7.3 MAGNETIC FORCE

7.3.1 Magnetic Force on a Moving Electric Charge

In the last few chapters we have studied electric force which is the force on an electrically charged object independent of the velocity of the object. We found that the electric force on an electric charge was best described by introducing the concept of electric field at each space point. We found that Coulomb's law says that the electric force $\vec{F}_{\text{elec}}$ on a charge Q placed at a point where the electric field is $\vec{E}_P$ is given by

$$\text{Electric force on } Q: \quad \vec{F}_{\text{elec}} = Q\vec{E}_P.$$

When you shoot a charged particle between the poles of a magnet you find that a charged particle also experiences another force that depends on the velocity of the charge. We call this force the magnetic force. The magnetic force on a particle can be used to quantify the magnetic field in a region and find a map of the magnetic field just as the Coulomb force was used to map the electric field.

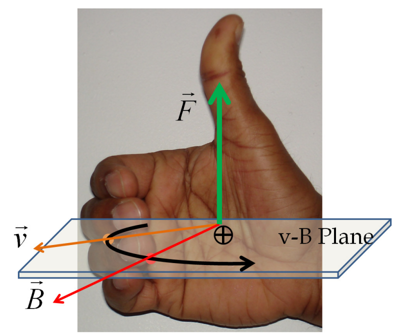

Figure 7.12: Right-hand rule for magnetic force on a positive charge. Place the velocity and magnetic field vectors in one plane and sweep from velocity towards magnetic field vector. The magnetic force is directed towards the thumb and perpendicular to the plane of velocity and magnetic field. The force on a negatively charged particle would be in the opposite direction.

Direction of Magnetic Force

The direction of the magnetic force is not as simple as the direction of the electric force. It turns out that the magnetic force is always perpendicular to the velocity of the particle. Furthermore, at an particular point in space the magnetic force is perpendicular to a fixed direction in the space regardless of the direction of motion of the particle at that point as displayed in Figs. 7.13 and 7.12.

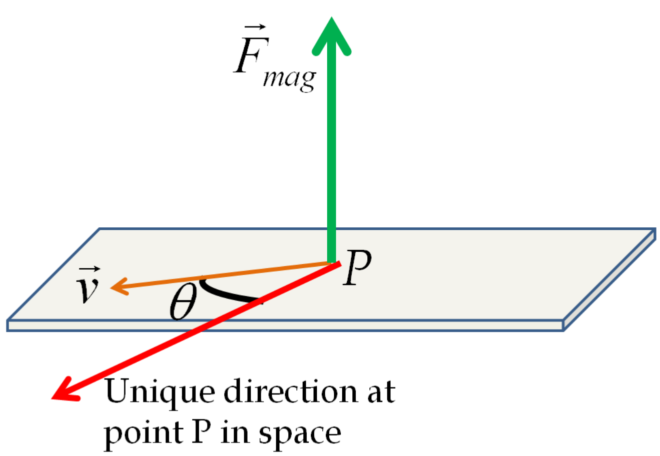

Figure 7.13: The direction of the magnetic force on a charge at point P is perpendicular to the direction of the velocity of the charge and a unique direction at point Pin space.

Magnitude of Magnetic Force

The magnitude of the magnetic force is proportional to the component of the velocity vector perpendicular to this unique direction in space and the magnitude of the electric charge. Let θ be the angle between the velocity vector $\vec{v}$ and this unique direction in space at point P. Then, experimentally, the magnitude of the magnetic force is give by

$$F_m \propto |Q|\ v\ sin\theta \tag{7.1}$$

To describe this force on the charge in terms of an equality, we can define a magnetic field vector at point P, to be denoted by $\vec{B}$ and making use of the vector product rule.

$$\vec{F}_m = Q\ \vec{v} \times \vec{B}. \tag{7.2}$$

The magnitude of the vector $\vec{B}$ corresponds to the proportionality constant in Eq. 7.1 and the direction of $\vec{B}$ when used in Eq. 7.2 gives the correct direction of the magnetic force on a positively charged particle. We often use the right-hand rule of vector product to remember the direction relations of the velocity, magnetic field and magnetic force for a positive charge as illustrated in Fig. 7.12

Clearly, magnetic force on a moving charge has complicated directional characteristics. For instance, if you shoot a positive particle from the bottom as in Fig. 7.14, the an inward force on the particle will curve the trajectory to bring the particle inside. A negatively charged particle, on the other hand, will be pushed out on a curved trajectory.

