How to Optimally Cut An Onion: A Love Letter to the Jacobian

For each section in the paper, we provide Mathematica code that verifies the solutions presented.

Inspiration for the Solution

Jacobian for the h=0 case

```
In[3]:= x[r_, θ_] := r Cos[θ];
        y[r_, θ_] := r Sin[θ];
        J[r_, θ_] := Abs[D[x[r, θ], r] × D[y[r, θ], θ] - D[x[r, θ], θ] × D[y[r, θ], r]]
```

```
In[6]:= FullSimplify[Abs[J[r, θ]]]
```

```
Out[6]= Abs[r]
```

Onion cutting statistics in the h=0 case

```
In[7]:= Abar = Assuming[R > 0,
          Integrate[J[r, θ], {r, 0, R}, {θ, 0, Pi / 2}] / Integrate[1, {r, 0, R}, {θ, 0, Pi / 2}]]
```

Out[7]= $\frac{R}{2}$

```
In[8]:= σ2 = Assuming[R > 0, Integrate[(J[r, θ] - Abar) ^2, {r, 0, R}, {θ, 0, Pi / 2}] /
           Integrate[1, {r, 0, R}, {θ, 0, Pi / 2}]]
```

Out[8]= $\frac{R^2}{12}$

Compare to the uniform distribution:

```
In[∘]:= Mean[UniformDistribution[{0, R}]]
```

Out[∘]= $\frac{R}{2}$

```
In[∘]:= Variance[UniformDistribution[{0, R}]]
```

Out[∘]= $\frac{R^2}{12}$

A Strange Coordinate System

c_h, x_h, y_h, and J_h from the paper:

In[13]:=
```
ch[r_, θ_, h_] := h Cos[θ] + Sqrt[r^2 - h^2 Sin[θ]^2];
xh[r_, θ_, h_] := ch[r, θ, h] Sin[θ];
yh[r_, θ_, h_] := ch[r, θ, h] Cos[θ] - h;
Jh[r_, θ_, h_] :=
 D[xh[r, θ, h], r] × D[yh[r, θ, h], θ] - D[xh[r, θ, h], θ] × D[yh[r, θ, h], r]
```

In[17]:=
```
cm[r_, θ_, h_] := h Cos[θ] - Sqrt[r^2 - h^2 Sin[θ]^2];
```

Note that the absolute value of the Jacobian is the below expression with the absolute value removed, since r>h tan(θ) and tan(θ)>sin(θ) on this interval. Moreover, r>0,h>0 and cos(θ)>0.

In[18]:=
```
FullSimplify[Jh[r, θ, h]]
```

Out[18]=

$$r\left(-1-\frac{h\,\mathrm{Cos}[\theta]}{\sqrt{r^2-h^2\,\mathrm{Sin}[\theta]^2}}\right)$$

So, let’s redefine Jh to be positive:

In[19]:=

$$\mathrm{Jh}[r\_,\theta\_,h\_] := r+\frac{h\,r\,\mathrm{Cos}[\theta]}{\sqrt{r^2-h^2\,\mathrm{Sin}[\theta]^2}}$$

Calculating Statistics

This code is for the numerator of Abar(h) in the paper.

The integral can be calculated numerically for any h between 0 and 1:

In[ ]:=

$$\mathrm{Manipulate}\left[\mathrm{NIntegrate}\left[r+\frac{h\,r\,\mathrm{Cos}[\theta]}{\sqrt{r^2-h^2\,\mathrm{Sin}[\theta]^2}},\{\theta,0,\mathrm{ArcTan}[1/h]\},\{r,\ h\,\mathrm{Tan}[\theta],1\}\right],\{h,.01,1\}\right]$$

Out[ ]=

h

```
0.785398
```

Which compares well to $\pi/4$.

In[•]:=
```
N[Pi / 4]
```
Out[•]=
```
0.785398
```

Symbolically, we cannot arrive at $\pi/4$, for

In[•]:=
```
Assuming[0 < h < 1,
  Integrate[r + (h r Cos[θ])/Sqrt[r^2 - h^2 Sin[θ]^2], {θ, 0, ArcTan[1 / h]}, {r, h Tan[θ], 1}]]
```
Out[•]=

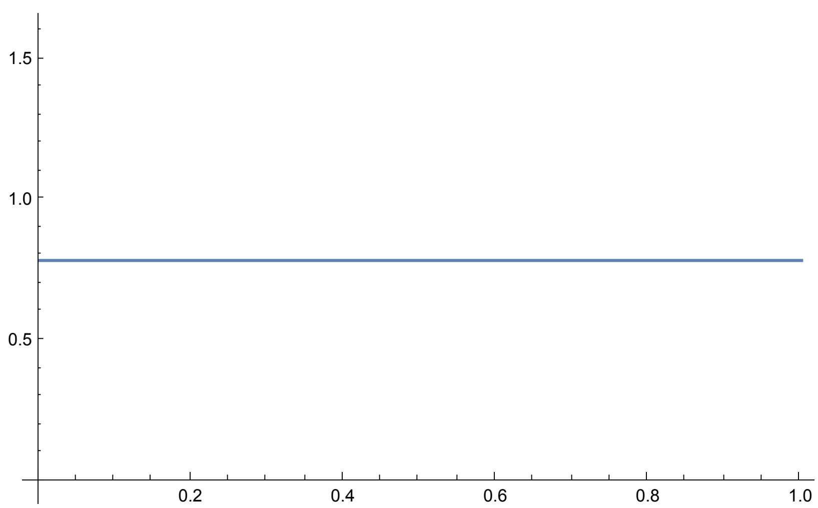
$$\frac{1}{2}\left(\mathrm{ArcSin}\left[h\,\mathrm{Cosh}\left[\frac{1}{2}\,(\mathrm{Log}[i-h]-\mathrm{Log}[i+h])\right]\right]+\mathrm{ArcTan}\left[\frac{1}{h}\right]\right)$$

