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Probabilistic Programming
Exercise Sheet 3 Solutions
Group 7

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1. (a) Check sheet03_ex1a.wppl for the code.

   (b) Check sheet03_ex1b.wppl for the code.

   (c) We can understand the impact of the parameter σ^2 of the proposal distribution through the samples accepted during our run of the algorithm.

   A really small variance of the proposal distribution leads to new samples which are almost always accepted as acceptance rate r is always close to 1 for continuous distributions since proposed moves are small. This issue can also be noticed with our proposed distribution in this problem.

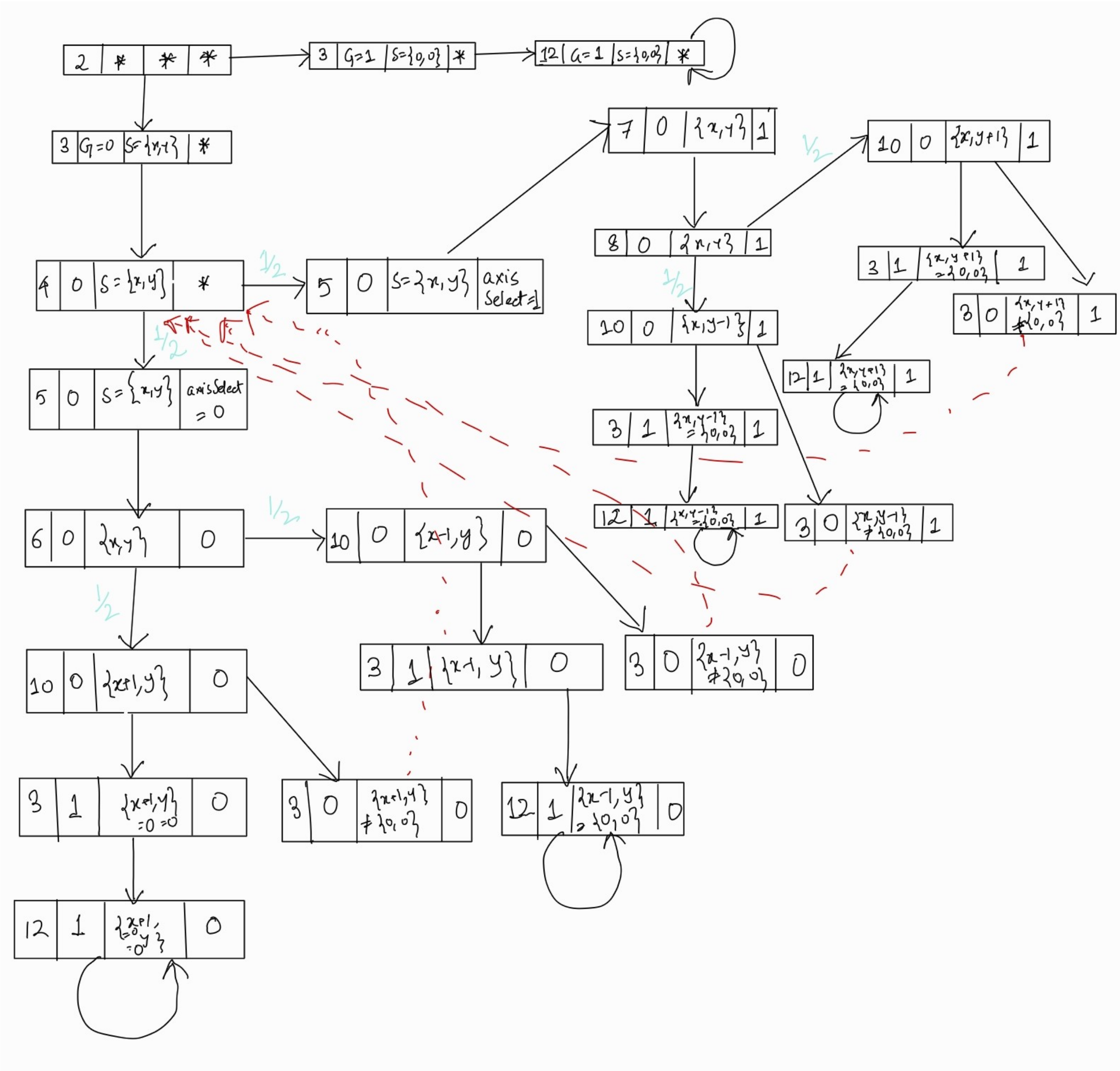
   Also, since the movements in each step is small, the transitions are concentrated into a specific region leading to poor coverage of the distribution globally. So, the Markov chain might not cover the entire distribution space especially if we have multiple modes, in this situation the samples might converge to one of the modes and never move to other ones.

