1.

(a) Given, a Markov Chain $D = (\Sigma, \sigma_I, P)$ with goal set $G \subseteq \Sigma$

Define $\Sigma_? = \Sigma \setminus G$, matrix $A = (a_{ij}) = (P(\sigma,\tau))_{\sigma,\tau \in \Sigma_?}$ and vector $\underset{\sim}{b} = (b_\sigma)_{\sigma \in \Sigma_?}$

where $b_\sigma = \sum_{\gamma \in G} P(\sigma, \gamma)$

We define a partial order on the vectors in V, where these vectors have entries indexed by the states in $\Sigma_?$ with functions of type $\Sigma_? \to [0,1]$.

The partial order is as follows —

$(V, "\sqsubseteq")$, where $\underset{\sim}{v}_1 \sqsubseteq \underset{\sim}{v}_2 \Rightarrow$ every element of $\underset{\sim}{v}_1 \leq$ every element of $\underset{\sim}{v}_2$;

$\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in V$.

Note that, every vector in V is of order $|\Sigma_?| \times 1$.

<u>Proof that $(V, "\sqsubseteq")$ is a partial order</u> :—

- For all $\underset{\sim}{x} \in V$, $\underset{\sim}{x} = \underset{\sim}{x}$; hence reflexive
- For any $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \underset{\sim}{x}_3 \in V$,

$x_{1i} \leq x_{2i} \;\; \forall i = 1(1)|\Sigma_?|$
and, $x_{2i} \leq x_{3i} \;\; \forall i = 1(1)|\Sigma_?|$ $\Rightarrow x_{1i} \leq x_{3i} \;\; \forall i = 1(1)|\Sigma_?|$

$\therefore$ $\underset{\sim}{x}_1 \sqsubseteq \underset{\sim}{x}_2$ and $\underset{\sim}{x}_2 \sqsubseteq \underset{\sim}{x}_3 \Rightarrow \underset{\sim}{x}_1 \sqsubseteq \underset{\sim}{x}_3$; hence transitive

- For $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in V$,

if $x_{1i} \leq x_{2i} \;\; \forall i = 1(1)|\Sigma_?|$ and $x_{2i} \leq x_{1i} \;\; \forall i = 1(1)|\Sigma_?|$
then $x_{1i} = x_{2i} \;\; \forall i = 1(1)|\Sigma_?|$

ie, $\underset{\sim}{x}_1 \sqsubseteq \underset{\sim}{x}_2$ and $\underset{\sim}{x}_2 \sqsubseteq \underset{\sim}{x}_1 \Rightarrow \underset{\sim}{x}_1 = \underset{\sim}{x}_2$; hence anti-symmetric

Therefore, $(V, "\sqsubseteq")$ is a partial order.

<u>To prove</u> :—

$f : V \to V ;\; x \mapsto Ax + b$ has a least fixed point in this order

By Kleene's fixed point theorem, f ~~to~~ has a lfp if—

- $(V, "\sqsubseteq")$ is a complete lattice
- f is Scott continuous

Complete lattice —

Here, V is a set of vectors where the vectors have dimension $|\Sigma_?| \times 1$ and entries in $[0,1]$. Therefore, there exists a well defined order between every pair of elements under $(V, "\sqsubseteq")$. Since D is a finite state MC, therefore every subset S has a supremum and infimum in D wrt the relation "$\sqsubseteq$".

Hence, $(V, \sqsubseteq)$ is a complete lattice.

Scott continuity —

We have $f : V \to V : x \mapsto Ax + b$, where A and b are as defined above.

Let, $x_0 = \bigsqcup S$, where $S \subseteq V$ is a non-empty chain.

Then, $f(x_0) = Ax_0 + b \sqsubseteq Ax + b \ \forall x \neq x_0$ and $x \in V$, since A is positive semi-definite and every element of b is non-negative.

Therefore, order of elements in S are preserved under f. In fact, S is clearly finite. Hence—

$$f(\bigsqcup S) = \bigsqcup f(S) \Rightarrow f \text{ is Scott-continuous}$$

w.r.t. this order $(V, "\sqsubseteq")$. $\blacksquare$

$\therefore$ By Kleene's fixpoint theorem, f has lfp

(b). By Kleene's fixed point theorem, we can approximate the lfp of f by iteratively applying f ~~to~~ as follows—

0 : Apply f to $\perp = \underset{\sim}{0}$

1 : Calculate $f(\perp) = b$

2. : Calculate $f(f(\perp)) = f(b) = A\underset{\sim}{b} + \underset{\sim}{b}$

3 : Calculate $f^3(\perp)$, $f^4(\perp)$ and so on

4 : For a large enough $n \in \mathbb{N}$, we should be able to reach $\text{lfp } f := \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$

In fact, we essentially calculate—

$$f^n(\perp) = A^{n-1}\underset{\sim}{b} + A^{n-2}\underset{\sim}{b} + \dots + A\underset{\sim}{b} + \underset{\sim}{b}$$

and take $\bigsqcup_{n \in \mathbb{N}} f^n(\perp)$.