Geometry and dynamics of non-standard billiard systems

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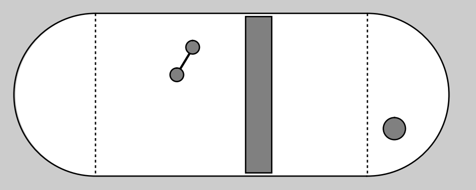
Overview

Joint work with **Chris Cox** and **Hong-Kun Zhang**

Definition (General billiard system)
Mechanical system made of rigid moving parts that interact through collisions.



Above: example of a billiard system with 7 degrees of freedom.

Overview

Question

What are the physically well-motivated ways to specify outcome of collisions?

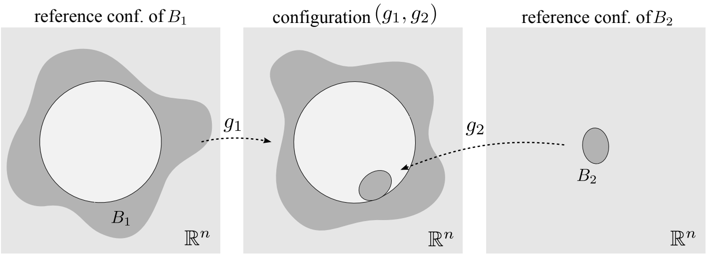
- Geometric/physical background on classification of rigid collisions
- No-slip billiards in dimension 2

Question

What kind of dynamics do we get for no-slip billiards?

- Compare and contrast no-slip billiards with standard type. (Examples)
- Extending results by Broomhead and Gutkin (1993): polygonal billiards
- Extending results by Wojtkowski (1994): linear stability for the Sinai billiard

Collisions of rigid bodies (geometry and physics)



- $B_1, B_2 \subset \mathbb{R}^n$ material bodies (dimension n, smooth boundaries)
- $G = SE(n) = SO(n) \ltimes \mathbb{R}^n$ rigid motions in $\mathbb{R}^n$
- $M_0 = \{(g_1, g_2) \in G \times G : g_1(B_1) \cap g_2(B_2) = \emptyset\}$
- $M = \overline{M_0}$ the configuration manifold of the system
- ∂M set of contact (or collision) configurations.

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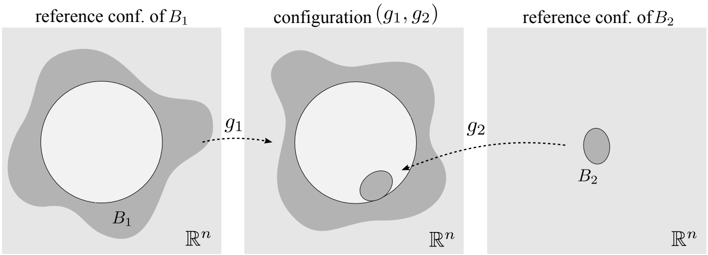

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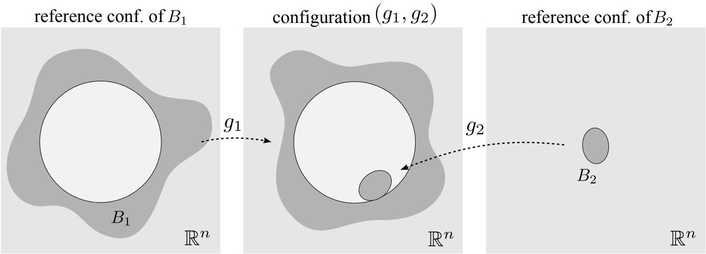

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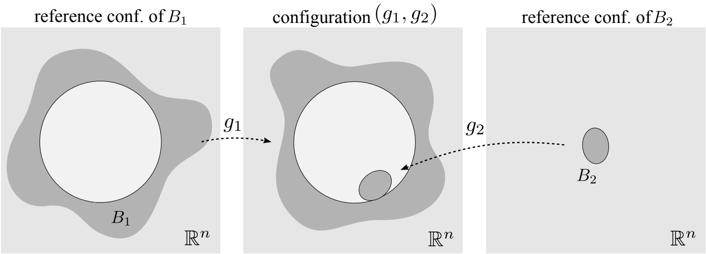

