

Post Quantum Cryptography

Lattice Based Cryptography

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SYSTEM OF m LINEAR EQUATIONS WITH n VARIABLES

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = t_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = t_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = t_m \end{cases} \pmod{q}$$

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We can write it in the matrix form as follows.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}}_{\mathbf{t}} \pmod{q}$$

(IN)-HOMOGENEOUS SYSTEM OF EQUATION

- We use \mathbb{Z}_q to denote $\mathbb{Z}/q\mathbb{Z}$ and use the notation $a \in_R A$ to mean that a is chosen randomly from A .
- Vectors are always treated as column vectors and they are denoted by small letter bold font.
- Capital bold letter will represent a matrix and \mathbf{A}^T will be used to present the transpose of matrix \mathbf{A} .

$$\mathbf{Ax} = \mathbf{t} \pmod{q}, \text{ where } \mathbf{A} \in \mathbb{Z}_q^{m \times n} \quad (1)$$

- If $\mathbf{t} = \mathbf{0}$, the system in Eq. (1) is called a homogeneous system of linear equations. For $\mathbf{t} \neq \mathbf{0}$, it is termed as inhomogeneous system of linear equations.

SHORTEST INTEGER SOLUTION PROBLEM

Definition 0.1

Given $n, m, q, \beta \in \mathbb{Z}_{>0}$ and

$$\mathbf{A}^T := \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \\ | & | & & | \end{bmatrix} \leftarrow \mathsf{U}(\mathbb{Z}_q^{n \times m}), \quad (2)$$

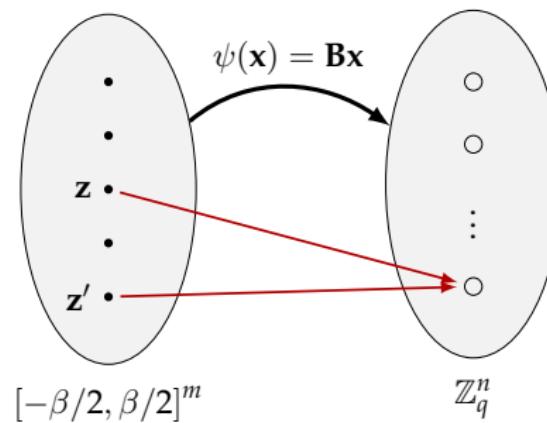
where $m > n$, $q = \text{poly}(n)$ and $\beta \ll q/2$. Find a nonzero vector $\mathbf{z} \in \mathbb{Z}^m$ with $\mathbf{z} \in [-\beta, \beta]^m$ such that

$$\mathbf{A}^T \mathbf{z} \equiv \mathbf{0} \pmod{q}. \quad (3)$$

Remark

1. In the above Definition 0.1, the use of \mathbf{A}^T is just for the notation convenience, which we leverage later. For simplicity we can use by $\mathbf{B} := \mathbf{A}^T$.
2. As the number of equations (n) is less than the number of variables (m), the system in Eq. (2) is under-determined system.
3. Although the SIS system has many solutions, but what is the guarantee of having such a small solution.
4. How is the SIS problem connected to lattice ?

SIS ..



- Size of domain: $(\beta + 1)^m$; size of co-domain: q^n .
- If $(\beta + 1)^m > q^n$, by Pigeonhole Principle, there exists $\mathbf{z}_1, \mathbf{z}_2 \in [-\beta/2, \beta/2]^m$ with $\mathbf{z}_1 \neq \mathbf{z}_2$ such that $\mathbf{Bz}_1 = \mathbf{Bz}_2 \pmod{q}$. Thus $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$ is a SIS solution.
- The SIS problem does not have unique solution. -z

INHOMOGENEOUS SIS PROBLEM (ISIS)

Definition 0.2

Given $n, m, q, \beta \in \mathbb{Z}_{>0}$ and

$$\mathbf{A}^T := \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_m \\ | & & | \end{bmatrix} \xleftarrow{\$} \mathbb{Z}_q^{n \times m}, \mathbf{b} := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \xleftarrow{\$} \mathbb{Z}_q^n, \quad (4)$$

where $m > n$ and $(2\beta + 1)^m \gg q^n$. Find a nonzero vector $\mathbf{z} \in \mathbb{Z}^m$ with $\mathbf{z} \in [-\beta, \beta]^m$ such that $\mathbf{A}^T \mathbf{z} \equiv \mathbf{b} \pmod{q}$.

