2.4 THE TWO-BODY PROBLEM FOR GRAVITATIONAL FORCE

2.4.1 CM and Relative Position Coordinates

In this section we will revisit the two-body problem discussed earlier in the book. This time we will discuss the problem in the context of a gravitational force between two bodies. Consider an isolated system consisting of two particles of masses m_1 and m_2 interacting with a gravitational force. Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of particles 1 and 2 respectively with respect to an inertial frame $Oxyz$ as shown in Fig. 2.5.

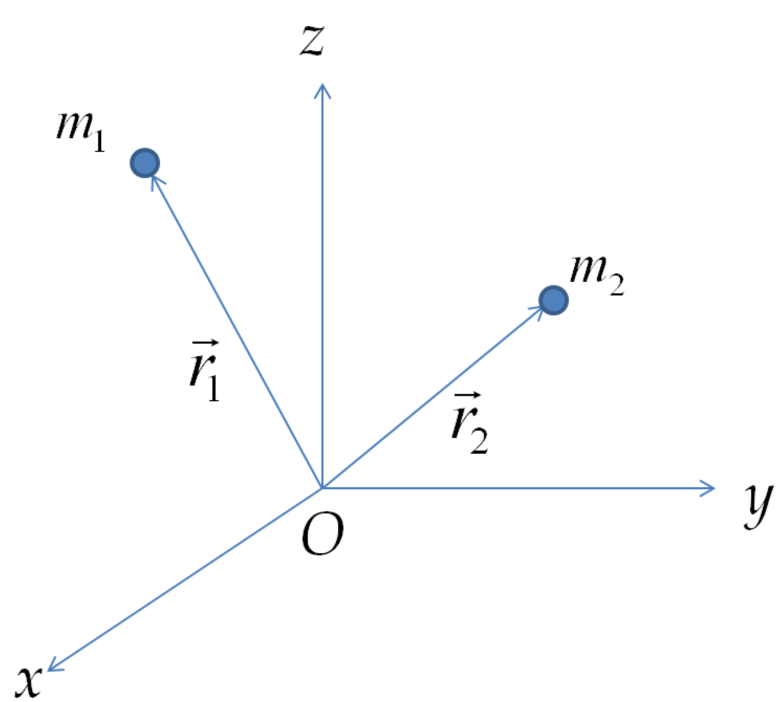

Figure 2.5: Position vectors of two objects of a two-body system in an inertial frame.

Here, the only force on either of the body is the gravitational force by the other body. The force on m_1 is directed from 1 to 2, and on m_2, it is directed in the opposite direction. The force on m_1 written in the vector notation takes the following form.

$$\vec{F}_1 = -\frac{G_N m_1 m_2}{r^2}\hat{u}_{2\to 1} \tag{2.5}$$

where $\hat{u}_{2\to 1}$ is a unit vector from position of m_2 towards the position of m_1. Therefore, the equations of motion for the six coordinates of two bodies, viz., $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$ are as follows.

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -\frac{G_N m_1 m_2}{r^2} \hat{u}_{2\to 1} \tag{2.6}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = +\frac{G_N m_1 m_2}{r^2} \hat{u}_{2\to 1} \tag{2.7}$$

There is an alternate set of six coordinates that is easier to work with than the set of Cartesian coordinates of the two bodies. They consist of the three coordinates of the center of mass $\vec{R}$ and three for the relative position vector $\vec{r}$ shown in Fig. 2.6.

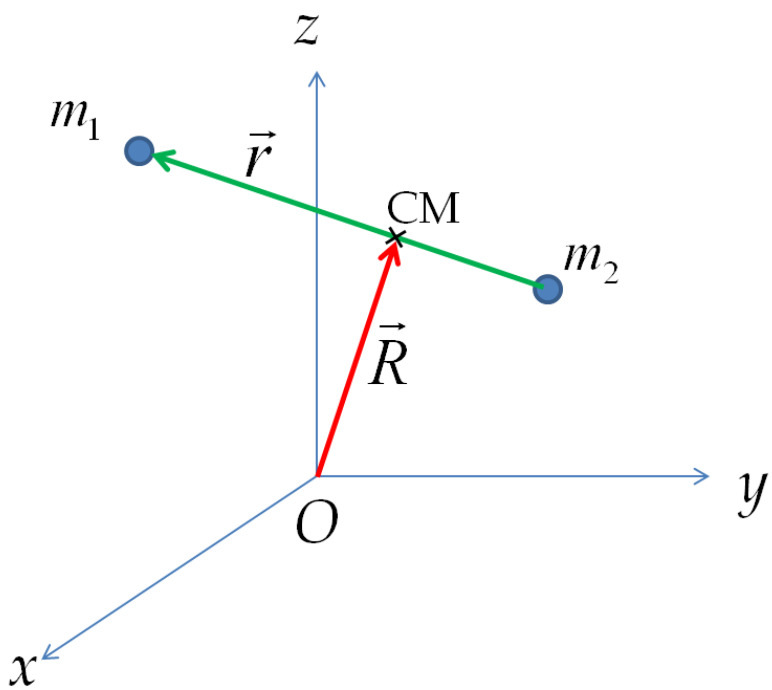

Figure 2.6: Position vectors of two objects of a two-body system.

The CM and the relative coordinates are defined as follows in terms of the Cartesian coordinates $\vec{r}_1$ and $\vec{r}_2$ of the two particles.

