3.

(a)

Markov Chain with rewards

We consider a six-sided fair die. We propose the reward Markov chain (D, r), where D is a Markov chain with state space $\Sigma = \{1,2,3,4,5,6\}$ and, $r : \Sigma \to \mathbb{R}$ is the reward function s.t.

$$r(\sigma) = 1 \quad \forall \sigma \in \Sigma$$

and goal state $G = \{6\}$ since we want to compute the expected # trials needed to see a six for the first time.

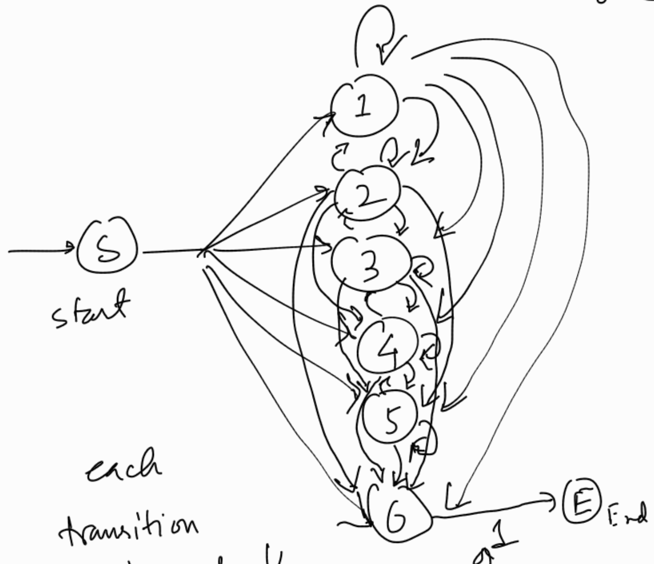
We know that expected rewards in finite Markov chain can be computed in polynomial time by solving a system of linear equations.

We also have a finite Markov chain in this case.

also $P(\sigma, \sigma') = 1/6 \quad \forall \sigma, \sigma' \in \Sigma$.

For $\Pr(\sigma \models \Diamond G) = 1 \quad \forall \sigma \in \Sigma$, we take the variable $y_\sigma \in \mathbb{R}$ for $\sigma \in \Sigma$

$$y_\sigma = \begin{cases} 0 & \text{if } \sigma \in G \\ r(\sigma) + \sum_{\sigma' \in \Sigma} P(\sigma, \sigma') \cdot y_{\sigma'} & \text{o.w.} \end{cases}$$



each transition with prob 1/6 except

Let N be the # trials until the MC hits 6 for the first time. If it hits 6 at the first step (with prob 1/6) we stop or with prob 5/6 it hits any other state we continue. So the expected # trials =

$$\frac{1}{6} \times 1 + 2 \times \frac{5}{6} \times \frac{1}{6} + 3 \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \cdots$$

$$= \frac{1}{6} \sum_{i \geq 1} i \left(\frac{5}{6}\right)^{i-1}$$

$$= \frac{1}{6} \sum_{k \geq 0} (k+1) \left(\frac{5}{6}\right)^{k} \quad \text{(substituting } k = i-1 \text{)}$$

$$= \frac{1}{6} \sum_{k \geq 0} k \left(\frac{5}{6}\right)^{k} + \frac{1}{6} \sum_{k \geq 0} \left(\frac{5}{6}\right)^{k}$$

$$= \cancel{\frac{1}{6}} \times \frac{5/6}{\cancel{(1/6)^2}\ 1/6} + \frac{1}{6} \times \frac{1}{1 - 5/6}$$

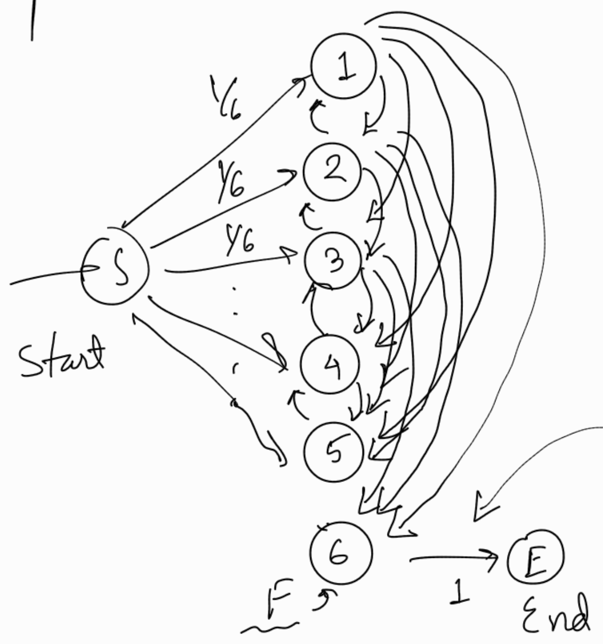
[By using the hint given and property of sum of Geometric Progression].

$$= 5 + \cancel{\frac{1}{6}} \times \frac{1}{\cancel{1/6}}$$

$$= 5 + 1$$

$$= \boxed{6}$$

3. b) The only major condition changed in this case is that we can't go the forbidden state 6 in our first step. So the Markov chain changes accordingly (say D')



All transition probability is $1/6$ except

Since first trial was no six, the minimum # trials to get a six is 2.

So all the paths π such that

$\pi = \{ \sigma_0 \ldots \sigma_n \mid \sigma_0 = 6 \} \notin Paths(D')$