Remark

The condition $(2\beta + 1)^m \gg q^n$ is required for a solution to likely exist.

EQUIVALENCE OF SIS AND ISIS

Theorem

The following statements hold.

- $\text{SIS} \leq \text{ISIS}$.
- $\text{ISIS} \leq \text{SIS}$.

NORMAL-FORM ISIS (NF-ISIS)

Definition 0.3 (nf-ISIS_{n,m,q,β})

Given $m, n, q, \beta, \mathbf{B} \xleftarrow{\$} \mathbb{Z}_q^{n \times m}$ and $\mathbf{b} \xleftarrow{\$} \mathbb{Z}_q^n$, find $\mathbf{z} \in \mathbb{Z}_q^{m+n}$ such that $[\mathbf{B} | \mathbf{I}_n] \mathbf{z} \equiv \mathbf{b} \pmod{q}$.

Claim 1

nf-ISIS_{n,m,q,β} and ISIS_{n,m+n,q,β} are equivalent.

SIS PROBLEM AND LATTICE

Definition 0.4

- (i) A ***q*-ary lattice** Λ of dimension m is a lattice satisfying

$$q\mathbb{Z}^m \subseteq \Lambda \subseteq \mathbb{Z}^m.$$

- (ii) Let $\mathbf{B} \in \mathbb{Z}^{n \times m}$, we define a *q*-ary lattice $\Lambda_q^\perp(\mathbf{B})$, called the **kernel lattice of \mathbf{B}** , as

$$\Lambda_q^\perp(\mathbf{B}) = \{\mathbf{x} \in \mathbb{Z}^m : \mathbf{B} \cdot \mathbf{x} = 0 \pmod{q}\}.$$

- (iii) We define a lattice called, **row lattice of \mathbf{B}** , as

$$\Lambda_q(\mathbf{B}) = \left\{ \mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{B}^T \mathbf{s} \pmod{q}, \text{ for some } \mathbf{s} \in \mathbb{Z}^n \right\}.$$

HARDNESS OF SIS

Theorem 1.1 (Corollary of Theorem 3.8). Let n and $m = \text{poly}(n)$ be integers, let $\beta \geq \beta_\infty \geq 1$ be reals, let $Z = \{\mathbf{z} \in \mathbb{Z}^m : \|\mathbf{z}\|_2 \leq \beta \text{ and } \|\mathbf{z}\|_\infty \leq \beta_\infty\}$, and let $q \geq \beta \cdot n^\delta$ for some constant $\delta > 0$. Then solving (on the average, with non-negligible probability) SIS with parameters n, m, q and solution set $Z \setminus \{\mathbf{0}\}$ is at least as hard as approximating lattice problems in the worst case on n -dimensional lattices to within $\gamma = \max\{1, \beta \cdot \beta_\infty/q\} \cdot \tilde{O}(\beta\sqrt{n})$ factors.

1

Remark

For $m > n \log q$, the function $f_{\mathbf{B}} : \{0, 1\}^m \mapsto \mathbb{Z}_q^n$ defined as $f_{\mathbf{B}}(\mathbf{x}) := \mathbf{Bx} \pmod{q}$ is a **collision resistant** hash function.

¹<https://web.eecs.umich.edu/~cpeikert/pubs/LWe.pdf>

LWE PROBLEM

Given a system, for example,

$$\left. \begin{array}{l} 14s_1 + 15s_2 + 5s_3 + 2s_4 \approx 8 \pmod{17} \\ 13s_1 + 14s_2 + 14s_3 + 6s_4 \approx 16 \pmod{17} \\ 6s_1 + 10s_2 + 13s_3 + 1s_4 \approx 3 \pmod{17} \\ 10s_1 + 4s_2 + 12s_3 + 16s_4 \approx 12 \pmod{17} \\ 9s_1 + 5s_2 + 9s_3 + 6s_4 \approx 9 \pmod{17} \\ 3s_1 + 6s_2 + 4s_3 + 5s_4 \approx 16 \pmod{17} \\ \vdots \\ 6s_1 + 7s_2 + 16s_3 + 2s_4 \approx 3 \pmod{17} \end{array} \right\}, \quad (5)$$

where the approximation error is random and small in size, we look for a solution, i.e. $\mathbf{s} = (s_1, s_2, s_3, s_4)^T$ satisfying above.

