Collision map

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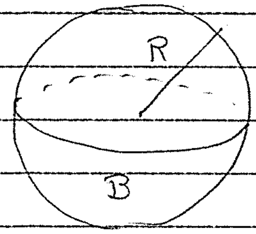


mass m $\quad l_{rs} = \frac{1}{m}\int_B b_r b_s \underbrace{d\mu(b)}_{\rho(|b|)\, dVol(b)}$

$$= \frac{\delta_{rs}}{m}\int_0^R b_r^2\, \rho(|b|)\, \underbrace{S(|b|)}_{}\, d|b|$$

hypersurface area of sphere of radius $|b|$

$$= \frac{S(1)}{n}\,\frac{\delta_{rs}}{m}\int_0^R x^{n+1}\rho(x)\,dx$$

$$= \frac{V_n}{m}\,\delta_{rs}\int_0^R x^{n+1}\rho(x)\,dx$$

V_n = volume of unit ball in dim. n.

Set $\lambda = \frac{V_n}{m}\int_0^R x^{n+1}\rho(x)\,dx$

<u>Example</u> $\rho = \text{const} = \frac{m}{V(R)}$, $\lambda = \frac{\cancel{V_n}}{\cancel{m}}\,\frac{\cancel{m}}{R^n \cancel{V_n}}\int_0^R x^{n+1}\,dx = \frac{R^2}{n+2}$

$\boxed{\text{Uniform mass dist.}: \lambda = R^2/n+2}$

$L = \lambda I$, $\mathcal{L} = 2\lambda I$, $\gamma = \frac{\sqrt{2\lambda}}{R}$

<u>Example</u> $\rho = \text{const.} \Rightarrow \gamma = \sqrt{\frac{2}{n+2}}$

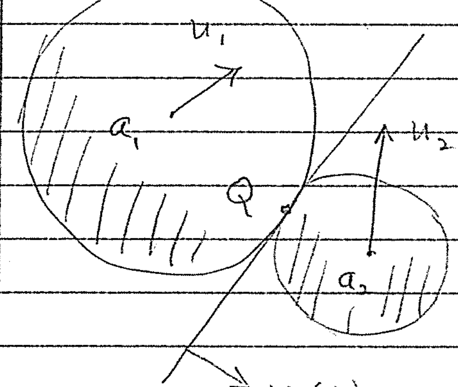
<u>Note</u>: In general, $0 \leq \lambda \leq R^2/n$.

Pre-collision state $(U_1, u_1), (U_2, u_2), (A_1, a_1), (A_2, a_2)$

$$A_1 = I, \quad A_2 = I$$

$U_i \in SO(n)$ $\qquad |a_1 - a_2| = R_1 + R_2$

$u_i \in \mathbb{R}^n$



x in body i $(x \in B_i + a_i)$ Velocity:

$$V_i(x) = U_i(x - a_i) + u_i$$

$$V_i(Q) = R_i U_i \nu_i(Q) + u_i$$

$$\nu_1(Q) = -\nu_2(Q) = \nu$$

Given two states at $q = (I, a_1, I, a_2)$:

$$\xi = (U_1^\xi, u_1^\xi, U_2^\xi, u_2^\xi), \quad \eta = (U_1^\eta, u_1^\eta, U_2^\eta, u_2^\eta)$$

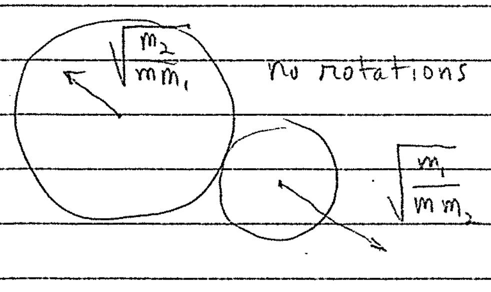
Kinetic energy Riemannian metric:

$$\langle \xi, \eta \rangle_q = \sum_{j=1,2} m_j \left[\lambda_j \operatorname{Tr}\left(U_j^\xi U_j^{\eta\dagger}\right) + u_j^\xi \cdot u_j^\eta \right]$$

Normal vector $\quad n_q = -\frac{1}{\sqrt{m}} \left(0, \sqrt{\frac{m_2}{m_1}}\, \nu_1(Q), 0, \sqrt{\frac{m_1}{m_2}}\, \nu_2(Q) \right)$

$m = m_1 + m_2$

$$= \frac{1}{\sqrt{m}} \left(0, -\sqrt{\frac{m_2}{m_1}}\, \nu, 0, \sqrt{\frac{m_1}{m_2}}\, \nu \right)$$

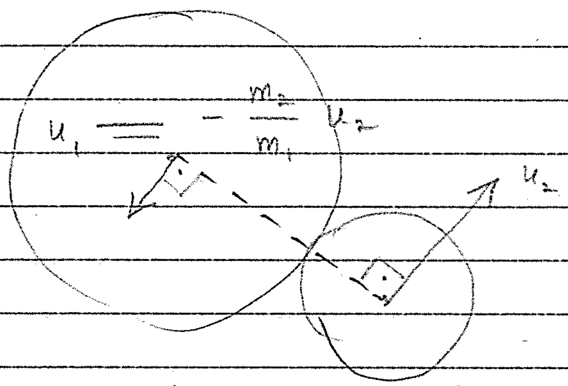


<u>Impulse vectors</u> $\overline{\mathfrak{C}}_q = \mathfrak{C}_q \ominus \mathbb{R}\, n_q$

$$\left\{ \left(\left(\frac{R_1}{2\lambda_1} \nu_1(Q) \wedge u_1 , u_1 \right), \left(\frac{R_2}{2\lambda_2} \nu_2(Q) \wedge u_2 , u_2 \right) \right) : \begin{array}{l} m_1 u_1 + m_2 u_2 = 0 \\ \nu \cdot u_j = 0 \end{array} \right\}$$

$\dim \overline{\mathfrak{C}}_q = n-1$

$\mathfrak{C}_q$ = same as $\overline{\mathfrak{C}}_q$ but without the condition $\nu \cdot u_j = 0$



<u>No-slip vectors</u>

$$\mathfrak{S}_q = \Big\{ (U_1, u_1, U_2, u_2) : \underbrace{\overbrace{U_1 \underbrace{(Q - a_1)}_{R_1 \nu} + u_1}^{V_1(Q)} = \overbrace{U_2 \underbrace{(Q - a_2)}_{-R_2 \nu} + u_2}^{V_2(Q)}}_{R_1 U_1 \nu + u_1 = -R_2 U_2 \nu + u_2} \Big\}$$

<u>Orthogonal decomposition</u>

$$T_q M = \mathfrak{S}_q \oplus \overline{\mathfrak{C}}_q \oplus \mathbb{R}\, n_q$$

Let $\xi = (U_1, u_1, U_2, u_2)$ be a state at $q = (I_1\, a_1, I_2, a_2)$

$$\xi = \underbrace{(W_1, w_1, W_2, w_2)}_{\in \mathfrak{S}_q} + \left(\frac{R_1}{2\lambda_1} \nu \wedge y_1,\ y_1,\ -\frac{R_2}{2\lambda_2} \nu \wedge y_2,\ y_2\right)$$

$$+ \underbrace{s \frac{1}{\sqrt{m}} \left(0, -\sqrt{\frac{m_2}{m_1}}\nu, 0, \sqrt{\frac{m_1}{m}}\nu\right)}_{\mathfrak{N}_q}$$

$\nu \cdot y_j = 0, \quad m_1 y_1 + m_2 y_2 = 0$

$R_1 W_1 \nu + w_1 = -R_2 W_2 \nu + w_2 = \boxtimes$

Sanity check:

(1) $\mathfrak{S}_q \perp \mathfrak{T}_q$

$$\sum_j m_j \left\{ \underbrace{\lambda_j \operatorname{Tr}\left(\frac{R_j}{2\lambda_j} \nu_j \wedge y_j W_j^T\right)}_{= R_j W_j \nu_j \cdot y_j} + w_j \cdot y_j \right\}$$

$$-\frac{R_j}{2} \underbrace{\operatorname{Tr}(\nu_j \wedge y_j W_j)}_{} = \frac{R_j}{2}\left(W_j \nu_j \cdot y_j - W_j y_j \cdot \nu_j\right)$$

$$\sum_K e_K \cdot \underbrace{(\nu_j \wedge y_j W_j e_K)}_{\nu_j \cdot W_j e_K\, y_j - y_j \cdot W_j e_K\, \nu} = \sum_K \nu_j \cdot W_j e_K\, y_j \cdot e_K - \sum_K y_j \cdot W_j e_K\, \nu_j e_K$$

$$= \sum_j m_j \underbrace{\{R_j W_j \nu_j + w_j\}}_{\boxtimes \text{ independ. of } j} \cdot y_j = \boxtimes \cdot \underbrace{\sum_j m_j y_j}_{=0} = 0$$

(2) $\bar{\xi}_q \perp n_q$.

$$\left\langle \left(\frac{R_1}{2\lambda_1} \nu \wedge y_1, y_1, -\frac{R_2}{2\lambda_2} \nu \wedge y_2, y_2 \right), \left(0, -\sqrt{\frac{m_2}{m_1}}\nu, 0, \sqrt{\frac{m_1}{m_2}}\nu \right) \right\rangle$$

$$= -m_1 y_1 \cdot \sqrt{\frac{m_2}{m_1}}\nu + m_2 y_2 \cdot \sqrt{\frac{m_1}{m_2}}\nu = \sqrt{m_1 m_2}\,(y_2 - y_1)\cdot\nu = 0.$$

Normal component of ξ

$$s = \langle \xi, n_q \rangle = \left\langle (U_1, u_1, U_2, u_2), \frac{1}{\sqrt{m}}\left(0, -\sqrt{\frac{m_2}{m_1}}\nu, 0, \sqrt{\frac{m_1}{m_2}}\nu\right) \right\rangle$$

$$\boxed{s = \sqrt{\frac{m_1 m_2}{m}}\,(u_2 - u_1)\cdot\nu}$$

Π orthogonal projection to $T_q \partial M$

$$\Pi\xi = \xi - \langle \xi, n_q \rangle n_q = (U_1, u_1, U_2, u_2) - \sqrt{\frac{m_1 m_2}{m}}(u_2 - u_1)\cdot\nu \frac{1}{\sqrt{m}}\left(0, -\sqrt{\frac{m_2}{m_1}}\nu, 0, \sqrt{\frac{m_1}{m_2}}\nu\right)$$

$$= \Big(U_1, \underbrace{u_1 + \frac{m_2}{m}(u_2 - u_1)\cdot\nu\,\nu}, U_2, \underbrace{u_2 - \frac{m_1}{m}(u_2 - u_1)\cdot\nu\,\nu} \Big)$$

$$\bar{u}_1 + u_1\cdot\nu\,\nu + \frac{m_2}{m}(u_2 - u_1)\cdot\nu\,\nu \qquad \bar{u}_2 + \frac{m_1 u_1 + m_2 u_2}{m}\cdot\nu\,\nu$$

$$\bar{u}_1 + \frac{m_1 u_1 + m_2 u_2}{m}\cdot\nu\,\nu$$

orth. proj. to $\nu^\perp$ $\longrightarrow$ $\bar{u}_1$

$$\boxed{\Pi\xi = \left(U_1, \bar{u}_1 + \frac{m_1 u_1 + m_2 u_2}{m}\cdot\nu\,\nu, U_2, \bar{u}_2 + \frac{m_1 u_1 + m_2 u_2}{m}\cdot\nu\,\nu \right)}$$

$$u_c = \frac{m_1 u_1 + m_2 u_2}{m} = \text{velocity of c.m.}$$

Check: $\langle \Pi \xi, n_q \rangle = \left\langle \left(U_1, \bar{u}_1 + u_c \cdot \nu\nu, U_2, \bar{u}_2 + u_c \cdot \nu\nu\right), \left(0, -\sqrt{\frac{m_2}{m_1}}\nu, 0, \sqrt{\frac{m_1}{m_2}}\nu\right)\right\rangle$

$$= -m_1 u_c \cdot \nu \sqrt{\frac{m_2}{m_1}} + m_2 u_c \cdot \nu \sqrt{\frac{m_1}{m_2}} = 0$$

Projection to $\mathfrak{S}_q$

$(Z_1, z_1, Z_2, z_2) \in n_q^{\perp}$

$$(Z_1, z_1, Z_2, z_2) - \underbrace{(W_1, w_1, W_2, w_2)}_{\mathfrak{S}_q} \in \bar{\mathfrak{C}}_q$$

$$R_1 W_1 \nu + w_1 = -R_2 W_2 \nu + w_2 \implies (w_1 - w_2) \cdot \nu = 0 \quad \circledast$$

$$\Big(\underbrace{Z_1 - W_1}_{\frac{R_1\, \nu \wedge (z_1 - w_1)}{2\lambda_1}}, z_1 - w_1, \underbrace{Z_2 - W_2}_{-\frac{R_2\, \nu \wedge (z_2 - w_2)}{2\lambda_2}}, z_2 - w_2\Big)$$

$$\left.\begin{array}{l} m_1 (z_1 - w_1) + m_2 (z_2 - w_2) = 0 \\ \\ \nu \cdot (z_i - w_i) = 0 \end{array}\right\} \text{condition on vectors in } \bar{\mathfrak{C}}_q$$

Checking: $\langle (W_1, w_1, W_2, w_2), n_q \rangle = \frac{1}{\sqrt{m}} \left(m_1 \left(-\sqrt{\frac{m_2}{m_1}}\right) w_1 \cdot \nu + m_2 \left(\sqrt{\frac{m_1}{m_2}}\right) w_2 \cdot \nu \right)$

$$= \sqrt{\frac{m_1 m_2}{m}} (w_2 - w_1) \cdot \nu = 0 \quad \text{by } \circledast$$

$$\Pi^{\perp} : \mathbb{R}^n \longrightarrow \nu^{\perp} \qquad \frac{R_i}{2\lambda_i} = \frac{1}{R_i \gamma_i^2}$$
$$u \longmapsto \overline{u}$$

1. $W_1 = Z_1 - \dfrac{1}{R_1\gamma_1^2}\, \nu \wedge (z_1 - w_1)$

2. $W_2 = Z_2 + \dfrac{1}{R_2\gamma_2^2}\, \nu \wedge (z_2 - w_2)$

3. $m_1 w_1 + m_2 w_2 = m_1 z_1 + m_2 z_2$

4. $w_1 \cdot \nu = z_1 \cdot \nu$

5. $w_2 \cdot \nu = z_2 \cdot \nu$

6. $R_1 W_1 \nu + w_1 = - R_2 W_2 \nu + w_2$

$\Downarrow$ $\qquad\qquad (\nu \wedge a)\nu = a - a\cdot\nu\nu$

7. $w_1 \cdot \nu = w_2 \cdot \nu \qquad (= z_1 \cdot \nu = z_2 \cdot \nu)$

1 & 2 & 6 $\Longrightarrow$

$$R_1 \underbrace{\left(Z_1 - \frac{1}{R_1\gamma_1^2}\, \nu \wedge (z_1 - w_1)\right)\nu}_{} + w_1 = - R_2 \left(Z_2 + \frac{1}{R_2\gamma_2^2}\, \nu \wedge (z_2 - w_2)\right)\nu + w_2$$

$$R_1 Z_1 \nu - \frac{1}{\gamma_1^2}(z_1 - w_1) + w_1 = - R_2 Z_2 \nu - \frac{1}{\gamma_2^2}(z_2 - w_2) + w_2$$

$$(R_1 Z_1 + R_2 Z_2)\nu - \frac{1}{\gamma_1^2}(z_1 - w_1) + \frac{1}{\gamma_2^2}(z_2 - w_2) = w_2 - w_1$$

