12.4 POWER IN AC CIRCUITS

Power of the AC source in the series RLC circuit

In the last chapter we had analyzed the power into the resistor of the RLC circuit. Here, we will look at the power of the EMF source itself.

The instantaneous power P of any element is obtained by the product of the current through the element and the voltage across the element. Thereofore, the power of the driving EMF would be equal to the product of its voltage $V(t)$ and the current $I(t)$ through it.

$$P(t) = I(t)V(t) = I_0 V_0 \cos(\omega t)\cos(\omega t + \phi).$$

where I_0 and ϕ are given in terms of V_0, ω, R, C and L as above. The instantaneous power is time-dependent. The average power P_{ave} obtained from $P(t)$ by averaging the later over a complete cycle of time is often of more interest that the instantaneous power $P(t)$. The average power of the source is sometimes called the **apparent power**. The product of cosines in $P(t)$ can be rewritten as sum of cosines.

$$P(t) = \frac{I_0 V_0}{2}\left[\cos(2\omega t + \phi) + \cos(\phi)\right].$$

The average of $\cos(2\omega t + \phi)$ over a period will be zero. Therefore P_{ave} is

$$P_{\text{ave}} = \frac{I_0 V_0}{2}\,\cos(\phi), \tag{12.44}$$

where

$$\cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \tag{12.45}$$

is called the **power factor** of the AC circuit. In purely resistive circuits, i.e, in the absence of inductor and capacitor, the power factor is 1. The power factor is important in highly inductive circuits such as motors, compressors and transformers and in high frequency signals. When the circuit is driven at the resonance frequency $\omega = \omega_R$, with $\omega_R = 1/\sqrt{LC}$, the power factor will be equal to 1, the maximum value.

From the voltage $V(t)$ and the current $I(t)$ we can construct time-averaged quantities. The time averages of $V(t)$ and $I(t)$ are zero. However, if we do the time averaging after squaring them then we get a non-zero value, called the root-mean squared V_{rms} and I_{rms} as follows.

$$V_{\text{rms}} = \sqrt{\langle V(t)^2 \rangle} = \frac{V_0}{\sqrt{2}} \tag{12.46}$$

$$I_{\text{rms}} = \sqrt{\langle I(t)^2 \rangle} = \frac{I_0}{\sqrt{2}} \tag{12.47}$$

where $\langle \cdots \rangle$ stands for time averaging operation. The average power of the source voltage is often expressed in terms of the V_{rms} and I_{rms} and the power factor as follows.

$$P_{\text{ave}} = V_{\text{rms}}\, I_{\text{rms}}\, \cos(\phi). \tag{12.48}$$

Meaning of the Power Factor

We find that the power of the source disappears when the resistance in the circuit is zero. Thus a purely reactive circuit does not use any energy. But you need a current for a reactive device such as a motor to function. Therefore, although no energy is used by a purely reactive device, the power supply company must supply a current to it. The product $(V_{\text{rms}}.I_{\text{rms}})$ is sometimes called the **"real power"** for which the power supplier must charge the power user, and not just the **"apparent power"**. The power factor is the ratio of the power of the source, i.e. the "apparent power", to the actual power used in the resistor, i.e., the "real power".

$$\text{Power Factor} = \frac{\text{Apparent Power}}{\text{Real Power}}. \tag{12.49}$$

The power factor therefore gives a measure of the reactive part of the power usage. The power factor should be maximized so as to make the delivery of power more efficient. The power factor of a circuit with large inductors can be reduced by adding a capacitor to the circuit.

Power dissipated in the RLC circuit and energy balance

We will calculate here the energy used by the resistor, inductor and capacitor in each cycle and show that only the resistor uses energy, and the inductor and the capacitor exchanges energy back and forth with the source.