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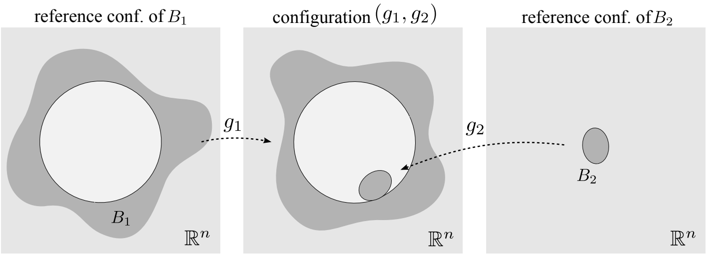

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The Lie algebra $\mathfrak{g}$ of G (notation)

- Elements of G: $g = (A, a) \in SO(n) \times \mathbb{R}^n$.
- Elements of $\mathfrak{g} = \mathfrak{so}(n) \times \mathbb{R}^n$ (semidirect sum): (Z, z).
- For $q = (g_1, g_2) \in M$, $T_qM \cong \mathfrak{g} \oplus \mathfrak{g}$, with elements written as

$$u = (Z_1, z_1, Z_2, z_2).$$

- For $a, b \in \mathbb{R}^n$ define $a \wedge b \in \mathfrak{so}(n)$ by

$$(a \wedge b)x = (a \cdot x)b - (b \cdot x)a$$

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The Riemannian metric

- μ_j a positive (mass distribution) measure on B_j ($j = 1, 2$). Assume:

$$m_j := \mu_j(B_j), \quad \int_{B_j} b \, d\mu_j(b) = 0 \text{ (= center of mass).}$$

  ($m_j = \infty$ also of interest.)

- $L_j = (l_{rs})$ is defined by

$$l_{rs} = \frac{1}{m_j} \int_{B_j} b_r b_s \, d\mu_j(b).$$

  We call L_j the *inertia matrix* of body B_j.

- Define $\mathcal{L}_j(Z) = L_j Z + Z L_j$ for $Z \in \mathfrak{so}(n)$.

Definition (Kinetic energy Riemannian metric on M)

$$\langle u, v \rangle_q = \sum_j m_j \left[\frac{1}{2} \mathrm{Tr} \left(\mathcal{L}_j(Z_j^u) Z_j^{v\dagger} \right) + z_j^u \cdot z_j^v \right] \text{ for } u, v \in T_q M$$

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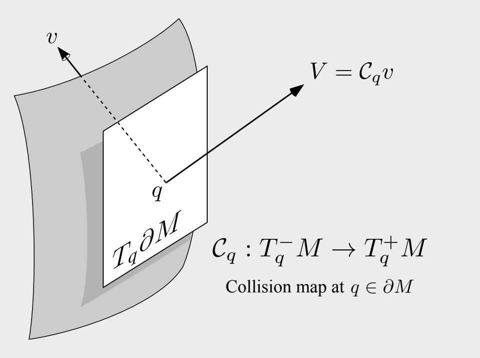
Collision maps

- Assume: motion between collisions is free (no potentials; geodesic).



- Boundary condition: Choice of $\mathcal{C}_q$ for each $q \in \partial M$.
- Standard example: mirror reflection: $\mathcal{C}_q v = v - 2\langle v, \mathbb{n}_q \rangle \mathbb{n}_q$.

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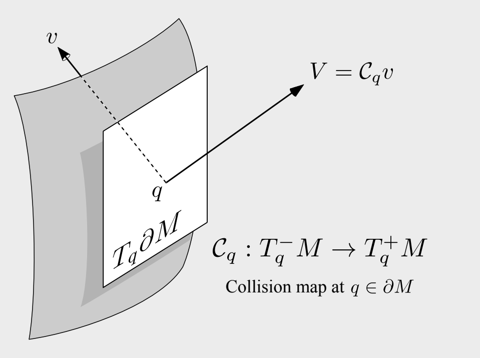

