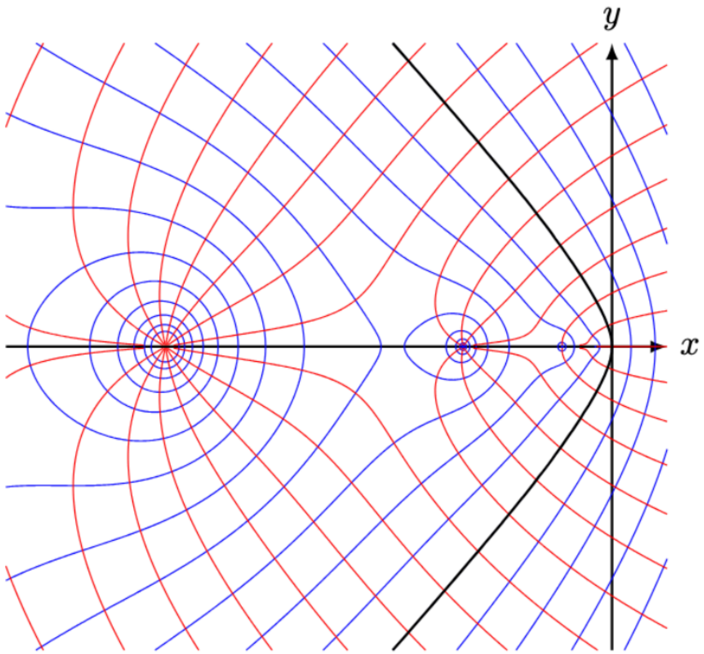
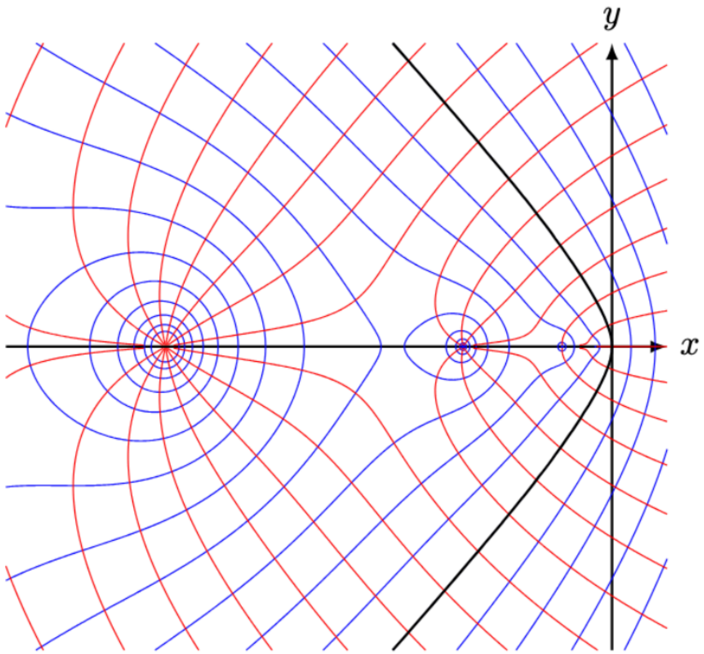


Fourier Series on Discrete Periodic Time Scales

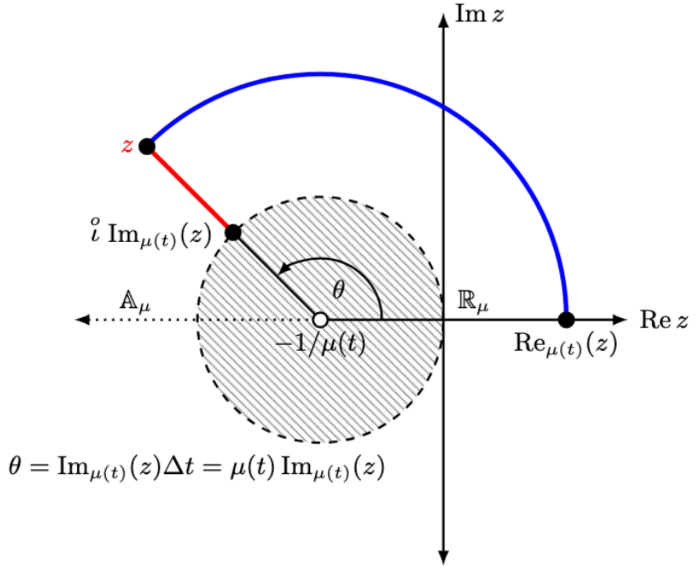
Dylan Poulsen
Washington College

Outline



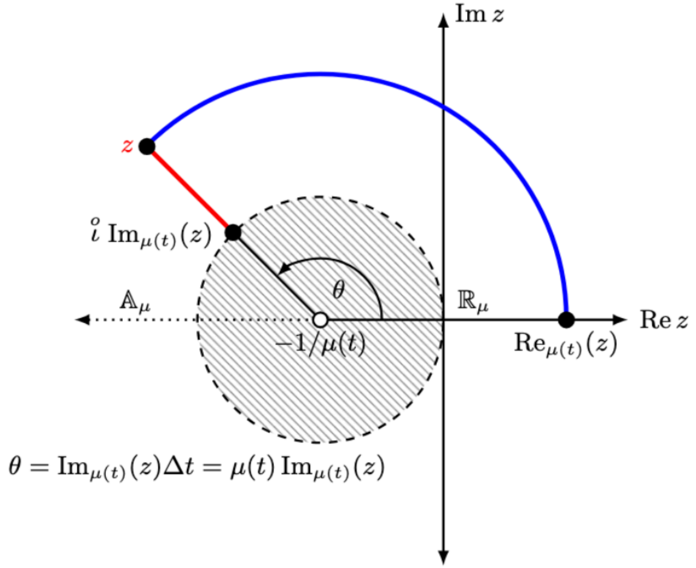
- Hilger's Complex Plane
- Ergodic Complex Plane
- Review Fourier Series
- Fourier Series on $\mathbb{T}$
- Antarctic Ice Core Example

Hilger's Local Plane



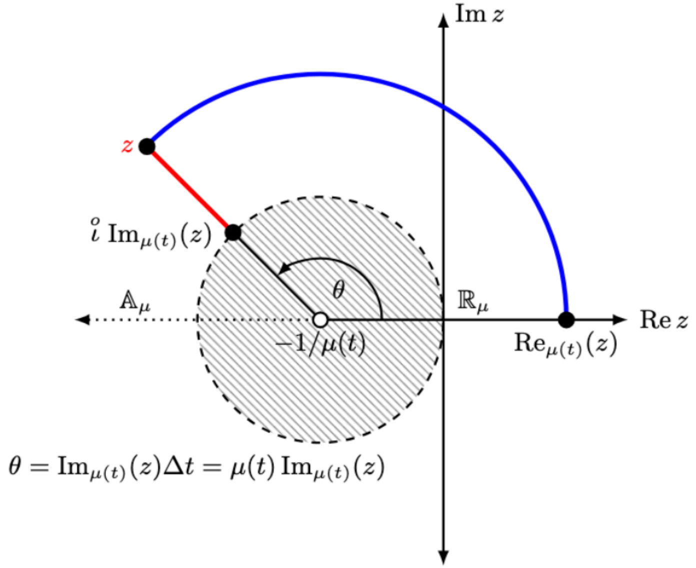
- Local behavior of $e_z(t, t_0)$ at time $t \in \mathbb{T}$ in terms of $z \in \mathbb{C} \setminus \{-1/\mu(t)\} = \mathbb{C}_{\mu(t)}$.
- Central to Hilger's plane is the *cylinder transformation* $\xi_{\mu(t)} : \mathbb{C}_{\mu(t)} \to \mathbb{Z}_{\mu(t)}$ defined by $\xi_{\mu(t)}(z) = \dfrac{\operatorname{Log}(1+\mu(t)z)}{\mu(t)}$, where $\mathbb{Z}_{\mu(t)} := \left\{z \in \mathbb{C} \mid -\frac{\pi}{\mu(t)} < \operatorname{Im}(z) \leq \frac{\pi}{\mu(t)}\right\}$.

Hilger's Local Plane



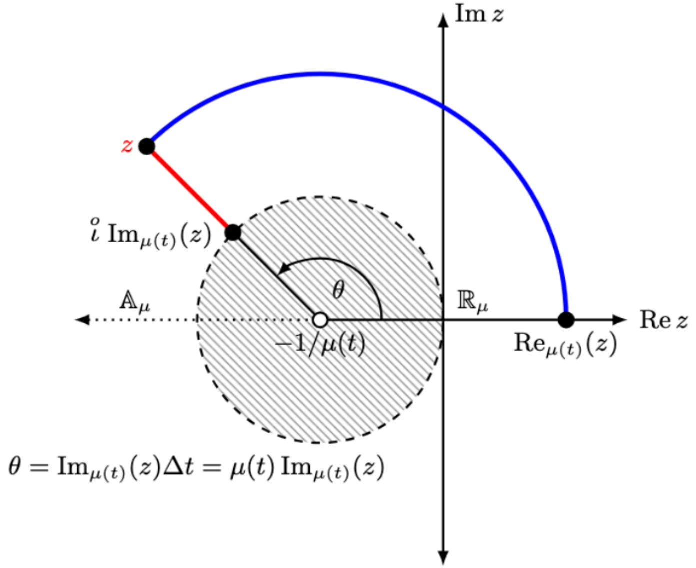
$$\operatorname{Re}_{\mu(t)}(z) := \frac{|1 + z\mu(t)| - 1}{\mu(t)}$$
$$= \xi^{-1}_{\mu(t)}(\operatorname{Re}(\xi_{\mu(t)}(z)))$$

