6.6 CIRCUIT ANALYSIS

6.6.1 Kirchhoff's Rules

When a circuit has only one power source, it can often be analyzed by constructing an equivalent circuit by successively employing series-parallel-series-etc to the circuit elements. If the circuit has more than one power source in different branches, or if employing series-parallel-series-etc method becomes too difficult or impossible to carry out, we appeal to the conservation principles of charge and energy in the circuit. These conservation principles lead to two rules called the Kirchhoff's rules or the Kirchhoff's laws.

The Node Rule: The sum of the incoming currents must equal the sum of the outgoing currents at any node in a circuit. Stated differently, the algebraic sum of currents at any node is zero if opposite signs are assigned to the incoming and outgoing currents.

$$\boxed{\sum_{\text{at any node}} I = 0.} \tag{6.65}$$

The node rule is also called **Kirchhoff's Current Law (KCL)**. Actually, this law must hold at any space point where charges are neither accumulating nor depleting.

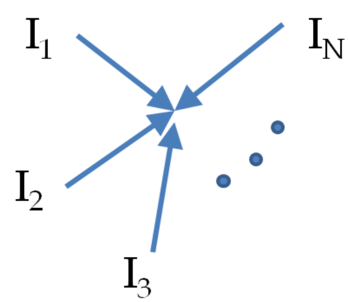


Figure 6.34: Kirchhoff's Node Rule: The algebraic sum of all currents at a node is equal to zero. $\sum_{i=1}^{N} I_i = 0$ with the convention that incoming currents and outgoing currents have opposite signs.

The Loop Rule: The sum of the potential drops and potential rises along any loop must add up to zero for the complete loop.

$$\boxed{\sum_{\text{Loop}} \Delta V = 0.} \tag{6.66}$$

The loop rule is also called **Kirchhoff's Voltage Law (KVL)**. This rule follows from the conservation of energy. Suppose you carry a test charge Q around a loop in space balancing electric force on the charge all the time so that the the kinetic energy of the charge does not change. In some parts of the loop the charge will gain potential energy and in other parts, it will lose potential energy, given by $Q\Delta V$ whenever potential changes. When you complete the loop, you return to the original point. Since the electric force is a conservative force, you expect that when you return to the original place, the net work will be zero regardless of the path which is the loop here. Therefore, you get

$$\Delta U = \sum_{Loop} (Q\Delta V) = 0 \implies \sum_{Loop} \Delta V = 0$$

Dividing out the charge of the test charge we obtain Kirchhoff's Loop Rule for the voltage. This law is modified for dynamic field as we will see when we study electrodynamics in a later chapter.