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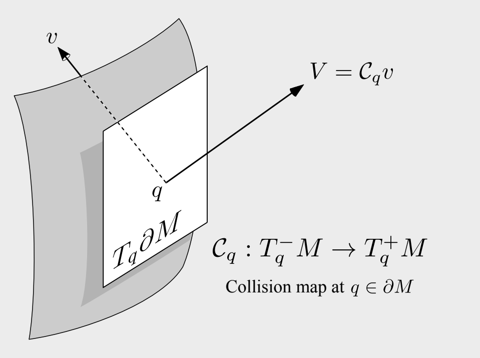

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Physically motivated requisites for $\mathcal{C}_q$

Definition (Strict collision maps)

For $q \in \partial M$, a *(strict) collision map* at q is a map

$$\mathcal{C}_q : T_q^- M \to T_q^+ M$$

satisfying:

1. Linearity ($\mathcal{C}_q$ is restriction of a linear map on $T_q M$)
2. Conservation of energy ($\mathcal{C}_q$ is an orthogonal map)
3. Conservation of momentum (vacuous if one body is kept fixed.)
4. Time reversibility ($\mathcal{C}_q$ is an involution)
5. Impulse at point of contact. (More on this soon.)

Remark: Momentum conservation

Definition (Momentum map)

The momentum map (for both bodies free) is the map $\mathcal{P} : TM \to \mathfrak{g}^*$ such that

$$\mathcal{P}(q, v)(\xi) := \langle v, \tilde{\xi}_q \rangle_q$$

for $(q, v) \in TM$ and $\xi \in \mathfrak{g}$, where $\mathfrak{g}^*$ is the dual to $\mathfrak{g}$ and

$$\tilde{\xi}_q := \left. \frac{d}{dt} \right|_{t=0} e^{t\xi} q.$$

If bodies not necessarily free, $\mathcal{P}$ maps into $\mathfrak{h}^*$ where $\mathfrak{h}$ is the Lie algebra of the symmetry group $H \subset G$.

<u>Momentum conservation</u>:

$$\mathcal{P}(q, \mathcal{C}_q v) = \mathcal{P}(q, v).$$

Special subbundles of $T(\partial M)$

Boundary state $q = (g_1, g_2)$, $\xi = (Z_1, z_1, Z_2, z_2)$, $g_j = (A_j, a_j)$.

$$T_q(\partial M) \cong \{\xi \in \mathfrak{g} \times \mathfrak{g} : \nu_1 \cdot (Z_1 b_1 + z_1) + \nu_2 \cdot (Z_2 b_2 + z_2) = 0\}.$$

($b_i \in B_i$ is the contact point; ν_i is unit normal to ∂B_i at b_i.)

Definition (No-slip subbundle)

The no-slip subbundle of $T(\partial M)$ is defined for each $q \in \partial M$ by

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Definition (Impulse subbundle)

The impulse subbundle of TM is defined for each $q \in \partial M$ by

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Classification of collision maps

Let $\mathbb{n}_q$ be the unit normal vector to ∂M at q, pointing into M.

Theorem (C. Cox, F. 2015)

The impulse subspace $\mathfrak{C}_q$ is the orthogonal complement of the no-slip subspace $\mathfrak{S}_q$ and contains the unit normal vector $\mathbb{n}_q$. Therefore,

$$T_qM = \mathfrak{S}_q \oplus (\mathfrak{C}_q \ominus \mathbb{R}\mathbb{n}_q) \oplus \mathbb{R}\mathbb{n}_q$$

is an orthogonal direct sum.

Note: conservation of momentum $\Leftrightarrow$ $\mathcal{C}_q$ is the identity on a certain (canonical) subspace of $\mathfrak{S}_q$ for each $q \in \partial M$.

Theorem (C. Cox, F. 2015)

Strict collision maps are the linear orthogonal involutions of T_qM, $q \in \partial M$, that restrict to the identity on the no-slip subspace $\mathfrak{S}_q$ and map $\mathbb{n}_q$ to $-\mathbb{n}_q$.

