The Dynamics of No-slip Billiards

The 11^{th} AIMS International Conference on Dynamical Systems, Differential Equations and Applications

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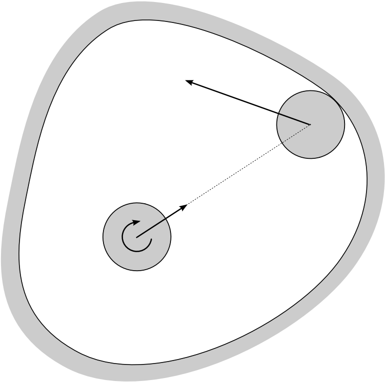
Mathematical Billiards with English



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- Endow M with the Riemannian metric whose quadratic form gives the system's kinetic energy function.
- Points $q \in \partial M$ represent configurations in which the bodies have a single contact point.
- At a boundary point q define a *collision map* as a linear map $\mathcal{C} : T_qM \to T_qM$ that sends vectors pointing out of M into vectors pointing inward.

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1. Energy is conserved. That is, $\mathcal{C}$ is an orthogonal linear map.
2. Linear and angular momentum are conserved in the unconstrained motion.
3. Time reversibility. This amounts to $\mathcal{C}$ being a linear involution.
4. Impulse forces at collision are applied only at the single point of contact. (This property may be regarded as a generalized momentum conservation law that is generally non-trivial and highly restrictive.)

Strict Collision Classification

Theorem (C., Feres, Ward)

At each boundary point of the configuration manifold of the system of two rigid bodies, assuming the boundary is differentiable at that point, the set of all strict collision maps can be expressed as the disjoint union of orthogonal Grassmannian manifolds $\mathrm{Gr}(k, n-1)$, $k = 0, \ldots, n-1$, *of all* k *dimensional planes in* $\mathbb{R}^{n-1}$.

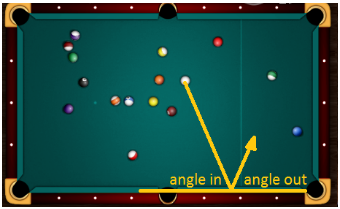
Corollary

When $n = 2$, the set of strict collision maps is a two-point set consisting of the specular reflection and the no-slip collision.

No-slip collisions are an alternative to specular collisions, a second ideal model in which linear and angular momentum may be exchanged.

Standard Billiard Model: Specular Collisions

- Fluid models, such as Lorentz gases
- Brownian motion
- Heat transfer (How do things cool off?)



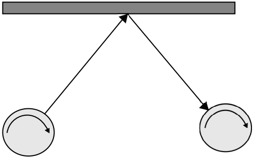
- Diffusion (How do mixtures spread out?)
- Even: a proof of Ohm's Law

Work on no-slip billiards:

- Broomhead and Gutkin (1993): Two dimensional no-slip model and no-slip strip example
- Wojtkowski (1994): Found a dispersing curvature limit in a special case
- Cox, Feres (2015): Model appears in any dimension as a special case of a general rigid body collision model
- Physicists using a version of the three dimensional model for super-balls and tennis balls

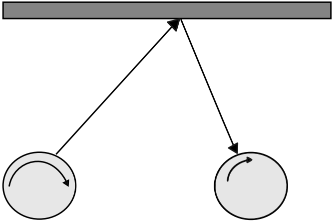
Intuitively:

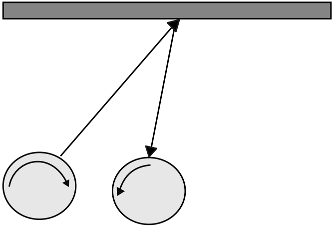
Standard



The standard model (left) is extensively used, while little is known about the no-slip model (below).

