Post Quantum Cryptography

Lattice Based Cryptography

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November 17, 2025

System of m linear equations with n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = t_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = t_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = t_m \end{cases} \pmod{q}$$

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We can write it in the matrix form as follows.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}}_{\mathbf{t}} \pmod{q}$$

(In)-homogeneous system of equation

- We use $\mathbb{Z}_q$to denote $\mathbb{Z}/q\mathbb{Z}$ and use the notation $a \in_R A$ to mean that a is chosen randomly from A.
- Vectors are always treated as column vectors and they are denoted by small letter bold font.
- Capital bold letter will represent a matrix and $\mathbf{A}^T$ will be used to present the transpose of matrix $\mathbf{A}$.

$$\mathbf{Ax} = \mathbf{t} \pmod q, \text{ where } \mathbf{A} \in {\mathbb{Z}_q}^{m \times n} \tag{1}$$

- If $\mathbf{t} = \mathbf{0}$, the system in Eq. (1) is called a homogeneous system of linear equations. For $\mathbf{t} \neq \mathbf{0}$, it is termed as inhomogeneous system of linear equations.

Shortest Integer Solution Problem

Definition 0.1
Given $n, m, q, \beta \in \mathbb{Z}_{>0}$ and

$$\mathbf{A}^T := \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \\ | & | & & | \end{bmatrix} \longleftarrow \mathsf{U}(\mathbb{Z}_q^{n \times m}), \tag{2}$$

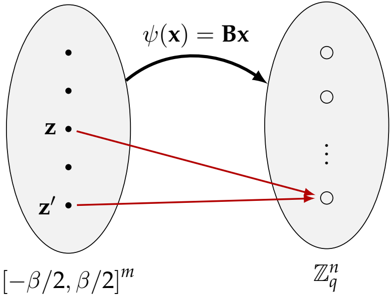
where $m > n$, $q = \mathrm{poly}(n)$ and $\beta \ll q/2$. Find a nonzero vector $\mathbf{z} \in \mathbb{Z}^m$ with $\mathbf{z} \in [-\beta, \beta]^m$ such that

$$\mathbf{A}^T \mathbf{z} \equiv \mathbf{0} \pmod q. \tag{3}$$

Remark

1. In the above Definition 0.1, the use of $\mathbf{A}^T$ is just for the notation convenience, which we leverage later. For simplicity we can use by $\mathbf{B} := \mathbf{A}^T$.
2. As the number of equations (n) is less than the number of variables (m), the system in Eq. (2) is underdetermined system.
3. Although the SIS system has many solutions, but what is the guarantee of having such a small solution.
4. How is the SIS problem connected to lattice ?

SIS ..



- Size of domain: $(\beta+1)^m$; size of co-domain: q^n.
- If $(\beta+1)^m > q^n$, by Pigeonhole Principle, there exists $\mathbf{z}_1, \mathbf{z}_2 \in [-\beta/2, \beta/2]^m$ with $\mathbf{z}_1 \neq \mathbf{z}_2$ such that $\mathbf{B}\mathbf{z}_1 = \mathbf{B}\mathbf{z}_2 \pmod{q}$. Thus $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$ is a SIS solution.
- The SIS problem does not have unique solution. $-\mathbf{z}$

Inhomogeneous SIS Problem (ISIS)

Definition 0.2
Given $n, m, q, \beta \in \mathbb{Z}_{>0}$ and

$$\mathbf{A}^T := \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_m \\ | & & | \end{bmatrix} \xleftarrow{\$} \mathbb{Z}_q^{n \times m}, \mathbf{b} := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \xleftarrow{\$} \mathbb{Z}_q^n, \tag{4}$$

where $m > n$ and $(2\beta + 1)^m \gg q^n$. Find a nonzero vector $\mathbf{z} \in \mathbb{Z}^m$ with $\mathbf{z} \in [-\beta, \beta]^m$ such that $\mathbf{A}^T\mathbf{z} \equiv \mathbf{b} \pmod q$.

