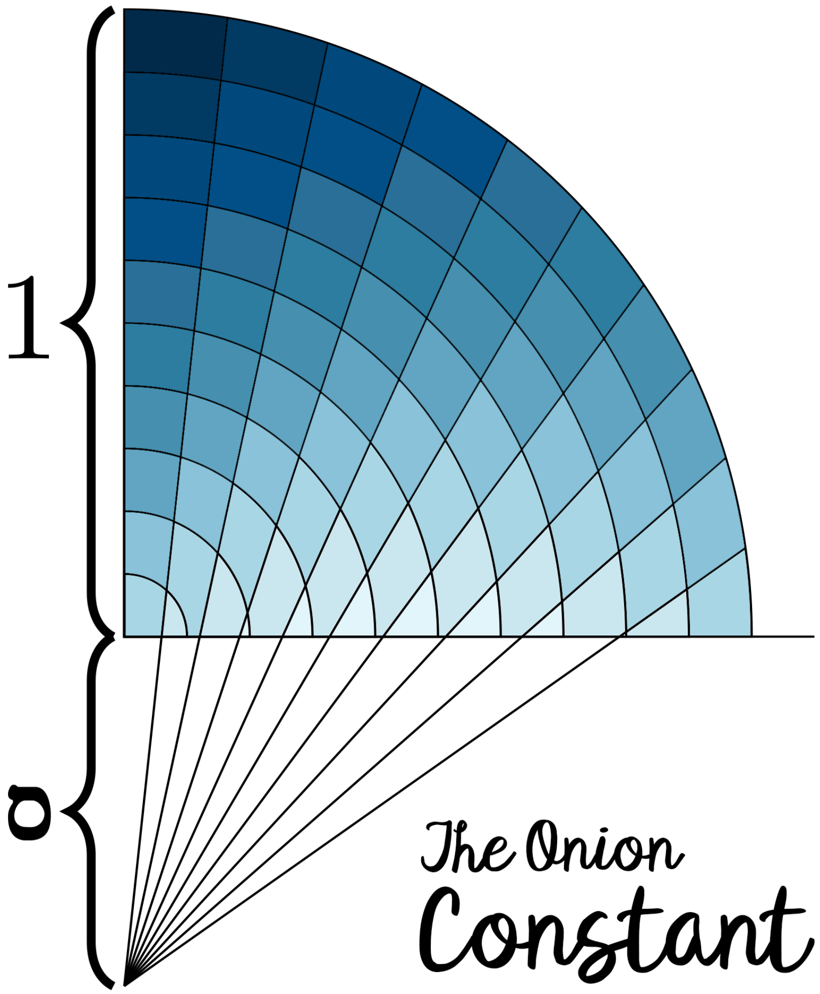
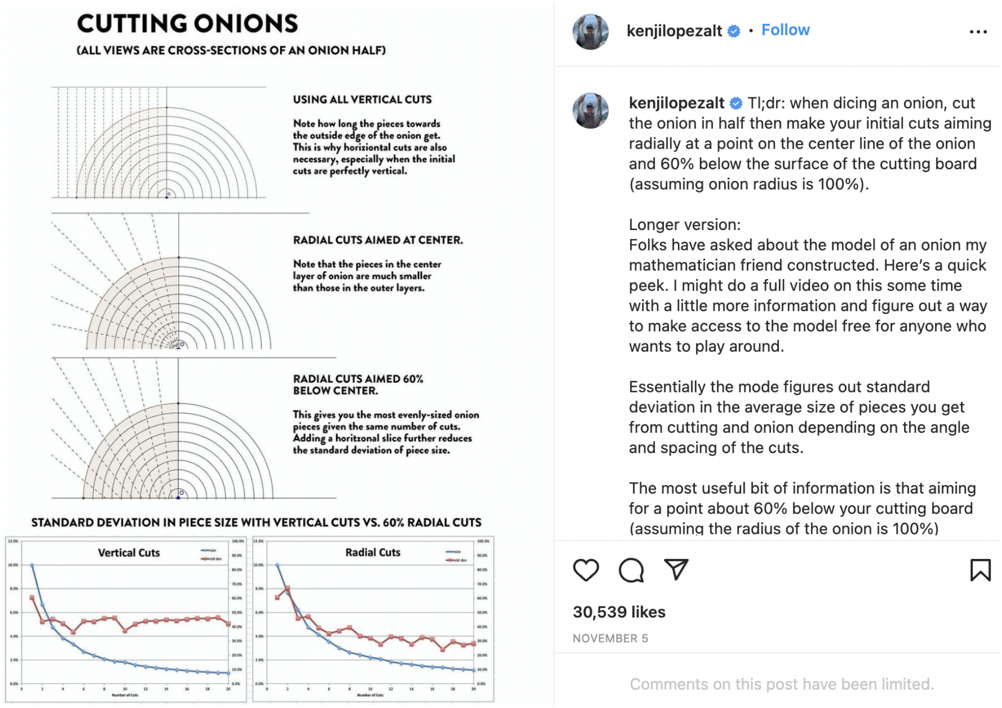


Finding Mathematical Joy Cutting Onions

Dr. Dylan Poulsen
Washington College

Origin of the Problem



Mathematical Set Up

We want to find the depth below the onion to cut towards in order to minimize the **variance** of the volume of each onion slice.

Mathematical Set Up

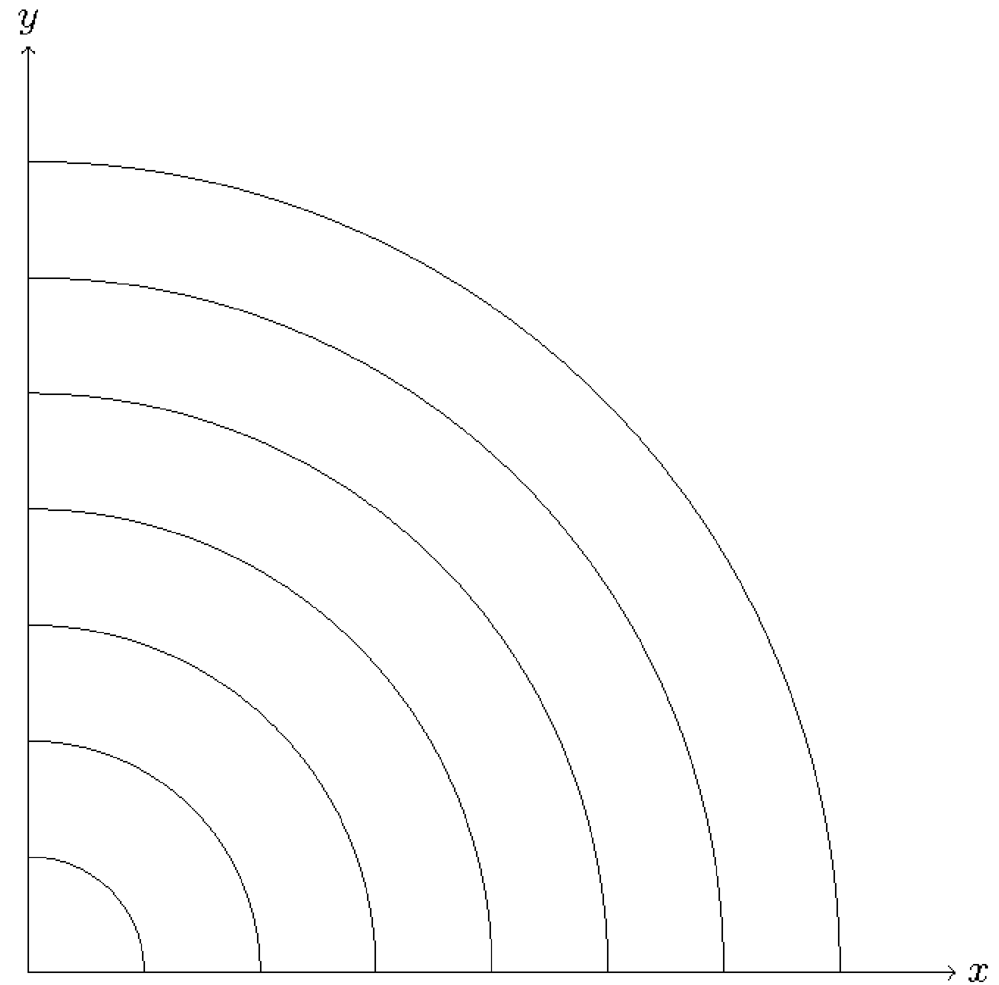
The **variance**, σ^2, of a set of n numbers $S = \{x_1, x_2, \ldots, x_n\}$ whose average value is $\overline{x}$ is

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2.$$

That is, the variance is the average of the square deviations from the mean (this will be important later).

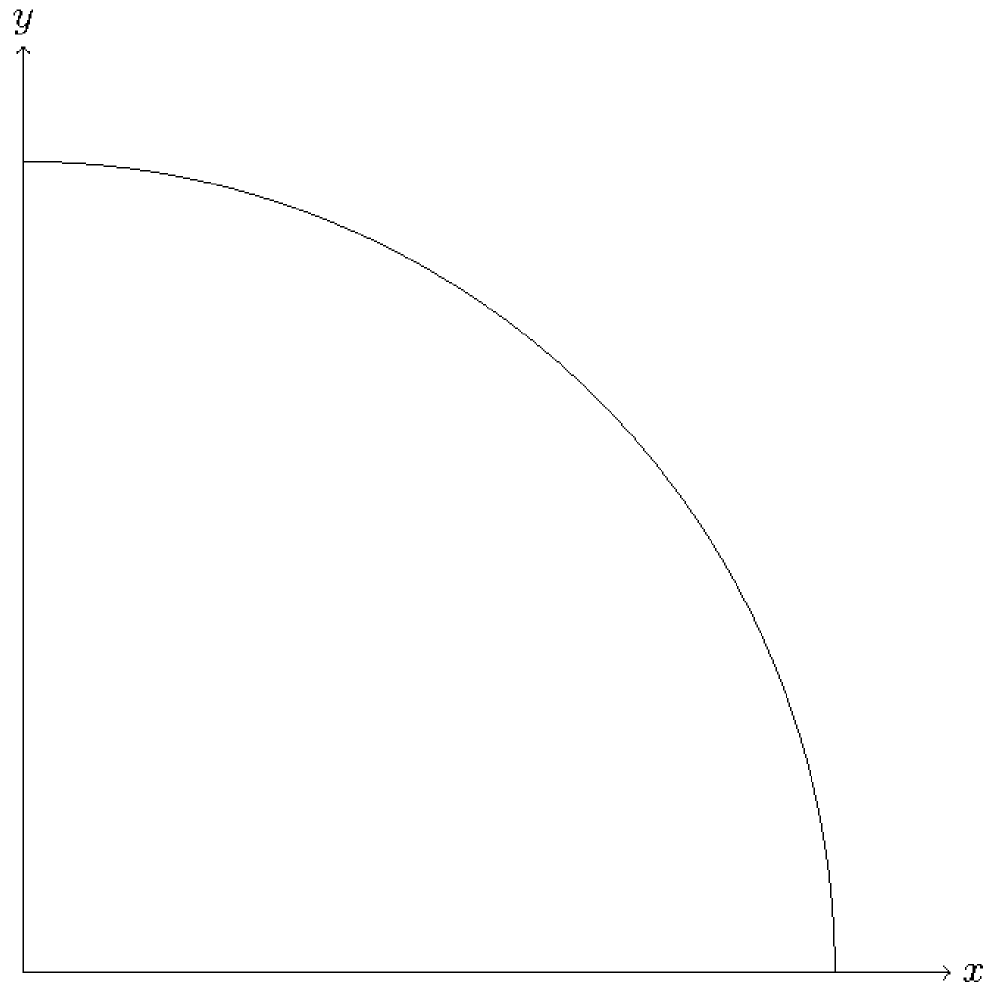
Simplifying the Problem

For simplicity, consider a two-dimensional onion.