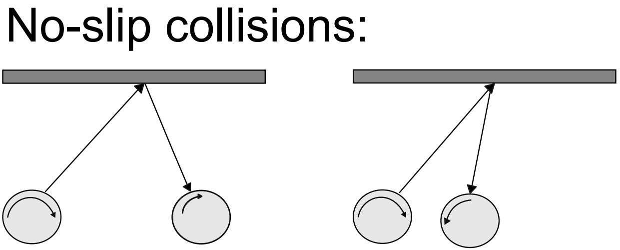
No-slip collisions:





Assumptions

- Model *does not* include frictional dissipation
- *Does* depend on mass distribution and scaling
- We will normalize and (generally) assume uniform distribution

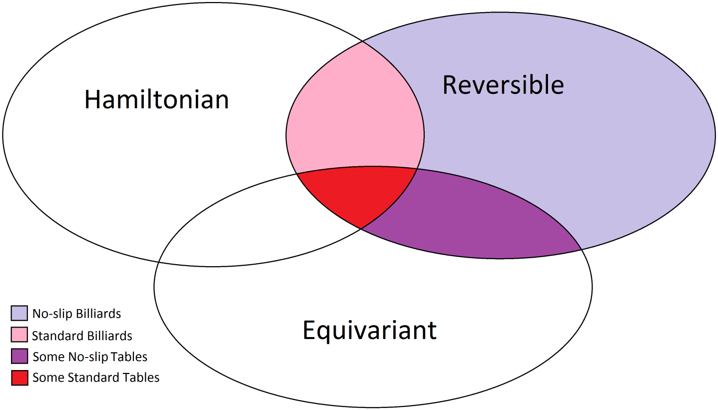


Remarks

- We are most interested in applying this collision model to billiard systems, with one rigid body (the billiard table) fixed in place. The focus is then on the motion of the billiard particle.[1]
- Unlike in the case of standard billiards, the symplectic form is not preserved.

[1] We may without essentially changing the theory adjust by the radius and look at the motion of the center of mass.

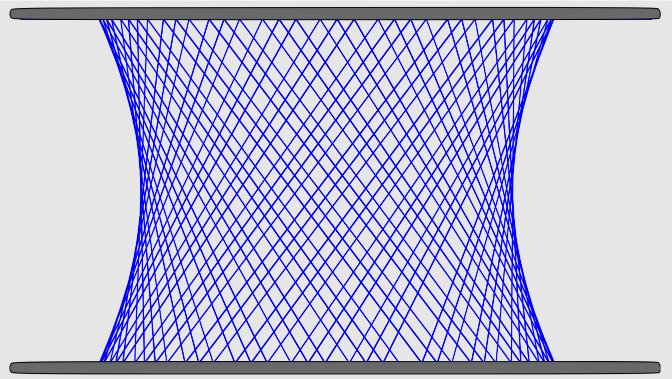
Types of dynamical systems



Example 1: The No-Slip Strip

Theorem (Broomhead, Gutkin 1993)

Any non-horizontal trajectory between parallel boundaries will have a bounded orbit.



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2. *At the kth collision, the horizontal position is given by*

$$h_n = \frac{\sqrt{3}r}{\sqrt{2}\sqrt{1-r^2}} \sin\left(\frac{k}{2}(\beta+\pi)\right) \cos\left(\psi + \frac{k-1}{2}(\beta+\pi) + \frac{\pi}{2}\right) \quad (1)$$

where β is an inertial constant and r and ϕ depend on the starting conditions.

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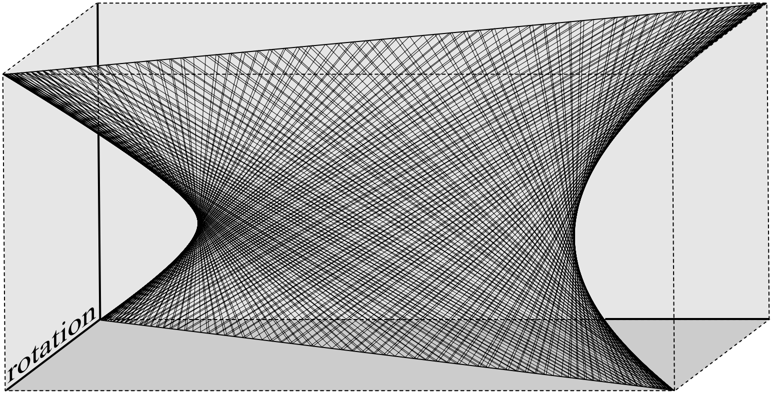
   where β is an inertial constant and r and ϕ depend on the starting conditions.
3. *The no-slip strip is never (non-trivially) periodic.*
4. *The three dimensional generalization also holds.*

