Dynamics and differential geometry of non-standard billiard models

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Dedicated to the memory of Professor Nikolai Chernov

Overview

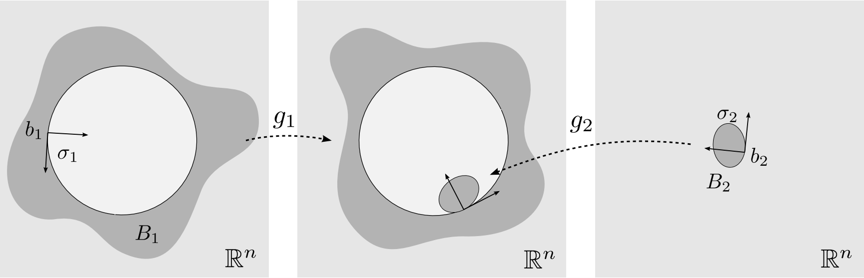
This is joint work with Chris Cox and Will Ward.

- Differential geometry of collisions of rigid bodies
- Structures on the boundary of the configuration manifold
- Classification of billiard boundary conditions
- Examples in dimension 2
- First steps in dynamics of non-standard 2-dimensional billiards.

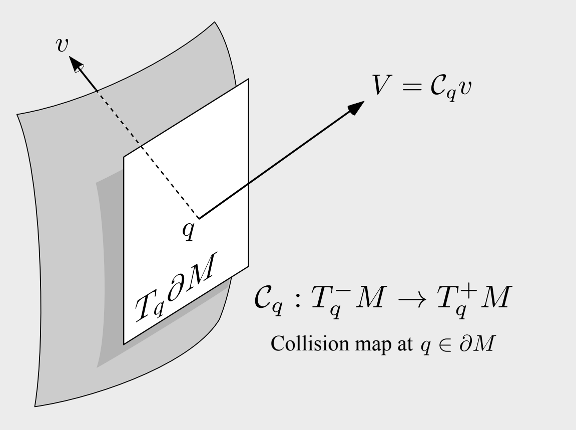
The last item: Chris Cox greatly extends earlier work by D.S. Broomhead and E. Gutkin (1993) about no-slip billiards in dimension 2.

Basic set-up: collisions of two rigid bodies

- $B_1, B_2 \subset \mathbb{R}^n$ material bodies (dimension n, smooth boundaries)
- $G = SE(n) = SO(n) \ltimes \mathbb{R}^n$ rigid motions in $\mathbb{R}^n$
- $M_0 = \{(g_1, g_2) \in G \times G : g_1(B_1) \cap g_2(B_2) = \emptyset\}$
- $M = \overline{M_0}$ the configuration manifold of the system
- ∂M set of collision configurations.



Collision maps



Standard example: mirror reflection.

$$\mathcal{C}_q v = v - 2\langle v, \mathbb{n}_q\rangle \mathbb{n}_q$$

Physically motivated requisites for $\mathcal{C}_q$

For $q \in \partial M$, a *(strict) collision map* at q is a map

$$\mathcal{C}_q : T_q^- M \to T_q^+ M$$

satisfying:

1. Linearity
2. Conservation of energy
3. Conservation of momentum
4. Time reversibility
5. Impulse at point of contact.

Definition

Space of collision maps at q will be denoted $\mathcal{CM}(q)$.

Definition

Boundary condition: choice of map $q \in \partial M \mapsto \mathcal{C}_q \in \mathcal{CM}(q)$.

The space of collision maps at $q \in \partial M$

System of two rigid bodies in $\mathbb{R}^n$.

Theorem (Cox, F., Ward, 2015)

The space of strict collision maps (at each q) in dimension n is isomorphic to

$$\begin{aligned}\mathcal{CM}(q) &\cong \left\{ C \in O(n-1) : C^2 = I \right\} \\ &= \bigcup_{k=0}^{n-1} \underbrace{O(n-1)/(O(n-k-1) \times O(k))}_{dim(\mathcal{CM}_k) = k(n-k-1)}\end{aligned}$$

where k is the roughness rank at q.