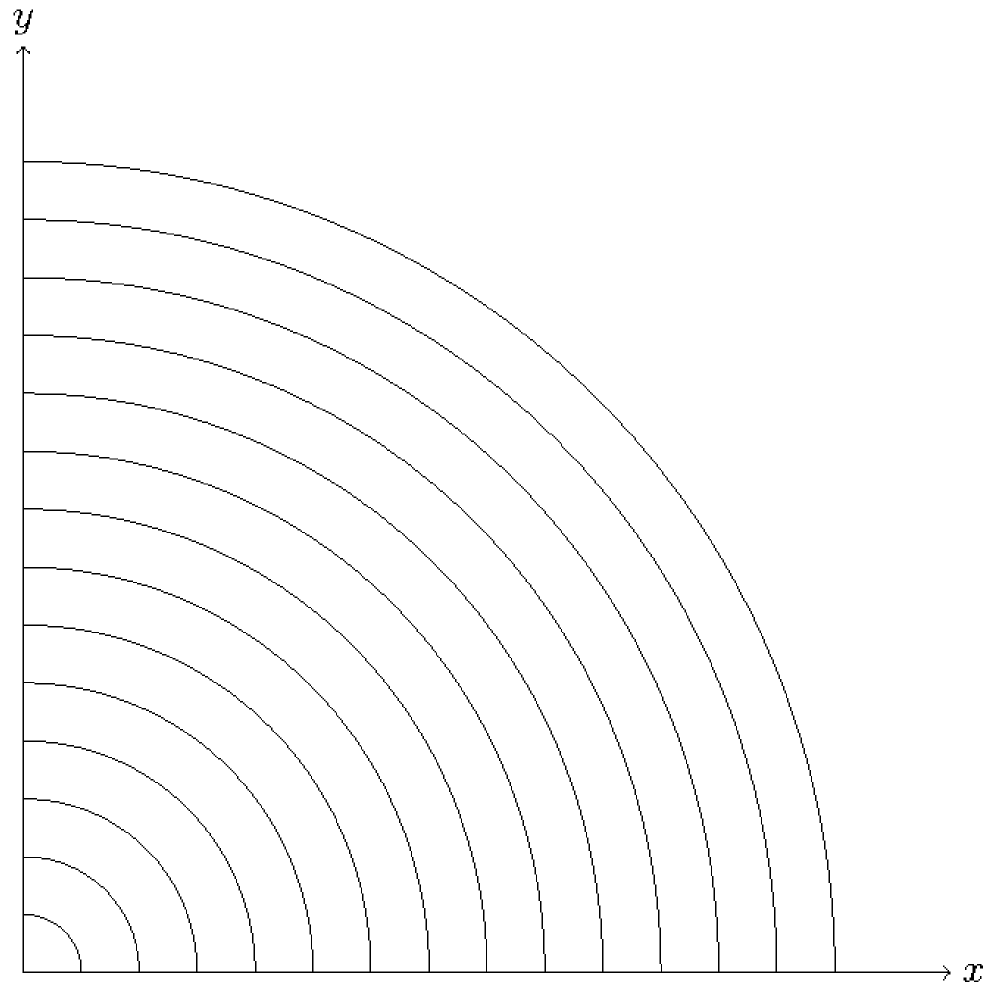
Simplifying the Problem

Insight: The depth to which you have to aim your knife for radial cuts depends on the number of layers.



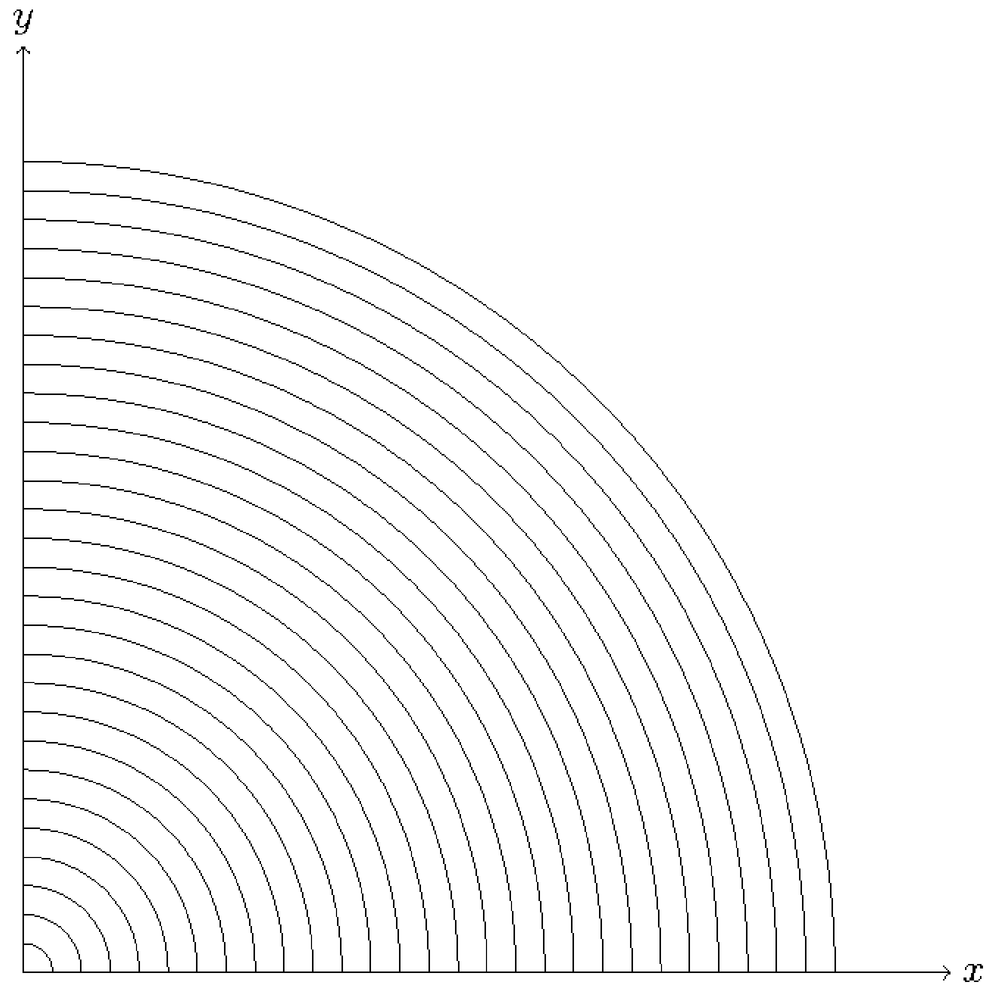
Simplifying the Problem

So, we might as well consider the limiting case as the number of layers approaches infinity.



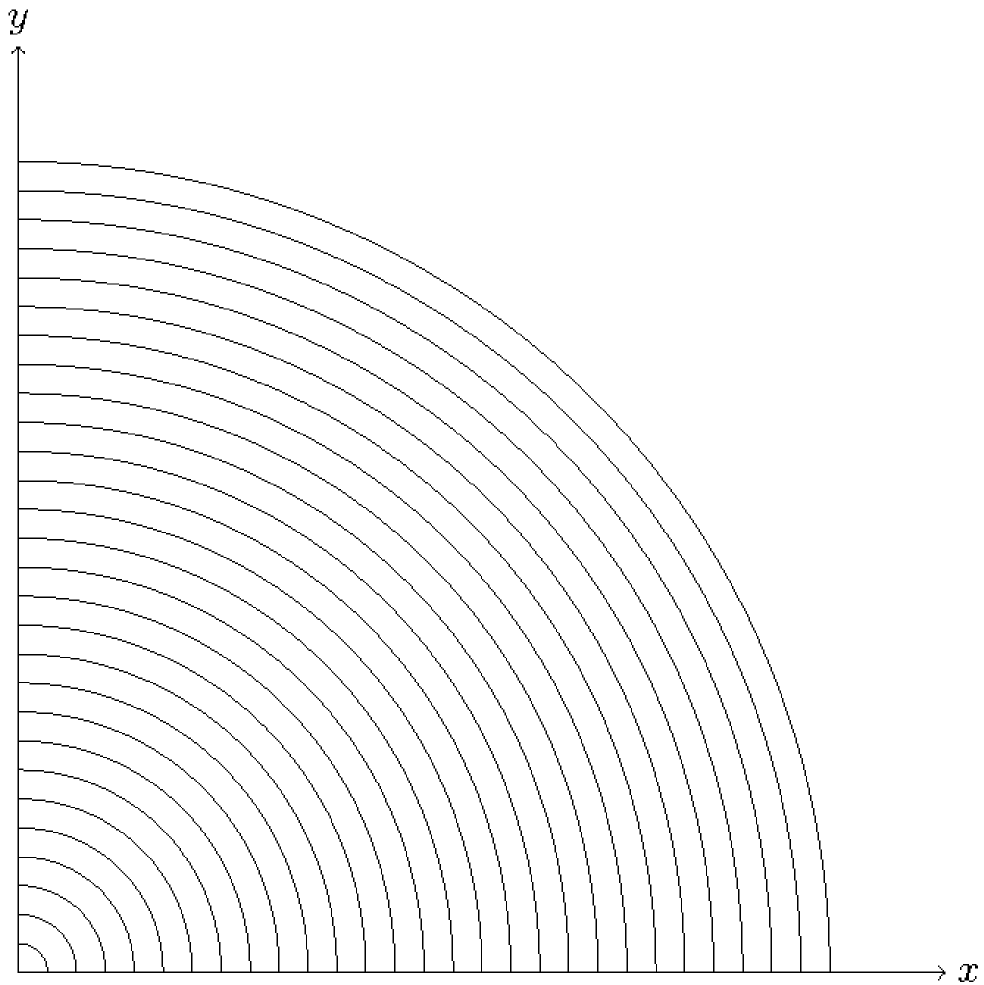
Simplifying the Problem

So, we might as well consider the limiting case as the number of layers approaches infinity.