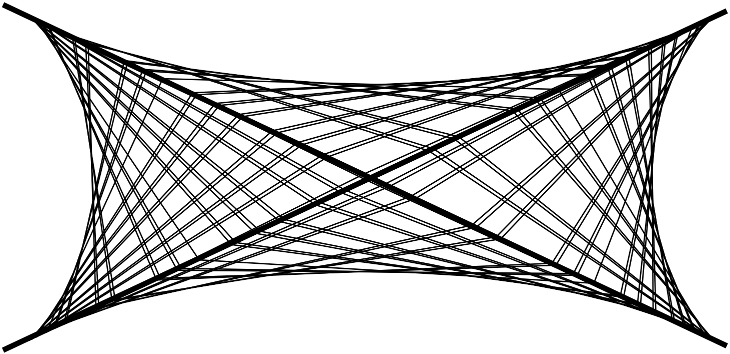
Example 1: The No-Slip Strip

Extending the boundaries in the rotational direction, the contact points occur on lines.

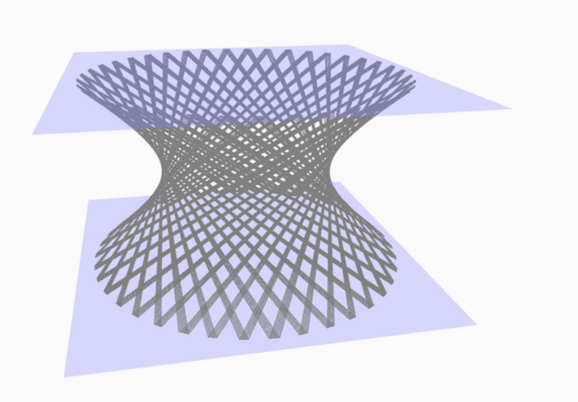


Example 1: The No-Slip Strip

Projected into the rotational-horizontal plane, each segment has the same length and goes from one line to the next.

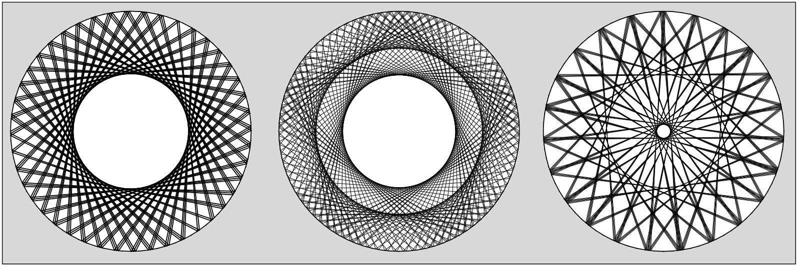


Rough collisions between parallel planes in three (spacial) dimensions yield bounded trajectories.



Example 3: No-Slip Circles

Standard circular billiards (left) have a circular caustic; no-slip billiards (center, right) have a double circular caustic.