Remark
The condition $(2\beta + 1)^m \gg q^n$ is required for a solution to likely exist.

Equivalence of SIS and ISIS

Theorem

The following statements hold.

- SIS $\leq$ ISIS.
- ISIS $\leq$ SIS.

NORMAL-FORM ISIS (NF-ISIS)

Definition 0.3 (nf-$\mathrm{ISIS}_{n,m,q,\beta}$)

Given m, n, q, β, $\mathbf{B} \xleftarrow{\$} \mathbb{Z}_q^{n\times m}$ and $\mathbf{b} \xleftarrow{\$} \mathbb{Z}_q^n$, find $\mathbf{z} \in \mathbb{Z}_q^{m+n}$ such that $[\mathbf{B}|\mathbf{I}_n]\,\mathbf{z} \equiv \mathbf{b} \pmod{q}$.

Claim 1

nf-$\mathrm{ISIS}_{n,m,q,\beta}$ and $\mathsf{ISIS}_{n,m+n,q,\beta}$ are equivalent.

SIS PROBLEM AND LATTICE

Definition 0.4

(i) A *q*-ary lattice Λ of dimension m is a lattice satisfying

$$q\mathbb{Z}^m \subseteq \Lambda \subseteq \mathbb{Z}^m.$$

(ii) Let $\mathbf{B} \in \mathbb{Z}^{n\times m}$, we define a q-ary lattice $\Lambda_q^\perp(\mathbf{B})$, called the kernel lattice of $\mathbf{B}$, as

$$\Lambda_q^\perp(\mathbf{B}) = \{\mathbf{x} \in \mathbb{Z}^m : \mathbf{B}\cdot\mathbf{x} = 0 \pmod q\}.$$

(iii) We define a lattice called, row lattice of $\mathbf{B}$, as

$$\Lambda_q(\mathbf{B}) = \left\{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{B}^T\mathbf{s} \pmod q, \text{ for some } \mathbf{s} \in \mathbb{Z}^n\right\}.$$

Hardness of SIS

Theorem 1.1 (Corollary of Theorem 3.8). *Let n and $m = \mathrm{poly}(n)$ be integers, let $\beta \geq \beta_\infty \geq 1$ be reals, let $Z = \{\mathbf{z} \in \mathbb{Z}^m : \|\mathbf{z}\|_2 \leq \beta$ and $\|\mathbf{z}\|_\infty \leq \beta_\infty\}$, and let $q \geq \beta \cdot n^\delta$ for some constant $\delta > 0$. Then solving (on the average, with non-negligible probability)* SIS *with parameters n, m, q and solution set $Z \setminus \{\mathbf{0}\}$ is at least as hard as approximating lattice problems in the worst case on n-dimensional lattices to within $\gamma = \max\{1, \beta \cdot \beta_\infty / q\} \cdot \tilde{O}(\beta\sqrt{n})$ factors.*

[1]

Remark

For $m > n \log q$, the function $f_{\mathbf{B}} : \{0,1\}^m \mapsto \mathbb{Z}_q^n$ defined as $f_{\mathbf{B}}(\mathbf{x}) := \mathbf{B}\mathbf{x} \pmod q$ is a collision resistant hash function.

[1] https://web.eecs.umich.edu/~cpeikert/pubs/LWsE.pdf

LWE Problem

Given a system, for example,

$$\left.\begin{aligned} 14s_1 + 15s_2 + 5s_3 + 2s_4 &\approx 8 \pmod{17} \\ 13s_1 + 14s_2 + 14s_3 + 6s_4 &\approx 16 \pmod{17} \\ 6s_1 + 10s_2 + 13s_3 + 1s_4 &\approx 3 \pmod{17} \\ 10s_1 + 4s_2 + 12s_3 + 16s_4 &\approx 12 \pmod{17} \\ 9s_1 + 5s_2 + 9s_3 + 6s_4 &\approx 9 \pmod{17} \\ 3s_1 + 6s_2 + 4s_3 + 5s_4 &\approx 16 \pmod{17} \\ &\vdots \\ 6s_1 + 7s_2 + 16s_3 + 2s_4 &\approx 3 \pmod{17} \end{aligned}\right\}, \tag{5}$$

where the approximation error is random and small in size, we look for a solution, i.e. $\mathbf{s} = (s_1, s_2, s_3, s_4)^T$ satisfying above.

