Exercise Sheet 3

General remarks:

- **Due date:** November 11th 12:30 (before the exercise class).
- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes (they will also be recorded).
- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.

Exercise 1 (Metropolis-Hastings) **25P**

(a) [10P] Write a WEBPPL program that given a function f and a *symmetric* sampling kernel $g(x_{t+1}|x_t)$, implements the Metropolis-Hastings algorithm.

(b) [10P] Extend the program from task (a) such that you can visualize the histogram for $n = 20000$ samples, starting at $x_0 = 0.2$ with:

1. $f(x) = \frac{1}{2}e^{-\frac{(x-2)^2}{2}} + \frac{1}{2}e^{-\frac{(x+2)^2}{2}}$, with $g(x_{t+1}|x_t) \sim \mathcal{N}(x_t, 0.3)$.
2. $f(x) = \begin{cases} -0.1319x^4 + 1.132x^2 + 0.5, & x \in (-3, 3) \\ 0, & \text{o.w.} \end{cases}$, with $g(x_{t+1}|x_t) \sim \mathcal{N}(x_t, 0.3)$

(c) [5P] Describe, what impact the parameter σ^2 of the proposal distribution has.

Exercise 2 (pGCL) **25P**

(a) [10P] Consider the following description of a stochastic process:

> *You live in the $\mathbb{Z}^2$ plane and are currently standing on position (x_0, y_0). You choose a direction (North, South, West, East) uniformly at random and take a step in this direction. You repeat this process until you have reached $(0, 0)$.*

Design a pGCL program that exactly implements this behavior.

(b) [15P] Give the operational semantics of the pGCL program defined in (a) as a Markov chain for one loop iteration.

Exercise 3 (Markov chains with rewards) **25P**

(a) [15P] Consider a six-sided fair die. Compute the *expected* number of trials needed to see a six for the first time by modeling the problem with a Markov chain with rewards.

Hint: The following variant of the geometric series might be helpful: For $0 < q < 1$,

$$\sum_{k \geq 0} kq^k = \frac{q}{(1-q)^2} \,.$$

(b) [10P] Modify your reward Markov chain from part (a) such that it models the *conditional* expected number of trials needed to see a six for the first time *given that the first trial was no six*. Indicate the set F of "forbidden" states. You do not have to compute the conditional expected reward.

Exercise 4 (Complete lattices) $\boxed{\textbf{25P}}$

Let $(D, \sqsubseteq)$ be a complete lattice. Further, let $D \to D$ be the set of all monotone functions between elements of D. We lift the order $\sqsubseteq$ to the domain $D \to D$ by pointwise application, i.e., we define $(D \to D, \preceq)$ where $\preceq$ is given by:

$$f \ \preceq \ g \qquad \Longleftrightarrow \qquad \forall d \in D. \quad f(d) \ \sqsubseteq \ g(d).$$

(a) [5P] Show that $(D \to D, \preceq)$ is a partial order.

(b) [5P] Show that $(D \to D, \preceq)$ is a complete lattice.

(c) [15P] Now assume that D satisfies the *ascending chain condition*(ACC, for short), i.e., for all chains $S \subseteq D$, there exists a positive integer n such that $s_n = s_{n+1} = s_{n+2} = \ldots$, or in other words *no infinite strictly ascending sequence of elements in D exists.*

Prove or disprove that fixed points of chains are continuous, i.e.,

$$\mathrm{lfp}\left(\bigsqcup \mathcal{F}\right) \quad = \quad \bigsqcup \{\mathrm{lfp}(f) \mid f \in \mathcal{F}\}$$

holds for all chains $\mathcal{F} \subseteq (D \to D)$.

Hint: If $(D, \sqsubseteq)$ is a complete lattice satisfying the ACC, $S \subseteq D$ a chain and $\Phi : D \to D$ is a monotone function, then $\Phi\left(\bigsqcup S\right) = \bigsqcup \Phi[S]$.