LWE PROBLEM..

- If the System 5 has no solution, it is of no use.
- So we generate such a system starting from a solution, i.e. from a fixed value of $\mathbf{s} = (s_1, s_2, s_3, s_4)^T$, and pick the coefficients randomly and compute the LHS of Eq. (5) and add a small random error to obtain the value of the RHS.
- Irrespective of the number of equations in such a system, a solution is always guaranteed, because we construct the system using a solution.

LWE PROBLEM..

We can convert the System (5) into the following exact system by introducing the error variables.

$$\left. \begin{array}{l} 14s_1 + 15s_2 + 5s_3 + 2s_4 + e_1 = 8 \pmod{17} \\ 13s_1 + 14s_2 + 14s_3 + 6s_4 + e_2 = 16 \pmod{17} \\ 6s_1 + 10s_2 + 13s_3 + 1s_4 + e_3 = 3 \pmod{17} \\ 10s_1 + 4s_2 + 12s_3 + 16s_4 + e_4 = 12 \pmod{17} \\ 9s_1 + 5s_2 + 9s_3 + 6s_4 + e_5 = 9 \pmod{17} \\ 3s_1 + 6s_2 + 4s_3 + 5s_4 + e_6 = 16 \pmod{17} \\ \vdots \\ 6s_1 + 7s_2 + 16s_3 + 2s_4 + e_7 = 3 \pmod{17} \end{array} \right\} \quad (6)$$

LWE PROBLEM..

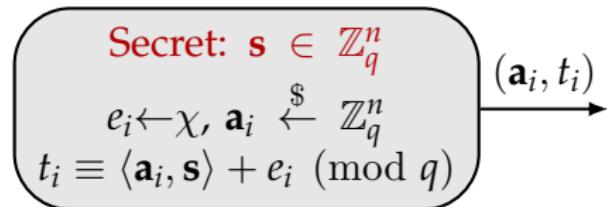
We can formally write the above system in the matrix form as follow:

$$\mathbf{A}\mathbf{s} + \mathbf{e} = \mathbf{b} \pmod{q}, \text{ where } \mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}, \mathbf{e} \xleftarrow{\chi} \mathbb{Z}_q^n, \mathbf{s} \in \mathbb{Z}_q^n. \quad (7)$$

- Given $(\mathbf{A}, \mathbf{b}) \in \mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m$, satisfying Eq. (7), the LWE problem is to find the secret \mathbf{s} .

LWE PROBLEM..

- The LWE instance is generated using a LWE-sampler, which on input \mathbf{s} , picks a random vector \mathbf{a}_i from \mathbb{Z}_q^n and samples an error e_i for the distribution χ and outputs $(\mathbf{a}_i, b_i := \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i)$.



- We run the LWE-sampler and collect m LWE-samples and set

$$\mathbf{A} := \begin{bmatrix} \text{---} & \mathbf{a}_1 & \text{---} \\ \text{---} & \mathbf{a}_2 & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & \mathbf{a}_m & \text{---} \end{bmatrix}, \mathbf{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{e} := \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}. \quad (8)$$

LWE PARAMETERS

$$\mathbf{A}\mathbf{s} + \mathbf{e} = \mathbf{b} \pmod{q}, \text{ where } \mathbf{A} \leftarrow \mathbb{Z}_q^{m \times n}, \mathbf{e} \leftarrow \chi, \mathbf{s} \in \mathbb{Z}_q^n.$$

- The notation (n, q, χ) -LWE is used to mean a computational LWE with following parameters n, q and χ , where
 - n represent the length of secret vector. It is called LWE dimension.
 - $q \in \text{poly}(n)$ is called LWE modulus.
 - χ is a probability distribution. The errors (noise), i.e. \mathbf{e} , are sampled from χ .
- We write (n, q, α) -LWE to mean $(n, q, D_{\alpha q})$ -LWE, where $D_{\alpha q}$ is discrete gaussian distribution with standard deviation αq . The error parameter α is typically $1/\text{poly}(n)$ and $\alpha q > \sqrt{n}$.
- The number of samples(LWE-equations), i.e. m , is usually not very important. It does not affect much the hardness of LWE.

