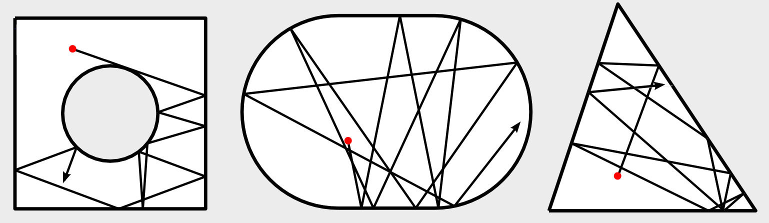
From billiards to thermodynamics

Renato Feres
Washington University
Department of Mathematics

Billiards

Engraving from Charles Cotton's 1674 book, *The Complete Gamester*

Math billiards - different shapes

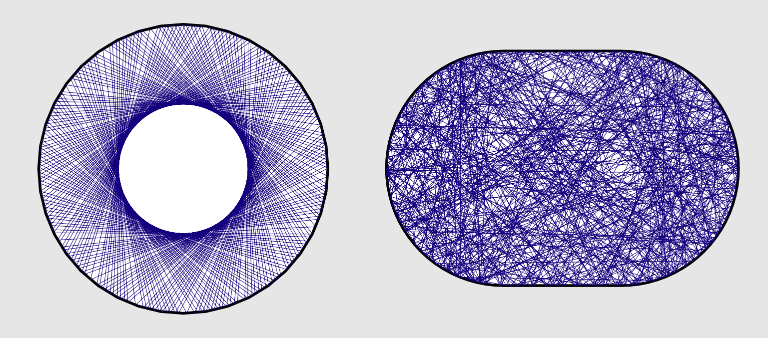


Mathematical billiards: simple models of mechanical systems that

- help to explore the foundations of statistical mechanics,
- used to develop the mathematical theory of dynamical systems
- has deep connections with many areas of mathematics.

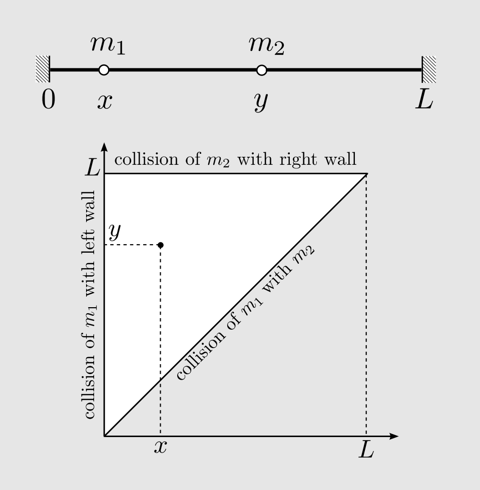
We don't think of it as a game of skill ...

... but a game of observation. We set the ball in motion and try to understand what happens to it over long periods of time and how what it does is affected by the shape of the table.



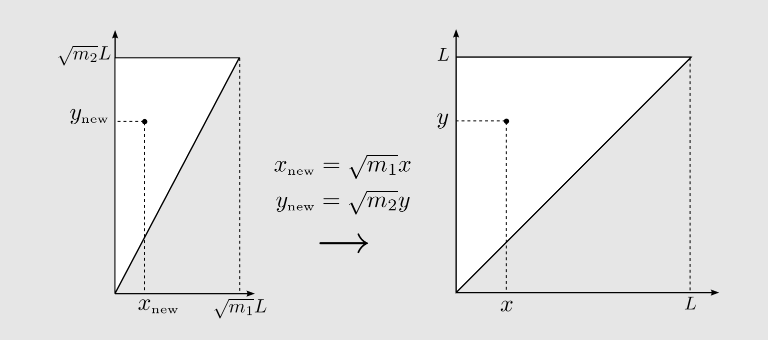
Very different long term behavior of trajectories for different shapes. The example on the left shows fairly regular behavior. The one on the right is very unstable and "chaotic." The stadium billiard is said to be **ergodic**.

