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The dynamics of billiards with no-slip collisions

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We propose a new model of a gas of rough hard spheres that includes inelastic effects. The spheres interact with each other and with the container walls through inelastic collisions. The impulsive friction without slipping at the impacts conserves the total energy of the system, but mixes the tangential velocity components with the angular velocities of colliding spheres. We explore the – sometimes surprising – consequences of this in a simple billiard model.

1. Introduction

The simplest physical picture we have of a gas is the classical one of a collection of hard spheres which interact with each other and with the container walls through elastic collisions [1]. In various forms, and in various degrees of abstraction, this has been a basic model studied by the physicists and mathematicians concerned with the fundamentals of statistical mechanics.

Since the particles of a gas are not, in reality, hard spheres, the assumption of perfectly elastic collisions is an approximation: in general there is transfer of energy between the translational motion of the gas particles and various internal degrees of freedom. For molecular gases at room temperatures the most accessible degrees of freedom are associated with the rotational motion of the molecules. However, vibrational and even electronic states can become involved depending on the physical and chemical details of the system. The coupling between these modes and the translational motion takes place at the collision and depends strongly on the specific geometrical arrangement of the molecules as they collide. In particular, coordinates, trivial in the hard sphere case, which specify the orientation (and conjugate momenta) of the molecules have to be included. These more realistic models are, therefore, considerably more complicated than the basic hard sphere gas.

The work described in this paper concerns a simple model which is intermediate between the hard sphere models and the more complicated, realistic models incorporating inelastic effects. The basic idea is to retain the conceptual simplicity of the hard sphere gas while modifying the interaction to include inelastic effects. We do this by dealing with rough hard spheres, i.e. we impose the *no-slip condition* on the impulsive frictional force at the point of impact (eq. (11)). Since the impulsive force at the collision does not act radially through the centres of mass of the two colliding spheres, a torque is generated which couples the translational and rotational degrees of freedom. On the other hand, the no-slip

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friction means that the total energy is conserved, and the resulting dynamics of the system of rough hard spheres is Hamiltonian.

The simplest idealization of a system of rough hard spheres is the *no-slip billiard*: one ball moving inside a domain bounded by rough hard walls. As we shall show, even this simple model produces interesting (and unexpected) effects.

In section 2 we set up the problem mathematically. For simplicity of exposition we restrict ourselves to the *homogeneous rough hard spheres* in 2D, i.e., colliding discs. We show that the requirement of conservation of the total energy imposes a unique solution for the impulsive force at the impact, and compute the resulting transformation of the generalized velocities (see eqs. (12), (13)).

In section 3 we consider the situation when a homogeneous rough hard sphere moves freely in a plane domain bounded by a rough hard wall. Treating collisions with the wall as a limiting case of the situation considered in section 2, we obtain the transformation of the generalized velocity of the sphere resulting from a collision with the wall (eqs. (16), (17)). Thus we obtain a new class of dynamical systems: *billiards with no-slip collisions* (we also call them *inelastic billiards*).

The dynamics of inelastic billiards is completely determined by the shape of the bounding curve. In section 4 we treat in considerable detail a simple example: *no-slip billiard in a strip*. An analysis of the inelastic billiard ball map reveals some surprising features of no-slip billiards. For instance, the trajectories of the rough hard ball in a strip are bounded: the ball moving along the strip *eventually reverses its direction*. We summarize the qualitative features of the dynamics of inelastic billiards in section 5.

The motion of rigid bodies subject to nonholonomic constraints, e.g., the rolling of a sphere on a surface without slipping, has been extensively discussed in the literature (see, e.g., [2,3] or [4].) There is also a vast body of literature on the impact dynamics of rigid bodies. Friction plays an important role in impact dynamics, and researchers studied the effects of many kinds of friction (see, e.g., [5–7]), but the impact with no-slip friction has not been considered. In this paper we begin a study of the dynamics of rigid bodies with no-slip collisions, and, in particular, the dynamics of no-slip billiards.