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{2.8}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{2.9}$$

The relative position vector $\vec{r}$, defined this way, gives the position of the particle 1 with respect to the position of the particle 2. The equations of motion for these abstract coordinates can be obtained by taking time derivative of the $\vec{R}$ and $\vec{r}$ vectors, and then using the equations of motions for the position coordinates $\vec{r}_1$ and $\vec{r}_2$ of the two particles based on Newton's second law of motion given in Eqs. 2.6

and 2.7. After some algebra which I leave to the student to complete, we find that the following equations of motion for the position of the center of mass and the relative position of particle 1 with respect to particle 2.

$$\boxed{\frac{d^2\vec{R}}{dt^2} = 0} \tag{2.10}$$

$$\boxed{\frac{d^2\vec{r}}{dt^2} = -\frac{G_N(m_1+m_2)}{r^2}\hat{u}_{2\to 1}} \tag{2.11}$$

These equations show that the motion of the CM is decoupled from the motion about the CM, similar to the decoupling of translation and rotational motion we have seen before. The acceleration of the CM of a two-body system in an inertial frame is zero, as expected, because the two bodies form an isolated system whose total momentum is conserved.

The position of m_1 with respect to m_2, given by $\vec{r}$, changes in a way that it is possible to recast the problem in terms of one hypothetical body. Let us divide both sides of Eq. 2.11 by $m_1 + m_2$, and multiply the result by $m_1 m_2$.

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)\frac{d^2\vec{r}}{dt^2} = -\frac{G_N m_1 m_2}{r^2}\hat{u}_{2\to 1} \tag{2.12}$$

The quantity in parenthesis on the left side of the equation has the units of mass, and is called the **reduced mass**. The reduced mass is often denoted by the Greek letter μ (pronounced mu).

$$\boxed{\mu = \frac{m_1 m_2}{m_1+m_2} \quad \Longleftrightarrow \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}.} \tag{2.13}$$

Note that when one of the masses is much greater than the other, the reduced mass is very close to the smaller mass.

$$\mu \approx m_2 \ \text{ if } \ m_1 >> m_2.$$

Let us write M for the total mass $m_1 + m_2$.

$$M = m_1 + m_2.$$

We also note that the unit vector $\hat{u}_{2\to 1}$ from 2 to 1 is a unit vector in the direction of the relative position vector $\vec{r}$. Therefore, for simpler notation, we use $\hat{u}_r$ for this unit vector.

$$\hat{u}_r \equiv \hat{u}_{2\to 1}.$$

Now, the equation of motion of the relative position can be written in terms of the total mass and the reduced mass in the following form, which "looks like" the equation of a single particle of mass μ in the gravitational force of another particle of mass M.

$$\boxed{\mu \frac{d^2\vec{r}}{dt^2} = -\frac{G_N M \mu}{r^2}\hat{u}_r.} \tag{2.14}$$

We say that the original equations of motion Eqs. 2.6 and 2.7, in which the motions of the particles m_1 and m_2 are coupled, has been decoupled into motions of two fictitious particles:

One of these fictitious particles has a mass M and moves at constant velocity with respect to the origin of the inertial frame. The second fictitious particle has a mass μ and moves in the gravitational field of another particle of mass M which is fixed at the origin.

These two de-coupled variables are easier to work than the original coordinates. Therefore, we will focus solving then and deducing the original coordinates from the solutions of $\vec{R}$ and $\vec{r}$.

2.4.2 Equations of Motion in the CM Frame

Since there are no external forces on an isolated two-body system, the CM does not accelerate. Therefore, it is possible to choose our inertial frame fixed to the CM. In that case $\vec{R}$ will be zero, $\vec{R} = 0$ (CM frame), and will remain zero for all time. The original particles and the relative position coordinates are shown in Fig. 2.7. Using the definition of CM in terms of the coordinates of the individual objects in the CM frame, we find the following relation between the coordinates of the two bodies in the CM frame.

$$\vec{R} = \frac{m_1\vec{r}_1{}' + m_2\vec{r}_2{}'}{m_1+m_2} = 0 \implies m_1\vec{r}_1{}' + m_2\vec{r}_2{}' = 0. \tag{2.15}$$

The relative position vector of the two masses in the CM-frame is also equal to $\vec{r}$ since a displacement vector is independent of the choice of origin.

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = \left(\vec{R} + \vec{r}_1{}'\right) - \left(\vec{R} + \vec{r}_2{}'\right) = \vec{r}_1{}' - \vec{r}_2{}'. \tag{2.16}$$

Combining Eqs. 2.15 and 2.16 we find that if we work in the CM frame, we will be able to easily obtain the motion of each body from the motion of the relative coordinate only by the following relations.

$$\vec{r}_1{}' = \frac{m_2}{m_1+m_2}\vec{r} = \left(\frac{\mu}{m_1}\right)\vec{r} \tag{2.17}$$

$$\vec{r}_2{}' = \frac{m_1}{m_1+m_2}\vec{r} = \left(\frac{\mu}{m_2}\right)\vec{r} \tag{2.18}$$

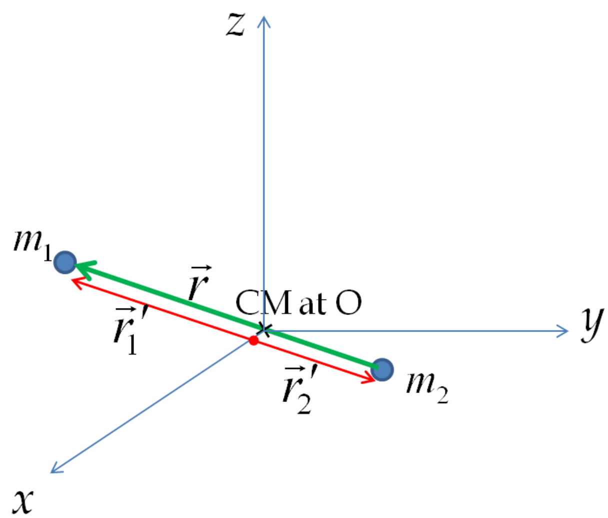

Figure 2.7: The CM frame. The positions of m_1 and m_2 in the CM frame are denoted by $\vec{r}_1{}'$ and $\vec{r}_2{}'$, respectively. The relative position of 2 with respect to 1, $\vec{r} = \vec{r}_1{}' - \vec{r}_2{}'$, which has the same formal appearance in this coordinate as it had in the inertial frame whose origin was at some other place than at the CM.

