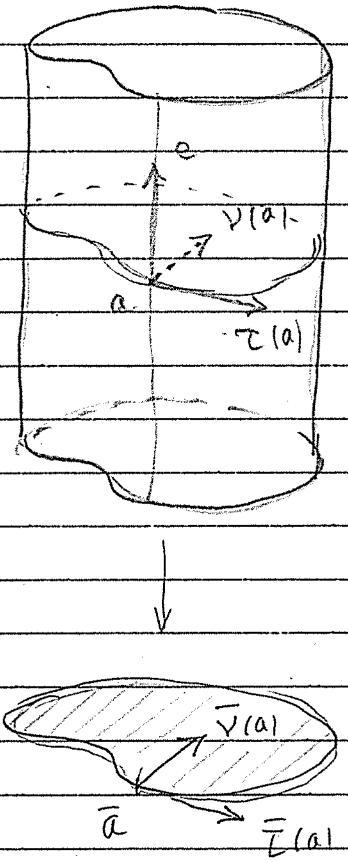
No-slip billiards in cylinders

Cylinder: $\underbrace{\mathcal{B}}_{\text{billiard domain in } \mathbb{R}^2} \times \mathbb{R} = M$



Phase space: $TM\big|_{\partial M} \times SO(3) =: \mathcal{N}$

states $(a, u, U) \in \partial M \times T_aM \times SO(3)$

metric on $\mathcal{N}$: $\frac{1}{m}\langle \xi, \eta\rangle_a = \frac{R^2\gamma^2}{2} Tr\left(U_\xi U_\eta^t\right) + u_\xi \cdot u_\eta$

Adapted frame at a $\quad \sigma_a : \mathbb{R}^6 \to T_aM \times SO(3)$

$$\sigma_a : \begin{cases} e_1 \longmapsto (1/R\gamma)\, \tau(a) \wedge \nu(a) \\ e_2 \longmapsto \tau(a) \\ e_3 \longmapsto \nu(a) \\ e_4 \longmapsto e \\ e_5 \longmapsto (1/R\gamma)\, \nu(a) \wedge e \\ e_6 \longmapsto (1/R\gamma)\, e \wedge \tau(a) \end{cases}$$

σ_a is a linear isometry: $(\mathbb{R}^6, \cdot) \to \left(T_aM \times SO(3), \frac{1}{m}\langle\cdot,\cdot\rangle\right)$

Define $\quad c = \cos\beta = \frac{1-\gamma^2}{1+\gamma^2}, \quad s = \sin\beta = \frac{2\gamma}{1+\gamma^2}$

Collision map $\quad C_a : \begin{pmatrix} u^- \\ U^- \end{pmatrix} \longmapsto \begin{pmatrix} u^+ \\ U^+ \end{pmatrix} = \begin{pmatrix} c u^- - \frac{s}{\gamma} u^- \cdot \nu(a)\, \nu(a) + R\gamma s\, U^- \nu(a) \\ \frac{s}{R\gamma} \nu(a) \wedge u^- + U^- - \frac{s}{\gamma} \nu(a) \wedge U^- \nu(a) \end{pmatrix}$

$C = [C_a] = \sigma_a^{-1} C_a \sigma_a : \mathbb{R}^6 \to \mathbb{R}^6$

$[C_a]_{ij} = e_i \cdot \left(\sigma_a^{-1} C_a \sigma_a e_j\right) = \frac{1}{m}\left\langle \sigma_a e_i, \sigma_a e_j \right\rangle$

②

$$[C_a]_{i1} = \frac{1}{m}\langle \sigma e_i, C_a (1/R\gamma)\, \tau(a)\wedge\nu(a)\rangle = \frac{1}{m}\langle \sigma e_i, \left(-s\tau, -c\,\frac{\tau\wedge\nu}{R\gamma}\right)\rangle$$

$$\frac{1}{R\gamma}\left(R\gamma s \underbrace{(\tau\wedge\nu)\nu}_{-\tau},\ \tau\wedge\nu - \frac{s}{\gamma}\,\nu\wedge\underbrace{(\tau\wedge\nu)\nu}_{-\tau} \right)$$

$$= \frac{1}{m}\langle \sigma e_i, -s\,\sigma e_2 - c\,\sigma e_1\rangle = -s\, e_i\cdot e_2 - c\, e_i\cdot e_1$$

$$[C_a]_{i2} = \frac{1}{m}\langle \sigma e_i, C_a\tau\rangle = \frac{1}{m}\langle \sigma e_i, \left(c\tau, -s\,\frac{\tau\wedge\nu}{R\gamma}\right)\rangle$$