Hilger's Local Plane



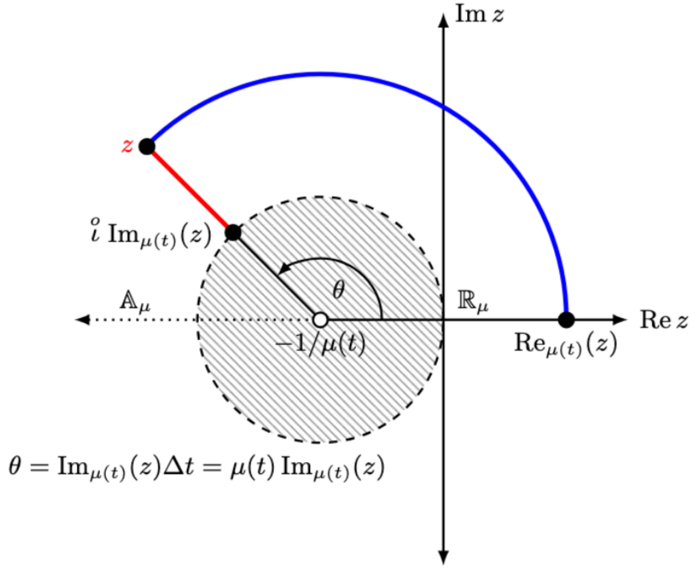
$$
\begin{aligned}
\operatorname{Re}_{\mu(t)}(z) &:= \frac{|1 + z\mu(t)| - 1}{\mu(t)} \\
&= \xi_{\mu(t)}^{-1}(\operatorname{Re}(\xi_{\mu(t)}(z))) \\
\operatorname{Im}_{\mu(t)}(z) &:= \frac{\operatorname{Arg}(1 + z\mu(t))}{\mu(t)} \\
&= \operatorname{Im}(\xi_{\mu(t)}(z))
\end{aligned}
$$

Hilger's Local Plane



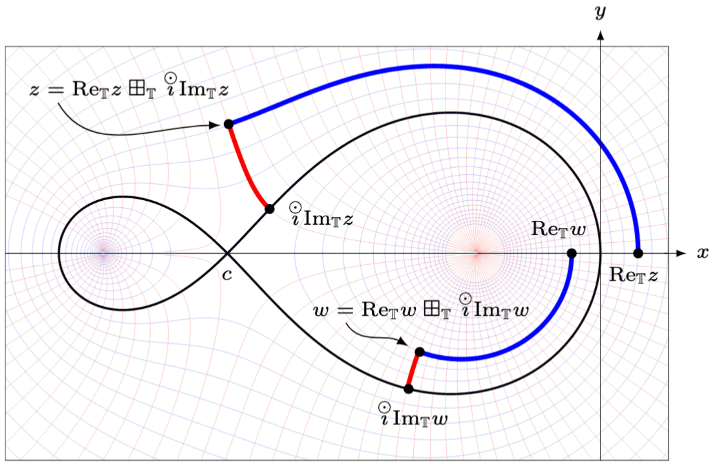
$$\mathrm{Re}_{\mu(t)}(z) := \frac{|1+z\mu(t)|-1}{\mu(t)}$$
$$= \xi^{-1}_{\mu(t)}(\mathrm{Re}(\xi_{\mu(t)}(z)))$$
$$\mathrm{Im}_{\mu(t)}(z) := \frac{\mathrm{Arg}(1+z\mu(t))}{\mu(t)}$$
$$= \mathrm{Im}(\xi_{\mu(t)}(z))$$
$$\overset{\circ}{\iota}\,\mathrm{Im}_{\mu(t)}(z) := \frac{e^{i\mathrm{Im}_{\mu(t)}(z)\mu(t)}-1}{\mu(t)}$$
$$= \xi^{-1}_{\mu(t)}(i\mathrm{Im}(\xi_{\mu(t)}(z)))$$

Hilger's Local Plane



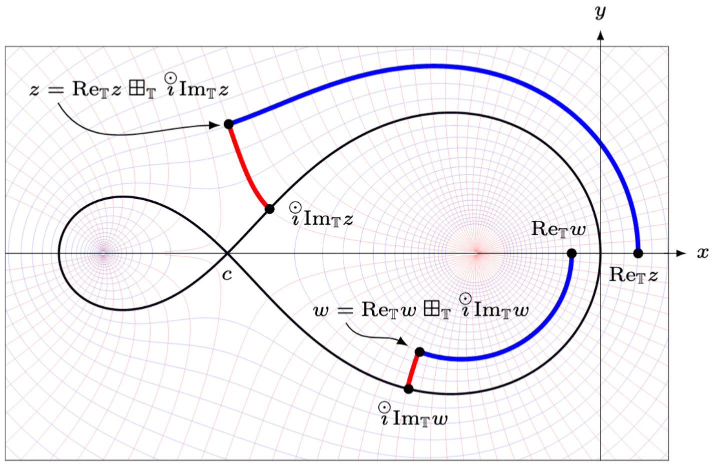
- With $a \oplus_{\mu(t)} b := a + b + \mu(t)ab$, Hilger's local plane is isomorphic to the cylinder strip $\mathbb{Z}_{\mu(t)}$ with addition mod $2\pi i/\mu(t)$.
- $z \oplus_{\mu(t)} w = \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(z) + \xi_{\mu(t)}(w))$
- Hilger's decomposition: $z = \mathrm{Re}_{\mu(t)}(z) \oplus_{\mu(t)} \overset{o}{\iota}\,\mathrm{Im}_{\mu(t)}(z)$

Ergodic Global Plane



- Global behavior of $e_z(t, t_0)$ in terms of $z \in \mathbb{C} \setminus \{-1/\mu(t) \mid t \in \mathbb{T}\} := \mathbb{C}_{\mu(\mathbb{T})}$.
- On a periodic (or finite) time scale of length L, defined by the *average* of the cylinder transformation $\overline{\xi}(z) = \dfrac{1}{L}\displaystyle\int_0^L \xi_{\mu(\tau)}(z)\Delta\tau$ (at least off the branch cut).

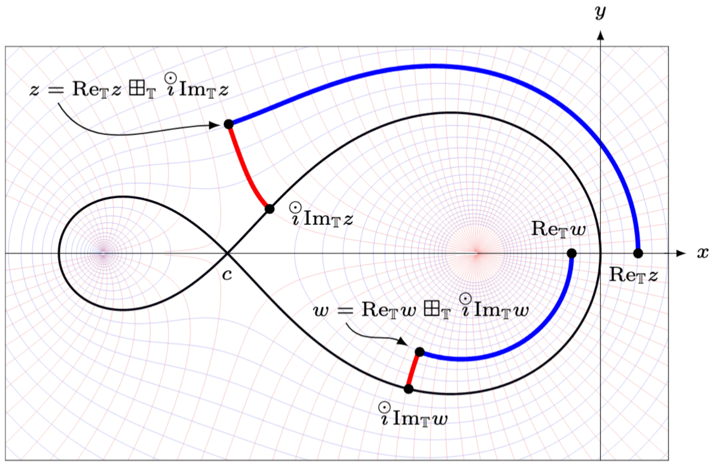
Ergodic Global Plane



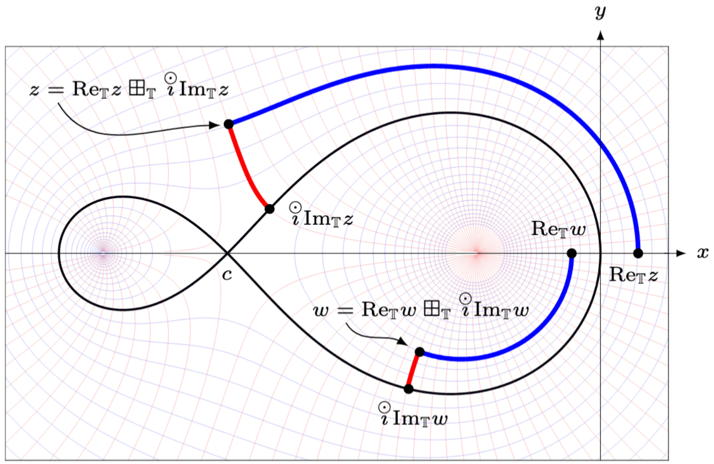
- The generalization of the cylinder strip is the set $\mathbb{C}^{\Omega} = \{z \in \mathbb{C} \mid -\Omega < \mathrm{Im}(z) \leq \Omega\}$, where Ω is the Nyquist frequency.
- We have carefully defined $\overline{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}^{\Omega}$ on the branch cut and shown that the resulting map is globally univalent.
- $z \boxplus_{\mathbb{T}} w = \overline{\xi}^{-1}(\overline{\xi}(z) + \overline{\xi}(w)(\mathrm{mod} 2i\Omega))$

Ergodic Global Plane

$$\mathrm{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z)))$$



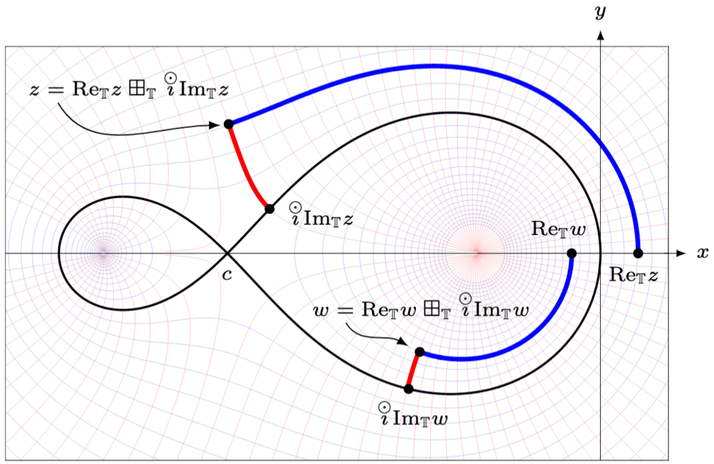
Ergodic Global Plane



$$\mathrm{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z)))$$

$$\mathrm{Im}_{\mathbb{T}}(z) = \mathrm{Im}(\overline{\xi}(z))$$

Ergodic Global Plane

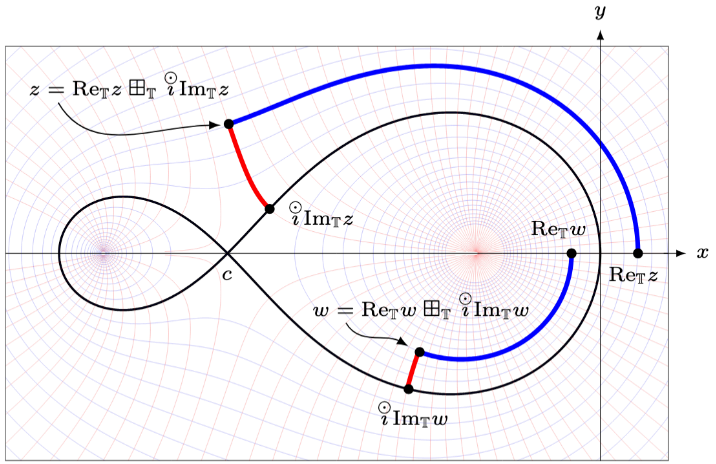


$$\mathrm{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z)))$$

$$\mathrm{Im}_{\mathbb{T}}(z) = \mathrm{Im}(\overline{\xi}(z))$$

$$\overset{\odot}{i}\mathrm{Im}_{\mathbb{T}}(z) := \overline{\xi}^{-1}(i\mathrm{Im}(\overline{\xi}(z)))$$