The upshot

There exists a canonical (given the Riemannian metric) subbundle

$$q \mapsto \overline{\mathfrak{C}}_q := \mathfrak{C}_q \ominus \mathbb{R}\mathfrak{n}_q \subset T_q(\partial M).$$

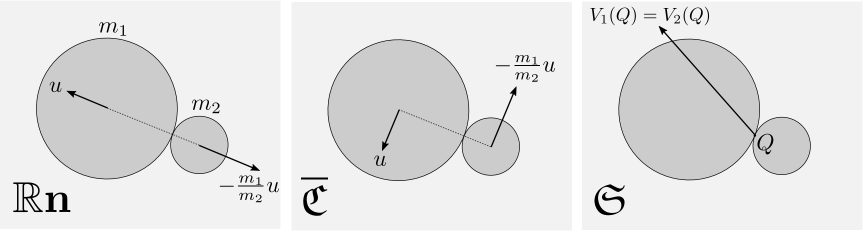
For each boundary configuration $q \in \partial M$,

1. $\{$Collision maps at $q\} \leftrightarrow \{$subspaces $\mathfrak{U}$ of $\overline{\mathfrak{C}}_q\}$, $\quad \mathfrak{U} \leftrightarrow \mathcal{C}_q = I|_{\mathfrak{U}^\perp} - I|_{\mathfrak{U}}$.
2. Mirror reflection $\leftrightarrow \mathfrak{U} = 0$
3. In $\mathbb{R}^2$ we have $\dim(\overline{\mathfrak{C}}_q) = 1$.
4. Thus only two possibilities in dimension 2: <u>mirror reflection</u> and <u>no-slip</u>.

Note (Billiard systems)

Often assume B_1 fixed (table); B_2 is sphere with symmetric mass distribution.

The orthogonal decomposition for spherical bodies



Typical elements of $\overline{\mathfrak{C}}$ are

$$\left(\left(\frac{R_1}{2\lambda_1}\nu_1(Q) \wedge u_1, u_1\right), \left(\frac{R_2}{2\lambda_2}\nu_1(Q) \wedge u_2, u_2\right)\right)$$

where

$$m_1 u_1 + m_2 u_2 = 0 \quad \text{and} \quad \nu_j(Q) \cdot u_j = 0.$$

Invariant measure

- Constant energy submanifold at boundary:

$$N^{\mathcal{E}} = \left\{ (q, v) \in TM : q \in \partial M, \frac{1}{2}\|v\|^2 = \mathcal{E} \right\}.$$

- $\omega =$ canonical symplectic form on $N^{\mathcal{E}} \setminus T(\partial M)$.

Theorem

If $q \in S \mapsto \mathfrak{C}_q$ is piecewise smooth and parallel (where smooth) then

$$\Omega = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-1}$$

is, up to sign, invariant under the billiard map.

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Corollary

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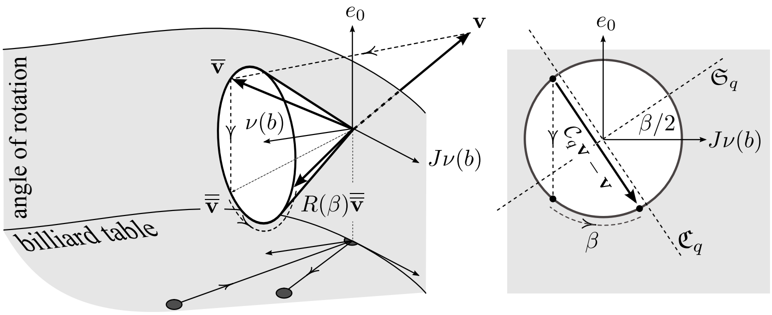
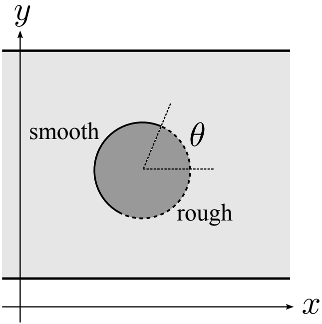
No-slip billiards in dimension 2 preserve the canonical billiard measure.