$$= \frac{1}{m}\langle \sigma e_i, c\,\sigma e_2 - s\,\sigma e_1\rangle = c\, e_i\cdot e_2 - s\, e_i\cdot e_1$$

$$[C_a]_{i3} = \frac{1}{m}\langle \sigma e_i, C_a\nu\rangle = \frac{1}{m}\langle \sigma e_i, (\underbrace{c\nu - \tfrac{s}{\gamma}\nu}_{-\nu}, 0)\rangle = \frac{1}{m}\langle \sigma e_i, -\sigma e_3\rangle$$

$$= -e_i\cdot e_3$$

$$[C_a]_{i4} = \frac{1}{m}\langle \sigma e_i, C_a e\rangle = \frac{1}{m}\langle \sigma e_i, \left(ce, s\,\frac{\nu\wedge e}{R\gamma}\right)\rangle = \frac{1}{m}\langle \sigma e_i, c\sigma e_4 + s\sigma e_5\rangle$$

$$= c\, e_i\cdot e_4 + s\, e_i\cdot e_5 .$$

$$[C_a]_{i5} = \frac{1}{m}\langle \sigma e_i, C_a \frac{\nu\wedge e}{R\gamma}\rangle = \frac{1}{m}\langle \sigma e_i, \left(se, -c\,\frac{\nu\wedge e}{R\gamma}\right)\rangle = \frac{1}{m}\langle \sigma e_i, s\sigma e_4 - c\sigma e_5\rangle$$

$$= s\, e_i\cdot e_4 - c\, e_i\cdot e_5$$

$$[C_a]_{i6} = \frac{1}{m}\langle \sigma e_i, C_a \frac{e\wedge\tau}{R\gamma}\rangle = \frac{1}{m}\langle \sigma e_i, (0, \underbrace{\tfrac{e\wedge\tau}{R\gamma}}_{\sigma e_6})\rangle = e_i\cdot e_6$$

$$
C = [C_a] = \begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & -c & -s & 0 & 0 & 0 & 0 \\
2 & -s & c & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & -1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & c & s & 0 \\
5 & 0 & 0 & 0 & s & -c & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

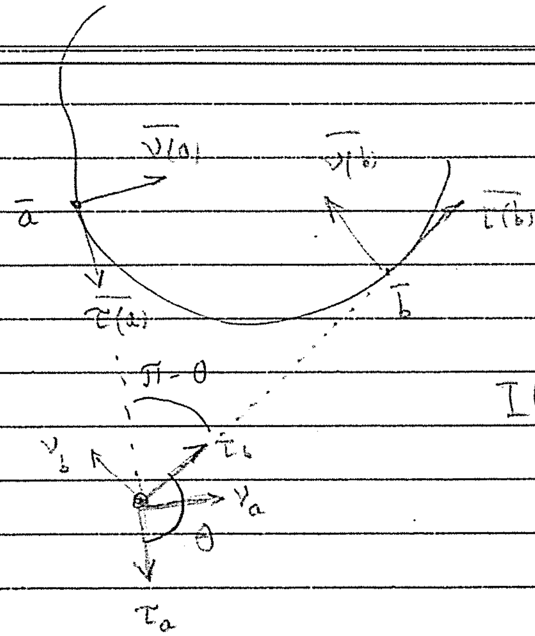
<u>Change of frame from</u> a to $b \in \partial M$

$$I(a,b) := \sigma_b^{-1}\sigma_a$$

$$I(a,b)_{ij} = e_i \cdot (\sigma_b^{-1}\sigma_a e_j) = \frac{1}{m}\langle \sigma_b e_i, \sigma_b e_j \rangle$$

<u>Note</u>: $\boxed{\frac{1}{2}\mathrm{Tr}\left((a \wedge b)(c \wedge d)^t\right) = b\cdot d\; a\cdot c - a\cdot d\; b\cdot c}$

$$
I(a,b) = \begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \tau(b)\cdot\tau(a) & \tau(b)\cdot\nu(a) & 0 & 0 & 0 \\
3 & 0 & \nu(b)\cdot\tau(a) & \nu(b)\cdot\nu(a) & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & \nu(b)\cdot\nu(a) & -\nu(b)\cdot\tau(a) \\
6 & 0 & 0 & 0 & 0 & -\tau(b)\cdot\nu(a) & \tau(b)\cdot\tau(a)
\end{array}
$$