If one of the masses, say m_1 is very much larger than m_2, i.e. $m_1 >> m_2$, as is the case for instance for the Sun/Earth system, then the relative coordinate is simply the coordinate of the smaller mass with respect to the CM. We will make this assumption in the rest of the chapter.

$$\text{When } m_1 >> m_2, \text{ then } \mu \approx m_1,\ \vec{r}_1{}' \approx 0 \text{ and } \vec{r}\,'2 \approx \vec{r}.$$

2.4.3 Angular Momentum Conservation and Kepler's Second Law

Further simplification of analysis of a two-body system takes place by exploiting the fact that for an isolated system, the total angular momentum is conserved. The angular momentum $\vec{L}$ of the two objects is the sum of their individual angular momenta. Writing the angular momentum about the CM we have

$$\vec{L} = \vec{r}_1{}' \times m_1\vec{v}_1{}' + \vec{r}_2{}' \times m_2\vec{v}_2{}'. \text{ [About the CM]} \tag{2.19}$$

By a series of algebraic manipulations you can show that this equation simplifies to the following.

$$\boxed{\vec{L} = \vec{r} \times \mu\frac{d\vec{r}}{dt}. \text{ [About the CM]}} \tag{2.20}$$

Therefore, the angular momentum of the two bodies about their CM is simply equal to the angular momentum of a hypothetical object of mass equal to the reduced mass and the position equal to the relative position of the two objects.

Since neither the magnitude nor the direction of the conserved angular momentum $\vec{L}$ can change with time, the motion of the effective body of mass μ must be in a plane perpendicular to the direction of $\vec{L}$ as illustrated in Fig. 2.8.

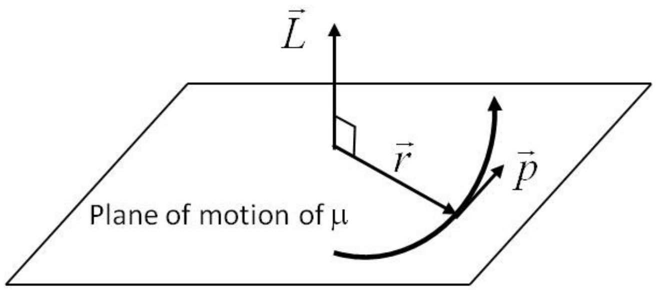

Figure 2.8: The reduced mass moves in a plane.

Let us choose the coordinate system so that the z axis is pointed along the conserved angular momentum direction, and the position of reduced mass is measured from the origin of this coordinate system. Let l be the magnitude of the angular momentum. Then, the angular momentum vector of the system is written as

$$\vec{L} = l\hat{u}_z. \tag{2.21}$$

The magnitude of the conserved angular momentum places the following constraint on the relative position.

$$\left|\vec{r} \times \mu\frac{d\vec{r}}{dt}\right| = l. \tag{2.22}$$

Kepler's second law of equal areas is a particular consequence of the conservation of angular momentum as seen by expressing this equation in polar coordinates (r, θ). This calculation here is a repeat of a similar calculation we have done in the chapter on rotation.

$$\mu r^2 \left|\frac{d\theta}{dt}\right| = l. \tag{2.23}$$

Here both r and θ are functions of time. From Fig. 2.9, it is clear that the area covered by the orbit of the reduced mass μ in time Δt is $r^2\Delta\theta/2$. We can write the area covered in a duration in terms of the conserved angular momentum of the object.

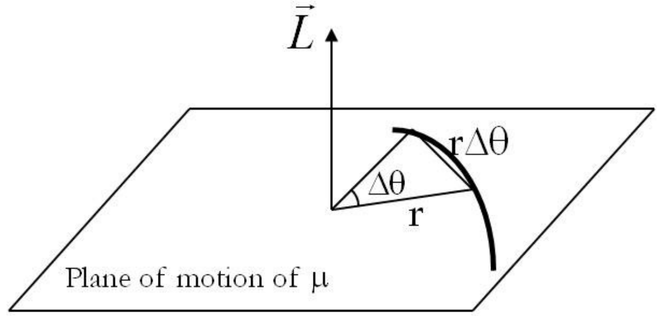

Figure 2.9: Finding area spanned by the reduced mass in its orbit.

$$\text{Area covered during } \Delta t \equiv \Delta A = \frac{r^2 \Delta\theta}{2} = \frac{l}{2\mu}\Delta t.$$

Therefore, the rate at which area swept by a line from the origin to the position of the particle is

$$\frac{\Delta A}{\Delta t} = \frac{l}{2\mu}, \text{ constant.}$$

Therefore, we can say that the radial vector from the Sun to a planet sweeps out an equal area in an equal time. This is Kepler's second law of planetary motion which was originally obtained empirically from examining Tycho Brahe's data on the motion of the planet Mars. Thus, Kepler's second law of motion is simply a consequence of the conservation of angular momentum of the combined Sun/Planet system.