MATRIX LWE (MULTI-SECRET EXTENSION OF LWE)

- Similar to the matrix-SIS problem, the matrix-LWE problem replaces secret vector of LWE with a **secret matrix**. Each column of secret matrix correspond to the classical LWE secret.
- For a fixed secret matrix $\mathbf{S} \in_{\phi} \mathbb{Z}_q^{n \times k}$, $\mathbf{A} \in_U \mathbb{Z}_q^{m \times n}$ and error matrix $\mathbf{E} \in_{\chi} \mathbb{Z}_q^{m \times k}$, we compute

$$\mathbf{T} = \mathbf{AS} + \mathbf{E} \pmod{q}.$$

The matrix-LWE problem is to find (\mathbf{S}, \mathbf{E}) , given (\mathbf{A}, \mathbf{T}) .

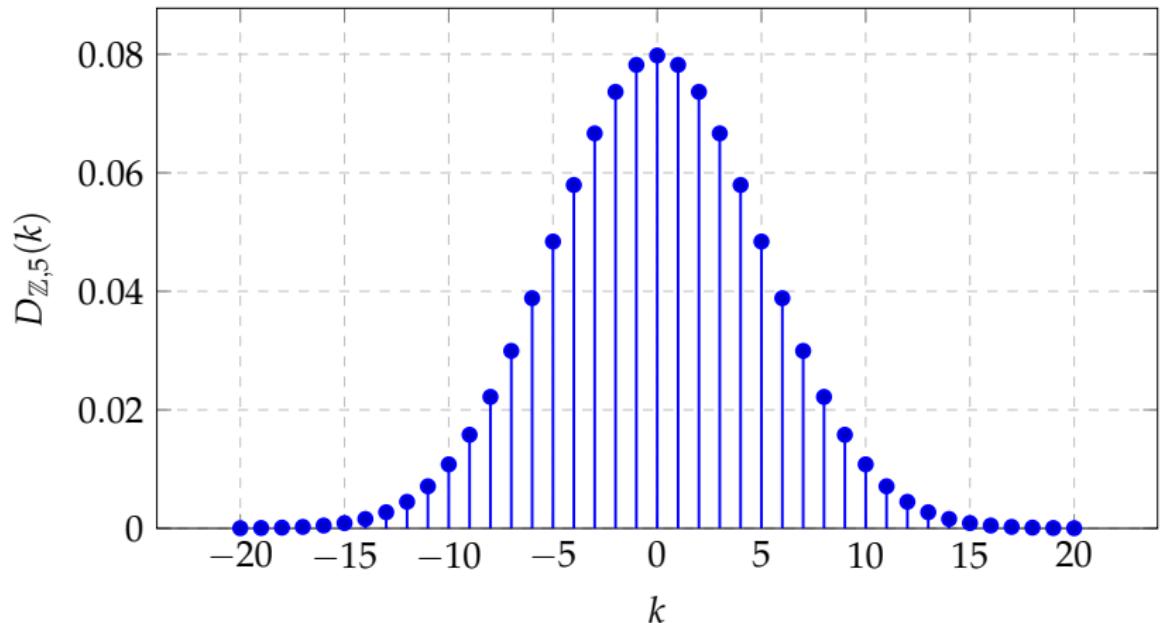
- The decision version of matrix-LWE problem is to distinguish $(\mathbf{A}, \mathbf{AS} + \mathbf{E})$ from (\mathbf{A}, \mathbf{R}) , where $\mathbf{R} \in_U \mathbb{Z}_q^{m \times k}$.

MATRIX LWE (MULTI-SECRET EXTENSION OF LWE)..