In[1]:=
```
Assuming[0.1 < h < .9,
  Integrate[r + (h r Cos[θ])/Sqrt[r^2 - h^2 Sin[θ]^2], {θ, 0, ArcTan[1 / h]}, {r, h Tan[θ], 1}]]
```
Out[1]=

$$\frac{1}{2}\left(\mathrm{ArcSin}\left[h\,\mathrm{Cosh}\left[\frac{1}{2}\,(\mathrm{Log}[i-h]-\mathrm{Log}[i+h])\right]\right]+\mathrm{ArcTan}\left[\frac{1}{h}\right]\right)$$

Which appears to be a constant as a function of h in this range.

```
Plot[1/2 (ArcSin[h Cosh[1/2 (Log[i - h] - Log[i + h])]] + ArcTan[1/h]), {h, 0, 1}]
```
Out[•]=



But, we cannot get the answer of $\pi/4$ Despite trying a few different simplifications...

In[•]:=
```
Assuming[0 < h < 1,
  FullSimplify[1/2 (ArcSin[h Cosh[1/2 (Log[i - h] - Log[i + h])]] + ArcTan[1/h])]]
```
Out[•]=

$$\frac{1}{2}\left(\mathrm{ArcCot}[h]+\mathrm{ArcSin}\left[h\,\mathrm{Cosh}\left[\frac{1}{2}\,(\mathrm{Log}[i-h]-\mathrm{Log}[i+h])\right]\right]\right)$$

In[ ]:=
```
Assuming[0 < h < 1,
 Simplify[TrigToExp[1/2 (ArcCot[h] + ArcSin[h Cosh[1/2 (Log[i - h] - Log[i + h])]])]]]
```
Out[ ]=

$$\frac{1}{4}\, i \left(\text{Log}[-i + h] - \text{Log}[i + h] - 2\,\text{Log}\left[\frac{-\sqrt{i - h} + h\,\sqrt{-i + h}}{(-1 - i\,h)\,\sqrt{i + h}}\right]\right)$$

In[ ]:=
```
Assuming[0 < h < 1,
 ComplexExpand[1/4 i (Log[-i + h] - Log[i + h] - 2 Log[(-Sqrt[i - h] + h Sqrt[-i + h])/((-1 - i h) Sqrt[i + h])])]]
```
Out[ ]=

$$-\frac{1}{4}\,\text{Arg}[-i + h] + \frac{1}{4}\,\text{Arg}[i + h] + \frac{1}{2}\,\text{Arg}\left[\frac{-\sqrt{i - h} + h\,\sqrt{-i + h}}{(-1 - i\,h)\,\sqrt{i + h}}\right] - \frac{1}{2}\, i\,\text{Log}\Bigg[\frac{1}{(1 + h^2)^{3/4}}\Bigg(\sqrt{\Bigg(\Big(-(1 + h^2)^{1/4}\,\text{Cos}\Big[\frac{1}{2}\,\text{Arg}[i - h]\Big] + h\,(1 + h^2)^{1/4}\,\text{Cos}\Big[\frac{1}{2}\,\text{Arg}[-i + h]\Big]\Big)^2 + \Big(-(1 + h^2)^{1/4}\,\text{Sin}\Big[\frac{1}{2}\,\text{Arg}[i - h]\Big] + h\,(1 + h^2)^{1/4}\,\text{Sin}\Big[\frac{1}{2}\,\text{Arg}[-i + h]\Big]\Big)^2\Bigg)}\Bigg)\Bigg]$$

The next two calculations are for the denominator of Abar(h) in the paper. We do the inner integral, then the outer.

In[ ]:=
```
Integrate[1, {r, h Tan[θ], 1}]
```
Out[ ]=

$$1 - h\,\text{Tan}[\theta]$$

In[ ]:=
```
Assuming[h > 0, Integrate[1 - h Tan[θ], {θ, 0, ArcTan[1/h]}]]
```
Out[ ]=

$$\text{ArcTan}\left[\frac{1}{h}\right] - \frac{1}{2}\,h\,\text{Log}\left[1 + \frac{1}{h^2}\right]$$

This allows us to define Abar(h)

In[ ]:=
```
Abarh[h_] = Pi / (4 (ArcTan[1/h] - 1/2 h Log[1 + 1/h^2]))
```
Out[ ]=

$$\frac{\pi}{4\left(\text{ArcTan}\left[\frac{1}{h}\right] - \frac{1}{2}\,h\,\text{Log}\left[1 + \frac{1}{h^2}\right]\right)}$$

This next code calculates the integrand of f(h) in the paper.