Simplifying the Problem

Similarly, the number of cuts being made has an effect on the answer. So, for simplicity, we can think of making infinitely many cuts as well.



Live Mathematics

Dylan 11/07/2021
I'm starting to envision a proof of this by taking the limit as the number of cuts goes to infinity, and looking at the infinitesimal volumes

Sort of like Jacobians

I'd really like to see if 1/phi comes out as the "true" best depth to cut towards

Gabe 11/07/2021
🤔

Maybe there's some other onion constant

Dylan 11/07/2021
The constant might also be a function of the number of layers. The true onion constant would then be in the limit as the number of layers goes to infinity...

Gabe 11/07/2021
Ah the great onion in the sky

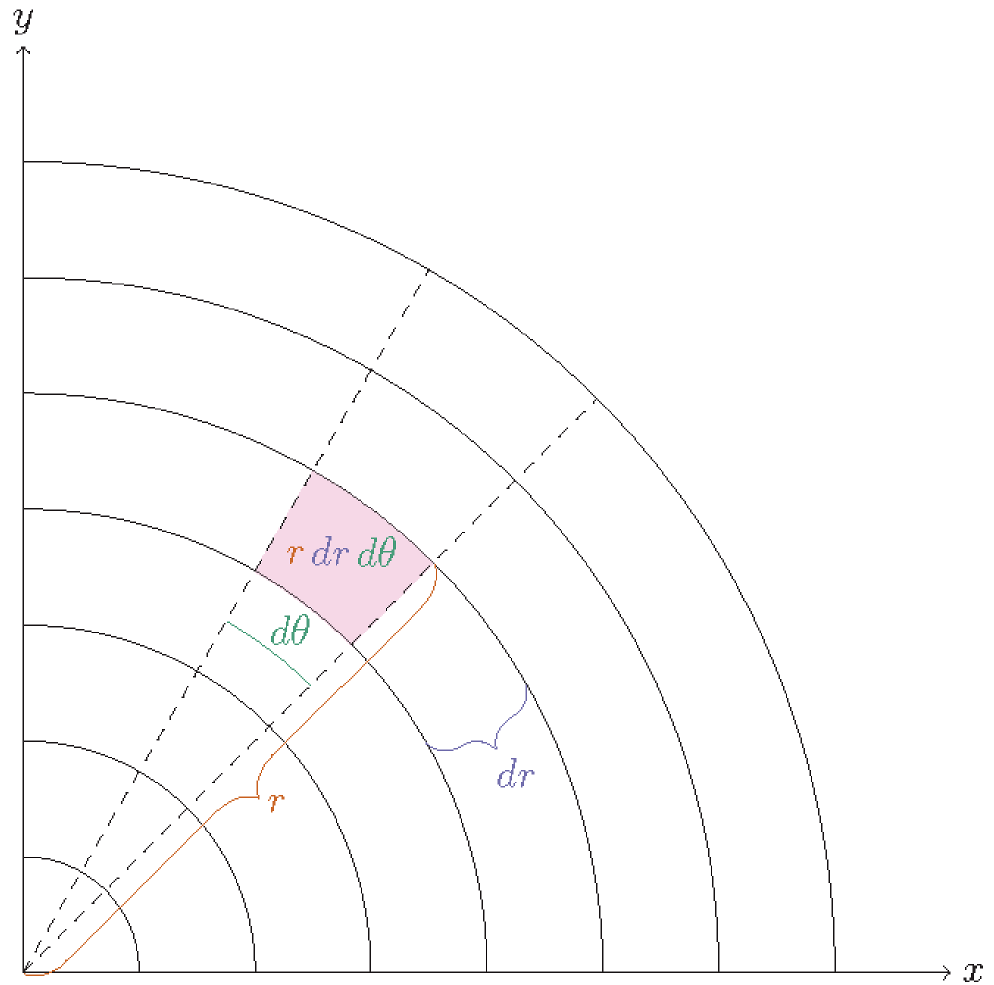
😂 1

Inspiration: The Jacobian!

Rectangular $\to$ Polar:

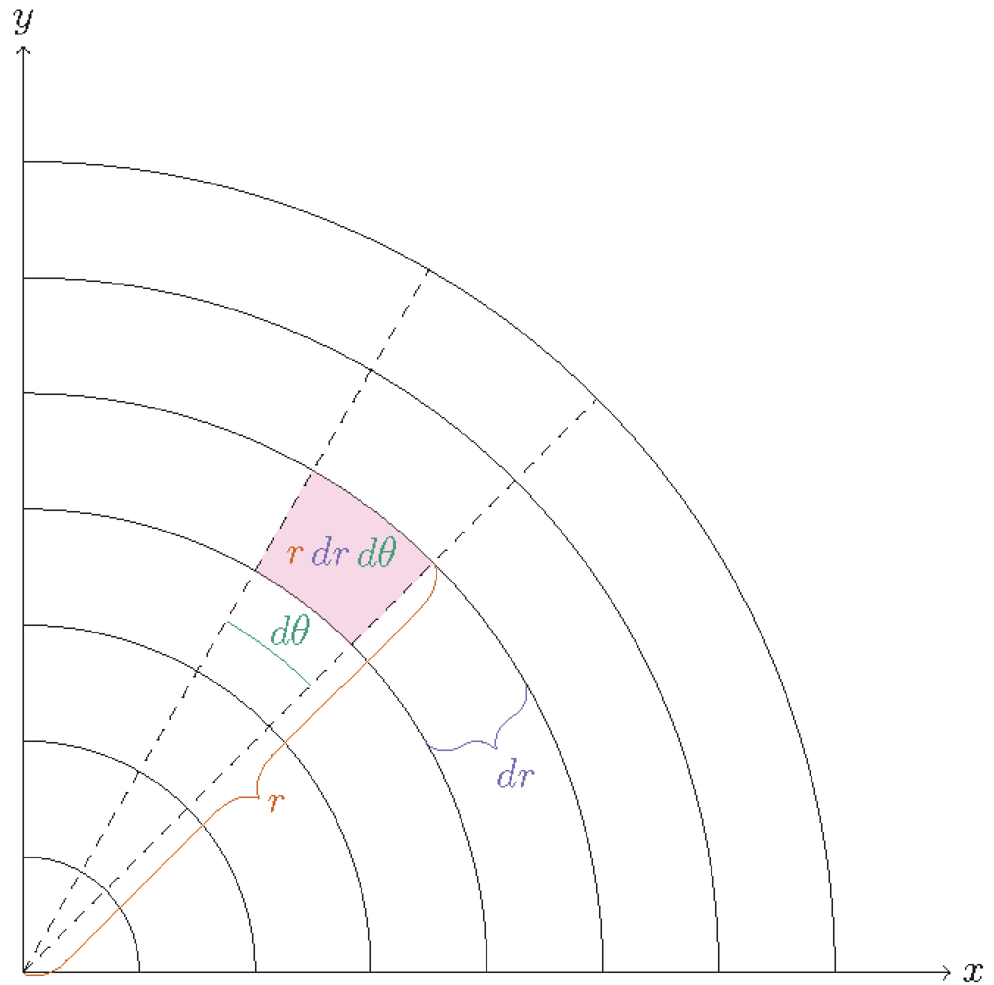
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$J(r,\theta) = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}$$
$$= r\cos^2(\theta) + r\sin^2(\theta)$$
$$= r$$



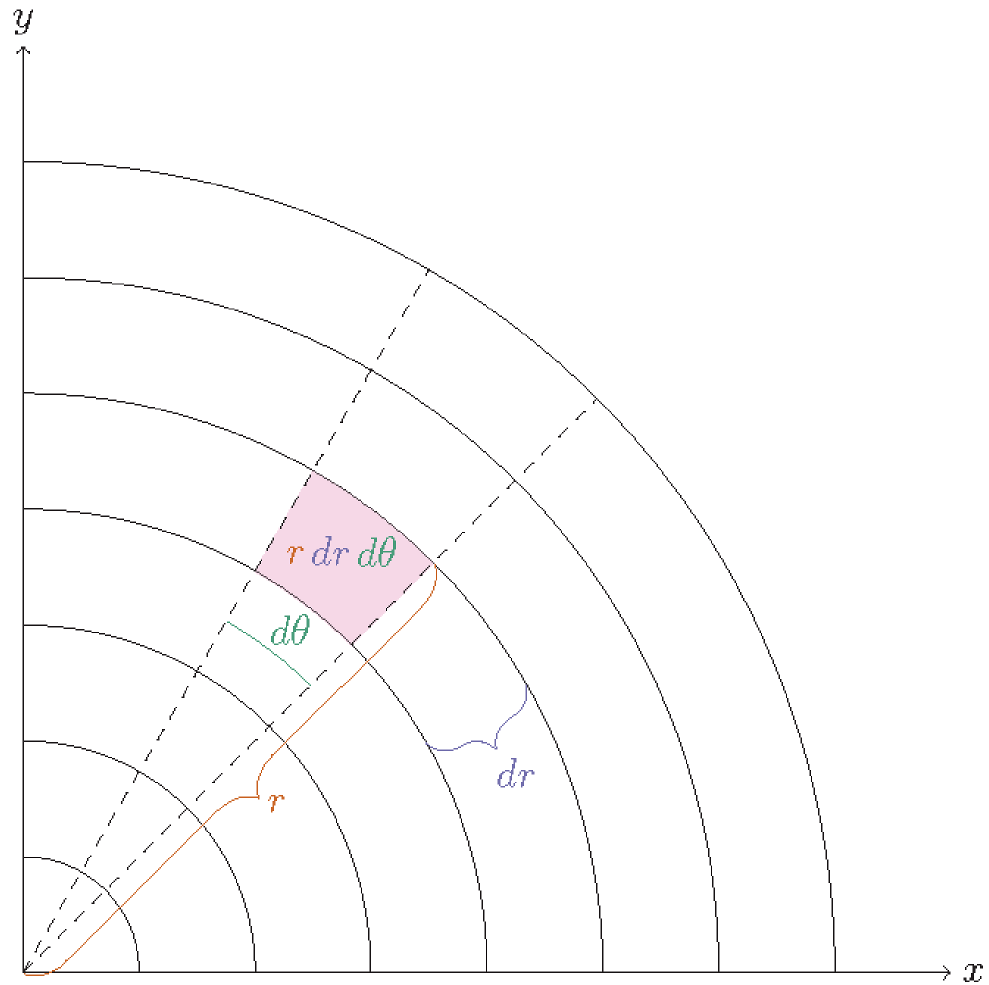
Inspiration: The Jacobian!

