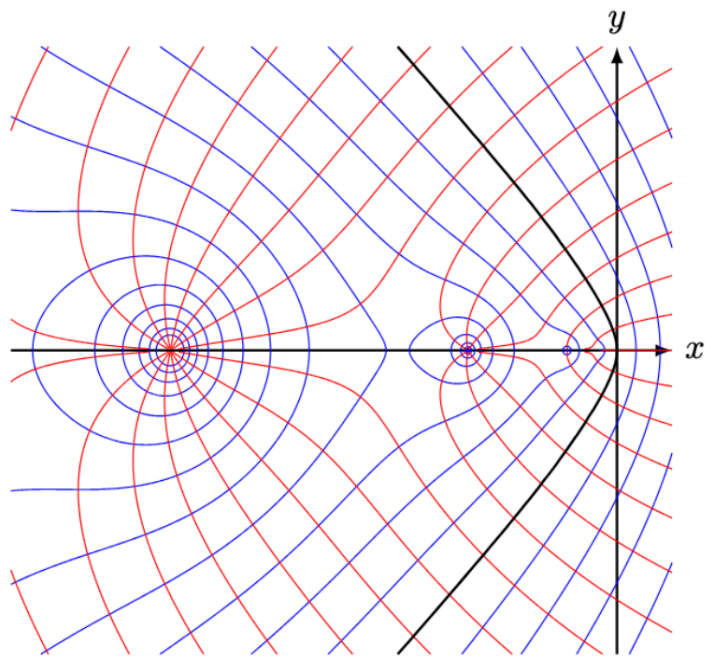


# Fourier Series on Discrete Periodic Time Scales

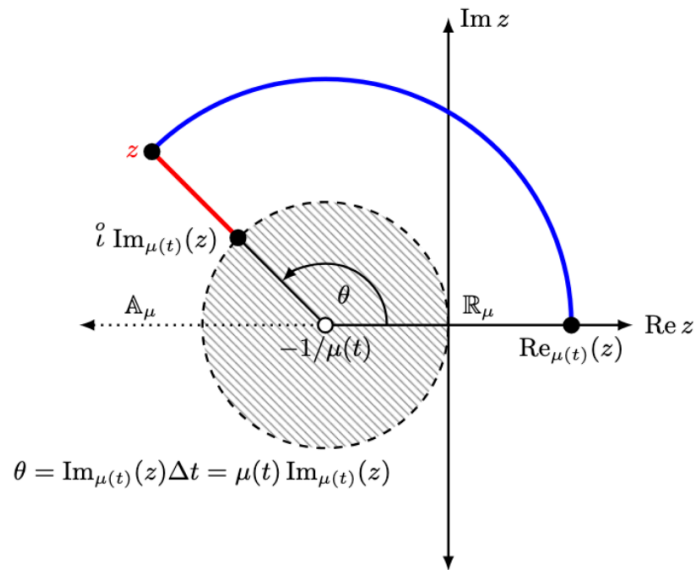
Dylan Poulsen  
Washington College



# Outline

- Hilger's Complex Plane
- Ergodic Complex Plane
- Review Fourier Series
- Fourier Series on  $\mathbb{T}$
- Antarctic Ice Core Example

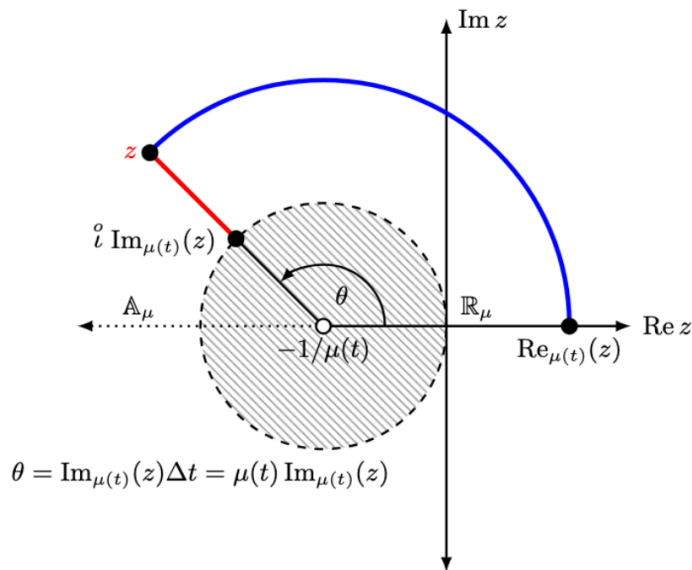
# Hilger's Local Plane



- Local behavior of  $e_z(t, t_0)$  at time  $t \in \mathbb{T}$  in terms of  $z \in \mathbb{C} \setminus \{-1/\mu(t)\} = \mathbb{C}_{\mu(t)}$ .
- Central to Hilger's plane is the *cylinder transformation*  $\xi_{\mu(t)} : \mathbb{C}_{\mu(t)} \rightarrow \mathbb{Z}_{\mu(t)}$  defined by  $\xi_{\mu(t)}(z) = \frac{\text{Log}(1 + \mu(t)z)}{\mu(t)}$ , where  

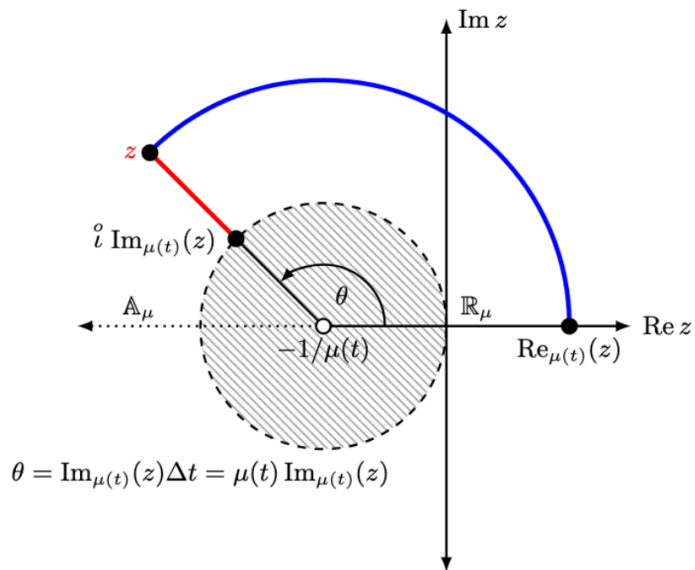
$$\mathbb{Z}_{\mu(t)} := \left\{ z \in \mathbb{C} \mid -\frac{\pi}{\mu(t)} < \text{Im}(z) \leq \frac{\pi}{\mu(t)} \right\}.$$

# Hilger's Local Plane



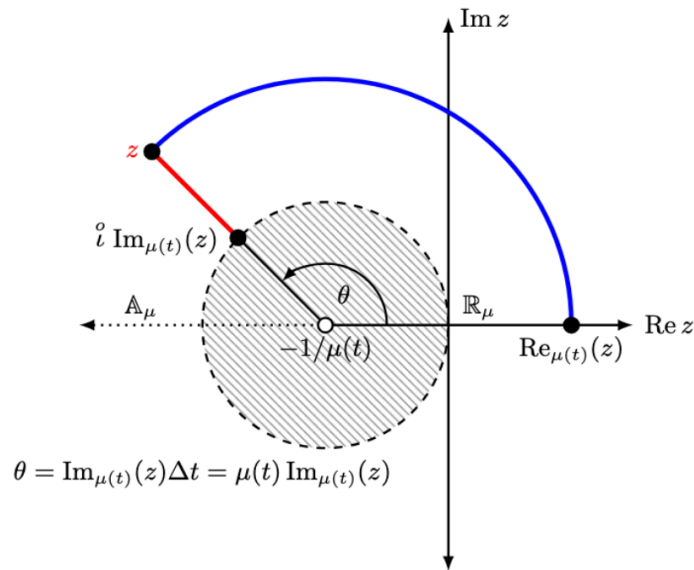
$$\begin{aligned} \text{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z))) \end{aligned}$$

# Hilger's Local Plane



$$\begin{aligned} \text{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z))) \\ \text{Im}_{\mu(t)}(z) &:= \frac{\text{Arg}(1 + z\mu(t))}{\mu(t)} \\ &= \text{Im}(\xi_{\mu(t)}(z)) \end{aligned}$$

# Hilger's Local Plane



$$\text{Re}_{\mu(t)}(z) := \frac{|1 + z\mu(t)| - 1}{\mu(t)}$$

$$= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z)))$$

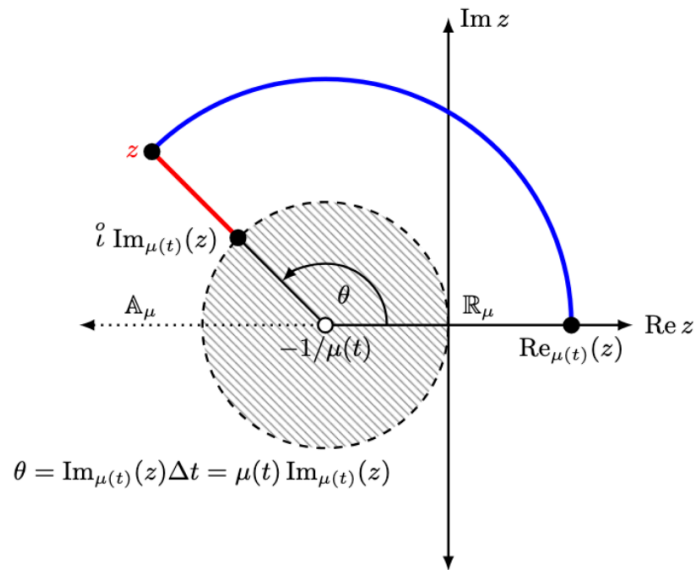
$$\text{Im}_{\mu(t)}(z) := \frac{\text{Arg}(1 + z\mu(t))}{\mu(t)}$$

$$= \text{Im}(\xi_{\mu(t)}(z))$$

$${}^o\text{Im}_{\mu(t)}(z) := \frac{e^{i\text{Im}_{\mu(t)}(z)\mu(t)} - 1}{\mu(t)}$$