First, this is trying to evaluate f(h) without any change of coordinates. I aborted this computation after one hour on my 2020 Macbook Pro.

In[2]:=

```
Assuming[0 < h < 1,
  Integrate[(r + h r Cos[θ]/Sqrt[r^2 - h^2 Sin[θ]^2])^2, {θ, 0, ArcTan[1/h]}, {r, h Tan[θ], 1}]]
```

$$\text{Assuming}\left[0<h<1,\ \text{Integrate}\left[\left(r+\frac{h\,r\,\text{Cos}[\theta]}{\sqrt{r^2-h^2\,\text{Sin}[\theta]^2}}\right)\text{^}2,\ \{\theta,0,\text{ArcTan}[1/h]\},\ \{r,\ h\,\text{Tan}[\theta],1\}\right]\right]$$

Out[2]= $Aborted

when we change coordinates from the onion coordinate system to rectangular coordinates

In[ ]:=

```
Clear[x, y]
```

In[ ]:=

```
Assuming[x > 0 && y > 0 && h > 0, FullSimplify[Jh[Sqrt[x^2 + y^2], ArcTan[x / (y + h)], h]]]
```

Out[ ]=

$$\frac{\sqrt{x^2+y^2}\,\left(x^2+(h+y)^2\right)}{x^2+y\,(h+y)}$$

In[ ]:=

$$\text{integrand}[x\_,\ y\_,\ h\_] := \frac{\sqrt{x^2+y^2}\,\left(x^2+(h+y)^2\right)}{x^2+y\,(h+y)}$$

In[ ]:=

```
Assuming[s > 0, FullSimplify[integrand[s Cos[ϕ], s Sin[ϕ], h]]]
```

Out[ ]=

$$\frac{h^2+s^2+2\,h\,s\,\text{Sin}[\phi]}{s+h\,\text{Sin}[\phi]}$$

Finally, we calculate f(h) by performing the double integral in polar coordinates from the paper

In[ ]:=

$$\text{Assuming}\left[0<h<1,\ \left(\int_0^1\int_0^{\pi/2}\left(\frac{h^2+s^2+2\,h\,s\,\text{Sin}[\phi]}{s+h\,\text{Sin}[\phi]}\right)s\,d\phi\,ds\right)\right]$$

Out[ ]=

$$\frac{1}{6}\left(h+2\,\pi+4\,\left(1-h^2\right)^{3/2}\,\text{ArcSin}[h]-4\,\left(1-h^2\right)^{3/2}\,\text{ArcTan}\left[\frac{1+h}{\sqrt{1-h^2}}\right]+h^3\,\text{Log}\left[4\,h^2\right]\right)$$

so, we can define f(h)

In[ ]:=

$$f[h\_] = \frac{1}{6}\left(h+2\,\pi+4\,\left(1-h^2\right)^{3/2}\,\text{ArcSin}[h]-4\,\left(1-h^2\right)^{3/2}\,\text{ArcTan}\left[\frac{1+h}{\sqrt{1-h^2}}\right]+h^3\,\text{Log}\left[4\,h^2\right]\right);$$

So, by the formula for $\sigma^2(h)$ in the paper,

In[ ]:=

$$\sigma 2h[h\_] = f[h] \Big/ \left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)-\text{Abarh}[h]\text{^}2$$

Out[ ]=

$$-\frac{\pi^2}{16\left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)^2}+\frac{h+2\,\pi+4\,\left(1-h^2\right)^{3/2}\,\text{ArcSin}[h]-4\,\left(1-h^2\right)^{3/2}\,\text{ArcTan}\left[\frac{1+h}{\sqrt{1-h^2}}\right]+h^3\,\text{Log}\left[4\,h^2\right]}{6\left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)}$$

And, making different assumptions, we can calculate the limit as h->infinity:

In[ ]:=
$$\text{Assuming}\left[h>1,\left(\int_0^1\int_0^{\pi/2}\left(\frac{h^2+s^2+2\,h\,s\,\text{Sin}[\phi]}{s+h\,\text{Sin}[\phi]}\right)s\,d\phi\,ds\right)\right]$$

Out[ ]=
$$\frac{1}{6}\left(h+2\pi+2h^3\,\text{Log}[2h]-2\sqrt{-1+h^2}\left(\left(1+2h^2\right)\text{Log}\left[h-\sqrt{-1+h^2}\right]+3h^2\,\text{Log}\left[h+\sqrt{-1+h^2}\right]\right)\right)$$

In[ ]:=
$$f[h\_]=\frac{1}{6}\left(h+2\pi+2h^3\,\text{Log}[2h]-2\sqrt{-1+h^2}\left(\left(1+2h^2\right)\text{Log}\left[h-\sqrt{-1+h^2}\right]+3h^2\,\text{Log}\left[h+\sqrt{-1+h^2}\right]\right)\right);$$