LWE PROBLEM..

- If the System 5 has no solution, it is of no use.
- So we generate such a system starting from a solution, i.e. from a fixed value of $\mathbf{s} = (s_1, s_2, s_3, s_4)^T$, and pick the coefficients randomly and compute the LHS of Eq. (5) and add a small random error to obtain the value of the RHS.
- Irrespective of the number of equations in such a system, a solution is always guaranteed, because we construct the system using a solution.

LWE PROBLEM..

We can convert the System (5) into the following exact system by introducing the error variables.

$$\left.\begin{aligned}
14s_1 + 15s_2 + 5s_3 + 2s_4 + e_1 &= 8 \pmod{17} \\
13s_1 + 14s_2 + 14s_3 + 6s_4 + e_2 &= 16 \pmod{17} \\
6s_1 + 10s_2 + 13s_3 + 1s_4 + e_3 &= 3 \pmod{17} \\
10s_1 + 4s_2 + 12s_3 + 16s_4 + e_4 &= 12 \pmod{17} \\
9s_1 + 5s_2 + 9s_3 + 6s_4 + e_5 &= 9 \pmod{17} \\
3s_1 + 6s_2 + 4s_3 + 5s_4 + e_6 &= 16 \pmod{17} \\
&\vdots \\
6s_1 + 7s_2 + 16s_3 + 2s_4 + e_7 &= 3 \pmod{17}
\end{aligned}\right\} \tag{6}$$

LWE PROBLEM..

We can formally write the above system in the matrix form as follow:

$$\mathbf{A}\mathbf{s} + \mathbf{e} = \mathbf{b} \pmod q, \text{ where } \mathbf{A} \xleftarrow{\$} {\mathbb{Z}_q}^{m\times n}, \mathbf{e} \leftarrow \chi, \mathbf{s} \in {\mathbb{Z}_q}^{n}. \quad (7)$$

– Given $(\mathbf{A}, \mathbf{b}) \in {\mathbb{Z}_q}^{m\times n} \times {\mathbb{Z}_q}^{m}$, satisfying Eq. (7), the LWE problem is to find the secret $\mathbf{s}$.

LWE Problem..

- The LWE instance is generated using a LWE-sampler, which on input $\mathbf{s}$, picks a random vector $\mathbf{a}_i$ from ${\mathbb{Z}_q}^n$ and samples an error e_i for the distribution χ and outputs $(\mathbf{a}_i, b_i := \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i)$.

Secret: $\mathbf{s} \in \mathbb{Z}_q^n$

$$e_i \leftarrow \chi,\ \mathbf{a}_i \overset{\$}{\leftarrow} \mathbb{Z}_q^n$$

$$t_i \equiv \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i \pmod{q} \xrightarrow{(\mathbf{a}_i, t_i)}$$

- We run the LWE-sampler and collect m LWE-samples and set

$$\mathbf{A} := \begin{bmatrix} \text{———} & \mathbf{a}_1 & \text{———} \\ \text{———} & \mathbf{a}_2 & \text{———} \\ \text{———} & \vdots & \text{———} \\ \text{———} & \mathbf{a}_m & \text{———} \end{bmatrix}, \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{e} := \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}. \tag{8}$$

LWE PARAMETERS

$$\mathbf{A}\mathbf{s} + \mathbf{e} = \mathbf{b} \pmod{q}, \text{ where } \mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}, \mathbf{e} \leftarrow \chi, \mathbf{s} \in \mathbb{Z}_q^{n}.$$