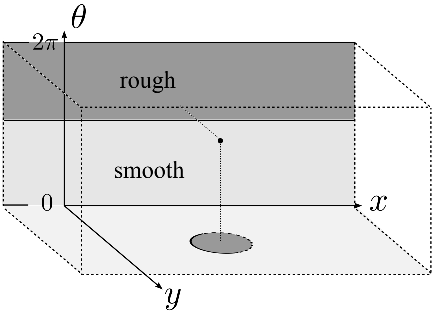
In dimension 2, either standard mirror reflection or:



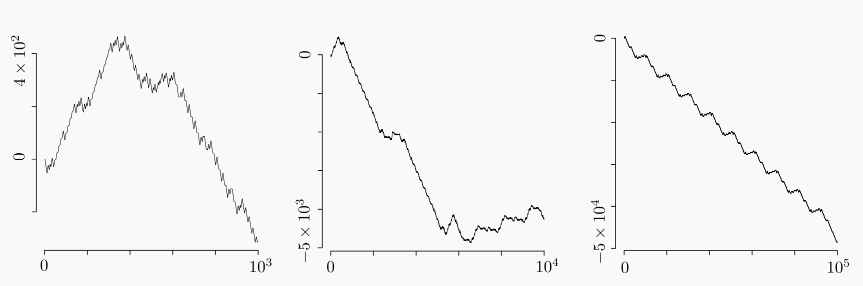
$\beta = \arccos(1/3)$ for uniform mass distribution on disc.

Example of non-standard boundary condition



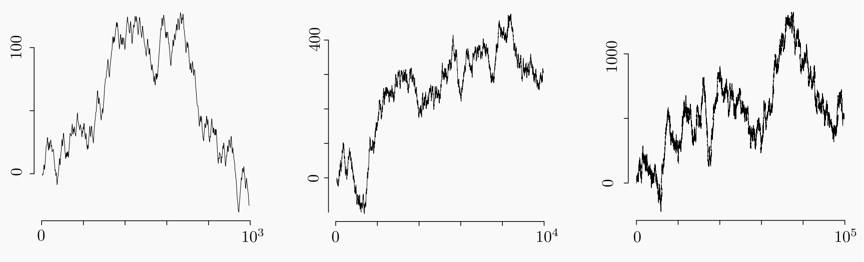


One trajectory of previous example



Single orbit of the motion of a disc between parallel plates in dimension 2. Half of the boundary circle is rough and the other half is smooth. The horizontal axis indicates the step number (time). The vertical axis gives the distance of the center of mass along the axis of the channel.

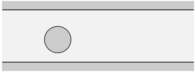
Random boundary condition



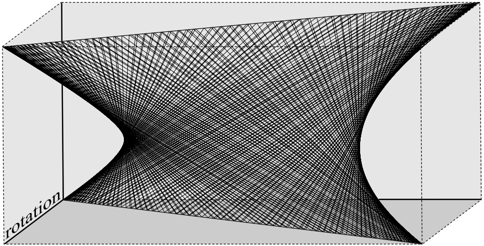
Single orbit of the motion of a disc between parallel plates in dimension 2. Each boundary point of the disc is smooth or rough with probability 1/2.

Actual billiard system is in dimension 3

- Actual billiard system is in 3D: disc center of mass and angle of rotation.



- Below is typical trajectory of no-slip billiard between two parallel lines.



For this example, orbits are bounded (Broomhead and Gutkin, 93).

- In examples below, only motion of center of mass is shown.

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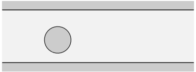

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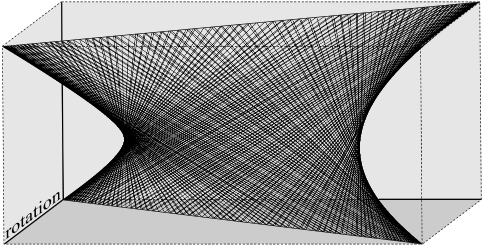