Definition

Roughness rank at q: dimension of the eigenspace of $\mathcal{C}_q$ for eigenvalue -1.

Mirror reflection: $k = 0$, $C = I$.

Dimension of component $\mathcal{CM}_k$

$$\dim \mathcal{CM}_k = \left[\begin{array}{l|ccccc} \quad\;\; k & 0 & 1 & 2 & 3 & 4 \\ n & & & & & \\ \hline 1 & 0 & & & & \\ 2 & 0 & 0 & & & \\ 3 & 0 & 1 & 0 & & \\ 4 & 0 & 2 & 2 & 0 & \\ 5 & 0 & 3 & 4 & 3 & 0 \end{array}\right]$$

The Lie algebra $\mathfrak{g}$ of G (notation)

- Elements of G will be written $g = (A, a) \in SO(n) \times \mathbb{R}^n$.
- Elements of $\mathfrak{g} = \mathfrak{so}(n) \times \mathbb{R}^n$ (semidirect sum) will be written (Z, z).
- For $q = (g_1, g_2) \in M$, $T_q M \cong \mathfrak{g} \oplus \mathfrak{g}$, with elements written as

$$(Z_1, z_1, Z_2, z_2).$$

- For $a, b \in \mathbb{R}^n$ define $a \wedge b \in \mathfrak{so}(n)$ by

$$(a \wedge b)u = (a \cdot u)b - (b \cdot u)a.$$

Remark

If the system is in *state* $(g_1, \xi_1; g_2, \xi_2)$ where $g_j = (A_j, a_j)$, $\xi_j = (Z_j, z_j)$ then

$$A_j(Z_j b + z_j)$$

is the velocity of material point $b \in B_j$ in the given state.

Special subbundles of $T(\partial M)$

For a boundary state $q = (g_1, g_2)$, $\xi = (Z_1, z_1, Z_2, z_2)$, with $g_j = (A_j, a_j)$, define:

R_1 : $\nu_1 \cdot (Z_1 b_1 + z_1) = \nu_2 \cdot (Z_2 b_2 + z_2)$

R_2 : $A_1(Z_1 b_1 + z_1) = A_2(Z_2 b_2 + z_2)$

R_3 : $Ad_{A_j} Z_j = W + \nu_j \wedge w_j$ for $W \in \mathfrak{so}(n)$ and $w_j \in T_{b_j} N_j$, $j = 1, 2$

R_4 : $Ad_{A_1} Z_1 = Ad_{A_2} Z_2$.

Then

$$T_q(\partial M) \cong \{\xi \in \mathfrak{g} \times \mathfrak{g} : R_1\}.$$

Now define the subspaces

$$\begin{aligned} \mathfrak{S}_q &\cong \{\xi \in \mathfrak{g} \times \mathfrak{g} : R_2\} \\ \mathfrak{R}_q &\cong \{\xi \in \mathfrak{g} \times \mathfrak{g} : R_2, R_3\} \\ \mathfrak{D}_q &\cong \{\xi \in \mathfrak{g} \times \mathfrak{g} : R_2, R_3, R_4\}. \end{aligned}$$

Special subbundles of $T(\partial M)$

$$\begin{aligned} \mathfrak{S} &: \text{ the no-slip subbundle} \\ \mathfrak{R} &: \text{ the rolling subbundle} \\ \mathfrak{D} &: \text{ the diagonal subbundle.} \end{aligned}$$

Remark

$\mathfrak{D}_q$ is the tangent space at $q = (g_1, g_2) \in \partial M$ to the orbit of the action of G on M on the left: $g(g_1, g_2) = (gg_1, gg_2)$.

Remark (Kinematic subbundles of $T(\partial M)$)

$$\mathfrak{D} \subset \mathfrak{R} \subset \mathfrak{S} \subset T(\partial M)$$

These subbundles do not depend on choice of mass distributions.

