Exercise Sheet 5

General remarks:

- **Due date:** November 25th 12:30 (before the exercise class).
- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes (they will also be recorded).
- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.

Exercise 1 (Weakest pre-expectation calculus) — 25P

(a) [5P] Determine $wp(P, x^2)$ for program P: $\{\ x := 2\ \}\ [1/3]\ \{\ \texttt{diverge}\ \}$.

(b) [10P] In the present and the next exercise part, prove or disprove whether the programs P_1 and P_2 are equivalent w.r.t. the post-expectation $f = x$:

P_1: $y := 5;\ \texttt{if}\ (y < 0)\ \{\ \texttt{skip}\ \}\ \texttt{else}\ \{\ \{\ x := 1\ \}\ [1/2]\ \{\ \texttt{skip}\ \}\ \}$

P_2: $\{\ \{\ x := x + 3\ \}\ [1/3]\ \{\ x := x\ \}\ \}\ [1/2]\ \{\ x := 0\ \}$

(c) [10P]

P_1: $\texttt{while}\ (x \neq x)\ \{\ \{\ x := y + 1\ \}\ [1/2]\ \{\ x := y - 1\ \}\ \}$

P_2: $\texttt{while}\ (\mathit{true})\ \{\ \texttt{skip}\ \}$

Exercise 2 (Continuity of weakest pre-expectations) — 25P

Fix a pGCL loop $P = \texttt{while}\ (G)\ \{\ P'\ \}$. For all and expectations $X, f \in \mathbb{E}$ we define

$$\Psi_{P,f}(X) = [G] \cdot wp(P', X) + [\neg G] \cdot f\ .$$

(a) [5P] Fix expectation $X \in \mathbb{E}$. Prove that $\Psi_{P,f}(X)$ is continuous *as a function in* f.
Hint: You can use without proof that the functions $\mu_g, \alpha_g \colon \mathbb{E} \to \mathbb{E}$ where $\mu_g(f) = g \cdot f$ and $\alpha_g(f) = g + f$ are continuous for all $g \in \mathbb{E}$.

(b) [20P] This time fix an arbitrary post-expectation f. Prove that $\Psi_{P,f}$ is continuous *as a function in* X.
Hint: We want you to focus on loops in this exercise. Therefore you can use all of the following without proof:

- For $P = \texttt{skip}$, $P = \texttt{diverge}$, $P = x := E$ it holds that $wp(P, f)$ is continuous.
- If P_1, P_2 are such that $wp(P_1, f)$ and $wp(P_2, f)$ are continuous, then $wp(P, f)$ is continuous for $P = P_1; P_2$, $P = \texttt{if}\ (H)\ \{\ P_1\ \}\ \texttt{else}\ \{\ P_2\ \}$ and $P = \{\ P_1\ \}\ [p]\ \{\ P_2\ \}$ for all guards H and probabilities $p \in [0, 1]$.

Exercise 3 (Reasoning with invariants) **25P**

Let P be a pGCL program, I an expectation, $s \in \mathbb{S}$ a state and x be a program variable. In this exercise, Φ_f (Ψ_f, resp.) denotes the $wp(wlp$, resp.)-characteristic function of P w.r.t. to a post-expectation f. For each of the colloquial specifications (1) – (5) below do the following: Either select at least one of the formal conditions (a) – (g) such that the specification holds if and only if the condition holds or indicate that no such condition exists!

Colloquial descriptions:

(1) P terminates almost-surely on input s.
(2) P diverges almost-surely on input state s.
(3) If P terminates almost-surely on input s, then expected value of x after termination is at most 1.
(4) P terminates with probability at least $1/2$ on all inputs.
(5) The probability that P on input s terminates in a state with $x = 1$ is zero.

Formal conditions:

(a) $I \sqsubseteq \Phi_x(I)$ and $I(s) = 1$.
(b) $I \sqsubseteq \Psi_1(I)$ and $I \geq 1/2$.
(c) $I \sqsubseteq \Psi_{[x\neq 1]}(I)$ and $I(s) = 1$.
(d) $\Phi_1(I) \sqsubseteq I$ and $I(s) = 0$.
(e) $I \sqsubseteq \Phi_1(I)$ and $I(s) = 1$.
(f) $I \sqsubseteq \Psi_0(I)$ and $I(s) = 1$.
(g) $\Phi_{[x\leq 1]}(I) \sqsubseteq I$ and $I(s) = s(x)$.

Exercise 4 (A syntax for expectations) **25P**

(a) [5P] Write the (semantic) expectation $f = \sqrt[3]{x}$ as a syntactic expectation in Exp.
(b) [5P] Write the (semantic) expectation $f = \frac{2}{2 \cdot x^2 + 7}$ as a syntactic expectation in Exp.
(c) [7P] Let a be an arithmetic expression with a free variable x. We write $a(y)$ to mean a where x is substituted by y. Write a syntactic expectation $\mathsf{f} \in \mathsf{Exp}$ that evaluates to 1 if and only if a is constant in x.
(d) [8P] Let a be an arithmetic expression with a free variable x. We write $a(y)$ to mean a where x is substituted by y. Write a syntactic expectation $\mathsf{f} \in \mathsf{Exp}$ that evaluates to 1 if and only if a represents a monotonic function in x.