$$\tau_b \cdot \tau_a = \cos\theta$$

$$\tau_b \cdot \nu_a = \cos\left(\theta - \tfrac{\pi}{2}\right) = \sin\theta$$

$$\nu_b \cdot \tau_a = \cos\left(\theta + \tfrac{\pi}{2}\right) = -\sin\theta$$

$$\nu_b \cdot \nu_a = \cos\theta$$

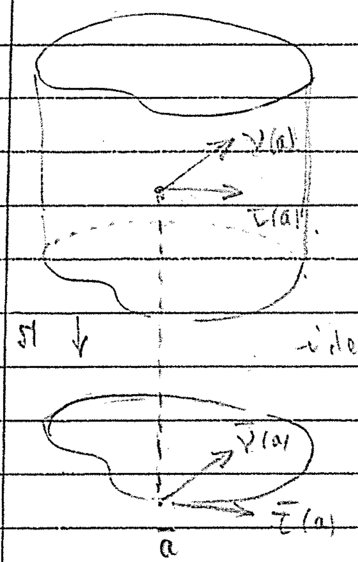
$$I(a,b) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_\theta & S_\theta & 0 & 0 & 0 \\ 0 & -S_\theta & C_\theta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_\theta & S_\theta \\ 0 & 0 & 0 & 0 & -S_\theta & C_\theta \end{array}\right]$$

$$\Theta = \theta(a,b) \qquad I(a,b) = \left[\begin{array}{c|c} \begin{array}{c|c} 1 & 0 \quad 0 \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \boxed{R_{-\theta}} \end{array} & O \\ \hline O & \begin{array}{c|c} 1 & 0 \quad 0 \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \boxed{R_{-\theta}} \end{array} \end{array}\right]$$

<u>Define projection</u> $\Pi : \mathcal{M} = TM|_{\partial M} \times SO(3) \to TB|_{\partial B} \times SO(2)$

$$\Pi(a, u, U) = (\bar{a}, \bar{u}, \bar{U}), \quad \bar{U} = \pi U|_{e^\perp}$$

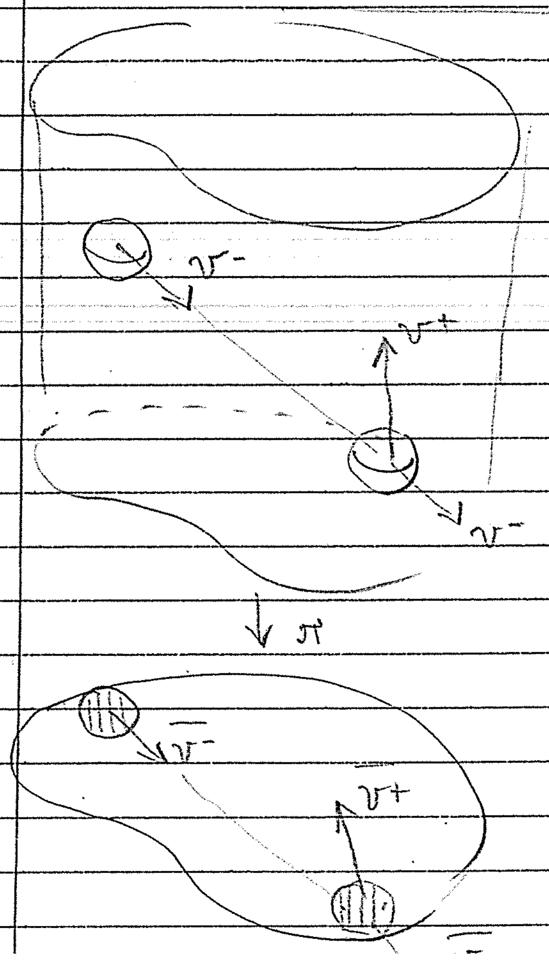
$$\Pi_a(u, U) = (\bar{u}, \bar{U})$$



identify $\tau = \bar{\tau}$
$\nu = \bar{\nu}$

collision map of projected plane billiard

<u>Note</u> $\boxed{\Pi_a C_a = C_{\bar{a}} \Pi_a}$



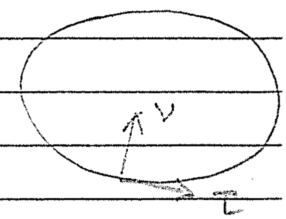
Trajectories of 3-D billiard project down to $\mathcal{B}$ to trajectories of 2-D billiard (for disc having same (mass and) <u>moment of inertia</u> as spherical particles.)

Vertical motion (along e)

a_n = position of n^{th} collision

t_n = time interval of free motion from n^{th} to $n+1st$ coll

$\begin{bmatrix} v_n \\ U_n \end{bmatrix}$ = post-coll. velocity at coll. n



$a_n = a_{n-1} + t_{n-1} v_{n-1}$

$\bar{a}_n = \bar{a}_{n-1} + t_{n-1} \bar{v}_{n-1}$ proj. to $\mathcal{B}$.

