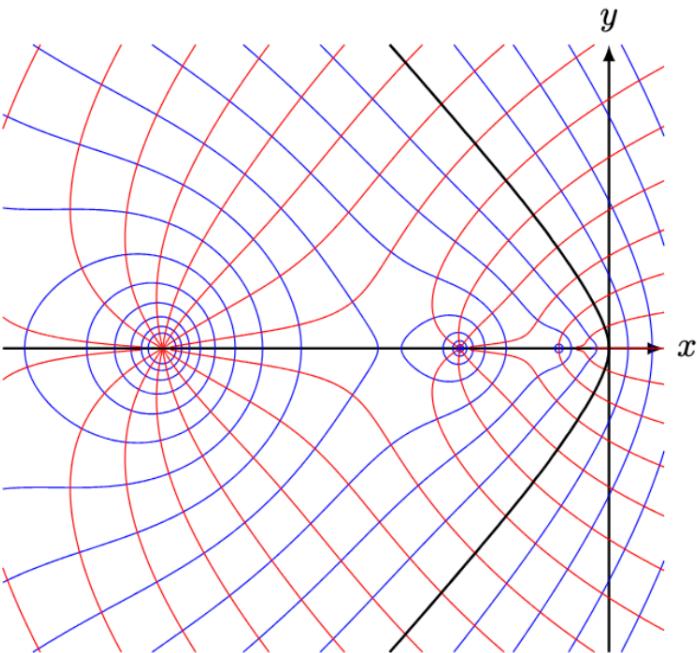


Fourier Series on Discrete Periodic Time Scales

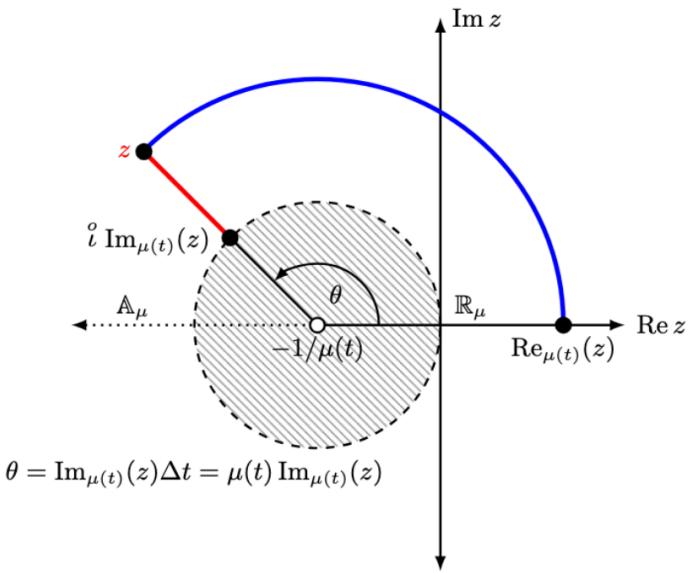
Dylan Poulsen
Washington College

Outline



- Hilger's Complex Plane
- Ergodic Complex Plane
- Review Fourier Series
- Fourier Series on \mathbb{T}
- Antarctic Ice Core Example

Hilger's Local Plane

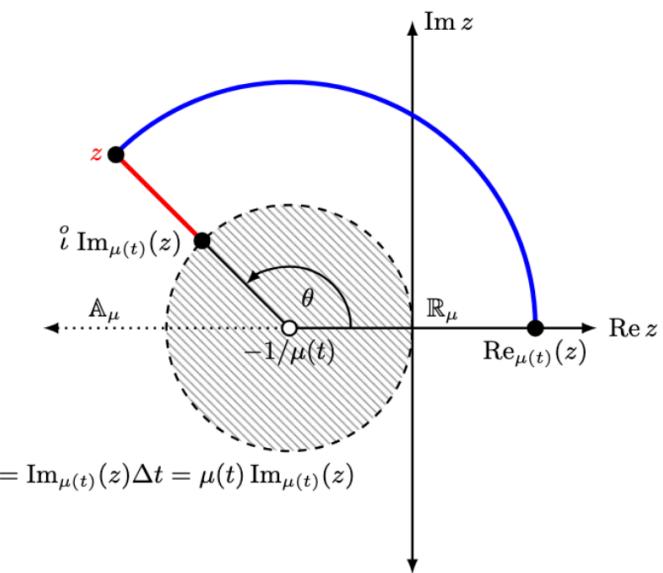


- Local behavior of $e_z(t, t_0)$ at time $t \in \mathbb{T}$ in terms of $z \in \mathbb{C} \setminus \{-1/\mu(t)\} = \mathbb{C}_{\mu(t)}$.
- Central to Hilger's plane is the *cylinder transformation* $\xi_{\mu(t)} : \mathbb{C}_{\mu(t)} \rightarrow \mathbb{Z}_{\mu(t)}$ defined by $\xi_{\mu(t)}(z) = \frac{\text{Log}(1 + \mu(t)z)}{\mu(t)}$,

where

$$\mathbb{Z}_{\mu(t)} := \left\{ z \in \mathbb{C} \mid -\frac{\pi}{\mu(t)} < \text{Im}(z) \leq \frac{\pi}{\mu(t)} \right\}.$$

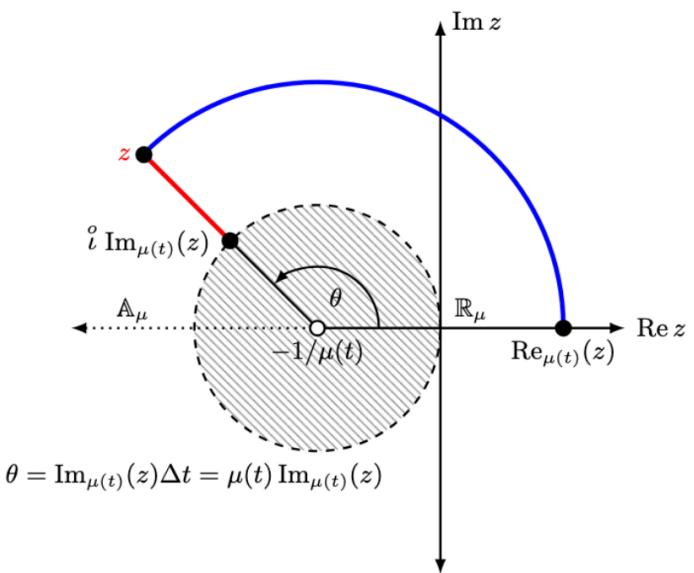
Hilger's Local Plane



$$\theta = \text{Im}_{\mu(t)}(z)\Delta t = \mu(t) \text{Im}_{\mu(t)}(z)$$

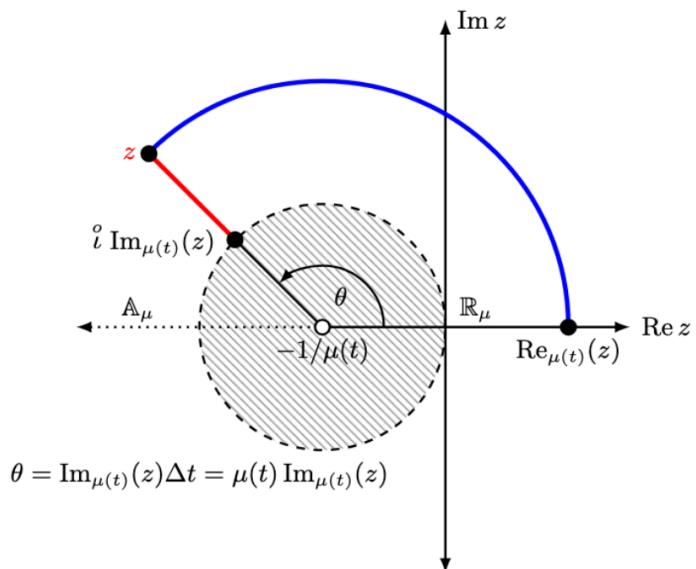
$$\begin{aligned}\text{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z)))\end{aligned}$$

Hilger's Local Plane



$$\begin{aligned}\text{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z))) \\ \text{Im}_{\mu(t)}(z) &:= \frac{\text{Arg}(1 + z\mu(t))}{\mu(t)} \\ &= \text{Im}(\xi_{\mu(t)}(z))\end{aligned}$$

Hilger's Local Plane



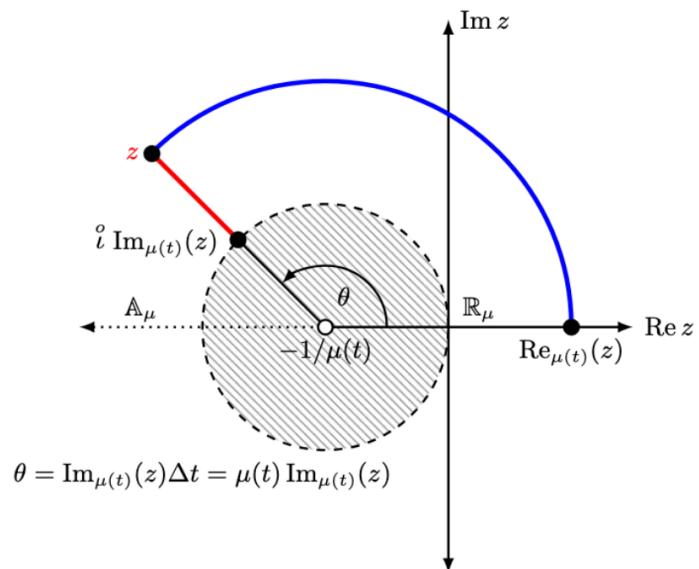
$$\begin{aligned} \text{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(\text{Re}(\xi_{\mu(t)}(z))) \end{aligned}$$

$$\begin{aligned} \text{Im}_{\mu(t)}(z) &:= \frac{\text{Arg}(1 + z\mu(t))}{\mu(t)} \\ &= \text{Im}(\xi_{\mu(t)}(z)) \end{aligned}$$

$$\begin{aligned} {}^o \text{Im}_{\mu(t)}(z) &:= \frac{e^{i\text{Im}_{\mu(t)}(z)\mu(t)} - 1}{\mu(t)} \\ &= \xi_{\mu(t)}^{-1}(i\text{Im}(\xi_{\mu(t)}(z))) \end{aligned}$$