Operational Definition and Unit of Magnetic Field

Based on the magnetic force law we can give an operational definition of the magnetic field $\vec{B}$. The magnetic field at a point in space will be equal to one unit if the force on a particle of charge 1 C moving at speed 1 m/s perpendicularly to the unique direction described above experiences a force of magnitude 1 N. This unit of magnetic field is called Tesla, which is denoted by letter T. Do not confuse this letter with temperature or time. We can write the Tesla in terms of fundamental units as kg/A.s.

$$[B] = \frac{[F_m]}{[Q][v]} = \frac{\text{kg.m/s}^2}{\text{C. m/s}} = \frac{\text{kg}}{\text{A. s}}.$$

You can express the Tesla in various other ways, such as V.s/m^2. It turns out that 1 Tesla is too large a unit for geophysical measurements, and another unit called Gauss (G) is in common use, where 10,000 Gauss = 1 Tesla.

$$1\ \text{T} = 10^4\ \text{G}.$$

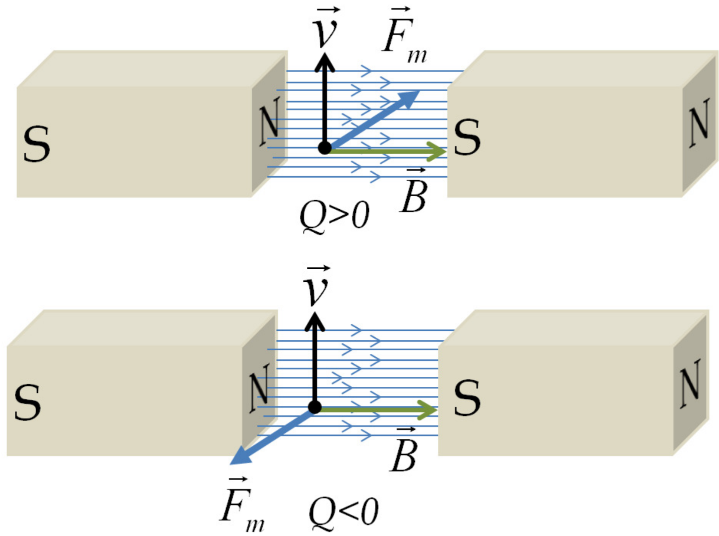

Figure 7.14: Magnetic force on a moving charged particle has the direction perpendicular to both velocity $\vec{v}$ and magnetic field $\vec{B}$. The force is has opposite directions for positive and negative charges. The magnitude of the force on the charge is equal to $|Q|vB\sin\theta$ where θ is the angle between $\vec{v}$ and $\vec{B}$.

Table 7.1: Typical Magnetic Fields (Various sources)

Location	Magnetic Field (Tesla)
Neutron star surface	10^8
Superconducting magnet	5
Small bar magnet	1
Earth's Surface	0.5×10^{-4}
Interstellar space	10^{-10}

The earth's magnetic field is approximately 0.5 Gauss while magnetic field of a refrigerator magnet is around 1 Tesla. Magnetic field used in an Magnetic Resonance Imaging (MRI) machine is around 2 T. Some typical values of magnetic fields are given in Table 7.1 so that you can get a feel for the order of magnitude in various situations. As you can see magnetic field in nature varies over 18 orders of magnitude, from less than 10^{-10} T in the interstellar space to more than 10^8 T in neutron stars.

Work by magnetic force on a charge

Since the magnetic force on a charged particle is perpendicular to its velocity the magnetic force can change only the direction of the velocity leaving the speed unchanged. Therefore, kinetic energy of a particle cannot be changed by magnetic force. We say that magnetic force does not do any work on a moving charge.

