightarrow I_2$, the magnitude of the force on wire 1 per unit length of wire 1 will be given by the same formula.

The effective force per unit length is proportional to the product of the currents in the two wires and inversely proportional to the distance between the wires. This result was found experimentally by Ampere.

If current in the wire 2 was in the same direction as current in 1, the cross product for the effective force on the wire would have given us the direction of the effective force on wire number 2 as $(-\hat{u}_x)$ in place of $\hat{u}_x$. This would have meant an attractive effective force between the wires.

Example 8.1.5. Magnetic Field on the Symmetry Axis of a Circular Loop.

Solution. Consider a circular loop of radius R carrying current I placed in the xy-plane with its center at the origin of coordinates. We want to find magnetic field at point P on the symmetry axis which is the z-axis in Fig. 8.6. Because of the circular geometry of the current structure, it is advantageous to use cylindrical coordinate system here.

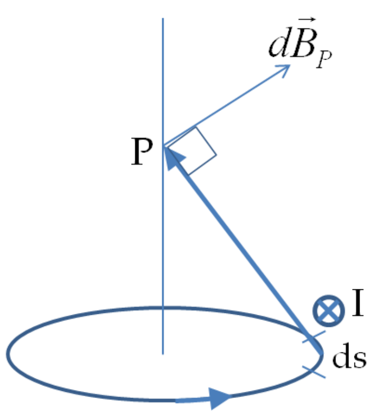

Figure 8.6: Example 8.1.5.

We note several simplifying features of this problem.

- The problem has a useful symmetry: all points of the ring are at an equal distance from the field point P.

- Since the direction of the current at any point of the ring is along the tangent to the circle, the angle between the current direction in any element and the direction from the element to the field point P is 90°.

Using these simplifying aspects of the problem, we find that magnetic field from each element ds of the current loop will have the same magnitude given by

$$dB_P = \frac{\mu_0}{4\pi}\frac{Ids}{R^2+z^2}. \tag{8.9}$$

However, the directions of the field at point P from current in different elements of the ring are different. Using the right-hand rule for the direction of magnetic field, we find that the directions from different parts of the ring form a cone at point P as shown in Fig. 8.7. Therefore, horizontal components will cancel out and we will be

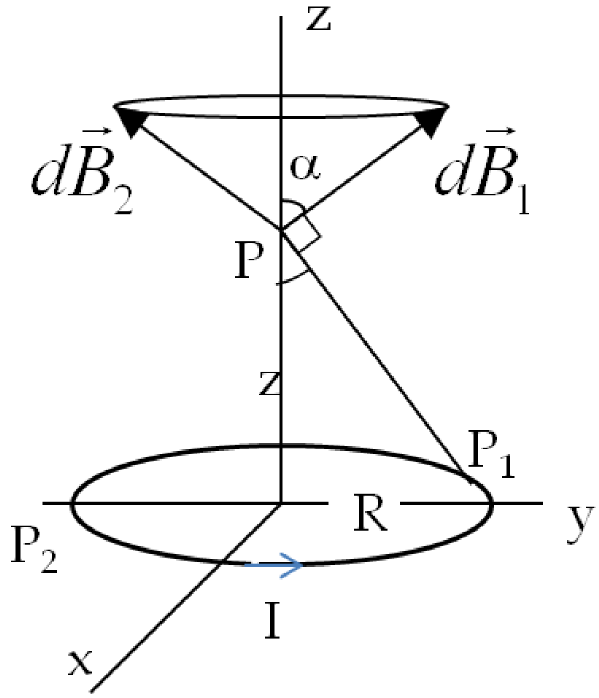

Figure 8.7: Magnetic fields $d\vec{B}_1$ and $d\vec{B}_2$ by current elements at P_1 and P_2, which are oppositely placed on the ring, result in the net magnetic field that has only the vertical component nonzero.

left with only the vertical components of magnetic field. We seek the vertical component of a vector of magnitude dB that makes an angle α with the vertical (see figure).