$$= \xi_{\mu(t)}^{-1}(i\text{Im}(\xi_{\mu(t)}(z)))$$

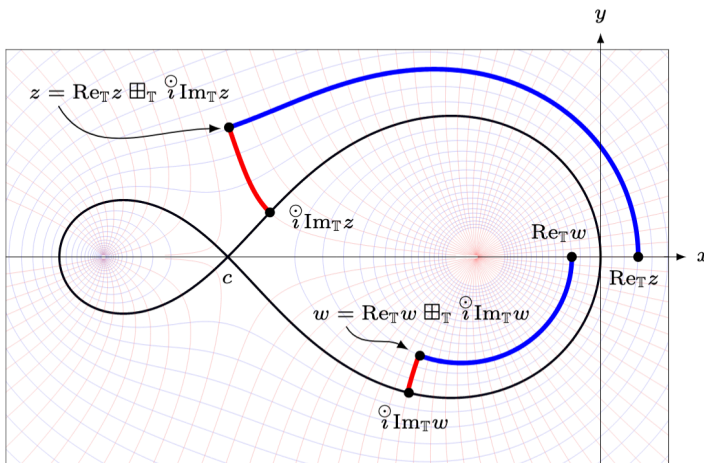
# Hilger's Local Plane



- With  $a \oplus_{\mu(t)} b := a + b + \mu(t)ab$ , Hilger's local plane is isomorphic to the cylinder strip  $\mathbb{Z}_{\mu(t)}$  with addition mod  $2\pi i/\mu(t)$ .
- $z \oplus_{\mu(t)} w = \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(z) + \xi_{\mu(t)}(w))$
- Hilger's decomposition:  

$$z = \text{Re}_{\mu(t)}(z) \oplus_{\mu(t)} i \text{Im}_{\mu(t)}(z)$$

# Ergodic Global Plane

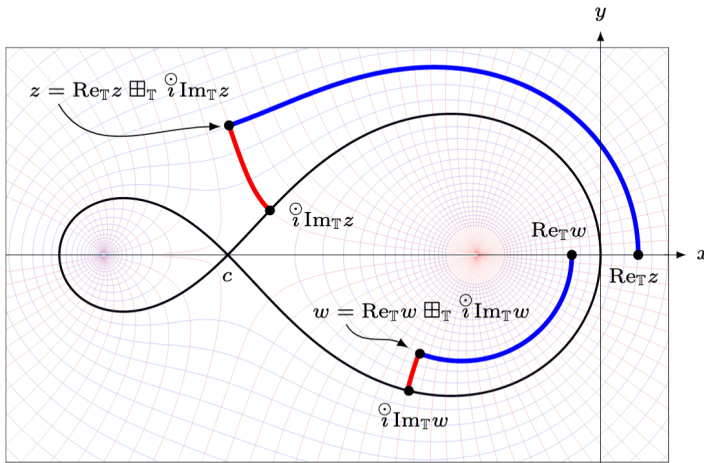


- Global behavior of  $e_z(t, t_0)$  in terms of  $z \in \mathbb{C} \setminus \{-1/\mu(t) \mid t \in \mathbb{T}\} := \mathbb{C}_{\mu(\mathbb{T})}$ .
- On a periodic (or finite) time scale of length  $L$ , defined by the *average* of the cylinder transformation  

$$\bar{\xi}(z) = \frac{1}{L} \int_0^L \xi_{\mu(\tau)}(z) \Delta\tau$$
 (at least off the branch cut).



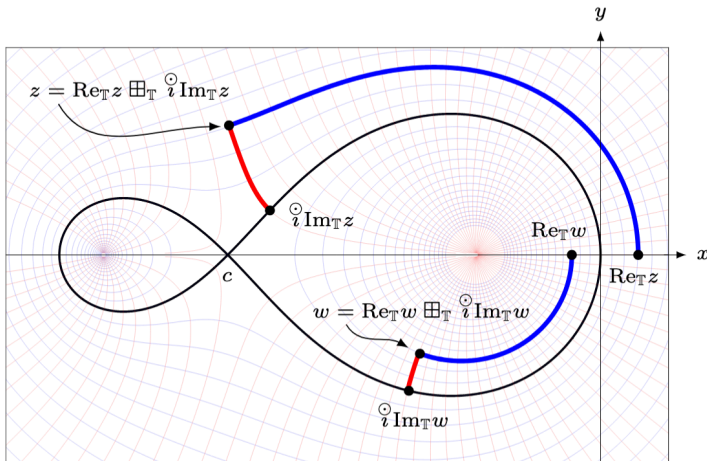
# Ergodic Global Plane



- The generalization of the cylinder strip is the set  $\mathbb{C}^\Omega = \{z \in \mathbb{C} \mid -\Omega < \text{Im}(z) \leq \Omega\}$ , where  $\Omega$  is the Nyquist frequency.
- We have carefully defined  $\bar{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \rightarrow \mathbb{C}^\Omega$  on the branch cut and shown that the resulting map is globally univalent.
- $z \boxplus_{\mathbb{T}} w = \bar{\xi}^{-1}(\bar{\xi}(z) + \bar{\xi}(w) \pmod{2i\Omega})$

# Ergodic Global Plane

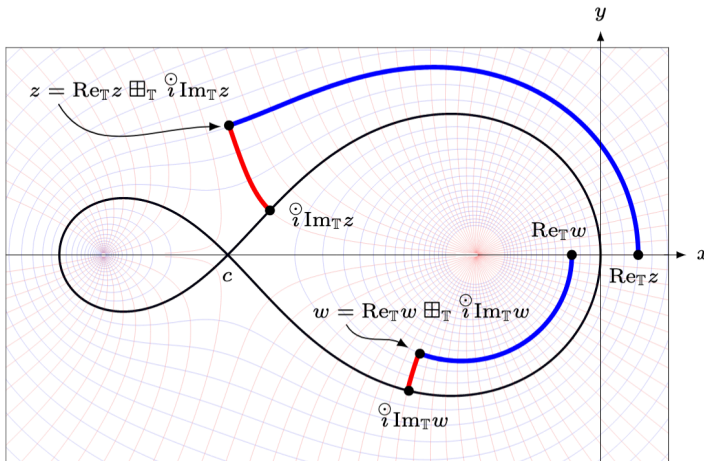
$$\operatorname{Re}_{\mathbb{T}}(z) = \bar{\xi}^{-1}(\operatorname{Re}(\bar{\xi}(z)))$$



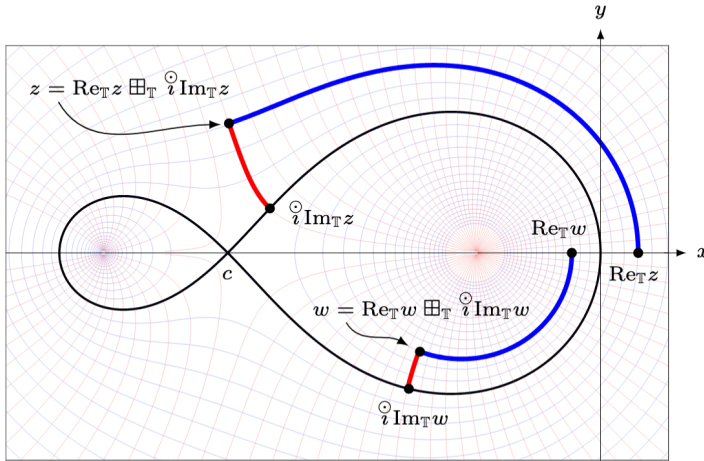
# Ergodic Global Plane

$$\operatorname{Re}_{\mathbb{T}}(z) = \bar{\xi}^{-1}(\operatorname{Re}(\bar{\xi}(z)))$$

$$\operatorname{Im}_{\mathbb{T}}(z) = \operatorname{Im}(\bar{\xi}(z))$$



# Ergodic Global Plane

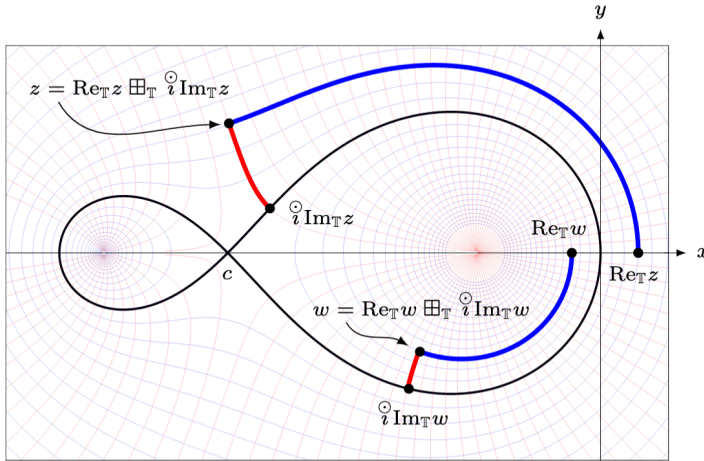


$$\text{Re}_{\mathbb{T}}(z) = \bar{\xi}^{-1}(\text{Re}(\bar{\xi}(z)))$$

$$\text{Im}_{\mathbb{T}}(z) = \text{Im}(\bar{\xi}(z))$$

$$\circ \text{Im}_{\mathbb{T}}(z) := \bar{\xi}^{-1}(i\text{Im}(\bar{\xi}(z)))$$

# Ergodic Global Plane



$$\text{Re}_{\mathbb{T}}(z) = \bar{\xi}^{-1}(\text{Re}(\bar{\xi}(z)))$$

$$\text{Im}_{\mathbb{T}}(z) = \text{Im}(\bar{\xi}(z))$$

$$\circledast \text{Im}_{\mathbb{T}}(z) := \bar{\xi}^{-1}(i \text{Im}(\bar{\xi}(z)))$$

$$z = \text{Re}_{\mathbb{T}}(z) \boxplus_{\mathbb{T}} \circledast \text{Im}_{\mathbb{T}}(z)$$