- The notation (n, q, χ)-LWE is used to mean a computational LWE with following parameters n, q and χ, where
  - n represent the length of secret vector. It is called LWE dimension.
  - $q \in \text{poly}(n)$ is called LWE modulus.
  - χ is a probability distribution. The errors (noise), i.e. $\mathbf{e}$, are sampled from χ.
- We write (n, q, α)-LWE to mean $(n, q, D_{\alpha q})$-LWE, where $D_{\alpha q}$ is discrete gaussian distribution with standard deviation αq. The error parameter α is typically $1/\text{poly}(n)$ and $\alpha q > \sqrt{n}$.
- The number of samples(LWE-equations), i.e. m, is usually not very important. It does not affect much the hardness of LWE.

MATRIX LWE (MULTI-SECRET EXTENSION OF LWE)

- Similar to the matrix-SIS problem, the matrix-LWE problem replaces secret vector of LWE with a secret matrix. Each column of secret matrix correspond to the classical LWE secret.
- For a fixed secret matrix $\mathbf{S} \in_{\phi} \mathbb{Z}_q^{n \times k}$, $\mathbf{A} \in_U \mathbb{Z}_q^{m \times n}$ and error matrix $\mathbf{E} \in_{\chi} \mathbb{Z}_q^{m \times k}$, we compute

$$\mathbf{T} = \mathbf{AS} + \mathbf{E} \pmod{q}.$$

  The matrix-LWE problem is to find $(\mathbf{S}, \mathbf{E})$, given $(\mathbf{A}, \mathbf{T})$.
- The decision version of matrix-LWE problem is to distinguish $(\mathbf{A}, \mathbf{AS} + \mathbf{E})$ from $(\mathbf{A}, \mathbf{R})$, where $\mathbf{R} \in_U \mathbb{Z}_q^{m \times k}$.

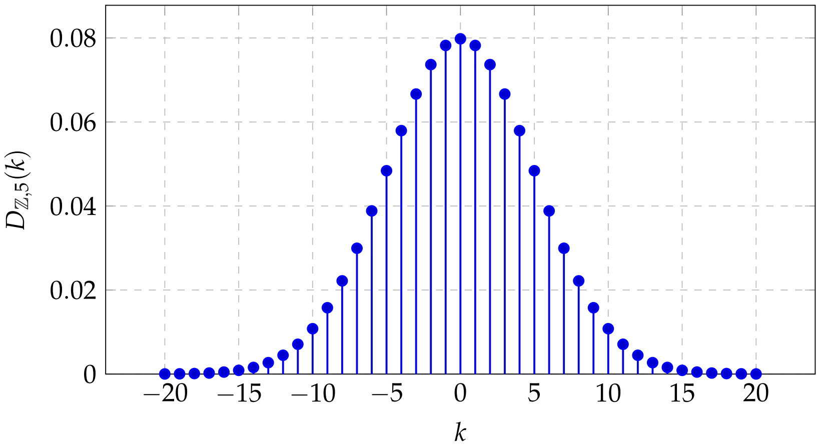
Matrix LWE (multi-secret extension of LWE)..

Let

$$\mathbf{S} = \begin{bmatrix} | & | & & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_k \\ | & | & & | \end{bmatrix} \text{ and } \mathbf{E} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_k \\ | & | & & | \end{bmatrix}$$

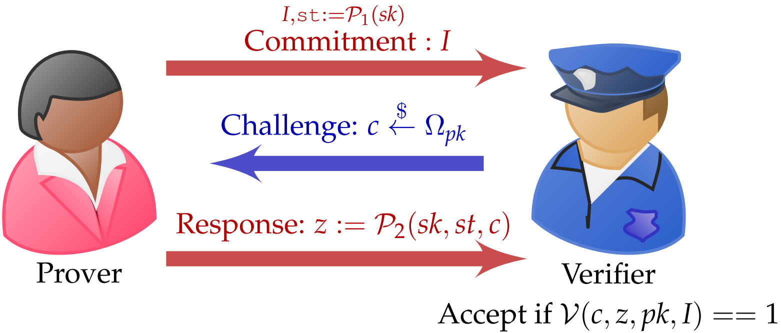
- The matrix-LWE problem can be treated as k parallel LWE instances $(\mathbf{A}, \mathbf{A}\mathbf{s}_i + \mathbf{e}_i)$, with a shared coefficient matrix $\mathbf{A}$.
- LWE $\leq$ Matrix-LWE i.e., matrix-LWE problem is at least as hard as ordinary LWE problem.