Ergodic Global Plane



$$\mathrm{Re}_{\mathbb{T}}(z) = \overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z)))$$

$$\mathrm{Im}_{\mathbb{T}}(z) = \mathrm{Im}(\overline{\xi}(z))$$

$$\overset{\odot}{i}\mathrm{Im}_{\mathbb{T}}(z) := \overline{\xi}^{-1}(i\mathrm{Im}(\overline{\xi}(z)))$$

$$z = \mathrm{Re}_{\mathbb{T}}(z) \boxplus_{\mathbb{T}} \overset{\odot}{i}\mathrm{Im}_{\mathbb{T}}(z)$$

Fourier Series on $\mathbb{R}$

On $\mathbb{R}$, the Fourier series for an L-periodic function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}t} \\
&= \sum_{n=-\infty}^{\infty} c_n e_{\xi_0^{-1}\left(i\frac{2\pi n}{L}\right)}(t,0), \text{ with} \\
c_n &= \frac{1}{L}\int_0^L f(t) e^{-i\frac{2\pi n}{L}t}\,dt.
\end{aligned}
$$

Fourier Series on $\mathbb{Z}$

On $\mathbb{Z}$, the discrete Fourier series for an L-periodic function $f : \mathbb{Z} \to \mathbb{C}$ is given by

$$\begin{aligned} f(t) &= \sum_{n=-\lfloor (L-1)/2 \rfloor}^{\lceil (L-1)/2 \rceil} c_n e^{i \frac{2\pi n}{L} t} \\ &= \sum_{n=-\lfloor (L-1)/2 \rfloor}^{\lceil (L-1)/2 \rceil} c_n e_{\xi_1^{-1}\left(i \frac{2\pi n}{L}\right)}(t, 0), \;\; \text{with} \\ c_n &= \frac{1}{L} \sum_{t=0}^{L-1} f(t) e^{-i \frac{2\pi n}{L} t}. \end{aligned}$$

Fourier Series on $\mathbb{T}$

Key Insight: Basis functions should be exponentials with zero growth rate and a harmonic frequency. That is, they should be time scale exponentials with subscript

$$\overline{\xi}^{-1}(i\omega_n) = \overset{\odot}{i}\omega_n,$$

where ω_n is a harmonic frequency.

Fourier Series on $\mathbb{T}$

- To calculate the values of the subscripts, we could sample the Nyquist interval uniformly and numerically calculate $\overline{\xi}^{-1}$ at each of these sample points. There is an easier way, however - eigenvalues!
- Let $\mathbb{T} = \{t_0, t_1, t_2, \ldots, t_{m-1}, t_m, t_1 + L, t_2 + L, \ldots t_0 + 2L, t_1 + 2L, \ldots\}$ be a periodic time scale of length L.
- We can write the dynamic equation on $\mathbb{T}$, $x^{\Delta} = \lambda x$, with a periodic condition $x(t_0) = x(t_m)$ as a system of equations.

Fourier Series on $\mathbb{T}$

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \qquad 0 \le k \le m-2$$

$$\frac{x(t_m) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

Fourier Series on $\mathbb{T}$

$$\frac{x(t_{k+1}) - x(t_k)}{\mu(t_k)} = \lambda x(t_k) \qquad 0 \leq k \leq m-2$$

$$\frac{x(t_0) - x(t_{m-1})}{\mu(t_{m-1})} = \lambda x(t_{m-1})$$

Fourier Series on $\mathbb{T}$

$$\begin{pmatrix} -1/\mu(t_0) & 1/\mu(t_0) & 0 & \cdots & 0 & 0 \\ 0 & -1/\mu(t_1) & 1/\mu(t_1) & \cdots & 0 & 0 \\ 0 & 0 & -1/\mu(t_2) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1/\mu(t_{m-3}) & 0 \\ 0 & 0 & \cdots & 0 & -1/\mu(t_{m-2}) & 1/\mu(t_{m-2}) \\ 1/\mu(t_{m-1}) & 0 & \cdots & 0 & 0 & -1/\mu(t_{m-1}) \end{pmatrix} \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix} = \lambda \begin{pmatrix} x(t_0) \\ x(t_1) \\ x(t_2) \\ \vdots \\ x(t_{m-2}) \\ x(t_{m-1}) \end{pmatrix}$$

The eigenvalues are $\overset{\odot}{\imath}\omega_k, 0 \leq k \leq m-1$.

The associated eigenvectors are $e_{\overset{\odot}{\imath}\omega_k}(t, 0)$ evaluated at each point in one period of the time scale.

Fourier Series on $\mathbb{T}$

Now, we calculate the Fourier coefficients in

$$f(t) = \sum_{n=0}^{m-1} c_n e_{\overset{\odot}{\imath\omega_n}}(t, 0).$$

Theorem

Suppose $\mathbb{T}$ is a periodic time scale with $t_0 = 0$. Suppose also that $\overset{\odot}{\imath\omega_n}$ exists for each $n, 0 \leq n \leq m-1$, and are distinct. Then, the sets $\{e_{\overset{\odot}{\imath\omega_n}}(t,0)\}$ and $\left\{e_{\ominus\overset{\odot}{\imath\omega_n}}(\sigma(t),0)\right\}$ form a biorthogonal system, that is

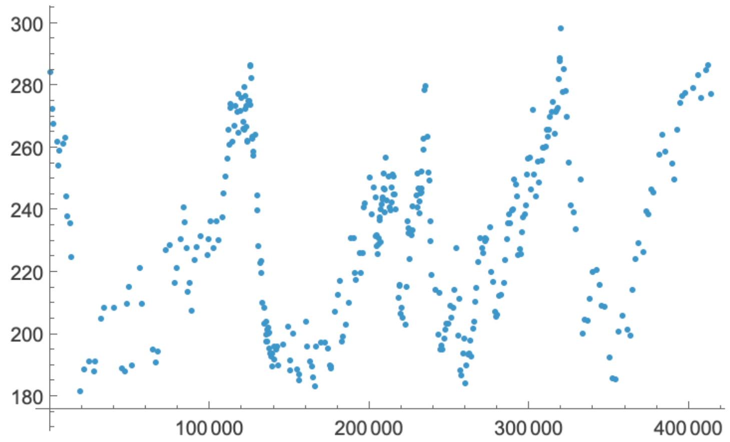
$$\int_0^L e_{\overset{\odot}{\imath\omega_n}}(t,0)e_{\ominus\overset{\odot}{\imath\omega_l}}(\sigma(t),0)\Delta t = K_n\delta_{n,l}.$$

Fourier Series on $\mathbb{T}$

Therefore,

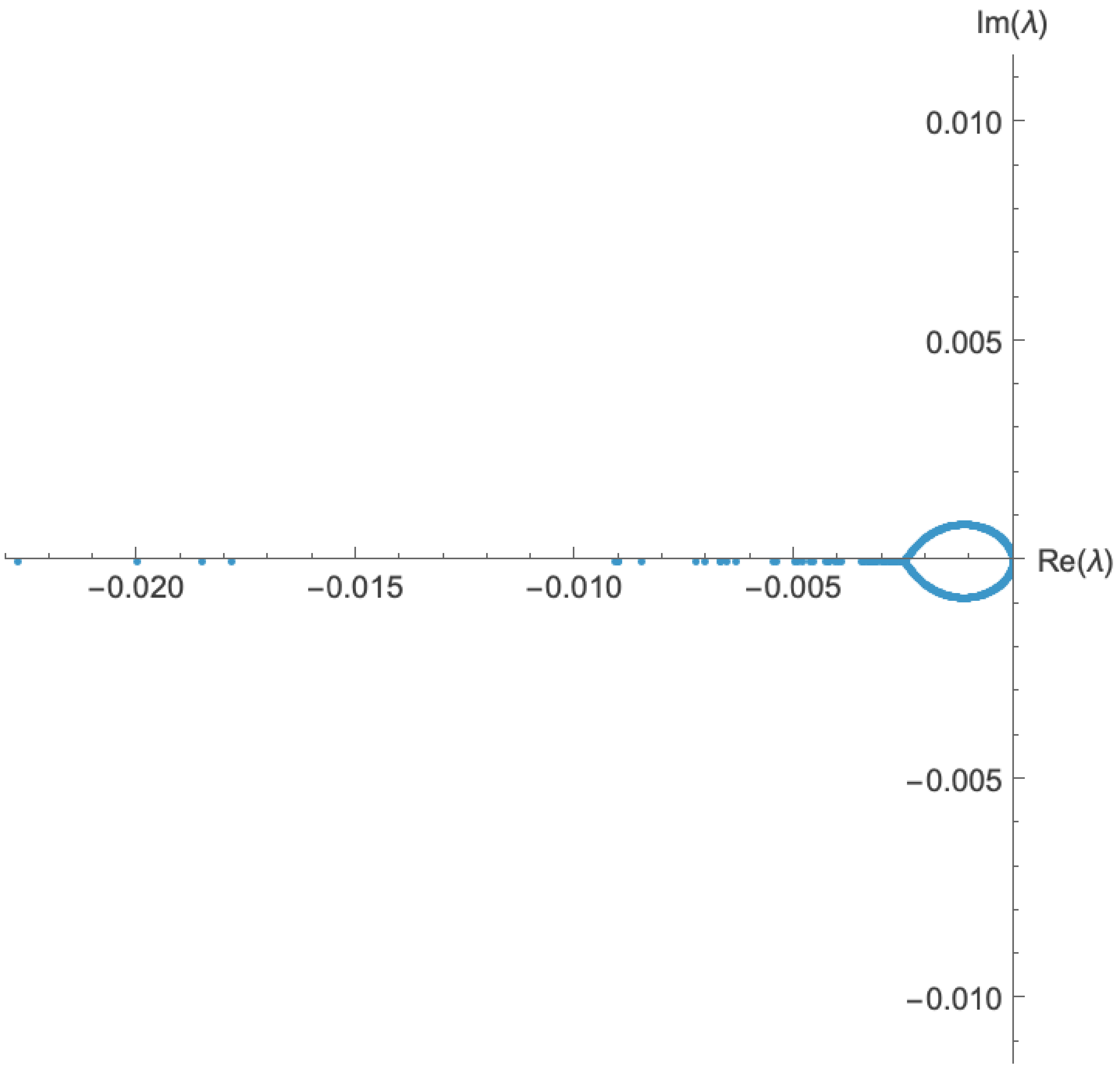
$$c_n = \frac{\int_0^L f(t) \cdot e_{\ominus \overset{\odot}{\imath} \omega_n}(\sigma(t), 0) \, \Delta t}{\int_0^L e_{\overset{\odot}{\imath} \omega_n}(t, 0) \cdot e_{\ominus \overset{\odot}{\imath} \omega_n}(\sigma(t), 0) \, \Delta t}.$$