Energy used by the resistor in a period T:

$$\Delta E_R(\text{in one cycle}) = \int_0^T P_{\text{in R}} dt = \int_0^T (IV)_{\text{in R}} dt = \frac{1}{2} I_0^2 R T. \tag{12.50}$$

Energy used by the inductor in a period T:

$$\Delta E_L(\text{in one cycle}) = \int_0^T P_{\text{in L}} dt = \int_0^T (I\ V)_{\text{in L}} dt$$
$$= -LI_0^2\omega \int_0^T \sin(\omega t+\phi)\cos(\omega t+\phi)dt = 0. \tag{12.51}$$

Energy used by the capacitor in a period T

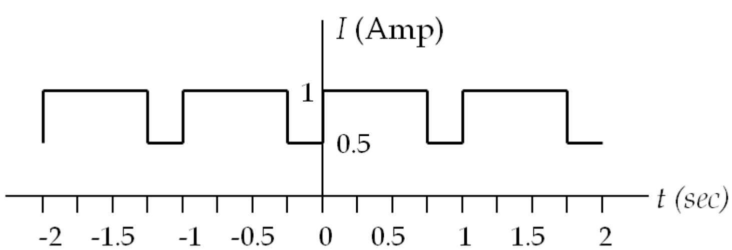
$$\Delta E_C(\text{in one cycle}) = \int_0^T V_C dq = \int_0^T V_C \frac{dq}{dt} dt$$
$$= \int_0^T V_C\ I\ dt = 0. \tag{12.52}$$

Hence, the energy in an AC circuit is used by the resistor only! From the energy conservation, the energy produced must equal the energy dissipated in the resistor in each cycle.

$$\text{Energy Used} = \text{Energy Delivered}$$
$$\Longrightarrow \quad \frac{1}{2} I_0^2 RT = \frac{1}{2} V_0 I_0 \cos(\phi) T \quad \Longrightarrow \quad V_0 = I_0 |Z|, \tag{12.53}$$

which is precisely the voltage-current relation in the circuit.

Example 12.4.1. Average and root-mean square values. Find the average and root-mean square value of the following current that is periodic time.



Solution. The period, T= 1 sec.

$$I_{\text{ave}} = \frac{1}{T}\int_0^T I(t)dt$$
$$= \frac{1}{1\text{ sec}}(0.75\text{ sec} \times 1\text{ A} + 0.25\text{ sec} \times 0.75\text{ A}) = 0.94\text{ A}.$$

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T I(t)^2 dt}$$
$$= \sqrt{\frac{1}{1\text{ sec}}[0.75\text{ sec} \times (1\text{ A})^2 + 0.25\text{ sec} \times (0.75\text{ A})^2]} = 0.94\text{ A}.$$

Example 12.4.2. Power Factor of a Large Motor. An AC circuit has a large motor, modeled as a resistance of 100 Ω and inductance of 8 H in series connected to a 120 V (rms) AC line that fluctuates at a 60 Hz frequency. Find the amplitude of the impedance of the circuit, the peak voltage, RMS current, the power factor, real power and apparent power.

Solution. This is simply a numerical example for an RL circuit and we can use the formulas already derived for that circuit. The amplitude of the impedance of the circuit is

$$\begin{aligned} |Z| &= \sqrt{R^2 + (\omega L)^2} \\ &= \sqrt{(100\ \Omega)^2 + (2\pi \times 60\ \text{Hz} \times 8\ \text{H})^2} = 318\ \Omega. \end{aligned}$$

Peak Voltage, $V_{\text{peak}} = \sqrt{2} \times V_{\text{rms}} = \sqrt{2} \times 120\ \text{V} = 169.7\ \text{V}$.

RMS current, $I_{\text{rms}} = \frac{V_{\text{rms}}}{|Z|} = \frac{120\ \text{rms}}{318\ \Omega} = 0.377\ \text{A}$.

The power factor $= \cos(\phi) = \frac{R}{Z} = \frac{100\ \Omega}{318\ \Omega} = 0.31$.

Real power $= I_{\text{rms}}.V_{\text{rms}} = 0.377\ \text{A} \times 120\ \text{V} = 45.2\ \text{W}$.

Apparent Power $= I_{\text{rms}} V_{\text{rms}} \cos(\phi) = 45.2\ \text{W} \times 0.31 = 14\ W$.

Although the unit Volt.Amp(V.A) is equal to unit Watt (W), one expresses the apparent power in VA and the real power in W. Note that the apparent power is what is happening as far as energy conservation is concerned, and the real power is what a utility company bases its charges on.