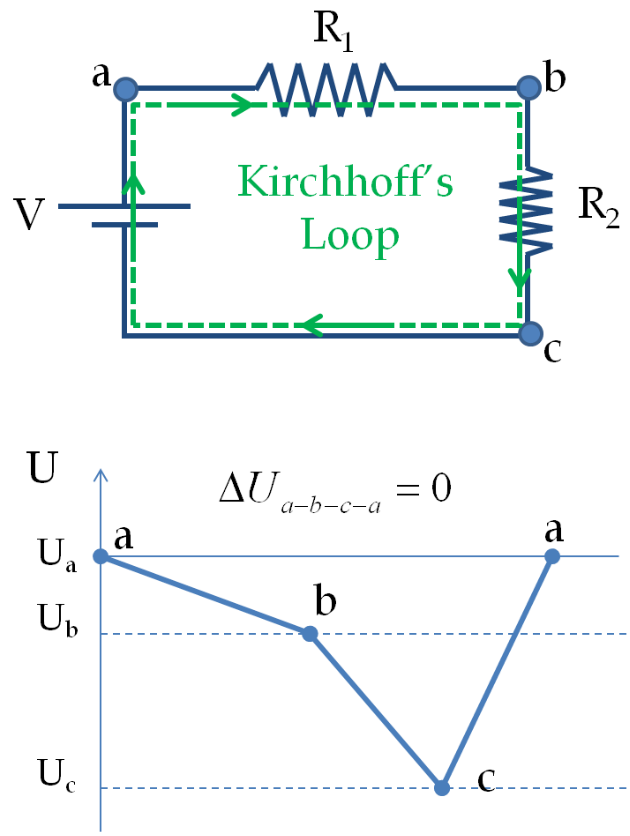


Figure 6.35: Kirchhoff's Loop Rule: A loop shown dashed (green) alongside the circuit is chosen to write the expression for Kirchhoff's loop rule, $\sum(\Delta V) = 0$. When you take a charge from some point around the Kirchhoff's loop, the net change in energy will be zero as shown for a positive charge here. The diagram for negatively charged electrons can be obtained from the figure given here by changing the sign of energy for each step.

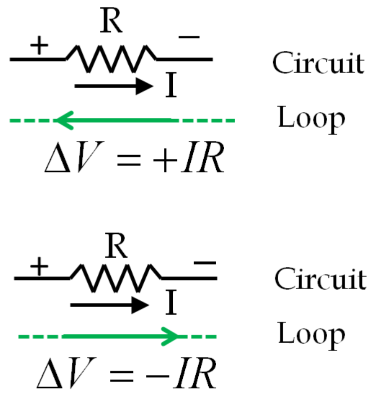
6.6.2 Sign Conventions

To apply the Loop Rule, it is often helpful to follow a sign convention for currents and voltages for consistency in algebraic manipulations.

For potentials across a voltage source, we note that potential drops from a higher value to a lower value when you go from the positive terminal of a battery to its negative terminal, but if you go in the other direction, the potential will increase across the battery.

Similarly, when we assume a direction of current in a resistor, we are actually assuming one side of the resistor as being at a higher potential than the other side since the current flows from the higher potential towards the lower potential.

In a circuit we may not know the direction of a current a-priori, but we must pick a direction in order to proceed in calculations. Therefore, we label the two ends of a resistor with + and − to indicate the sides presumed to be at higher and lower voltages respectively. Although such labeling is not necessary but it oftenhelps avoid silly mistakes.



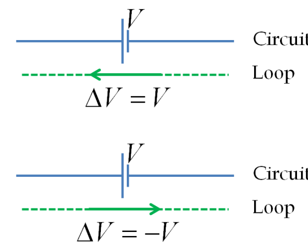


Figure 6.36: Sign Convention: In the Kirchhoff's Loop, when we go from the negative to the positive of a voltage source, the change in potential is positive and when we go from positive to the negative of the source the change in potential is negative.

6.6.3 Examples of Kirchhoff's Rules

In this section we will illustrate applications of Kirchhoff's rules to selective circuits. Kirchhoff's method can be used to solve any circuit problem. However, it is specially beneficial in cases where the series/parallel method does not work. A simple example of such circuits is a circuit with two voltage sources. A bridge circuit is another example that cannot be solved by series-parallel-series method since resistors are neither in series nor in parallel. We will illustrate how Kirchhoff's method can be applied to even in these problems.

Example 6.6.1. A Circuit With Two Voltage Sources. Given the values of the voltages V_1 and V_2 and resistances in the circuit, find the currents through the three resistors shown in Fig. 6.37.

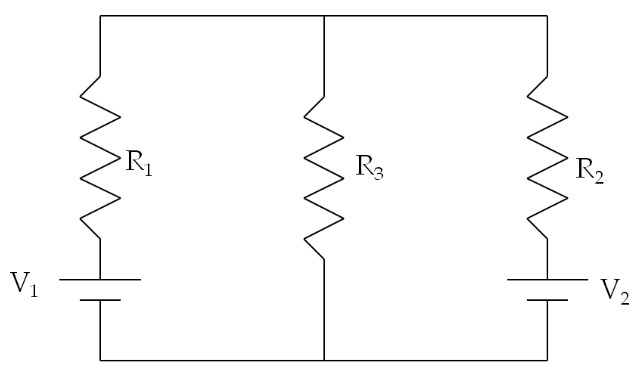


Figure 6.37: Example 6.6.1.