Antarctic Ice Core Data

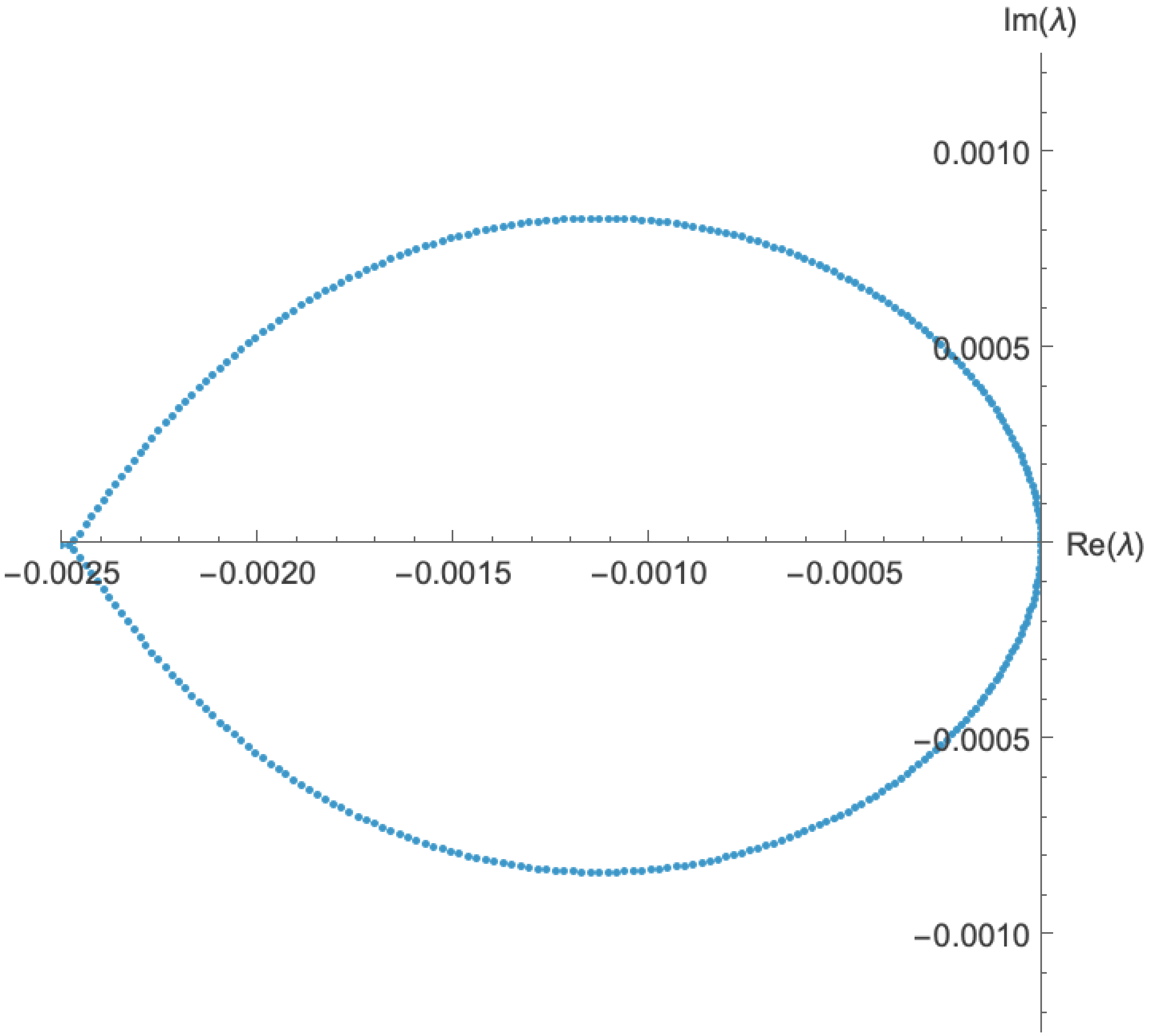


- Record of 400,000+ years of atmospheric CO2 concentration
- Irregularly sampled in time because the ice core is sampled uniformly along the length of the ice core.

Spectrum

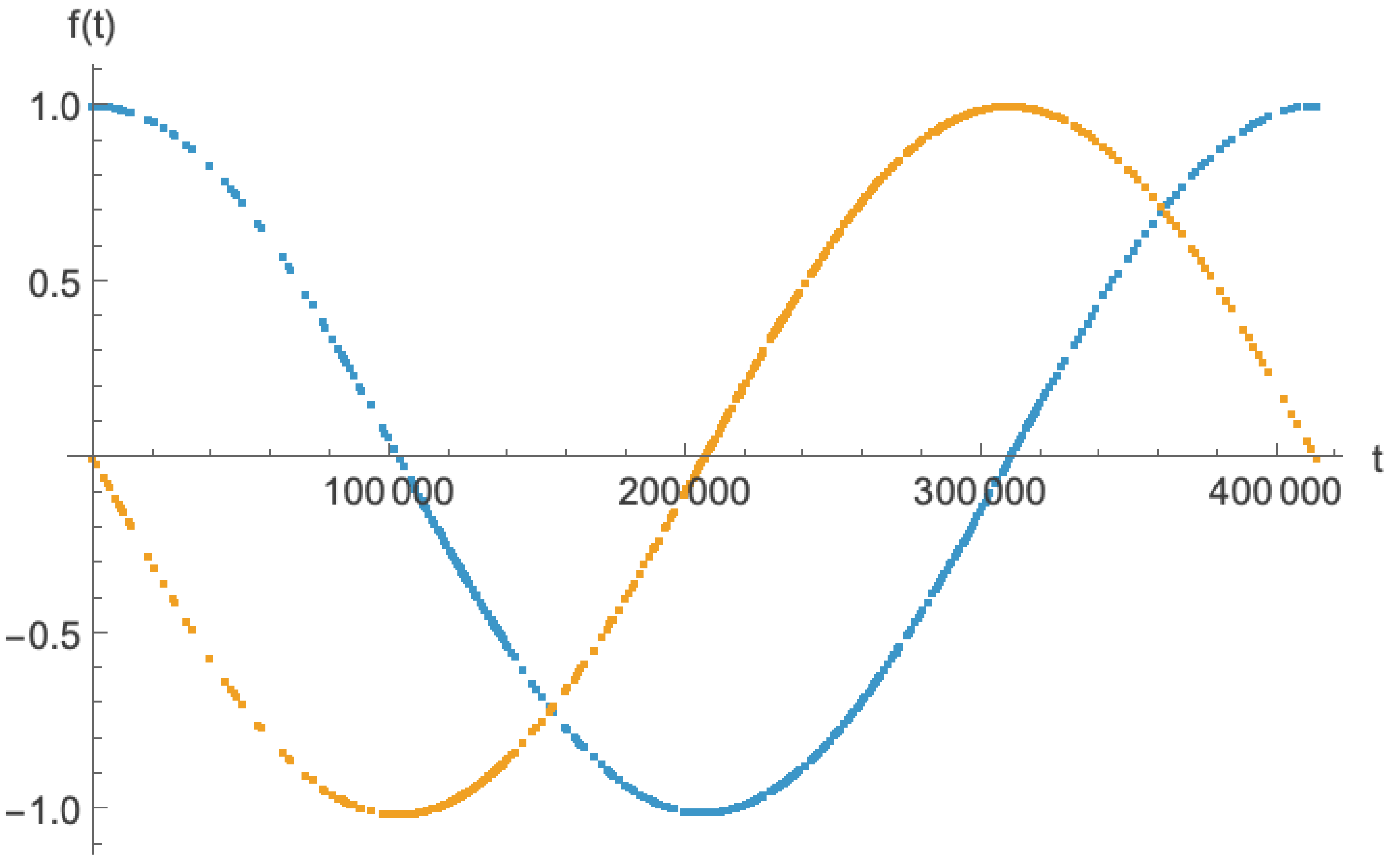


Spectrum



Im(λ)
0.0010
0.0005
−0.0005
−0.0010
Re(λ)
−0.0025
−0.0020
−0.0015
−0.0010
−0.0005