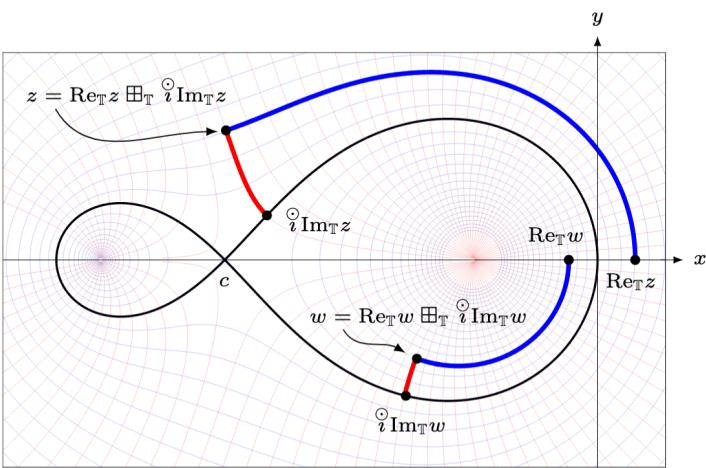
Hilger's Local Plane

- With $a \oplus_{\mu(t)} b := a + b + \mu(t)ab$,
Hilger's local plane is isomorphic to the cylinder strip $\mathbb{Z}_{\mu(t)}$ with addition mod $2\pi i/\mu(t)$.



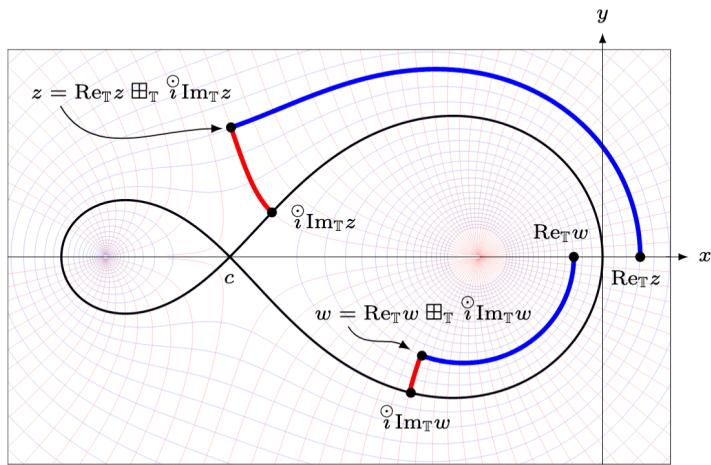
- $z \oplus_{\mu(t)} w = \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(z) + \xi_{\mu(t)}(w))$
- Hilger's decomposition:
$$z = \text{Re}_{\mu(t)}(z) \oplus_{\mu(t)} \overset{o}{\text{Im}}_{\mu(t)}(z)$$

Ergodic Global Plane



- Global behavior of $e_z(t, t_0)$ in terms of $z \in \mathbb{C} \setminus \{-1/\mu(t) \mid t \in \mathbb{T}\} := \mathbb{C}_{\mu(\mathbb{T})}$.
- On a periodic (or finite) time scale of length L , defined by the *average* of the cylinder transformation
$$\bar{\xi}(z) = \frac{1}{L} \int_0^L \xi_{\mu(\tau)}(z) \Delta \tau$$
 (at least off the branch cut).

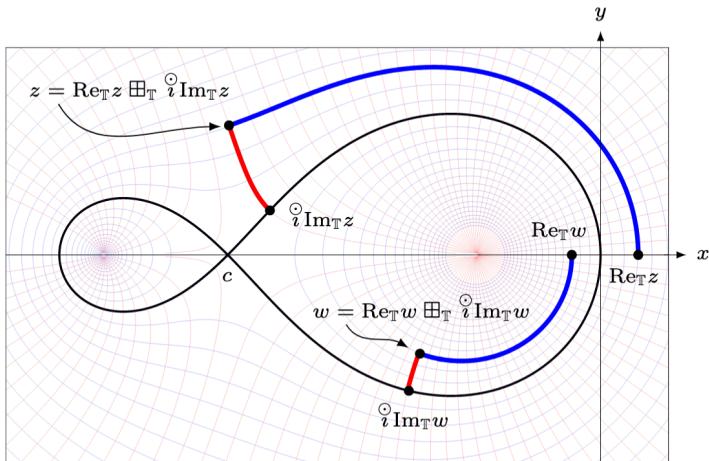
Ergodic Global Plane



- The generalization of the cylinder strip is the set $\mathbb{C}^\Omega = \{z \in \mathbb{C} \mid -\Omega < \text{Im}(z) \leq \Omega\}$, where Ω is the Nyquist frequency.
- We have carefully defined $\bar{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \rightarrow \mathbb{C}^\Omega$ on the branch cut and shown that the resulting map is globally univalent.
- $z \boxplus_{\mathbb{T}} w = \bar{\xi}^{-1}(\bar{\xi}(z) + \bar{\xi}(w)(\text{mod } 2i\Omega))$

Ergodic Global Plane

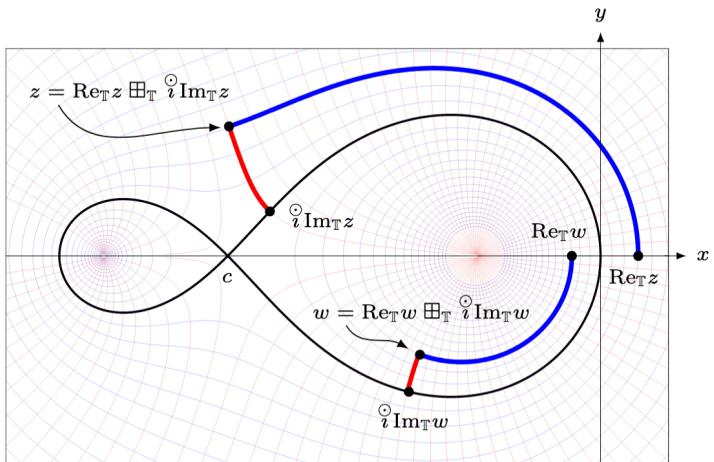
$$\text{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\text{Re}(\overline{\xi}(z)))$$



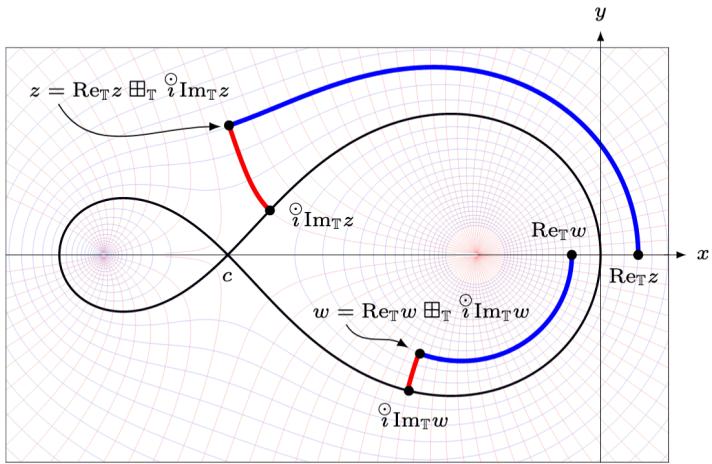
Ergodic Global Plane

$$\text{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\text{Re}(\overline{\xi}(z)))$$

$$\text{Im}_{\mathbb{T}}(z) = \text{Im}(\overline{\xi}(z))$$



Ergodic Global Plane

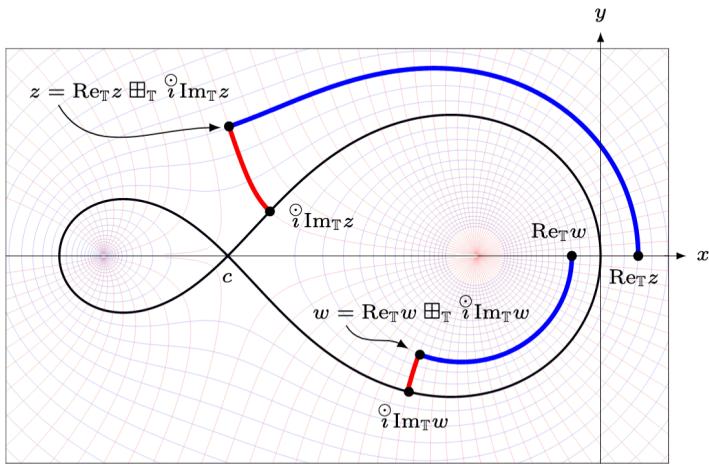


$$\text{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\text{Re}(\overline{\xi}(z)))$$

$$\text{Im}_{\mathbb{T}}(z) = \text{Im}(\overline{\xi}(z))$$

$$\overset{\circ}{i}\text{Im}_{\mathbb{T}}(z) := \overline{\xi}^{-1}(i\text{Im}(\overline{\xi}(z)))$$

Ergodic Global Plane



$$\text{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\text{Re}(\overline{\xi}(z)))$$

$$\text{Im}_{\mathbb{T}}(z) = \text{Im}(\overline{\xi}(z))$$

$$\overset{\circ}{i}\text{Im}_{\mathbb{T}}(z) := \overline{\xi}^{-1}(i\text{Im}(\overline{\xi}(z)))$$

$$z = \text{Re}_{\mathbb{T}}(z) \boxplus_{\mathbb{T}} \overset{\circ}{i}\text{Im}_{\mathbb{T}}(z)$$

Fourier Series on \mathbb{R}

On \mathbb{R} , the Fourier series for an L -periodic function $f : \mathbb{R} \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi n}{L} t} \\ &= \sum_{n=-\infty}^{\infty} c_n e_{\xi_0^{-1}}(i \frac{2\pi n}{L})(t, 0), \text{ with} \\ c_n &= \frac{1}{L} \int_0^L f(t) e^{-i \frac{2\pi n}{L} t} dt. \end{aligned}$$

Fourier Series on \mathbb{Z}

On \mathbb{Z} , the discrete Fourier series for an L -periodic function $f : \mathbb{Z} \rightarrow \mathbb{C}$ is given by

$$\begin{aligned} f(t) &= \sum_{n=-\lfloor(L-1)/2\rfloor}^{\lceil(L-1)/2\rceil} c_n e^{i\frac{2\pi n}{L}t} \\ &= \sum_{n=-\lfloor(L-1)/2\rfloor}^{\lceil(L-1)/2\rceil} c_n e_{\xi_1^{-1}(i\frac{2\pi n}{L})}(t, 0), \text{ with} \\ c_n &= \frac{1}{L} \sum_{t=0}^{L-1} f(t) e^{-i\frac{2\pi n}{L}t}. \end{aligned}$$

Fourier Series on \mathbb{T}

Key Insight: Basis functions should be exponentials with zero growth rate and a harmonic frequency. That is, they should be time scale exponentials with subscript

$$\bar{\xi}^{-1}(i\omega_n) = \overset{\circ}{i}\omega_n,$$

where ω_n is a harmonic frequency.

