Exercise 2

1. (a) From the given markov chain, we have $\Sigma_?$ as the set of states that can reach G by >0 steps. Therefore, we have—

$$\Sigma_? = \{1, 2, 3, 4\} \quad \text{for } G = \{6\}$$

(b) As per definition, we have—

- $A = (P(\sigma, \tau))_{\sigma, \tau \in \Sigma_?}$, ie, the transition probabilities in $\Sigma_?$
- $b = (b_\sigma)_{\sigma \in \Sigma_?}$, ie the probabilities to reach G in 1 step, ie $b_\sigma = \sum_{\gamma \in G} P(\sigma, \gamma)$

~~As~~ such ~~we have~~, for $G = \{6\}$, we have —

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1/2 & 0 & 1/4 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 4/5 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1/3 \\ 0 \\ 0 \end{pmatrix}$$

(c) For this problem, we have $G = \{5, 6\}$.

For the new Markov Chain, with $G = \{5, 6\}$, we have—

$$\Sigma_? = \{1, 2, 3, 4, 7, 8\}$$

So, we have—

→

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\
\begin{bmatrix}
0 & 1/2 & 0 & 1/2 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 0 & 0 \\
0 & 3/4 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 4/5 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 0 & 1/2 & 0
\end{bmatrix} \end{array}
\quad \text{and} \quad
b = \begin{pmatrix} 0 \\ \frac{1}{3}+0 \\ 0 \\ \frac{1}{5}+0 \\ \frac{1}{3}+0 \\ 0 \end{pmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}
= \begin{pmatrix} 0 \\ 1/3 \\ 0 \\ 1/5 \\ 1/3 \\ 0 \end{pmatrix}
$$

Therefore, we now solve the following system of equations—

$(I-A)\underset{\sim}{x} = \underset{\sim}{b}$, where $x = (x_\sigma)_{\sigma \in \Sigma_?}$ with $x_\sigma = \Pr(\sigma \models \Diamond G)$ as the unique solution.

Here, I is the identity matrix with dimension 6×6.

So we have—

$$
\left\{ \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
0 & 1/2 & 0 & 1/2 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 0 & 0 \\
0 & 3/4 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 4/5 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 0 & 1/2 & 0
\end{bmatrix} \right\} \cdot \underset{\sim}{x} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/5 \\ 1/3 \\ 0 \end{bmatrix}
$$

$$
\Rightarrow \underbrace{\begin{bmatrix}
1 & -1/2 & 0 & -1/2 & 0 & 0 \\
-1/3 & 1 & -1/3 & 0 & 0 & 0 \\
0 & -3/4 & 1 & -1/4 & 0 & 0 \\
0 & 0 & -4/5 & 1 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & 1 & -1/3 \\
0 & 0 & 0 & 0 & -1/2 & 1
\end{bmatrix}}_{= C \text{ (say)}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/5 \\ 1/3 \\ 0 \end{bmatrix}
$$

$$
\Rightarrow \underset{\sim}{x} = C^{-1} \underset{\sim}{b}
$$

$$
\Rightarrow \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 0.8 \\ 0.4 \end{pmatrix}}
$$

(Answer)

Therefore, the probability of reaching G from initial position 7 is $\underline{\underline{0.8}}$ (Ans)

$\downarrow$

$\Pr(\Diamond G)$