Discrete Gaussian $D_{\mathbb{Z},5}$ (normalized over $k = -20 \dots 20$)



Formal definition of ID Protocol

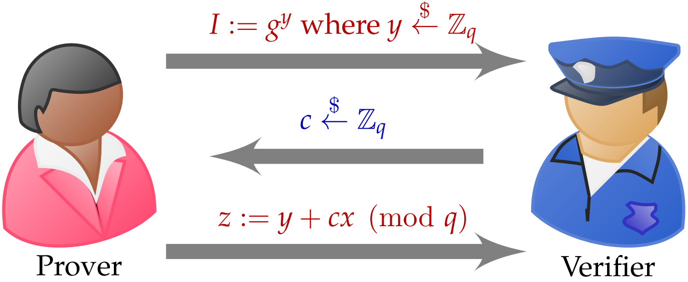
Formally an ID protocol Π is a collection of PPT algorithms $(\texttt{KeyGen}, \mathcal{P}, \mathcal{V})$, where $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$.



The Schnorr Identification Scheme

Let $(G, \cdot) = \langle g \rangle$ be a cyclic group of order q. It is known that the discrete logarithm problem in G is computationally hard.

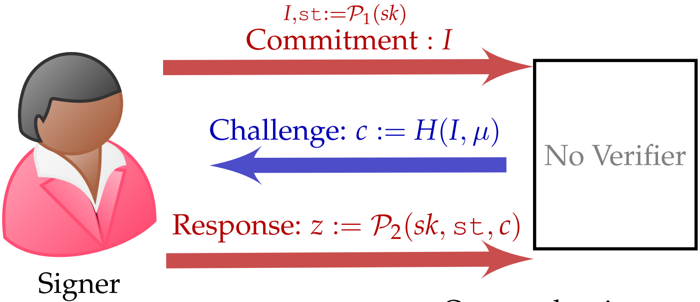
$$\boxed{sk := x, \quad pk := g^x \text{ where } x \xleftarrow{\$} \mathbb{Z}_q}$$



Accept if $g^z \cdot pk^{-c} \stackrel{?}{=} I$

Fiat-Shamir transform

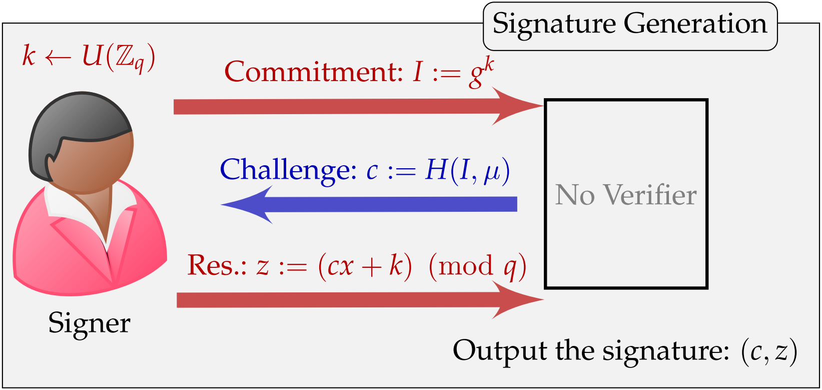
Construction of Signature from ID Scheme



Output the signature: (c, z)

SCHNORR SIGNATURE SCHEME

$$(G, \cdot) = \langle g \rangle, |G| = q, sk := x \leftarrow U(\mathbb{Z}_q) \text{ and } pk := g^x$$