2.4.4 Energy Conservation

Since gravitational force is a conservative force, the total energy of the two-body interacting via the gravitational force is conserved. The total energy E of the two-body system is given by the following.

$$E = K + U = \left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right) - G_N \frac{m_1 m_2}{r} \tag{2.24}$$

By using polar coordinates and substituting various quantities, you should show that this expression can also be written as follows.

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\mu r^2\left(\frac{d\theta}{dt}\right)^2 - G_N\frac{m_1 m_2}{r} \\ &= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \left(\frac{1}{2}\frac{l^2}{\mu r^2} - G_N\frac{m_1 m_2}{r}\right) \end{aligned} \tag{2.25}$$

In polar coordinates, we notice that the kinetic energy of the angular motion, i.e. the centrifugal energy, acts like an additional term for the potential energy. We combine the actual potential energy (the last term) with the second term on the right side in the above equation and define an effective potential energy $U_{eff}(r)$.

$$\boxed{U_{\text{eff}} = \frac{1}{2}\frac{l^2}{\mu r^2} - G_N \frac{m_1 m_2}{r}.} \tag{2.26}$$

Let us now rewrite the energy of a planet of mass m moving in a space where it has only the gravitational force of a Sun of mass M as

$$\boxed{E = \frac{1}{2}\mu v_r^2 + U_{\text{eff}} \equiv K_r + U_{\text{eff}},} \tag{2.27}$$

where K_r stands for the kinetic energy of the planet, given by $\frac{1}{2}\mu v_r^2$ with v_r for the radial speed of the planet. Note that K_r is not the kinetic energy of the planet since the angular part of the kinetic energy has been lumped with the potential energy to obtain the effective potential energy. We will see below that the accounting of energy in this modified way helps with the analysis of the motion of the planet.

This equation "looks like" an equation of energy of a particle of mass μ, not the mass m of the planet. This fictitious particle appears to move in a "one dimensional world" of radial coordinate, whose value ranges from zero to positive infinity. Of course, in the real world, the planet is moving in a plane with both the radial coordinate and the angular coordinate. But, as far as Eq. 2.27, both K_r and U_{eff} depend only on one coordinate r. Therefore, we may pretend that there is only one coordinate r to worry about. But, this coordinate is weird since it has only positive values, unlike the x, y, z-coordinates.

Interpreting the Effective Potential Energy

Now, we examine Eq. 2.27 for various possible values of energy E. Note that K_r cannot be negative in Eq. 2.27. Therefore, $E - U_{\text{eff}}$ must always be greater than or equal to zero.

$$E - U_{\text{eff}}(r) \geq 0, \quad \text{since } K_r \geq 0, \tag{2.28}$$

where I have indicated the dependence of U_{eff} on r explicitly. For given masses M, m and magnitude of the angular momentum l, the effective potential takes a definite functional form. To explore the implications of these inequalities, it is helpful to write the equality in Eq. 2.28, viz. $E = U_{\text{eff}}(r)$ more explicitly using the expression for $U_{\text{eff}}(r)$. We will also write $U_{\text{eff}}(r)$ as $a/r^2 - b/r$ for simplicity.

$$U_{\text{eff}} = \frac{1}{2}\frac{l^2}{\mu r^2} - G_N \frac{m_1 m_2}{r} \Leftrightarrow a/r^2 - b/r.$$

Equation $E = U_{\text{eff}}(r)$ is then a quadratic equation in r.

$$Er^2 + br - a = 0.$$

For $E = 0$ this equation has the solution

$$r = \frac{a}{b} = \frac{l^2}{2G_N m_1^2 m_2^2} \quad [E = 0].$$

and for non-zero E, the solution is

$$r = \frac{1}{E}\left(-b \pm \sqrt{b^2 + 4aE}\right) \quad [E \neq 0]. \tag{2.29}$$

If $E > 0$, then we see that the radical in this solution gives a larger value than b. Therefore, only plus of $\pm$ will be physical since $r > 0$ here. This gives one physical solution for $E > 0$.

$$r = \frac{1}{E}\left(-b + \sqrt{b^2 - 4aE}\right), \quad [E > 0]. \tag{2.30}$$

If $E < 0$, then, the radical will be real if $b^2 - 4a|E| > 0$. Let us write the solution with absolute value of energy $|E|$ so that we have easy time making sure $r > 0$ in the final analysis.

$$r = \frac{1}{|E|}\left(b \mp \sqrt{b^2 - 4a|E|}\right). \tag{2.31}$$

Now, we note that since $b^2 - 4a|E| > 0$, we will have $b^2 > 4a|E| > 0$. Therefore, we will always have $b > \sqrt{b^2 - 4a|E|}$, which would give two values for r.

$$r_1 = \frac{1}{|E|}\left(b - \sqrt{b^2 - 4a|E|}\right) \quad [E < 0] \tag{2.32}$$

$$\text{and,} \quad r_2 = \frac{1}{|E|}\left(b + \sqrt{b^2 - 4a|E|}\right) \quad [E < 0] \tag{2.33}$$

Finally, the negative value for E cannot be below the minimum of U_{eff}. This show up in the requirement of the reality of the radical in the solution when $E < 0$. We find that when $E < 0$, then

$$|E| \leq \frac{b^2}{4a}.$$

At the minimum, $|E| = \frac{b^2}{4a}$. Using this condition in the solution we find that r has only one value.

$$r = \frac{2a}{b} \quad E = E_{\text{min}}. \tag{2.34}$$

A plot of U_{eff} shown in Fig. 2.10 helps visualize these different types of solutions. In this plot the four values of interest of the total