Let

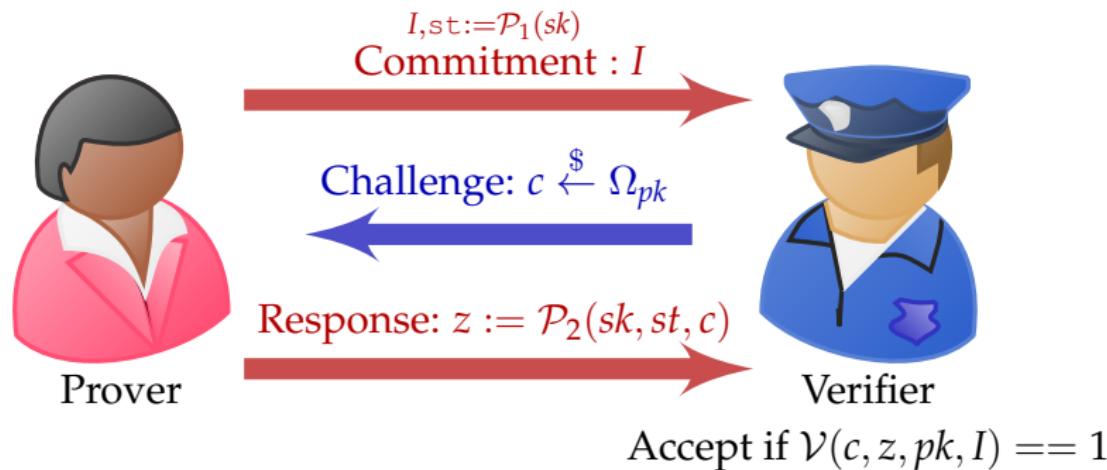
$$\mathbf{S} = \begin{bmatrix} | & | & & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_k \\ | & | & & | \end{bmatrix} \text{ and } \mathbf{E} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_k \\ | & | & & | \end{bmatrix}$$

- The matrix-LWE problem can be treated as k parallel LWE instances $(\mathbf{A}, \mathbf{As}_i + \mathbf{e}_i)$, with a shared coefficient matrix \mathbf{A} .
- $\text{LWE} \leq \text{Matrix-LWE}$ i.e., matrix-LWE problem is at least as hard as ordinary LWE problem.
- Matrix-SIS and Matrix-SIS problems can be similarly defined. We can safely assume that these problems are at least as hard as their classical counterparts.

Discrete Gaussian $D_{\mathbb{Z},5}$ (normalized over $k = -20 \dots 20$)

FORMAL DEFINITION OF ID PROTOCOL

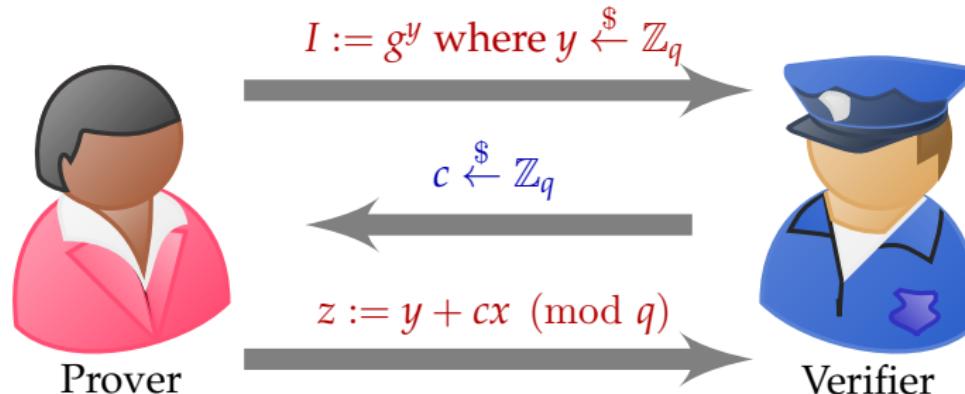
Formally an ID protocol Π is a collection of PPT algorithms $(\mathbf{KeyGen}, \mathcal{P}, \mathcal{V})$, where $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2)$.



THE SCHNORR IDENTIFICATION SCHEME

Let $(G, \cdot) = \langle g \rangle$ be a cyclic group of order q . It is known that the discrete logarithm problem in G is computationally hard.