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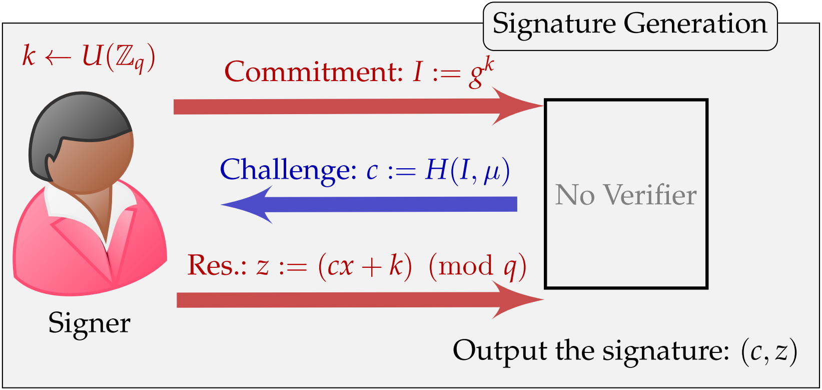

Verification

$$H\left(g^z.(pk)^{-c}, \mu\right) \stackrel{?}{=} c$$

A SIS BASED DIGITAL SIGNATURE ALGORITHM

Algorithm 1 — **KeyGen**(n, m, k, q)

Input: n, m, k, q
Output: sk, pk
$\mathbf{S} \xleftarrow{\$} \{-1, 0, 1\}^{m \times k}$
$\mathbf{A}^T \xleftarrow{\$} \mathbb{Z}_q^{n \times m}$, $\mathbf{T} = \mathbf{A}^T \cdot \mathbf{S} \pmod{q}$
return $sk := \mathbf{S}$ and $pk := \left(\mathbf{A}^T, \mathbf{T}\right)$

- Given $\left(\mathbf{A}^T, \mathbf{T}\right)$, it is computationally hard to recover $\mathbf{S}$.This is multi-secret (matrix) variant of the ISIS problem.
- Corresponding column vectors of $\mathbf{T}$ and $\mathbf{S}$ together with $\mathbf{A}^T$ will represnt the individual ISIS problems.
- In fact there are k ISIS instances with a shared coefficient matrix $\mathbf{A}^T$.

Algorithm 2 — Sign(μ, sk, pk)

Input: Signing key sk and message μ

Output: Signature

1. $\mathbf{y} \leftarrow D_\sigma^m$ ▷ Commitment: $\mathbf{A}^T\mathbf{y}$
2. $\mathbf{c} = \mathsf{H}\left(\mathbf{A}^T \cdot \mathbf{y}, \mu\right) \in \{-1, 0, +1\}^k$ ▷ Challenge: $\mathbf{c}$
3. $\mathbf{z} = \mathbf{Sc} + \mathbf{y}$ ▷ Response: $\mathbf{z}$
4. **return** $(\mathbf{z}, \mathbf{c})$ with probability $\min\left(\frac{D_\sigma^m}{MD_{\mathbf{Sc},\sigma}^m}, 1\right)$, where $M \in \mathbb{R}$ is such that $\Pr\left[MD_{\mathbf{Sc},\sigma}^m(\mathbf{z}) \geq D_\sigma^m(\mathbf{z}) : \mathbf{z} \leftarrow D_\sigma^m\right] \geq 1 - \varepsilon.$

A SIS based Digital Signature Algorithm..