Fourier Series on \mathbb{T}

- To calculate the values of the subscripts, we could sample the Nyquist interval uniformly and numerically calculate $\bar{\xi}^{-1}$ at each of these sample points. There is an easier way, however - eigenvalues!
- Let
$$\mathbb{T} = \{t_0, t_1, t_2, \dots, t_{m-1}, t_m, t_1 + L, t_2 + L, \dots, t_0 + 2L, t_1 + 2L, \dots\}$$
be a periodic time scale of length L .
- We can write the dynamic equation on \mathbb{T} , $x^\Delta = \lambda x$, with a periodic condition $x(t_0) = x(t_m)$ as a system of equations.

Fourier Series on \mathbb{T}

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \quad 0 \leq k \leq m-2$$

$$\frac{x(t_m) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

Fourier Series on \mathbb{T}

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \quad 0 \leq k \leq m-2$$

$$\frac{x(t_0) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

Fourier Series on \mathbb{T}

$$\begin{pmatrix} -1/\mu(t_0) & 1/\mu(t_0) & 0 & \cdots & 0 & 0 \\ 0 & -1/\mu(t_1) & 1/\mu(t_1) & \cdots & 0 & 0 \\ 0 & 0 & -1/\mu(t_2) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1/\mu(t_{m-3}) & 0 \\ 0 & 0 & \cdots & 0 & -1/\mu(t_{m-2}) & 1/\mu(t_{m-2}) \\ 1/\mu(t_{m-1}) & 0 & \cdots & 0 & 0 & -1/\mu(t_{m-1}) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix} = \lambda \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix}$$

The eigenvalues are $\overset{\odot}{i}\omega_k$, $0 \leq k \leq m - 1$.

The associated eigenvectors are $e_{\overset{\odot}{i}\omega_k}(t, 0)$ evaluated at each point in one period of the time scale.

Fourier Series on \mathbb{T}

Now, we calculate the Fourier coefficients in

$$f(t) = \sum_{n=0}^{m-1} c_n e_{\odot_{i\omega_n}}(t, 0).$$

Theorem

Suppose \mathbb{T} is a periodic time scale with $t_0 = 0$. Suppose also that $i\omega_n$ exists for each n , $0 \leq n \leq m - 1$, and are distinct. Then, the sets $\{e_{\ominus i\omega_n}(t, 0)\}$ and $\{e_{\ominus i\omega_n}(\sigma(t), 0)\}$ form a biorthogonal system, that is

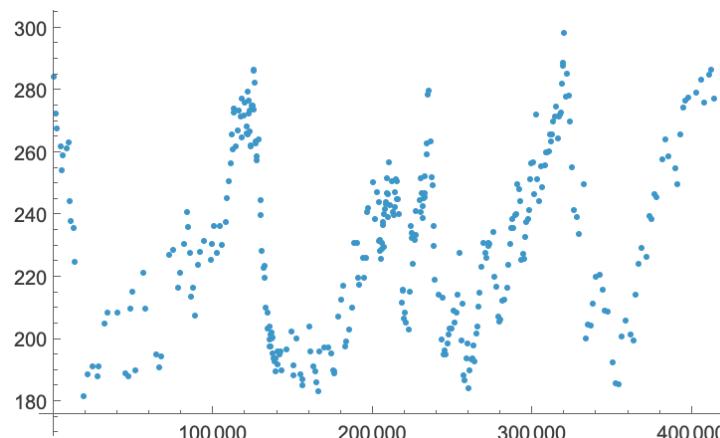
$$\int_0^L e_{\ominus i\omega_n}(t, 0) e_{\ominus i\omega_l}(\sigma(t), 0) \Delta t = K_n \delta_{n,l}.$$

Fourier Series on \mathbb{T}

Therefore,

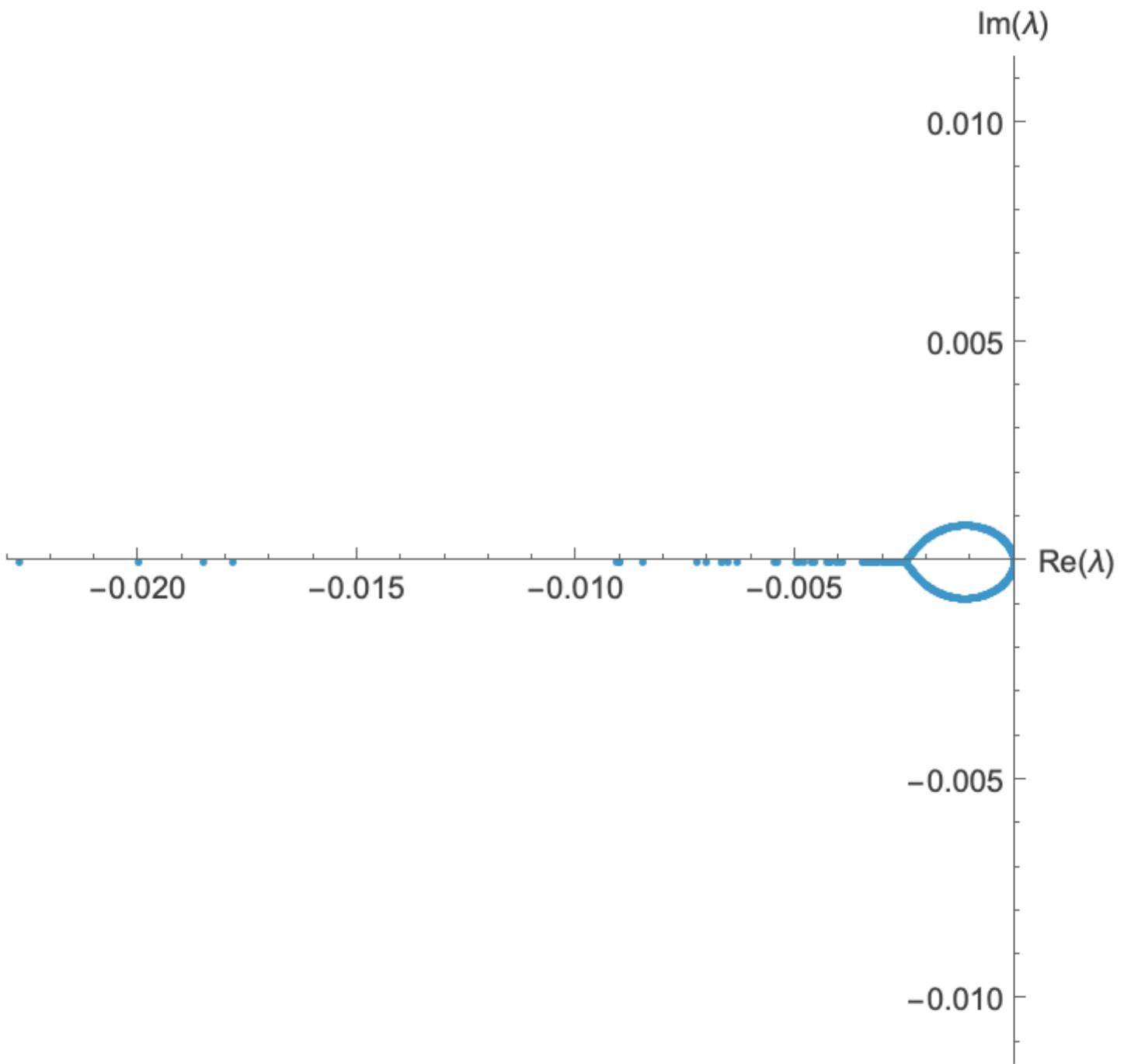
$$c_n = \frac{\int_0^L f(t) \cdot e_{\ominus i\omega_n}(\sigma(t), 0) \Delta t}{\int_0^L e_{i\omega_n}(t, 0) \cdot e_{\ominus i\omega_n}(\sigma(t), 0) \Delta t}.$$