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{r} = 0, \quad \text{since, } \vec{F}_{mag} \perp \vec{v} \text{ or } d\vec{r}/dt.$$

Magnetic fields do not do work on moving charges.

7.3.2 Lorentz Force

A charged particle can experience two forces, electric and magnetic forces, that are proportional to the amount of charge. The net force on an electric charge is called Lorentz force, or electromagnetic force, which is just a vector sum of the electric and magnetic forces on the charge.

Thus, net electromagnetic force $\vec{F}_{em}$ on a charged particle of charge Q and velocity $\vec{v}$ at a particular location, where electric field is $\vec{E}$ and magnetic field $\vec{B}$, is given by

$$\vec{F}_{\text{em}} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{7.3}$$

Note that the sign of charge Q is important here for determining the direction of the force. The motion of a charged particle in an environment, where both electric and magnetic fields exist, is determined as usual by setting up Newton's second law of motion. In order to study the motion of an electric charge Q in a given electromagnetic environment, you can use Lorentz force in Newton's second law of motion if its speed is not relativistic, i.e., $v << c$, where c is the speed of light in vacuum.

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{em}} + \vec{F}_{\text{other}} \tag{7.4}$$

where $\vec{F}_{other}$ denotes the net non-electromagnetic force on the particle and $\vec{p}$ is the momentum of the particle.

7.3.3 Motion of Charged Particles in a Uniform Magnetic Field

In this section we will study motion of a charged particle in an environment where only magnetic field is present. This simplifies equation of motion given above, Eq. 7.4, to only magnetic force on the right side.

$$\frac{d\vec{p}}{dt} = Q\vec{v} \times \vec{B}. \tag{7.5}$$

To further simplify the problem, we consider the motion of a charged particles in a uniform magnetic field. This case has important applications in ion and particle identifications.

For an analytic treatment of the problem, we choose Cartesian axes so that the z-axis id in the direction of the uniform magnetic field, whose magnitude will be indicated by attaching a subscript 0 to the symbol for magnetic field, B_0.

$$\vec{B} = B_0 \hat{u}_z. \tag{7.6}$$

Let $(x(t), y(t), z(t))$ be the coordinates of the particle at time t. The velocity $\vec{v}$ has the components $(v_x(t), v_y(t), v_z(t))$. Performing the cross product between the velocity vector and the magnetic field vector we find that the magnetic force has the following components.

$$F_x = QB_0 v_y \tag{7.7}$$
$$F_y = -QB_0 v_x \tag{7.8}$$
$$F_z = 0 \tag{7.9}$$

Then, equation of motion, viz. Eq. 7.5, can be separated into three equations corresponding to the axes.

$$m\frac{dv_x}{dt} = QB_0 v_y \tag{7.10}$$
$$m\frac{dv_y}{dt} = -QB_0 v_x \tag{7.11}$$
$$m\frac{dv_z}{dt} = 0 \tag{7.12}$$

It helps to collect together constants into one symbol.

$$\Omega^2 = \frac{QB_0}{m}, \tag{7.13}$$

which has the units of $1/[\text{time}]^2$. The equations of motion in Eq. 7.10 - Eq.7.12 can be simplified by taking time derivatives of the x- and y-equations and combining them.

$$\frac{d^2 v_x}{dt^2} = -\Omega^2 v_x \tag{7.14}$$
$$\frac{d^2 v_y}{dt^2} = -\Omega^2 v_y \tag{7.15}$$
$$\frac{dv_z}{dt} = 0 \tag{7.16}$$

The x- and y-equations are similar to the equation of displacement of a simple harmonic oscillator we have worked out before. Note that the velocity components must also satisfy the original equations, Eqs. 7.10-7.12. The general solution of these equations can be written as

$$v_x(t) = A_1 \cos(\Omega t) + B_1 \sin(\Omega t) \tag{7.17}$$
$$v_y(t) = B_1 \cos(\Omega t) - A_1 \sin(\Omega t) \tag{7.18}$$
$$v_z(t) = v_{0z} \quad \text{(constant)} \tag{7.19}$$

These equations take definite form for different initial conditions which are usually specified by the initial position and velocity. The trajectory of the particle can be deduced from the final solutions that incorporate the initial conditions also. We will illustrate two initial conditions of common interest.