2. Collisions of homogeneous rough hard spheres

In this section the basic equations specifying the rough hard spheres model are derived. It is assumed that there are no long-range forces, thus the system consists of a finite number of homogeneous spheres each with specified mass m_i, and radius r_i (i taken from an indexing set), which move freely between collisions. In deriving the equations governing the collisions the possibility of a frictional force tangential to the spheres at their point of contact is included. The form of this force is determined by imposing energy conservation at the collision.

For simplicity of exposition, we restrict ourselves to 2 dimensions, i.e., our spheres are in fact discs on the plane. Fig. 1 shows the notation and coordinates used for a mathematical description of a collision between two such discs. We take the convention that Φ is the impulsive force applied to the right hand sphere during the collision (labeled v on fig. 1). By Newton's law of the reaction force, the impulsive force applied to left hand sphere is $-\Phi$.

The impulsive force Φ can be resolved into a component Φ_x, normal to the spheres at the point of contact and the tangential component Φ_y. In the absence of friction, $\Phi_y = 0$. In the case that $\Phi_y \neq 0$, there is an impulsive torque, $-r_v \Phi_y$, applied to the sphere labeled v and a corresponding torque, $-r_V \Phi_y$, applied to the sphere labeled V. The changes of linear momentum and angular momentum that

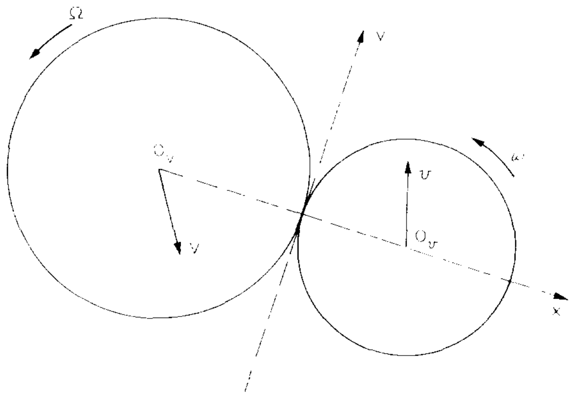

Fig. 1. Collision between two rough hard discs. Collision frame.

result from the collision of a pair of spheres are given by the appropriate impulsive force or torque:

$$m_v[v(2) - v(1)] = \Phi \,, \tag{1}$$

$$m_V[V(2) - V(1)] = -\Phi \,, \tag{2}$$

$$I_v[\omega(2) - \omega(1)] = -r_v \Phi_y \,, \tag{3}$$

$$I_V[\Omega(2) - \Omega(1)] = -r_V \Phi_y \,, \tag{4}$$

where $I = \frac{1}{2}mr^2$ is the moment of inertia of the appropriately labeled sphere, $v(1)$, $V(1)$, $\omega(1)$ and $\Omega(1)$ indicate respectively the velocities and angular velocities before the collision, and $v(2)$, $V(2)$, $\omega(2)$ and $\Omega(2)$ are the corresponding quantities after the collision.

Assuming free motion between collisions, the total energy of a pair of spheres is simply

$$E = \tfrac{1}{2}m_v\|v\|^2 + \tfrac{1}{2}m_V\|V\|^2 + \tfrac{1}{2}I_v\omega^2 + \tfrac{1}{2}I_V\Omega^2 \,. \tag{5}$$

Eqs. (1)–(4) then give the following expression for the change in the total energy, $E(2) - E(1)$, as a result of the collision:

$$E(2) - E(1) = \Phi_x\left(v_x(1) - V_x(1) + \frac{1}{2\mu}\,\Phi_x\right) \tag{6}$$

$$+\Phi_y\left(v_y(1) - V_y(1) - r_v\omega(1) - r_V\Omega(1) + \frac{3}{2\mu}\,\Phi_y\right) , \tag{7}$$

where $\mu = m_v m_V/(m_v + m_V)$, is the reduced mass of the pair. It follows, from conservation of the total energy, that the normal component of the impulse must satisfy

$$\Phi_x = 0 \,, \tag{8}$$

or

$$\Phi_x = 2\mu[V_x(1) - v_x(1)] \,. \tag{9}$$

Similarly, the tangential component of the impulse must satisfy

$$\Phi_y = 0\,, \tag{10}$$

or

$$\Phi_y = \tfrac{2}{3}\mu\{V_y(1) + r_V\Omega(1) - [v_y(1) - r_v\omega(1)]\}\,. \tag{11}$$