In[ ]:=
$$\sigma 2h[h\_]=f[h]\Big/\left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)-\text{Abarh}[h]\,\hat{}\,2$$

Out[ ]=
$$-\frac{\pi^2}{16\left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)^2}+\frac{h+2\pi+2h^3\,\text{Log}[2h]-2\sqrt{-1+h^2}\left(\left(1+2h^2\right)\text{Log}\left[h-\sqrt{-1+h^2}\right]+3h^2\,\text{Log}\left[h+\sqrt{-1+h^2}\right]\right)}{6\left(\text{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)}$$

In[ ]:=
```
Limit[σ2h[h], h → Infinity]
```

Out[ ]=
$$\infty$$

The Optimization Problem

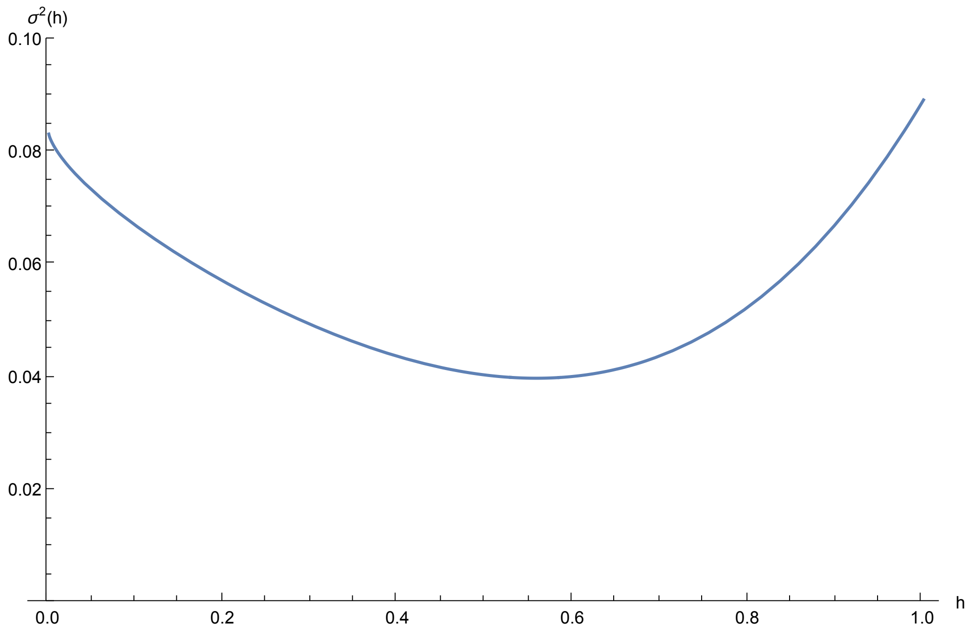
In[ ]:=
```
Assuming[h > 0, FullSimplify[D[σ2h[h], h]]]
```

Out[ ]=
$$\left(-3\pi^2\,\text{Log}\left[1+\frac{1}{h^2}\right]-2\,\text{Log}\left[1+\frac{1}{h^2}\right]\left(-2\,\text{ArcCot}[h]+h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)\left(h+2\pi-4\left(1-h^2\right)^{3/2}\text{ArcCot}\left[\sqrt{-1+\frac{2}{1+h}}\right]+4\left(1-h^2\right)^{3/2}\text{ArcSin}[h]+h^3\,\text{Log}\left[4h^2\right]\right)+6\left(-2\,\text{ArcCot}[h]+h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)^2\left(1+h\left(4\sqrt{1-h^2}\left(\text{ArcCot}\left[\sqrt{-1+\frac{2}{1+h}}\right]-\text{ArcSin}[h]\right)+h\,\text{Log}\left[4h^2\right]\right)\right)\right)\Big/\left(48\left(\text{ArcCot}[h]-\frac{1}{2}h\,\text{Log}\left[1+\frac{1}{h^2}\right]\right)^3\right)$$

We can plot $\sigma^2(h)$ vs h and see that there is a minimum around h=.557

In[ ]:=
```
Plot[σ2h[h], {h, 0, 1}, AxesLabel → {"h", "σ²(h)"}, PlotRange → {0, .1}]
```

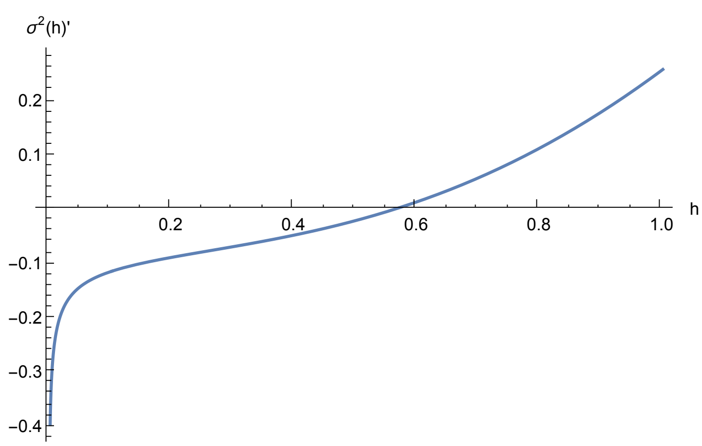
Out[ ]=



We can plot σ^2 (h) ' vs h and see that there is a zero with a sign change from negative to positive around h=.557