$h_n = a_n \cdot e$ vertical proj.

$h_n = h_0 + \sum_{j=0}^{n-1} t_j \sigma_j$

Define: $w_n = R\gamma U_n e$, $\sigma_n = v_n \cdot e$, $\nu_n = \nu(a_n)$

From collision equations; (Note: $w_n \in e^{\perp}$)

$w_n = w_n \cdot \tau_n \tau_n + w_n \cdot \nu_n \nu_n$

$$\begin{cases} \sigma_n = c\,\sigma_{n-1} - s\, w_{n-1} \cdot \nu_n \\ w_n = -s\,\sigma_{n-1} \nu_n + \underbrace{w_{n-1} - \frac{s}{\gamma} w_{n-1} \cdot \nu_n \nu_n}_{w_{n-1} \cdot \tau_n \tau_n + \left(1 - \frac{s}{\gamma}\right) w_{n-1} \cdot \nu_n \nu_n} \end{cases}$$

$$\underbrace{\left(1 - \frac{s}{\gamma}\right)}_{\frac{1+\gamma^2 - 2}{1+\gamma^2} = -\frac{1-\gamma^2}{1+\gamma^2} = -c}$$

$w_n = -s\,\sigma_{n-1} \nu_n - c\, w_{n-1} \cdot \nu_n \nu_n + w_{n-1} \cdot \tau_n \tau_n$

$$\begin{bmatrix} \sigma_n \\ \\ w_n \end{bmatrix} = \underbrace{\begin{bmatrix} c & -s\,\nu_n^t \\ \\ -s\,\nu_n & -c\,\nu_n\nu_n^t + \tau_n\tau_n^t \end{bmatrix}}_{\mathcal{A}_n} \begin{bmatrix} \sigma_{n-1} \\ \\ w_{n-1} \end{bmatrix}$$

Note: $\mathcal{A}_n$ is a symmetric orthogonal matrix:

$$\mathcal{A}\mathcal{A} = \begin{bmatrix} c & -s\nu^t \\ \\ -s\nu & -c\nu\nu^t + \tau\tau^t \end{bmatrix} \begin{bmatrix} c & -s\nu^t \\ \\ -s\nu & -c\nu\nu^t + \tau\tau^t \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{c^2+s^2}_{1} & \underbrace{-cs\nu^t + sc\underbrace{\nu^t\nu}_{1}\nu^t}_{0} - s\underbrace{\nu^t\tau}_{0}\tau^t \\ \\ \underbrace{-sc\nu + cs\,\nu\underbrace{\nu^t\nu}_{1}}_{0} - s\tau\underbrace{\tau^t\nu}_{0} & \underbrace{s^2\nu\nu^t + \underbrace{(-c\nu\nu^t+\tau\tau^t)(-c\nu\nu^t+\tau\tau^t)}_{c^2\nu\nu^t+\tau\tau^t}}_{\nu\nu^t+\tau\tau^t = Id \text{ on } e^\perp} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \\ 0 & I \end{bmatrix}$$

Note:

$$\overbrace{\begin{bmatrix} c & -s\nu^t \\ -s\nu & -c\nu\nu^t + \tau\tau^t \end{bmatrix}}^{\mathcal{A}(\nu)} \begin{bmatrix} 0 \\ \tau \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} c & -s\nu^t \\ -s\nu & -c\nu\nu^t + \tau\tau^t \end{bmatrix} \begin{bmatrix} \gamma \\ \nu \end{bmatrix} = \begin{bmatrix} c\gamma - s \\ (-s\gamma - c)\nu \end{bmatrix} = -\begin{bmatrix} \gamma \\ \nu \end{bmatrix}$$

$$c\gamma - s = \frac{\gamma - \gamma^3 - 2\gamma}{1+\gamma^2} = -\gamma$$

$$s\gamma + c = \frac{2\gamma^2 + 1 - \gamma^2}{1+\gamma^2} = 1$$

$$\begin{bmatrix} c & -s\nu^t \\ -s\nu & -c\nu\nu^t + \tau\tau^t \end{bmatrix} \begin{bmatrix} -1 \\ \gamma\nu \end{bmatrix} = \begin{bmatrix} -c - s\gamma \\ (s - \gamma c)\nu \end{bmatrix} = \begin{bmatrix} -1 \\ \gamma\nu \end{bmatrix}$$