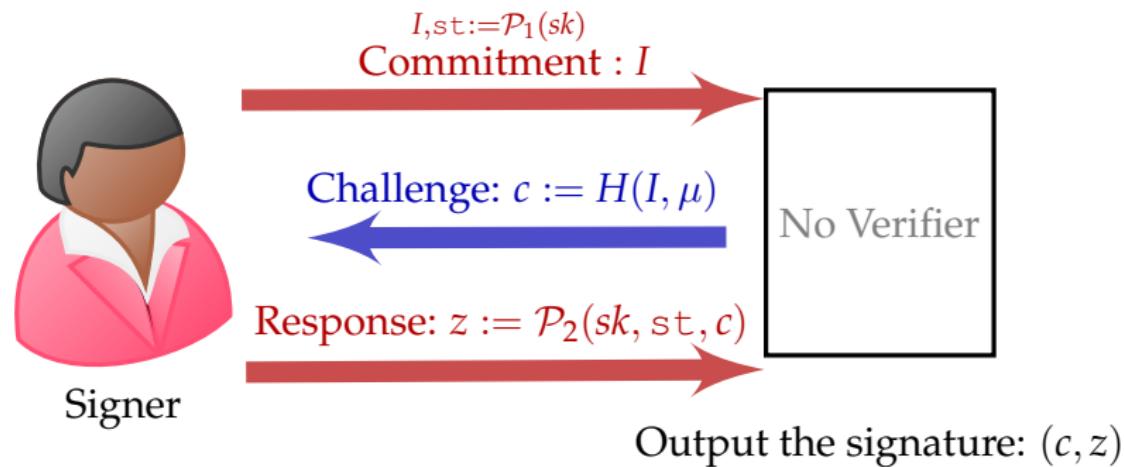
$$sk := x, \quad pk := g^x \text{ where } x \xleftarrow{\$} \mathbb{Z}_q$$