The Riemannian metric

- μ_j a positive (mass distribution) measure on B_j. Assume:

$$m_j := \mu_j(B_j) < \infty, \quad \int_{B_j} b \, d\mu_j(b) = 0 \text{ (= center of mass)}.$$

- $L_j = (l_{rs})$ is defined by

$$l_{rs} = \frac{1}{m_j} \int_{B_j} b_r b_s \, d\mu_j(b).$$

  We call L_j the *inertia matrix* of body B_j.

- For $Z \in \mathfrak{so}(n)$, define $\mathcal{L}_j(Z) = L_j Z + Z L_j$.

Definition (Kinetic energy Riemannian metric on M)

$$\langle u, v \rangle_q = \sum_j m_j \left[\frac{1}{2} \mathrm{Tr} \left(\mathcal{L}_j(Z_j^u) Z_j^{v\dagger} \right) + z_j^u \cdot z_j^v \right]$$

The Riemannian metric is G-invariant.

Momentum and Newton's equation

- For $u \in \mathfrak{g}$ define vector field $q \mapsto \tilde{u}_q \in T_q M$, $q = (g_1, g_2)$, by

$$\tilde{u}_q := \left.\frac{d}{dt}\right|_{t=0} e^{tu} q.$$

- Define the momentum map $\mathcal{P}^{\mathfrak{g}} : TM \to \mathfrak{g}^*$ by

$$\mathcal{P}^{\mathfrak{g}}(q, \dot{q})(u) = \langle \dot{q}, \tilde{u}_q \rangle_q.$$

- Define a force field as a bundle map $F : TM \to T^*M$.
- Newton's equation:

$$\frac{d}{dt}\mathcal{P}^{\mathfrak{g}}_j(q, \dot{q}) = R^*_{g_j} F_j.$$

- Impulsive forces:

$$\mathcal{P}^{\mathfrak{g}}_j(q, \dot{q}_+) - \mathcal{P}^{\mathfrak{g}}_j(q, \dot{q}_-) = \int_{t_-}^{t_+} R^*_{g_j} F_j \, ds = \text{ impulse at } t.$$

Impulses at a single point of contact

Let $q = (g_1, g_2) \in \partial M$ be a collision configuration. Let

$$(Z_1^{\pm}, z_1^{\pm}, Z_2^{\pm}, z_2^{\pm}) \in T_q M$$

be the post- $(+)$ and pre- $(-)$ collision velocities of the two rigid bodies.

Proposition (Collision map due to impulse at contact point)

Given pre-collision velocity $(Z_1^-, z_1^-, Z_2^-, z_2^-)$ *there exist* $u_1, u_2 \in \mathbb{R}^n$ *such that*

$$\begin{aligned} z_j^+ &= z_j^- + u_j \\ Z_j^+ &= Z_j^- + \mathcal{L}_j^{-1}(b_j \wedge u_j). \end{aligned}$$

Under conservation of linear momentum: $m_1 A_1 u_1 + m_2 A_2 u_2 = 0$.

Definition (Impulse subbundle)

The *impulse subbundle* of TM (over the base manifold ∂M) is defined for each $q \in \partial M$ by the subspace $\mathfrak{C}_q \subset T_q M$ such that

$$\mathfrak{C}_q = \left\{ ((\mathcal{L}_1^{-1}(b_1 \wedge u_1), u_1), (\mathcal{L}_2^{-1}(b_2 \wedge u_2), u_2)) : u_j \in \mathbb{R}^n, m_1 A_1 u_1 + m_2 A_2 u_2 = 0 \right\}.$$

Spelling out the conditions on $\mathcal{C}_q$

1. Linearity: $\mathcal{C}_q$ extends to a linear map $T_qM \to T_qM$.
2. Conservation of energy: $\mathcal{C}_q$ is an isometry for the Riemannian metric
$$\langle \mathcal{C}_q v, \mathcal{C}_q u\rangle_q = \langle v, u\rangle_q.$$
3. Conservation of momentum:
$$\mathcal{P}^{\mathfrak{g}}(q, \mathcal{C}_q v) = \mathcal{P}^{\mathfrak{g}}(q, v).$$
Equivalently, $\mathcal{C}_q$ restricts to the identity map on $\mathfrak{D}_q$.
4. Time reversibility: $\mathcal{C}_q^2 = \mathrm{Id}$ (a linear involution).
5. Impulse at the point of contact: for all $v \in T_q^- M$
$$\mathcal{C}_q v - v \in \mathfrak{C}_q.$$
Equivalently, $\mathcal{C}_q$ is the identity on $\mathfrak{C}_q^\perp$.