# Fourier Series on $\mathbb{R}$

On  $\mathbb{R}$ , the Fourier series for an  $L$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is given by

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n}{L} t} \\ &= \sum_{n=-\infty}^{\infty} c_n e_{\xi_0^{-1}(i \frac{2\pi n}{L})}(t, 0), \text{ with} \\ c_n &= \frac{1}{L} \int_0^L f(t) e^{-i \frac{2\pi n}{L} t} dt. \end{aligned}$$

# Fourier Series on $\mathbb{Z}$

On  $\mathbb{Z}$ , the discrete Fourier series for an  $L$ -periodic function  $f : \mathbb{Z} \rightarrow \mathbb{C}$  is given by

$$\begin{aligned} f(t) &= \sum_{n=-\lfloor (L-1)/2 \rfloor}^{\lceil (L-1)/2 \rceil} c_n e^{i \frac{2\pi n}{L} t} \\ &= \sum_{n=-\lfloor (L-1)/2 \rfloor}^{\lceil (L-1)/2 \rceil} c_n e_{\xi_1^{-1}(i \frac{2\pi n}{L})}(t, 0), \text{ with} \\ c_n &= \frac{1}{L} \sum_{t=0}^{L-1} f(t) e^{-i \frac{2\pi n}{L} t}. \end{aligned}$$

# Fourier Series on $\mathbb{T}$

**Key Insight:** Basis functions should be exponentials with zero growth rate and a harmonic frequency. That is, they should be time scale exponentials with subscript

$$\bar{\xi}^{-1}(i\omega_n) = i\omega_n,$$

where  $\omega_n$  is a harmonic frequency.



# Fourier Series on $\mathbb{T}$

- To calculate the values of the subscripts, we could sample the Nyquist interval uniformly and numerically calculate  $\xi^{-1}$  at each of these sample points. There is an easier way, however - eigenvalues!
- Let
$$\mathbb{T} = \{t_0, t_1, t_2, \dots, t_{m-1}, t_m, t_1 + L, t_2 + L, \dots, t_0 + 2L, t_1 + 2L, \dots\}$$
be a periodic time scale of length  $L$ .
- We can write the dynamic equation on  $\mathbb{T}$ ,  $x^\Delta = \lambda x$ , with a periodic condition  $x(t_0) = x(t_m)$  as a system of equations.

## Fourier Series on $\mathbb{T}$

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \quad 0 \leq k \leq m-2$$
$$\frac{x(t_m) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

# Fourier Series on $\mathbb{T}$

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \quad 0 \leq k \leq m-2$$

$$\frac{x(t_0) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

# Fourier Series on $\mathbb{T}$

$$\begin{pmatrix} -1/\mu(t_0) & 1/\mu(t_0) & 0 & \cdots & 0 & 0 \\ 0 & -1/\mu(t_1) & 1/\mu(t_1) & \cdots & 0 & 0 \\ 0 & 0 & -1/\mu(t_2) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1/\mu(t_{m-3}) & 0 \\ 0 & 0 & \cdots & 0 & -1/\mu(t_{m-2}) & 1/\mu(t_{m-2}) \\ 1/\mu(t_{m-1}) & 0 & \cdots & 0 & 0 & -1/\mu(t_{m-1}) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix} = \lambda \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix}$$

The eigenvalues are  $i\omega_k^\odot, 0 \leq k \leq m-1$ .

The associated eigenvectors are  $e_{i\omega_k^\odot}(t, 0)$  evaluated at each point in one period of the time scale.

# Fourier Series on $\mathbb{T}$

Now, we calculate the Fourier coefficients in

$$f(t) = \sum_{n=0}^{m-1} c_n e_{i\omega_n}^{\odot}(t, 0).$$

# Theorem

Suppose  $\mathbb{T}$  is a periodic time scale with  $t_0 = 0$ . Suppose also that  $i\omega_n$  exists for each  $n, 0 \leq n \leq m - 1$ , and are distinct. Then, the sets  $\{e_{i\omega_n}(t, 0)\}$  and  $\{e_{\ominus i\omega_n}(\sigma(t), 0)\}$  form a biorthogonal system, that is

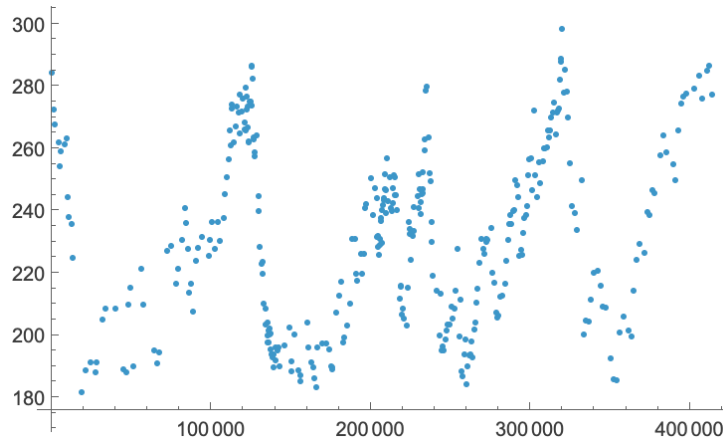
$$\int_0^L e_{i\omega_n}(t, 0) e_{\ominus i\omega_l}(\sigma(t), 0) \Delta t = K_n \delta_{n,l}.$$

# Fourier Series on $\mathbb{T}$

Therefore,

$$c_n = \frac{\int_0^L f(t) \cdot e_{\ominus i\omega_n}(\sigma(t), 0) \Delta t}{\int_0^L e_{i\omega_n}(t, 0) \cdot e_{\ominus i\omega_n}(\sigma(t), 0) \Delta t}.$$