Initial Condition # 1

We start with an initial condition such that velocity has zero component in the z-direction. Consider the following set of initial conditions.

$$x(0) = R,\ v_x(0) = 0; \tag{7.20}$$
$$y(0) = 0,\ v_y(0) = v; \tag{7.21}$$
$$z(0) = 0,\ v_z(0) = 0; \tag{7.22}$$

Using the initial velocity components in Eqs. 7.17 to 7.19 we find that

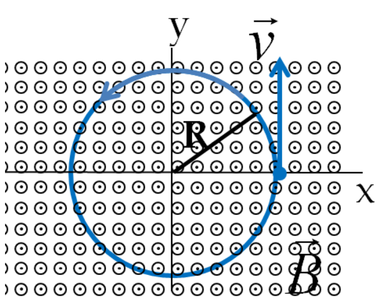

Figure 7.15: Circular motion of a positive charge in a uniform magnetic field $\vec{B}$ with initial condition $x(0) = R$, $y(0) = 0$, $v_x(0) = 0$, and $v_y(0) = v$.

$$v_x(t) = v \sin(\Omega t) \tag{7.23}$$
$$v_y(t) = v \cos(\Omega t) \tag{7.24}$$
$$v_z(t) = 0 \quad \text{(constant)} \tag{7.25}$$

These can be integrated and initial position components used to yield

$$x(t) = R \cos(\Omega t) \tag{7.26}$$
$$y(t) = R \sin(\Omega t) \tag{7.27}$$
$$z(t) = 0 \quad \text{(constant)} \tag{7.28}$$

where v/Ω was replaced with R which is equal to v/Ω since the trajectory is a circular motion. The equation of the trajectory of the particle can be obtained by eliminating t from these equations.

$$x^2 + y^2 = R. \tag{7.29}$$

Eqs. 7.37-7.39 show that the particle moves in the xy-plane in a circle of radius R with angular frequency given by Ω, which is also called angular cyclotron frequency. The particle will undergo uniform circular motion with frequency f given by

$$\text{Cyclotron frequency,} \quad f = \frac{\Omega}{2\pi} = \frac{1}{2\pi}\frac{QB_0}{m}. \tag{7.30}$$

The period of the circular motion of the particle T, which is inverse of frequency f, can also be related to the speed of the particle v and the radius of the circle since the motion is a uniform circular motion. The particle covers a distance $2\pi R$ with speed v. Therefore, period is

$$T = \frac{2\pi R}{v} \tag{7.31}$$

Relating this to cyclotron frequency, Eq. 7.30, by $T = 1/f$, we have

$$\frac{v}{R} = \frac{QB_0}{m}, \tag{7.32}$$

which can be solved for charge-to-mass ratio of the particle.

$$\frac{Q}{m} = \frac{v}{RB_0}. \tag{7.33}$$

Initial Condition # 2

What happens when the initial velocity has a non-zero component of velocity in the z-direction? Consider the following set of initial conditions.

$$x(0) = R, \ v_x(0) = 0; \tag{7.34}$$
$$y(0) = 0, \ v_y(0) = v; \tag{7.35}$$
$$z(0) = 0, \ v_z(0) = v_{0z}; \tag{7.36}$$

Satisfying these conditions in the solution given in Eqs. 7.17 to 7.19 leads to

$$x(t) = R\cos(\Omega t) \tag{7.37}$$
$$y(t) = R\sin(\Omega t) \tag{7.38}$$
$$z(t) = v_{0z}t \ \ (\text{constant}) \tag{7.39}$$

The particle moves in a helical path perpendicular to the z-axis with speed $|v_{0z}|$ along the axis and angular frequency Ω in a circular projection in the xy-plane (Fig. 7.16).