Bases



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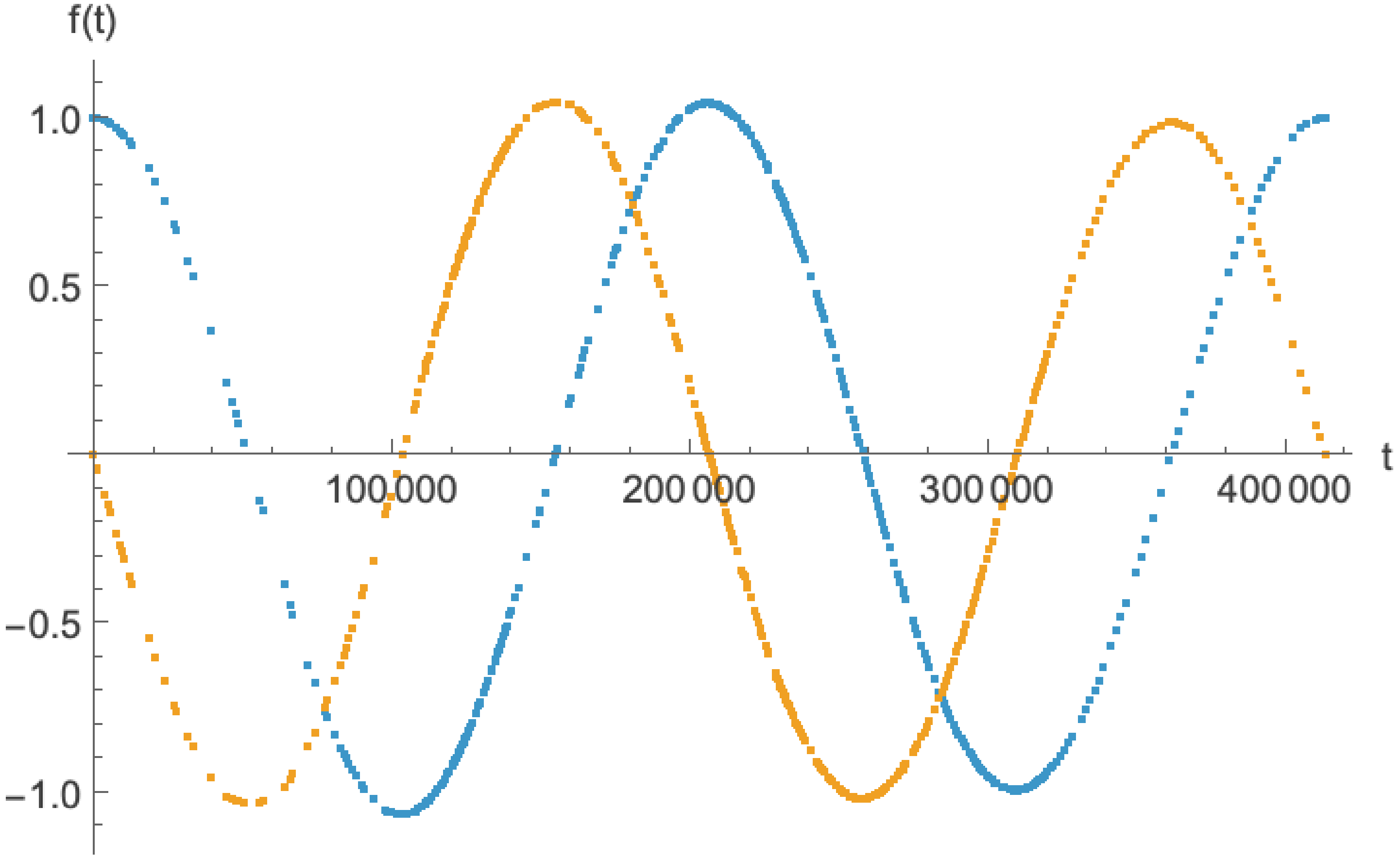

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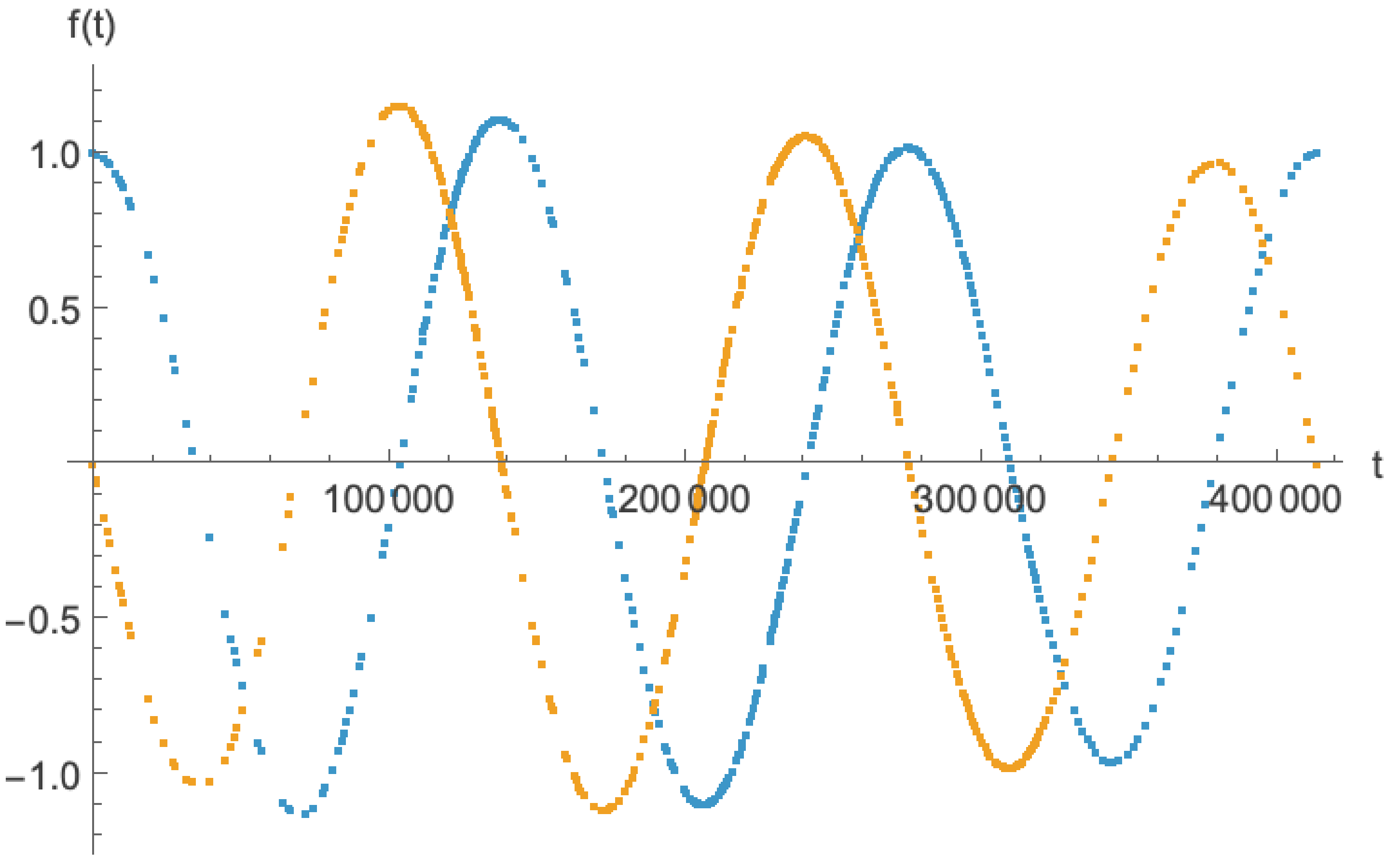