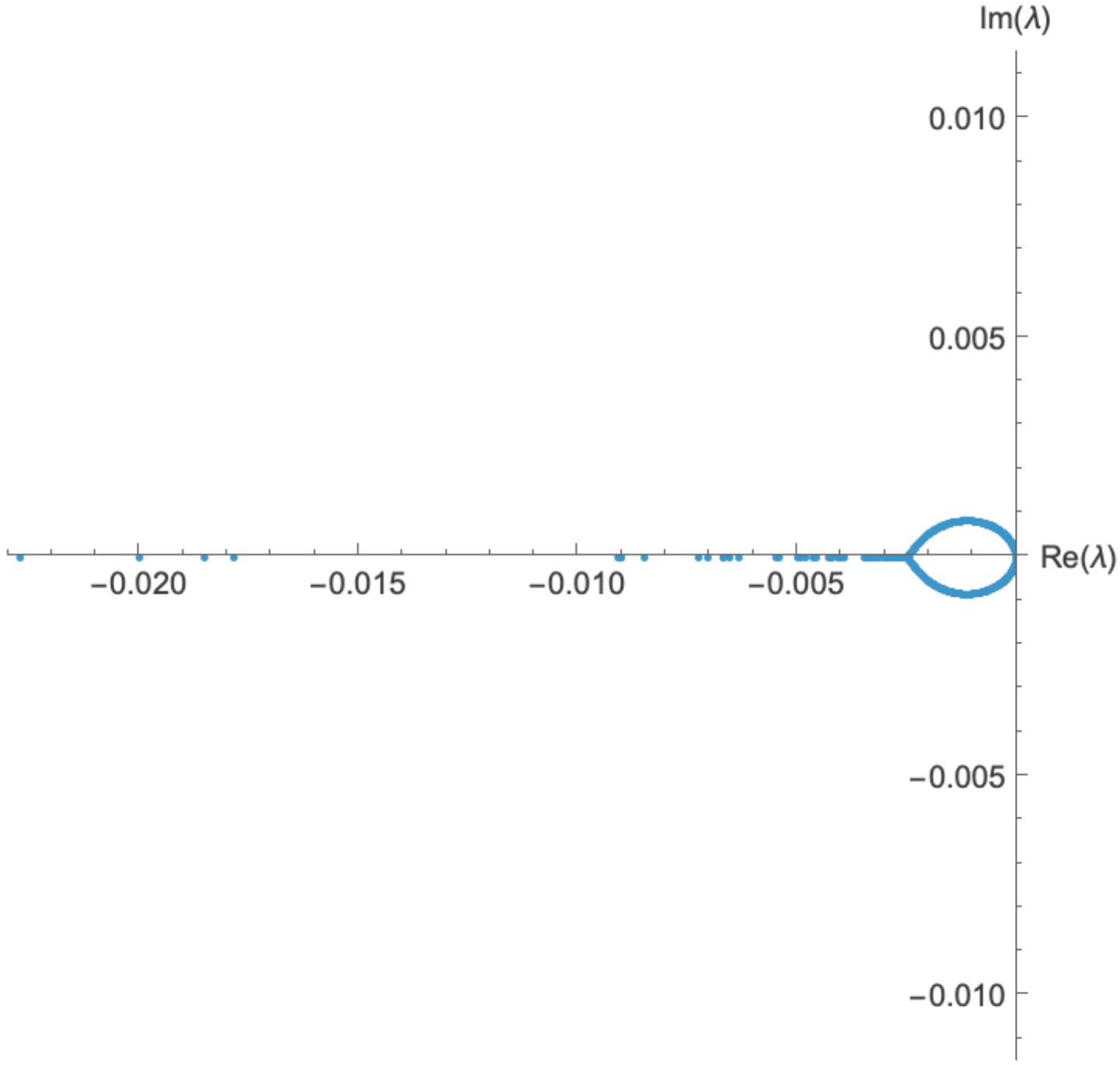
# Antarctic Ice Core Data



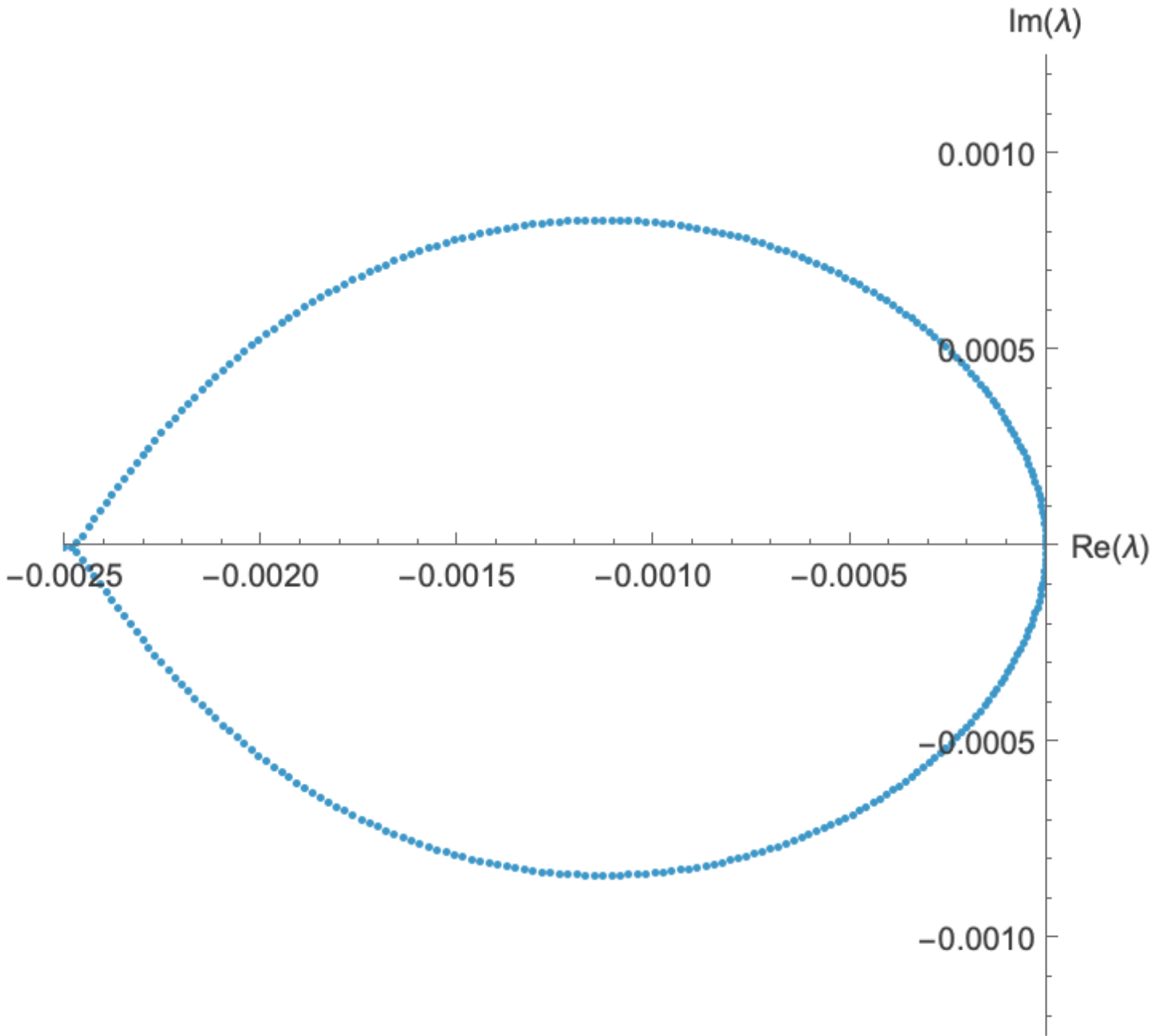
- Record of 400,000+ years of atmospheric CO2 concentration
- Irregularly sampled in time because the ice core is sampled uniformly along the length of the ice core.



