③ Monotonicity of weakest pre-expectations

ⓐ $(D, \sqsubseteq)$ - a complete lattice and $f, g : D \to D$ monotonic such that $\forall\, d \in D$, $f(d) \sqsubseteq g(d)$.

Using extended Knaster-Tarski Theorem, we can directly write $\text{lfp}\, f = \sqcap S$, where $S = \{d \in D \mid f(d) \sqsubseteq d\}$.

Also, $\text{lfp}\, g = \sqcap S'$, where $S' = \{d \in D \mid g(d) \sqsubseteq d\}$

Consider any such S' as above. $\forall\, d \in S' \subseteq D$, $f(d) \sqsubseteq g(d) \sqsubseteq d$. This means all elements d in any set S' also satisfies $\{d \in D \mid f(d) \sqsubseteq d\}$.

So, we have more sets $S = \{d \in D \mid f(d) \sqsubseteq d\}$ than sets S' satisfying $S' = \{d \in D \mid g(d) \sqsubseteq d\}$.

Now, since $\sqcap S$ or $\sqcap S'$ represents intersections of such states and we have more sets like S than S', by basic properties of intersection $\sqcap S \sqsubseteq \sqcap S'$ which gives us

$$\text{lfp}\, f = \sqcap S \sqsubseteq \sqcap S' = \text{lfp}\, g$$

$$\Rightarrow \text{lfp}\, f \sqsubseteq \text{lfp}\, g \qquad \square$$

(b) For any GCL program P, the function $wp(P) : \mathbb{P} \to \mathbb{P}$ takes a predicate as postcondition and returns the predicate which is the weakest precondition of P w.r.t to the given postcondition.

We consider the ordering $\sqsubseteq \; = \; \Rightarrow$ on $\mathbb{P}$.

So, we have $F \sqsubseteq G$ iff $F \Rightarrow G$.

Now, $wp[\![P]\!](F)$ refers to the precondition that reaches/terminates in a state $t \models F$

and hence $wp[\![P]\!](F) \Rightarrow F$

By transitivity of $\Rightarrow$

$wp[\![P]\!](F) \Rightarrow G$ . . . . . . (1)

Similarly, $wp[\![P]\!](G)$ refers to the pre-condition that reaches a state $s \models G$, which gives us $wp[\![P]\!](G) \Rightarrow G$ . . . (2)

So if $H = wp[\![P]\!](G)$, then P starts in state $r \models H$ and terminates in a state $s \models G$.

Now for the post condition G, H can be wp iff. P starts in a state in H and terminates in G. Considering that P can also start from a state in $wp[\![P]\!](F)$ and terminate at G (by ①), either H and $wp[\![P]\!](F)$ are same or $wp[\![P]\!](F)$ can reach H, meaning

$$wp[\![P]\!](F) \Rightarrow H = wp[\![P]\!](G)$$

which proves that $wp(P) : \mathbb{P} \to \mathbb{P}$ is monotonic. $\square$