$$(R_1 Z_1 + R_2 Z_2)\nu + \frac{1}{\gamma_2^2} z_2 - \frac{1}{\gamma_1^2} z_1 = w_2 - w_1 + \left(\frac{1}{\gamma_2^2} w_2 - \frac{1}{\gamma_1^2} w_1\right)$$

$$
\begin{cases}
-\left(1+\dfrac{1}{\gamma_1^2}\right)w_1+\left(1+\dfrac{1}{\gamma_2^2}\right)w_2=\left(R_1Z_1+R_2Z_2\right)v-\dfrac{1}{\gamma_1^2}Z_1+\dfrac{1}{\gamma_2^2}Z_2\\[2ex]
m_1w_1+m_2w_2=m_1Z_1+m_2Z_2
\end{cases}
$$

$$
\begin{bmatrix} w_1\\ w_2\end{bmatrix}=\underbrace{\begin{bmatrix}-\left(1+\dfrac{1}{\gamma_1^2}\right) & 1+\dfrac{1}{\gamma_2^2}\\[2ex] m_1 & m_2\end{bmatrix}^{-1}}\begin{bmatrix}\left(R_1Z_1+R_2Z_2\right)v-\dfrac{Z_1}{\gamma_1^2}+\dfrac{Z_2}{\gamma_2^2}\\[2ex] m_1Z_1+m_2Z_2\end{bmatrix}
$$

$$
\frac{1}{m_2\left(1+\dfrac{1}{\gamma_1^2}\right)+m_1\left(1+\dfrac{1}{\gamma_2^2}\right)}\begin{bmatrix}-m_2 & 1+\dfrac{1}{\gamma_2^2}\\[2ex] m_1 & 1+\dfrac{1}{\gamma_1^2}\end{bmatrix}
$$

$$
\circledast := m+\frac{m_2}{\gamma_1^2}+\frac{m_1}{\gamma_2^2}
$$

$$
\begin{bmatrix} w_1\\ w_2\end{bmatrix}=\frac{1}{\circledast}\begin{bmatrix}-m_2\left[\left(R_1Z_1+R_2Z_2\right)v-\dfrac{Z_1}{\gamma_1^2}+\dfrac{Z_2}{\gamma_2^2}\right]+\left(1+\dfrac{1}{\gamma_2^2}\right)\left(m_1Z_1+m_2Z_2\right)\\[2ex] m_1\left[\left(R_1Z_1+R_2Z_2\right)v-\dfrac{Z_1}{\gamma_1^2}+\dfrac{Z_2}{\gamma_2^2}\right]+\left(1+\dfrac{1}{\gamma_1^2}\right)\left(m_1Z_1+m_2Z_2\right)\end{bmatrix}
$$

$$
=\frac{1}{\circledast}\begin{bmatrix}-m_2\left(R_1Z_1+R_2Z_2\right)v+\dfrac{m_2Z_1}{\gamma_1^2}-\cancel{\dfrac{m_2Z_2}{\gamma_2^2}}+m_1Z_1+m_2Z_2+\dfrac{m_1}{\gamma_2^2}Z_1+\cancel{\dfrac{m_2Z_2}{\gamma_2^2}}\\[2ex] m_1\left(R_1Z_1+R_2Z_2\right)v-\cancel{\dfrac{m_1Z_1}{\gamma_1^2}}+\dfrac{m_1Z_2}{\gamma_2^2}+m_1Z_1+m_2Z_2+\cancel{\dfrac{m_1Z_1}{\gamma_1^2}}+\dfrac{m_2Z_2}{\gamma_1^2}\end{bmatrix}
$$

$$
=\frac{1}{\circledast}\begin{bmatrix}-m_2\left(R_1Z_1+R_2Z_2\right)v+\circledast Z_1+m_2\left(Z_2-Z_1\right)\\[2ex] m_1\left(R_1Z_1+R_2Z_2\right)v+\circledast Z_2-m_1\left(Z_2-Z_1\right)\end{bmatrix}
$$

$$\begin{cases} w_1 = z_1 + \dfrac{m_2}{\circledast}\left[z_2 - z_1 - (R_1 Z_1 + R_2 Z_2) v \right] \\[2ex] w_2 = z_2 - \dfrac{m_1}{\circledast}\left[z_2 - z_1 - (R_1 Z_1 + R_2 Z_2) v \right] \\[2ex] W_1 = Z_1 + \dfrac{m_2}{\circledast R_1 \gamma_1^2} \, v \wedge \left[z_2 - z_1 - (R_1 Z_1 + R_2 Z_2) v \right] \\[2ex] W_2 = Z_2 + \dfrac{m_1}{\circledast R_2 \gamma_2^2} \, v \wedge \left[z_2 - z_1 - (R_1 Z_1 + R_2 Z_2) v \right] \end{cases}$$

$$\overline{z_2} - \overline{z_1} - (R_1 Z_1 + R_2 Z_2) v$$

Sanity check:

$$[\cdots] \cdot v = 0$$

$$R_1 W_1 v + w_1 - (-R_2 W_2 v + w_2) = (R_1 W_1 + R_2 W_2) v + w_1 - w_2$$

$$= (R_1 Z_1 + R_2 Z_2) v + \left(\frac{m_2}{\gamma_1^2 \circledast} + \frac{m_1}{\gamma_2^2 \circledast} \right) \underbrace{(v \wedge [\cdots]) v}_{} + z_1 - z_2 + \frac{m}{\circledast}[\cdots]$$

$$\overline{[\cdots]} = [\cdots] - [\cdots] \cdot v\, v = [\cdots]$$

$$= (R_1 Z_1 + R_2 Z_2) v + \left(\frac{m_2}{\gamma_1^2} + \frac{m_1}{\gamma_2^2} \right) \frac{1}{\circledast} \cdot \overline{[\cdots]} + z_1 - z_2 + \frac{m}{\circledast}[\cdots]$$

$$= \underbrace{(R_1 Z_1 + R_2 Z_2) v + \overline{z_1} - \overline{z_2}}_{-[\cdots]} + [\cdots] = 0$$

Orthogonal decomposition continued

Let $\xi = (U_1, u_1, U_2, u_2)$ be a state at boundary configuration $q = (I, a_1, I, a_2)$ s.t. $\langle \xi, n_q \rangle < 0$. (a pre-collision state).

$$T_q \partial M \ni \Pi_q \xi = \Big(\underset{Z_1}{\underset{\shortparallel}{U_1}}, \underbrace{\bar{u}_1 + u_c \cdot \nu\nu}_{z_1}, \underset{Z_2}{\underset{\shortparallel}{U_2}}, \bar{u}_2 + u_c \cdot \nu\nu \Big)$$

$$\Pi_q^{\mathfrak{S}} \xi = \Big(U_1 + \frac{m_2}{R_1 \gamma_1^2 \circledast} \nu \wedge \left[\bar{u}_2 - \bar{u}_1 - (R_1 U_1 + R_2 U_2)\nu \right],$$

$$\bar{u}_1 + u_c \cdot \nu\nu + \frac{m_2}{\circledast} \underbrace{\left[\bar{u}_2 - \bar{u}_1 - (R_1 U_1 + R_2 U_2)\nu \right]}_{[\cdots]},$$

$$U_2 + \frac{m_1}{R_2 \gamma_2^2 \circledast} \nu \wedge \left[\cdots\cdots \right],$$

$$\bar{u}_2 + u_c \cdot \nu\nu - \frac{m_1}{\circledast} \left[\cdots \right] \Big)$$

$$\xi - \Pi_q^{\mathfrak{S}} \xi = \Big(-\frac{m_2}{R_1 \gamma_1^2 \circledast} \nu \wedge [\cdots], (u_1 - u_c)\cdot \nu\nu - \frac{m_2}{\circledast}[\cdots],$$

$$-\frac{m_1}{R_2 \gamma_2^2 \circledast} \nu \wedge [\cdots], (u_2 - u_c)\cdot \nu\nu + \frac{m_1}{\circledast}[\cdots] \Big) \in \mathfrak{C}_q$$