Problem: With infinitely many layers and cuts, the area of each piece of onion is zero. So, it is hard to measure variance.



Inspiration: The Jacobian!

Solution: Recognize that the Jacobian $J(r, \theta) = r$ gives a measure of how big the infinitely small pieces are relative to each other. So, we can use the average value of the function $J(r, \theta) = r$ as a stand-in for the average area.



Average of a Function

Fact from Integral Calculus: the average value, $\overline{f}$, of a function f over a region Ω is

$$\overline{f} = \frac{\int_\Omega f \, dV}{\int_\Omega 1 \, dV}.$$

Here, over a quarter onion of radius 1, the average "relative area", $\overline{A}$, is given by (any guesses?)

Average of a Function

$$\overline{A} = \frac{\int_0^{\pi/2} \int_0^1 r \; dr \; d\theta}{\int_0^{\pi/2} \int_0^1 1 \; dr \; d\theta} = \frac{1}{2}$$

Variance of a Function

To generalize the variance we saw earlier, we recall the variance is the average of the square deviations from the mean! So, the variance of our relative area is

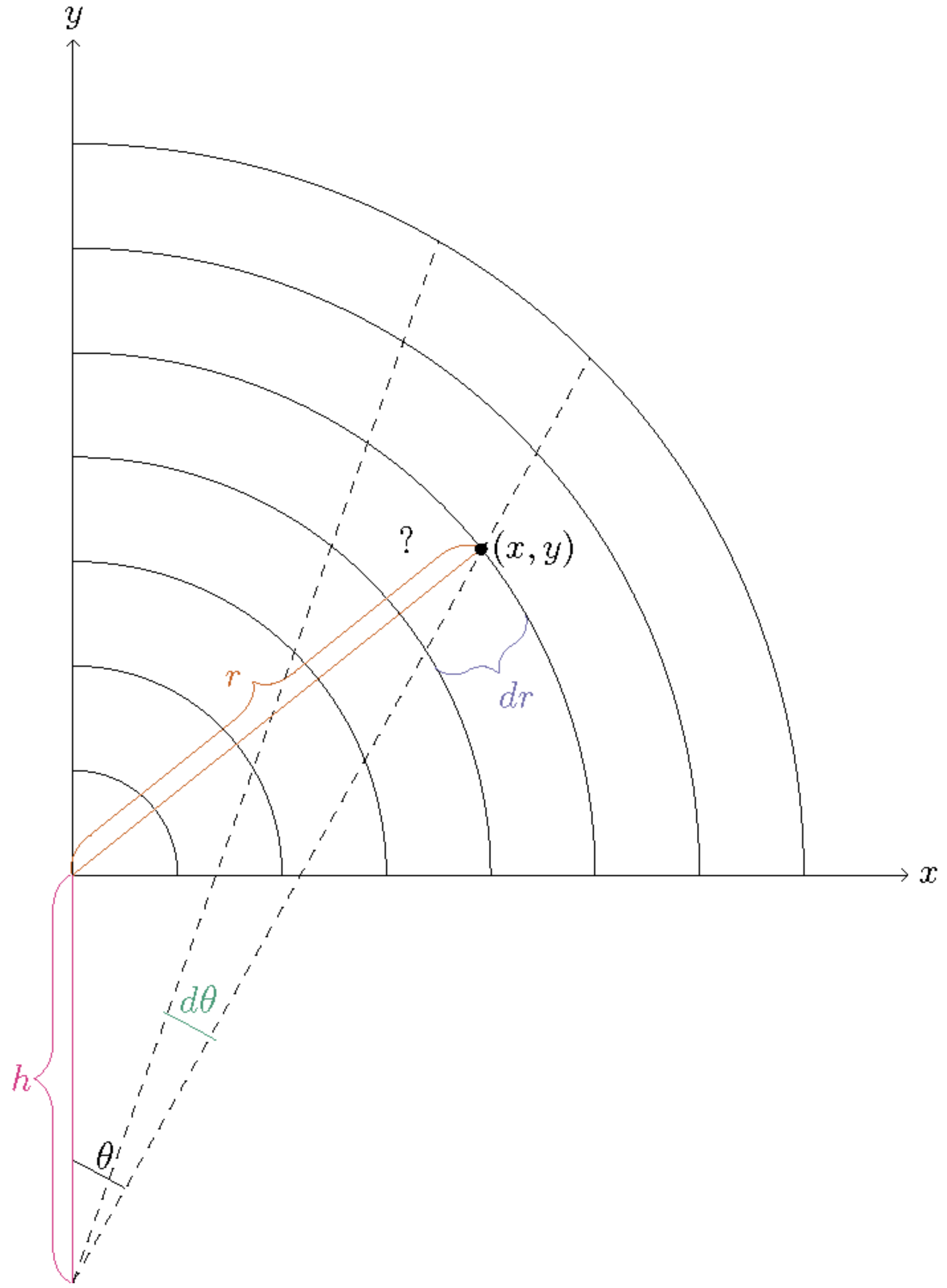
$$\sigma^2 = \frac{\int_0^{\pi/2}\int_0^1 (r - \overline{A})^2 \, dr \, d\theta}{\int_0^{\pi/2}\int_0^1 1 \, dr \, d\theta} = \frac{\int_0^{\pi/2}\int_0^1 (r - 1/2)^2 \, dr \, d\theta}{\int_0^{\pi/2}\int_0^1 1 \, dr \, d\theta} = \frac{1}{12}$$

Rest and Reflect

- All of this is great, but it doesn't answer the question!
- What allowed all this to work was a coordinate system whose axes cut the onion.
- Can we find a coordinate system that cuts the onion in the way described by Chef Kenji Lopez-Alt?

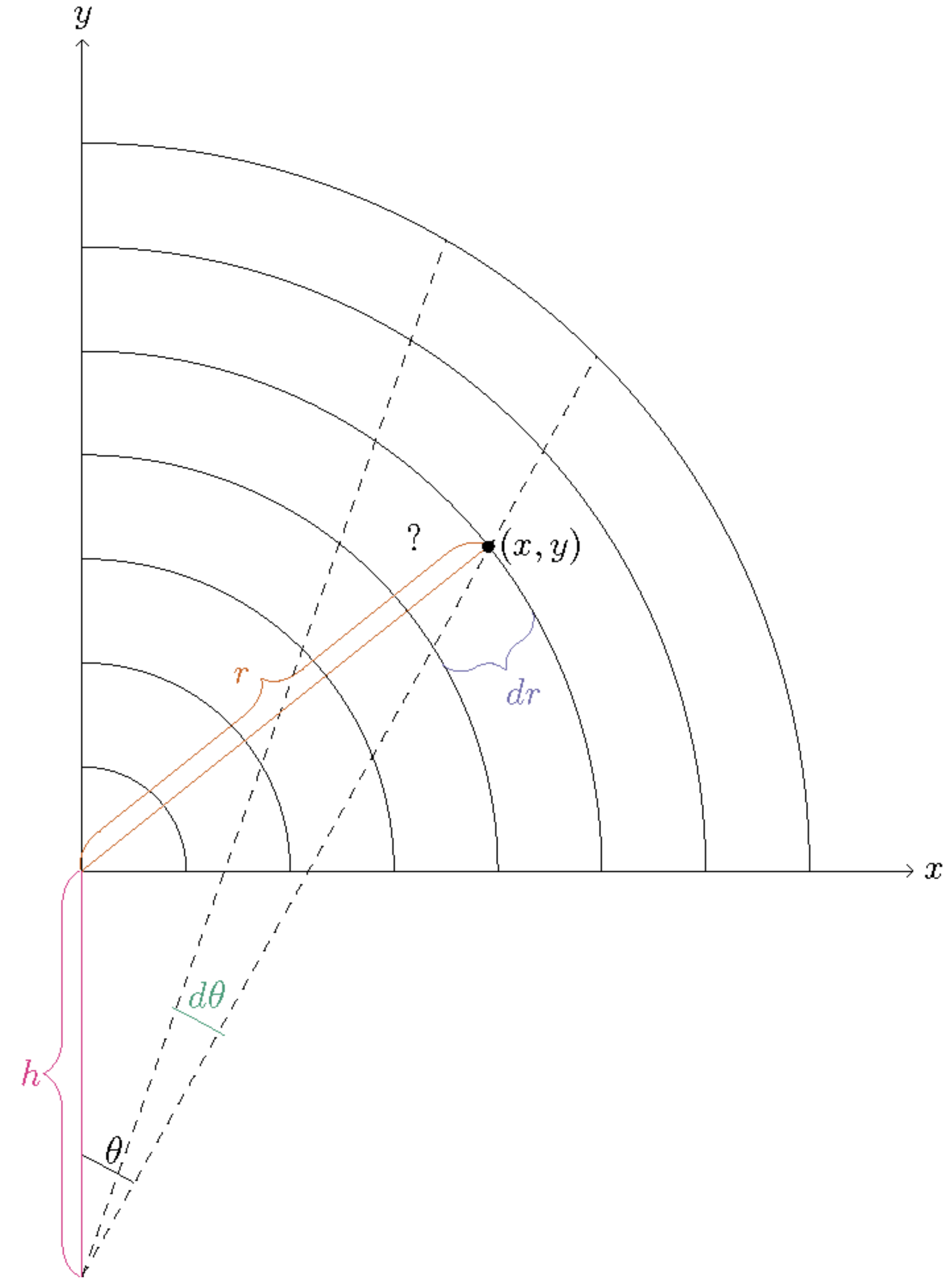
New Coordinate System

We make a coordinate system for cutting towards a point a distance $h > 0$ below the center of the onion. In this coordinate system, we measure the angle θ from the point $(0, -h)$, while we measure the radius from the



New Coordinate System

This coordinate system only works for the upper half plane, as there are now technically two points in the plane for a given point (r, θ).