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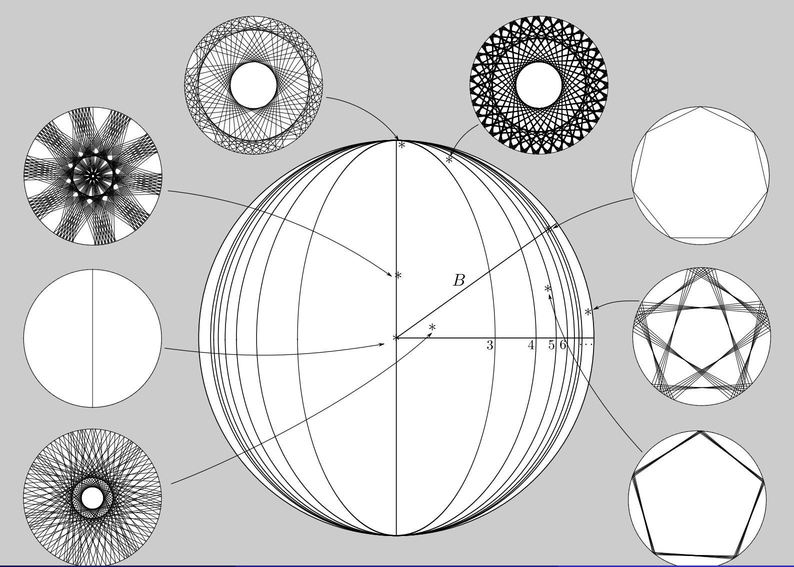
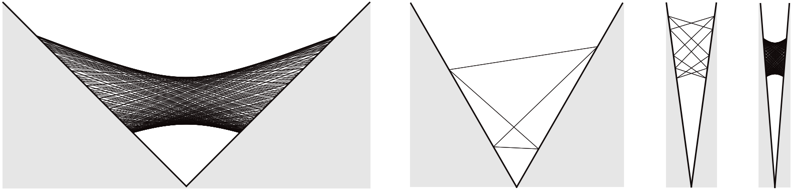


Example 4: The No-slip Wedge

Consider the no-slip billiard composed of two rays meeting at angle θ, forming an open wedge in two dimensions.



For most angles, no-slip open wedges give non-periodic orbits (far left and far right), but $\theta = \frac{\pi}{3}$ (center left) and $\theta \approx .27$ (center right) give period four and period ten orbits, which persist when initial velocities are varied.

Example 4: The No-slip Wedge

Theorem (C.)

The wedge angle θ and the rotational angle α_θ of $S \in SO(3)$, the corresponding transformation of the velocity after one complete cycle of two no-slip collisions, are related by

$$\frac{32}{9}\cos^4\left(\frac{\theta}{2}\right) - \frac{16}{3}\cos^2\left(\frac{\theta}{2}\right) + 1 = \cos(\alpha_\theta)$$

Example 4: The No-slip Wedge

The proposition can be confirmed numerically.

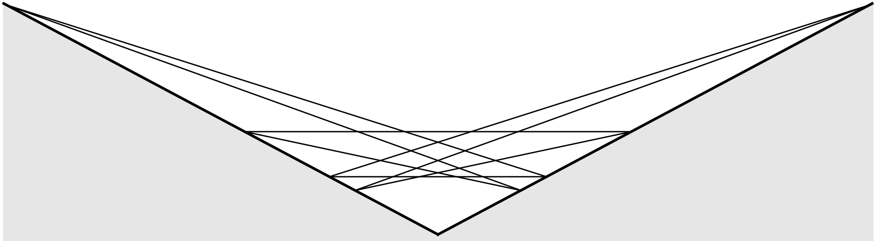


Figure: Wedge angle $\theta \approx 2.16598$ gives period 8 orbits, as predicted.

Theorem (C.)