Accept if $g^z \cdot pk^{-c} \stackrel{?}{=} I$

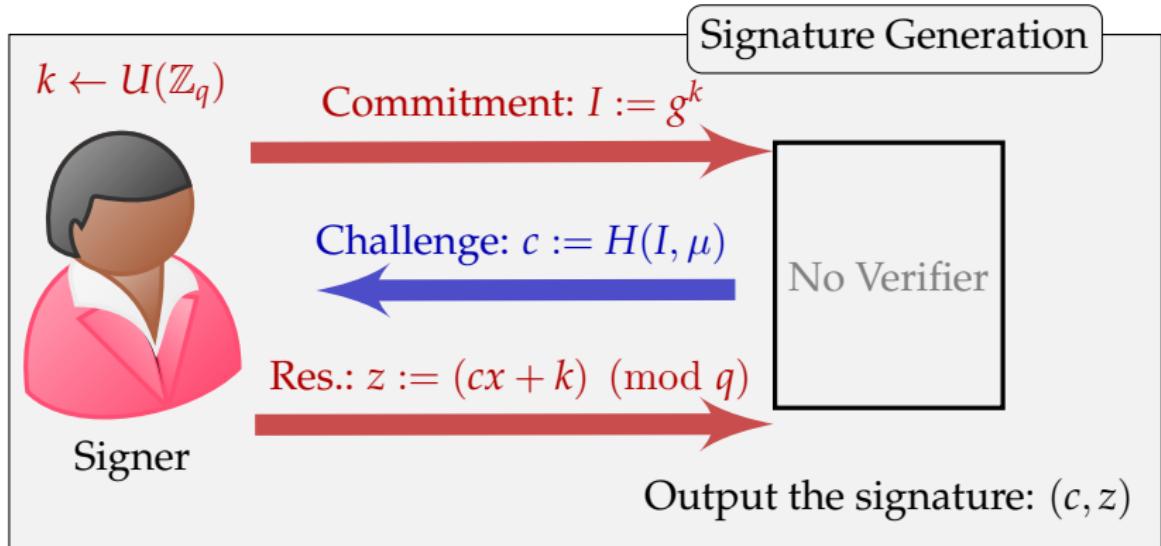
FIAT-SHAMIR TRANSFORM

CONSTRUCTION OF SIGNATURE FROM ID SCHEME



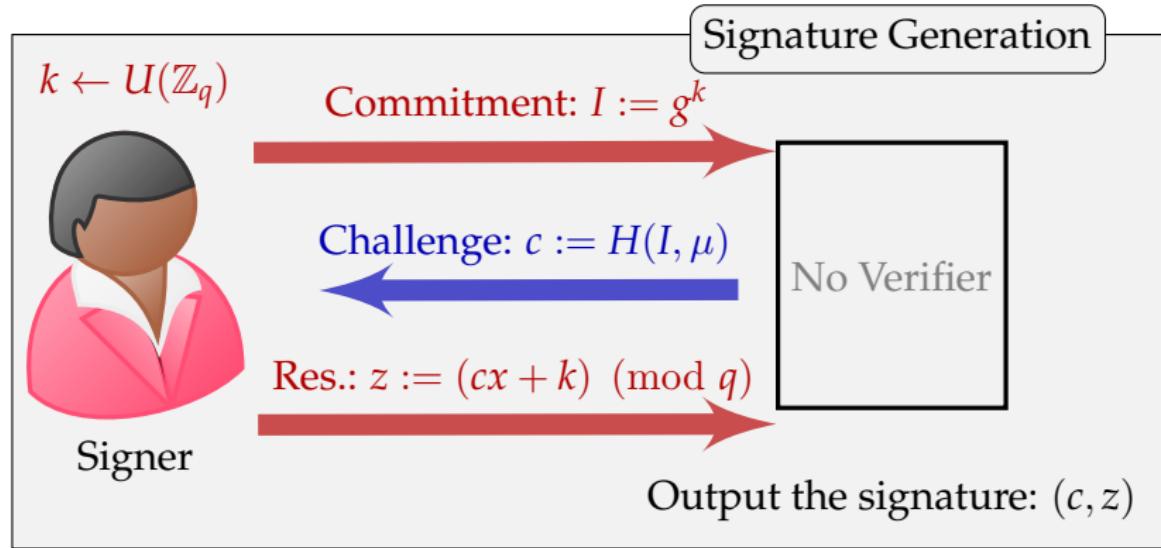
SCHNORR SIGNATURE SCHEME

$(G, \cdot) = \langle g \rangle, |G| = q, sk := x \leftarrow U(\mathbb{Z}_q)$ and $pk := g^x$



SCHNORR SIGNATURE SCHEME

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Verification

$$H(g^z \cdot (pk)^{-c}, \mu) \stackrel{?}{=} c$$

A SIS BASED DIGITAL SIGNATURE ALGORITHM

Algorithm 1 — **KeyGen**(n, m, k, q)

Input: n, m, k, q

Output: sk, pk

1 $\mathbf{S} \xleftarrow{\$} \{-1, 0, 1\}^{m \times k}$

2 $\mathbf{A}^T \xleftarrow{\$} \mathbb{Z}_q^{n \times m}, \mathbf{T} = \mathbf{A}^T \cdot \mathbf{S} \pmod{q}$

3 **return** $sk := \mathbf{S}$ and $pk := (\mathbf{A}^T, \mathbf{T})$

- Given $(\mathbf{A}^T, \mathbf{T})$, it is computationally hard to recover \mathbf{S} . This is multi-secret (matrix) variant of the ISIS problem.
- Corresponding column vectors of \mathbf{T} and \mathbf{S} together with \mathbf{A}^T will represent the individual ISIS problems.
- In fact there are k ISIS instances with a shared coefficient matrix \mathbf{A}^T .

A SIS BASED DIGITAL SIGNATURE ALGORITHM..

FIAT-SHAMIR WITH ABORTS FRAMEWORK

Algorithm 2 — $\text{Sign}(\mu, sk, pk)$

Input: Signing key sk and message μ

Output: Signature

- 1 $\mathbf{y} \leftarrow D_{\sigma}^m$ ▷ Commitment: $\mathbf{A}^T \mathbf{y}$
- 2 $\mathbf{c} = \mathsf{H}(\mathbf{A}^T \cdot \mathbf{y}, \mu) \in \{-1, 0, +1\}^k$ ▷ Challenge: \mathbf{c}
- 3 $\mathbf{z} = \mathbf{Sc} + \mathbf{y}$ ▷ Response: \mathbf{z}
- 4 **return** (\mathbf{z}, \mathbf{c}) with probability $\min\left(\frac{D_{\sigma}^m}{MD_{\mathbf{Sc}, \sigma}^m}, 1\right)$,

where $M \in \mathbb{R}$ is such that

$$\Pr \left[MD_{\mathbf{Sc}, \sigma}^m (\mathbf{z}) \geq D_{\sigma}^m (\mathbf{z}) : \mathbf{z} \leftarrow D_{\sigma}^m \right] \geq 1 - \varepsilon.$$

A SIS BASED DIGITAL SIGNATURE ALGORITHM..