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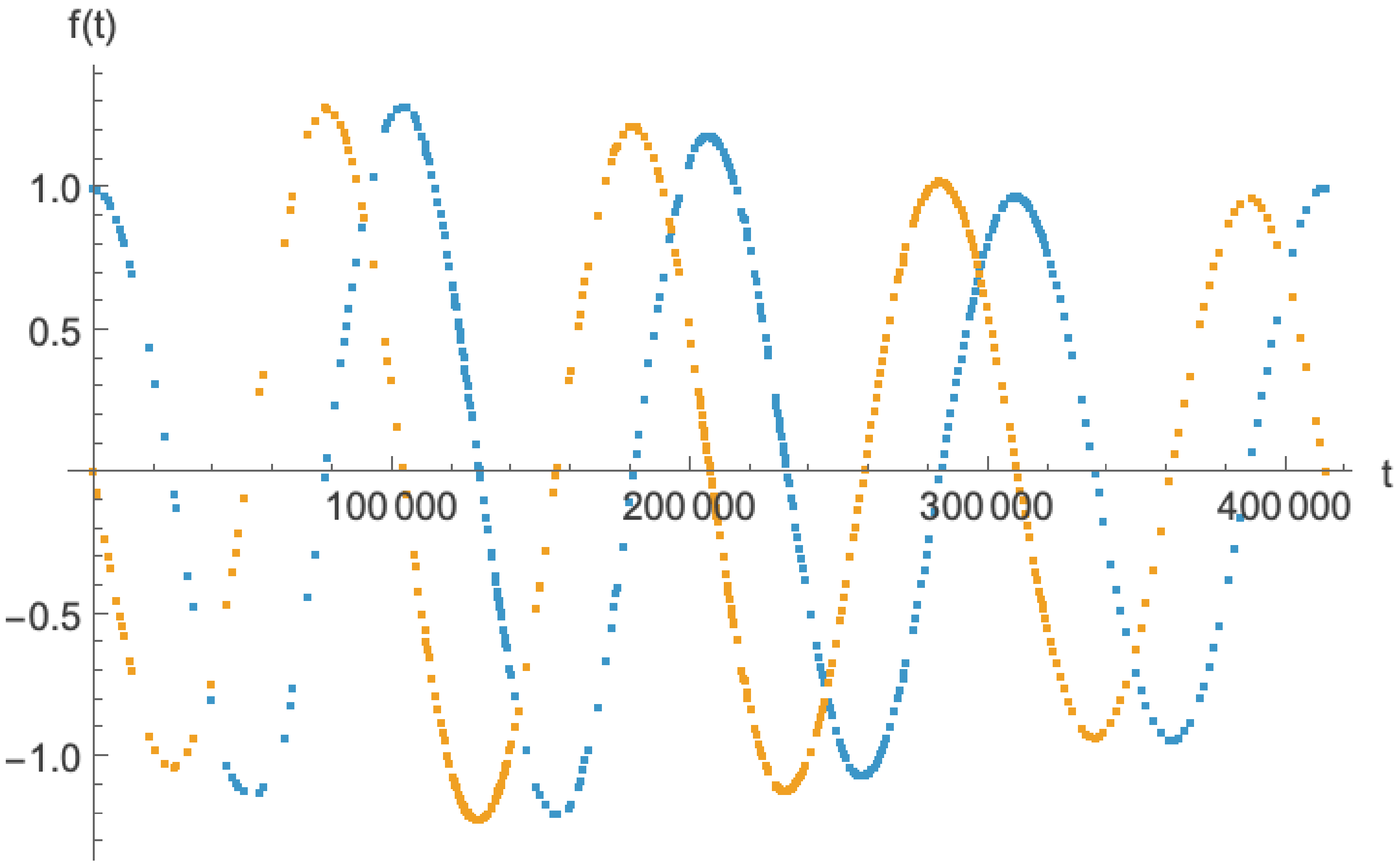
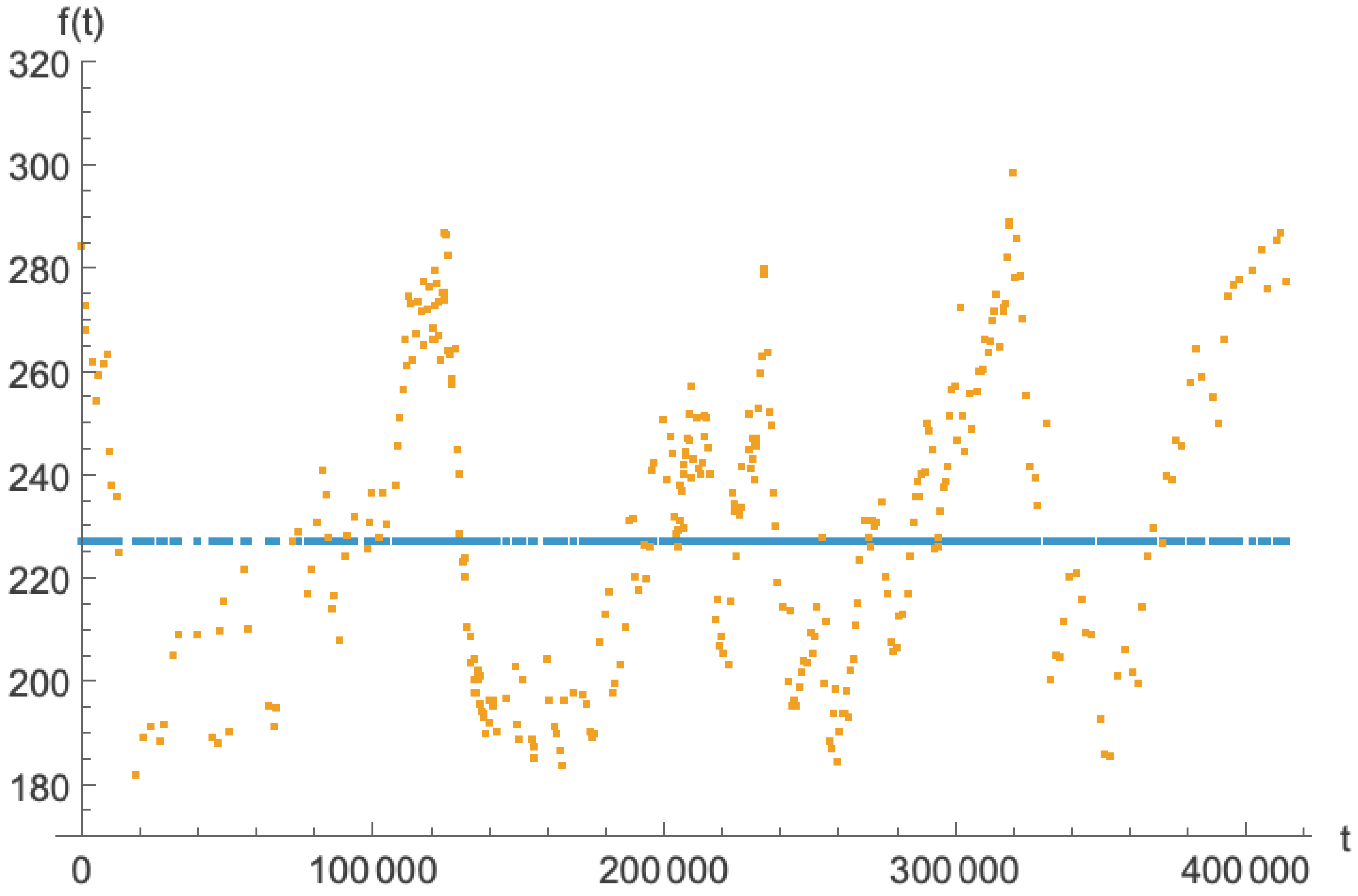
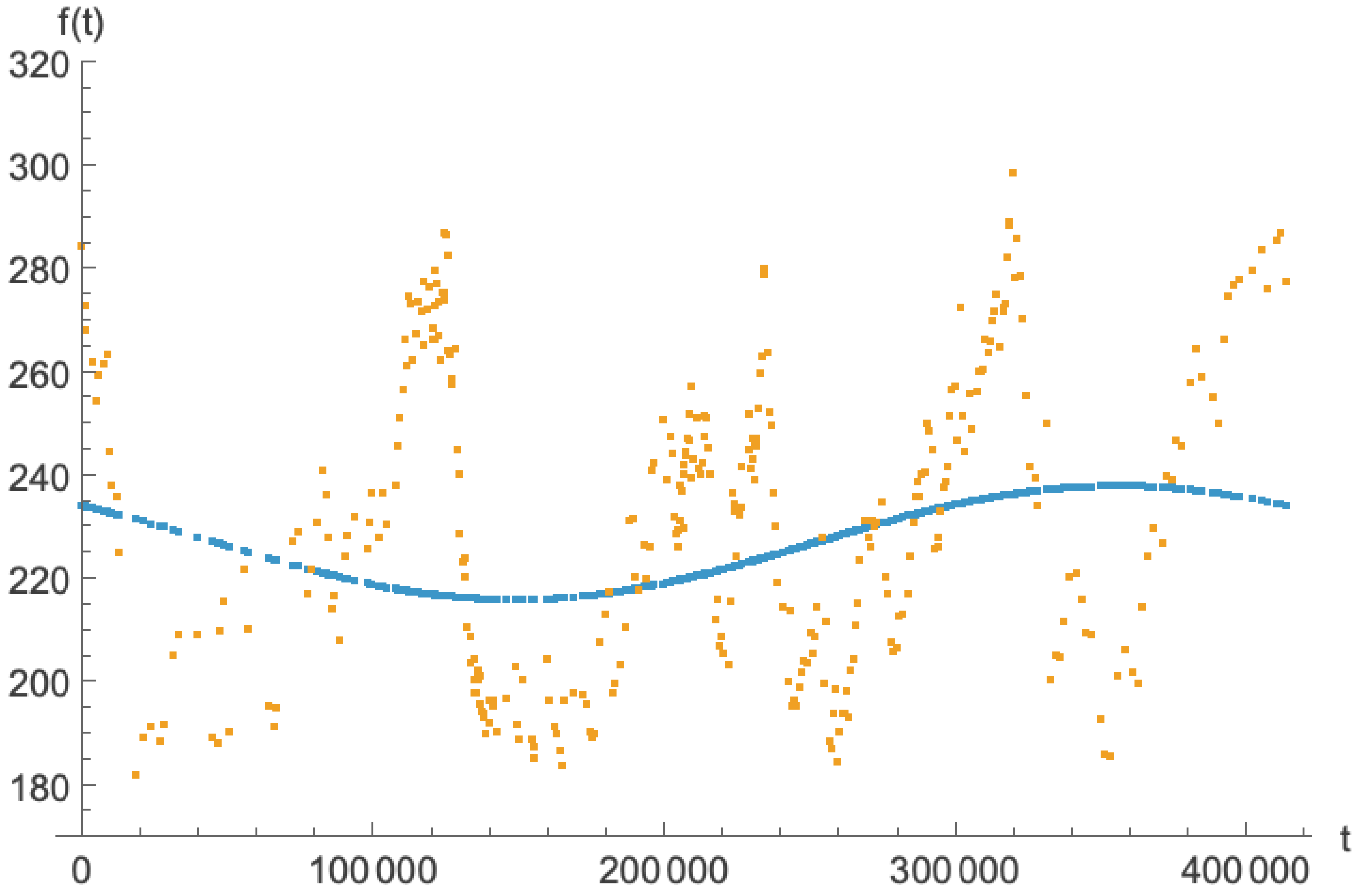