Of these possibilities eq. (8) is physically uninteresting, while eq. (9) is the usual law of elastic collisions for point masses. The two possibilities for the tangential component, Φ_y, correspond to either the elastic collision without friction, eq. (10), or, in the case of eq. (11), to the frictional impulse being proportional to the relative tangential velocity of the spheres at the point of contact. These two extremes correspond, respectively, to frictionless slip and static *no-slip friction* between the spheres [2]. Physically, either of these possibilities conserves energy, see eq. (5)

The equations governing the collision of a pair of rough spheres are obtained by substituting eqs. (9) and (11) into eqs. (1)–(4). The result is the following *collision transformation*:

$$U(2) = TU(1)\,, \tag{12}$$

where $U = (v_x, V_x, v_y, V_y, \tilde{\omega}, \tilde{\Omega})$, $\tilde{\omega} = r_v\omega/\sqrt{2}$, $\tilde{\Omega} = r_V\Omega/\sqrt{2}$, is the vector of *total generalized velocity*, and the linear transformation T is given in our *collision coordinates* by the following *collision matrix*:

$$T = \begin{pmatrix} 1-2\mu_v & 2\mu_v & 0 & 0 & 0 & 0 \\ 2\mu_V & 1-2\mu_V & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{2}{3}\mu_v & \frac{2}{3}\mu_v & \frac{2}{3}\sqrt{2}\mu_v & \frac{2}{3}\sqrt{2}\mu_v \\ 0 & 0 & \frac{2}{3}\mu_V & 1-\frac{2}{3}\mu_V & -\frac{2}{3}\sqrt{2}\mu_V & -\frac{2}{3}\sqrt{2}\mu_V \\ 0 & 0 & \frac{2}{3}\mu_v & -\frac{2}{3}\sqrt{2}\mu_v & 1-\frac{4}{3}\mu_v & -\frac{4}{3}\mu_v \\ 0 & 0 & \frac{2}{3}\sqrt{2}\mu_V & -\frac{2}{3}\sqrt{2}\mu_V & -\frac{4}{3}\mu_V & 1-\frac{4}{3}\mu_V \end{pmatrix}, \tag{13}$$

where $\mu_v = m_V/(m_v + m_V)$ and $\mu_V = m_v/(m_v + m_V)$.

The block diagonal form of T in eq. (13) reflects the fact that the collision transformation does not mix the normal velocity components with either the tangential components or the angular velocities. However, the effect of the friction is that T does mix the tangential velocity components with the angular velocities.

In order to calculate the products of the collision transformations T corresponding to successive collisions it is necessary to write the collision matrices in a fixed coordinate frame. In this frame the collision matrix will have the form $R_\theta^{-1}TR_\theta$, where R_θ rotates the fixed frame into the collision frame:

$$R_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

3. Billiards with no-slip impacts

To apply these ideas to billiard models we need to consider the collision of a rough hard sphere with a rough wall. Let the wall be represented by a smooth curve Γ. To analyse a collision with the wall we replace Γ at the point of impact by a stationary rough hard sphere of infinite mass and specialise the equations of section 2 to this case. Our dynamical system is then a *billiard with no-slip impacts*. We will also call it the *inelastic billiard* in Γ, as opposed to the usual *elastic billiard* (eq. (10)).

Let (v_x, v_y) be the linear velocity and ω the angular velocity of the ball of mass m and radius r. Specialising eqs. (9) and (11) to the limiting case $V=0$, $\Omega=0$, and $m_V=\infty$, we obtain

$$\Phi = \begin{pmatrix} -2mv_x(1) \\ -\frac{2}{3}m[v_y(1) - r\omega(1)] \end{pmatrix} \tag{15}$$

for the impulsive force on the ball. At the moment of collision with the wall the *total velocity* $V=(v_x, v_y, r\omega/\sqrt{2})$ of the ball is changed by the *collision transformation*:

$$V(2) = TV(1)\,, \tag{16}$$

which is given in collision frame by the following 3×3 matrix obtained from eq. (13).