Antarctic Ice Core Data

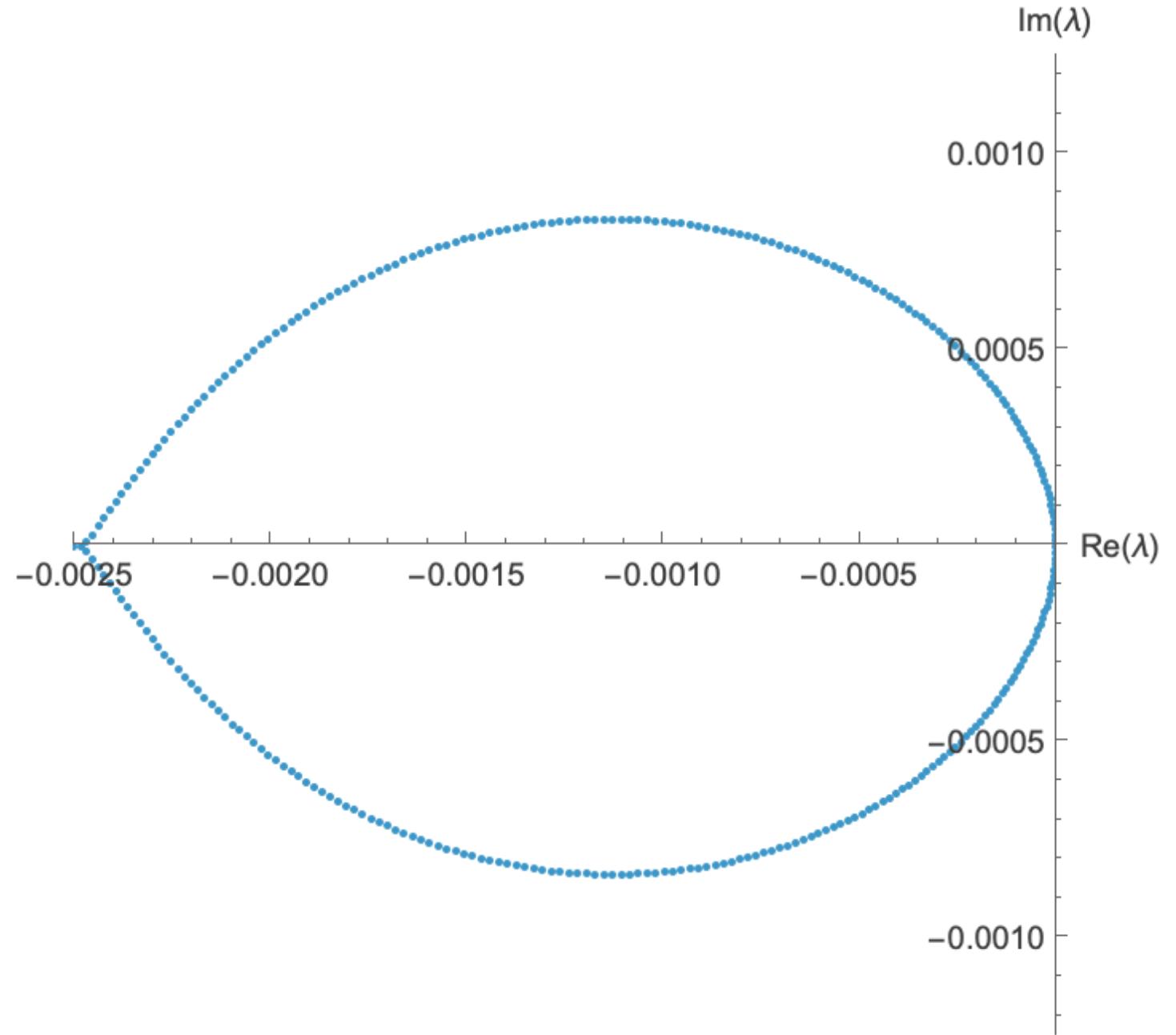


- Record of 400,000+ years of atmospheric CO₂ concentration
- Irregularly sampled in time because the ice core is sampled uniformly along the length of the ice core.

