Sheet 05

1

a

// $4/3$

// $1/3 \cdot 4 + 2/3 \cdot 0$

{

// $2^2 = 4$

$x := 2$

// x^2

{ $x := 2$ }

{

// 0

diverge

// x^2

}

// x^2

b

P_1: $y := 5$; if $(y < 0)$ { skip } else { { $x := 1$ } [$1/2$] { skip } }

P_2: { { $x := x + 3$ } [$1/3$] { $x := x$ } } [$1/2$] { $x := 0$ }.

$P_1 \equiv P_2$

P_1

// $[5 < 0]x + [5 \geq 0]\left(\frac{x+1}{2}\right) = \frac{x+1}{2}$

$y := 5$

// $[y<0]x + [y \geq 0]\left(\frac{1}{2} + \frac{1}{2} \cdot x\right)$

if $(y<0)$ {

skip

} else {

{ // $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot x$

// 1

$x := 1$

}

[$1/2$]

{ // x

skip

}

}

// x

P_2: $\frac{1}{2}\left[\frac{x}{3} + 1 + \frac{2}{3}x\right] + \frac{1}{2} \cdot 0 = \frac{x+1}{2}$

{

// $\frac{1}{3}[x+3] + \frac{2}{3}x$

{ // $x+3$

$x := x+3$

// x

}

[$1/3$]

{

// x

$x := x$

// x

}

}

[$1/2$]

{ // 0

$x := 0$

} // x

// x

c

while $(x \neq x)$ {

{ $x := y + 1$ }

[$1/2$]

{ $x := y - 1$ }

}

// x

→ check our solution!

it's correct & better so didn't write theirs

(3) None of them has iff relationship

(1) P terminates . . . . — None

(2) (d), (f)

$\Phi_1(I) \sqsubseteq I$ $\quad I(s) = 0$

$\uparrow$ prob. of termination

$\Rightarrow$ If $p\ \Phi \sqsubseteq I$. $\quad \Rightarrow \Leftarrow$

$\Rightarrow$ wp (while, 1) $\quad$ can't terminate

$I \sqsubseteq \Psi_0(I)$

$\Rightarrow$

wlp (P, 0) $\longrightarrow$ $f(\tau_1) f(\tau_2) f(\tau_3)$ . . . $= 0$

wp[P](σ)

prob. of divergence

$\Rightarrow I \sqsubseteq gfp\ \Psi_0$, $I(s) = 1$.

$\Rightarrow$ diverges with prob. 1

(3) None

(4) None

(5) (c) $\quad I \sqsubseteq \Psi_{[x \neq 1]}(I)$ $\quad I(s) = 1$.

$x \neq 1$ $\forall$ states do hold or doesn't terminate. or if diverges.

$\rightarrow P(x=1) = 0$

② ⓐ $wp(P, f) = \mathrm{lfp}\, X.\ \Phi_{P,f}(X) \overset{\text{Kleene}}{=} \bigsqcup_{n>0} \underline{\Phi_{P,f}}\,(0)$.

continuous : prove

Let $S \subseteq \mathbb{E}$ be a chain

$S = \{f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots\}$ $\qquad f_i \in \mathbb{E} = \{S \to \mathbb{R}^{\infty}_{\geq 0}\}$

Have to show: $\bigsqcup_{f \in S} \Phi_{P,f}(X) = \Phi_{P, \sqcup S}(X)$. $\quad \sqcup S = \bigsqcup_{f \in S} f$.

$$\bigsqcup_{f \in S} \Phi_{P,f}(X) = \bigsqcup_{f \in S} \left([G] \cdot wp(P', X) + [\neg G] \cdot f \right)$$

$$= [G] \cdot wp(P' X) + \underbrace{\bigsqcup_{f \in S} [\neg G] \cdot f}\ (\alpha \text{ is const.})$$

$$= [G] \cdot wp(P' X) + [\neg G] \cdot \bigsqcup_{f \in S} f \ (\mu \text{ is const.})$$

$\mu \to [\neg G]$

$$= \Phi_{P, \sqcup S}(X).$$

ⓑ Let $S \subseteq \mathbb{E}$ be a chain

Have to show: $\bigsqcup_{X \in S} \Phi_{P,f}(X) = \Phi_{P,f}(\sqcup S)$.

$$\bigsqcup_{X \in S} \Phi_{P,f}(X) = \bigsqcup_{X \in S} \left([G]\ wp(P', X) + [\neg G] \cdot f \right)$$

$$= \left(\bigsqcup_{X \in S} [G] \cdot wp(P', X) \right) + [\neg G] \cdot f \quad (\alpha \text{ is const})$$

$$= [G] \bigsqcup_{X \in S} wp(P', X) + [\neg G] \cdot f \quad (\mu \text{ is const.})$$

Show: $wp(P', \cdot)$ is continuous.

By induction over structure of P'.

- P' not a while loop $\Rightarrow$ Use Hint.
- $P' = \text{while } (G') \{P''\}$

show: $wp(P', \cdot)$ is continuous.

For all $g \in \mathbb{E}$, $P' = \text{loop}$

$$\Phi_{P', g}(X) = [G'] \, wp(P'', X) + [\neg G] \cdot g$$

S is a chain.

$$\bigsqcup_{X \in S} wp(P', X) = \bigsqcup_{X \in S} \text{lfp } Y. \, \Phi_{P', X}(Y)$$

$$= \bigsqcup_{X \in S} \bigsqcup_{n \geq 0} \Phi^n_{P', X}(0) \quad (\text{Kleene F.P.T.})$$

$$= \bigsqcup_{n \geq 0} \bigsqcup_{X \in S} \Phi^n_{P', X}(0) \quad (\text{suprema commutative})$$

$$= \bigsqcup_{n \geq 0} \Phi^n_{P', \sqcup S}(0) \quad (\text{using part (a)})$$

$$= \text{lfp } Y. \, \Phi_{P', \sqcup S}(Y) = wp(P', \sqcup S).$$

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(a) $\exists y \; [y \cdot y \cdot y < x] \cdot y$ TF!!

(b) $\exists y \; [y \cdot (2x^2 + 7) < 2) \cdot y.$ TF!!

(c) $(\forall y \, \forall z : a(y) = a(z))$