$$d\vec{B}_P\Big|_{\perp} = dB_P\ \cos\alpha, \tag{8.10}$$

where $\cos\alpha$ is

$$\cos\alpha = \frac{R}{\sqrt{R^2+z^2}}.$$

Hence, the magnetic field at P can be written conceptually as

$$\vec{B}_P = \left[\int dB_P\ \cos\alpha,\ \text{(pointed up)}\right]$$

Substituting for dB_P, $\cos\alpha$, and $ds = Rd\phi$, we find that integration variable is the polar angle ϕ. Since the integrand does not have dependence on ϕ, the integration over ϕ from 0 to 2π give the following result for the magnetic field at point P.

$$\vec{B}_P = \left[\frac{\mu_0}{2}\frac{IR^2}{(R^2+z^2)^{3/2}},\ \text{(pointed up in the figure)}\right] \tag{8.11}$$

Further Remarks:

You should make a note of the limitations of what we have done here. We have found the magnetic field at points on the symmetry axis. If you want the magnetic field at other points in space, you will have to set up the appropriate integrals and carry out the difficult integrals in a brute force way since you would not have the benefits of the symmetry for other points in space.

8.1.3 Magnetic Dipole Field and Magnetic Dipole Moment

The magnetic field of a current in a circular loop given in Eq. 8.11 has interesting limiting behaviors. The field is constant near the center but drops off as 1/ distance-cubed ($1/r^3$) similar to the electric field of an electric dipole. In the last chapter we have defined magnetic dipole moment from a consideration of the torque on a current loop as current times area of the loop. We will show here that far away from a loop of wire, the magnetic field of the current in a loop varies as 1/distance-cubed similar to the way electric field of a electric dipole varies with distance. We will see that the magnetic field of a current loop in this limit can be written in terms of the magnetic dipole moment of the current loop.

The formula of magnetic field of a loop of current far away from the loop can be obtained by calculating the leading behavior as $z >> R$ in the exact result given in Eq. 8.11.

$$\lim_{z>>R}\vec{B}_P = \hat{u}_z\frac{\mu_0 I_R^2}{2}\lim_{z>>R} \sim \hat{u}_z\frac{\mu_0 I_R^2}{2}\frac{1}{z^3}$$

The magnetic field of a circular loop of current goes as $1/z^3$ for large z. The magnetic field varying as $1/z^3$ is called a magnetic dipole field, which is denoted by $\vec{B}_{\text{dipole}}$. Introducing factors of π and rewriting the dipole field in the following way.

$$\vec{B}_{\text{dipole}} = \hat{u}_z \frac{\mu_0}{2\pi} \frac{I\pi R^2}{z^3} \tag{8.12}$$

which can be rewritten in terms of the area A of the loop, which is $A = \pi R^2$.

$$\vec{B}_{\text{dipole}} = \hat{u}_z \frac{\mu_0}{2\pi} \frac{IA}{z^3} \quad \text{(on the } z\text{-axis)} \tag{8.13}$$

The product of current and the area of the loop IA is called magnetic dipole moment of the current loop, denoted by μ

$$\text{Magnetic dipole moment of a current loop:} \quad \mu = IA, \tag{8.14}$$

and the field given by Eq. 8.15 is called magnetic dipole field.

$$\vec{B}_{\text{dipole}} = \hat{u}_z \frac{\mu_0}{2\pi} \frac{\mu}{z^3} \quad \text{(on the } z\text{-axis)} \tag{8.15}$$

We interpret this formula to give us two equivalent views of an infinitesimal current loop: *think of the loop as a loop of current or alternately think of the current loop as a magnetic dipole moment of the current loop. Conversely, we can represent a magnetic dipole moment, such as a magnetic dipole moment of a tiny magnet by a current loop where the product of the area of the loop and the current is equal to the magnetic dipole moment.*

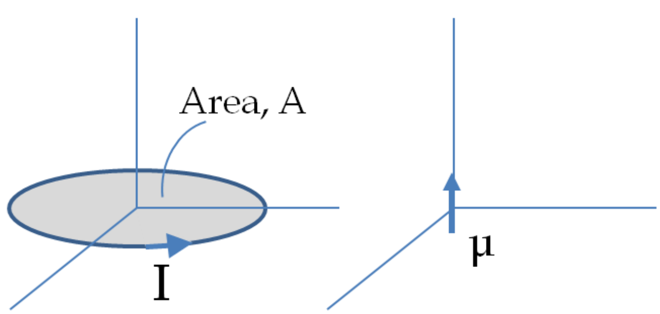

Figure 8.8: Two equivalent pictures of an infinitesimal current loop: either think of the current loop as current loop or a magnetic dipole of magnetic dipole moment $\mu = IA$ in the direction as displayed here.