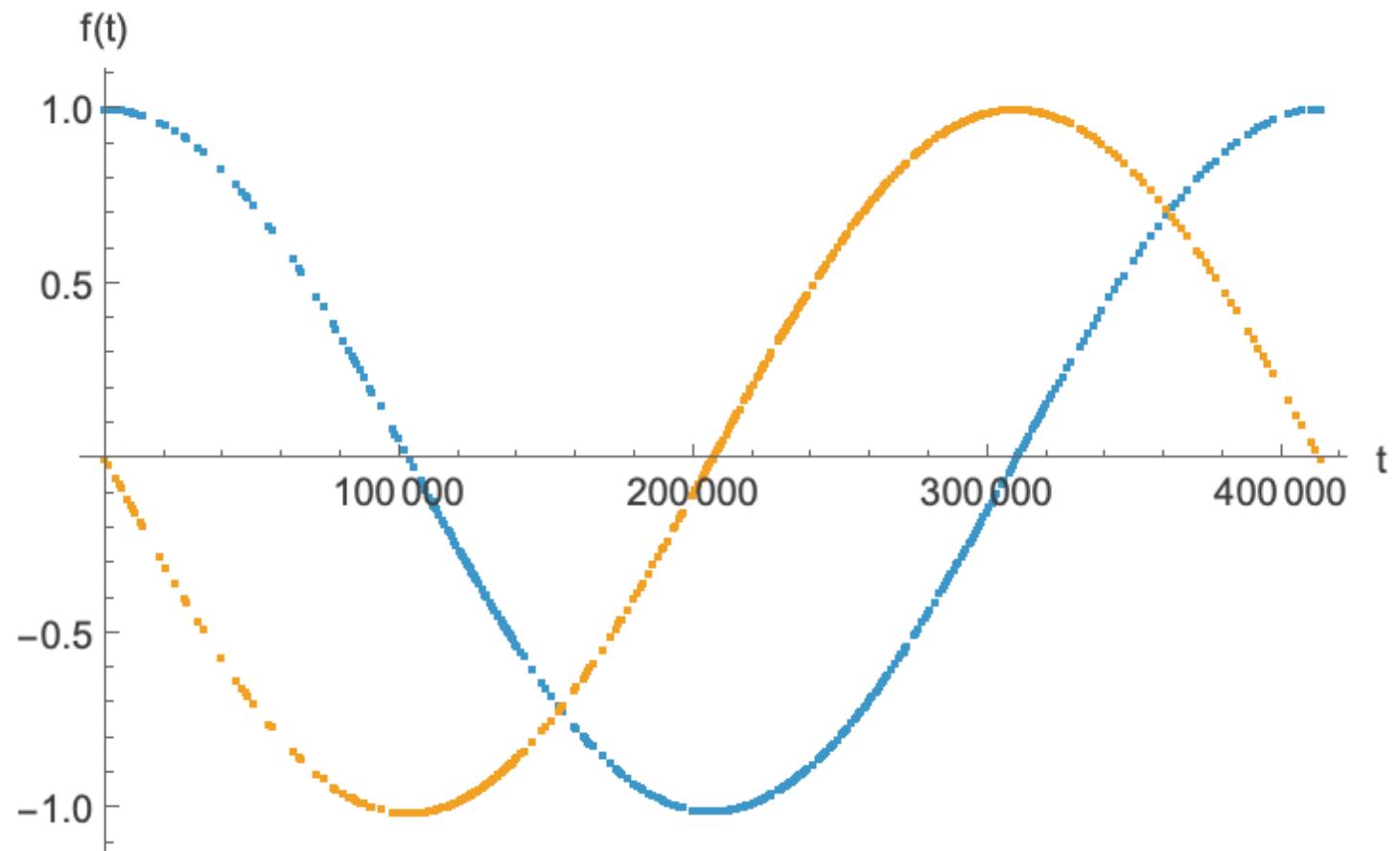
Spectrum



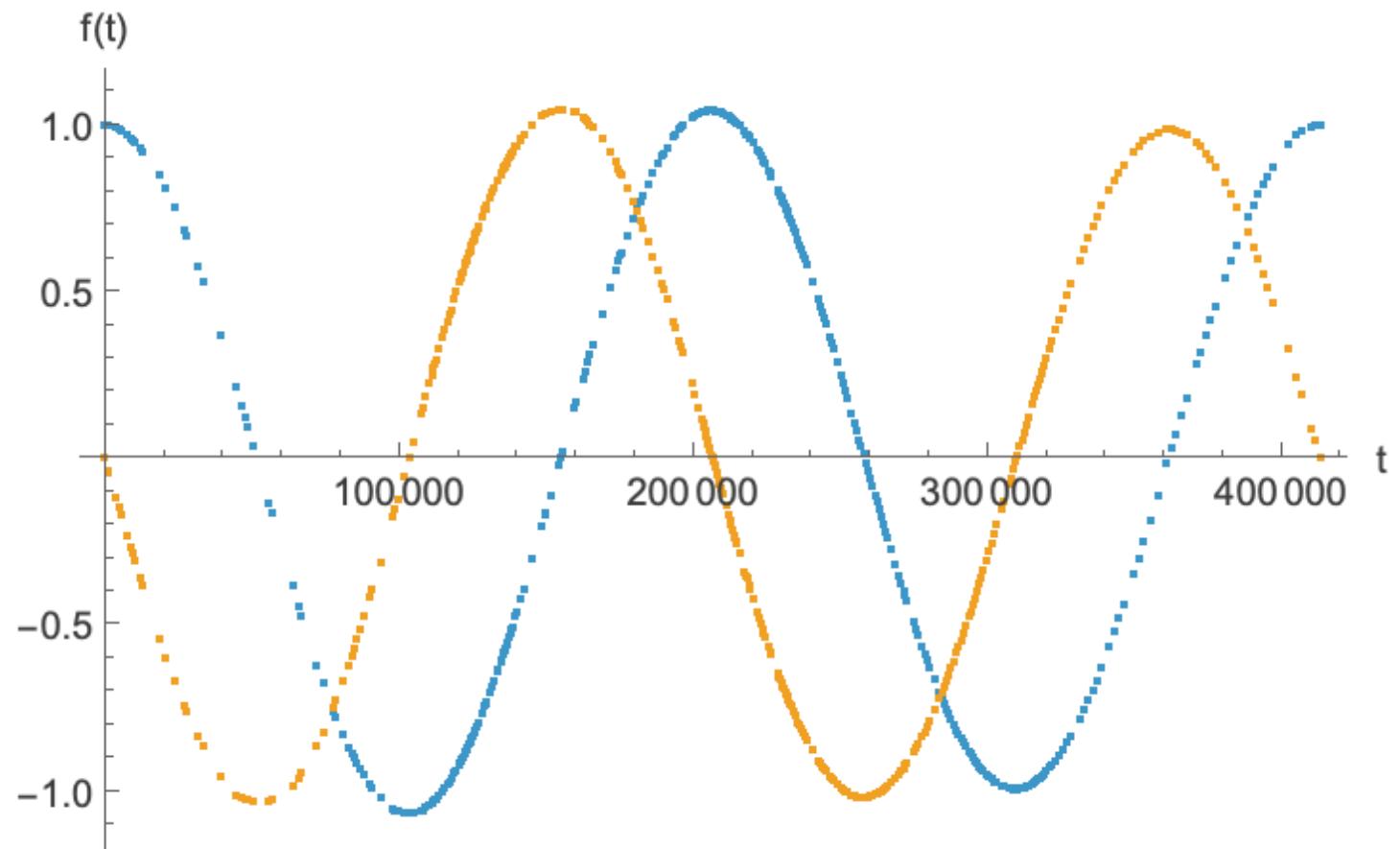
Spectrum



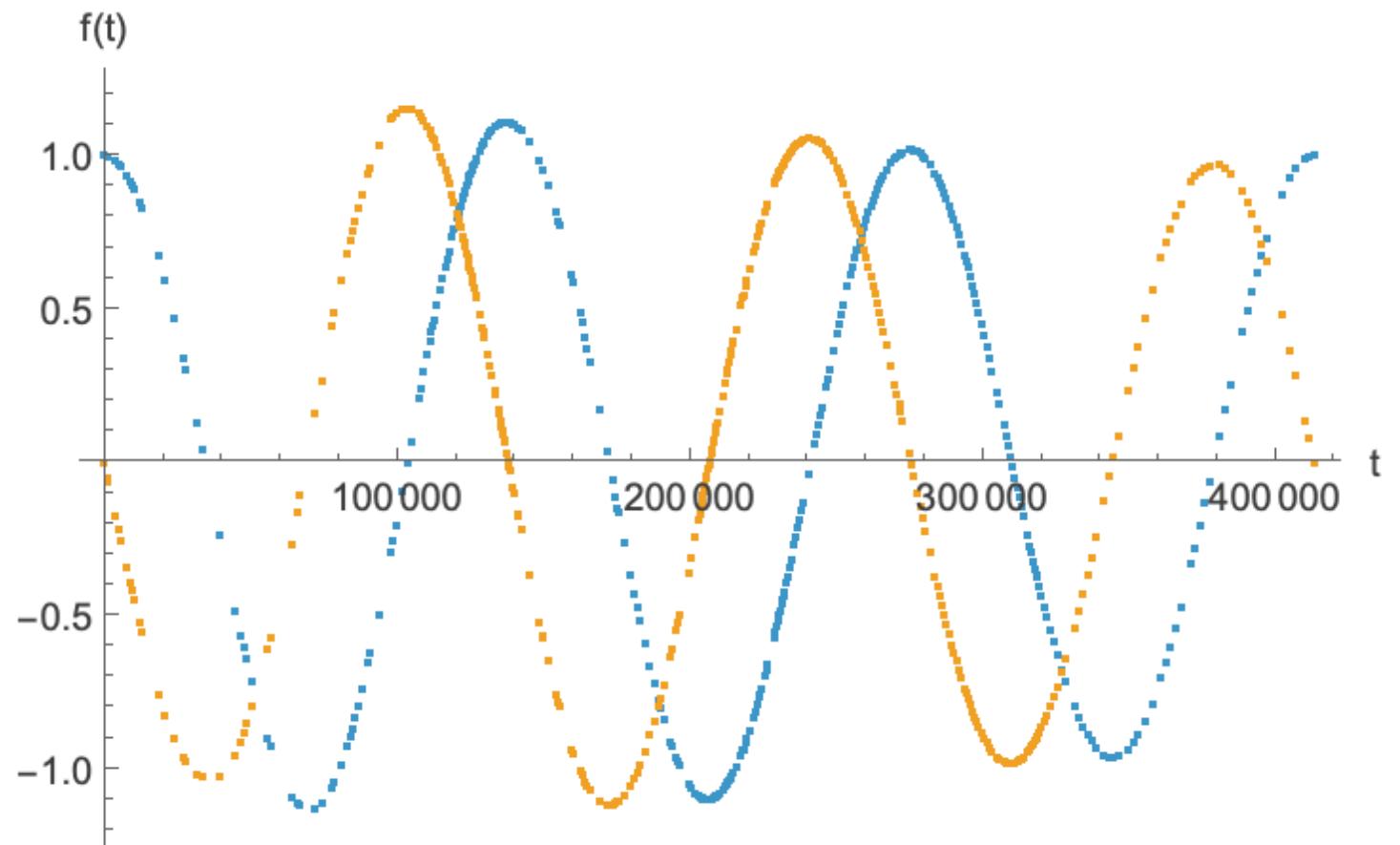
Bases



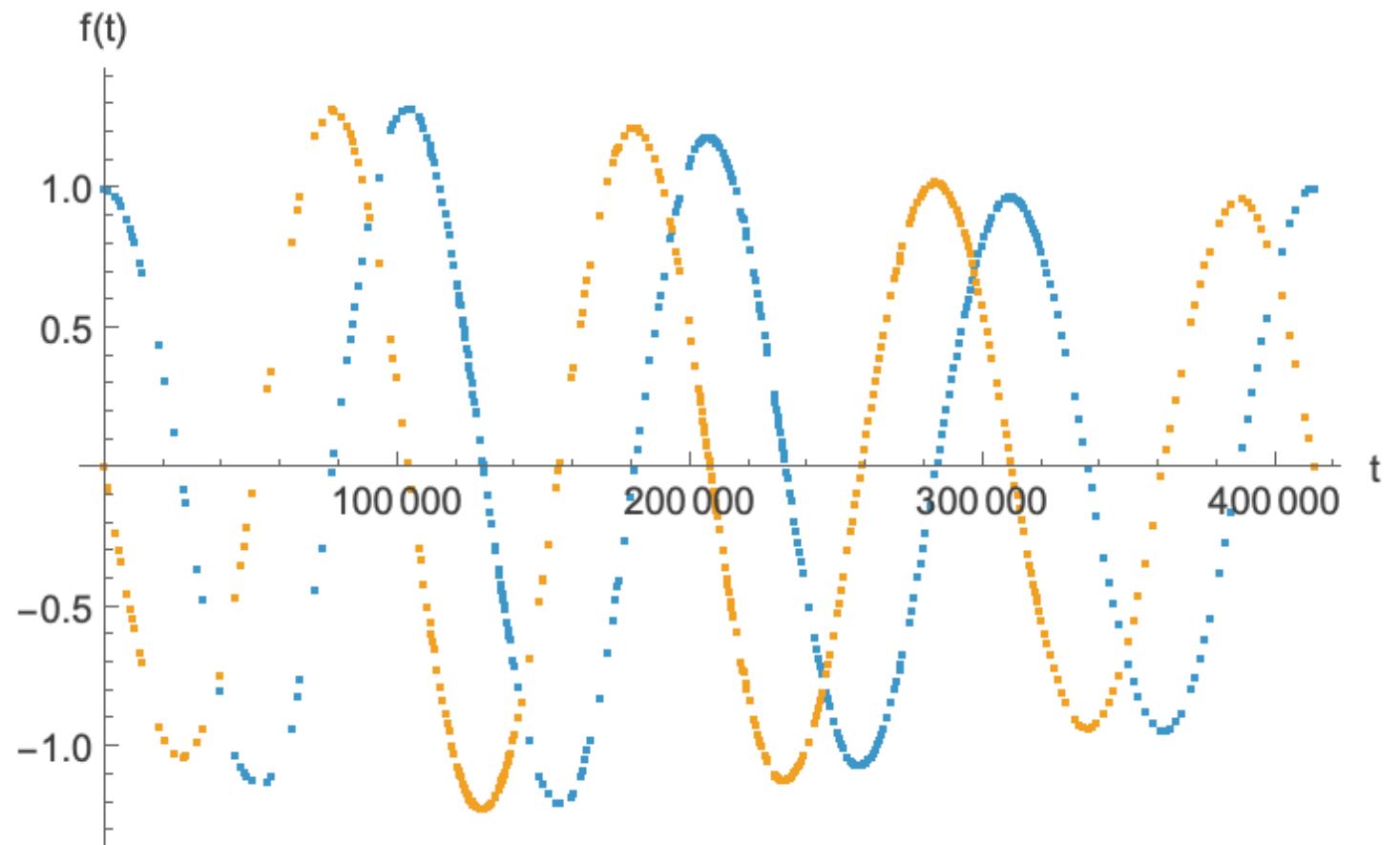