For a no-slip billiard wedge of angle $\theta \in (0, \pi)$, let x_0 be the rotational axis, x_2 the direction of the wedge bisector, and x_1 the perpendicular spacial direction.

i There exists a periodic axis, a direction in which all trajectories are periodic. Specifically, for velocity $(\dot{x}_0, \dot{x}_1, \dot{x}_2)$, the orbit will be periodic whenever

$$\frac{\dot{x}_0}{\dot{x}_1} = -\sqrt{2} \sin \frac{\theta}{2}.$$

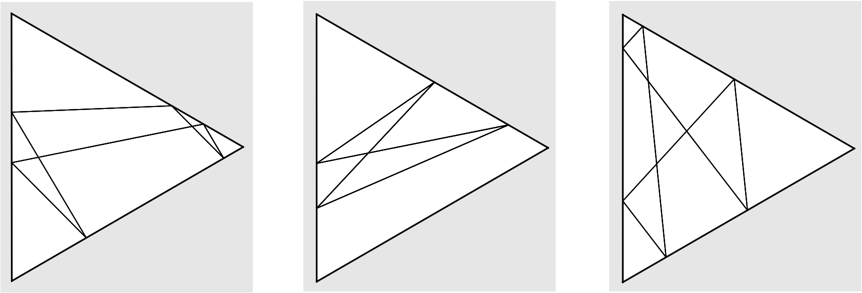
ii For any $n \in \mathbb{Z}^+$, wedge angle θ_n can be chosen so that all non-escape velocities yield $2n$-periodic orbits. Furthermore, the set of all such θ_n is dense in $(0, \pi)$.

iii The angle ψ between the velocity and the axis of periodicity is invariant throughout an orbit, remaining unchanged after collisions.

Example 8: The No-slip Equilateral Triangle

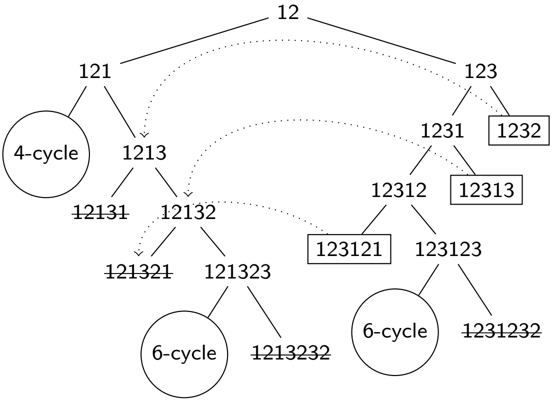
Proposition (C.)

The equilateral triangle is periodic for all initial conditions, of period 2, 3, 4, or 6.



Example 8: The No-slip Equilateral Triangle

Idea of proof:



Theorem (C., Feres, Ward)

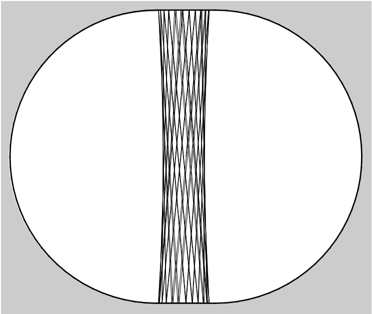
Suppose that the field of collision maps $q \in S \mapsto \mathcal{C}_q$ is piecewise smooth and parallel (where it is smooth) with respect to the Levi-Civita connection associated to the kinetic energy Riemannian metric. Let $\Omega = d\theta \wedge \cdots \wedge d\theta$ be the form (of degree $2n-2$, where n is the dimension of the ambient Euclidean space) derived from the canonical symplectic form $d\theta$ on $N^{\mathcal{E}} \setminus TS$. Then Ω is, up to sign, invariant under the billiard map.

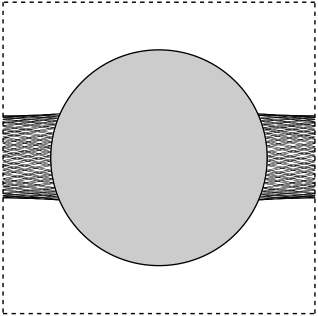
Corollary

Rough billiards in dimension 2 preserve the canonical billiard measure.

With this prerequisite, we may consider questions of ergocity of no-slip billiards in two dimensions.