   On the other hand, if we used large variance in the proposal distribution then most transitions might have been proposed in regions of very little probability and many moves are rejected, leading to the samples stuck with a single value for many iterations. So, we need to find a middle ground.

2. (a) The corresponding pGCL program for the stochastic process defined is as follows :

```
int TravelZ2(int x_0, y_0) {
    bool G := (x_0 = 0) ∧ (y_0 = 0);
    while(¬G) {
        int axisSelect := 0 [0.5] axisSelect := 1;
        if(axisSelect=0){
            (x_0 := (x_0 + 1) [0.5] x_0 := (x_0 − 1));
        } else {
            (y_0 := (y_0 + 1) [0.5] y_0 := (y_0 − 1));
        }
        G := (x_0 = 0) ∧ (y_0 = 0);
    }
    return G;
}
```

(b) To get the operational semantics of the pGCL program defined in (a) as a Markov chain for one loop iteration we assume initial states $x_0 = xandy_0 = y$. See the figure below for the corresponding operational semantics:



3.

(a)

Markov Chain with rewards

We consider a six-sided fair die. We propose the reward Markov chain (D, r), where D is a Markov chain with state space $\Sigma = \{1,2,3,4,5,6\}$ and, $r : \Sigma \to \mathbb{R}$ is the reward function s.t.

$$r(\sigma) = 1 \quad \forall \sigma \in \Sigma$$

and goal state $G = \{6\}$, since we want to compute the expected # trials needed to see a six for the first time.

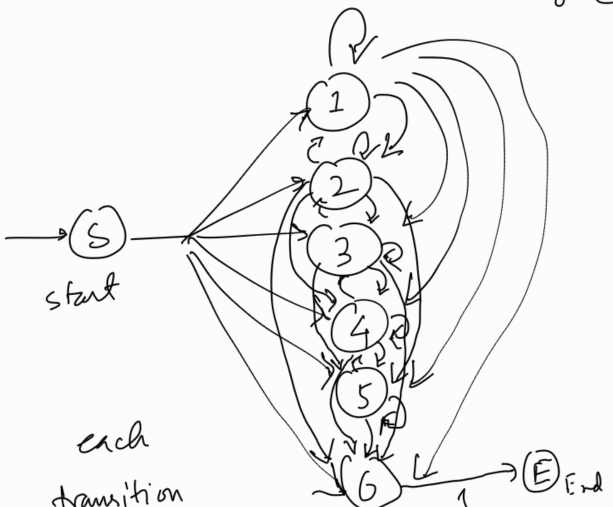
We know that expected rewards in finite Markov chain can be computed in polynomial time by solving a system of linear equations.

We also have a finite Markov chain in this case.

also $P(\sigma, \sigma') = 1/6 \quad \forall \sigma, \sigma' \in \Sigma$.

For $Pr(\sigma \models \Diamond G) = 1 \quad \forall \sigma \in \Sigma$, we take the variable $y_\sigma \in \mathbb{R}$ for $\sigma \in \Sigma$

$$y_\sigma = \begin{cases} 0 & \text{if } \sigma \in G \\ r(\sigma) + \sum_{\sigma' \in \Sigma} P(\sigma, \sigma') \cdot y_{\sigma'} & \text{o.w.} \end{cases}$$



each transition with prob 1/6 except

Let N be the # trials until the MC hits 6 for the first time. If it hits 6 at the first step (with prob $1/6$) we stop or with prob $5/6$ it hits any other state we continue. So the expected # trials =

$$\frac{1}{6} \times 1 + 2 \times \frac{5}{6} \times \frac{1}{6} + 3 \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \cdots$$

$$= \frac{1}{6} \sum_{i \geq 1} i \left(\frac{5}{6}\right)^{i-1}$$

$$= \frac{1}{6} \sum_{k \geq 0} (k+1) \left(\frac{5}{6}\right)^{k}$$ (substituting $k = i-1$)

$$= \frac{1}{6} \sum_{k \geq 0} k \left(\frac{5}{6}\right)^{k} + \frac{1}{6} \sum_{k \geq 0} \left(\frac{5}{6}\right)^{k}$$

$$= \cancel{\frac{1}{6}} \times \frac{5/6}{\cancel{(1/6)^2}\ 1/6} + \frac{1}{6} \cdot \frac{1}{1 - 5/6}$$