Bases



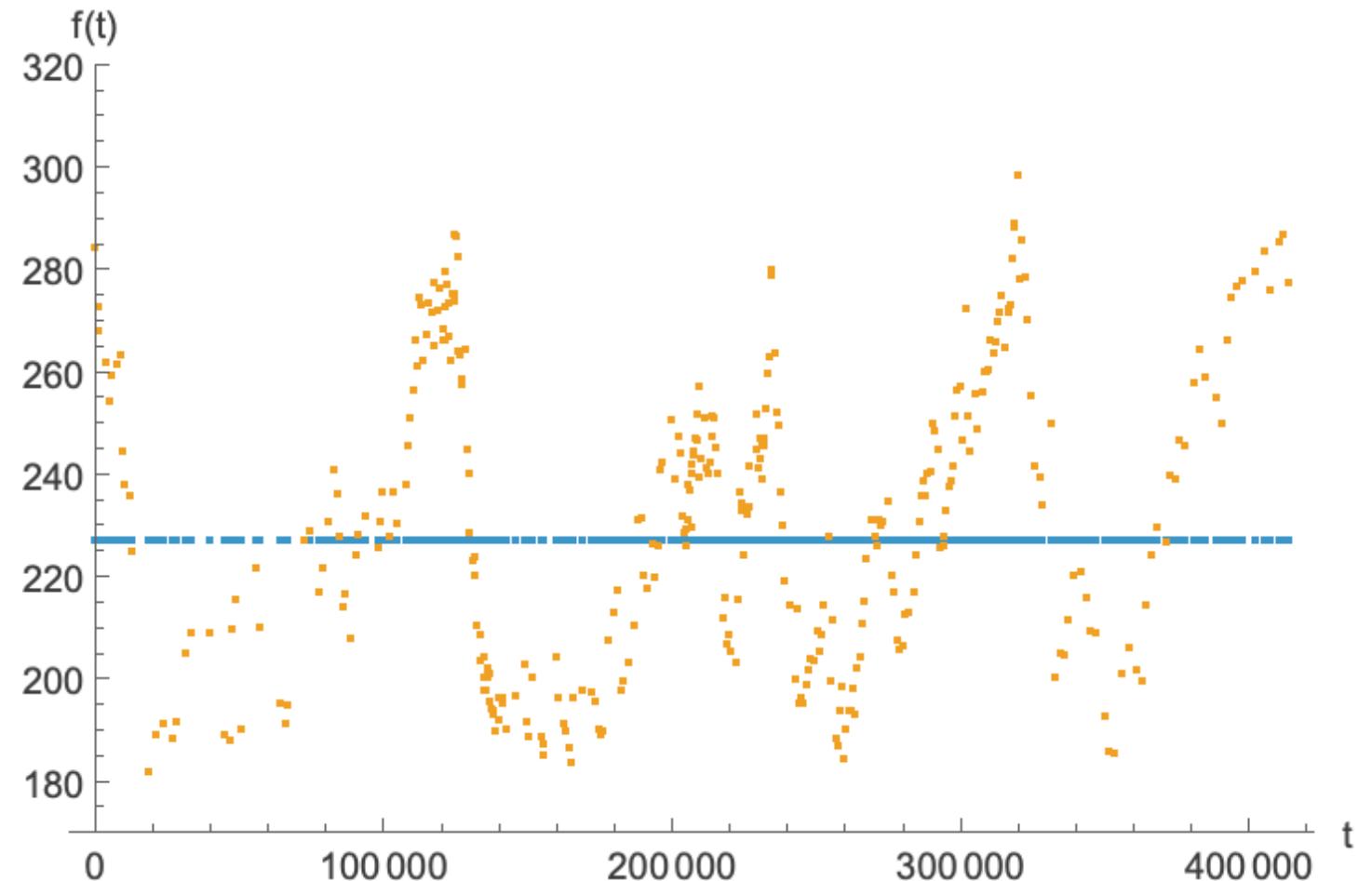
Bases



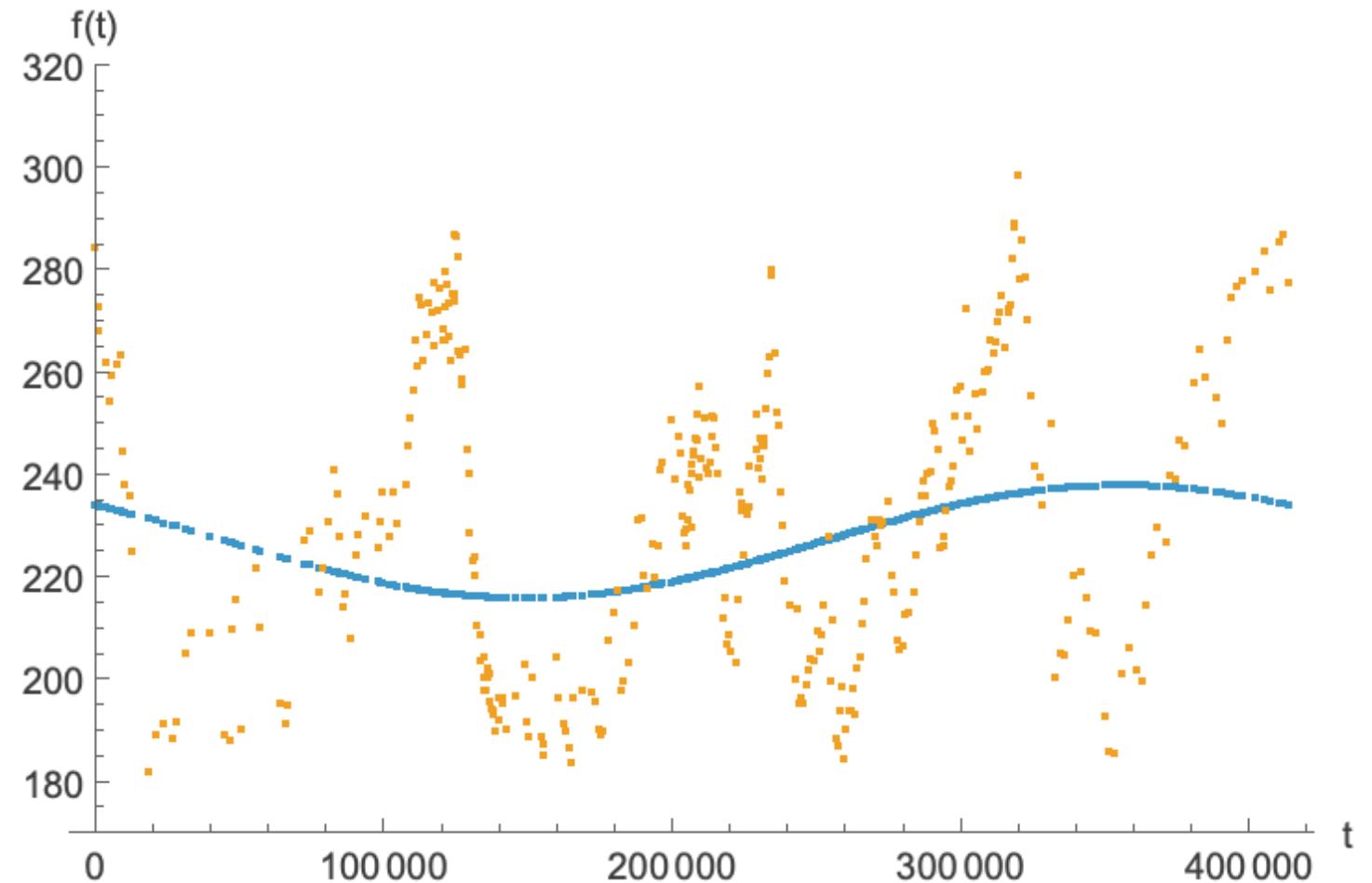
Bases



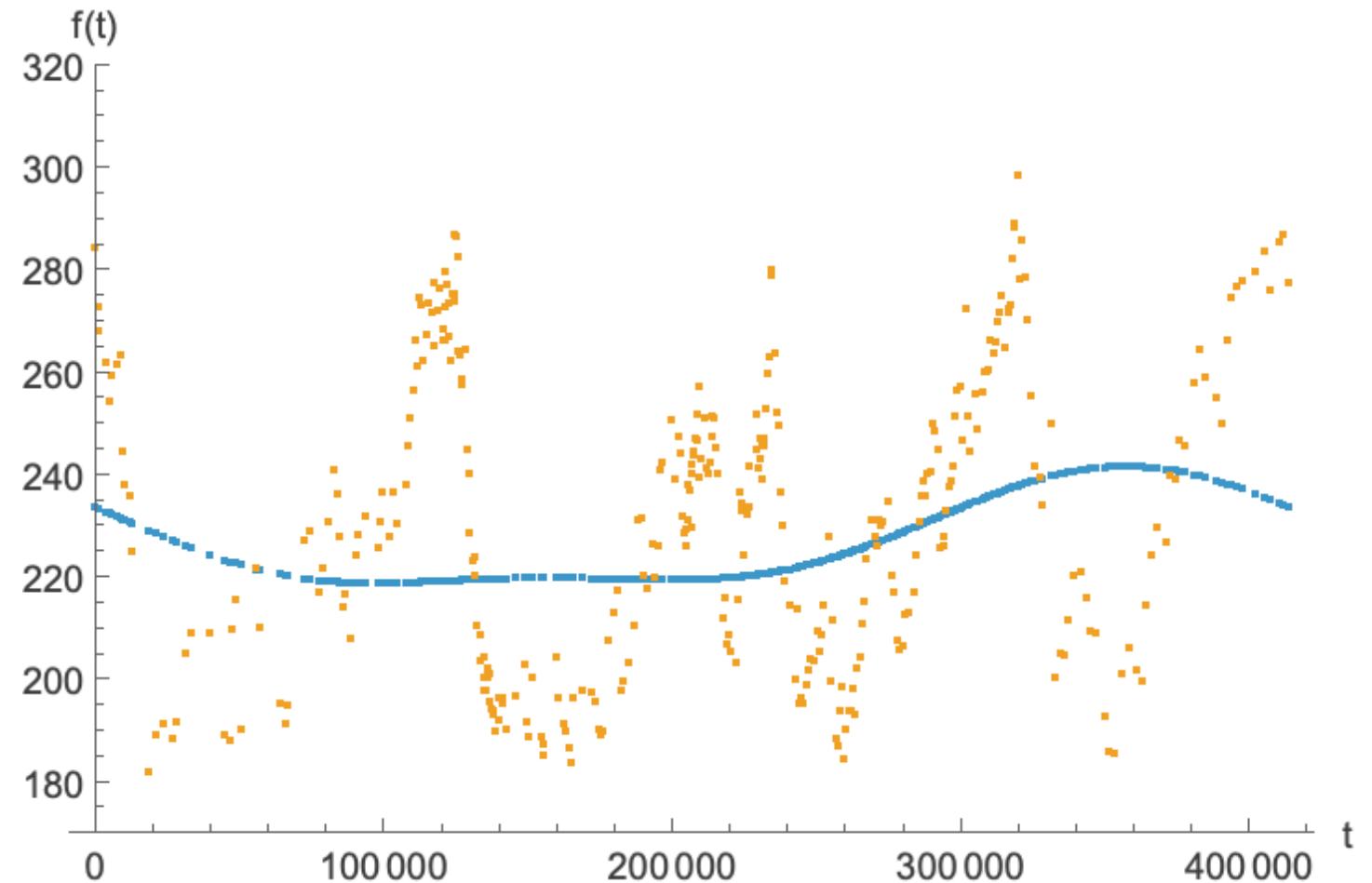
Approx