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3}\sqrt{2} \\ 0 & \frac{2}{3}\sqrt{2} & -\frac{1}{3} \end{pmatrix}. \tag{17}$$

The rotation matrix R_θ of eq. 14 becomes:

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

To describe the dynamics completely, we need to determine the phase space Y and to define the *inelastic billiard flow* B^t on it. A point in Y is given by its *generalized position* $Q=(q_1, q_2, q_3)$ and the *generalized velocity* $V=(v_1, v_2, v_3)$. Here $P=(q_1, q_2)$ is the position of the centre of the ball, and $q_3=(r/\sqrt{2})\alpha$ is its normalized angular coordinate (α is the angle of intrinsic rotation of the ball). Analogously, $v=(v_1, v_2)$ is the linear velocity, and $v_3=(r/\sqrt{2})\omega$ is the normalized angular velocity of the ball. The flow B^t between collisions is given by

$$B^t(Q, V) = (Q + tV, V)\,. \tag{19}$$

The result of a collision (at $P\in\Gamma$) is that the generalized velocity V changes instantaneously by $V\mapsto V' = TV$ according to eqs. (16), (17). Note the eq. (17) gives the matrix of the linear transformation T in collision coordinates (x, y, z) determined by the point of impact. Then the flow of eq. (19) resumes until the next collision.

Since $\|TV\| = \|V\|$, the speed $\|V\|$ is preserved by B^t, and we restrict the flow to the 5-dimensional manifold $H=\{(Q, V): \|V\| = 1\}$. We call the flow B^t on H the billiard flow with no-slip impacts inside Γ, or, simply, *no-slip billiard flow* in Γ.

The generalized angle variable q_3 is defined modulo $2\pi r/\sqrt{2}$. We shall assume that Γ is such that the value of r plays no role in our model, so without loss of generality we set $r = \sqrt{2}$, and hence q_3 takes values in the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Denote by G the closed planar domain bounded by Γ. If Γ is a simple closed curve, G is homeomorphic to the unit disc D. Denote by S^2 the unit sphere of the generalized velocities V. The closed 5-dimensional manifold H is obtained from $G \times S^1 \times S^2$ by identifications on its boundary $\Gamma \times S^1 \times S^2$.

As with the usual elastic billiard [8], our no-slip billiard flow B^t has a *natural cross-section* $C \subset H$ formed by phase points describing the data "immediately after impact". Let $(\boldsymbol{t}(P), \boldsymbol{n}(P), \boldsymbol{k})$ be the canonical orthonormal frame defined by Γ at $P \in \Gamma$. For a point $Q = (q_1, q_2, q_3)$ in the configuration space we set $P = (q_1, q_2) \in G$, $\theta = q_3 (\mathrm{mod}\, 2\pi) \in S^1$. Then $C \doteq \{(Q, V) : P \in \Gamma,\ V \in S^2,\ \langle V | \boldsymbol{n}(P) \rangle \geq 0\}$ where $\langle V | \boldsymbol{n}(p) \rangle$ is the scalar product in $\mathbb{R}^3$. Identifying the semisphere $\{V : \langle V | \boldsymbol{n}(P) \rangle \geq 0\}$ with the unit disc D, we obtain $C = \{\Gamma \times S^1 \times D\}$. When Γ is a simple closed curve, C is homeomorphic to $\mathbb{T}^2 \times D$, the direct product of the 2-torus and the disc.

The billiard flow B^t on H determines the *first return map* $F : C \to C$, which is essentially equivalent to B^t, and is often more convenient to work with. In what follows we call F the *"no-slip" billiard ball map*.