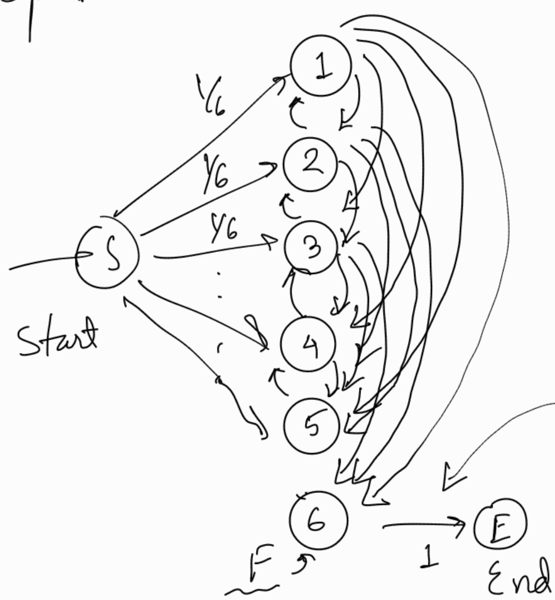
[By using the hint given and property of sum of Geometric Progression].

$$= 5 + \cancel{\frac{1}{6}} \times \frac{1}{\cancel{1/6}}$$

$$= 5 + 1$$

$$= \boxed{6}$$

3. b) The only major condition changed in this case is that we can't go the forbidden state 6 in our first step. So the Markov chain changes accordingly (say D')



All transition probability is 1/6 except

Since first trial was no six, the minimum # trials to get a six is 2.

So all the paths π such that

$\pi = \{\sigma_0 \ldots \sigma_n \mid \sigma_0 = 6\} \notin Paths(D')$

④ a) In order to prove that $(D \to D, \preceq)$ is a partial order we will show:

i) <u>monotonicity</u>:

let f be a monotonic function in D, $f \in D \to D$, for $\forall d \in D$ $f(d) \in D$, since $(D, \sqsubseteq)$ is a complete lattice (thus a partial order) we know $f(d) \sqsubseteq f(d)$ {from monotonicity of $(D, \sqsubseteq)$}

$\Longleftrightarrow$ $f \preceq f$

by defenition

ii) <u>transitivity</u>:

let $f, g, h \in D \to D$ such that $f \preceq g$, $g \preceq h$, thus, by defenition, $\forall d \in D$ $f(d) \sqsubseteq g(d)$, $g(d) \sqsubseteq h(d)$. by transitivity of $(D, \sqsubseteq)$ we get $f(d) \sqsubseteq h(d)$ for $\forall d \in D$, thus $f \preceq g$.

iii) <u>antisymmetry</u>:

let $f, g \in D \to D$, such that $f \preceq g$, $g \preceq f$.

thus, by defenition: $\forall d \in D$ $f(d) \sqsubseteq g(d)$, $g(d) \sqsubseteq f(d)$.

from antisymmetry of $(D, \sqsubseteq)$ we get $f(d) = g(d)$.

since this is true for $\forall d \in D$ we can conclude $f = g$!

b) we're seen that $(D \to D, \sqsubseteq)$ is indeed a partial order, let there B a subset $S \subseteq D \to D$ of monotonic functions in D.

we construct a function f, which for any given $d \in D$ $s(d) \sqsubseteq f(d)$ for $\forall s \in S$, such a function exist since we can just define: $f(d) := \max\{s(d) \mid s \in S\}$. such a function is also still monotonic since $\forall s \in S$ are monotonic as well $\Rightarrow$ if $d_1, d_2 \in D$, $d_1 \sqsubseteq d_2$ let $s_i(d_1)$ B the max value for d_1 out of S and $s_j(d_2)$ B the max value for d_2 out of S, we get $s_j(d_1) \sqsubseteq s_i(d_1) \sqsubseteq s_i(d_1) \sqsubseteq s_j(d_2)$

from transitivity: $s_i(d_1) \sqsubseteq s_j(d_2) \Rightarrow s(d_1) \sqsubseteq f(d_2)$!

f is monotonic as well $\Rightarrow f \in D \to D$.

by defenition f is an upper bound of S.

now lets assume there exist g which is also an upper bound of S and $g \sqsubseteq f$, thus $\exists d \in D$ $\forall s_i \in S$ $s_i(d) \sqsubseteq g(d) \sqsubseteq f[d]$ contradicting the fact that $f(d)$ is defined as the max of $\{s(d) \mid s \in S\}$

$\Rightarrow f$ is lub \ supremum of S !