Generalized momentum conservation

Let $\mathbb{n}_q$ be the unit normal vector to ∂M at q, pointing into M.

Theorem

The impulse subspace $\mathfrak{C}_q$ is the orthogonal complement of the no-slip subspace $\mathfrak{S}_q$ and contains the unit normal vector $\mathbb{n}_q$. Therefore,

$$T_qM = \mathfrak{S}_q \oplus (\mathfrak{C}_q \ominus \mathbb{R}\mathbb{n}_q) \oplus \mathbb{R}\mathbb{n}_q$$

is an orthogonal direct sum.

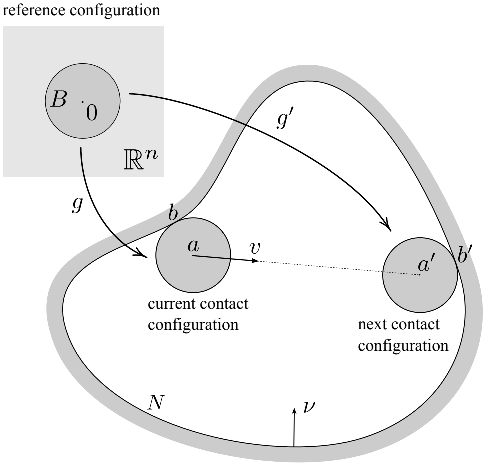
Thus momentum conservation and impulse at a single contact point are together equivalent to $\mathcal{C}_q$ being the identity on $\mathfrak{S}_q$.

Corollary

Strict collision maps are the linear isometric involutions of T_qM, $q \in \partial M$, that restrict to the identity on the no-slip subspace.

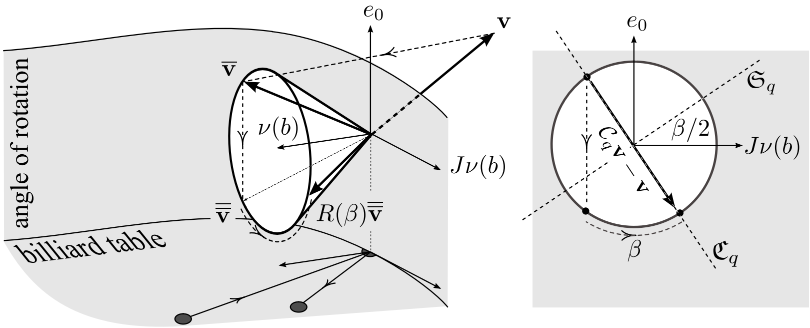
The fact that the collision map restricts to the identity on $\mathfrak{S}_q$ may be interpreted as a *generalized momentum conservation*.

Spherical billiard balls



In dimension 2 only two possibilities for collision maps.

In dimension 2, either standard mirror reflection or:



$\beta = \arccos(1/3)$

Invariant measure

- Constant energy submanifold at boundary:

$$N^{\mathcal{E}} = \left\{ (q, v) \in TM : q \in \partial M, \frac{1}{2}\|v\|^2 = \mathcal{E} \right\}.$$

- $\omega =$ canonical symplectic form on $N^{\mathcal{E}} \setminus T(\partial M)$.