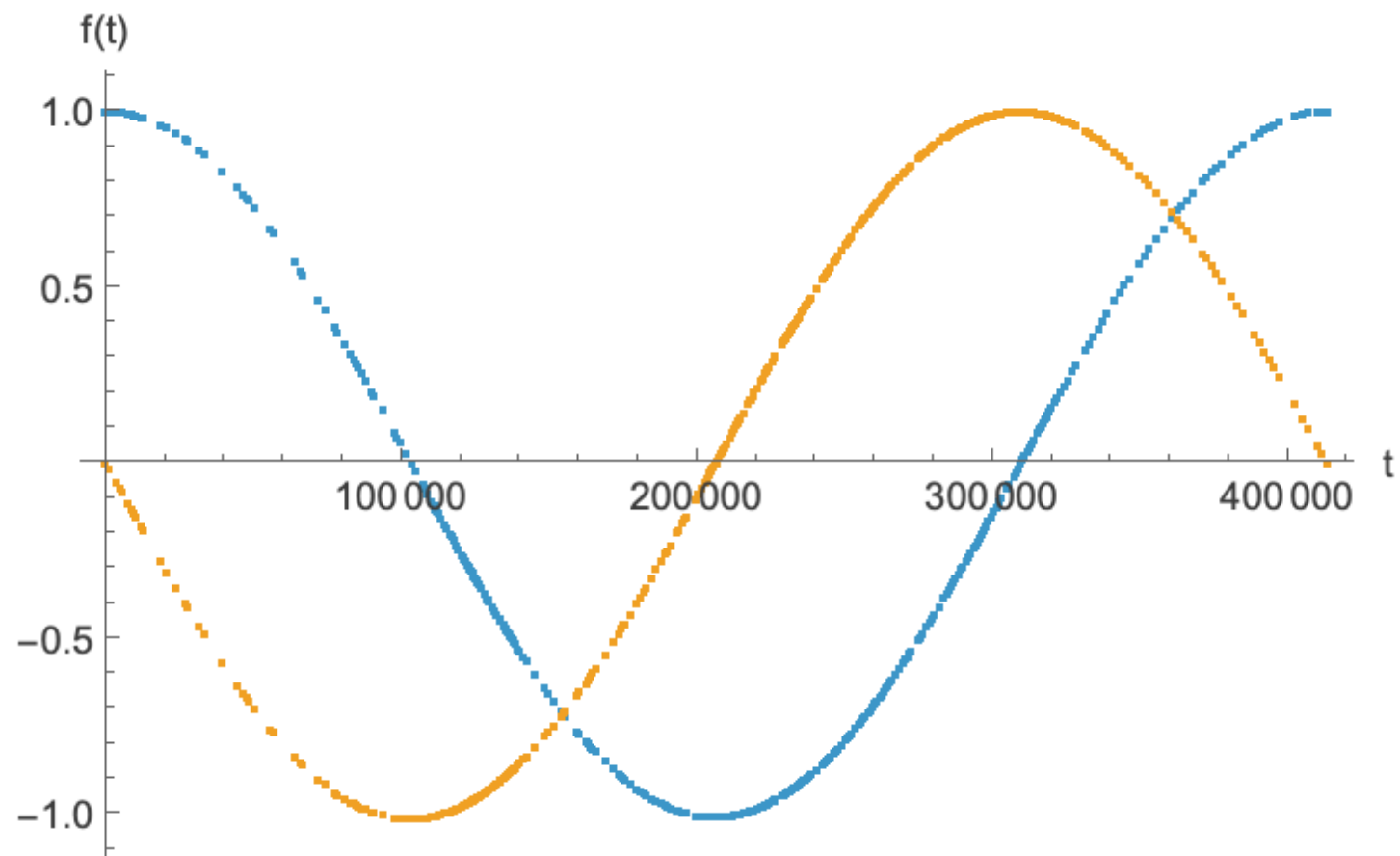
# Spectrum



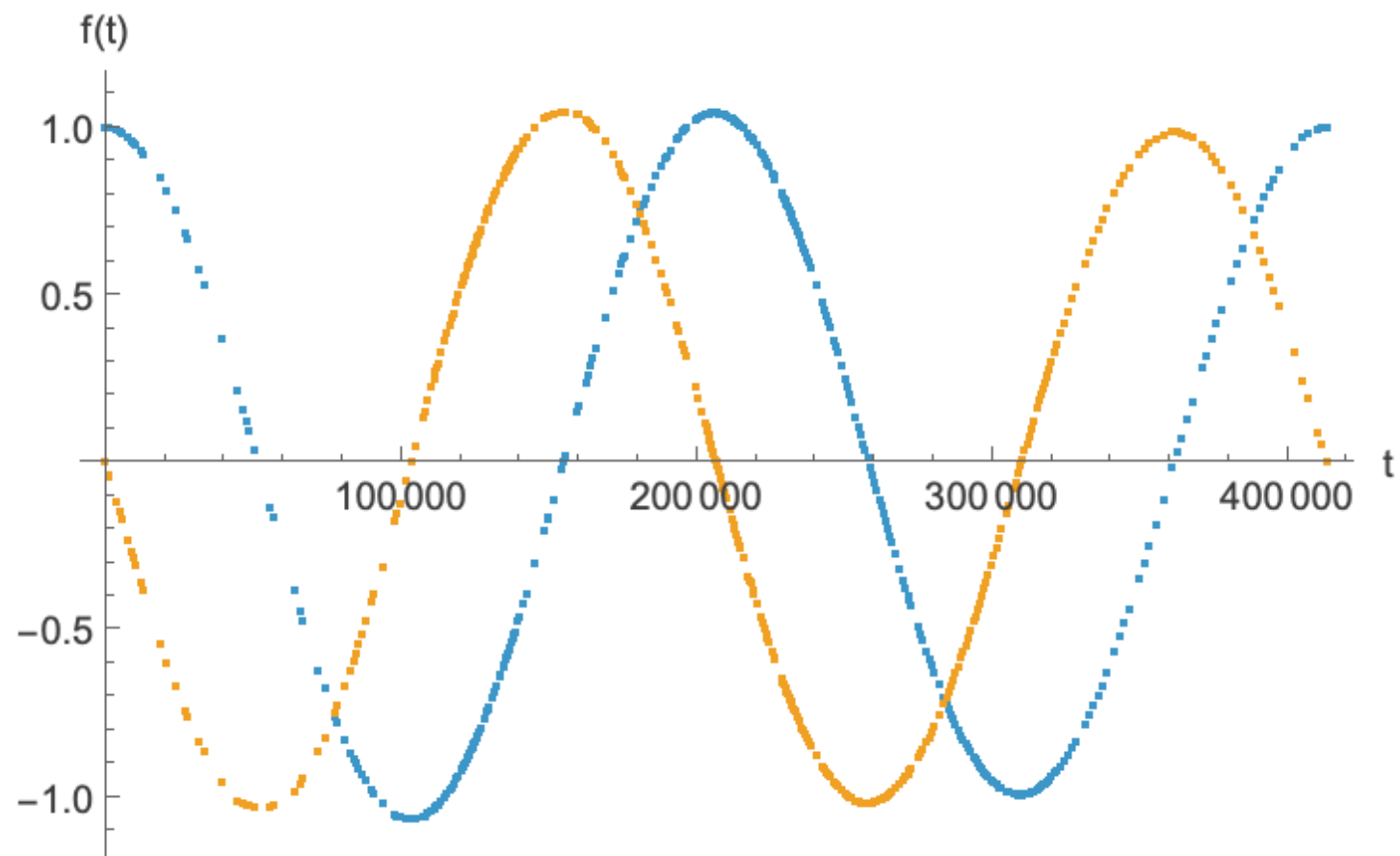
# Spectrum



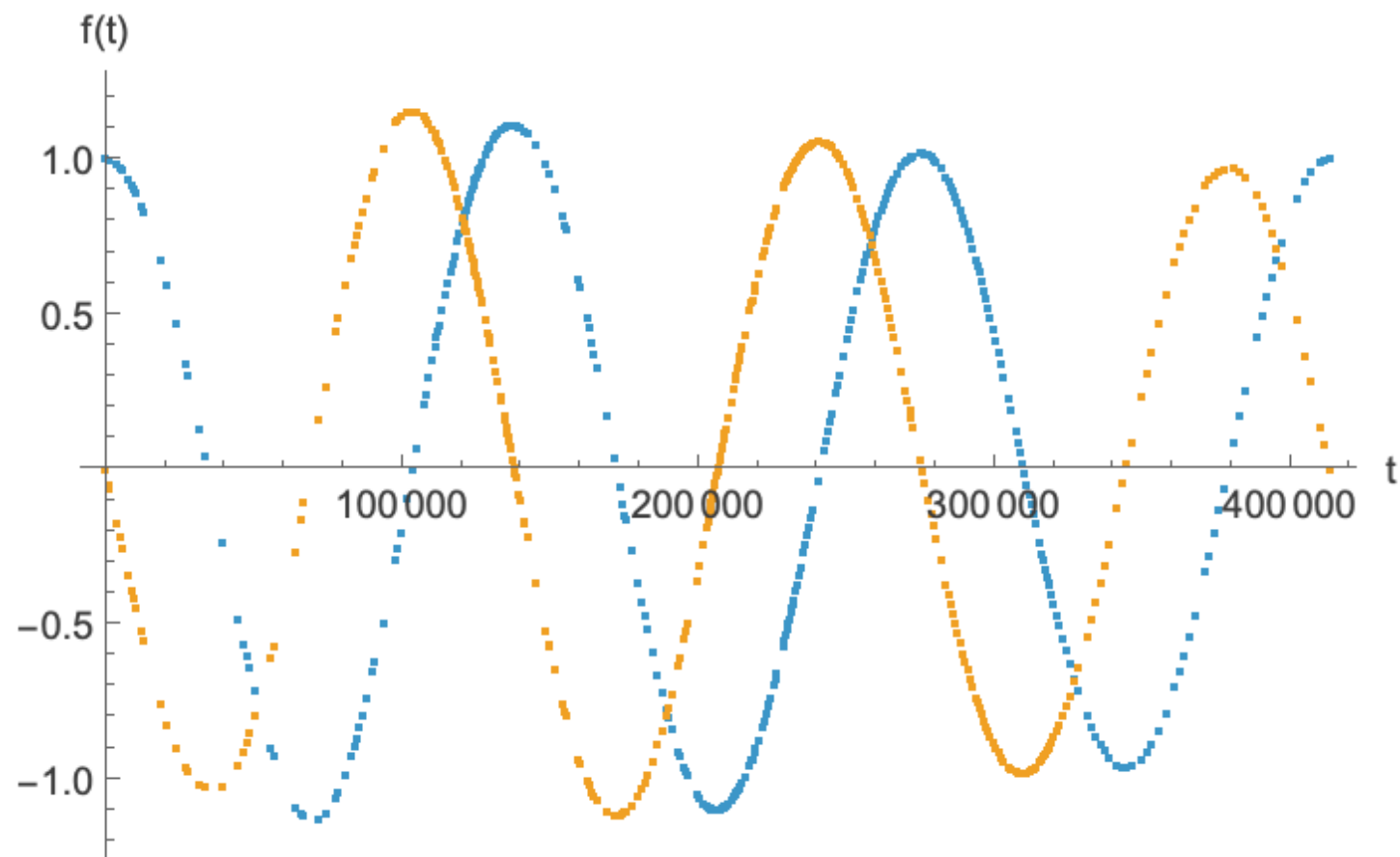
# Bases



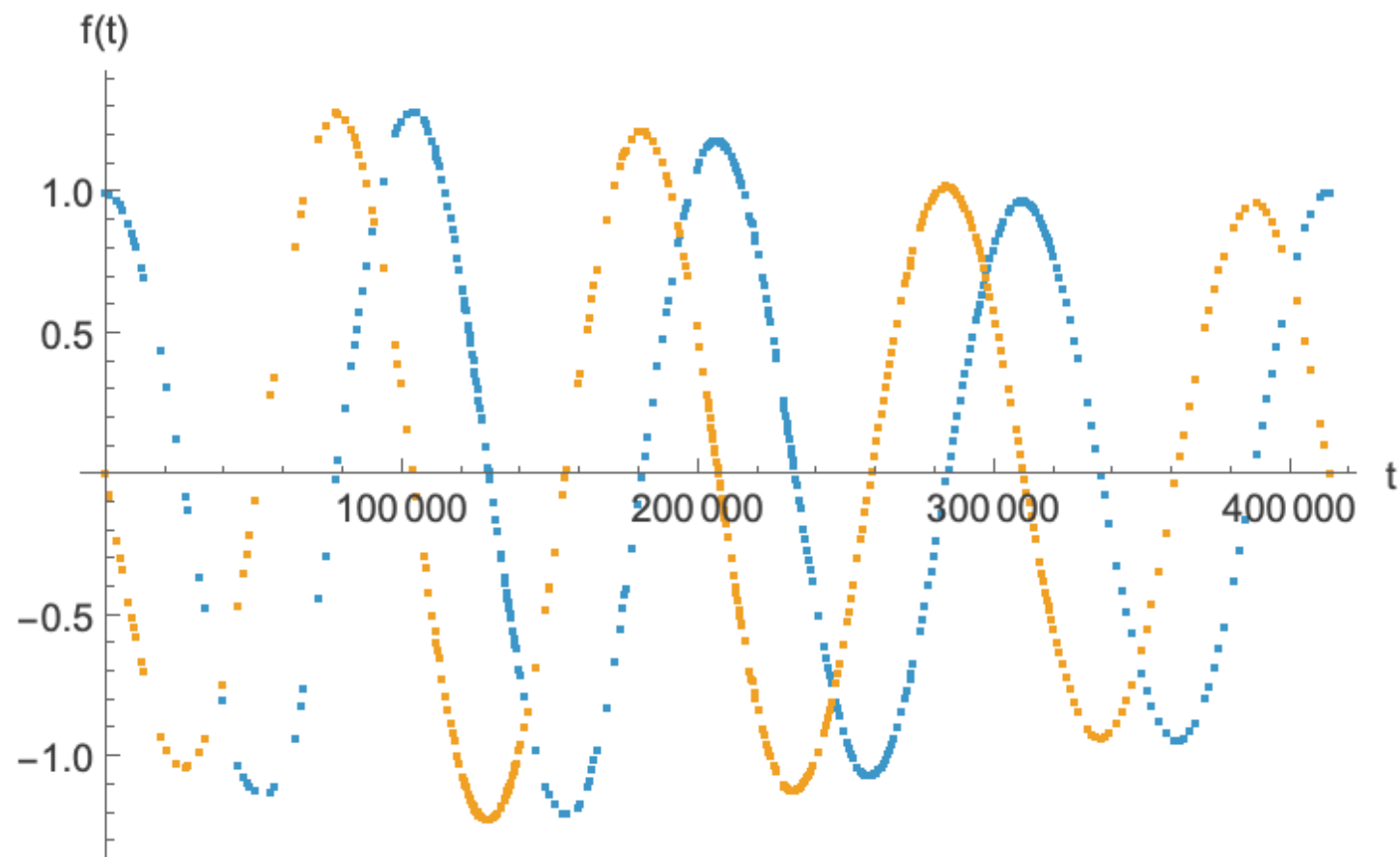