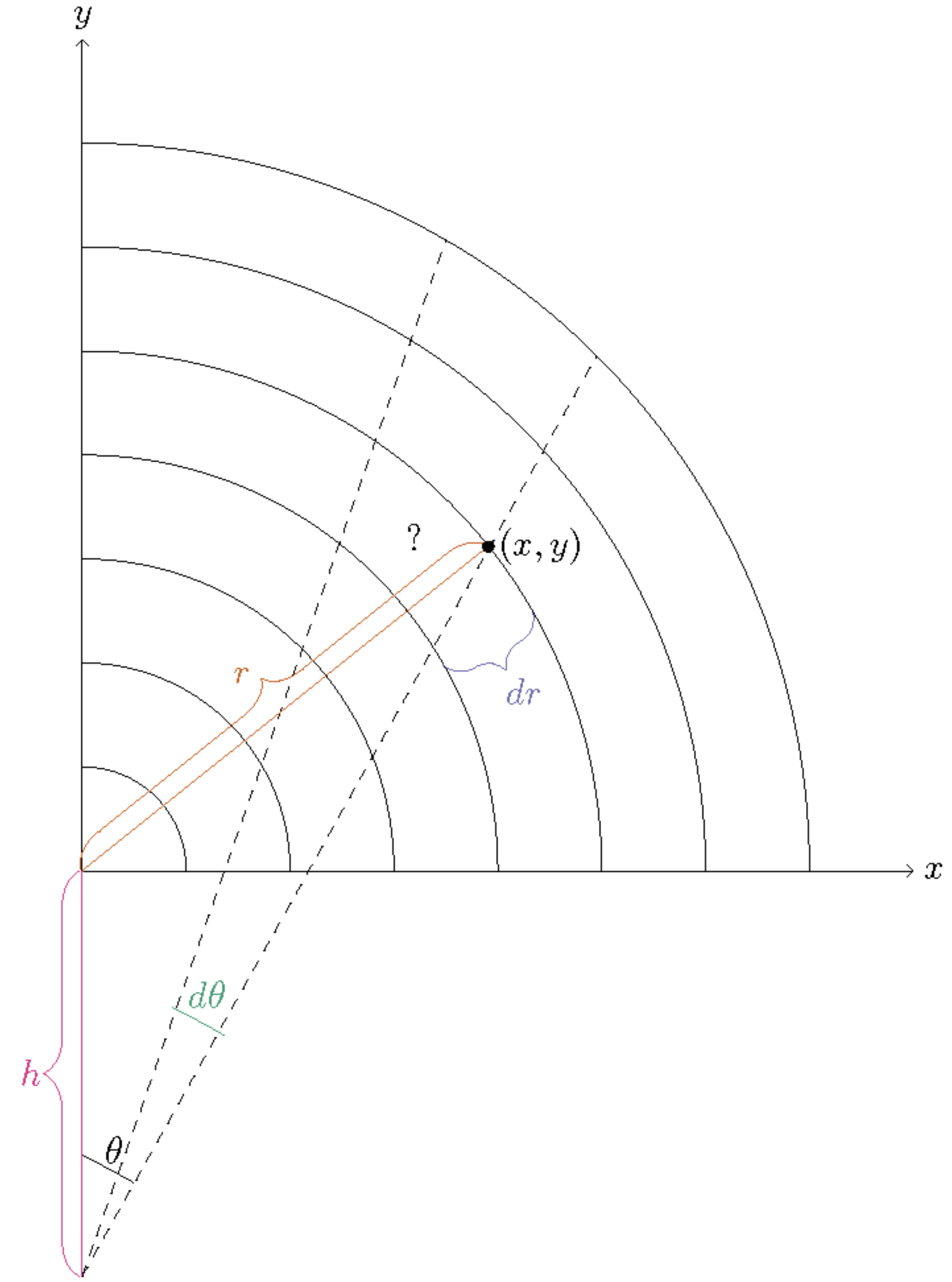
Game Plan

In order to mimic our computation for polar coordinates, we need to

1. define the region and
2. compute the Jacobian.

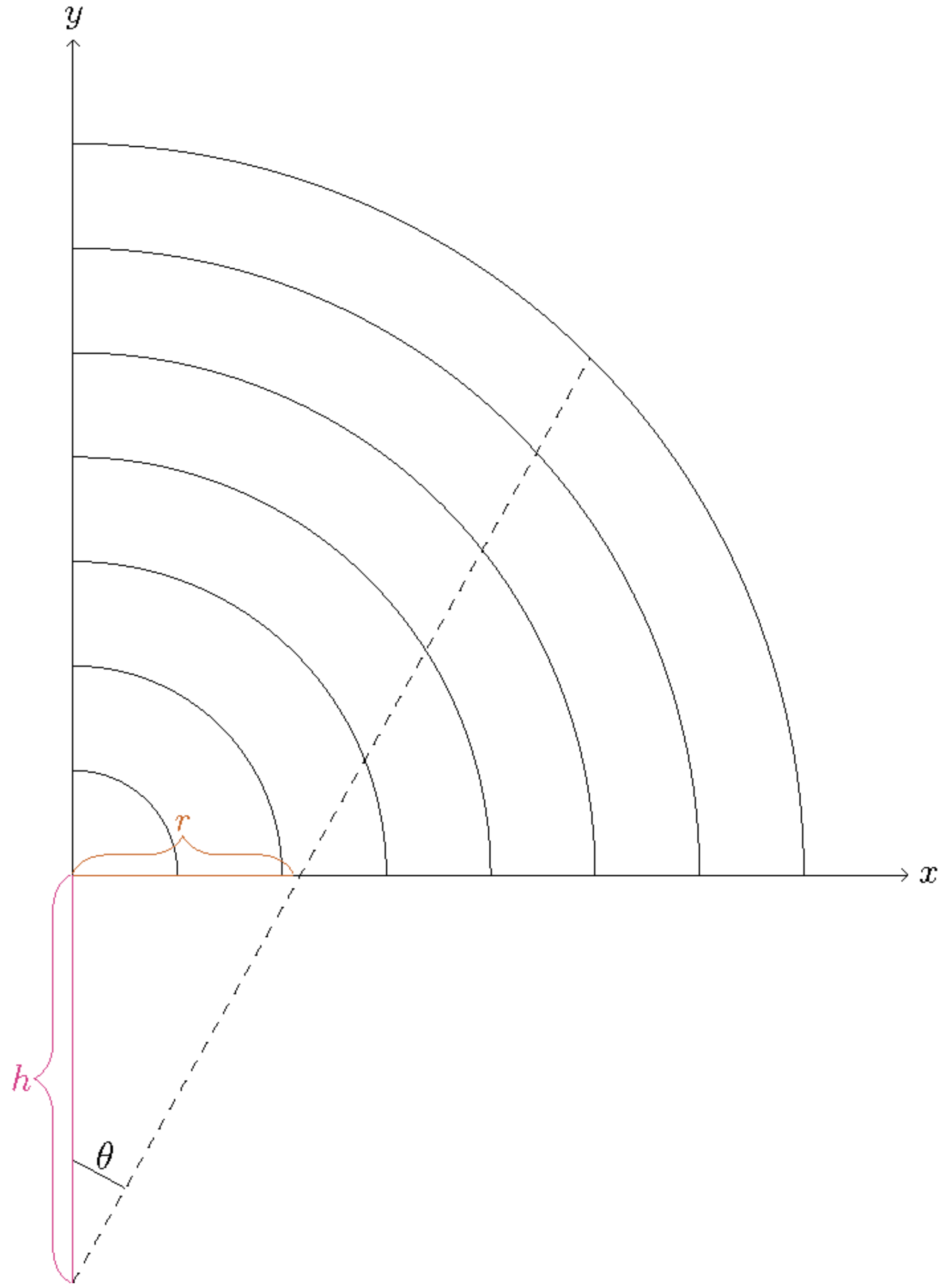
Define the Region

Fix θ. What is the range of r?