In[ ]:=
```
Plot[σ2h'[h], {h, 0, 1}, AxesLabel → {"h", "σ²(h)'"}]
```

Out[ ]=



In[ ]:=
```
FindRoot[σ2h'[h], {h, .5}]
```

Out[ ]=

{h → 0.557307}

The New York Times and Mathematical Community

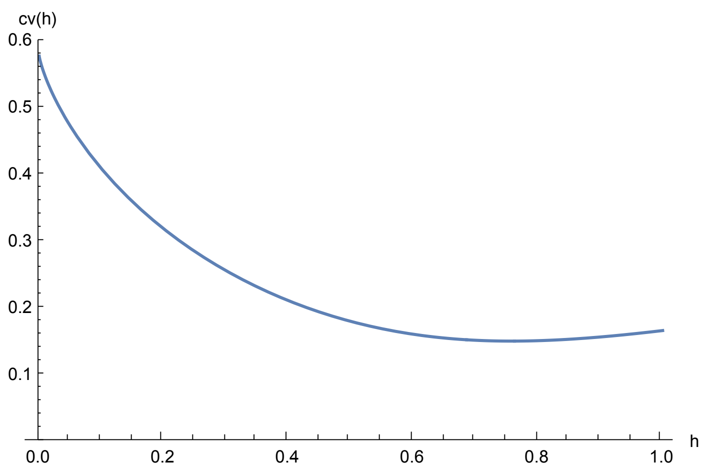
In[∘]:= `cv[h_] = Sqrt[σ2h[h]] / Abarh[h]`

Out[∘]=

$$\frac{1}{\pi}\,4\left(\mathrm{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\mathrm{Log}\left[1+\frac{1}{h^2}\right]\right)\sqrt{\left(-\frac{\pi^2}{16\left(\mathrm{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\mathrm{Log}\left[1+\frac{1}{h^2}\right]\right)^2}+\frac{h+2\pi+4\left(1-h^2\right)^{3/2}\mathrm{ArcSin}[h]-4\left(1-h^2\right)^{3/2}\mathrm{ArcTan}\left[\frac{1+h}{\sqrt{1-h^2}}\right]+h^3\,\mathrm{Log}\left[4\,h^2\right]}{6\left(\mathrm{ArcTan}\left[\frac{1}{h}\right]-\frac{1}{2}\,h\,\mathrm{Log}\left[1+\frac{1}{h^2}\right]\right)}\right)}$$

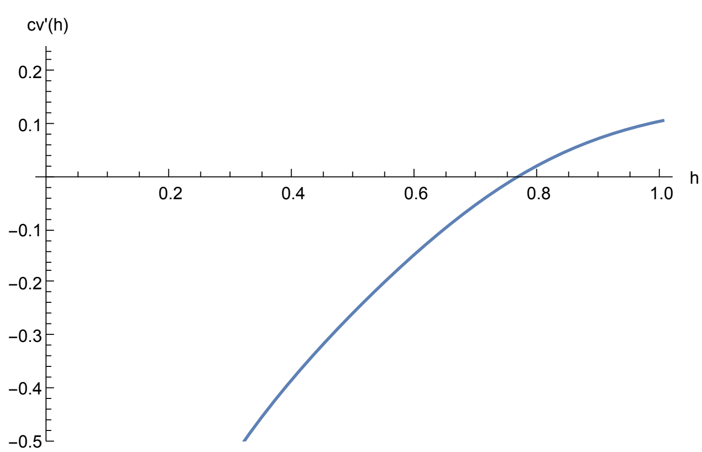
In[∘]:= `Plot[cv[h], {h, 0, 1}, AxesLabel → {"h", "cv(h)"}, PlotRange → {0, .6}]`

Out[∘]=



In[∘]:= `Plot[cv'[h], {h, 0, 1}, AxesLabel → {"h", "cv'(h)"}, PlotRange → {-.5, .25}]`

Out[∘]=



In[∘]:= `FindRoot[cv'[h], {h, .5}]`

Out[∘]=

{h → 0.757426}

Three Dimensional Onions

c_h, x_h, y_h, and J_h from the paper:

In[ ]:=
```
ch[r_, ϕ_, z_, h_] := h Cos[ϕ] + Sqrt[r^2 - z^2 - h^2 Sin[ϕ]^2];
xh[r_, ϕ_, z_, h_] := ch[r, ϕ, z, h] Sin[ϕ];
yh[r_, ϕ_, z_, h_] :=  ch[r, ϕ, z, h] Cos[ϕ] - h;
Jh[r_, ϕ_, z_, h_] :=
 D[xh[r, ϕ, z, h], r] × D[yh[r, ϕ, z, h], ϕ] - D[xh[r, ϕ, z, h], ϕ] × D[yh[r, ϕ, z, h], r]
```

Note that the absolute value of the Jacobian is the below expression with the absolute value removed, since r>h tan(θ) and tan(θ)>sin(θ) on this interval. Moreover, r>0,h>0 and cos(θ)>0.