<u>No-slip collision</u>: $\xi^+ = \xi^- - 2 \Pi_q^{\mathfrak{C}} \xi^-$

$$\xi^+ = -\xi^- + 2\left[\xi^- - \Pi_q^{\mathfrak{C}} \xi^- \right] = -\xi^- + 2 \Pi_q^{\mathfrak{S}} \xi$$

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$$(U_1^+, u_1^+, U_2^+, u_2^+) = (U_1^-, u_1^-, U_2^-, u_2^-)$$

$$-2\Big(-\frac{m_2}{R_1\gamma_1^2 \circledast}\,\nu\wedge[\cdots]^-,\ (u_1^-\cdot\nu - u_c^-\cdot\nu)\nu - \frac{m_2}{\circledast}[\cdots]^-,$$

$$\frac{m_1}{R_2\gamma_2^2\circledast}\,\nu\wedge[\cdots]^-,\ (u_2^-\cdot\nu - u_c^-\cdot\nu)\nu + \frac{m_1}{\circledast}[\cdots]^-\Big)$$

$$\boxed{\begin{aligned}
U_1^+ &= U_1^- + \frac{2m_2}{R_1\gamma_1^2\circledast}\,\nu\wedge\left[\bar{u}_2^- - \bar{u}_1^- - (R_1U_1^- + R_2U_2^-)\nu\right]\\
U_2^+ &= U_2^- + \frac{2m_1}{R_2\gamma_2^2\circledast}\,\nu\wedge\left[\bar{u}_2^- - \bar{u}_1^- - (R_1U_1^- + R_2U_2^-)\nu\right]\\
u_1^+ &= u_1^- - 2\left\{(u_1^-\cdot\nu - u_c^-\cdot\nu)\nu - \frac{m_2}{\circledast}\left[\bar{u}_2^- - \bar{u}_1^- - (R_1U_1^- + R_2U_2^-)\nu\right]\right\}\\
u_2^+ &= u_2^- - 2\left\{(u_2^-\cdot\nu - u_c^-\cdot\nu)\nu + \frac{m_1}{\circledast}\left[\bar{u}_2^- - \bar{u}_1^- - (R_1U_1^- + R_2U_2^-)\nu\right]\right\}
\end{aligned}}$$

$$[\cdots]^- = \underbrace{\bar{u}_2^- - R_2U_2^-\nu}_{\bar{V}_2(Q)} - \underbrace{(\bar{u}_1^- + R_1U_1^-\nu)}_{\bar{V}_1(Q)} = \bar{V}_2(Q) - \bar{V}_1(Q)$$

$\bar{V}_i(Q)$ proj. of velocity to $\nu^\perp$

$$\circledast = m + \frac{m_2}{\gamma_1^2} + \frac{m_1}{\gamma_2^2}$$

$$u_1 - u_c = u_1 - \frac{m_1u_1 + m_2u_2}{m} = \frac{m_2}{m}(u_1 - u_2)$$

$$u_2 - u_c = -\frac{m_1}{m}(u_1 - u_2)$$

Dim = 2 & 3.

Dim 2 $\quad \mathbb{U} = c\,J \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad c = \dot\theta$

$$v = (\underset{\dot x}{\underset{\|}{v_1}},\ \overbrace{\underset{\dot y}{\underset{\|}{v_2}},\ \underset{\dot z}{\underset{\|}{v_3}}}^{u}) \qquad \dot x = \gamma R c$$

$$V(x) = \mathcal{U}(x-a) + u$$

Kinetic energy $\quad K(\mathcal{U}, u) = \frac{1}{2}\int_{B_R} |V(x)|^2 \underbrace{d\mu(x)}_{\rho(x)\,dx}$ ~~area measure~~

translated body

$$= \frac{1}{2}\int_B \Big(|\underset{cJ}{\underset{\|}{\mathcal{U}}}(x-a)|^2 + |u|^2 \Big)\rho(x)\,dx$$

$$= \frac{1}{2}\Big[c^2 \int_B |x-a|^2 \rho(x)\,dx + m|u|^2 \Big] = \frac{m}{2}\Big\{ \underbrace{2\lambda c^2}_{\gamma^2 R^2 c^2} + |u|^2 \Big\}$$

$$= \frac{m}{2}\left(\dot x^2 + |u|^2 \right) = \frac{1}{2} m |v|^2$$

$\mathcal{J}$ = moment of inertia $= R^2 \gamma^2 c^2$

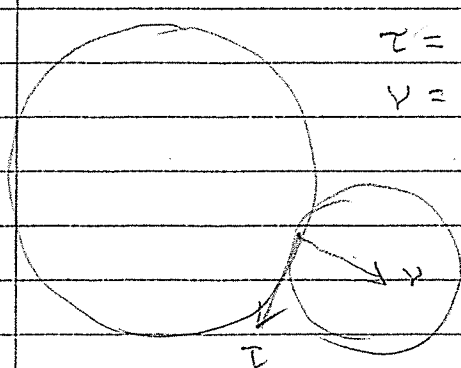
Two (disc) bodies $\quad \mathcal{U}_i^{\pm} = c_i^{\pm} J \longrightarrow v_i^{\pm}\cdot e_1 = \gamma_i R_i c_i^{\pm}$

$$v_i^{\pm} = \left(v_i^{\pm}\cdot e_1,\ u_i^{\pm}\cdot e_2,\ u_i^{\pm}\cdot e_3 \right) \qquad (\mathbb{U}_i e_3)\cdot e_2 = c_i \underbrace{(J\nu)\cdot\tau}_{-1} = -c_i$$

$$= -\gamma_i R_i c_i / \gamma_i R_i = -\frac{v_i\cdot e_1}{\gamma_i R_i}$$

$\tau = e_2 \qquad \tau\wedge\nu = e_1$

$\nu = e_3$



$$[\![u]\!] = \left[\bar u_2 - \bar u_1 - (R_1\mathbb{U}_1 + R_2\mathbb{U}_2)\nu \right]$$

$$[\![u]\!]\cdot e_3 = \bar u_2\cdot e_3 - \bar u_1\cdot e_3 = 0$$

$$[\![u]\!]\cdot e_2 = u_2\cdot e_2 - u_1\cdot e_2 - \underbrace{\left[(R_1 U_1 + R_2 U_2)\nu\right]\cdot \overset{e_3}{\overset{\|}{e_2}}}_{-\,v_1\cdot e_1\ -\ v_2\cdot e_1}$$

(13)

$$[\cdots]\cdot e_2 = u_2\cdot e_2 - u_1\cdot e_2 + \frac{v_1\cdot e_1}{\gamma_1} + \frac{v_2\cdot e_1}{\gamma_2}$$

$$= v_2\cdot e_2 - v_1\cdot e_2 + \frac{v_1\cdot e_1}{\gamma_1} + \frac{v_2\cdot e_1}{\gamma_2}$$

$$[\cdots]\cdot e_3 = v_2\cdot e_3 - v_1\cdot e_3$$

$$u_i^+ = u_i^- + \frac{2(m-m_i)}{R_i\gamma_i^2 \circledast}\, \nu \wedge \underbrace{[\cdots]^-}_{\substack{\| \\ \underbrace{e_3\wedge\left([\cdots]^-\cdot e_2\, e_1\right)} \\ -[\cdots]^-\cdot e_2\, (e_2\wedge e_3) = J}}$$

$$= u_i^- - \frac{2(m-m_i)}{R_i\gamma_i^2 \circledast}[\cdots]^-\cdot e_2\, J$$

$$c_i^+ = c_i^- - \frac{2(m-m_i)}{R_i\gamma_i^2 \circledast}[\cdots]^-\cdot e_2$$

$$\underbrace{\gamma_i R_i c_i^+} = \underbrace{\gamma_i R_i c_i^-} - \frac{2(m-m_i)}{\gamma_i \circledast}[\cdots]^-\cdot e_2$$

$$\boxed{\begin{aligned} v_i^+\cdot e_1 &= v_i^-\cdot e_1 - \frac{2(m-m_i)}{\gamma_i \circledast}\left(v_2^-\cdot e_2 - v_1^-\cdot e_2 + \frac{v_1^-\cdot e_1}{\gamma_1} + \frac{v_2^-\cdot e_1}{\gamma_2}\right) \\ u_i^+ &= u_i^- - 2\left\{\left(u_i^-\cdot e_3 - u_c^-\cdot e_3\right)e_3 + (-1)^i\frac{(m-m_i)}{\circledast}[\cdots]\right\} \end{aligned}}$$