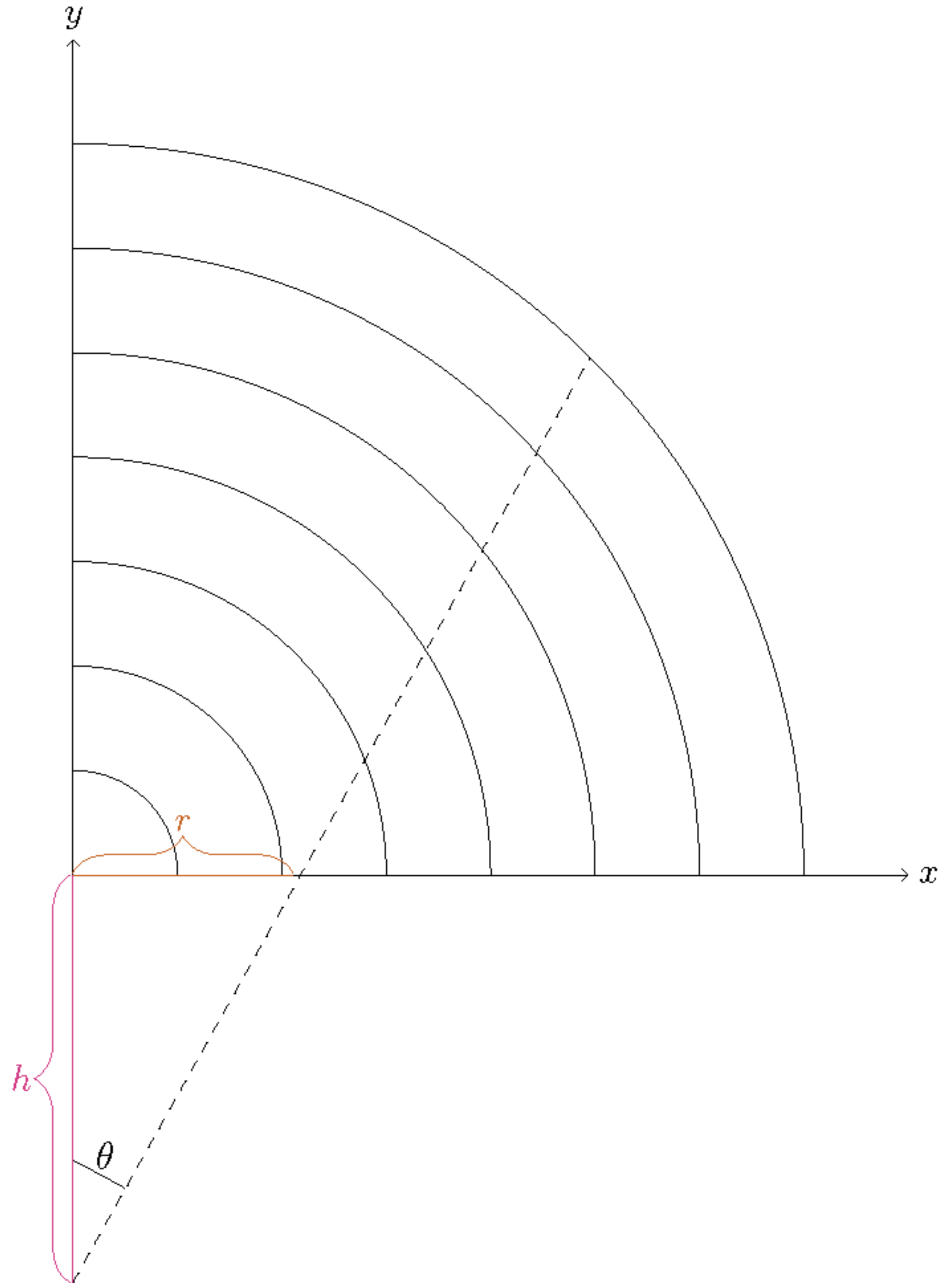
Define the Region

$$\tan(\theta) = \frac{r}{h}$$



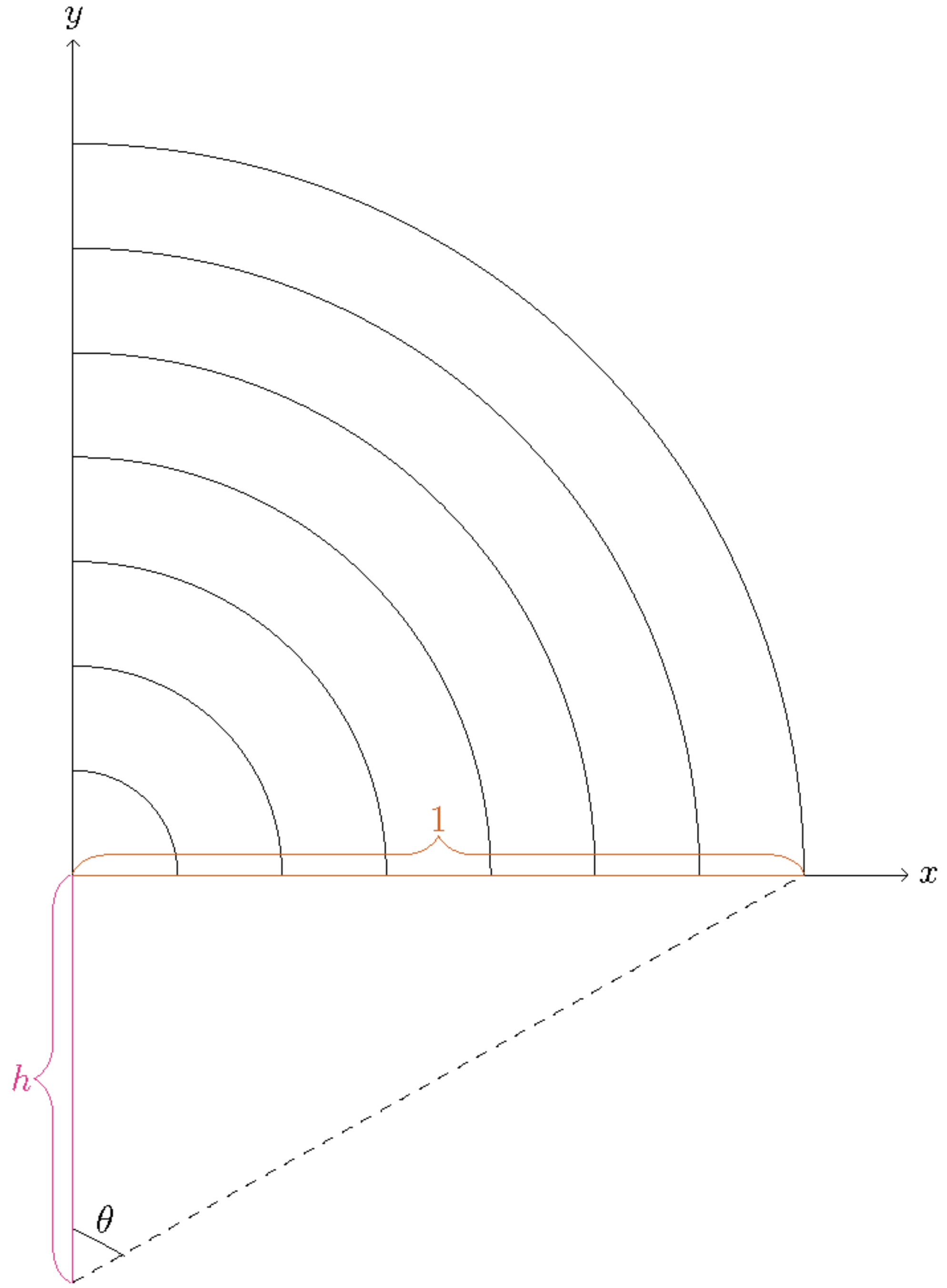
Define the Region

So, r ranges from $h\tan(\theta)$ to 1.



Define the Region

Also, θ ranges from 0 to $\arctan(1/h)$



The Jacobian

In order to compute the Jacobian

$$J(r, \theta) = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r},$$

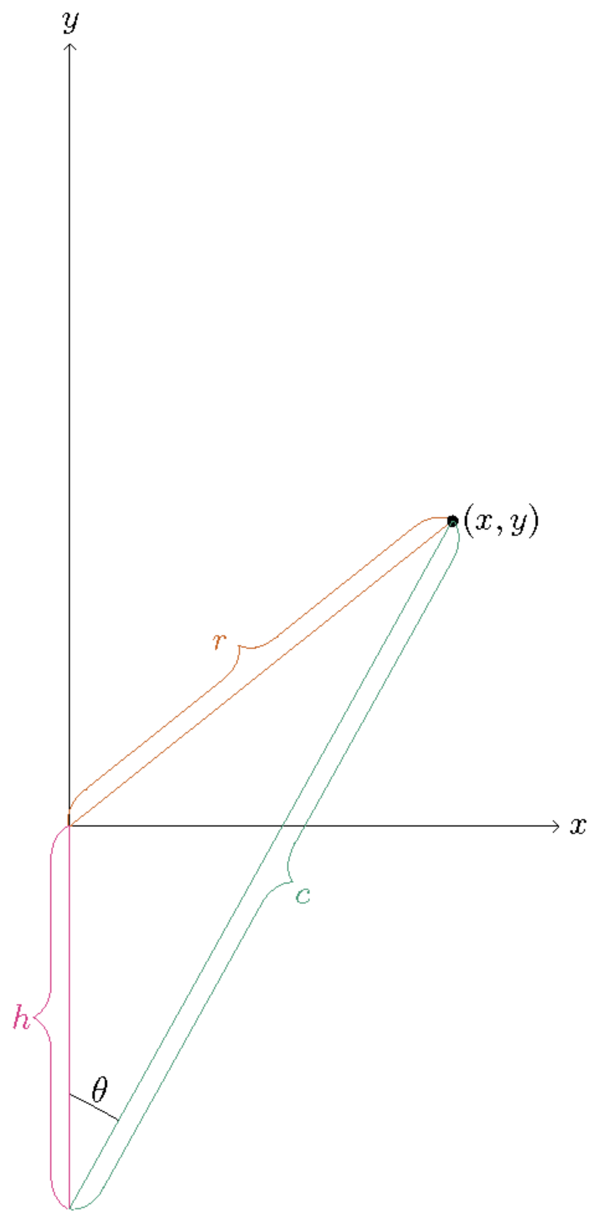
We need to know what is the relationship between (x, y) and (r, θ) in this coordinate system?

The Jacobian

Define c to be the distance from the point $(0, -h)$ to a given point (x, y) (both in the rectangular coordinate system).