In[ ]:=
```
FullSimplify[Jh[r, ϕ, z, h]]
```

Out[ ]=

$$r + \frac{h\, r\, \mathrm{Cos}[\phi]}{\sqrt{r^2 - z^2 - h^2\, \mathrm{Sin}[\phi]^2}}$$

Let’s redefine Jh to be positive:

In[ ]:=

$$\mathrm{Jh}[r\_, \phi\_, z\_, h\_] := r\left(1 + \frac{h\, \mathrm{Cos}[\phi]}{\sqrt{r^2 - z^2 - h^2\, \mathrm{Sin}[\phi]^2}}\right)$$

Simplifying the denominator doing the integrals from inside to outside.

In[ ]:=
```
Assuming[w > 0 && w < 1, Integrate[1, {r, Sqrt[h^2 Tan[θ]^2 + w^2], 1}]]
```

Out[ ]=

$$1 - \sqrt{w^2 + h^2\, \mathrm{Tan}[\theta]^2}$$

$$\mathrm{Assuming}\left[h > 0\, \&\&\, w > 0\, \&\&\, w < 1,\ \mathrm{Integrate}\left[1 - \sqrt{w^2 + h^2\, \mathrm{Tan}[\phi]^2},\ \{\phi, 0, \mathrm{ArcTan}[\mathrm{Sqrt}[1 - w\verb|^|2] / h]\}\right]\right]$$

Out[ ]=

```
$Aborted
```

In[∘]:=
```
Assuming[h > 0, Integrate[
  ArcCot[h/Sqrt[1 - w^2]] + Sqrt[(h - w) (h + w)] (ArcCoth[(h Sqrt[(h - w) (h + w)])/(1 + h^2 - w^2 - Sqrt[1 - w^2])] - ArcTanh[h/Sqrt[h^2 - w^2]]) +
   h Log[(1 - Sqrt[1 - w^2])/w], {w, 0, 1}]]
```

Out[∘]=

$$\int_0^1 \left(\text{ArcCot}\left[\frac{h}{\sqrt{1-w^2}}\right] + \sqrt{(h-w)\ (h+w)}\left(\text{ArcCoth}\left[\frac{h\sqrt{(h-w)\ (h+w)}}{1+h^2-w^2-\sqrt{1-w^2}}\right] - \text{ArcTanh}\left[\frac{h}{\sqrt{h^2-w^2}}\right]\right) + h\,\text{Log}\left[\frac{1-\sqrt{1-w^2}}{w}\right]\right) dw$$

```
denom3D[h_] := NIntegrate[(ArcCot[h/Sqrt[1 - w^2]] +
    Sqrt[(h - w) (h + w)] (ArcCoth[(h Sqrt[(h - w) (h + w)])/(1 + h^2 - w^2 - Sqrt[1 - w^2])] - ArcTanh[h/Sqrt[h^2 - w^2]]) +
    h Log[(1 - Sqrt[1 - w^2])/w]), {w, 0, 1}, Method → "GlobalAdaptive"]

Vbar[h_] := (Pi / 6) / denom3D[h]
```

The following code mimics the change to rectangular and then to polar of the integral of the square of the Jacobian.

Now, in anticipation of the square integral, we evaluate the Jacobian at r, θ, and w

In[∘]:=
```
Assuming[h > 0 && x > 0 && x < 1 && y > 0 && y < 1 && z > 0 && z < 1,
 J[Sqrt[x^2 + y^2 + z^2], ArcTan[x / (y + h)], z, h]]
```

Out[∘]=

$$\left(1 + \frac{h}{\sqrt{1+\frac{x^2}{(h+y)^2}}\ \sqrt{x^2+y^2-\frac{h^2\,x^2}{(h+y)^2\left(1+\frac{x^2}{(h+y)^2}\right)}}}\right)\sqrt{x^2+y^2+z^2}$$

J in rectangular coordinates

In[∘]:=
$$\text{JinRect}[x\_, y\_, z\_, h\_] = \left(1 + \frac{h}{\sqrt{1 + \frac{x^2}{(h+y)^2}}\ \sqrt{x^2 + y^2 - \frac{h^2 x^2}{(h+y)^2 \left(1 + \frac{x^2}{(h+y)^2}\right)}}}\right) \sqrt{x^2 + y^2 + z^2}$$

Out[∘]=
$$\left(1 + \frac{h}{\sqrt{1 + \frac{x^2}{(h+y)^2}}\ \sqrt{x^2 + y^2 - \frac{h^2 x^2}{(h+y)^2 \left(1 + \frac{x^2}{(h+y)^2}\right)}}}\right) \sqrt{x^2 + y^2 + z^2}$$

Define spherical coordiantes

In[∘]:=
```
x[ρ_, ω_, ϕ_] = ρ Sin[ω] Cos[ϕ];
y[ρ_, ω_, ϕ_] = ρ Sin[ω] Sin[ϕ];
z[ρ_, ω_, ϕ_] = ρ Cos[ω];
```