$i = 1, 2$

$$v_i^+ \cdot e_1 = v_i^- \cdot e_1 - \frac{2(m - m_i)}{\gamma_i \circledast}\left(\frac{v_1^- \cdot e_1}{\gamma_1} - v_1^- \cdot e_2 + \frac{v_2^- \cdot e_1}{\gamma_2} + v_2^- \cdot e_2\right)$$

$$v_i^+ \cdot e_2 = v_i^- \cdot e_2 + 2(-1)^i \frac{(m - m_i)}{\circledast}\left(\frac{v_1^- \cdot e_1}{\gamma_1} - v_1^- \cdot e_2 + \frac{v_2^- \cdot e_1}{\gamma_2} + v_2^- \cdot e_2\right)$$

$$v_i^+ \cdot e_3 = v_i^- \cdot e_3 - 2\left\{v_i^- \cdot e_3 - \frac{(m_1 v_1^- \cdot e_3 + m_2 v_2^- \cdot e_3)}{m} + (-1)^i \frac{(m - m_i)}{\circledast}\underbrace{(v_2^- \cdot e_3 - v_1^- \cdot e_3)}_{0}\right\}$$

$$\circledast = m + \frac{m_2}{\gamma_1^2} + \frac{m_1}{\gamma_2^2} = m_2\left[1 + \frac{1}{\gamma_1^2}\right] + m_1\left[1 + \frac{1}{\gamma_2^2}\right]$$

$$\left\{\begin{aligned}
v_1^+ \cdot e_1 &= \left[1 - \frac{2m_2}{\gamma_1^2 \circledast}\right] v_1^- \cdot e_1 + \frac{2m_2}{\gamma_1 \circledast} v_1^- \cdot e_2 - \frac{2m_2}{\gamma_1\gamma_2 \circledast} v_2^- \cdot e_1 - \frac{2m_2}{\gamma_1 \circledast} v_2^- \cdot e_2 \\
v_1^+ \cdot e_2 &= \frac{2m_2}{\gamma_1 \circledast} v_1^- \cdot e_1 + \left[1 - \frac{2m_2}{\circledast}\right] v_1^- \cdot e_2 + \frac{2m_2}{\gamma_2 \circledast} v_2^- \cdot e_1 + \frac{2m_2}{\circledast} v_2^- \cdot e_2 \\
v_1^+ \cdot e_3 &= \left[1 - \frac{2m_2}{m}\right] v_1^- \cdot e_3 + \frac{2m_2}{m} v_2^- \cdot e_3 + \underbrace{\frac{2m_2}{\circledast}(v_2^- - v_1^-) \cdot e_3}_{0} \\
v_2^+ \cdot e_1 &= -\frac{2m_1}{\gamma_1\gamma_2 \circledast} v_1^- \cdot e_1 + \frac{2m_1}{\gamma_2 \circledast} v_1^- \cdot e_2 + \left[1 - \frac{2m_1}{\gamma_2^2 \circledast}\right] v_2^- \cdot e_1 - \frac{2m_1}{\gamma_2 \circledast} v_2^- \cdot e_2 \\
v_2^+ \cdot e_2 &= -\frac{2m_1}{\gamma_1 \circledast} v_1^- \cdot e_1 + \frac{2m_1}{\circledast} v_1^- \cdot e_2 - \frac{2m_1}{\gamma_2 \circledast} v_2^- \cdot e_1 + \left[1 - \frac{2m_1}{\circledast}\right] v_2^- \cdot e_2 \\
v_2^+ \cdot e_3 &= \frac{2m_1}{m} v_1^- \cdot e_3 + \left[1 - \frac{2m_1}{m}\right] v_2^- \cdot e_3 \underbrace{- \frac{2m_1}{\circledast}(v_2^- - v_1^-) \cdot e_3}_{0}
\end{aligned}\right.$$

$$\frac{1}{⊛} = \frac{1}{m_1 m_2 \left[\frac{1}{m_1}\left(1+\frac{1}{\gamma_1^2}\right) + \frac{1}{m_2}\left(1+\frac{1}{\gamma_2^2}\right) \right]}$$

$$\frac{m_1}{⊛} = \frac{1}{\frac{m_2}{m_1}\left(1+\frac{1}{\gamma_1^2}\right) + 1 + \frac{1}{\gamma_2^2}} \xrightarrow{m_1 \to \infty} \frac{\gamma_2^2}{1+\gamma_2^2}$$

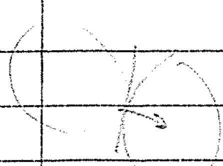
$$\frac{m_2}{⊛} = \frac{1}{1+\frac{1}{\gamma_1^2} + \frac{m_1}{m_2}\left(1+\frac{1}{\gamma_2^2}\right)} \xrightarrow{m_1 \to \infty} 0$$

In this case $(m_1 \to \infty)$ $\quad v_1^+ = v_1^-$ (set $= 0$)

$$v_2^+ \cdot e_1 = \underbrace{\left[1 - \frac{2}{\cancel{\gamma_2^2}} \frac{\cancel{\gamma_2^2}}{1+\gamma_2^2}\right]}_{\frac{1-\gamma_2^2}{1+\gamma_2^2}} v_2^- \cdot e_1 - \underbrace{\frac{2}{\cancel{\gamma_2}} \frac{\gamma_2^{\cancel{2}}}{1+\gamma_2^2}}_{\frac{2\gamma_2}{1+\gamma_2^2}} v_2^- \cdot e_2$$

$$v_2^+ \cdot e_2 = -\underbrace{\frac{2}{\cancel{\gamma_2}} \frac{\gamma_2^{\cancel{2}}}{1+\gamma_2^2}}_{\frac{2\gamma_2}{1+\gamma_2^2}} v_2^- \cdot e_1 + \underbrace{\left[1 - 2\frac{\gamma_2^2}{1+\gamma_2^2}\right]}_{\frac{1-\gamma_2^2}{1+\gamma_2^2}} v_2^- \cdot e_2$$

$$v_2^+ \cdot e_3 = + v_2^- \cdot e_3$$



Dim. 2

$$\frac{1}{\circledast} = \frac{1}{m_1 m_2 \left[\frac{1}{m_1}\left(1+\frac{1}{\gamma_1^2}\right) + \frac{1}{m_2}\left(1+\frac{1}{\gamma_2^2}\right) \right]} = \frac{\delta}{2 m_1 m_2}$$

$$\left[1 - \frac{2m_2}{\gamma_1^2 \circledast}\right] = 1 - \frac{\delta}{\gamma_1^2 m_1} \qquad \frac{2m_2}{\gamma_1 \circledast} = \frac{\delta}{\gamma_1 m_1}$$

$$\left[1 - \frac{2m_2}{\circledast}\right] = 1 - \frac{\delta}{m_1} \qquad \frac{2m_2}{\gamma_1 \gamma_2 \circledast} = \frac{\delta}{\gamma_1 \gamma_2 m_1}$$