4. No-slip billiard in a strip

The quantitative behavior of billiards with no-slip impacts differs considerably from the dynamics of the usual, elastic billiards [9]. To illustrate this, we consider the case of a ball bouncing between two straight, parallel walls. There are two collision frames, labelled "left" and "right" (L and R) in fig. 2. Take L as the fixed frame. With this choice, the reduced rotation matrix R_θ of eq. (18) for the collision with the left hand wall becomes the 3×3 identity matrix, and for the right hand wall it becomes

$$R_\pi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

These matrices, since they are diagonal, do not couple the parallel and rotational motions of the sphere with the motion perpendicular to the walls (thus $|v_x|$ is conserved and v_x reverses sign with each bounce). The part Y_0 of the phase space given by $\{|v_x| = 0\}$ is invariant under B^t. The phase points in Y_0 move along the strip with constant generalized velocity V. In what follows we investigate the billiard flow in the dense open B^t-invariant part of the phase space given by $\{|v_x| \neq 0\}$.

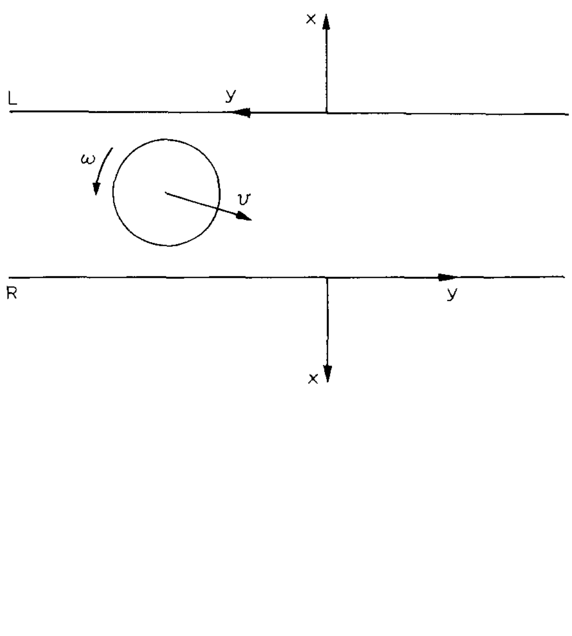

Fig. 2. Billiard with no-slip impacts in an infinite strip.

We consider the natural projection $(v_x, v_y, \omega) \mapsto (v_y, \omega)$ of the total velocity onto the (v_y, ω)-plane W. In this projection the collision matrix reduces to a matrix $\bar{T}$, which has the following form:

$$\bar{T} = s\bar{R}_\beta\,, \tag{21}$$

where s is a reflection in the v_y-axis and $\bar{R}_\beta$ is a plane rotation through β, the dihedral angle of a regular tetrahedron[#1]. Similarly R_π reduces to σ, a reflection in the ω-axis.

If Γ is a general polygon, each side can be labelled with a symbol, and trajectories will be coded by sequences of symbols according to the sides visited. In the present case this is rather trivial: there are two walls corresponding to the labels L and R. Associated with these labels we define a pair of operators, $\bar{T}_L, \bar{T}_R : W \to W$, which give the evolution of (v_y, ω) as a result of collisions at the walls:

$$\bar{T}_L = \bar{T}\,, \tag{22}$$

$$\bar{T}_R = \sigma \bar{T} \sigma\,. \tag{23}$$

Using elementary properties of s and σ, $\bar{T}_L$ and $\bar{T}_R$, we can write

$$\bar{T}_L = s\bar{R}_\beta\,, \tag{24}$$

$$\bar{T}_R = \bar{R}_\beta s\,. \tag{25}$$

To simplify the computations, we introduce the complex variables:

$$w = v_y + \mathrm{i}\omega\,, \quad z = y + \mathrm{i}q_3\,. \tag{26}$$

The "complex generalized position" z measures the position y along the wall and the "unfolded internal angle" q_3 of the ball. The "complex generalized velocity" w measures the velocity component along the wall and the angular velocity of the ball. Since s corresponds to complex conjugation, and $\bar{R}_\beta$ to multiplication by a complex constant, eqs. (22), (23) become

$$\bar{T}_L w = \mathrm{e}^{\mathrm{i}\beta} w^*\,, \quad \bar{T}_R w = \mathrm{e}^{-\mathrm{i}\beta} w^*\,, \tag{27}$$