What if there are many balls?



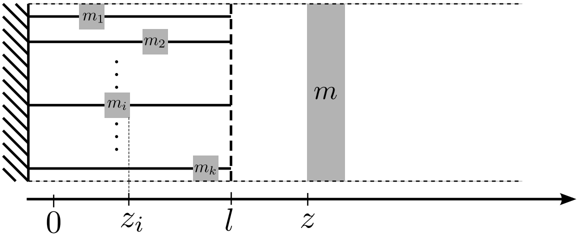
Single particle system in dimension 2 describes two-particle system in dimension 1. This idea applies to any number of particles in two or three dimensions.

How to make reflections mirror-like?



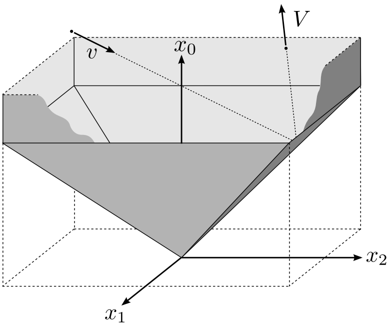
A linear change of coordinates makes the total kinetic energy proportional to the square of the length of a vector. This makes collisions mirror-like. This idea holds in any dimension, for any number of particles.

Example: a multi-particle scattering process



Mass m on the right can move freely along a rail track, while masses m_i on the left can only move freely over the interval $[0, l]$. We want to understand the motion of m as it approaches the wall-bound masses, collides with the m_i multiple times, then eventually turns around and moves aways to the right.

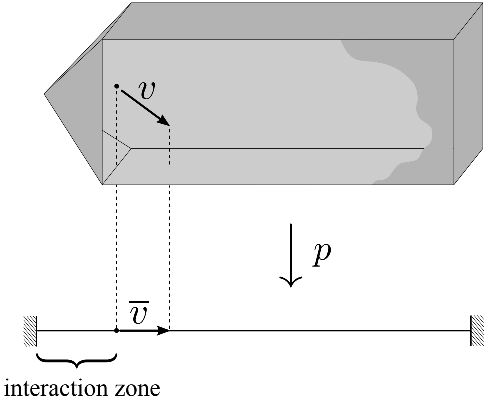
A billiard representation of the scattering process



The multi-particle system can be represented by a single particle billiard system in dimension $k+1$. The vertical axis x_0 gives the position of m. ($x_i := \sqrt{m_i}|z_i|$.)

What can we say about the motion of m?

We add a simple, reflecting wall on the right.



The motion of m is described by the projection to the axis of the rail track of m. This is a projection from an $(k+1)$-dimensional space to a 1-dimensional space.

Temperature comes into play

Theorem

Suppose that:

- *the* m_i *have random initial velocities with mean* 0 *and variance* σ_i^2 *such that*
$$\beta = 1/(m_i \sigma_i^2)$$
is the same for all i*;*
- *the polygonal billiard in dimension* $k+1$ *is ergodic;*
- k *is very large;*
- m *is small relative to the* m_i*.*

Let $v_1, v_2, v_3, \ldots$ *be the velocities of* m *as it emerges from the interaction zone after each scattering event. Then the long term distribution of these velocities is approximately given by:*

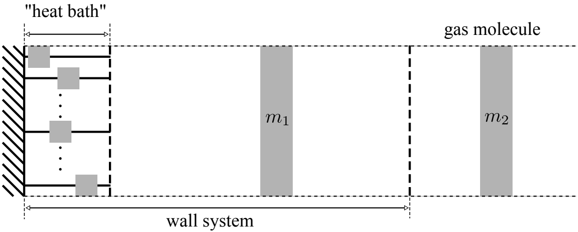
$$\varrho(v) = \beta m v \exp\left(-\beta \frac{mv^2}{2}\right).$$

Maxwell-Boltzmann distribution and temperature

- $\varrho(v)$ is the *Maxwell-Boltzmann* distribution of velocities of m.
- Equality of β for m and the m_i characterizes stationarity of the iterated scattering process.
- In physics textbooks
$$\beta = \frac{1}{\kappa T}$$
where κ is called the Boltzmann constant and T is the absolute temperature (of the wall).
- At equilibrium, the wall and m have the same temperature! (Thermal equilibrium.)
- The proof of the theorem is geometric. This is a theorem about geometry in very high dimensions!