$$\begin{array}{l} v_1^+ \cdot e_1 \\ v_1^+ \cdot e_2 \\ v_1^+ \cdot e_3 \\ v_2^+ \cdot e_1 \\ v_2^+ \cdot e_2 \\ v_2^+ \cdot e_3 \end{array}
\left(\begin{array}{ccc|ccc}
1-\frac{\delta}{\gamma^2 m_1} & \frac{\delta}{\gamma_1 m_1} & 0 & -\frac{\delta}{\gamma_1\gamma_2 m_1} & -\frac{\delta}{\gamma_1 m_1} & 0 \\
\frac{\delta}{\gamma_1 m_1} & 1-\frac{\delta}{m_1} & 0 & \frac{\delta}{\gamma_2 m_1} & \frac{\delta}{m_1} & 0 \\
0 & 0 & 1-\frac{2m_2}{m} & 0 & 0 & \frac{2m_2}{m} \\
\hline
-\frac{\delta}{\gamma_1\gamma_2 m_2} & \frac{\delta}{\gamma_2 m_2} & 0 & 1-\frac{\delta}{\gamma_2^2 m_2} & -\frac{\delta}{\gamma_2 m_2} & 0 \\
-\frac{\delta}{\gamma_1 m_2} & \frac{\delta}{m_2} & 0 & -\frac{\delta}{\gamma_2 m_2} & 1-\frac{\delta}{m_2} & 0 \\
0 & 0 & \frac{2m_1}{m} & 0 & 0 & 1-\frac{2m_1}{m}
\end{array}\right)$$

$$\frac{2m_1}{\circledast} = \frac{\delta}{m_2}$$

Same mass Distrib. on B_1, B_2 : $m_0 = m_1 = m_2$

$$\circledast = \frac{1}{2 m_0 \left(1+\frac{1}{\gamma^2}\right)} \qquad \delta = \frac{m_0}{1+\frac{1}{\gamma^2}} \qquad \frac{\delta}{m_0} = \frac{\gamma^2}{1+\gamma^2}$$

$$
\left(\begin{array}{ccc|ccc}
\frac{\gamma^2}{1+\gamma^2} & \frac{\gamma}{1+\gamma^2} & 0 & -\frac{1}{1+\gamma^2} & -\frac{\gamma}{1+\gamma^2} & 0 \\
\frac{\gamma}{1+\gamma^2} & \frac{1}{1+\gamma^2} & 0 & \frac{\gamma}{1+\gamma^2} & \frac{\gamma^2}{1+\gamma^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\hline
-\frac{1}{1+\gamma^2} & \frac{\gamma}{1+\gamma^2} & 0 & \frac{\gamma^2}{1+\gamma^2} & -\frac{\gamma}{1+\gamma^2} & 0 \\
-\frac{\gamma}{1+\gamma^2} & \frac{\gamma^2}{1+\gamma^2} & 0 & -\frac{\gamma}{1+\gamma^2} & \frac{1}{1+\gamma^2} & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right).
$$

$$m_1 = \infty, \quad \frac{\delta}{m_2} = \frac{2m_1}{\otimes} = \frac{2\gamma_2^2}{1+\gamma_2^2} \qquad \gamma = \gamma_2$$

$$
\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & -\frac{1-\gamma^2}{1+\gamma^2} & -\frac{2\gamma}{1+\gamma^2} & 0 \\
 & * & & -\frac{2\gamma}{1+\gamma^2} & \frac{1-\gamma^2}{1+\gamma^2} & 0 \\
 & & & 0 & 0 & -1
\end{array}\right)
$$

Dim 3 $\quad U \in so(3)$ $\qquad \omega(a)b = a\times b$

$a\wedge b = \omega(a\times b)$

$$\omega(u) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

$u = (u_1, u_2, u_3)^t$, $U = \omega(u)$

State : $\boxed{(u, v)}$ $\qquad$ velocity of $x \in B + a$:

$$V(x) = u \times (x-a) + v$$

$e_3 \wedge w = \omega(e_3 \times w)$

$\Updownarrow$

$e_3 \times w$ $\qquad e_3 \wedge \underbrace{U e_3}_{u\times e_3} \longleftrightarrow e_3 \times (u \times e_3) = \bar{u}$ (proj. to $e_3^\perp$)

From page 11 $\qquad \nu = e_3$

$$\begin{cases} u_1^+ = u_1^- + \dfrac{2m_2}{R_1\gamma_1^2 \circledast}\, \underbrace{e_3 \times \left[\bar{v}_2^- - \bar{v}_1^- - (R_1 u_1^- + R_2 u_2^-)\times e_3\right]}_{e_3\times(\bar{v}_2^- - \bar{v}_1^-) + R_1\bar{u}_1^- + R_2\bar{u}_2^-} \\[2ex] u_2^+ = u_2^- + \dfrac{2m_1}{R_2\gamma_2^2\circledast}\left\{ e_3\times(\bar{v}_2^- - \bar{v}_1^-) + R_1\bar{u}_1^- + R_2\bar{u}_2^-\right\} \\[2ex] v_1^+ = v_1^- - 2\left\{(v_1^- - v_c^-)\cdot e_3\, e_3 - \dfrac{m_2}{\circledast}\left[\bar{v}_2^- - \bar{v}_1^- - (R_1u_1^- + R_2u_2^-)\times e_3\right]\right\} \\[2ex] v_2^+ = v_2^- - 2\left\{(v_2^- - v_c^-)\cdot e_3\, e_3 + \dfrac{m_1}{\circledast}\left[\bar{v}_2^- - \bar{v}_1^- - (R_1u_1^- + R_2u_2^-)\times e_3\right]\right\} \end{cases}$$

$\dfrac{2m_2}{R_1\gamma_1^2\circledast} = \dfrac{\delta}{R_1\gamma_1^2 m_1}$, $\quad \dfrac{2m_1}{R_2\gamma_2^2\circledast} = \dfrac{\delta}{R_2\gamma_2^2 m_2}$, $\quad \dfrac{m_2}{\circledast} = \dfrac{1}{2}\dfrac{\delta}{m_1}$, $\quad \dfrac{m_1}{\circledast} = \dfrac{1}{2}\dfrac{\delta}{m_2}$

$$\begin{cases} u_1^+ = u_1^- + \dfrac{\delta}{m_1 R_1 \gamma_1^2}\left(R_1 \overline{u_1^-} + R_2 \overline{u_2^-}\right) + \dfrac{\delta}{m_1 R_1 \gamma_1^2}\, e_3 \times \left(\overline{v_2^-} - \overline{v_1^-}\right) \\[2ex] u_2^+ = u_2^- + \dfrac{\delta}{m_2 R_2 \gamma_2^2}\left(R_1 \overline{u_1^-} + R_2 \overline{u_2^-}\right) + \dfrac{\delta}{m_2 R_2 \gamma_2^2}\, e_3 \times \left(\overline{v_2^-} - \overline{v_1^-}\right) \\[2ex] v_1^+ = \dfrac{\delta}{m_1}\, e_3 \times \left(R_1 u_1^- + R_2 u_2^-\right) + v_1^- - 2\left\{\left(v_1^- - v_c^-\right)\cdot e_3\, e_3 - \dfrac{\delta}{2m_1}\left(\overline{v_2^-} - \overline{v_1^-}\right)\right\} \\[2ex] v_2^+ = -\dfrac{\delta}{m_2}\, e_3 \times \left(R_1 u_1^- + R_2 u_2^-\right) + v_2^- - 2\left\{\left(v_2^- - v_c^-\right)\cdot e_3\, e_3 - \dfrac{\delta}{2m_2}\left(\overline{v_2^-} - \overline{v_1^-}\right)\right\} \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c|c|c} I + \dfrac{\delta}{m_1\gamma_1^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} & \dfrac{R_2\delta}{R_1 m_1\gamma_1^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} & \dfrac{\delta}{m_1 R_1\gamma_1^2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & -\dfrac{\delta}{m_1 R_1\gamma_1^2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} \\ \hline \dfrac{R_1\delta}{R_2 m_2\gamma_2^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} & I + \dfrac{\delta}{m_2\gamma_2^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} & -\dfrac{\delta}{m_2 R_2\gamma_2^2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & \dfrac{\delta}{m_2 R_2\gamma_2^2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} \\ \hline \dfrac{R_1\delta}{m_1}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & \dfrac{R_2\delta}{m_1}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & \begin{matrix}1-\frac{\delta}{m_1}&0&0\\0&1-\frac{\delta}{m_1}&0\\0&0&1-\frac{2m_2}{m}\end{matrix} & \begin{matrix}\frac{\delta}{m_1}&0&0\\0&\frac{\delta}{m_1}&0\\0&0&\frac{2m_2}{m}\end{matrix} \\ \hline -\dfrac{R_1\delta}{m_2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & -\dfrac{R_2\delta}{m_2}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} & \begin{matrix}-\frac{\delta}{m_2}&0&0\\0&-\frac{\delta}{m_2}&0\\0&0&\frac{2m_1}{m}\end{matrix} & \begin{matrix}1+\frac{\delta}{m_2}&0&0\\0&1+\frac{\delta}{m_1}&0\\0&0&1-\frac{2m_1}{m}\end{matrix} \end{array}\right] \begin{bmatrix} u_1 \\ u_2^- \\ v_1^- \\ v_2^- \end{bmatrix}$$