- **An infinitesimal current loop as a magnetic dipole.**
- **A magnetic dipole an infinitesimal current loop.**

Since current has direction, magnetic dipole moment has direction as well. The vector form of magnetic dipole moment is defined by introducing a quantity called the area vector. The magnitude of the area vector of a planar surface is defined to be equal to the area of the surface, and the direction is along the normal given by the following right-hand rule. Sweep around the direction of the current with four finger of your right palm, then the direction in which your thumb points is the direction of the area vector. For your reference, area vectors are illustrated in Fig. 8.9.

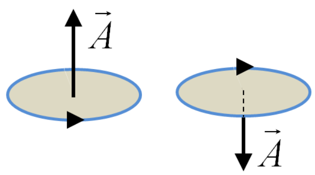

Figure 8.9: Direction of Area vectors using right-hand rule on the current loop.

$$\vec{A} = \begin{cases} \text{Magnitude} = \text{Area} \\ \text{Direction} = \text{Normal; Right-Hand Rule.} \end{cases}$$

With this definition of the area vector, the magnetic dipole moment vector is given as

$$\boxed{\vec{\mu} = I\vec{A}.} \tag{8.16}$$

8.1.4 Superposition Principle of Magnetic Field

Magnetic field has a superposition principle similar to that of electric field. It is present in Biot-Savart law where we were basically adding magnetic fields from segments of the wire carrying a steady current. Now, if we have a system that contains more than one current carrying subsystem, then the net magnetic field will be a vector sum of the magnetic fields of all subsystems. The magnetic field of each circuit can be separately worked out by Biot-Savart law before adding them together to obtain the net magnetic field as illustrated by example below.

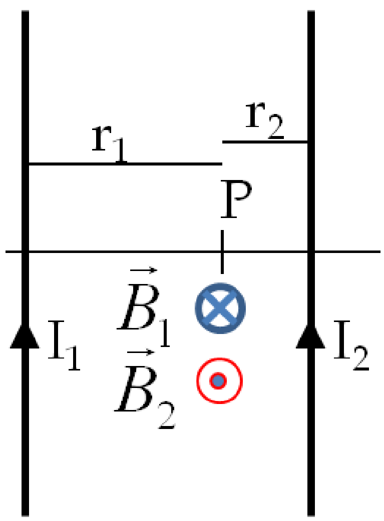

Figure 8.10: Magnetic field at P is superposition of magnetic fields $\vec{B}_1$ and $\vec{B}_2$ of currents $I_1$1 and I_2.

Example 8.1.6. Magnetic Field of Two Parallel Wires.

Solution. The net magnetic field at point P will be a vector sum of the magnetic fields by currents in each wire. Let z axis be out-of page, then, we have the following analytic forms of the magnetic fields $\vec{B}_1$ and $\vec{B}_2$ by currents I_1 and I_2 respectively. Refer to Fig. 8.10.

$$\vec{B}_1 = -\frac{\mu_0}{2\pi}\frac{I_1}{r_1}\hat{u}_z$$

$$\vec{B}_2 = \frac{\mu_0}{2\pi}\frac{I_2}{r_2}\hat{u}_z$$

We have used right-hand rule to get the direction of each magnetic field at P. Note the direction of magnetic field by the two currents will depend upon the location of P. For point P in the figure magnetic field from the two wires are in the opposite directions, so they will subtract. To the left of left wire and to the right of the right wire, the magnetic fields will be in the same direction and therefore they will add at those points.

Here we will work with the point P shown in Fig. 8.10. Therefore magnetic field at P is.

$$\vec{B}_P = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{2\pi}\left(\frac{I_2}{r_2} - \frac{I_1}{r_1}\right)\hat{u}_z$$

The magnetic field will be either out of page or into page depending upon the sign that multiplies the unit vector $\hat{u}_z$ here, which depend on the relative values of the two terms in parenthesis. If $I_2/r_2 > I_1/r_1$, then the magnetic field is coming out of page, and if $I_2/r_2 < I_1/r_1$, then the magnetic field is into the page.