Using the law of cosines, we can calculate

$$c = h\cos(\theta) + \sqrt{r^2 - h^2\sin^2(\theta)}.$$

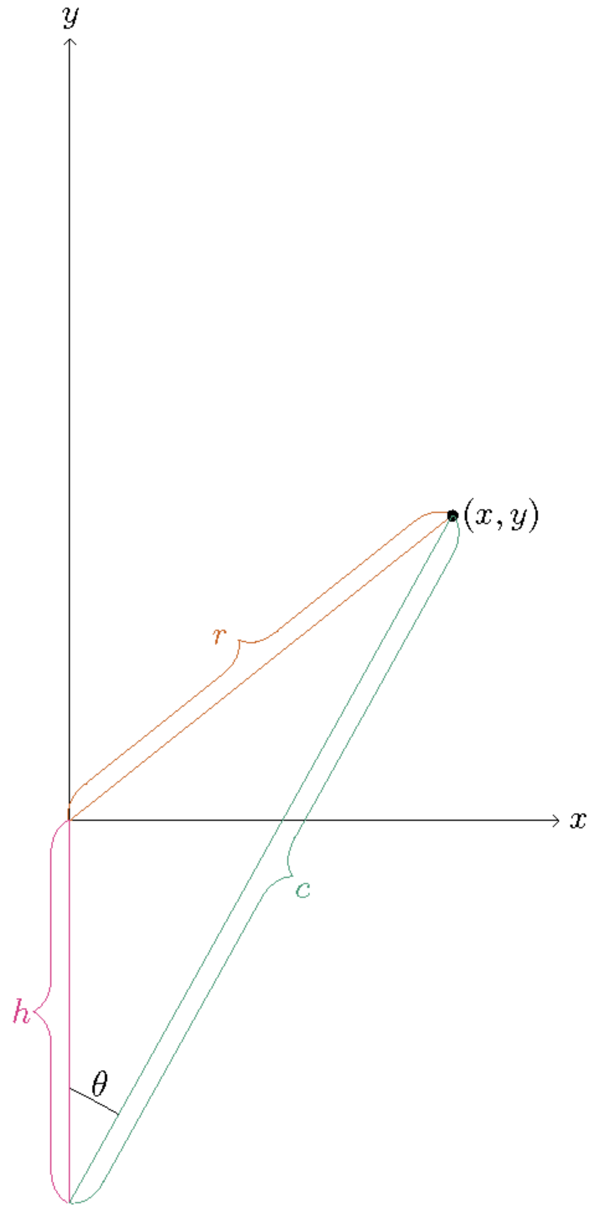


The Jacobian

With

$$c = h\cos(\theta) + \sqrt{r^2 - h^2\sin^2(\theta)}.$$

$$\begin{aligned} x &= c\sin(\theta) \\ y &= c\cos(\theta) - h \end{aligned}$$



The Jacobian

From this, for a given depth h, we can calculate the Jacobian as

$$J(r,\theta)=\frac{r\cos(\theta)\left(\sin(\theta)\left(-\frac{h^2\sin(\theta)\cos(\theta)}{\sqrt{r^2-h^2\sin^2(\theta)}}-h\sin(\theta)\right)+\cos(\theta)\left(\sqrt{r^2-h^2\sin^2(\theta)}+h\cos(\theta)\right)\right)}{\sqrt{r^2-h^2\sin^2(\theta)}}$$
$$-\frac{r\sin(\theta)\left(\cos(\theta)\left(-\frac{h^2\sin(\theta)\cos(\theta)}{\sqrt{r^2-h^2\sin^2(\theta)}}-h\sin(\theta)\right)-\sin(\theta)\left(\sqrt{r^2-h^2\sin^2(\theta)}+h\cos(\theta)\right)\right)}{\sqrt{r^2-h^2\sin^2(\theta)}}.$$

Yikes!

Executing the Plan

Despite how difficult the Jacobian looks, we can proceed as we did before. Putting the pieces our plan together, we need to first compute

$$\overline{A}(h) = \frac{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} J(r,\theta)\,dr\,d\theta}{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} 1\,dr\,d\theta}.$$

Amazingly, this is not as bad as it looks (even though Mathematica cannot do it).

Executing the Plan

Let's look at the numerator

$$\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^1 J(r,\theta)\; dr\; d\theta.$$

Executing the Plan

Let's look at the numerator

$$\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} J(r, \theta)\ dr\ d\theta.$$

If we go back to the rectangular coordinate system, this would simply be the integral of 1 over the region we define to be the onion. That is, it's the area of the quarter onion so this integral is just $\pi/4$.

Executing the Plan

Now let's look at the denominator

$$\int_{0}^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} 1 \, dr \, d\theta.$$

Executing the Plan

Now let's look at the denominator

$$\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} 1 \; dr \; d\theta.$$

This is fairly easy to do as well. We can see the denominator is equal to

$$\left(\arctan\left(\frac{1}{h}\right) - \frac{1}{2}h \ln\left(\frac{1}{h^2} + 1\right)\right).$$

Executing the Plan

In all, the average ``relative area'' of each piece is

$$\overline{A}(h) = \frac{\pi}{4\left(\arctan\left(\frac{1}{h}\right) - \frac{1}{2}h\ln\left(\frac{1}{h^2}+1\right)\right)}.$$

Executing the Plan

$$\sigma^2(h) = \frac{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} (J(r,\theta) - \overline{A}(h))^2 \, dr \, d\theta}{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} \, dr \, d\theta}.$$

$$= \frac{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} (J(r,\theta))^2 \, dr \, d\theta}{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} \, dr \, d\theta} - (\overline{A}(h))^2$$

$$= \frac{\int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} (J(r,\theta))^2 \, dr \, d\theta}{\arctan\left(\frac{1}{h}\right) - \frac{1}{2} h \ln\left(\frac{1}{h^2} + 1\right)} - (\overline{A}(h))^2.$$

Reducing the Problem

So, finding $\sigma^2(h)$ in a closed form reduces to evaluating

$$f(h) := \int_0^{\arctan(1/h)} \int_{h\tan(\theta)}^{1} (J(r,\theta))^2 \, dr \, d\theta.$$

Reducing the Problem

- This integral cannot be evaluated by Mathematica. We can be clever, though.
- We can go *back* to the Cartesian coordinate system using $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{x}{y+h}\right)$. The Jacobian for this transformation is $1/J(r, \theta)$.

Reducing the Problem

$$\begin{aligned} f(h) &= \int_0^1 \int_0^{\sqrt{1-x^2}} J(r,\theta)\,dydx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} J\left(\sqrt{x^2+y^2}, \arctan\left(\frac{x}{y+h}\right)\right)dydx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\sqrt{x^2+y^2}\left((h+y)^2+x^2\right)}{y(h+y)+x^2}dydx. \end{aligned}$$

But, here we are integrating over a quarter disc. Wouldn't polar coordinates be easier?

Let $x = s\cos(\varphi)$ and $y = s\sin(\varphi)$. Then, assuming $0 < h < 1$,

$$
\begin{aligned}
f(h) &= \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\left(h^2 + 2hs\sin(\varphi) + s^2\right)}{h\sin(\varphi) + s} s d\varphi ds \\
&= \frac{1}{6}\left(4\left(1-h^2\right)^{3/2}\arcsin(h) - 4\left(1-h^2\right)^{3/2}\arctan\left(\frac{h+1}{\sqrt{1-h^2}}\right)\right. \\
&\left. + h^3\log\left(4h^2\right) + h + 2\pi\right).
\end{aligned}
$$

So, we now have a closed form for $\sigma^2(h)$ (don't make me write it out though).

Rest and Reflect

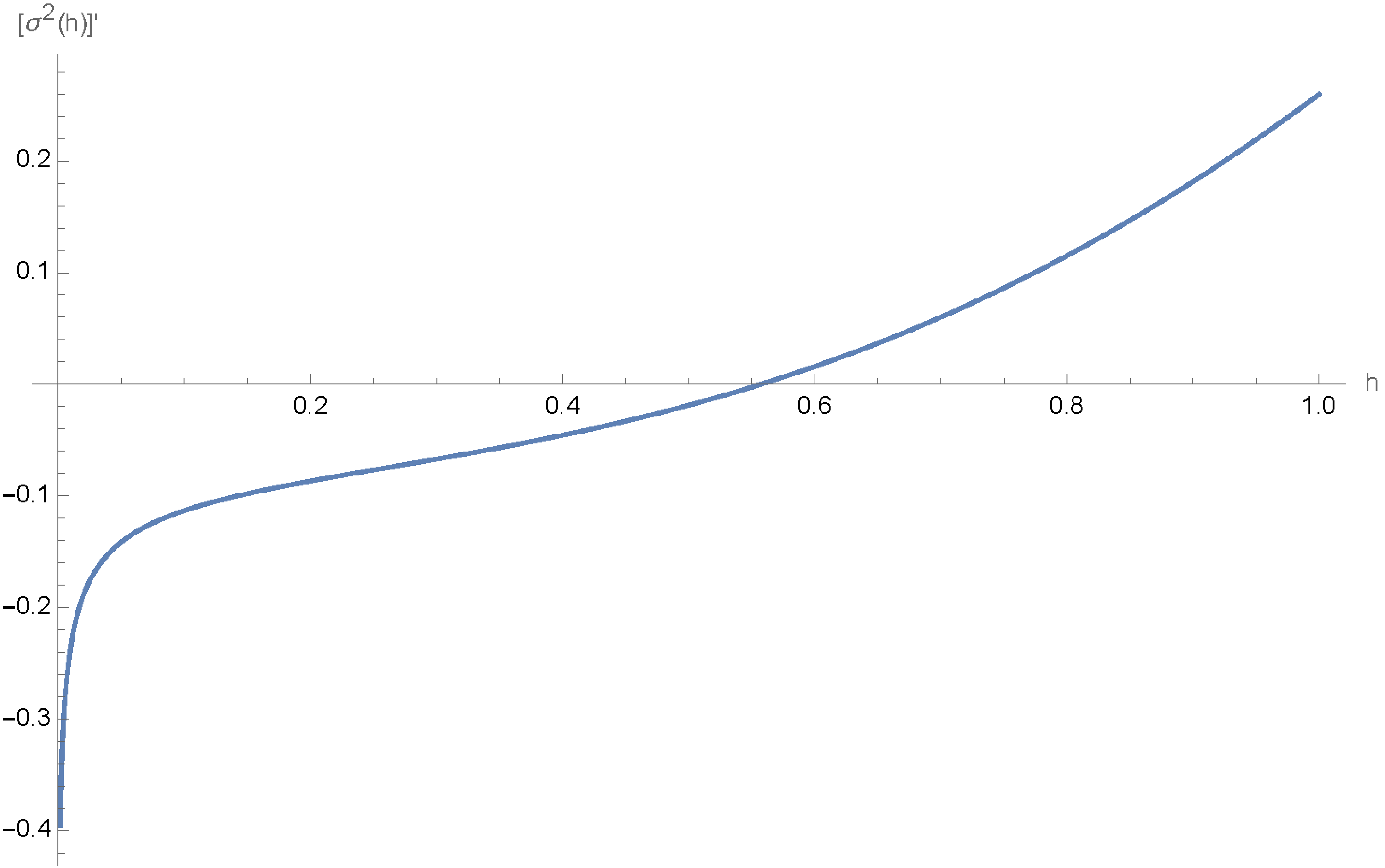
- Whew! We have done a lot of integration work with three different coordinate systems. What do we have at the end of it?
- We have an expression for the variance of the ``relative area'' as a function of the depth h which we are cutting towards.
- Our goal is to minimize variance. What should we do?
- Take the derivative and set it equal to zero! (then test).