On to building a heat engine!

But first, we wish to simplify the wall heat bath/thermostat.
Deterministic version:

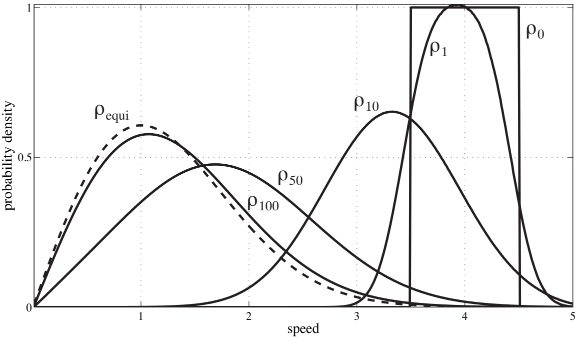


Probabilistic version:

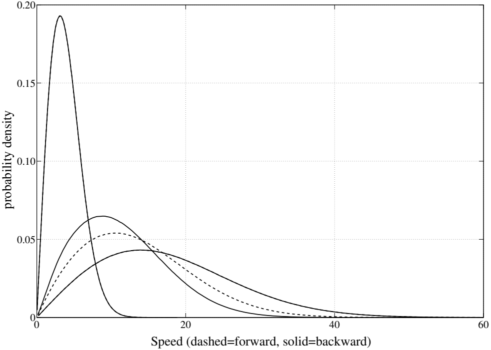
Assume that prior to each scattering of m_2, the velocity of m_1 is chosen randomly, with mean 0 and variance σ^2.

Approach to thermal equilibrium

This probabilistic model of collision can be iterated to show how thermal equilibrium is reached.

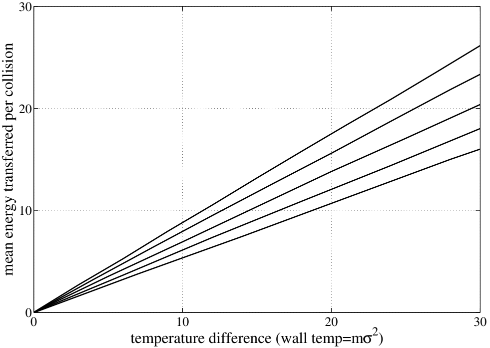


Two temperatures and heat flow



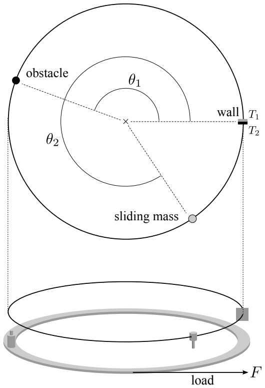
Relation between heat flow and temperature difference

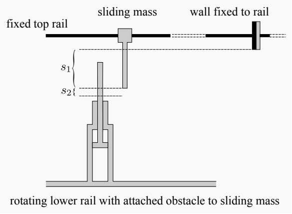
The heat flow Q is the mean energy transferred from the hot to the cold wall per collision with m_2.