Solution. As before, we start with labeling the circuit with nodes of unique potential values, which are shown as a, b, c, and d in Fig. 6.38. If a current splits in branches, then we label the point of split with one node, such as points a and b in the figure.

Next, we assign currents in each branch, i.e. same current segment of the circuit. For instance, the same current I_1 goes through both

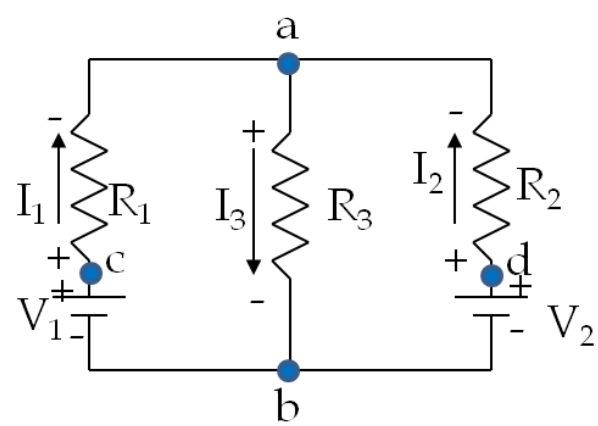


Figure 6.38: Example 6.6.1.

resistor R_1 and the source V_1. The directions of currents at this stage of solution are unknown and are picked arbitrarily. Of course, if the direction is obvious, it helps to pick the right direction from the start. If after solving the problem we find that a particular current has a negative numerical value, then it would mean that the assumed current direction was in the opposite direction to the actual current.

Finally, we use Kirchhoff's rules to write down as many independent equations as there are unknowns in the problem. The number of unknowns can be counted in terms of number of current values that are not known or number of node values that are not known. Here, there are three unknown currents and only one unknown node voltage. The reason we have only one unknown node voltage is that if we choose the node b to be the zero volt reference, then the nodes c and d have their potentials maintained at $V_c = V_1$ and $V_d = V_2$, the voltages of the two sources. This leaves only the voltage at the node a undetermined.

If we choose to work with currents, we will seek three independent equations, and if we choose to work with node voltages, then we will seek one equation in one unknown.

Working with currents

Since, we have three unknown currents we need to generate three independent equations. Two nodes a and b in the circuit bring these three currents together. However, we get only one equation from Kirchoff's node rule, because currents at the two nodes are related by being inverse of each other.

$$\text{NODE RULE: } I_1 + I_2 - I_3 = 0 \quad (\text{node } a \text{ or } b) \tag{6.67}$$

There are three unique Kirchhoff's loops we can draw in the circuit as shown in Fig. 6.39, but we need only two more equations.

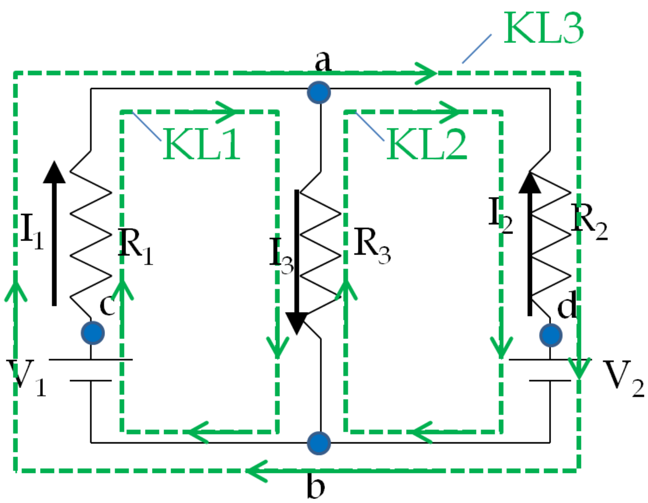


Figure 6.39: Example 6.6.1. Kirchhoff's loops KL1, KL2, and KL3 overlayed along the circuit. We use KL1 and KL3 for calculations in the text.