Standard techniques of generating chaos

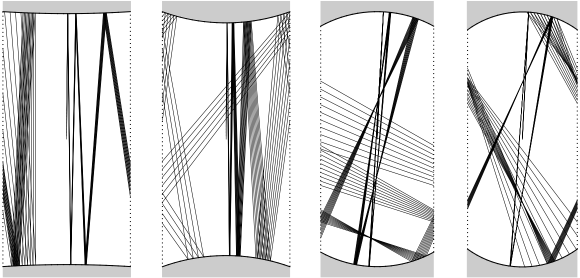




The no-slip stadium map is not ergodic, nor is this simple Sinai dispersing billiard.

Focusing and Dispersion

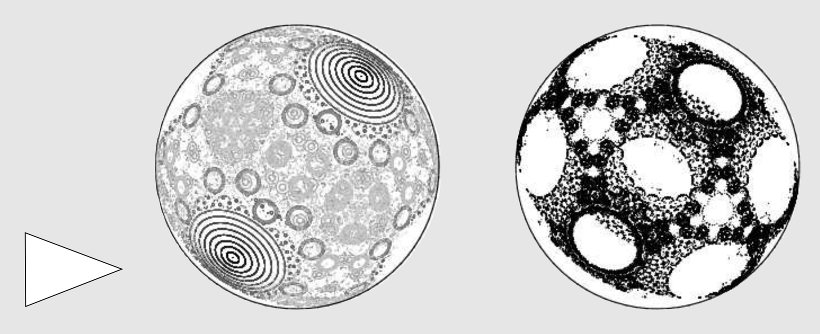
A small band of trajectories for (in order) concave standard and no-slip, convex standard and no-slip collisions. Sides are identified.



Phase Portraits

Quasiperiodic orbits are ubiquitous in polygons, like this equilateral triangle.

Left: a sampling of orbits. Right: One elaborate orbit.

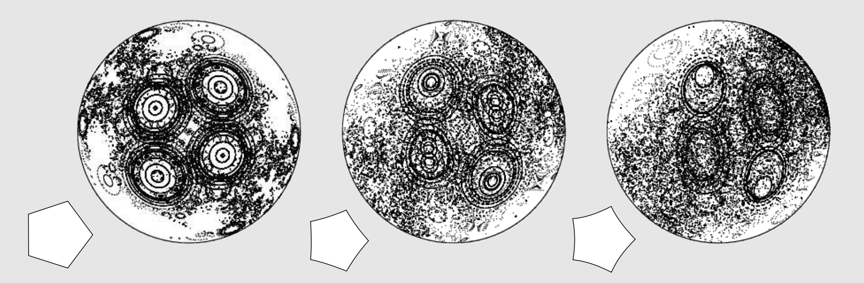


Theorem (Cox, Feres, Zhang)

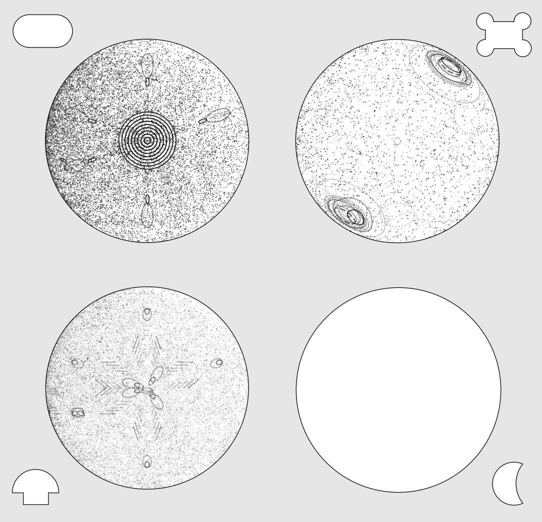
For any two dimensional no-slip billiard with an axis of periodicity, the no-slip map is non-ergodic. In particular, all polygons are non-ergodic.

Phase Portraits

Non-ergodic features persist for curvilinear polygons, unlike dispersing standard billiards.