Algorithm 3 — `Verify` $(pk := (\mathbf{A}, \mathbf{T}), \mu, (\mathbf{z}, \mathbf{c}))$

if $\mathbf{c} = \mathsf{H}\left(\mathbf{A}^T\mathbf{z} - \mathbf{Tc}, \mu\right)$ *and* $\|\mathbf{z}\| \leq \sigma\sqrt{m}$ then
  return 1

LWE BASED DIGITAL SIGNATURE ALGORITHM

Algorithm 4 — KeyGen(n, m, k, q)

$\mathbf{S} \in_{\phi} \mathbb{Z}_q^{n \times k}$

$\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}$, $\mathbf{E} \in_{\chi} \mathbb{Z}_q^{m \times k}$ and $\mathbf{T} = \mathbf{A} \cdot \mathbf{S} + \mathbf{E} \pmod q$

return $sk := \mathbf{S}$ and $pk := (\mathbf{A}, \mathbf{T})$

A framework for LWE based signature

Algorithm 5 — Sign(μ, sk, pk)

$\mathbf{y}_1, \mathbf{y}_2 \leftarrow D_{\mathbf{y}}^n$

$\mathbf{w} = \mathbf{A} \cdot \mathbf{y}_1 + \mathbf{y}_2$ ▷ Commitment: $\mathbf{w}$

$\mathbf{c} = \mathsf{H}(\mu || \mathbf{w}) \in \{-1, 0, +1\}^k$ ▷ Challenge: sparse $\mathbf{c}$

$\mathbf{z}_1 = \mathbf{S}\mathbf{c} + \mathbf{y}_1, \mathbf{z}_2 = \mathbf{E}\mathbf{c} + \mathbf{y}_2$ ▷ Response: $\mathbf{z}_1, \mathbf{z}_2$

if $(\mathbf{z}_1, \mathbf{z}_2)$ *leaks dist of* $\mathbf{S}$ **then** ▷ $\|\mathbf{z}_i\| > \mathtt{bd}_i$

  restart

return $(\mathbf{z}_1, \mathbf{z}_2, \mathbf{c})$

Algorithm 6 — Verify $(pk := (\mathbf{A}, \mathbf{T}), \mu, (\mathbf{z}_1, \mathbf{z}_2, \mathbf{c}))$

if $\|\mathbf{z}_i\| \leq \mathtt{bd}_i \; \forall i$ **then**

  return $H(\mu || \mathbf{A}\mathbf{z}_1 + \mathbf{z}_2 - \mathbf{T}\mathbf{c}) \stackrel{?}{=} \mathbf{c}$

A FRAMEWORK FOR LWE BASED SIGNATURE..

- The main draw back of this framework (Algorithm 5) is the signature size. In comparison to the SIS based schemes, the signature has an additional vector $\mathbf{z}_2$ of dimension n.
- This can be resolved using the Bai-Galbraith technique. We discuss this in the signature scheme given in the Algorithm 7 below.

LWE based digital signature algorithm..

Algorithm 7 — Sign(μ, sk, pk)

$\mathbf{y} \leftarrow D_\sigma^n; \mathbf{w} = \mathbf{A} \cdot \mathbf{y}$ ▷ Commitment: $\mathbf{w}$

$\mathbf{w}_1 = \mathrm{HighBits}(\mathbf{w})$

$\mathbf{c} = \mathsf{H}(\mu || \mathbf{w}_1) \in \{-1, 0, +1\}^k$ ▷ Challenge: sparse $\mathbf{c}$

if $\mathrm{LowBits}(\mathbf{w} - \mathbf{Ec}) > \mathrm{bd}$ **then**

  restart

$\mathbf{z} = \mathbf{Sc} + \mathbf{y}$ ▷ Response: $\mathbf{z}$

if $\mathbf{z}$ *leaks dist of* $\mathbf{S}$ **then** ▷ $\|\mathbf{z}\| > \mathrm{bd}_1$

  restart

return $(\mathbf{z}, \mathbf{c})$

LWE BASED GIGITAL SIGNATURE ALGORITHM..

Algorithm 8 — **Verify** $(pk := (\mathbf{A}, \mathbf{T}), \mu, (\mathbf{z}, \mathbf{c}))$

$\mathbf{w}_1' = \mathrm{HighBits}(\mathbf{AZ} - \mathbf{Tc})$
if $\mathbf{c} = \mathsf{H}\left(\mu || \mathbf{w}_1'\right)$ *and* $\|\mathbf{z}\| \leq \mathtt{bd}_1$ **then**
  return 1