# Bases



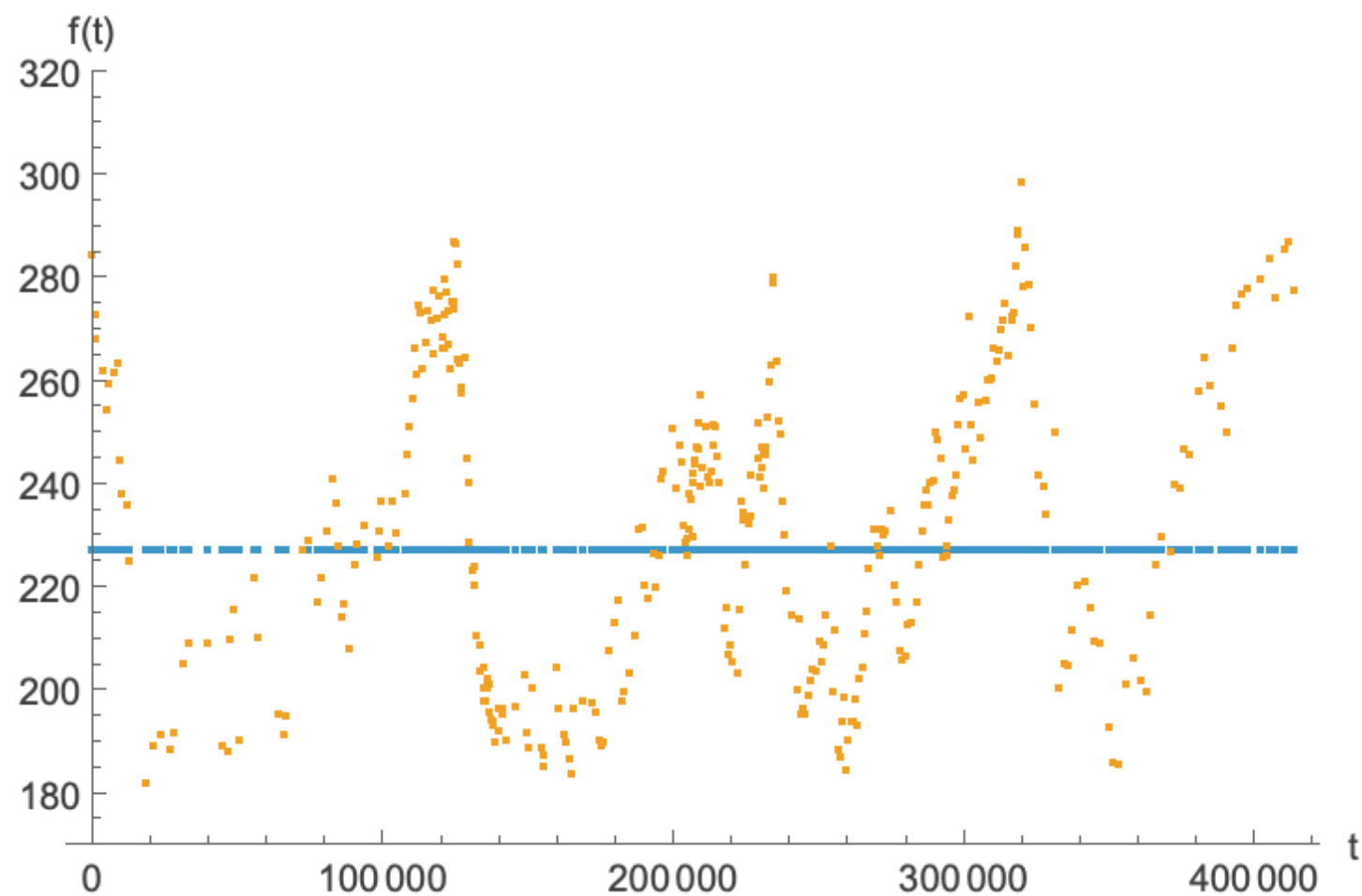
# Bases



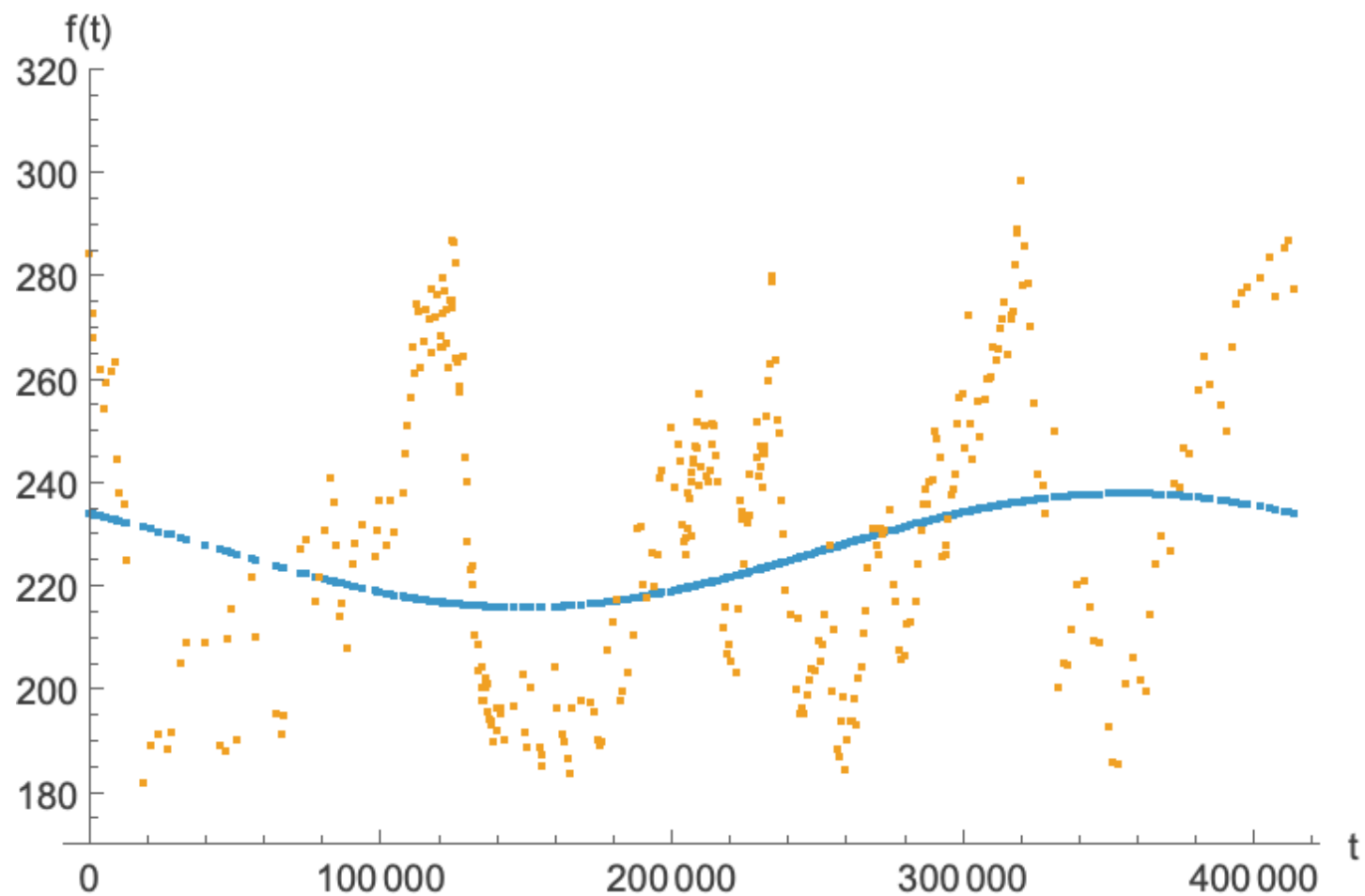
# Bases



# Approx

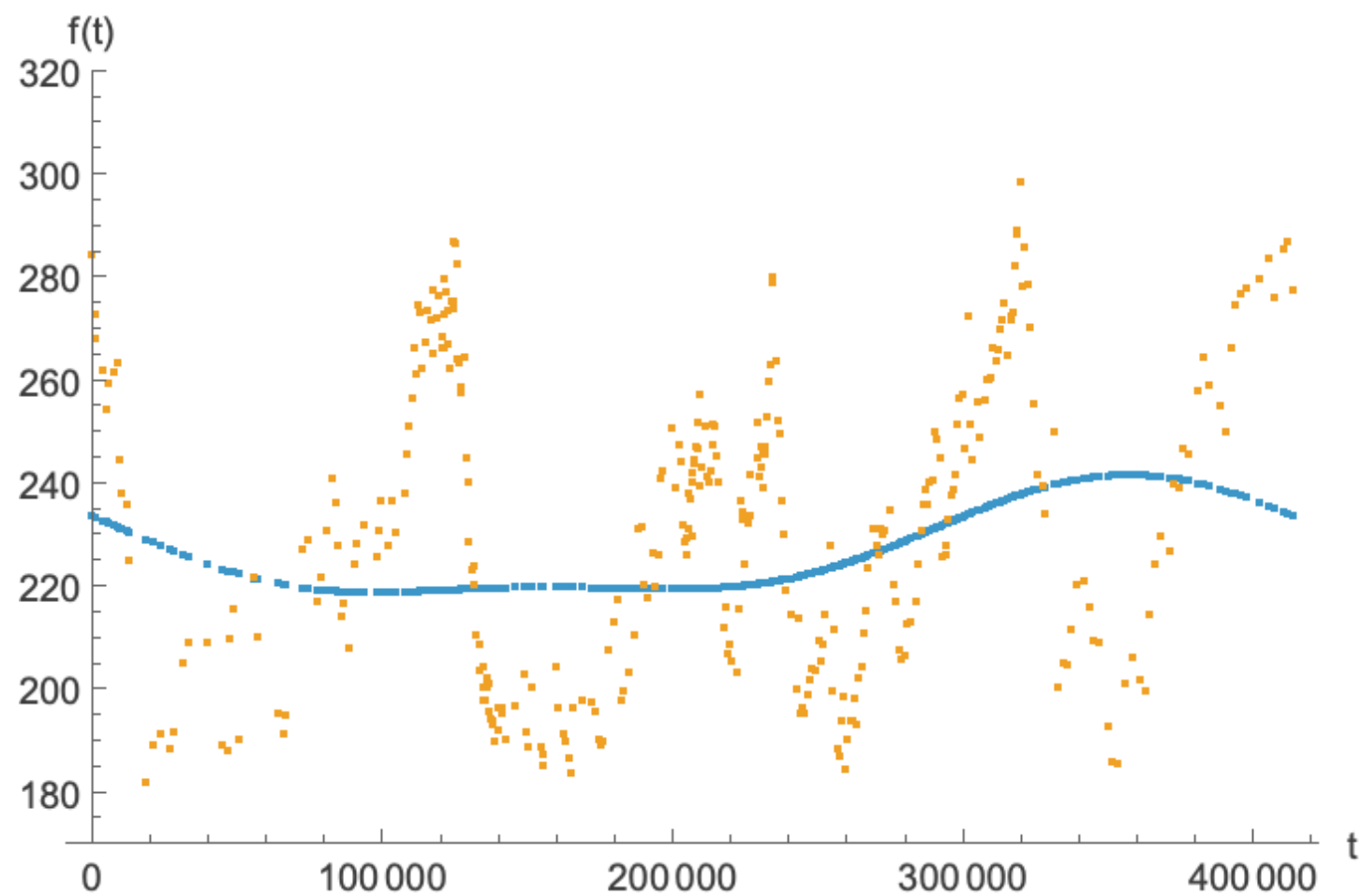