Example 8.1.7. Magnetic Field of Current in a Solenoid

Solution. A solenoid is a collection of tightly packed loops such that steady current in each loop travels in the same direction as shown in Fig. 8.11. You need shielded wires so that current travels in loops and does not jump from one loop to the other. It turns out that magnetic field inside of a long solenoid is uniform as we will see below. Because of uniform magnetic field, solenoids are very useful in physics.

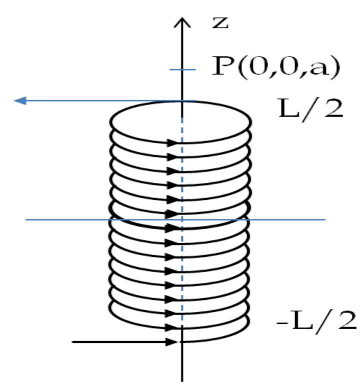

Figure 8.11: A solenoid consists of shielded wires wrapped repetitively either by itself or over a support structure. To find the magnetic field at space point P we superpose the magnetic fields of each loop there as shown in the text.

To find the magnetic field at points on the axis of the solenoid we can make use of the magnetic field at the axis of a current loop we have worked out above. We will demonstrate how we can add up the contributions of all loops to find the field at a particular point on the axis. For calculation purposes, we will choose the z-axis to coincide with the axis of the solenoid as shown in Fig 8.12. Each loop of current will contribute magnetic field at the field point P. Let the field point be located at coordinate $(0, 0, a)$ on the axis. Then, by Eq. 8.11, a loop located at z on the axis will contribute the following magnetic field at point P.

$$\vec{B}_{\text{one loop}} = \hat{u}_z \frac{\mu_0 I R^2}{2} \frac{1}{\left[R^2 + (z-a)^2\right]^{3/2}}$$

Let there be n loops per unit length in the solenoid. Then, number of loops dN in a thickness dz between z and $z + dz$ will be

$$dN = n dz$$

The contributions of these loops to magnetic field at point P will be magnetic field of one loop multiplied by dN. Therefore, we can write

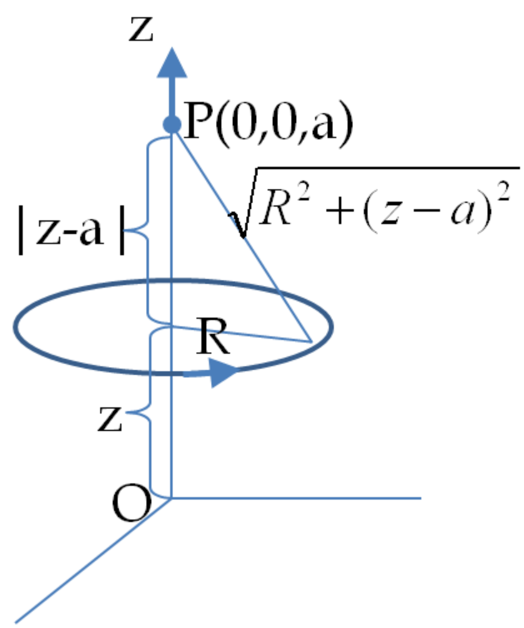

Figure 8.12: The magnetic field of one loop located a distance z from the xy-plane.

the magnetic field $d\vec{B}$ at point P from all loops between z and $z+dz$ to be

$$d\vec{B}_P = \vec{B}_{\text{one loop}} dN = \hat{u}_z \frac{\mu_0 I R^2}{2} \frac{1}{\left[R^2 + (z-a)^2\right]^{3/2}} n dz$$

We need to integrate this over the length of the solenoid to obtain the net field of the solenoid. As shown in Fig. 8.11, half of the solenoid is above $z = 0$ and the other half below. Then, we obtain the following for the net field at P.

$$\vec{B}_P = \hat{u}_z \frac{\mu_0 n I R^2}{2} \int_{-L/2}^{L/2} \frac{dz}{\left[R^2 + (z-a)^2\right]^{3/2}}$$

Normally, we are interested in magnetic field inside a very long, "infinitely long" or "ideal" solenoid. In that case, we can take $L \to \infty$ limit to obtain the following simple result.

$$\vec{B}_P = \hat{u}_z \mu_0 n I \quad (L >> a) \tag{8.17}$$

Although, here, we have derived the formula for magnetic field at a point on the axis, we will see below that the same formula gives magnetic field for all points inside an ideal solenoid.