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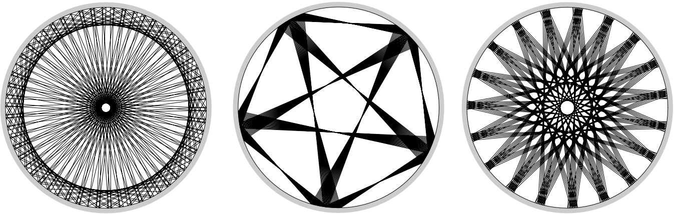
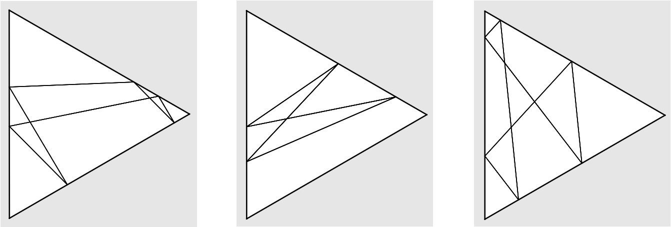
Example: Circular table and disc



Caustics of no-slip circular billiard consist of two circle components.

Example: Equilateral triangle



Proposition (C. Cox., R.F., 2016)

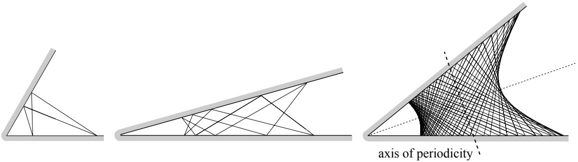
The no-slip equilateral triangle is periodic for all initial conditions. Every orbit is of one of the above three types or their degeneracies.

No-slip billiards on polygons are never ergodic

Theorem (C. Cox, R.F., H.-K. Zhang, 2016)

No-slip billiards on polygons are never ergodic.

- Trajectories on wedge billiards that do not escape to infinity in finitely many steps are bounded. (This extends main result by Broomhead and Gutkin (1993) on no-slip billiards on strips.)



- Initial conditions close to axis of periodicity (next slide) yield orbits contained in a band arbitrarily narrow around that axis.
- Dense set of wedge angles giving (all) periodic trajectories. (Classified.)

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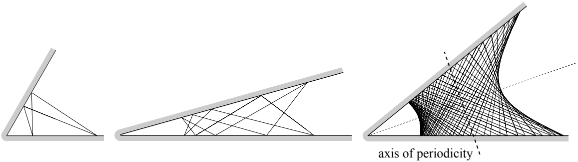

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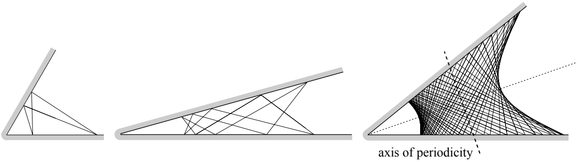

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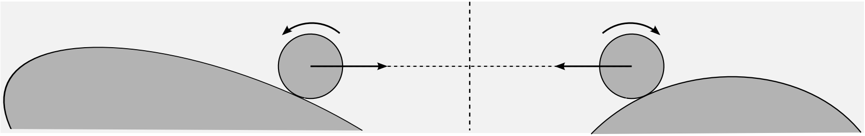
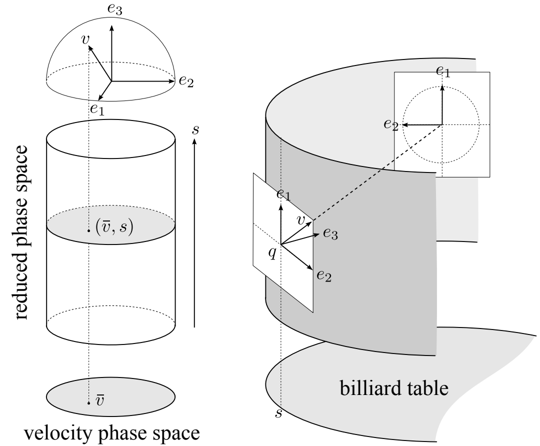
The period-2 periodic orbit

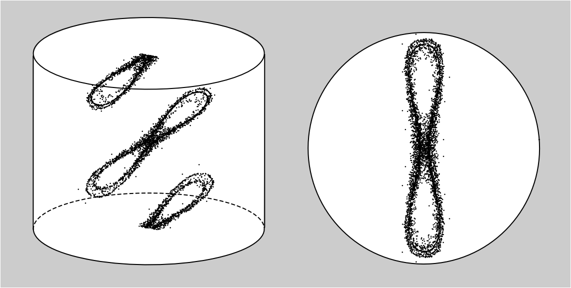


Reduced phase space and the billiard map - I

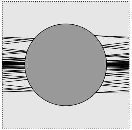


- Reduced phase space: 4-dim phase space modulo rotation.
- This is a solid torus (if ∂B is connected).