# Approx

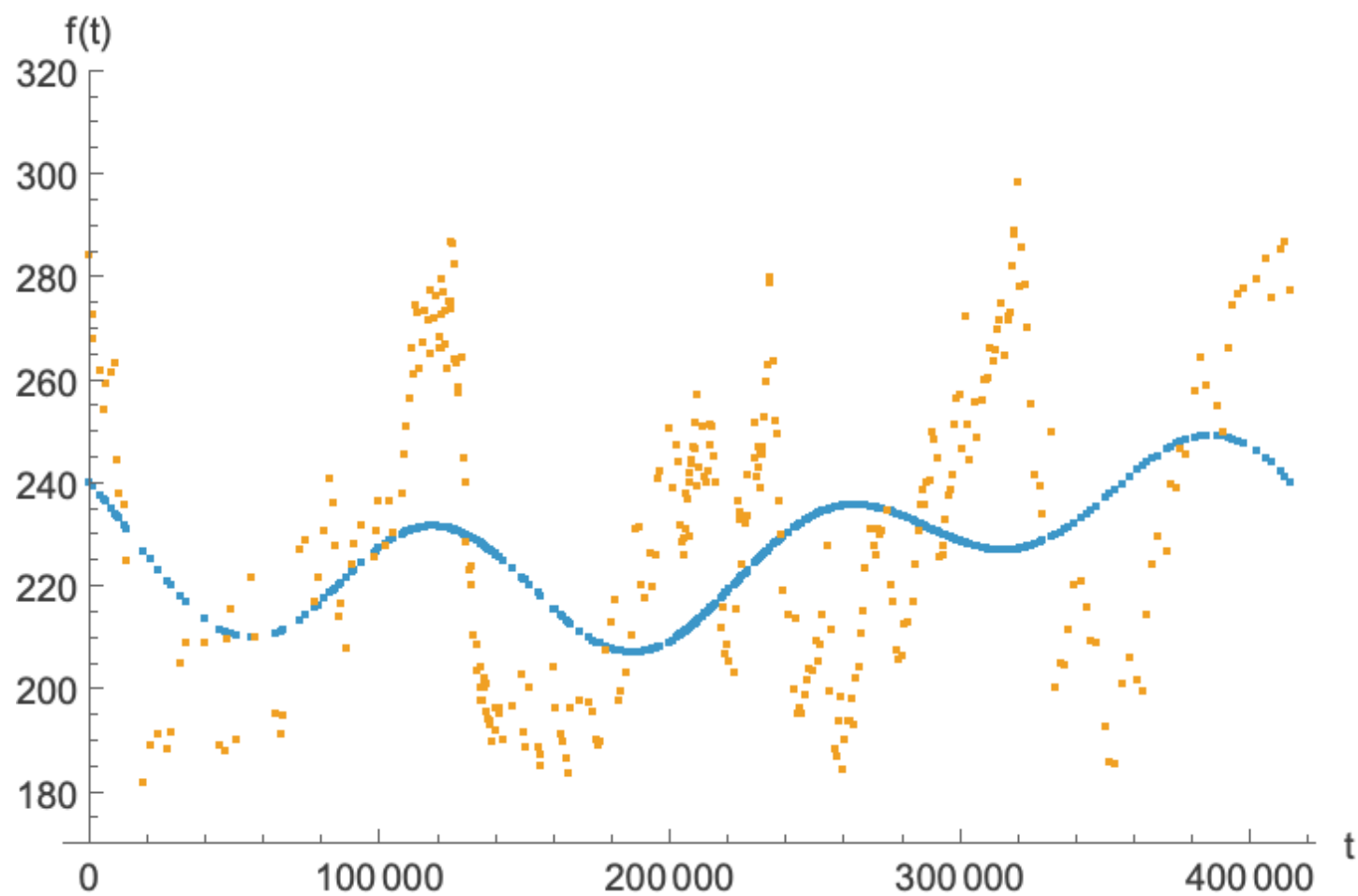




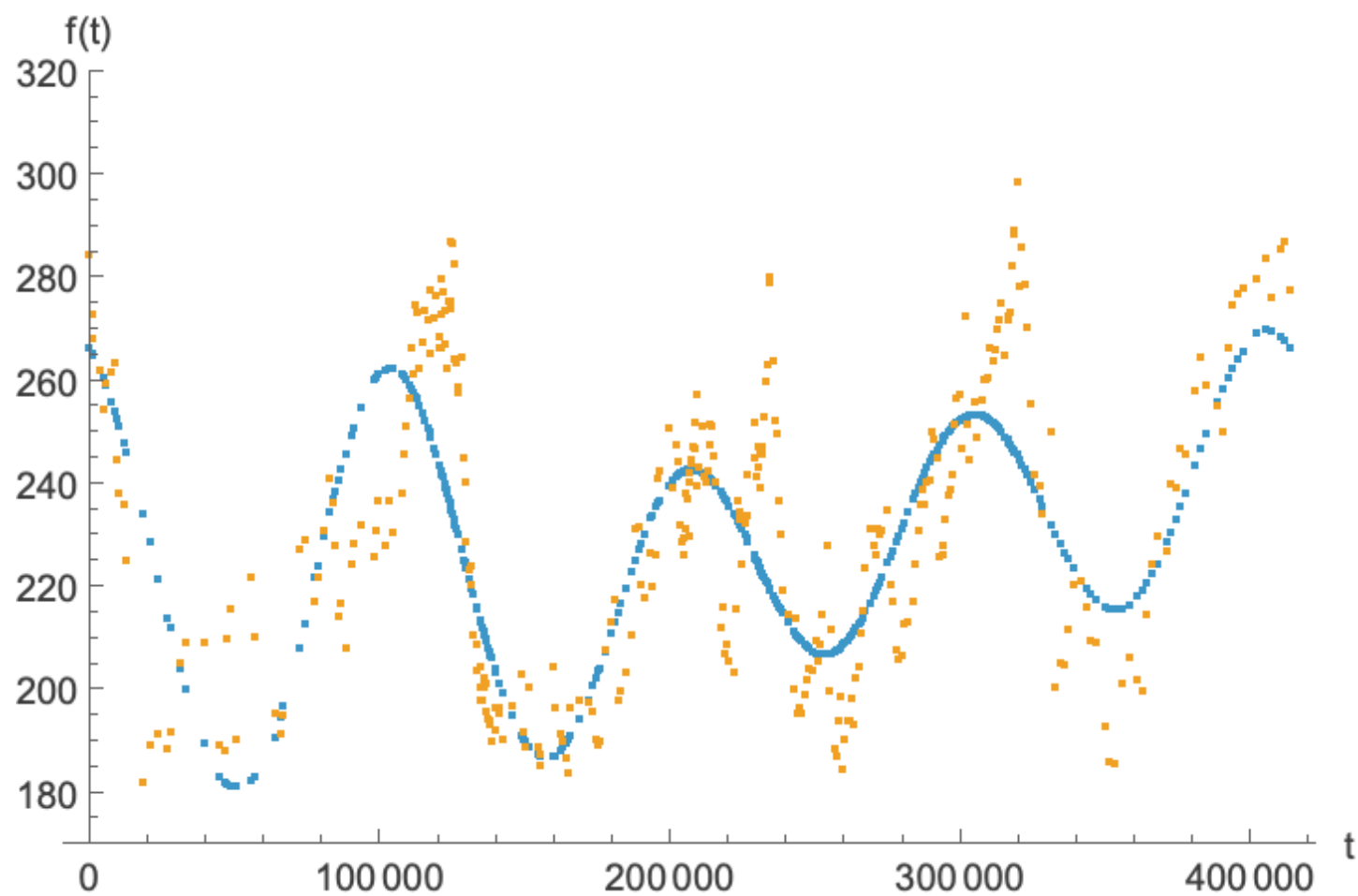
# Approx



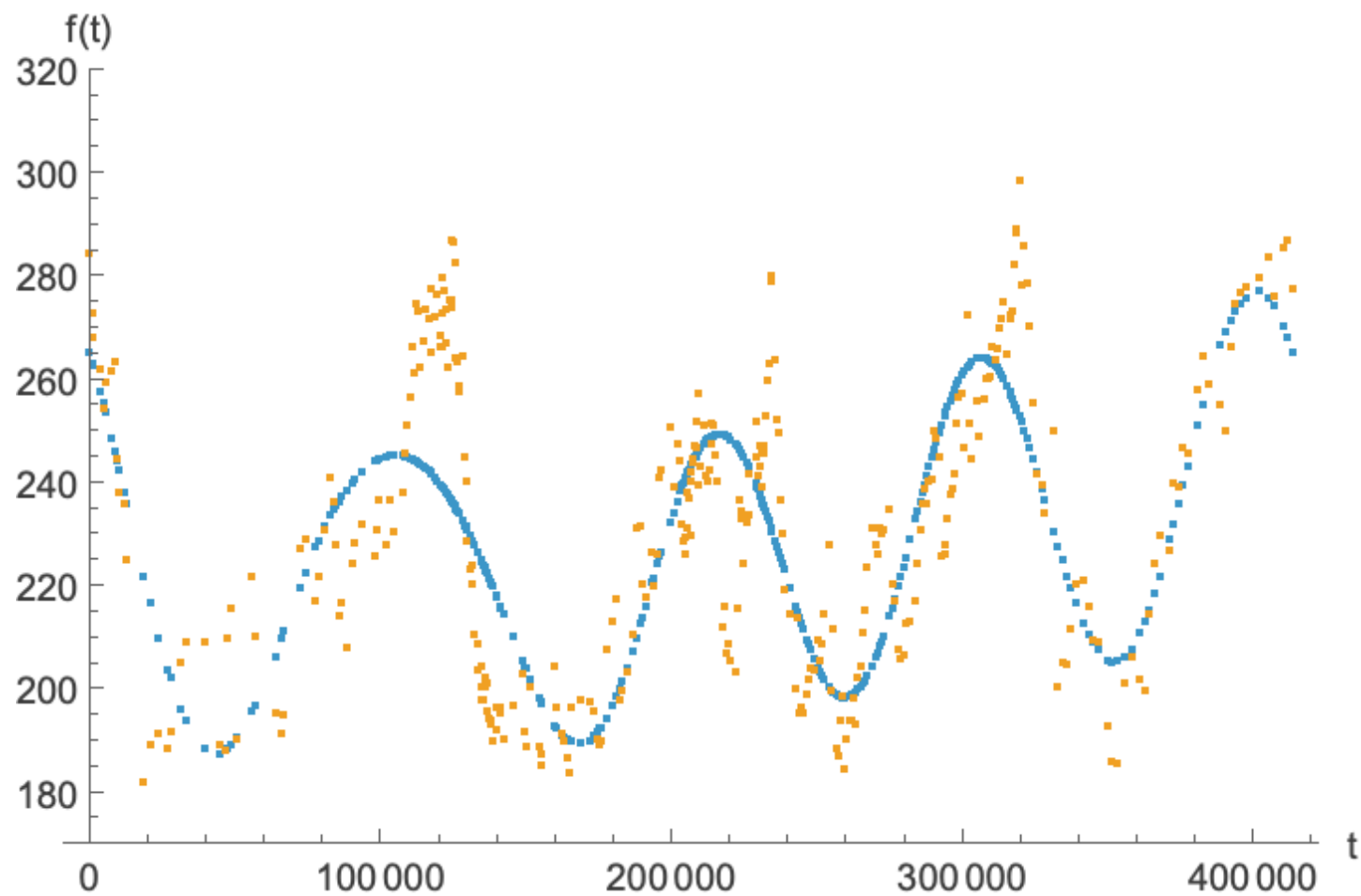
# Approx



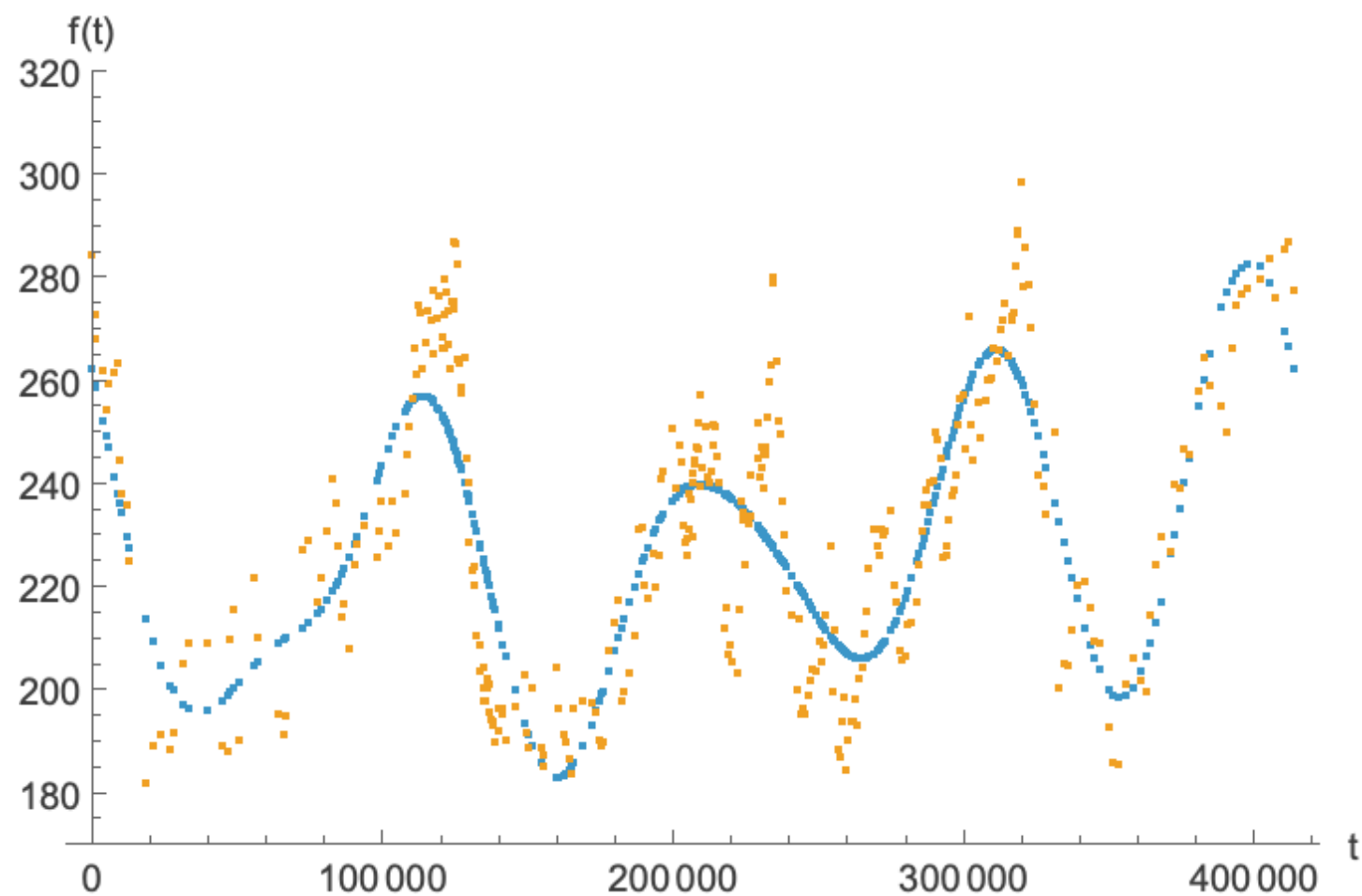