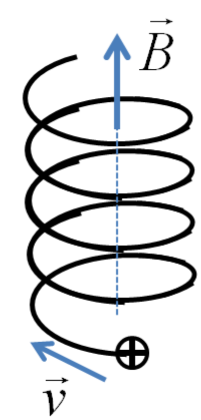

Figure 7.16: Helical trajectory for a positively charged particle that has non-zero velocity in the direction of the magnetic field.

Example 7.3.1. Identifying Elementary Particles. A particle of charge $+1e$ enters a region of uniform magnetic field of magnitude $2\ T$ with a velocity of $9.6 \times 10^5\ m/s$ perpendicular to the magnetic field. It is observed that the particle makes an arc of radius 5 mm before coming out of the region. Find the mass of the particle.

Solution. The problem states that the particle makes an arc, which is part of a circle. Therefore, we can claim that the particle describes a circular path when in the region of constant magnetic field and use the formula for Q/m obtained above to write mass in terms of the given quantities.

$$\frac{Q}{m} = \frac{v}{RB_0} \quad \Longrightarrow \quad m = \frac{QRB_0}{v}$$

Now, we put numbers in to obtain

$$m = \frac{1.6 \times 10^{-19}\text{C} \times 0.005\ \text{m} \times 2\ \text{T}}{9.6 \times 10^{5}\text{m/s}} = 1.67 \times 10^{-27}\ \text{kg}.$$

The particle with charge $+1\ e$ and mass 1.67×10^{-27} kg is proton.

Example 7.3.2. The Velocity Selector. Consider a region of space where electric field $\vec{E}$ and magnetic field $\vec{B}$ are perpendicular to each other. The perpendicular electric and magnetic fields are called crossed fields. A charged particle with a particular velocity $\vec{v}$ entering this region will exit with the same velocity undeviated, while particles of other velocities will be deviated away from the original path. The arrangement leads to a mechanism of selecting particles of particular velocities, and hence the name velocity selector. In the arrangement of electric field and magnetic field shown, find the velocity selected.

Solution. Since the velocity of the particle does not change, the electric and magnetic forces are equal in magnitude and opposite in direction. This gives the following condition.

$$QvB \sin 90^\circ = QE \quad \text{(ignoring gravity)}$$

Solving this equation for speed v we find that the speed for which no deviation of the path of the particle occurs is $v = E/B$, which is independent of the charge of the particle.

$$\boxed{\text{Speed selected:} \quad v = \frac{E}{B}.}$$

7.3.4 The Hall Effect

Consider a metal plate placed in a magnetic field as shown in Fig. 7.18. When no current is passing in the metal plate there is no voltage difference between the sides of the plate labeled + and − in the figure. When a current passes through the metal plate, electrons

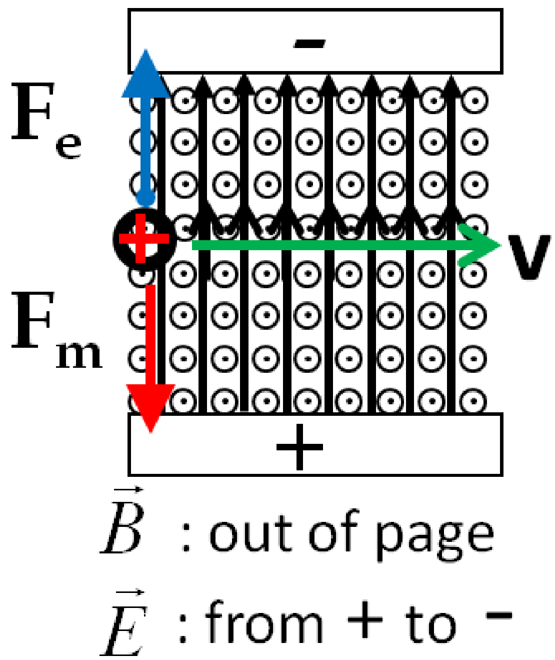

Figure 7.17: Crossed field for selecting particles with particular velocity. Only particle with speed $v = E/B$ pass through the region undeviated.

develop a drift velocity. Moving electrons are subject to magnetic force, which deflects them in the direction perpendicular to the current making one edge of the plate more negative than the other. This creates a potential difference $\Delta V_\perp$ across the plate perpendicular to the direction of the original current flow. This effect is called the Hall effect.