where w^* denotes the complex conjugate of w. Since $v_x \neq 0$, the geometry of the system ensures that the only coding permissible for trajectories is: . . . LRLRLR. . . . The cross-section C defined in section 3 is disjoint union of two identical sets C_L and C_R corresponding to the two walls of the strip. The billiard ball map satisfies $F: C_L \to C_R$, $C_R \to C_L$. Hence C_L and C_R are invariant under F^2. It suffices to analyze $F^2: C_R \to C_R$. Dynamically, this corresponds to the ball starting off at the right wall, colliding with the left wall, returning back to the right wall, hitting it, and about to repeat the cycle.

Let (z, w) be the generalized position and the generalized velocity of the ball immediately after a collision with the right wall, i.e. $(z, w) \in C_R$. Denote by (z_1, w_1) the generalized position and velocity of the ball immediately after the first collision with the left wall (measured in the "right wall coordinates"). The same way we define (z_2, w_2), etc. The sequence $\{(z_n, w_n), -\infty < n < \infty, (z_0, w_0) =$

[#1] The angle β satisfies $\sin \beta = \frac{2}{3}\sqrt{2}$ and $\cos \beta = \frac{1}{3}$, hence $\beta = \arctan 2\sqrt{2}$.

$(z, w)\}$ is, essentially, the trajectory of our no-slip billiard ball in the strip. Thus $F:(z, w) \to (z_1, w_1)$, $F^2:(z, w) \to (z_2, w_2)$. If d is the width of the strip, the time between consecutive collisions is $t = d/|v_x|$.

By eqs. (26), (27),

$$z_1 = z + tw\,, \quad w_1 = \mathrm{e}^{-\mathrm{i}\beta} w^*\,, \tag{28}$$

$$z_2 = z + t(w + \mathrm{e}^{-\mathrm{i}\beta}\, w)\,, \quad w_2 = \mathrm{e}^{2\mathrm{i}\beta}\, w\,. \tag{29}$$

Iterating these equations n times, we obtain

$$z_{2n} = z + t(w + \mathrm{e}^{-\mathrm{i}\beta}\, w^* + \mathrm{e}^{2\mathrm{i}\beta}\, w + \cdots + \mathrm{e}^{-\mathrm{i}(2n-1)\beta}\, w^* + \mathrm{e}^{2\mathrm{i}(n-1)\beta} w) \tag{30}$$

and

$$w_{2n} = \mathrm{e}^{2\mathrm{i}n\beta}\, w\,. \tag{31}$$

Summing the geometric progressions, we simplify eq. (30):

$$z_{2n} - z = t\left(w \frac{\mathrm{e}^{2\mathrm{i}n\beta} - 1}{\mathrm{e}^{2\mathrm{i}\beta} - 1} + \mathrm{e}^{-\mathrm{i}\beta}\, w^* \frac{\mathrm{e}^{-2\mathrm{i}n\beta} - 1}{\mathrm{e}^{-2\mathrm{i}\beta} - 1} \right). \tag{32}$$

After further simplifications, eq. (32) yields

$$z_{2n} - z = \frac{t\,\mathrm{e}^{-\mathrm{i}\beta/2}}{\sin\beta}\, \mathscr{I}[w\,\mathrm{e}^{-\mathrm{i}\beta/2}(\mathrm{e}^{2n\mathrm{i}\beta} - 1)]\,. \tag{33}$$

The quantity $z_{2n} - z$ in eq. (33) is the "complex displacement" of the ball at its nth return to the wall. For the real displacement $y_{2n} - y$ and the "angular increment" $\tilde{\theta}_{2n} - \tilde{\theta}$ (we denote by $\tilde{\theta}$ the unfolded intrinsic angle of the ball) we have

$$y_{2n} - y = \mathscr{R}(z_{2n} - z)\,, \quad \tilde{\theta}_{2n} - \tilde{\theta} = \mathscr{I}(z_{2n} - z)\,. \tag{34}$$