# Approx



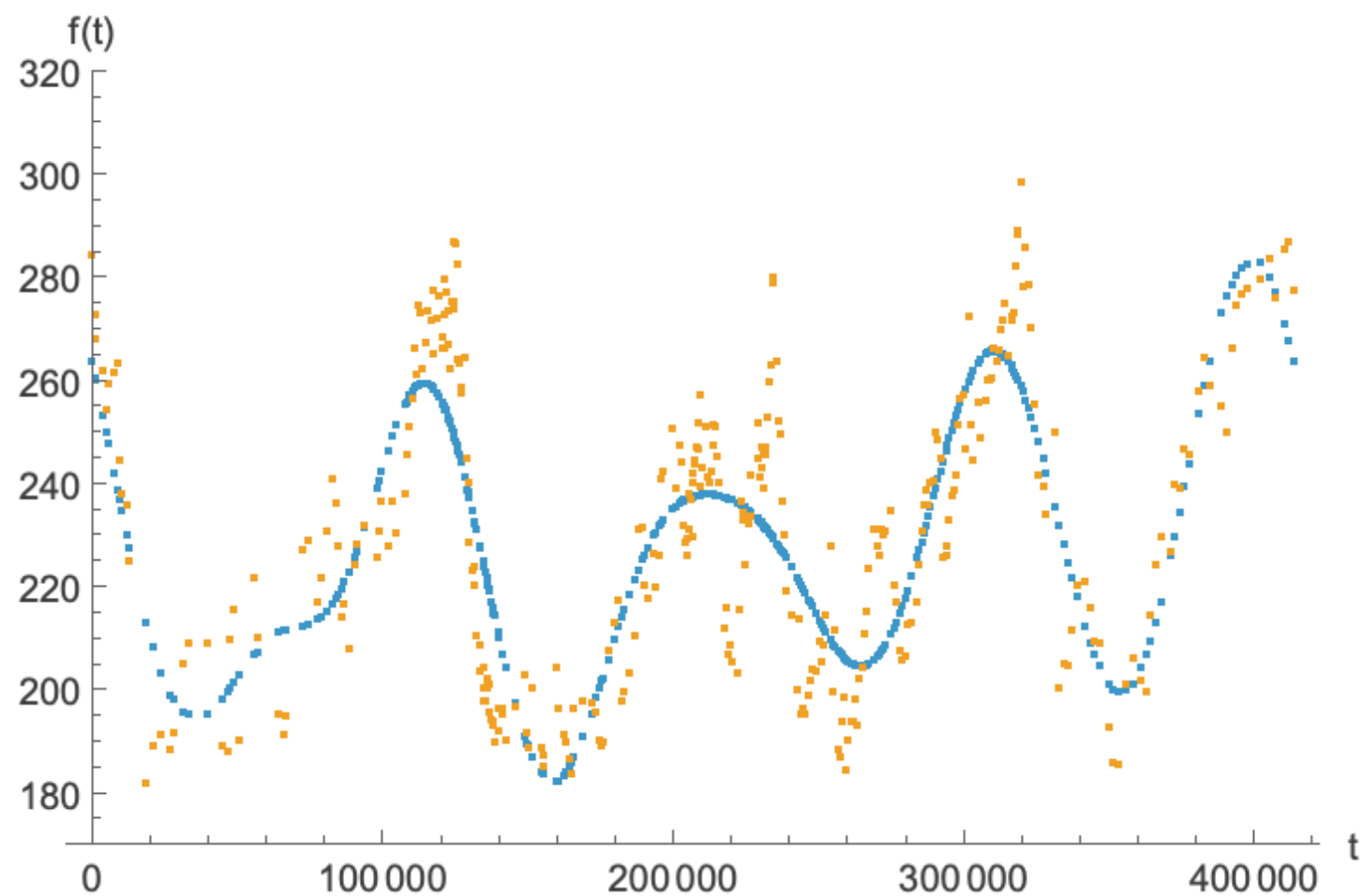
# Approx



# Approx



# Approx



# Questions?

Thank you for your attention. What questions do you have?