The potential difference $\Delta V_\perp$ across the plate corresponds to the electric field $\vec{E}_\perp$ in the direction perpendicular to the direction of drift velocity $\vec{v}_d$. Assuming uniform electric field in the direction perpendicular to the direction of drift velocity $\vec{v}_d$, we find the following relation between the potential difference and the magnitude of the electric field.

$$\Delta V_\perp = -\int_{\text{across plate}} \vec{E}_\perp \cdot d\vec{r} = E_\perp\ w. \tag{7.40}$$

At steady state, electric force perpendicular to the flow balances the magnetic force on each moving electron.

$$|-eE_\perp| = |-ev_d B|\,. \tag{7.41}$$

Replacing $E_\perp$ by $\Delta V_\perp / w$, and solving for B we obtain the following relation.

$$B = \frac{\Delta V_\perp}{v_d w}, \tag{7.42}$$

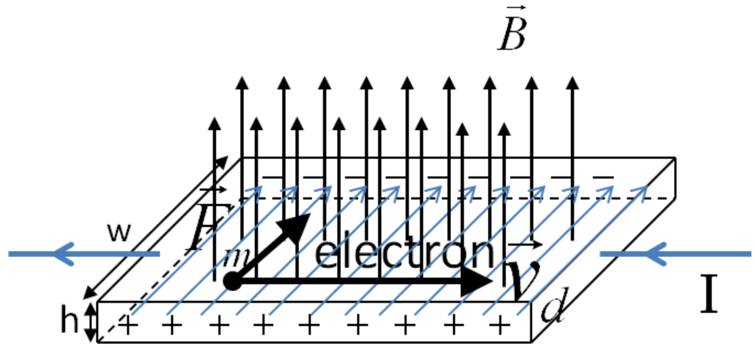

Figure 7.18: Hall Effect. The moving electrons are deflected to the far side of the plate by the magnetic force, making the near side positive and the far side negative. The accumulated charges produce electric field that applies an electric force on the drifting electrons in opposite direction to the magnetic force. The balances of the forces leads to a steady value of potential difference across the plate $\Delta V_\perp$, which is proportional to the magnitude of the magnetic field.

which we can write in terms of the current in the plate and the geometric dimensions of the plate as follows.

Let there be N mobile electrons per unit volume and w and h be the width and height of the cross-section area of the plate perpendicular to the current as shown in Fig. 7.18. Then, steady current I can be shown to be

$$|I| = |-eNv_d wh| \quad \Longrightarrow \quad v_d = \frac{|I|}{eNwh}. \tag{7.43}$$

Putting this v_d into Eq. 7.42 we obtain magnetic field in terms of two measurable quantities, current through the plate and potential difference across the plate.

$$B = \frac{\Delta V_\perp}{|I|}\left(eNwh\right). \tag{7.44}$$

This formula based on Hall effect is frequently used for measuring magnetic field by a device called a Hall probe. You first calibrate the instrument for a particular current, and then use formula given in Eq. 7.44 to determine an unknown magnitude B from the reading of $\Delta V_\perp$ by connecting a voltmeter across the plate. The direction of the magnetic field is obtained by the orientation of the area of the plate. The normal to the plate that gives the maximum reading of the Hall voltage is the direction of the magnetic field.

Note that if the mobile charges in the conductor are positively charged the potential direction will have just the opposite sign. For instance, in some semi-conductors the current is actually carried by positively charged entities called holes, where the Hall voltage has just opposite of what you would expect if current was due to movement of electrons. In this way, Hall effect also provides information about the nature of charge carriers in a material.