Algorithm 3 — **Verify** ($pk := (\mathbf{A}, \mathbf{T}) , \mu, (\mathbf{z}, \mathbf{c})$)

if $\mathbf{c} = \mathsf{H}(\mathbf{A}^T \mathbf{z} - \mathbf{T}\mathbf{c}, \mu)$ and $\|\mathbf{z}\| \leq \sigma\sqrt{m}$ then
 └ return 1

LWE BASED DIGITAL SIGNATURE ALGORITHM

Algorithm 4 — **KeyGen**(n, m, k, q)

- 1 $\mathbf{S} \in_{\phi} \mathbb{Z}_q^{n \times k}$
- 2 $\mathbf{A} \xleftarrow{\$} \mathbb{Z}_q^{m \times n}, \mathbf{E} \in_{\chi} \mathbb{Z}_q^{m \times k}$ and $\mathbf{T} = \mathbf{A} \cdot \mathbf{S} + \mathbf{E} \pmod{q}$
- 3 **return** $sk := \mathbf{S}$ and $pk := (\mathbf{A}, \mathbf{T})$

A FRAMEWORK FOR LWE BASED SIGNATURE

Algorithm 5 — $\text{Sign}(\mu, sk, pk)$

```
1  $\mathbf{y}_1, \mathbf{y}_2 \leftarrow D_{\mathbf{y}}^n$ 
2  $\mathbf{w} = \mathbf{A} \cdot \mathbf{y}_1 + \mathbf{y}_2$                                 ▷ Commitment:  $\mathbf{w}$ 
3  $\mathbf{c} = \mathsf{H}(\mu || \mathbf{w}) \in \{-1, 0, +1\}^k$       ▷ Challenge: sparse  $\mathbf{c}$ 
4  $\mathbf{z}_1 = \mathbf{S}\mathbf{c} + \mathbf{y}_1, \mathbf{z}_2 = \mathbf{E}\mathbf{c} + \mathbf{y}_2$     ▷ Response:  $\mathbf{z}_1, \mathbf{z}_2$ 
5 if ( $\mathbf{z}_1, \mathbf{z}_2$ ) leaks dist of  $\mathbf{S}$  then ▷  $\|\mathbf{z}_i\| > bd_i$ 
6   restart
7 return ( $\mathbf{z}_1, \mathbf{z}_2, \mathbf{c}$ )
```

Algorithm 6 — $\text{Verify}(pk := (\mathbf{A}, \mathbf{T}), \mu, (\mathbf{z}_1, \mathbf{z}_2, \mathbf{c}))$

```
1 if  $\|\mathbf{z}_i\| \leq bd_i \forall i$  then
2   return  $H(\mu || \mathbf{A}\mathbf{z}_1 + \mathbf{z}_2 - \mathbf{T}\mathbf{c}) \stackrel{?}{=} \mathbf{c}$ 
```

A FRAMEWORK FOR LWE BASED SIGNATURE..

- ❑ The main draw back of this framework (Algorithm 5) is the signature size. In comparison to the SIS based schemes, the signature has an additional vector \mathbf{z}_2 of dimension n .
- ❑ This can be resolved using the Bai-Galbraith technique. We discuss this in the signature scheme given in the Algorithm 7 below.

LWE BASED DIGITAL SIGNATURE ALGORITHM..

Algorithm 7 — **Sign**(μ, sk, pk)

```
1  $y \leftarrow D_\sigma^n; w = A \cdot y$                                 ▷ Commitment: w
2  $w_1 = \text{HighBits}(w)$ 
3  $c = H(\mu || w_1) \in \{-1, 0, +1\}^k$       ▷ Challenge: sparse c
4 if  $\text{LowBits}(w - Ec) > bd$  then
5   restart
6  $z = Sc + y$                                               ▷ Response: z
7 if  $z$  leaks dist of S then ▷  $\|z\| > bd_1$ 
8   restart
9 return (z, c)
```

LWE BASED DIGITAL SIGNATURE ALGORITHM..

Algorithm 8 — **verify** ($pk := (\mathbf{A}, \mathbf{T}) , \mu, (\mathbf{z}, \mathbf{c})$)

```
1 w'1 = HighBits(AZ – Tc)
2 if c =  $H(\mu || \mathbf{w}'_1)$  and  $\|\mathbf{z}\| \leq bd_1$  then
3   return 1
```