So <u>eigenvalues and eigenvectors</u>:

$$\lambda = 1 \qquad 1 \qquad -1$$

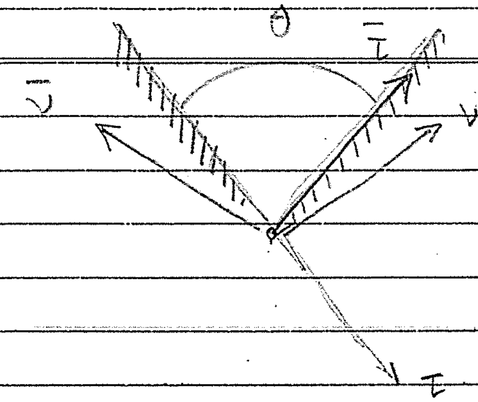
$$\begin{bmatrix} 0 \\ \tau \end{bmatrix}, \quad \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} -1 \\ \gamma\nu \end{bmatrix}, \quad \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} \gamma \\ \nu \end{bmatrix}$$

Define $$\mathcal{R}(\nu) = \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 0 & -1 & \gamma \\ \sqrt{1+\gamma^2}\,\tau & \gamma\nu & \nu \end{bmatrix} \in SO(3)$$

($\det \mathcal{R}(\nu) = 1$)

$$\mathcal{A}(\nu) = \mathcal{R}(\nu) \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \mathcal{R}(\nu)^t$$

Wedge



$\nu \cdot \bar{\nu} = \cos\theta$

$$\begin{bmatrix} \sigma_j \\ w_j \end{bmatrix} = \mathcal{A}_j \begin{bmatrix} \sigma_{j-1} \\ w_{j-1} \end{bmatrix} \qquad \mathcal{A}_{\text{even}} = \mathcal{A}(\nu)$$

$$\mathcal{A}_{\text{odd}} = \mathcal{A}(\bar{\nu})$$

$$h_n = h_0 + \underbrace{\sum_{j=0}^{n-1} \tau_j \sigma_j}_{\sim\, \tau_{\text{mean}} \sum_{j=0}^{n-1} \sigma_j}$$

Def. $Q = \mathcal{A}(\nu)\mathcal{A}(\bar{\nu}) \in SO(3)$

τ_j determined by projected 2-D billiard on $\mathcal{B}$

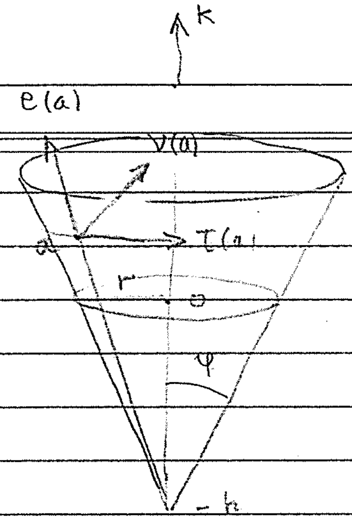
$$\begin{cases} \begin{bmatrix} \sigma_{2n} \\ w_{2n} \end{bmatrix} = \underbrace{(\mathcal{A}_{2n}\mathcal{A}_{2n-1})}_{Q} \cdots \underbrace{(\mathcal{A}_2 \mathcal{A}_1)}_{Q} \begin{bmatrix} \sigma_0 \\ w_0 \end{bmatrix} = Q^n \begin{bmatrix} \sigma_0 \\ w_0 \end{bmatrix} \\ \begin{bmatrix} \sigma_{2n+1} \\ w_{2n+1} \end{bmatrix} = \mathcal{A}(\bar{\nu})\, Q^n \begin{bmatrix} \sigma_0 \\ w_0 \end{bmatrix} \end{cases}$$

$$V_j = \begin{bmatrix} \sigma_j \\ w_j \end{bmatrix}$$

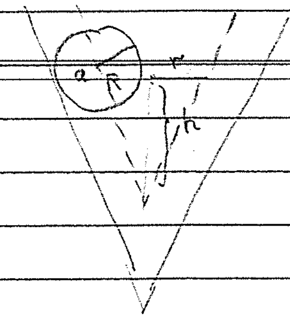
$$h_n = h_0 + [1, 0] \Big\{ \tau_0 I + \tau_1 \mathcal{A}(\bar{\nu}) + \tau_2 Q + \tau_3 \mathcal{A}(\bar{\nu}) Q + \tau_4 Q^2 + \tau_5 \mathcal{A}(\bar{\nu}) Q^2 + \tau_6 Q^3 + \cdots + \tau_n \mathcal{A}(\bar{\nu})^{\varepsilon} Q^{\lfloor n/2 \rfloor} \Big\} V_0$$

$$\varepsilon = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$$

① Cone



$$\frac{a}{|a|} \cdot k = \cos\varphi = \frac{h}{\sqrt{r^2+h^2}}$$



$$\begin{cases} e(a) = \dfrac{a}{|a|} \\[2ex] \nu(a) = \dfrac{k - k\cdot e(a)\, e(a)}{\sin\varphi} \\[2ex] \tau(a) = \nu(a) \times e(a) = \dfrac{k \times e(a)}{\sin\varphi} \end{cases}$$