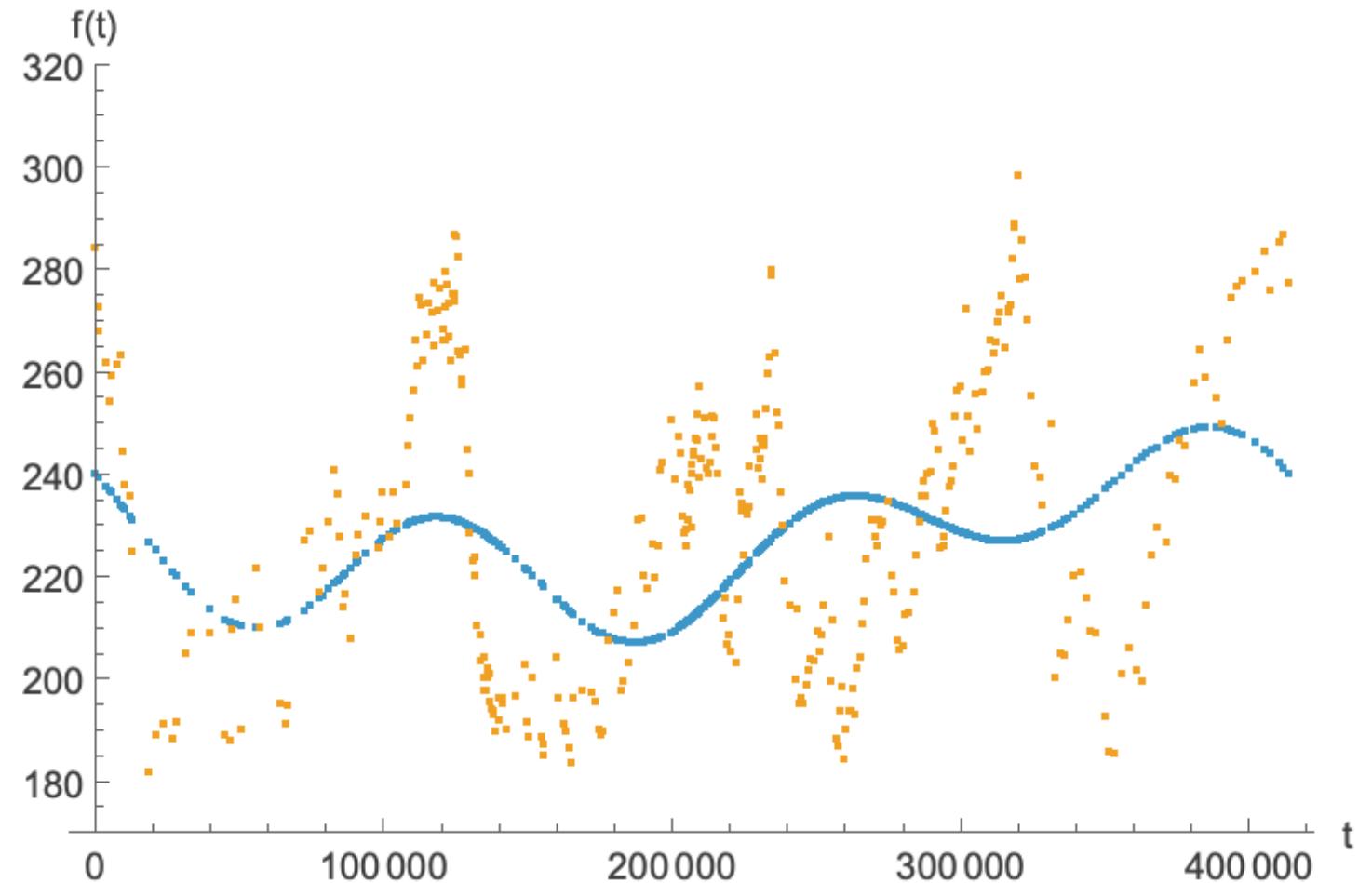
Approx



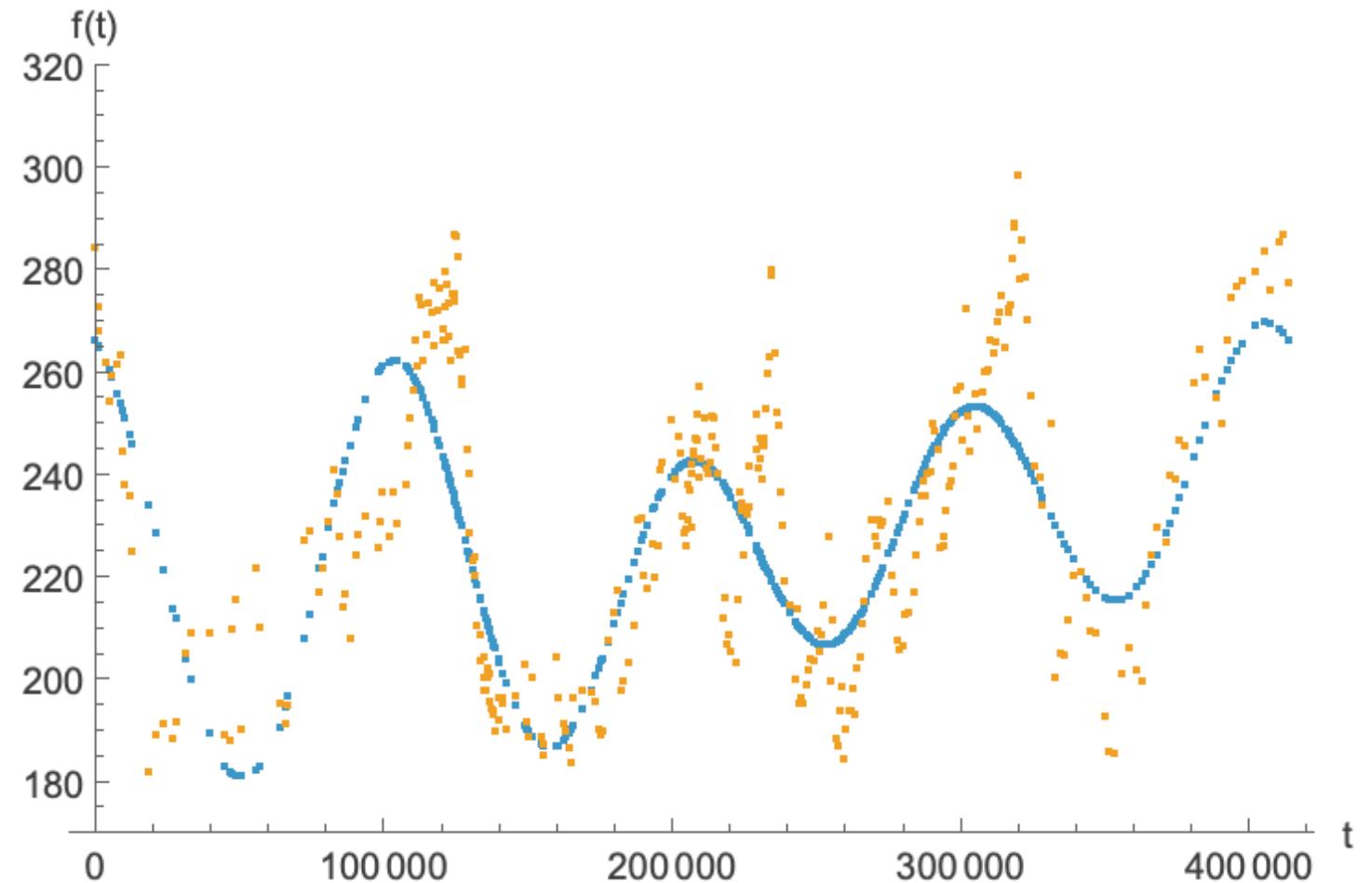
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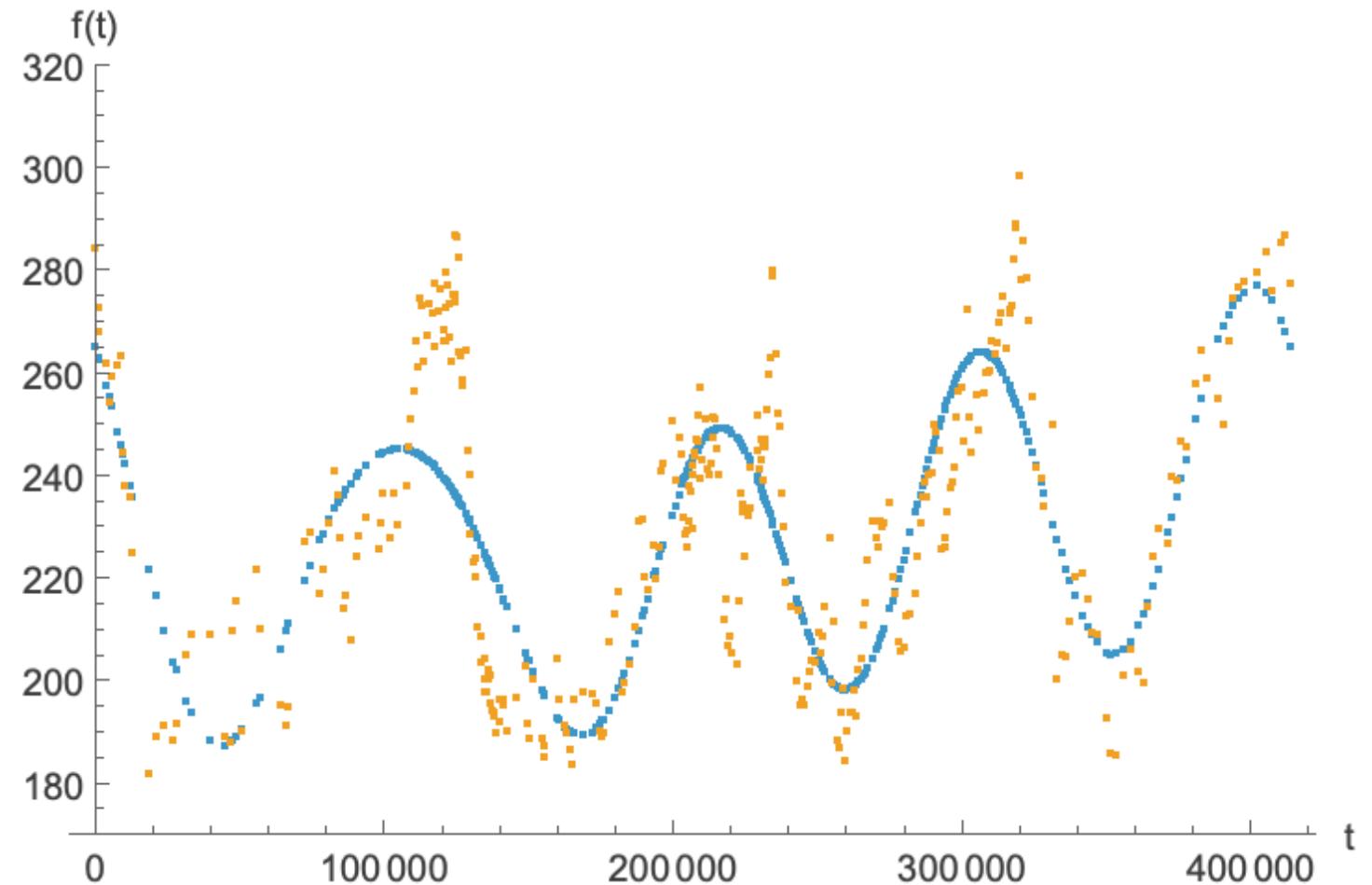
Approx