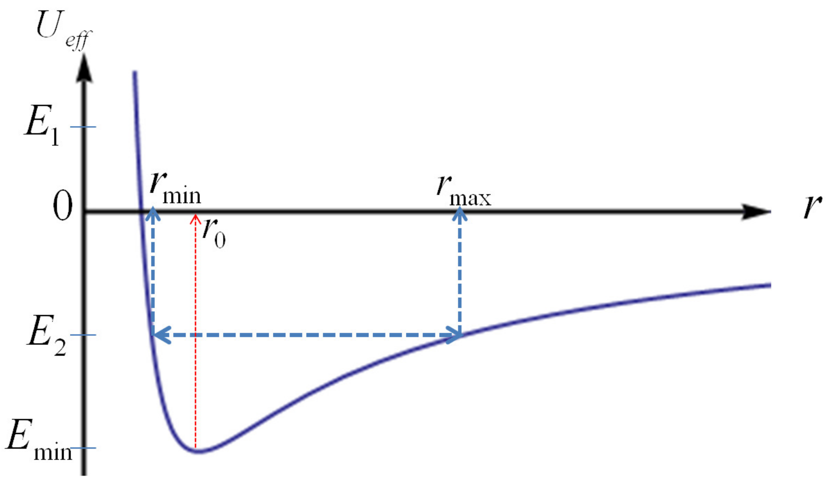

Figure 2.10: Plot of effective energy versus radial distance r. Four energy levels of interest are indicated on the axis for $U_{\rm eff}$. They are $E = 0, E_1, E_2, E_{\rm min}$.

energy of the system are indicated on the ordinate axis. The four solutions have the following meaning.

Case 1: $E = 0$. There is only one radial distance when $E = U_{\rm eff}$. At this point K_r must be zero. Below we will look at the equation of the trajectory and find that the trajectory of μ is parabolic. That is, the particle can move to a closest distance to the origin where the "force-exerting" fictitious particle of mass M is located. The position where $K_r = 0$ corresponds to $v_r = 0$, which is the point where the radial velocity will change from being positive to being negative and vice-versa. This point in space is called the turning point of the radial coordinate. Note that the radial velocity zero does not mean that the full velocity of the particle is zero there, since we also have angular velocity, which would be non-zero.

The picture of the original two-particle would be that the two particles approach one another to a closest approach and then return to be far apart. They are kept apart due to the centrifugal energy barrier. The motion is an unbounded one.

Case 3: $E > 0$. This case is similar to the case of $E = 0$ in the sense that there is only one turning point at the solution. The two bodies approach one another to a closest approach and then return to be far apart. They are kept apart due to the centrifugal energy barrier. The motion is an unbounded one. Although, this case appears to be similar to $E = 0$ case, but we will find below that the trajectory is not parabolic but has the shape of a hyperbola.

Case 3: $U_{eff}^{min} < E < 0$. In this case, there are two turning points corresponding to the two solutions, the smaller value of the solution is called $r_{\min}$, or closest approach, and the larger one $r_{\max}$, or furthest apart. In the Sun/Earth system, the $r_{\min}$ and $r_{\max}$ are called **perihelion** and **aphelion**. In the Earth/Satellite system, these points are called **perigee** and **apogee**. The motion is bounded with a maximum and a minimum separation between the two bodies. A rigorous calculation shows that the trajectory is an ellipse.

Case 4: $E = U_{\text{eff}}^{\min}$. This is a particular sub-case of case 3, and corresponds to a circular motion as is evident in only one solution to the condition $E = U_{\text{eff}}$ or $K_r = 0$. The radial component of the velocity is zero at all points of the trajectory, i.e. the velocity has only angular component in this case.

2.4.5 The Orbit Equation and Kepler's First Law

We saw above that the orbits of two bodies interacting with a gravitational force is more readily presented in the CM-frame. In the CM frame, it suffices to work out the equivalent problem of the orbit of a fictitious particle with mass equal to the reduced mass. Once, the orbit of the fictitious particle is known, we can immediately work out the orbits of two masses by the relation between the coordinates of the reduced mass and the coordinates of the individual masses.

Often, it is not necessary to work out the orbits of the individual masses since we often deal with systems where one of the bodies has much more mass then the other. For instance, in the Sun/Earth system, the mass of the Sun is much more than the mass of Earth. In these systems, we approximate the motion of the smaller body by the motion of the reduced mass and represent the larger mass as a point mass at the CM, where the origin is also placed.

If the masses of the two bodies are equal to each other or close to each other, as maybe the case of two stars in a binary star system, we cannot make the above simplification. The motion of the fictitious particle in these cases does not approximate to the motion of either body. We must deduce the motion of each mass either directly or indirectly. However, even in these cases, we find that the equation of motion of the reduced mass is very helpful in deducing the motion of each mass. We work out the motion of the reduced mass and deduce the motion of each mass indirectly from the solution.

Therefore, regardless of whether we are dealing with disparate mass system or closer mass system, it is always a better strategy to think in terms of the reduced mass. We will examine the motion of the reduced mass in this section more fully. We will continue to call the particle with the reduced mass a fictitious particle just to emphasize the fact that reduced mass is a deduced quantity, the actual particles being m_1 and m_2.

To obtain the orbit of the fictitious particle μ in a two-body system it is sufficient to examine the conservation of angular momentum and the conservation of energy in the system. We will place the origin at the CM and write the energy and the angular momentum conservation equations in the polar coordinates as follows.

$$\mu r^2 \frac{d\theta}{dt} = l, \text{ constant angular momentum} \tag{2.35}$$

$$\frac{1}{2}\mu \left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{l^2}{\mu r^2} - \frac{G_N m_1 m_2}{r} = E, \quad \text{constant energy} \tag{2.36}$$

Note that, although the angular momentum and the kinetic energy of the two bodies in the system can be written completely in terms of the one-body corresponding to the reduced mass particle, the potential energy still contains the original masses m_1 and m_2 as parameters in the equations.