We normalize the strip to be of unit width, i.e., $t = |v_x|^{-1}$. For the angle $\frac{1}{2}\beta$ (half the dihedral angle of the regular tetrahedron), we have

$$\sin\tfrac{1}{2}\beta = 1/\sqrt{3}\,, \quad \cos\tfrac{1}{2}\beta = \sqrt{2}/\sqrt{3}\,. \tag{35}$$

From eqs. (33)–(35), we obtain the positional and the angular displacements of the ball:

$$y_{2n} - y = (2\sqrt{3}|v_x|)^{-1}\mathscr{I}[w\,\mathrm{e}^{-\mathrm{i}\beta/2}(\mathrm{e}^{2\mathrm{i}n\beta} - 1)]\,, \tag{36}$$

$$\tilde{\theta}_{2n} - \tilde{\theta} = -(2\sqrt{6}|v_x|)^{-1}\mathscr{I}[w\,\mathrm{e}^{-\mathrm{i}\beta/2}(\mathrm{e}^{2\mathrm{i}n\beta} - 1)]\,. \tag{37}$$

Recall that the total speed $\|V\| = \|(v_x, v_y, w)\| = 1$. Hence $0 < |v_x| \le 1$ and

$$|w| = |v_y + \mathrm{i}\omega| = (1 - |v_x|^2)^{1/2} < 1\,. \tag{38}$$

By eqs. (36)–(38), for all $n = \pm 1, \pm 2, \ldots$:

$$|y_{2n} - y| < \frac{1}{\sqrt{3}} ((|v_x|)^{-2} - 1)^{1/2}, \tag{39}$$

$$|\tilde{\theta}_{2n} - \tilde{\theta}| < \frac{1}{\sqrt{6}} ((|v_x|)^{-2} - 1)^{1/2}. \tag{40}$$

The inequalities in eqs. (39)–(40) are strict because β is an irrational angle (i.e. incommensurable with π), hence

$$0 < |e^{2in\beta} - 1| < 2. \tag{41}$$

From eqs. (26) and (31) we compute the tangential velocity $(v_y)_{2n}$ and the angular velocity ω_{2n} of the ball after the nth return to the wall:

$$(v_y)_{2n} = \mathscr{R}[e^{2in\beta}(v_y i\omega)] = v_y \cos n\beta - \omega \sin 2n\beta, \tag{42}$$

$$\omega_{2n} = \mathscr{I}[e^{2in\beta}(v_y + i\omega)] = \omega \cos 2n\beta + v_y \sin 2n\beta. \tag{43}$$

5. Conclusions and discussion

An interesting consequence of eq. (39) is that the trajectories of the inelastic billiard between parallel walls are bounded (if $|v_x| \neq 0$). Moreover, by eq. (39), the span of a trajectory along the strip is arbitrarily small, provided the quantity $|v_x|$ is sufficiently close to one (in normalized coordinates). This is in direct contrast with the behaviour of the corresponding elastic billiard[#2], and clearly has implications for billiards in the tables containing parallel walls.

Let Γ be a closed billiard table containing two parallel sides opposite each other (fig. 3). Both the usual and the inelastic billiard in Γ have trajectories confined to these walls: those perpendicular to them. In the case of elastic billiard these trajectories are unstable: an arbitrarily small tangential component of the velocity of the ball will make it leave eventually the area between the parallel walls. Not so for the inelastic billiard (with no-slip impacts). Eq. (39) implies that any sufficiently small perturbation γ of a perpendicular trajectory γ_0 remains confined to a narrow region around γ_0.

Thus the inelastic billiard in the table with parallel walls can not be ergodic. We point out, for comparison, that the usual (elastic) billiard in the stadium is ergodic [10].

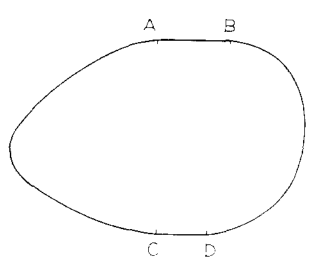

Fig. 3. A billiard table with parallel walls (AB and CD).

[#2] But will be a familiar effect to those who have tried bouncing a "superball" under a table.

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