$\Rightarrow$ similiraly we can define function h by the minimum values and get that h is the infimum of S

thus F, h are $\in D \to D$ and are the supremum and infimum of an arbitrary $S \subseteq D \to D$.

thus $(D \to D, \sqsubseteq)$ is a complete lattice. $\square$

c) The statement is correct:

well prove it by using the hint.

i) First well prove that because D is satisfying the ACC, $D \to D$ also satisfies the same condition. Let there be a function $f \in (D \to D)$ since D is ACC then f's Image and PreImage is also limited! Thus making a chain $F \subseteq (D \to D)$ follow the rules required from ACC.

ii) In our scenario where D satisfies ACC lFP is actually a monotone function: $f_1, f_2 \in (D \to D)$ $f_1 \sqsubseteq f_2$, since D is limited (and basically finite) both functions will have $d_1, d_2 \in D$ where $d_1 = f_1(d_1)$ $d_2 = f_2(d_2)$ (from pigeonhole principle), and since we assumed $F_1 \sqsubseteq f_2$ and the function are monotonic we can derive that $lfs(f_1) \sqsubseteq lfs(f_2)$.

iii) $\Rightarrow$ Now we can use the hint:

$$\phi\left(\bigsqcup S\right) = \bigsqcup \phi(S)$$

$$\Downarrow$$

plugging In:

$\phi = lfs$

$S = \mathcal{F} \subseteq (D \to D)$

$$\Downarrow$$

$$lfs\left(\bigsqcup \mathcal{F}\right) = \bigsqcup lfs(f \mid f \in \mathcal{F})$$

END.

NEXT

PAGE

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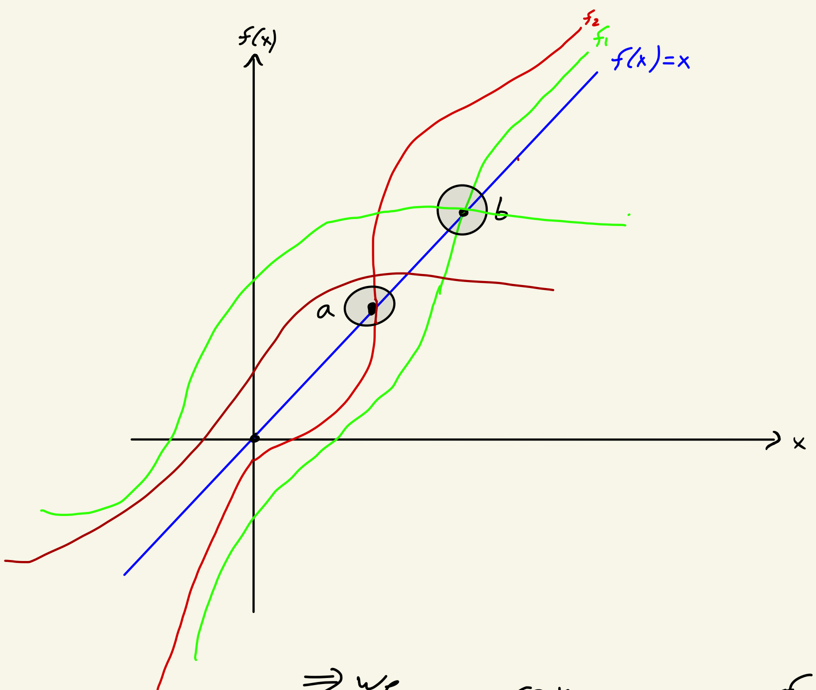
SKETCH.

c) we will disprove the statement by a counter example:

- $D := \mathbb{R}$
- $\sqsubseteq := \leq$

$\Big\} \Rightarrow (\mathbb{R}, \leq) \rightarrow$ a complete lattice

- lets look at the following function $f_1, f_2 \in \mathbb{R} \rightarrow \mathbb{R}$



$\Rightarrow$ we can see from the graph, that both of the Functions are monotone.

$\Rightarrow$ let's choose $\rightarrow \mathcal{F} = \{f_1 \leq f_2\}$.

In this case $\sqcup \mathcal{F} = f_2$, as described in (4.a)

thus we can see that the $lfp(\sqcup F) = lfp(f_2) = a$

$\Rightarrow$ now lets look at $\{lfp(f) \mid \in \mathcal{F}\} = \{a, b\}$

$\Rightarrow \sqcup\{lfp(f) \mid \in \mathcal{F}\} = \sqcup\{a, b\} = b$

$\Downarrow$

The equation does not hold ∇