Approx



f(t)
320
300
280
260
240
220
200
180
0
100 000
200 000
300 000
400 000
t

Approx



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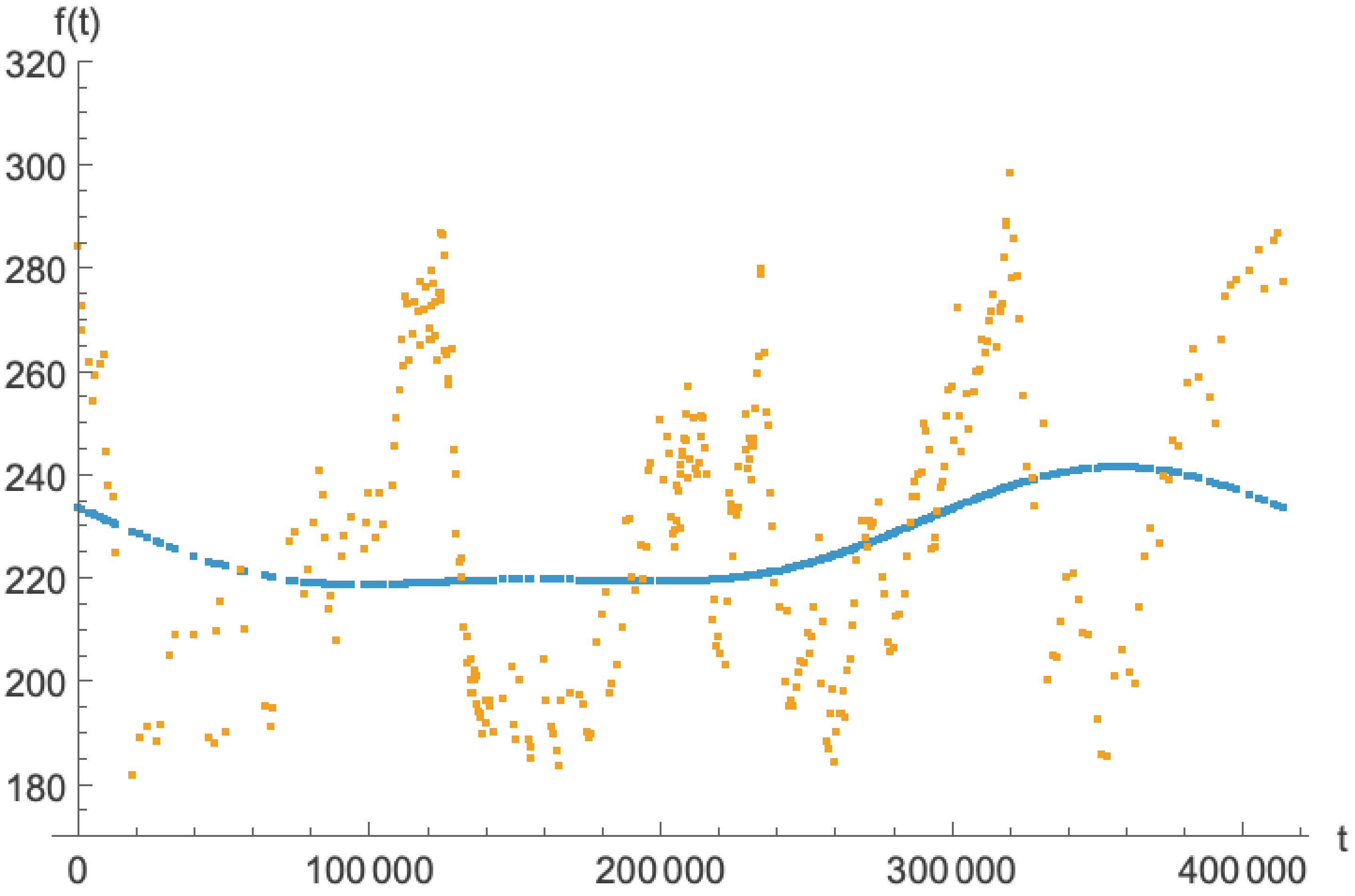
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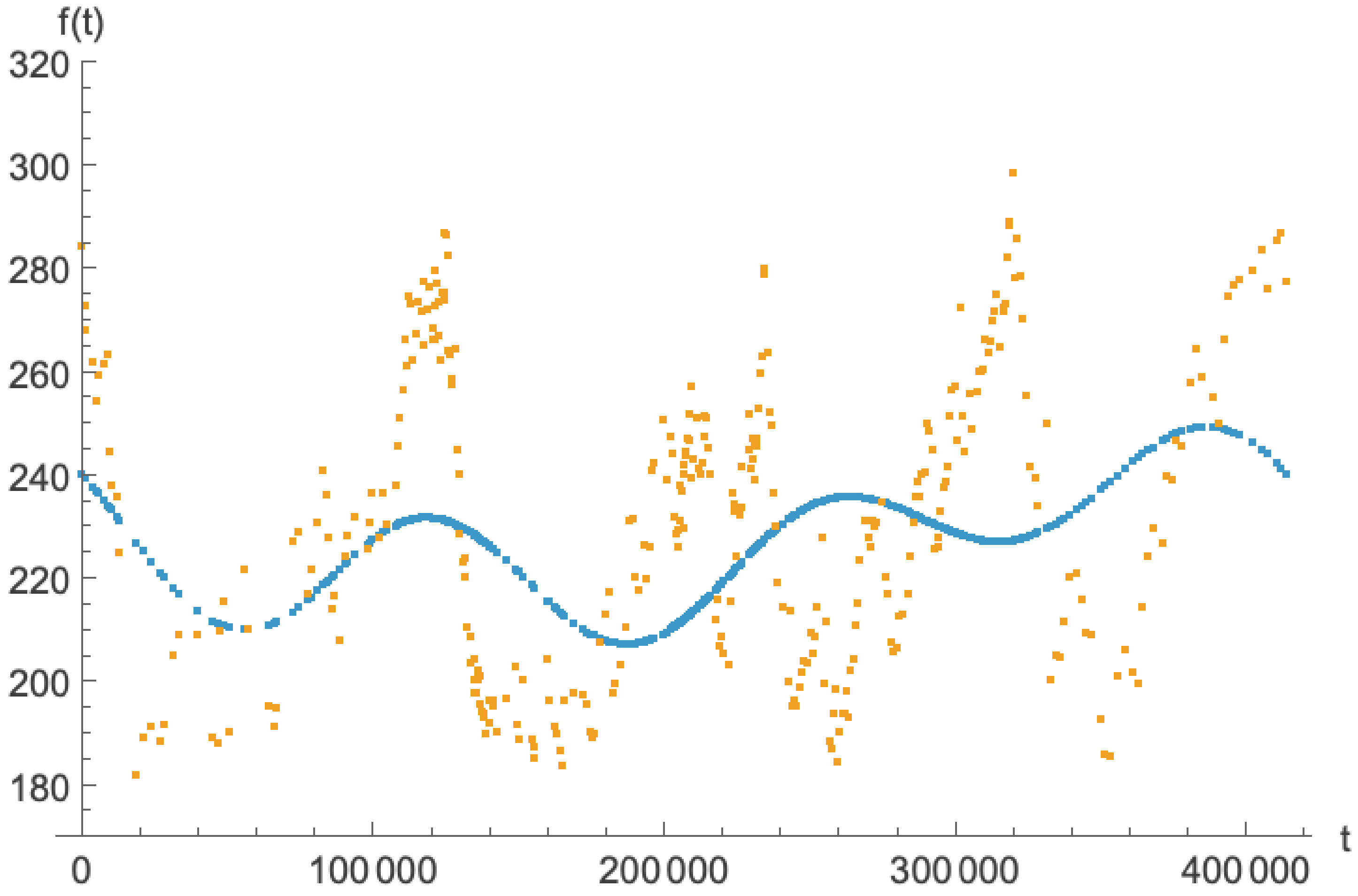

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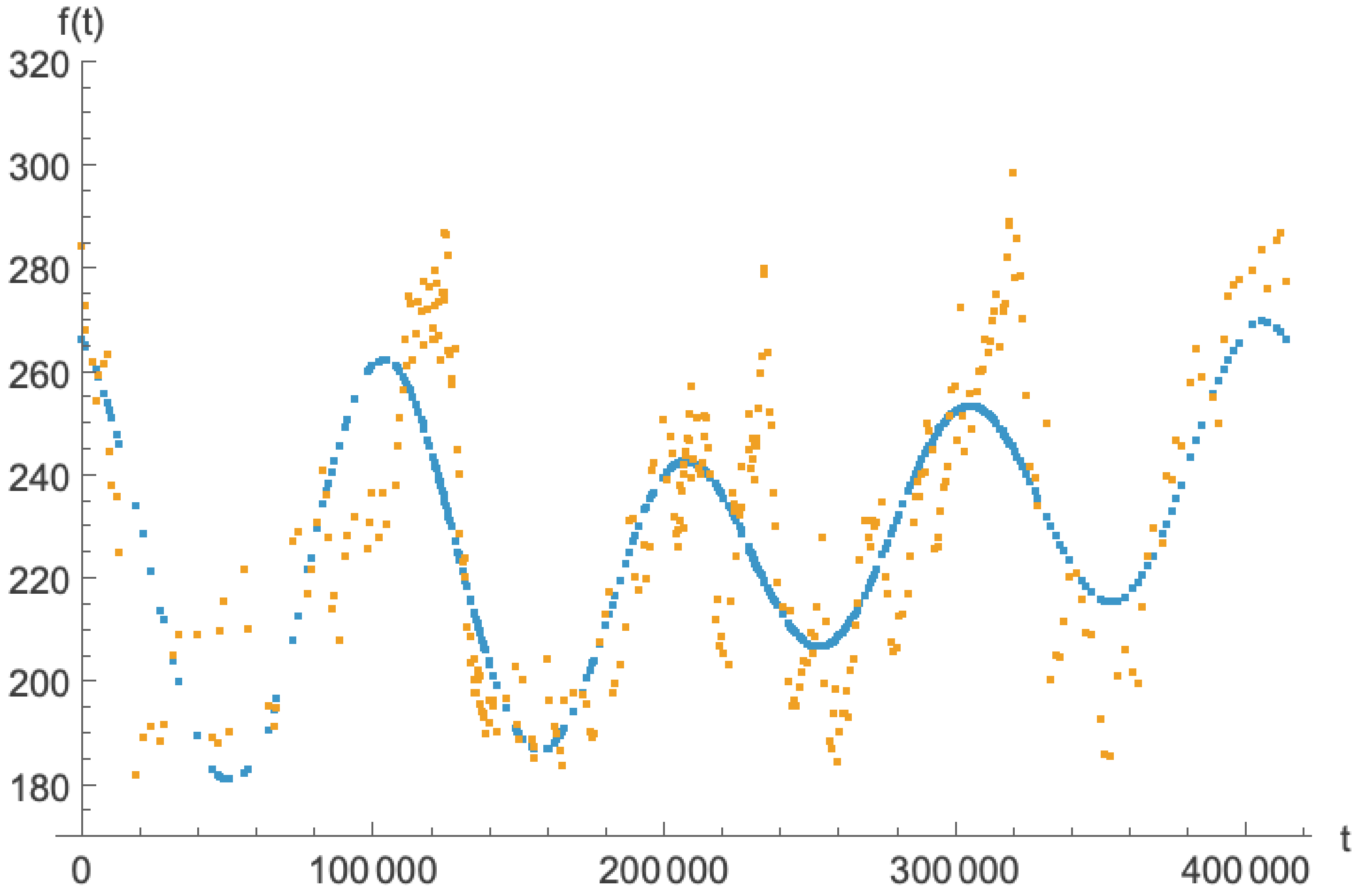