Phase Portraits of no-slip versions of ergodic billiards



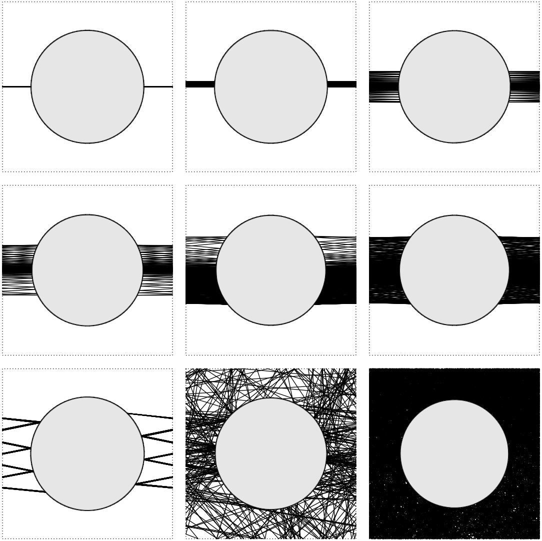
Theorem (Wojtkowski 1994)

A periodic orbit between no-slip collision points with parallel tangents, path length l and (common) curvature κ will be linearly stable if

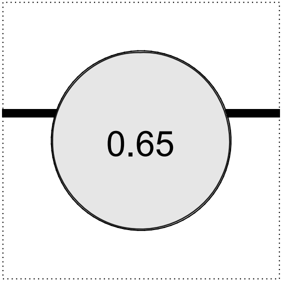
$$kl < 2\xi^2$$

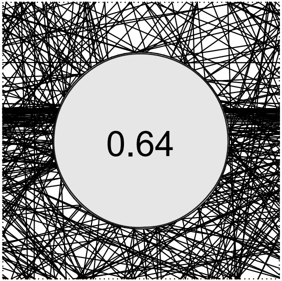
where ξ depends on the moment of inertia of the particle.

Dispersive Limit r=$\frac{2}{3}$ to r=0.664

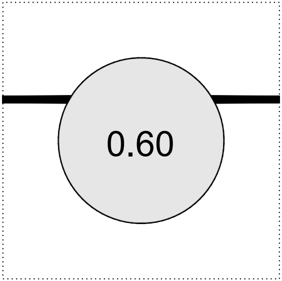


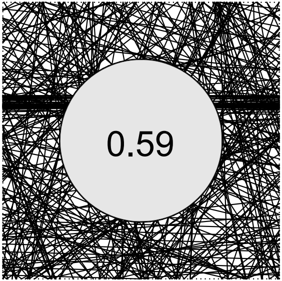
Small radius disperser y=0.2



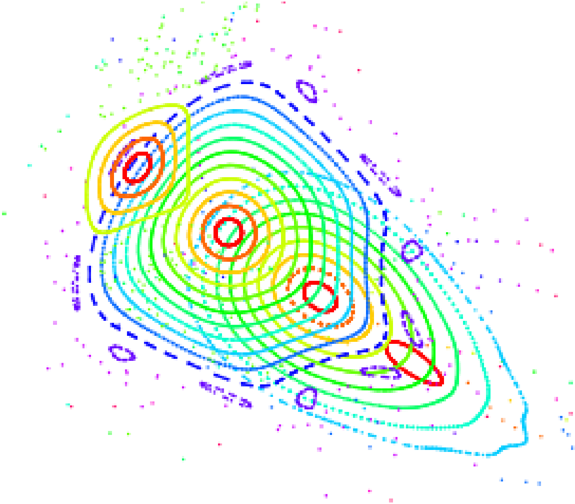


Small radius disperser y=0.3





Small radius disperser: r=.02



Open question

Conjecture: Two dimensional no-slip billiards are never ergodic.

References:

- *"No-slip billiards in dimension two,"* Cox and Feres, to appear in AMS Contemporary Mathematics Series
- *"Differential geometry of rigid bodies collisions and non-standard billiards,"* Cox and Feres, to appear in Discrete and Continuous Dynamical Systems - A.

Thank You!