The equations corresponding to the conservation of energy and angular momentum can be solved to gain information about the types of orbits possible for given l, E, m_1 and m_2. The equation of the orbit is obtained by dividing dr/dt by $d\theta/dt$ so that we can get an equation for $dr/d\theta$.

$$\frac{dr}{d\theta} = \frac{dr/dt}{dr/d\theta} = \frac{\text{From Eq. 2.36}}{\text{From Eq. 2.35}}. \tag{2.37}$$

This equation can be integrated to find radial coordinate as a function of the angular coordinate, $r(\theta)$, for the orbit. The calculus needed to perform these operations is somewhat beyond this course. If you are intrigued, you will have to look up a more advanced book on mechanics to find the necessary algebra, or better yet, carry out the calculations yourself. I will state the final answer here so that we can discuss the physical consequences of the result, which are important for understanding the motion of planets. The solution $r(\theta)$ is as follows.

$$\boxed{r = \frac{r_0}{1 - e\cos\theta}} \tag{2.38}$$

where

$$r_0 = \frac{l^2}{G_N m_1 m_2 \mu} \tag{2.39}$$

$$e = \sqrt{1 + \frac{2El^2}{G_N^2 m_1^2 m_2^2 \mu}} \tag{2.40}$$

The parameter r_0 corresponds to the radius of the circular orbit and e is called the eccentricity of the orbit, which characterizes the shape of the orbit. To identify the orbits as ellipses, circles, parabolas or hyperbolas, it is better to write the orbit equation in more familiar Cartesian coordinates. Using $x = r\cos\theta$ and $y = r\sin\theta$ we find the following equivalent form of the orbit equation.

$$\left(1-e^2\right)x^2 - 2r_0ex + y^2 = r_0^2. \tag{2.41}$$

Now, we can see how the four cases of different types of orbits depend on energies of the system, through the parameter e. To discuss the type of orbits that result for various values of energy as compared to the minimum effective potential energy, it is best to rewrite the parameters r_0 and e in terms of the minimum value of the effective potential energy $U_{\text{eff}}^{\min}$, which we will denote by symbol U_0. By an elementary calculation, you can show that the minimum value of the effective potential energy is

$$U_0 = -\frac{(G_N m_1 m_2)^2\mu}{2l^2} \tag{2.42}$$

Therefore, the parameters r_0 and e in the orbit equation are

$$r_0 = \frac{1}{\sqrt{-2\mu U_0}} \tag{2.43}$$

$$e = \sqrt{1-\frac{E}{U_0}}. \tag{2.44}$$

Case 1 $E = 0$. If $E = 0$, then $e = 1$. The coefficient of x^2 in Eq. 2.41 vanishes which leads to a parabola.

Case 2 $E > 0$. If $E > 0$, then $e > 1$ since $U_0 < 0$. This will give an opposite sign to the coefficient of x^2 than that of the y^2. Therefore, this is an equation of a hyperbola. The fictitious particle μ has a closest approach to the CM given in the figure by the point when the hyperbolic orbit crosses the x-axis. (Fig. 2.11). Setting $y = 0$ in the orbit equation, we solve for x, and choose the smaller of the two negative x where the closest approach occurs.

Case 3 $U_0 < E < 0$. This case corresponds to $0 < e < 1$. Both x^2 and y^2 have the same sign and unequal coefficients. Hence, the orbit is an ellipse. Using the Cartesian form of the orbit equation, the semi-major axis a and semi-minor axis b are given by

$$a = \left|\frac{r_0}{1-e^2}\right| = \frac{G_N m_1 m_2}{2|E|} \tag{2.45}$$

$$b = \frac{r_0}{\sqrt{1-e^2}} = \frac{l}{\sqrt{2\mu|E|}} \tag{2.46}$$

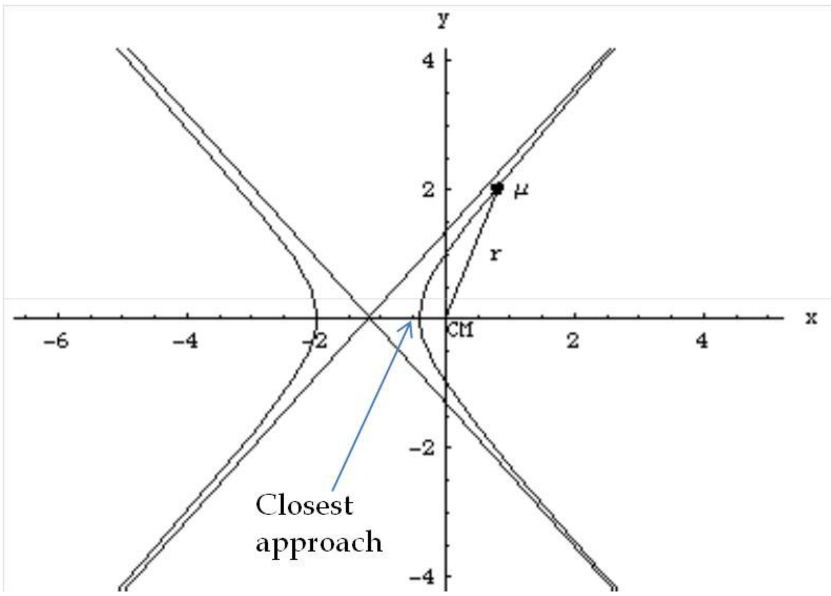

Figure 2.11: An elliptic orbit for $r_0 = 1$ unit and $e = 1.5$ for an attractive potential. The other branch is for a repulsive force since acceleration will be pointed away from the CM.