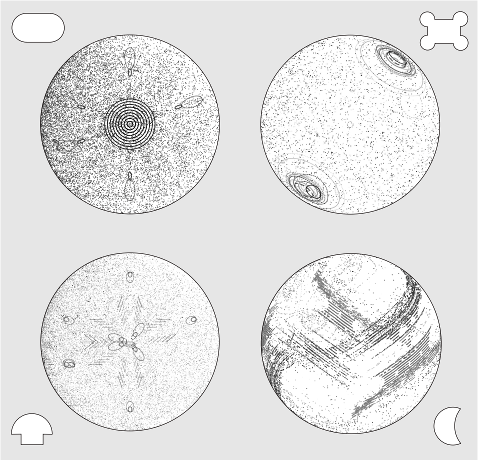
Reduced phase space and the billiard map - II



- Velocity phase portrait: projection to velocity unit disc.
- Billiard map on reduced phase space is volume preserving.
- Above orbit is from no-slip Sinai billiard. More later.



Examples of velocity phase portrait



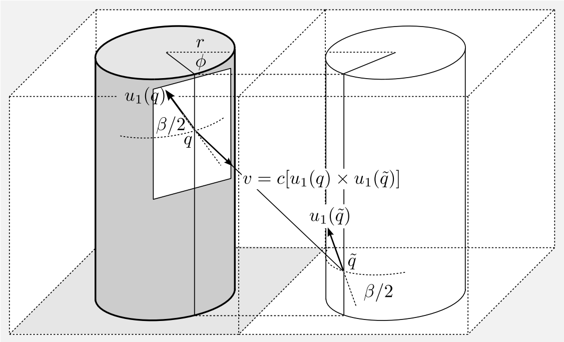
A question

Question

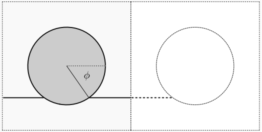
Are there any ergodic no-slip billiards?

- More precisely: are there examples of billiard shapes for which the billiard map on the 3-dimensional reduced phase space is ergodic for the canonical invariant measure (volume form)?
- No good candidates yet.
- Some apparently chaotic behavior nevertheless (numerics).

The no-slip Sinai billiard



- Figure shows one period 2 orbit for each ϕ.



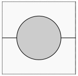
- This orbit is linearly stable if scatterer curvature is small enough.

Linear stability and the Sinai billiard

Conjecture

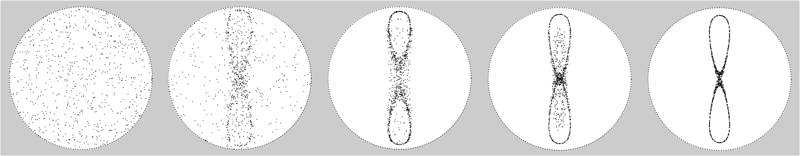
The no-slip Sinai billiard is not ergodic.

- (Wojtkowski 1994) The periodic trajectory



  is elliptic if radius greater than $1/3$.

- Sharp "chaotic" to "stable" transition around cut-off radius.

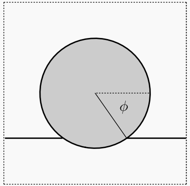


0.3240 0.3244 0.3252 0.3267 0.3272

Linear stability and the Sinai billiard

We show:

- As $\phi \to \pi/2$ cut-off radius $\to 0$: $R_{\text{critical}} = \frac{\cos\phi}{3+2(\cos\phi - \cos^2\phi)}$.



- Linearly stable periodic orbits always exist for no-slip Sinai billiards.
- Proof of non-ergodicity requires KAM argument (not symplectic!)
- Same argument should apply to all dispersing no-slip billiards.