We choose two of these loops to obtain the following relations.

LOOP RULE:

$$\begin{aligned} &a\text{-}b\text{-}c\text{-}a: \qquad -I_3R_3 + V_1 - I_1R_1 = 0 \\ &a\text{-}d\text{-}c\text{-}a: \qquad +I_2R_2 - V_2 + V_1 - I_1R_1 = 0 \end{aligned}$$

Note the sign of different terms in the loop equations: when you go from $-$ to $+$ of an element you add, and when you go from $+$ to $-$ you subtract. Solving these equations simultaneously for I_1, I_2 and I_3 yields the required currents in the circuit.

$$\begin{aligned} &I_1 = [(R_2 + R_3)\, V_1 - R_3V_2] \,/\, [R_1R_2 + R_2R_3 + R_3R_1] \\ &I_2 = [-\,(R_3 + R_1)\, V_2 + R_3V_1] \,/\, [R_1R_2 + R_2R_3 + R_3R_1] \\ &I_3 == [R_2V_1 + R_1V_2] \,/\, [R_1R_2 + R_2R_3 + R_3R_1] \end{aligned}$$

Working with node voltages

To work with only node voltages is often the easiest way to implement Kirchhoff's rules. In this method, we use node voltages and resistances of the resistors to write all currents in terms of the node voltages, and then use the Kirchhoff's node rule to find the relation among the node voltages.

Here, we set node b to be zero, then we have the following node values already known.

$$V_b = 0 \quad \text{(Choice of reference zero)}$$
$$V_c - V_b = V_1 \implies V_c = V_1 \quad \text{(Definition of voltage of source)}$$
$$V_d - V_b = V_2 \implies V_d = V_2 \quad \text{(Definition of voltage of source)}$$

The currents I_1, I_2, and I_3 in terms of node voltages are:

$$I_1 = \frac{V_c - V_a}{R_1} = \frac{V_1 - V_a}{R_1}$$
$$I_2 = \frac{V_d - V_a}{R_2} = \frac{V_2 - V_a}{R_2}$$
$$I_3 = \frac{V_a - V_b}{R_3} = \frac{V_a}{R_3}$$

At node a or b the node rule gives

$$I_1 + I_2 - I_3 = 0. \tag{6.68}$$

Replacing the currents in this equation we get one equation in one unknown V_a:

$$\frac{V_1 - V_a}{R_1} + \frac{V_2 - V_a}{R_1} - \frac{V_a}{R_3} = 0, \tag{6.69}$$

which can be solved for V_a.

$$V_c = R_P \left(\frac{V_1}{R_1} + \frac{V_2}{R_1} \right), \tag{6.70}$$

where

$$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

The currents now can be obtained from their relations to node voltages and resistances. This node voltage procedure appears easier to me than the procedure for the physical currents.

There is also a more abstract method called the "mesh current" method. In the mesh current method you assign currents to non-overlapping loops insted of physicsl currents in each branches. Then, you generate equations for these mesh currents by using the Kirchoff's loop rule. This gives as many equations as there are unknowns. Although this method is attractive, we will not use this method. An interested student may look this method up in a textbook on circuits.

Example 6.6.2. Bridge Circuit - Numerical. In a bridge circuit, resistors and other elements in a circuit form a bridge-like structure. Bridge circuits are used for many purposes in engineering and physics. Find currents in each wire of the bridge circuit given in Fig.6.40.