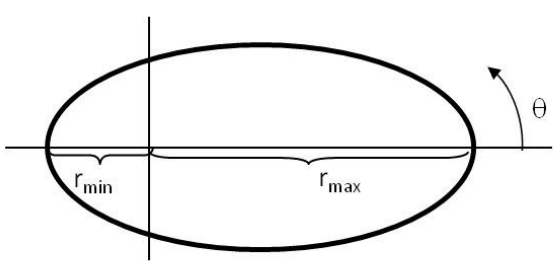

Figure 2.12: Closest and farthest approach of a object in elliptical orbit.

The closest and the farthest approaches occur on the semi-major axis, which is along the x-axis here, as shown in Fig. 2.12. The farthest distance r_{max} is obtained by setting $\theta = 0$ in the orbit equation.

$$r_{max} = r|_{\theta=0} = \frac{r_0}{1-e} \tag{2.47}$$

The closest distance occurs for $\theta = 180°$.

$$r_{min} = r|_{\theta=180°} = \frac{r_0}{1+e} \tag{2.48}$$

Of course, you could also obtain the closest and farthest distances from the CM by setting $y = 0$ in the Cartesian form of the orbit equations, and solving the quadratic equation for x. These are the perihelion and aphelion or perigee and apogee points for planets and satellites

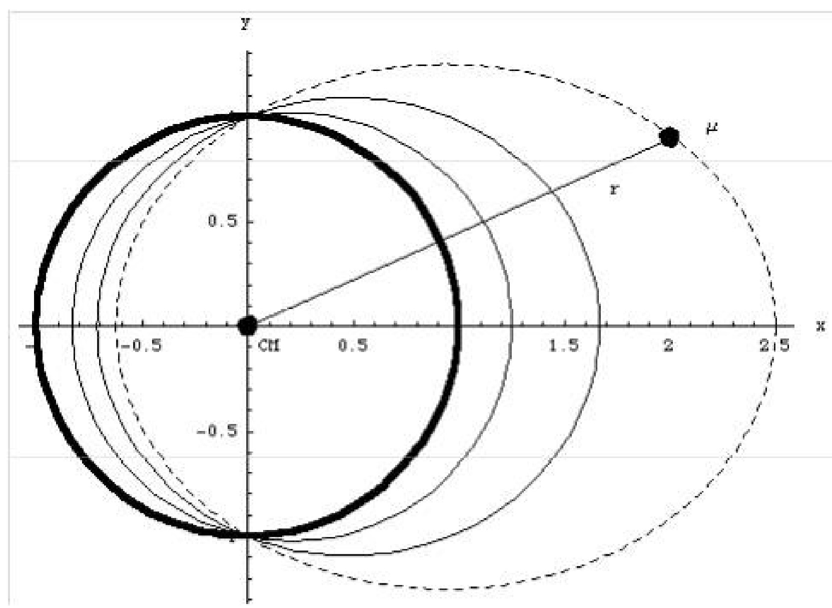

Figure 2.13: Elliptical orbits for three different eccentricities 0.2, 0.4, 0.6. The thick solid line is a circular orbit which corresponds to $e = 0$. The higher the eccentricity more elongated the ellipse. The value of r_0 was set to 1 for the plot.

Case 4 $E = U_0$. This makes $e = 0$. With $e = 0$, the orbit equation becomes $x^2 + y^2 = r_0^2$, an equation of a circle of radius r_0.

The eccentricity of orbits tells us about the elongation of the ellipse; the larger the eccentricity the more elongated the ellipse. Thus when $e = 0$, ellipse is same as a circle, and when $e = 0.6$, the ellipse is elongated as shown in Fig. 2.13.

Both cases 3 and 4 correspond to bound orbits. Hence planets (as well as comets) have either elliptical or circular orbits around the Sun confirming Kepler's first law. The orbits of different planets differ in their eccentricities. The orbit of Venus is almost a circle while the orbit of Pluto is quite elongated. Comets have much higher eccentricities. For instance, the orbit of Halley's comet has an eccentricity of 0.967. Recall that if $e > 1$ the orbit will not be bounded, but hyperbolic. Table 2.1 summarizes eccentricities of planets. Except for Mercury and Pluto, all other planets have minor eccentricities.

Table 2.1: Eccentricities of solar planets and planet-like objects

Planet	Eccentricity (e)
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.055
Uranus	0.051
Neptune	0.007
Pluto	0.252
Sedna	0.8

2.4.6 Kepler's Third Law From the Orbit Equation

We can deduce Kepler's third law if we apply orbit equation to two planets revolving around the Sun. Rather than directly deal with the orbit equation, we first work out a relation between the period of a planet of mass m in orbit around the Sun of mass M and the planet's distance r from the Sun. We work with polar coordinates with the planet in xy plane and the angular momentum pointed along z axis. Let l be the z component of the angular momentum. We have seen above that

$$l = \mu r^2 \frac{d\theta}{dt}$$

Here μ is the reduced mass of m and M. Rearrange the equation, and divide both sides by 2.

$$\frac{l}{2\mu} dt = \frac{1}{2} r (r d\theta)$$

The right side is a differential element of area enclosed in the ellipse centered about the focus. Therefore, upon integration over a whole period we will obtain the area of the ellipse, πab, on the right side. On the left side integration will give a quantity proportional to the period T.

$$\frac{l}{2\mu} T = \pi ab \tag{2.49}$$

From the formulas for a and b given above, we can express b in terms of a.

$$b^2 = \left(\frac{l^2}{G_N M m \mu} \right) a,$$

where the quantities within parenthesis are constants. Squaring both sides of Eq. 2.49, rearranging terms, and putting b in terms of a we find the following for a planet of mass m around the Sun of mass M.