Theorem

Suppose the field $q \in S \mapsto \mathcal{C}_q$ is piecewise smooth and parallel (where smooth) with respect to the Levi-Civita connection. Then

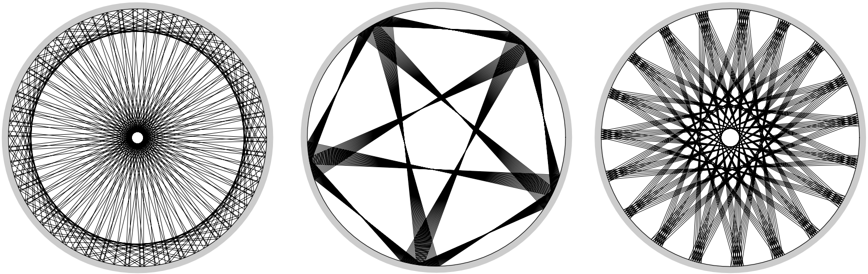
$$\Omega = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-1}$$

is, up to sign, invariant under the billiard map.

Corollary

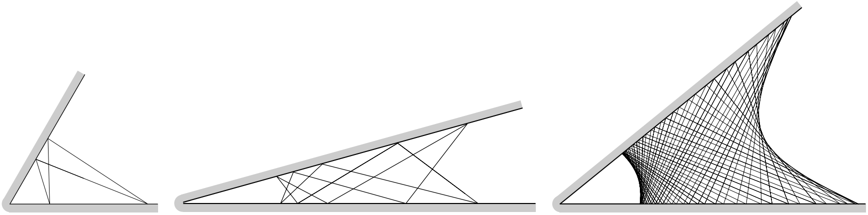
Rough billiards in dimension 2 preserve the canonical billiard measure.

Examples: Circular table and disc



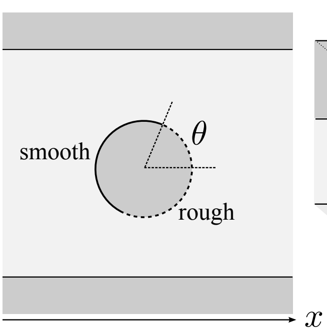
Three orbit segments with different initial conditions for the motion of the center of mass of a disc in a circular billiard table with rough contact.

Wedge tables and disc

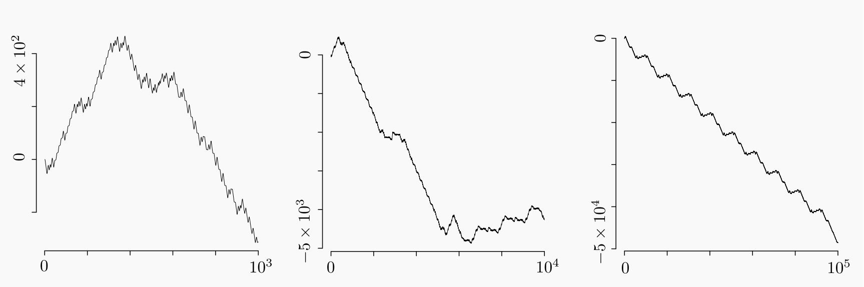


Motion of a disc in wedge shaped table. The typical segment of trajectory (motion of the center of mass only) is shown on the right. The two on the left are periodic.

Example of boundary condition

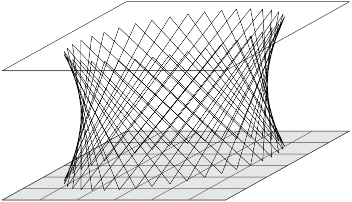


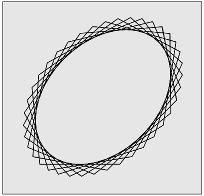
Disc between two parallel lines in $\mathbb{R}^2$



Single orbit of the motion of a disc between parallel plates in dimension 2.Half of the boundary circle is rough and the other half is smooth. The horizontal axis indicates the step number (time). The vertical axis gives the distance of the center of mass along the length of the channel.

Ball between two parallel plates in $\mathbb{R}^3$; rough rank 2





Orbit segment for the motion of the center of mass of a ball bouncing between two parallel plates in dimension 3 with rough rank equal to 2 (maximal). Note that orbits are bounded.