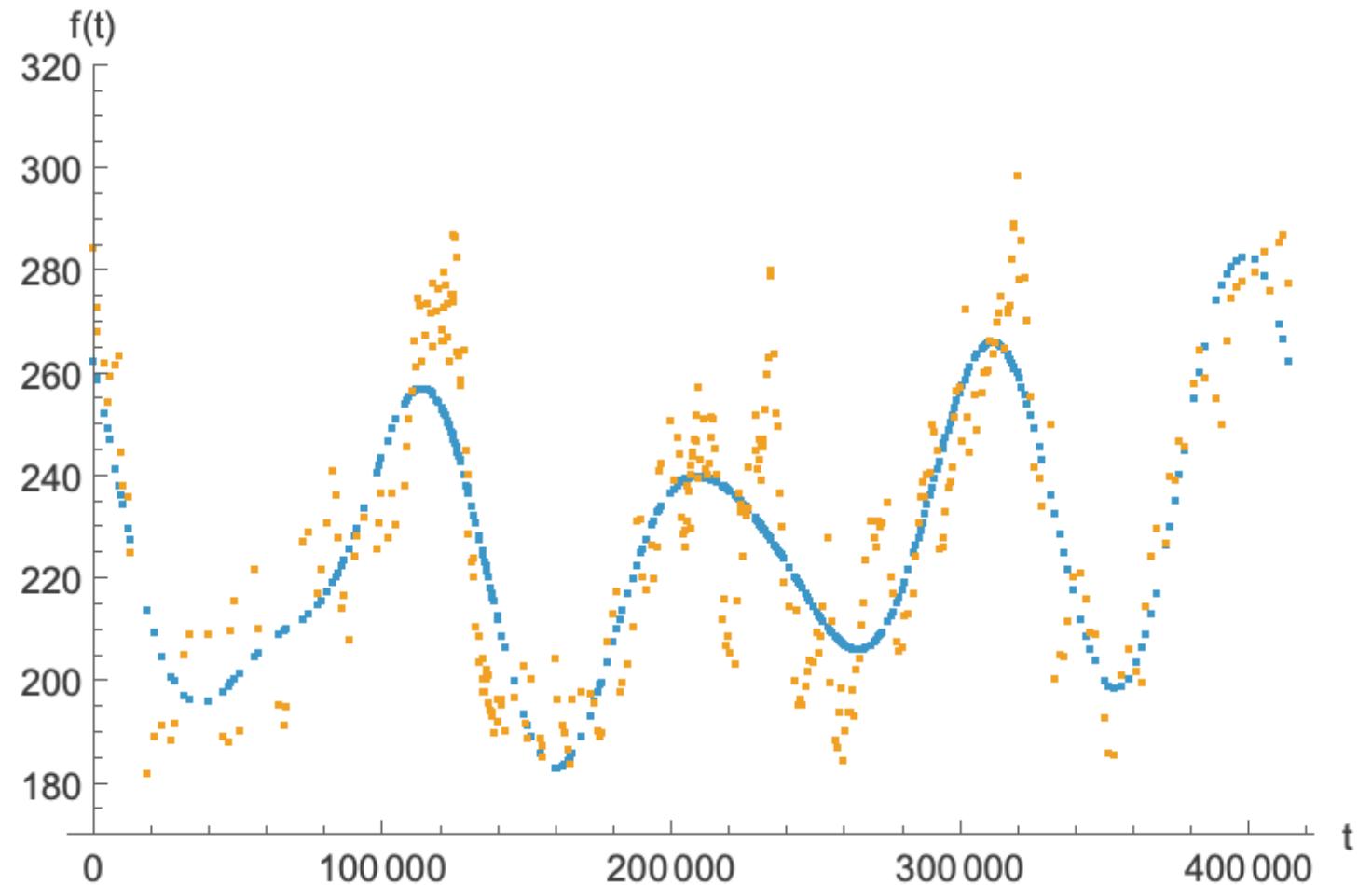
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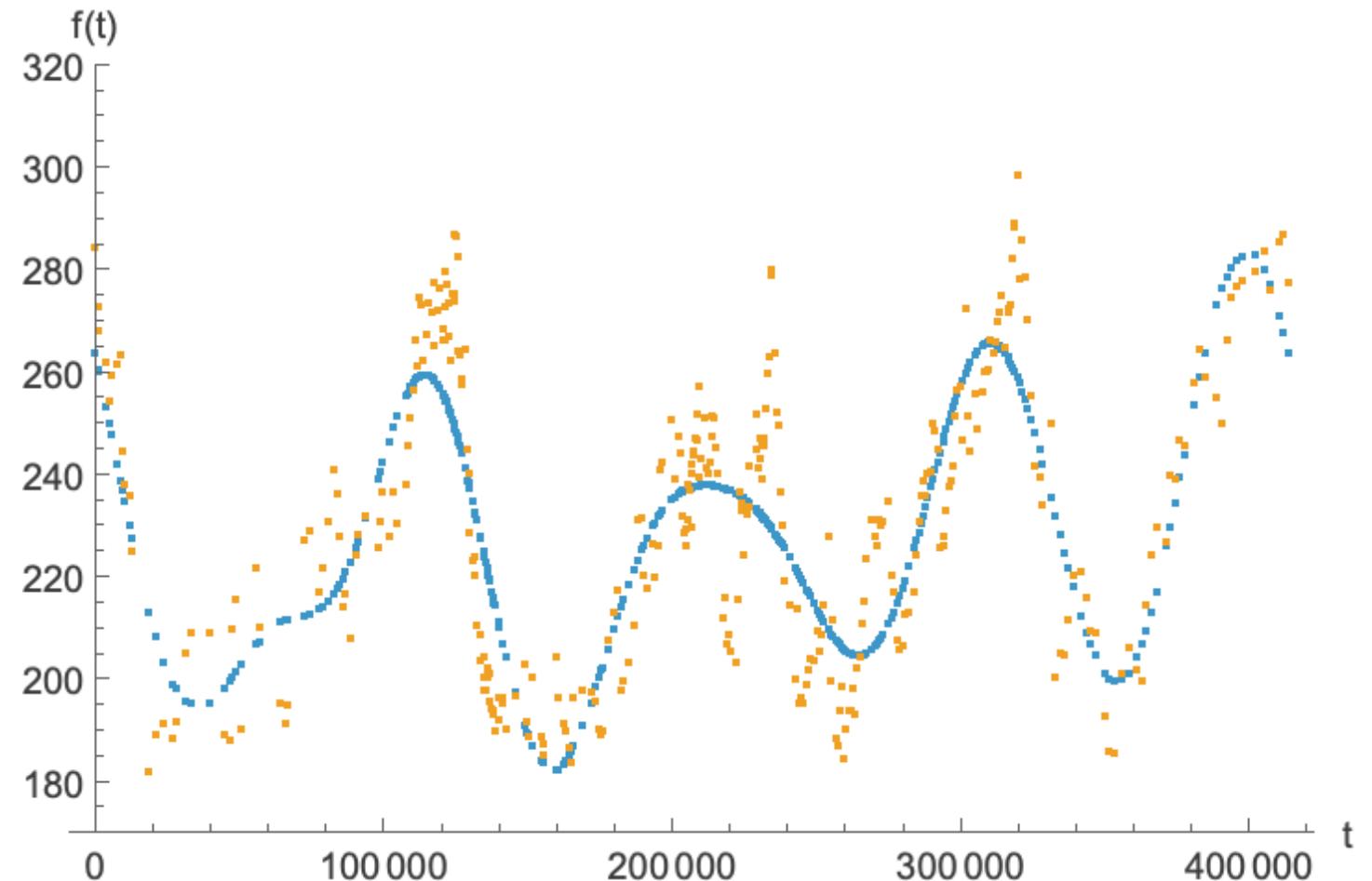
Approx



Approx



Approx



Questions?

Thank you for your attention. What questions do you have?