Adapted frame at a: $\sigma_a : (\mathbb{R}^6, \cdot) \to \left(T_a M \times SU(3), \frac{1}{m}\langle , \rangle\right)$

linear isometry

σ_a : $e_1 \longmapsto (1/R\gamma)\, \tau(a) \wedge \nu(a)$

$e_2 \longmapsto \tau(a)$

$e_3 \longmapsto \nu(a)$

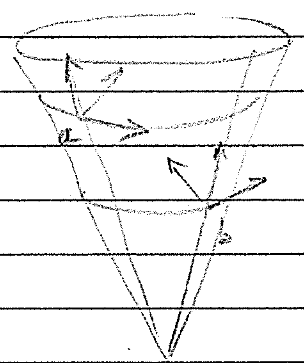
$e_4 \longmapsto e(a)$

$e_5 \longmapsto (1/R\gamma)\, \nu(a) \wedge e(a)$

$e_6 \longmapsto (1/R\gamma)\, e(a) \wedge \tau(a)$

Change of frame matrix $I(a,b) := \sigma_b^{-1} \sigma_a$

$$I(a,b)_{ij} = \frac{1}{m}\langle \sigma_b e_i, \sigma_a e_j \rangle$$



(2)

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
I(a,b) = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\left[\begin{array}{cccccc}
\circledast_{11} & 0 & 0 & 0 & \circledast_{15} & \circledast_{16} \\
0 & \tau(b)\cdot\tau(a) & \tau(b)\cdot\nu(a) & \tau(b)\cdot e(a) & 0 & 0 \\
0 & \nu(b)\cdot\tau(a) & \nu(b)\cdot\nu(a) & \nu(b)\cdot e(a) & 0 & 0 \\
0 & e(b)\cdot\tau(a) & e(b)\cdot\nu(a) & e(b)\cdot e(a) & 0 & 0 \\
\circledast_{51} & 0 & 0 & 0 & \circledast_{55} & \circledast_{56} \\
\circledast_{61} & 0 & 0 & 0 & \circledast_{65} & \circledast_{66}
\end{array} \right]
\end{array}
$$

$$\frac{1}{m}\langle \sigma_b' e_1, \sigma_a' e_1 \rangle = \frac{1}{2}\operatorname{Tr}\left(\tau(b)\wedge\nu(b)\ (\tau(a)\wedge\nu(a))^t \right) = \nu(b)\cdot\nu(a)\ \tau(b)\cdot\tau(a) - \tau(b)\cdot\nu(a)\ \nu(b)\cdot\tau(a)$$

a b c d

$$\circledast_{11} = \underbrace{\nu(b)\cdot\nu(a)}_{(3,3)}\ \underbrace{\tau(b)\cdot\tau(a)}_{(2,2)} - \underbrace{\tau(b)\cdot\nu(a)}_{(2,3)}\ \underbrace{\nu(b)\cdot\tau(a)}_{(3,2)}$$

$$= \det I[2:3, 2:3]$$

$$\circledast_{15} = \frac{1}{m}\langle \sigma_b' e_1, \sigma_a' e_5 \rangle = \frac{1}{2}\operatorname{Tr}\left(\tau(b)\wedge\nu(b)\ (\nu(a)\wedge e(a))^t \right) = \nu(b)\cdot e(a)\ \tau(b)\cdot\nu(a) - \tau(b)\cdot e(a)\ \nu(b)\cdot\nu(a)$$

$$\circledast_{16} = \frac{1}{m}\langle \sigma_b' e_1, \sigma_a' e_6 \rangle = \frac{1}{2}\operatorname{Tr}\left(\tau(b)\wedge\nu(b)\ (e(a)\wedge\tau(a))^t \right) = \nu(b)\cdot\tau(a)\ \tau(b)\cdot e(a) - \tau(b)\cdot\tau(a)\ \nu(b)\cdot e(a)$$

$$\circledast_{51} = \frac{1}{m}\langle \sigma_b' e_5, \sigma_a' e_1 \rangle = e(b)\cdot\nu(a)\ \nu(b)\cdot\tau(a) - e(b)\cdot\tau(a)\ \nu(b)\cdot\nu(a)$$