In[∘]:=
```
JinRect[x[ρ, ω, ϕ], y[ρ, ω, ϕ], z[ρ, ω, ϕ], h] ρ^2 Sin[ω]
```

Out[∘]=
$$\rho^2 \text{Sin}[\omega]\ \sqrt{\rho^2 \text{Cos}[\omega]^2 + \rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2 + \rho^2 \text{Sin}[\phi]^2 \text{Sin}[\omega]^2} \left(1 + h \middle/ \left(\sqrt{1 + \frac{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h + \rho \text{Sin}[\phi] \text{Sin}[\omega])^2}}\ \sqrt{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2 + \rho^2 \text{Sin}[\phi]^2 \text{Sin}[\omega]^2 - \frac{h^2 \rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h + \rho \text{Sin}[\phi] \text{Sin}[\omega])^2 \left(1 + \frac{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h+\rho \text{Sin}[\phi] \text{Sin}[\omega])^2}\right)}}\right)\right)$$

In[∘]:=
$$\text{Assuming}\Big[0 < h \,\&\&\, 0 < \rho \,\&\&\, \rho < 1 \,\&\&\, 0 < \omega \,\&\&\, \omega < \text{Pi}/2 \,\&\&\, 0 < \phi \,\&\&\, \phi < \text{Pi}/2,$$
$$\text{FullSimplify}\Big[\rho^2 \text{Sin}[\omega]\ \sqrt{\rho^2 \text{Cos}[\omega]^2 + \rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2 + \rho^2 \text{Sin}[\phi]^2 \text{Sin}[\omega]^2} \left(1 + h \middle/ \left(\sqrt{1 + \frac{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h + \rho \text{Sin}[\phi] \text{Sin}[\omega])^2}}\ \sqrt{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2 + \rho^2 \text{Sin}[\phi]^2 \text{Sin}[\omega]^2 - \frac{h^2 \rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h + \rho \text{Sin}[\phi] \text{Sin}[\omega])^2 \left(1 + \frac{\rho^2 \text{Cos}[\phi]^2 \text{Sin}[\omega]^2}{(h+\rho \text{Sin}[\phi] \text{Sin}[\omega])^2}\right)}}\right)\right)\Big]\Big]$$

Out[∘]=
$$\frac{\rho^2 \left(h^2 + \rho \text{Sin}[\omega]\ (2 h \text{Sin}[\phi] + \rho \text{Sin}[\omega])\right)}{h \text{Sin}[\phi] + \rho \text{Sin}[\omega]}$$

This is as good as Mathematica can symbolically manipulate the integral of J in spherical coordinates (which represents J^2 in the strange coordinate system)

```
Integrate[ρ^2 (h^2 + ρ Sin[ω] (2 h Sin[ϕ] + ρ Sin[ω])) / (h Sin[ϕ] + ρ Sin[ω]),
 {ρ, 0, 1}, {ω, 0, Pi / 2}, {ϕ, 0, Pi / 2}]
```

Out[ ]=

$$\int_0^1\int_0^{\frac{\pi}{2}}\rho^2\left(\sqrt{2}\left(\text{ArcTan}\left[\frac{\sqrt{2}\,h}{\sqrt{-2h^2+\rho^2-\rho^2\text{Cos}[2\omega]}}\right]-\text{ArcTan}\left[\frac{\sqrt{2}\,(h+\rho\,\text{Sin}[\omega])}{\sqrt{-2h^2+\rho^2-\rho^2\text{Cos}[2\omega]}}\right]\right)\sqrt{-2h^2+\rho^2-\rho^2\text{Cos}[2\omega]}+\pi\rho\,\text{Sin}[\omega]\right)d\omega\, d\rho$$

But, at least we have somewhat simplified the integral for the numerical algorithms...