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G_N (M + m)}. \tag{2.50}$$

Kepler's third law compares T^2/a^3 for two planets around the Sun. Their ratios will be

$$\boxed{\frac{(T^2/a^3)_1}{(T^2/a^3)_2} = \frac{M+m_2}{M+m_1}.} \tag{2.51}$$

Since planets' masses m_1 and m_2 are considerably smaller than the mass M of the Sun, we can expand $1/M+m_1$ in the Maclaurin series series and keep the leading terms to obtain

$$\frac{(T^2/a^3)_1}{(T^2/a^3)_2} = 1 + \frac{m_2 - m_1}{M} + \cdots \tag{2.52}$$

According to Kepler's third law the ratio of the square of the period to the cube of the semi-major axis of a planet's motion is independent of the planet, but we find here that T^2/a^3 does depend upon the masses m of the planets. Thus, **Kepler's third law is not an exact law**. The deviation can be expressed in terms of m/M of planets which we show in Table 2.2. Even for the most massive planet, Jupiter, m/M is of the order of 0.001% only.

Table 2.2: Ratio of Mass of Planets to Mass of Sun

Planet	m (10^{23} kg)	m/M $\times 10^{-07}$
Mercury	3	1.51
Venus	50	25.1
Earth	60	30.2
Mars	6	3.02
Jupiter	20000	10100
Saturn	6000	3020
Uranus	900	452
Neptune	1000	503

Example 2.4.1. Europa around Jupiter. Europa is a moon of Jupiter that is most like Earth. It has a mass of 4.8×10^{22} kg and moves in an ellipse of eccentricity 0.009 with a mean distance of 670, 900 km from the center of Jupiter whose mass is 1.9×10^{27} kg. What are the energy and the angular momentum of Europa? Ignore the motion around the Sun.

Solution. Since the mass of Europa is so much smaller than that of Jupiter, we can assume one-particle picture and set μ equal to the mass m of Europa. The data is given for the average distance between Europa and Jupiter which we will equate to r_0 since if Europa moved at the average distance its distance will not change and the orbit would be circular.

$$e = 0.009$$
$$r_0 = 6.7 \times 10^8\ m$$

We need $G_N Mm$ to find energy E. We will calculate the numerical value of this quantity so that we do not clutter the final formulas.

$$G_N Mm = 6.7\times 10^{-11}\frac{\text{N.m}^2}{\text{kg}^2}\times 1.9\times 10^{27}\text{kg}\times 4.8\times 10^{22}\text{kg} = 6.1\times 10^{39}\text{N.m}^2.$$

Hence, the energy of Europa will be

$$E = -\frac{G_N Mm}{2a} \approx -\frac{G_N Mm}{2r_0} = -4.6 \times 10^{30}\ \text{J}.$$

We use the energy to find the magnitude of the angular momentum.

$$l = b\sqrt{-2\mu E} \approx r_0\sqrt{-2\mu E} = 4.5 \times 10^{35}\ \text{kg.m}^2/\text{s}.$$

The direction of the angular momentum is pointed perpendicular to the plane of the orbit.

Example 2.4.2. Satellite about the Earth

A satellite of mass 3000 kg is put in an elliptic orbit about the Earth with the semi-major and the semi-minor axes of the orbit being 10,000 km and 8,000 km respectively. (a) What are the energy and angular momentum of the satellite? (b) How far away is the satellite from the center of the Earth at its nearest approach, perigee, and its farthest point, apogee? (c) What are the speed of the satellite at the perigee and the apogee? Ignore the effect of the Sun.

Solution. Once again, we have the situation $m << M$. Therefore, we will substitute m for μ.

(a) We have $G_N Mm = 1.2 \times 10^{18}$ N.m^2, $a = 1.0 \times 10^7$ m, and $b = 8.0 \times 10^6\ m$. Therefore, the energy of the satellite is

$$E = -\frac{G_N Mm}{2a} = -6.0 \times 10^{10}\ \text{J},$$

and the angular momentum has the following magnitude.

$$l = b\sqrt{-2\mu E} = 1.5 \times 10^{14}\ \text{kg.m}^2/\text{s}.$$

The direction of the angular momentum is pointed perpendicular to the plane of the orbit.

(b) We will use the orbit formula to answer these questions. The eccentricity is obtained from a and b.

$$e = \sqrt{1 - \frac{b^2}{a^2}} = 0.6.$$

The circular orbit radius is

$$r_0 = a(1 - e^2) = 6.4 \times 10^6\ \text{m}.$$

The perigee $r_{\min}$ occurs at $\theta = 180°$ and apogee $r_{\max}$ at $\theta = 0°$ in the orbit equation $r = r_0/(1 - e\cos\theta)$.

$$r_{\max} = 1.6 \times 10^7\ \text{m}$$
$$r_{\min} = 2a - r_{max} = 4.0 \times 10^6\ \text{m}$$

(c) Equating the energy E of the satellite to the sum of the kinetic and potential energies, we find the speed of the satellite.

$$E = \frac{1}{2}\mu v^2 + U \approx \frac{1}{2}mv^2 - \frac{G_N M m}{r} \implies v = \sqrt{\frac{2}{m}\left(E + \frac{G_N M m}{r}\right)}.$$

We use the value of energy from part (a) that is same for all r and obtain the value of speed at the perigee by setting $r = r_{\text{min}}$ and at apogee by setting $r = r_{\text{max}}$ in this equation. The values obtained are $v_{\text{perigee}} = 12,700$ m/s and $v_{\text{apogee}} = 3,200$ m/s.