$$v_1^- - v_c^- = \frac{m_2\left(v_1^- - v_2^-\right)}{m} \qquad v_2^- - v_c^- = -\frac{m_1\left(v_1^- - v_2^-\right)}{m}$$

$$(v_1^- - v_c^-)\cdot e_3\, e_3 = \frac{m_2}{m}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}(v_1^- - v_2^-)$$

$$(v_2^- - v_c^-)\cdot e_3\, e_3 = -\frac{m_1}{m}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}(v_1^- - v_2^-)$$

$$-\frac{\delta}{2m_1}(\bar{v}_2^- - \bar{v}_1^-) = \frac{\delta}{2m_1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}(v_1^- - v_2^-)$$

$$-\frac{\delta}{2m_2}(\bar{v}_2^- - \bar{v}_1^-) = \frac{\delta}{2m_2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}(v_1^- - v_2^-)$$

$$v_1^- - 2\left\{(v_1^- - v_c^-)\cdot e_3\, e_3 - \frac{\delta}{2m_1}(\bar{v}_2^- - \bar{v}_1^-)\right\}$$

$$= v_1^- - 2\left\{\frac{m_2}{m}(v_1^- - v_2^-)\cdot e_3\, e_3 + \frac{\delta}{2m_1}(\bar{v}_1^- - \bar{v}_2^-)\right\}$$

$$= \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2m_2}{m}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} - \frac{\delta}{m_1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)v_1^-$$

$$+ \left(\frac{2m_2}{m}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + \frac{\delta}{m_1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\right)v_2^-$$

$$= \begin{bmatrix}1-\frac{\delta}{m_1}&0&0\\0&1-\frac{\delta}{m_1}&0\\0&0&1-\frac{2m_2}{m}\end{bmatrix}v_1^- + \begin{bmatrix}\frac{\delta}{m_1}&0&0\\0&\frac{\delta}{m_1}&0\\0&0&\frac{2m_2}{m}\end{bmatrix}v_2^-$$

$$v_2^- - 2\left\{\underbrace{(v_2^- - v_c^-)\cdot e_3 e_3} + \frac{\delta}{2m_2}\left(\bar{v}_1^- - \bar{v}_2^-\right)\right\}$$

$$v. \qquad -\frac{m_1}{m}\left(v_1^- - v_2^-\right)\cdot e_3 e_3$$

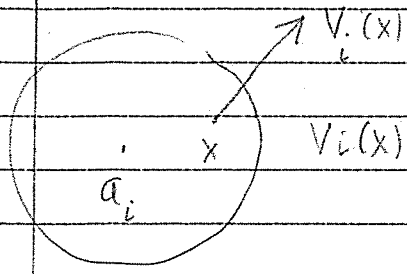
$$= \frac{2m_1}{m} v_1^-\cdot e_3 e_3 - \frac{\delta}{m_2}\bar{v}_1^- + v_2^- - 2\frac{m_1}{m} v_2^-\cdot e_3 e_3 + \frac{\delta}{m_2}\bar{v}_2^-$$

$$= \begin{bmatrix} -\frac{\delta}{m_1} & 0 & 0 \\ 0 & -\frac{\delta}{m_2} & 0 \\ 0 & 0 & \frac{2m_1}{m} \end{bmatrix} \bar{v}_1^- + \begin{bmatrix} 1+\frac{\delta}{m_2} & 0 & 0 \\ 0 & 1+\frac{\delta}{m_2} & 0 \\ 0 & 0 & 1-\frac{2m_1}{m} \end{bmatrix} \bar{v}_2^-$$

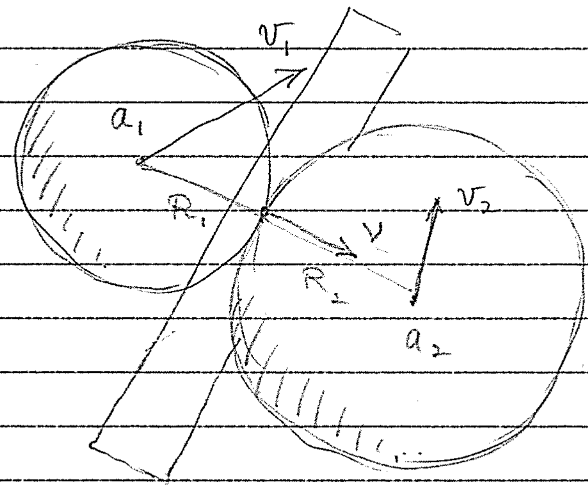
$$\left(\begin{array}{ccc|ccc|ccc|ccc}
1+\frac{\delta}{m_1\gamma_1^2} & 0 & 0 & \frac{R_2}{R_1}\frac{\delta}{m_1\gamma_1^2} & 0 & 0 & 0 & -\frac{\delta}{m_1R_1\gamma_1^2} & 0 & 0 & \frac{\delta}{m_1R_1\gamma_1^2} & 0 \\
0 & 1+\frac{\delta}{m_1\gamma_1^2} & 0 & 0 & \frac{R_2}{R_1}\frac{\delta}{m_1\gamma_1^2} & 0 & \frac{\delta}{m_1R_1\gamma_1^2} & 0 & 0 & -\frac{\delta}{m_1R_1\gamma_1^2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{R_1\delta}{R_2 m_2\gamma_2^2} & 0 & 0 & 1+\frac{\delta}{m_2\gamma_2^2} & 0 & 0 & 0 & \frac{\delta}{m_2R_2\gamma_2^2} & 0 & 0 & -\frac{\delta}{m_2R_2\gamma_2^2} & 0 \\
0 & \frac{R_1\delta}{R_2 m_2\gamma_2^2} & 0 & 0 & 1+\frac{\delta}{m_2\gamma_2^2} & 0 & -\frac{\delta}{m_2R_2\gamma_2^2} & 0 & 0 & \frac{\delta}{m_2R_2\gamma_2^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -\frac{R_1\delta}{m_1} & 0 & 0 & -\frac{R_2\delta}{m_1} & 0 & 1-\frac{\delta}{m_1} & 0 & 0 & \frac{\delta}{m_1} & 0 & 0 \\
\frac{R_1\delta}{m_1} & 0 & 0 & \frac{R_2\delta}{m_1} & 0 & 0 & 0 & 1-\frac{\delta}{m_1} & 0 & 0 & \frac{\delta}{m_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{2m_2}{m} & 0 & 0 & \frac{2m_2}{m} \\
\hline
0 & \frac{R_1\delta}{m_2} & 0 & 0 & \frac{R_2\delta}{m_2} & 0 & -\frac{\delta}{m_2} & 0 & 0 & 1+\frac{\delta}{m_2} & 0 & 0 \\
-\frac{R_1\delta}{m_2} & 0 & 0 & -\frac{R_2\delta}{m_2} & 0 & 0 & 0 & -\frac{\delta}{m_2} & 0 & 0 & 1+\frac{\delta}{m_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2m_1}{m} & 0 & 0 & 1-\frac{2m_1}{m}
\end{array}\right)$$

