④ Weakest liberal pre-conditions

ⓐ Without using the diverge statement a GCL program P along with a post-condition f can be such that $wp(P,f) \neq wlp(P,f)$ if we have a loop in the program P as $\forall$ loop-free P $\forall f$, $wlp[\![P]\!](f) = wp[\![P]\!](f)$ by construction.

Consider the following program:

P: while $(x \neq 0)$ { $x := x - 1$ } and

post-condition

f: $x := 0$.

We can see, to calculate $wp(P,f)$, $\phi(X) = (x=0 \wedge x=0) \vee (x \neq 0 \wedge wp[\![x := x-1]\!](X))$

$= (x=0) \vee (x \neq 0 \wedge wp[\![x := x-1]\!](X))$

$\phi^0(false) = false$, $\phi^1(false) = (x=0) \vee (x \neq 0 \wedge wp[\![x := x-1]\!](false))$

$= (x=0) \vee (x \neq 0 \wedge false)$

$= (x=0) \vee false = (x=0)$

$\phi^2(false) = (x=0) \vee (x \neq 0 \wedge wp[\![x := x-1]\!](x=0))$

$= (x=0) \vee (x \neq 0 \wedge x-1=0)$

$= (x=0) \vee (x=1)$

. . . .

$\phi^n(false) = (x=0) \vee (x=1) \ldots \vee (x=n)$

$\Rightarrow \sup_{n \in \mathbb{N}} \phi^n(false) = \bigvee_{k=0}^{n} (x=k) = x \geq 0,$ or $x \in \mathbb{N}$

So, starting with $x \geq 0$ terminates with $x=0$, starting with $x<0$ doesn't terminate.

While $wlp(P,f) = true$

So, $wp(P,f) \neq wlp(P,f)$.

(6) $P:$ while $(x \neq 0)\ \{x := x-2\}$

$$\underbrace{\varphi(X) = (x=0 \wedge f) \vee (x \neq 0 \wedge wlp[\![x:=x-2]\!](X))}_{\text{loop characteristics fn } \Psi_f(x)}$$

$$wlp\ [\![P]\!]\ (f) = gfp\ X.\ \Psi_f(x).$$

We know from construction of wlp and Kleene's fixed-point theorem that $gfp\ \Psi_f = \inf_{n \in N} \Psi_f^{\,n}(true)$

Let us perform the fixed point iteration for φ as:

0. $\varphi^0(true) = true$

1. $\varphi^1(true) = (x=0 \wedge f) \vee (x \neq 0 \wedge wlp[\![x := x-2]\!](true))$

$= (x=0 \wedge f) \vee (x \neq 0 \wedge true)$

$= (x=0 \wedge f) \vee (x \neq 0)$

We can consider $f: x := 2n+1$, $n \in$ Integer

i.e. $x := \pm 1, \pm 3, \pm 5, \ldots$

Assume $x := 1$ WLOG,

$\varphi^1(true) = (x := 0 \wedge x = 1) \vee (x \neq 0)$

$= (x \neq 0)$

$\varphi^2(true) = (x=0 \wedge f) \vee (x \neq 0 \wedge wlp[\![x := x-2]\!](x \neq 0))$

$= (x=0 \wedge x=1) \vee (x \neq 0 \wedge x-2 \neq 0)$

$=$ false $\vee (x(x-2) \neq 0)$

$= x(x-2) \neq 0.$

In this way we'll get $x(x-2) \cdots (x-2n) \neq 0.$

Now, we can see that P does not terminate if $x(x-2) \cdots (x-2n) \neq 0$, which is true for all $x :=$ odd numbers! i.e. $x := 2n+1$, $n \in I$.

(c) Rules for constructing the wlp (weakest liberal pre-condition) gives us that for atomic program lines skip & (assignment) $x := E$ the rules are exactly the same. For the remaining programs without the existence of loop, definitions differ only in the fact that we use wlp instead of wp on the program lines. So, in case of non-termination which only happens for loops both wp and wlp are structurally same.

So, for $\forall$ loop-free P $\forall$ post-condition f, $wp(P, f) = wlp(P, f)$.

Now when we look at while rule we can prove:

$$(\star)\begin{cases} lfp\, X.\ ((\varphi \wedge wlp[\![P]\!](X)) \vee (\neg\varphi \wedge f)) \\ \sqsubseteq\ gfp\, X.\ ((\varphi \wedge wlp[\![P]\!](X)) \vee (\neg\varphi \wedge f)) \end{cases}$$

which actually will correspond to what we want to prove: (by definition)

$$(1)\quad wp(P, f) \sqsubseteq wlp(P, f).$$

using the hint in the question we know that wlp is monotonic, thus by using Knaster Tarski, we know all the fixed points are forming a complete lattice thus

The order $(\star)$ holds proving the order (1) we were asked.