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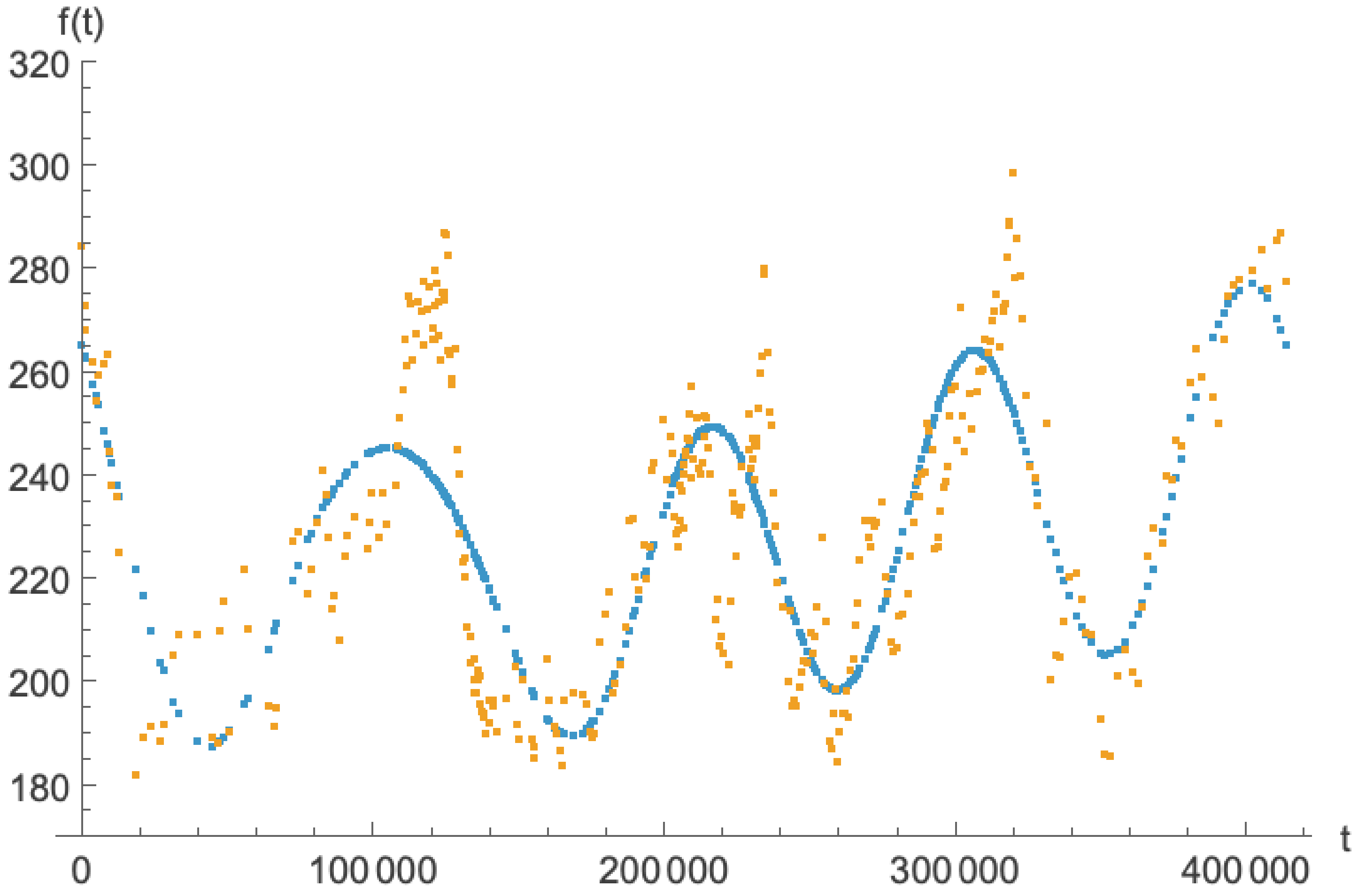

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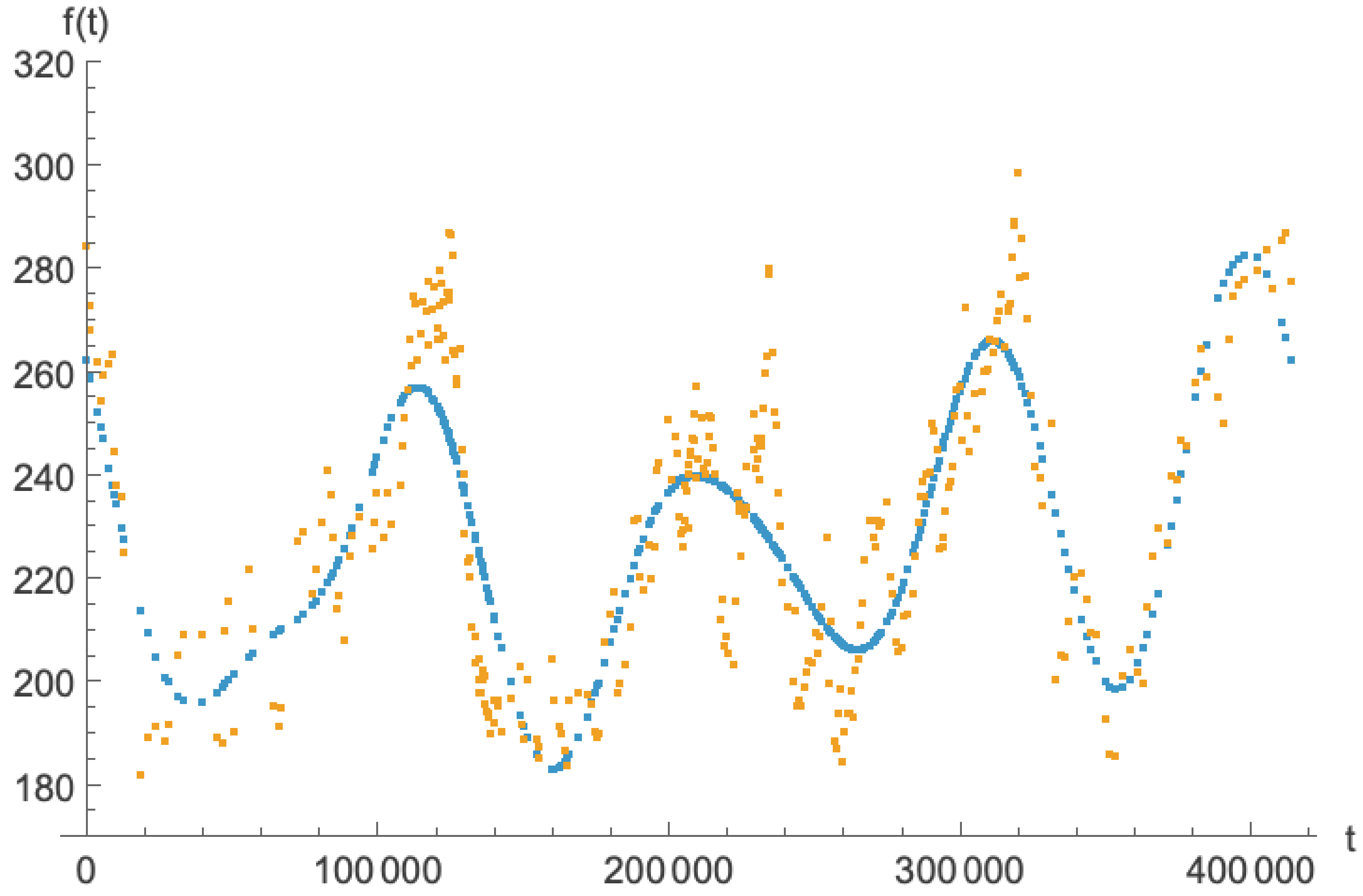
Approx



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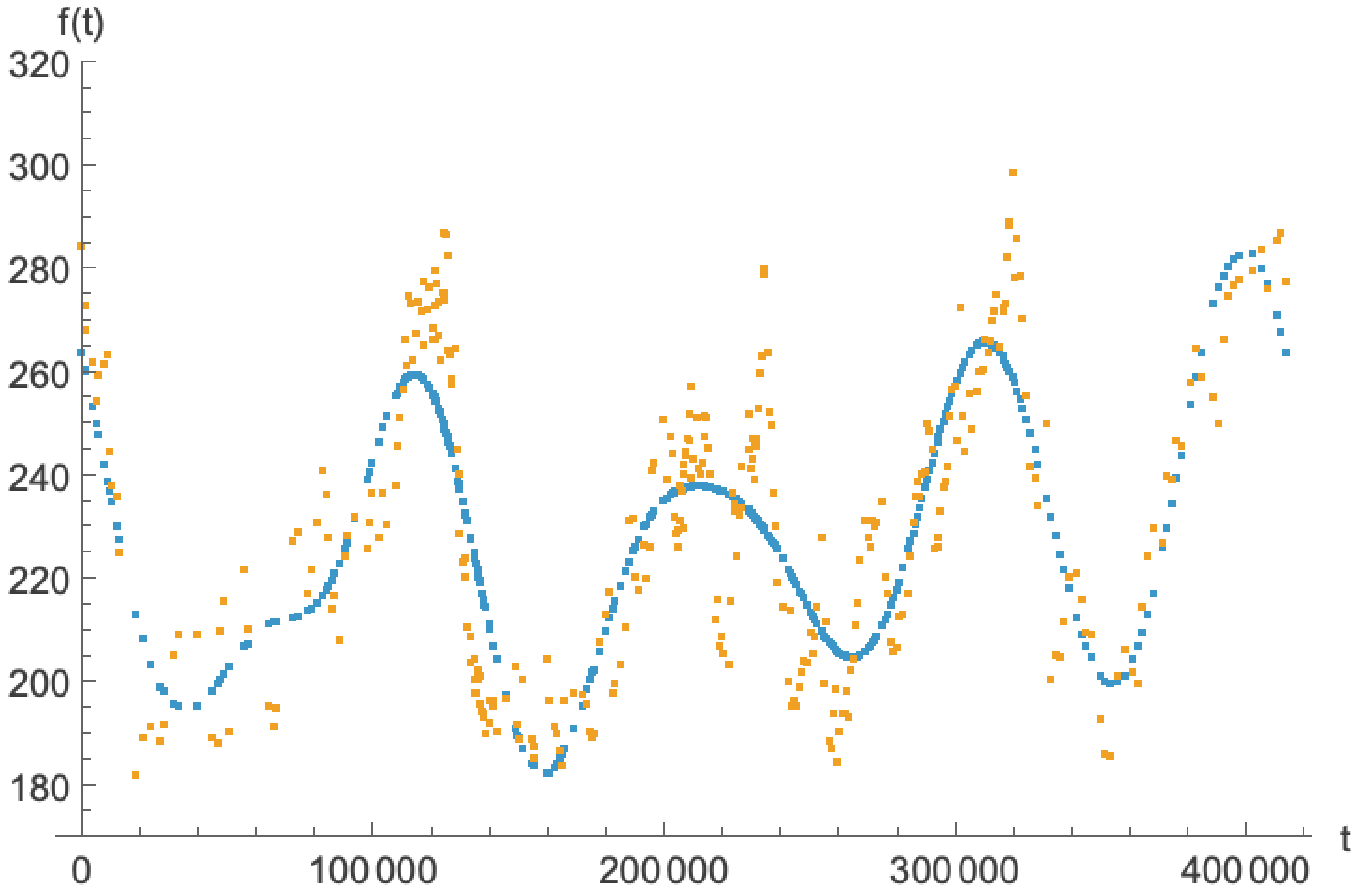

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Questions?

Thank you for your attention. What questions do you have?

Slide Link

Scan for slides:

https://drspoulsen.github.io/JMM_2026_Talk/