State of body B_i : $(a_i \in \mathbb{R}^n,\ U_i \in SO(n),\ v_i \in \mathbb{R}^n)$

center angular veloc. velocity of center.



$$V_i(x) = U_i(x - a_i) + v_i$$



$$\frac{2m_2}{R_1\gamma_1^2 \circledast} = \frac{\delta}{R_1\gamma_1^2 m_1} \qquad \frac{2m_1}{R_2\gamma_2^2 \circledast} = \frac{\delta}{R_2\gamma_2^2 m_2} \qquad \frac{m_1}{\circledast} = \frac{1}{2}\frac{\delta}{m_2}$$

$$\frac{m_2}{\circledast} = \frac{1}{2}\frac{\delta}{m_1}$$

$$\begin{cases} U_1^+ = U_1^- - \dfrac{\delta}{R_1\gamma_1^2 m_1}\,\nu \wedge (R_1U_1^- + R_2U_2^-)\nu + \dfrac{\delta}{R_1\gamma_1^2 m_1}\,\nu \wedge (v_2^- - v_1^-) \\[2ex] U_2^+ = U_2^- - \dfrac{\delta}{R_2\gamma_2^2 m_2}\,\nu \wedge (R_1U_1^- + R_2U_2^-)\nu + \dfrac{\delta}{R_2\gamma_2^2 m_2}\,\nu \wedge (v_2^- - v_1^-) \\[2ex] v_1^+ = v_1^- - \dfrac{\delta}{m_1}(R_1U_1^- + R_2U_2^-)\nu - \dfrac{2m_2}{m}(v_1^- - v_2^-)\cdot\nu\,\nu - \dfrac{\delta}{m_1}(\bar{v}_1^- - \bar{v}_2^-) \\[2ex] v_2^+ = v_2^- + \dfrac{\delta}{m_2}(R_1U_1^- + R_2U_2^-)\nu + \dfrac{2m_1}{m}(v_1^- - v_2^-)\cdot\nu\,\nu + \dfrac{\delta}{m_2}(\bar{v}_1^- - \bar{v}_2^-) \end{cases}$$

$\Pi : \mathbb{R}^n \longrightarrow \nu^\perp$

$\Pi^\nu : \mathbb{R}^n \longrightarrow \mathbb{R}\nu$

orthogonal projection

$\Lambda^\nu : so(n) \longrightarrow so(n) \qquad \Lambda^\nu(U) = \nu \wedge U\nu$

$\Lambda^\nu(U) = \nu \wedge U\nu \qquad \Lambda^\nu(U)_{ij} = \nu_j (U\nu)_i - \nu_i (U\nu)_j$

$\Gamma^\nu : so(n) \longrightarrow \mathbb{R}^n, \quad U \mapsto U\nu$

$E^\nu : \mathbb{R}^n \longrightarrow so(n) \qquad E^\nu(v) = \nu \wedge v$

$$\begin{cases}
U_1^+ = \left[I - \dfrac{\delta}{m_1\gamma_1^2}\Lambda^\nu\right] U_1^- - \dfrac{\delta}{m_1 R_1 \gamma_1^2} E^\nu v_1^- - \dfrac{R_2}{R_1}\dfrac{\delta}{m_1\gamma_1^2}\Lambda^\nu U_2^- + \dfrac{\delta}{m_1 R_1 \gamma_1^2} E^\nu v_2^- \\[2ex]
v_1^+ = -\dfrac{R_1\delta}{m_1}\Gamma^\nu U_1^- + \left(I - \dfrac{2m_2}{m}\Pi^\nu - \dfrac{\delta}{m_1}\Pi\right) v_1^- - \dfrac{R_2\delta}{m_1}\Gamma^\nu U_2^- + \left(\dfrac{2m_2}{m}\Pi^\nu + \dfrac{\delta}{m_1}\Pi\right) v_2^- \\[2ex]
U_2^+ = -\dfrac{R_1}{R_2}\dfrac{\delta}{m_2\gamma_2^2}\Lambda^\nu U_1^- - \dfrac{\delta}{m_2 R_2 \gamma_2^2} E^\nu v_1^- + \left[I - \dfrac{\delta}{m_2\gamma_2^2}\Lambda^\nu\right] U_2^- + \dfrac{\delta}{m_2 R_2 \gamma_2^2} E^\nu v_2^- \\[2ex]
v_2^+ = \dfrac{R_1\delta}{m_2}\Gamma^\nu U_1^- + \left(\dfrac{2m_1}{m}\Pi^\nu + \dfrac{\delta}{m_2}\Pi\right) v_1^- + \dfrac{R_2\delta}{m_2}\Gamma^\nu U_2^- + \left(I - \dfrac{2m_1}{m}\Pi^\nu - \dfrac{\delta}{m_2}\Pi\right) v_2^-
\end{cases}$$

$$
\begin{array}{c}
u_1 \\ \\ v_1 \\ \\ U_2 \\ \\ v_2
\end{array}
\left(
\begin{array}{cc|cc}
I - \frac{\delta}{m_1\gamma_1^2}\Lambda^\nu & -\frac{\delta}{m_1 R_1 \gamma_1^2} E^\nu & -\frac{R_2\,\delta}{R_1\, m_1\gamma_1^2}\Lambda^\nu & \frac{\delta}{m_1 R_1 \gamma_1^2} E^\nu \\[2ex]
-\frac{R_1\delta}{m_1}\Gamma^\nu & I - \frac{2m_2}{m}\Pi^\nu - \frac{\delta}{m_1}\Pi & -\frac{R_2\delta}{m_1}\Gamma^\nu & \frac{2m_2}{m}\Pi^\nu + \frac{\delta}{m_1}\Pi \\[2ex]
\hline
-\frac{R_1}{R_2}\frac{\delta}{m_2\gamma_2^2}\Lambda^\nu & -\frac{\delta}{m_2 R_2\gamma_2^2}E^\nu & I - \frac{\delta}{m_2\gamma_2^2}\Lambda^\nu & \frac{\delta}{m_2 R_2 \gamma_2^2}E^\nu \\[2ex]
\frac{R_1\delta}{m_2}\Gamma^\nu & \frac{2m_1}{m}\Pi^\nu + \frac{\delta}{m_2}\Pi & \frac{R_2\delta}{m_2}\Gamma^\nu & I - \frac{2m_1}{m}\Pi^\nu - \frac{\delta}{m_2}\Pi
\end{array}
\right)
$$

$m_1 \to \infty$

$$
\left(
\begin{array}{cc|cc}
I & 0 & 0 & 0 \\[2ex]
0 & I & 0 & 0 \\[2ex]
\hline
-\frac{R_1}{R_2}\frac{\delta}{m_2\gamma_2^2}\Lambda^\nu & -\frac{\delta}{m_2 R_2\gamma_2^2}E^\nu & I - \frac{\delta}{m_2\gamma_2^2}\Lambda^\nu & \frac{\delta}{m_2 R_2\gamma_2^2}E^\nu \\[2ex]
\frac{R_1\delta}{m_2}\Gamma^\nu & 2\Pi^\nu + \frac{\delta}{m_2}\Pi & \frac{R_2\delta}{m_2}\Gamma^\nu & I - 2\Pi^\nu - \frac{\delta}{m_2}\Pi
\end{array}
\right)
$$

$\frac{\delta}{m\gamma^2} = \frac{2}{1+\gamma^2}$, $\delta = \frac{2m\gamma^2}{1+\gamma^2}$

$u_1 = 0,\ v_1 = 0$

$u_2 = u,\ v_2 = v$

$R_2 = R$, etc.

$$
\begin{cases}
U^+ = \Big(I - \underbrace{\frac{\delta}{m\gamma^2}}_{=2/(1+\gamma^2)}\Lambda^\nu\Big)U^- + \frac{\delta}{mR\gamma^2}E^\nu v^- \\[2ex]
v^+ = \frac{R\delta}{m}\Gamma^\nu U^- + \Big(I - 2\Pi^\nu - \frac{\delta}{m}\Pi\Big)v^-
\end{cases}
$$