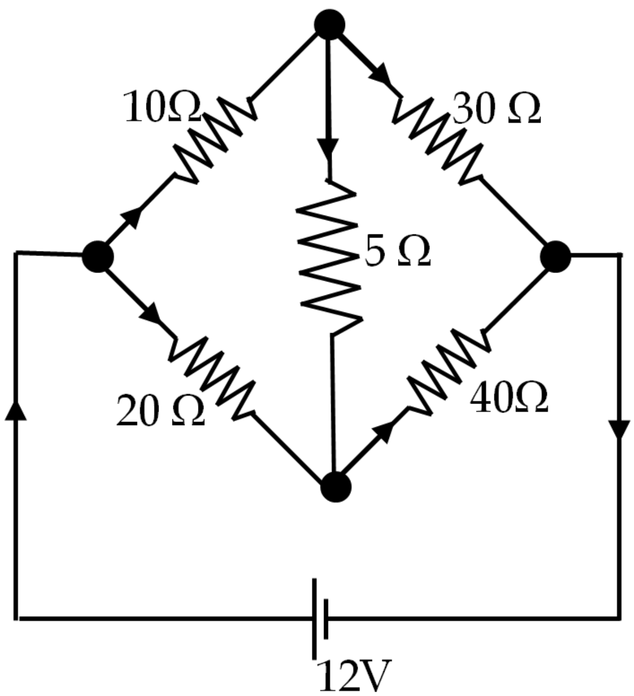


Figure 6.40: Example 6.6.2.

Solution. As in the last example, we start with labeling nodes of unique potentials. Figure 6.41 shows four nodes, labeled a, b, c, and d. Choosing the zero of potential at the negative of the voltage source makes $V_c = 0$ and $V_a = 12V$. This leaves two node voltages V_b and V_c to be determined to completely determine all voltages and currents in the circuit.

Next we pick the currents in each branch. We find that there are six branches. There will be one current in each branch, which we choose to be I, I_1, I_2, I_3, I_4, and I_5. All currents are unknown at this stage. We can determine six currents from the four node voltages and vice-versa. We will show the procedure for determining the six unknown currents by setting up six independent equations. The alternative method based on the node-voltages, which is illustrated in the last example, is left as an exercise for the student.

Now we are ready to write six equations for six unknown currents based on Kirchhoff's rules.

NODE RULE: We find that there are four modes various currents come together. We get only three independent relations from these four nodes, the fourth one being just the sum of the other three.

$$\begin{aligned} \text{node a:} &\quad I = I_1 + I_2 \\ \text{node b:} &\quad I_1 = I_3 + I_5 \\ \text{node d:} &\quad I_2 + I_5 = I_4 \end{aligned}$$

LOOP RULE: We need three independent equations from various Kirchhoff's loops in the circuit shown in Fig. 6.41.

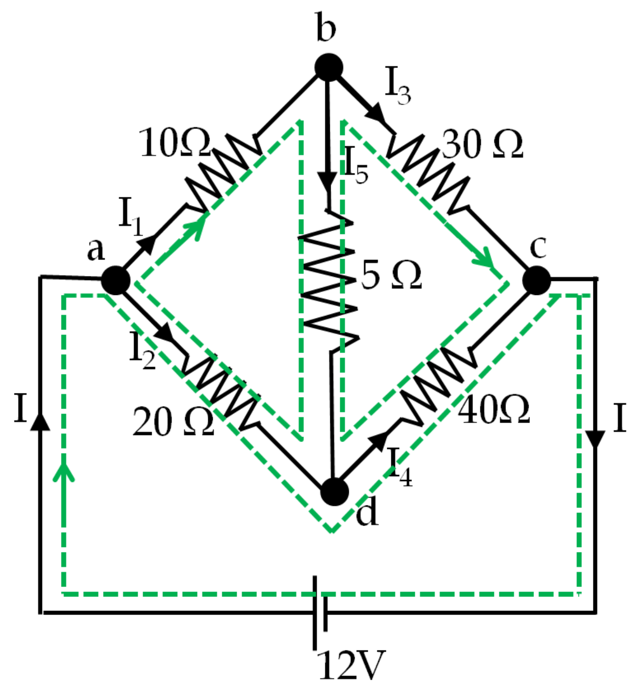


Figure 6.41: Example 6.6.2. Three Kirchhoff's loops used in the calculations are shown in the figure. Other Kirchhoff's loops are also possible. A student should practice using other loops.