$$Q = C(T_{\text{hot}} - T_{\text{cold}}).$$

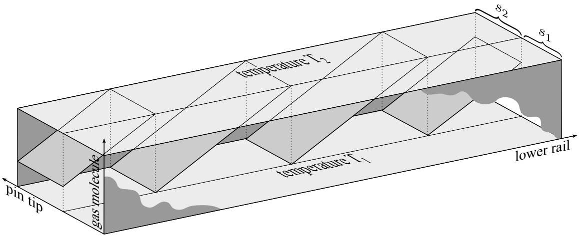
A minimalistic heat engine



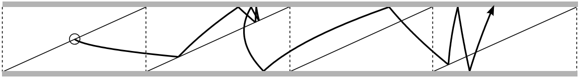


The engine's billiard representation

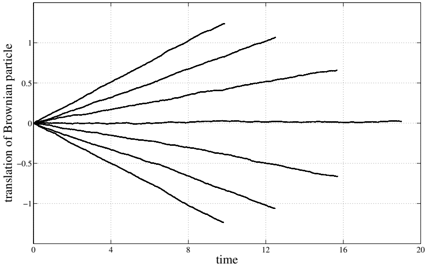
The billiard representation for the heat engine. The evolution of the engine is described by an accelerating billiard particle inside this channel.

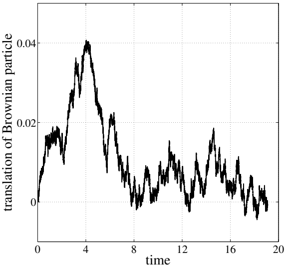


Side view of one trajectory.



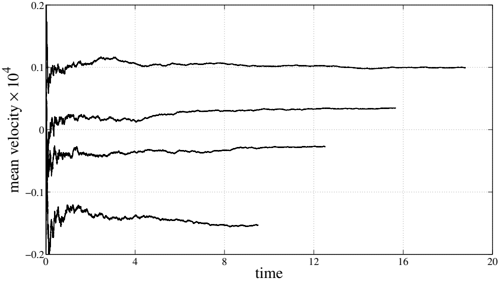
It works! (First, no load)





Left: position of the Brownian particle (zero load), as function of time. Parameters: mass of thermostat wall $m_0 = 10$, Brownian particle $m_1 = 100$, gas molecule $m_2 = 1$. Length of circular track $l = 10^{-4}$, the vertical axis measures translation along the track. Number of events $N = 10^6$. Temperatures, from middle graph to the top: $\sigma_1^2 = 1$ and $\sigma_2^2 = 1, 2, 4, 8$. For lower graphs the two parameters are reversed.

Mean rotation velocity, now with load

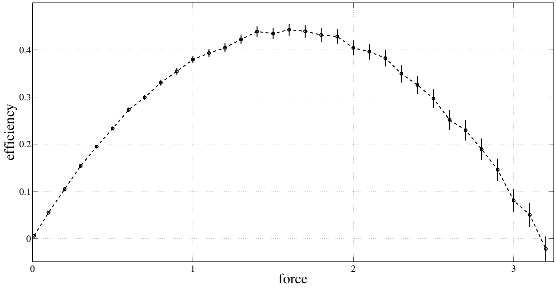


The working engine with a force load. The graphs show the mean velocity of the Brownian particle as a function of time. Force load = 1. From top to bottom: $\sigma_1^2 = 1, 2, 4, 8$.

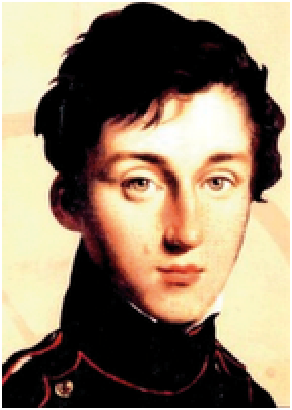
Efficiency. (Not much to brag about... but positive!)

The *efficiency* of the engine is the ratio

$$\frac{\text{mean mechanical work done by the engine (rotation against load)}}{\text{heat flow from hot source}}$$



A young Sadi Carnot



Nicolas Léonard Sadi Carnot (1 June 1796 - 24 August 1832). In his 1824 book *Reflections on the Motive Power of Fire*, he gave the first successful theoretical account of heat engines (Carnot cycle), laying the foundations for the second law of thermodynamics. (Wikipedia)