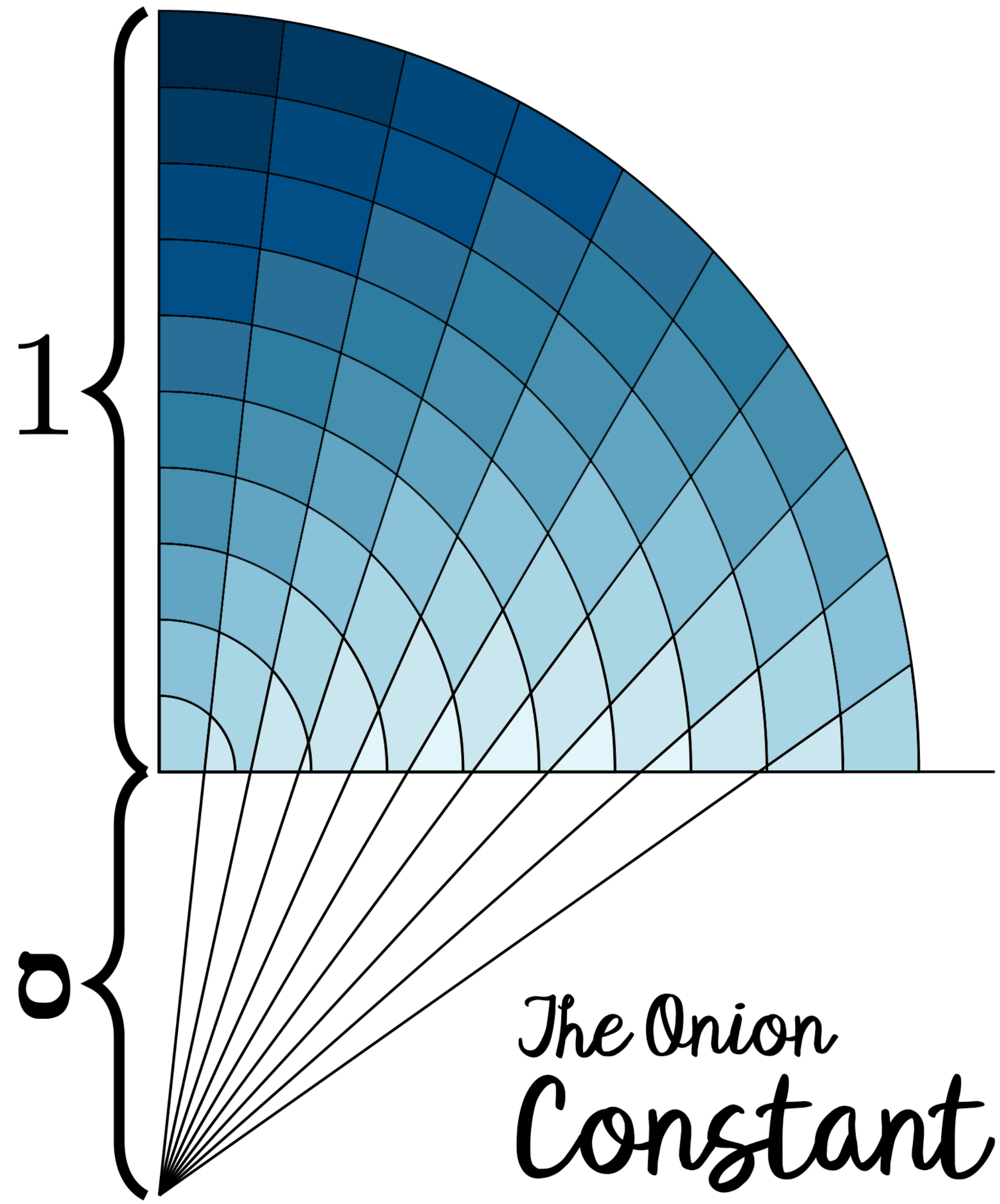
Putting it Together

$$[\sigma^2(h)]' = \frac{k(h)}{48\left(\cot^{-1}(h) - \frac{1}{2} h \log\left(\frac{1}{h^2} + 1\right)\right)^3},$$

where

$$\begin{aligned}
k(h) = &-3\pi^2 \log\left(\frac{1}{h^2}+1\right) \\
&+6\left(h\log\left(\frac{1}{h^2}+1\right)-2\cot^{-1}(h)\right)^2\left(h\left(h\log\left(4h^2\right)+4\sqrt{1-h^2}\left(\tan^{-1}\left(\frac{h+1}{\sqrt{1-h^2}}\right)-\sin^{-1}(h)\right)\right)+1\right) \\
&-2\log\left(\frac{1}{h^2}+1\right)\left(h\log\left(\frac{1}{h^2}+1\right)-2\cot^{-1}(h)\right) \\
&\times\left(4\left(1-h^2\right)^{3/2}\sin^{-1}(h)-4\left(1-h^2\right)^{3/2}\tan^{-1}\left(\frac{h+1}{\sqrt{1-h^2}}\right)+h^3\log\left(4h^2\right)+h+2\pi\right).
\end{aligned}$$


[σ²(h)]'
0.2
0.1
-0.1
-0.2
-0.3
-0.4
0.2
0.4
0.6
0.8
1.0
h



The (2D) Onion Constant

define **o** to be the unique root of $k(h)$ over the interval $0 < h < 1$. We call **o** the onion constant. To fifty decimal places its value is 0.55730669298566447885109305914592718083200030207273...

Three Dimensions

- We can also do a similar analysis in three dimensions. We model the onion as nested spheres. We first cut the onion in half, as before.
- Usually, we cut an onion into slices perpendicular to the root. If we add a z axis, the planes perpendicular to this access cut the onion in this way. So, we can use an analog of cylindrical coordinates (with our distorted "polar" coordinates) to answer the three-dimensional case.

Three Dimensions

- If we do the same analysis as before, we find the three-dimensional onion constant is approximately 0.484457.
- This is awfully close to 1/2 (an easy number for humans to estimate. What if we add in the fact that usually we cut off the ends of the onion?

Comprehensive Onion Cutting Guide

1. Cut off the ends of the onion, leaving approximately 82.8513% of the length of the onion in the vertical direction.

Comprehensive Onion Cutting Guide

2. Cut the onion in half perpendicular to the previous two cuts.

Comprehensive Onion Cutting Guide

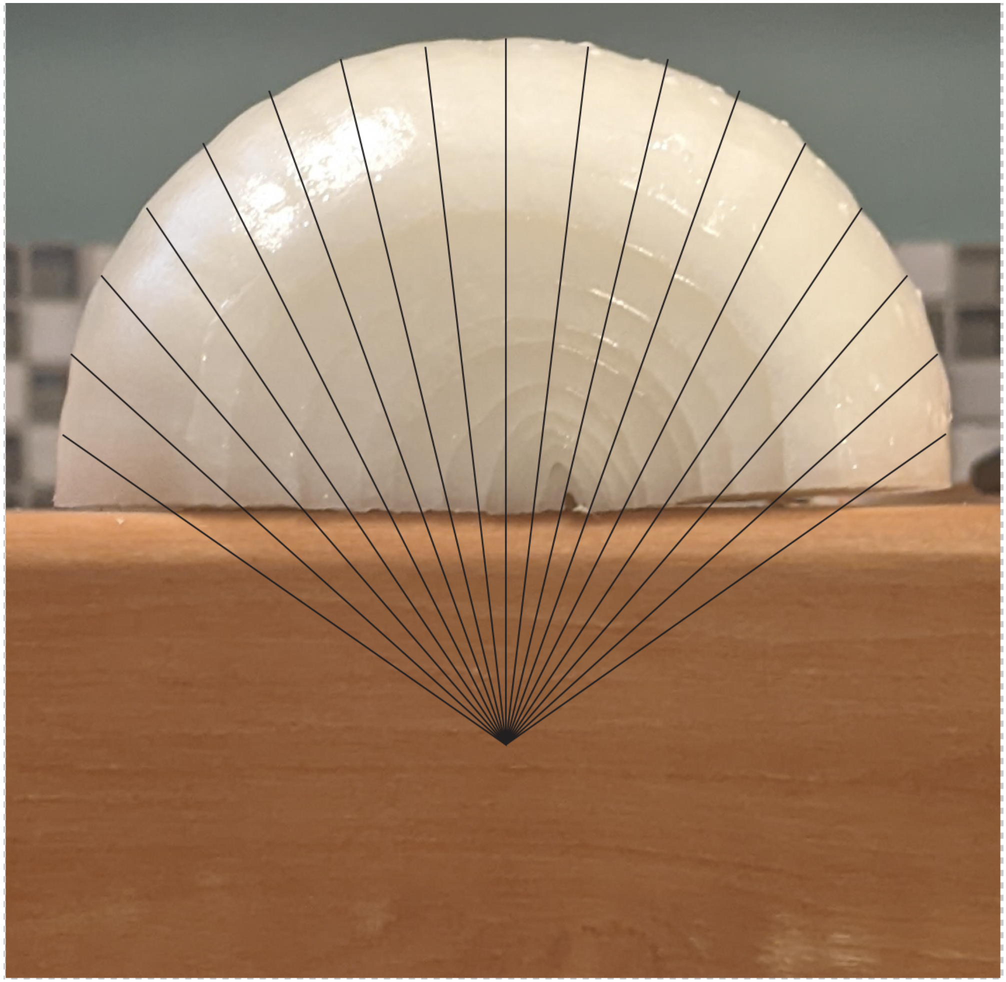
2. Cut the onion in half perpendicular to the previous two cuts.

Comprehensive Onion Cutting Guide

3. Slice the onion parallel to the direction of your initial cuts and perpendicular to the cutting board up the half-onion.

Comprehensive Onion Cutting Guide

4. For the final cuts, cut perpendicularly to the previous cuts, but for each cut aim towards a spot 50% of the radius of the onion below the center of the onion.



Comprehensive Onion Cutting Guide

5. Profit

Concluding Thoughts

- Why do this?
- What now?
- What questions do you have?

Pinned by J. Kenji López-Alt

Dylan Poulsen 1 month ago

I'm a mathematician who got really obsessed with this problem. I wrote up a solution that calculates the best depth to cut towards. The answer is 55.73066...%, which unfortunately does not seem to involve the golden ratio. You can read my solution here:
https://medium.com/@drspoulsen/a-solution-to-the-onion-problem-of-j-kenji-l%C3%B3pez-alt-c3c4ab22e67c

4 REPLY

Hide 2 replies

J. Kenji López-Alt 1 month ago

Hey thanks for the work and the link. We never calculated it to that many decimal places but the golden ratio thing was always just a shorthand way for me to remember it, I never meant that it was literally the golden ratio.

The problem gets even more complicated when you add horizontal cuts. Yikes.

If I ever make a longer video about this do you mind if I quote your medium piece?