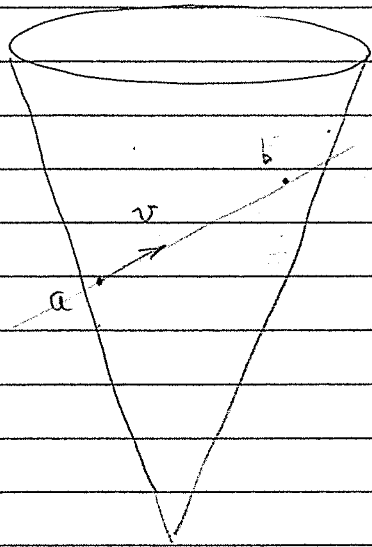
$$\circledast_{61} = \frac{1}{m}\langle \sigma_b' e_1, \sigma_a' e_6 \rangle = \tau(b)\cdot\nu(a)\ e(b)\cdot\tau(a) - \tau(b)\cdot\tau(a)\ e(b)\cdot\nu(a)$$

$$\circledast_{55} = \frac{1}{m}\langle \sigma_b' e_5, \sigma_a e_5 \rangle = \frac{1}{2}\operatorname{Tr}\left(\nu(b)\wedge e(b)\ (\nu(a)\wedge e(a))^t \right) = e(b)\cdot e(a)\ \nu(b)\cdot\nu(a) - \nu(b)\cdot e(a)\ e(b)\cdot\nu(a)$$

$$\circledast_{66} = \frac{1}{m}\langle \sigma_b e_6, \sigma_a e_6 \rangle = \frac{1}{2}\operatorname{Tr}\left(e(b)\wedge\tau(b)\ (e(a)\wedge\tau(a))^t \right) = \tau(b)\cdot\tau(a)\ e(b)\cdot e(a) - e(b)\cdot\tau(a)\ \tau(b)\cdot e(a)$$

$$\circledast_{56} = \frac{1}{m}\langle \sigma_1 e_5, \sigma_2 e_6\rangle = \frac{1}{2}\mathrm{Tr}\left(v(b)\wedge e(b)\cdot\left(e(a)\wedge \tau(a)\right)^{\dagger}\right) = e(b)\cdot\tau(a)\, v(b)\cdot e(a) - v(b)\cdot\tau(a)\, e(b)\cdot e(a)$$

$$\circledast_{65} = \frac{1}{m}\langle \sigma_1 e_6, \sigma_2 e_5\rangle = \frac{1}{2}\mathrm{Tr}\left(e(b)\wedge \tau(b)\left(v(a)\wedge e(a)\right)^{\dagger}\right) = \tau(b)\cdot e(a)\, e(b)\cdot v(a) - \tau(b)\cdot v(a)\, e(b)\cdot e(a)$$



$$\frac{a+tv}{|a+tv|}\cdot k = \cos\varphi \qquad a\cdot k = |a|\cos\varphi$$

$$\underbrace{a\cdot k}_{|a|\cos\varphi} + t\, v\cdot k = \left(|a|^2 + 2t\, a\cdot v + t^2|v|^2\right)^{1/2}\cos\varphi$$

$$\cancel{|a|^2\cos^2\varphi} + 2t\, a\cdot k\, v\cdot k + t^2 (v\cdot k)^2 = \cancel{|a|^2\cos^2\varphi} + 2t\, a\cdot v\cos^2\varphi + t^2|v|^2\cos^2\varphi$$

$$\left[(v\cdot k)^2 - (|v|\cos\varphi)^2\right] t + 2\left[a\cdot k\, v\cdot k - a\cdot v\cos^2\varphi\right] = 0$$

$$\boxed{t = 2\,\frac{a\cdot k\, v\cdot k - a\cdot v\cos^2\varphi}{(v\cdot k)^2 - (|v|\cos\varphi)^2}}$$

$$\boxed{\frac{v}{|v|}\cdot k > \cos\varphi \quad \text{if } v\cdot k > 0}$$

4

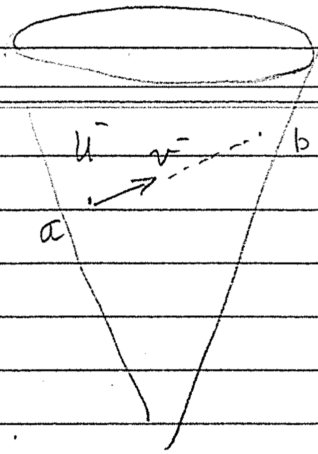
$$\xi = \begin{bmatrix} v \\ U \end{bmatrix} = \underbrace{\frac{1}{m}\langle \xi, \sigma_a e_i \rangle}_{R^2\gamma^2 \frac{}{2} Tr(U, E_i^t) + v \cdot \varepsilon_i} \sigma_a e_i \longrightarrow [\xi]_a = \begin{bmatrix} \frac{1}{m}\langle \xi, \sigma_a e_i \rangle \\ \cdots \\ \frac{1}{m}\langle \xi, \sigma_a e_2 \rangle \end{bmatrix}$$