$$\begin{aligned} \text{a-d-c-a:} &\quad -20I_2 - 40I_4 + 12 = 0 \\ \text{a-b-d-a:} &\quad -10I_1 - 5I_5 + 20I_2 = 0 \quad (\text{Note + for } 20I_2) \\ \text{b-c-d-b:} &\quad -30I_3 + 40I_4 + 5I_5 = 0 \end{aligned}$$

We have six simultaneous equations in six unknowns $I,\ I_1,\ I_2,\ I_3,\ I_4,\ I_5$ which can be solved by method of elimination giving the following values.

$$I = \frac{78}{155}A;\ I_1 = \frac{51}{155}A;\ I_2 = \frac{27}{155}A;\ I_3 = \frac{9}{31}A;\ I_4 = \frac{33}{155}A;\ I_5 = \frac{6}{155}A.$$

Example 6.6.3. The Wheatstone Bridge

Perhaps the most famous bridge circuit is the Wheatstone bridge, which is used to measure resistances accurately (see Fig. 6.42).

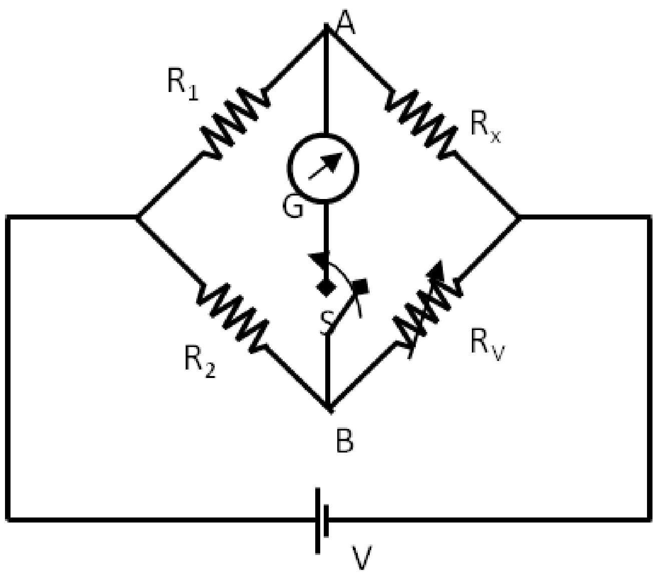


Figure 6.42: Wheatstone bridge.

The Wheatstone bridge circuit consists of four resistors connected to form a quadrilateral. One of the sides of the quadrilateral contains the unknown resistor $R_\times$, another contains a variable resistor R_V and the remaining two contain high precision resistors R_1 and R_2. A voltage source V is connected to two opposite corners and a detector of voltage difference such as a sensitive galvanometer G with a switch is connected between the other two corners. A galvanometer is a device that detects current using the magnetic property of current which we will study in a later chapter. When the resistors are related in a particular way, there is no potential difference between the points marked A and B, and the galvanometer shows no deflection. This condition is called the **null condition** and the circuit is said to be balanced.

When there is no current in the AGB branch, then the circuit becomes simply a circuit of two resistances $R_1 + R_\times$ and $R_2 + R_V$ in parallel, and we can use the simple parallel circuits analysis discussed above to figure out the relation at the null condition.

$$\frac{R_\times}{R_V} = \frac{R_1}{R_2} \quad \text{(Null condition)} \tag{6.71}$$

The derivation of this equation is left as an exercise.