1.4 PROPAGATION OF UNCERTAINTIES

In physics, many physical quantities are derived from other quantities that are directly measured in an experiment. How do uncertainties in the measured quantities propagate into derived quantities? Several different situations occur routinely.

Case 1: The derived quantity is a constant times the measured quantity. For instance, if you measure the diameter of a spherical ball to be $D \pm \Delta D$, what would be the radius R of the ball? We know that radius is 1/2 of the diameter. Therefore, we simply divide the diameter by 2.

$$R \pm \Delta R = \frac{D \pm \Delta D}{2} = \frac{D}{2} \pm \frac{\Delta D}{2}. \tag{1.2}$$

Therefore, the uncertainty in radius is 1/2 of the uncertainty in the diameter.

Case 2: Two measured quantities are added to get the composite quantity. For instance, suppose you measure the length and width of a rectangle to be $L \pm \Delta L$ and $W \pm \Delta W$ respectively. What would be the uncertainty in the perimeter? The perimeter P of a rectangle is given as 2 times the length plus width.

$$P \pm \Delta P = 2\left(L \pm \Delta L + W \pm \Delta W\right) = 2\left(L + W\right) \pm 2\left(\Delta L + \Delta W\right). \tag{1.3}$$

Therefore, the uncertainty in the perimeter would be two times the sum of the uncertainties in length and width.

Case 3: A measured quantity is raised to a power. For instance, suppose you measure the diameter of a circle to be $D \pm \Delta D$. What would be the area A?

$$A \pm \Delta A = \frac{\pi}{4}\left(D \pm \Delta D\right)^2 = \frac{\pi}{4} D^2 \pm \frac{\pi}{2} D \Delta D + \frac{\pi}{4} (\Delta D)^2. \tag{1.4}$$

Normally, the uncertainty will be much smaller than the main number. Therefore, the term containing the square of the uncertainty in the expansion on the right side can be ignored in comparison with other terms to yield the following.

$$A \pm \Delta A = \frac{\pi}{4} D^2 \pm \frac{\pi}{2} D \Delta D. \tag{1.5}$$

Therefore, the uncertainty in the area of the circle will be $\Delta A = \frac{\pi}{2} D \Delta D$, which shows that the uncertainty in area of a circle not only depends on the uncertainty in the diameter but also the diameter itself.

Case 4: Two or more measured quantities are multiplied together. This is the case when we try to calculate area of a rectangle or volume of a parallelepiped. Suppose you measure the length, width and height of a parallelepiped to be $L \pm \Delta L$, $W \pm \Delta W$ and $H \pm \Delta H$, and you wish to calculate its volume V with uncertainty in the volume ΔV, how would that work?

$$\begin{aligned} V \pm \Delta V &= (L \pm \Delta L) \times (W \pm \Delta W) \times (H \pm \Delta H) \\ &= LWH \pm (WH\Delta L + LH\Delta W + LW\Delta H), \end{aligned} \tag{1.6}$$

where, once again, we have assumed that the uncertainties are much smaller than the main readings so that we can neglect terms that have product of uncertainties with each other, and kept terms that have only one uncertainty factor in them. The largest deviation from the main value occur when all the $\pm$ are $+$ or $-$, that is $(WH\Delta L + LH\Delta W + LW\Delta H)$. Thus, the value of the volume ranges between $V_{min} = LWH - (WH\Delta L + LH\Delta W + LW\Delta H)$ and $V_{max} = LWH + (WH\Delta L + LH\Delta W + LW\Delta H)$. In practice, we add each of the contributions to the uncertainty in the composite quantity in quadrature, i.e. square them first and then take the square root of the sum as illustrated next. Let

$$\begin{aligned} \Delta V_L &= WH\Delta L \quad \text{(uncertainty in volume due to uncertainty in length).} \\ \Delta V_W &= LH\Delta W \quad \text{(uncertainty in volume due to uncertainty in width).} \\ \Delta V_H &= LW\Delta H \quad \text{(uncertainty in volume due to uncertainty in height).} \end{aligned}$$

Since each individual source of uncertainty is independent of other uncertainties, the net uncertainty in the volume is constructed by using Pythagorean theorem, giving a diagonal in the space of uncertainties.

$$\Delta V = \sqrt{(\Delta V_L)^2 + (\Delta V_W)^2 + (\Delta V_H)^2}. \tag{1.7}$$

Example 1.4.1. Propagation of uncertainties. The length, width and height of a metal rectangular parallelepiped were measured to be 2.54 cm $\pm$ 0.01 cm, 0.500 cm $\pm$ 0.005 cm, and 0.208 cm $\pm$ 0.001 cm respectively. Find the average volume, and absolute and relative uncertainties in volume.

Solution. The average volume is obtained by simply multiplying the average length, width and height and making sure to keep the appropriate significant figures.

$$\begin{aligned} \text{Volume} &= 2.54\ \text{cm} \times 0.500\ \text{cm} \times 0.208\ \text{cm} \\ &= 0.264\ \text{cm}^3 = 2.64 \times 10^{-1}\ \text{cm}^3. \end{aligned}$$

The uncertainty in volume will be constructed from uncertainty in volume due to uncertainty in each of the measured quantities. Thus the uncertainty in volume due to uncertainty in length is

$$\begin{aligned}
\Delta V_L &= WT\Delta L \\
&= 0.500 \text{ cm} \times 0.208 \text{ cm} \times 0.01 \text{ cm} \\
&= 0.001 \text{ cm}^3 \ (\textit{keeping only the most significant digit}).
\end{aligned}$$

Similarly, the uncertainty in volume due to the uncertainty in width is 0.003 cm^3, and the uncertainty in volume due to the uncertainty in thickness is 0.001 cm^3.

Now we add the three sources of uncertainties in the volume in quadrature to obtain the net uncertainty.

$$\Delta V = \sqrt{(\Delta V_L)^2 + (\Delta V_W)^2 + (\Delta V_T)^2} = 0.003 \text{ cm}^3.$$

Hence, the volume of the block is $(2.64 \pm 0.03) \times 10^{-1}$ cm^3.

The relative uncertainty in volume is obtained from the ratio of absolute uncertainty to the average value.

$$\text{Relative uncertainty} = \frac{\Delta V}{V_{ave}} = \frac{0.03}{2.64} = 1\%.$$