Exercise Sheet 7

General remarks:

- **Due date:** Thursday, December 15$^{\text{th}}$ 16:30 (before the exercise class).

- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.

- Solutions must be written in English.

- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.

- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.

Exercise 1 (The Arithmetical Hierarchy)   20P

Consider the following decision problem *INF*:

- Input: A (non-probabilistic) GCL program P with a single non-negative integer variable v.

- Output: *Yes*, if P terminates for infinitely many initial values of v; *No*, otherwise.

Identify a class A of the arithmetical hierarchy such that *INF* is A-complete. Prove that your answer is correct.

Exercise 2 (Proving Almost-Sure Termination)   35P

Consider the PGCL program P below:

$$
\begin{aligned}
&\texttt{while } (x \neq 10) \ \{ \\
&\qquad \texttt{if } (x \text{ is even}) \ \{ \\
&\qquad\qquad \{x := x - 2\} \ [^1\!/_2] \ \{x := x + 2\} \\
&\qquad \} \texttt{ else } \{ \\
&\qquad\qquad x := x + 1 \\
&\qquad \} \\
&\}
\end{aligned}
$$

Here, we assume that x is an integer variable. Use the proof rule for almost-sure termination from Lecture #15 (the rule involving the antitone functions p and d) to show that P terminates almost-surely for any given initial value of x.

Hint: Consider the expectation $V = 3 \cdot [x \text{ is odd}] + |x - 10|$ and choose *constant* functions p and d.

Exercise 3 (Positive Almost-Sure Termination)   20P

Consider a PGCL program P of the form

$$\texttt{while} \, (G) \, \{P'\} \ ,$$

where P' is a loop-free PGCL program. A clever student suggests the following scheme to prove positive almost-sure termination by weakest preexpectation reasoning:

1. Modify program P by introducing a fresh variable, say v, which is initialized with 0.
2. Increment v for every loop iteration by 1.

Hence, the modified program $\hat{P}$ is given by

$$v := 0; \ \texttt{while} \ (G) \ \{v := v + 1; P'\}.$$

Prove or disprove: $wp(\hat{P}, v)(s) < \infty$ implies that P terminates positive almost-surely on initial state s.