$$e_i = \begin{bmatrix} \varepsilon_i \\ E_i \end{bmatrix}$$

$$\frac{R^2\gamma^2}{2} Tr(U, E_i^t) + v \cdot \varepsilon_i$$

$$[\xi]_a = \begin{cases} \frac{1}{m}\langle U, (1/R\gamma)\,\tau(a)\wedge\nu(a)\rangle = \frac{1}{2}Tr(U(\tau\wedge\nu)^t) = (U\tau(a))\cdot\nu(a) \\ \frac{1}{m}\langle v, \tau(a)\rangle = v\cdot\tau(a) \\ \frac{1}{m}\langle v, \nu(a)\rangle = v\cdot\nu(a) \\ \frac{1}{m}\langle v, e(a)\rangle = v\cdot e(a) \\ \frac{1}{m}\langle U, (1/R\gamma)\,\nu(a)\wedge e(a)\rangle = \frac{1}{2}Tr(U(\nu\wedge e)^t) = (U\nu(a))\cdot e(a) \\ \frac{1}{m}\langle U, (1/R\gamma)\, e(a)\wedge\tau(a)\rangle = \frac{1}{2}Tr(U(e\wedge\tau)^t) = (U e(a))\cdot\tau(a) \end{cases}$$

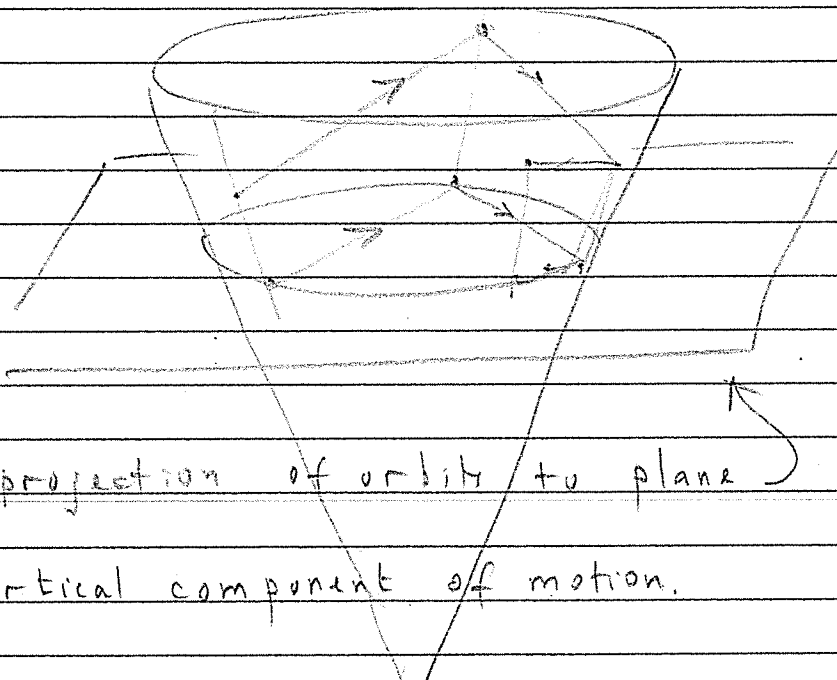
$$[\xi]_a = \begin{bmatrix} (U\tau(a))\cdot\nu(a) \\ v\cdot\tau(a) \\ v\cdot\nu(a) \\ v\cdot e(a) \\ (U\nu(a))\cdot e(a) \\ (U e(a))\cdot\tau(a) \end{bmatrix}$$



$b = a + tv$

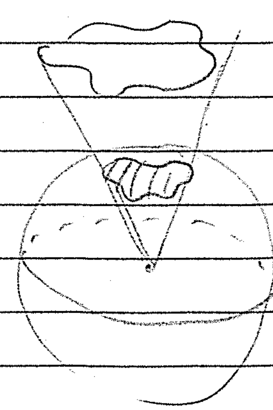
$$\begin{bmatrix} v^+ \\ u^+ \end{bmatrix} = \sigma_t \; C \begin{bmatrix} v^- \\ u^- \end{bmatrix}_b$$

$$v^- = \begin{bmatrix} v^- \\ u^- \end{bmatrix}_a (2)\,\tau(a) + \begin{bmatrix} \; \\ \; \end{bmatrix}_a (3)\,\nu(a) + \begin{bmatrix} \; \\ \; \end{bmatrix}_a (4)\,e(a)$$



Study projection of orbit to plane

and vertical component of motion.



Perhaps better to project on sphere?