In[ ]:=

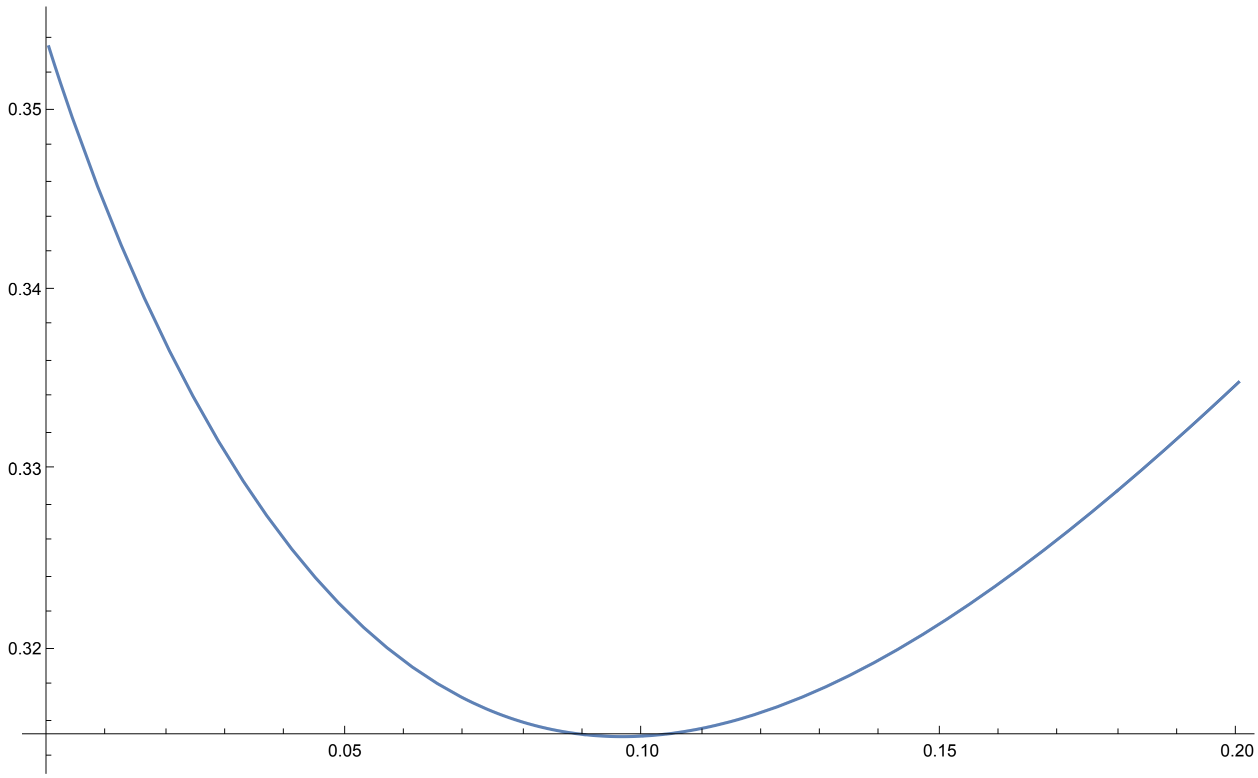
```
numericalJsquared[h_] := NIntegrate[
  ρ^2 (√2 (ArcTan[√2 h / √(-2 h^2 + ρ^2 - ρ^2 Cos[2 ω])] - ArcTan[√2 (h + ρ Sin[ω]) / √(-2 h^2 + ρ^2 - ρ^2 Cos[2 ω])])
       √(-2 h^2 + ρ^2 - ρ^2 Cos[2 ω]) + π ρ Sin[ω]),
  {ω, 0, Pi / 2}, {ρ, 0, 1}, Method → "GlobalAdaptive"]
```

Define the variance as before, with the formula mimicking the formula for variance in terms of the mean and the second moment.

```
var[h_] := numericalJsquared[h] / denom3DGlobal[h] - (Vbar[h]) ^2
```

In[ ]:= `Plot[Sqrt[var[h]] / Vbar[h], {h, 0, .2}]`

Out[ ]=



In[ ]:= `NMinimize[Sqrt[var[h]] / Vbar[h], {h, 0, .2}]`

Out[ ]=

`{0.315803, {h → 0.0816575}}`

This is the 3D onion constant for coefficient of variation!

The following code shows how we calculate for a cutoff. Here, cutoff =.75

```
cutoff = .75;
JSquaredCutoff[h_] := NIntegrate[(J[r, θ, w, h]) ^2 * Boole[r^2 - w^2 - h^2 Sin[θ]^2 > 0],
  {w, 0, cutoff}, {θ, 0, ArcTan[Sqrt[1 - w^2] / h]}, {r, Sqrt[h^2 Tan[θ]^2 + w^2], 1},
  Method → {"GlobalAdaptive", "MaxErrorIncreases" → 30 000},
  Exclusions → r^2 - w^2 - h^2 Sin[θ]^2 == 0]
```

Now the denominator (integral of 1 over the region)

$$\text{denom3DCutoff[h\_] := NIntegrate}\left[\left(\text{ArcCot}\left[\frac{h}{\sqrt{1-w^2}}\right]+\sqrt{(h-w)\ (h+w)}\ \left(\text{ArcCoth}\left[\frac{h\ \sqrt{(h-w)\ (h+w)}}{1+h^2-w^2-\sqrt{1-w^2}}\right]-\text{ArcTanh}\left[\frac{h}{\sqrt{h^2-w^2}}\right]\right)+h\ \text{Log}\left[\frac{1-\sqrt{1-w^2}}{w}\right]\right),\ \{w,\ 0,\ \text{cutoff}\},\ \text{Method}\to\text{"GlobalAdaptive"}\right]$$

We need to adjust the total volume of the onion by removing a spherical cap

```
sphericalCap[c_] = Pi (1 - c) ^2 (3 - (1 - c)) / 3
VbarCutoff[h_] := ((2 / 3 Pi - sphericalCap[cutoff]) / 4) / denom3DCutoff[h]
```

Out[ ]=

$$\frac{1}{3}(1-c)^2(2+c)\pi$$

This is slow to run...

```
ParallelTable[
 {h, Sqrt[JSquaredCutoff[h] / denom3DCutoff[h] - (VbarCutoff[h]) ^2] / VbarCutoff[h]},
 {h, 0.001, 0.75, 0.001}]
```

Out[ ]=

```
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