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Probabilistic Programming
Exercise Sheet 2 Solutions
Group 7

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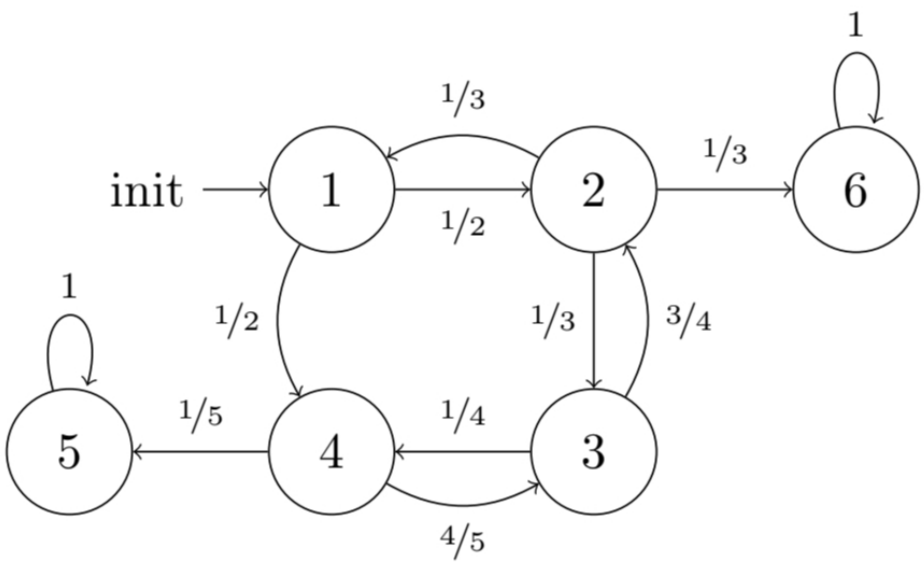
1. We consider the following Markov Chain with state space $\Sigma = \{1, 2, 3, 4, 5, 6\}$ and goal states G = $\{6\}$

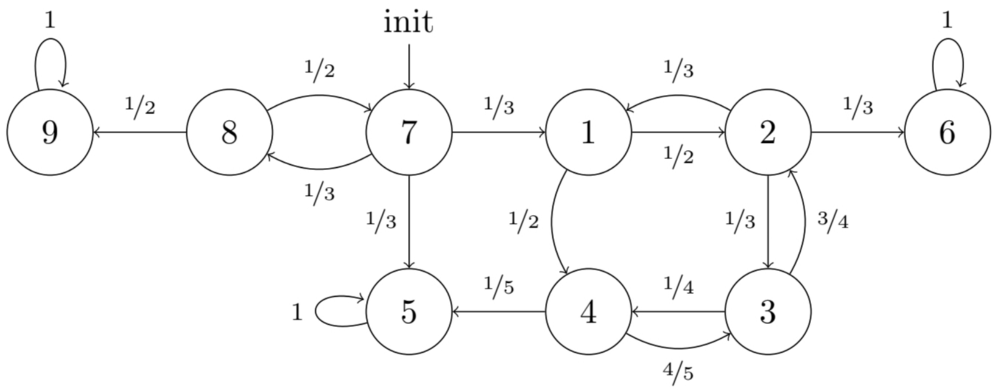


   (a) From the given Markov Chain above, we formulate $\Sigma_?$ as follows :
   $\Sigma_?$ is the set of states that can reach G by > 0 steps and we can reach goal state 6 from states 1, 2, 3 and 4 in at least 2, 1, 2 and 3 steps respectively. Also, 5 is an absorbing state from which we cannot leave. Therefore, we have, $Pre^*(G) = \{1, 2, 3, 4, 6\}$ and $\Sigma_? = Pre^*(G) \backslash G = \{1, 2, 3, 4\}$ for G = $\{6\}$

   (b) As per definition, we have -
   $A = (P(\sigma, \tau))_{\sigma,\tau\in\Sigma_?}$ which are the transition probabilities in $\Sigma_?$
   $b = (b_\sigma)_{\sigma\in\Sigma_?}$ which are the probabilities to reach G in 1 step, that is $b_\sigma = \sum_{\gamma\in G} P(\sigma, \gamma)$
   For G = $\{6\}$, we therefore have

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{4}{5} & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$

(c) We now have an extension of the previous Markov Chain as follows,



In this new MC, we are given G as $G = \{5, 6\}$. As such, for this new MC and new set of goal states, we have $\Sigma_? = \{1,\ 2,\ 3,\ 4,\ 7,\ 8\}$
In turn, we construct A and b as follows (where A and b are as defined above in Part (b).

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{4}{5} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{5} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

We now solve the following system of linear equations to obtain our desired probabilities
$(I - A)\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (x_\sigma)_{\sigma \in \Sigma_?}$ with $x_\sigma = Pr(\sigma| = \diamond G)$ as the unique solution, where, I is the identity matrix of order 6. So, we have -

$$\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & 0 & 0 \\ \frac{-1}{3} & 1 & \frac{-1}{3} & 0 & 0 & 0 \\ 0 & \frac{-3}{4} & 1 & \frac{-1}{4} & 0 & 0 \\ 0 & 0 & \frac{-4}{5} & 1 & 0 & 0 \\ \frac{-1}{3} & 0 & 0 & 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 0 & 0 & \frac{-1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{5} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

We solve the above system of linear equations and obtain the following solution,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 0.8 \\ 0.4 \end{bmatrix}$$

Therefore, the probability of reaching G from initial position 7 is 0.8.

(d) For any Markov Chain $D = (\Sigma, \sigma_I, P)$ with goal set $\emptyset \neq G \subseteq \Sigma$ such that $\Sigma_? \neq \emptyset$, we know that G is reachable from any state in $\Sigma_?$. Further, the transition probability matrix of a finite Markov Chain is stochastic, with every row summing up to 1.
Now, to construct the matrix A for non-null $\Sigma_?$, we eliminate the corresponding row(s) and column(s) from P for which we cannot reach G in > 0 steps. In fact, we also eliminate the row(s) and column(s) corresponding to the goal state(s). Since a state in G is reachable from all the states in $\Sigma_?$, there has to be a connection or one step path from a state in $\Sigma_?$ to a state in G with non-zero probability. Removing the states in G also removes the corresponding entry in the row for that state in $\Sigma_?$. As such, there will exist atleast one row which has a sum strictly less than 1.
Hence, the matrix A is always proper sub-stochastic.

2. (a) Check sheet02_ex2a-b.wppl for the code.

   (b) Check sheet02_ex2a-b.wppl for the code. After comparing the two different sampling techniques, the MCMC with Metropolis-Hasting ran in 62 ms, while the rejection sampling took 5658 ms, significantly slower then MCMC!

   (c)

2.c. As stated in the hint the volume of the d-dimensional ball is:

The acceptance rate of a sample is proportional to the ratio of volumes of the d-dimensional unit box ($1^d = 1$) and ball $\left(\frac{2\pi^{\frac{d}{2}}}{d \cdot (\frac{d}{2}-1)!} \right)$

$\Rightarrow$ meaning The acceptance probability is:

$$p = \frac{V_0}{V_{\square}} = \frac{2\pi^{\frac{d}{2}}}{d \cdot (\frac{d}{2}-1)!}$$

- $\underline{\text{for} \quad d = 10:}$

$$p = \frac{2\pi^{2.5}}{10 \cdot (4!)} = \frac{34.98}{240} = 0.145$$

$\Rightarrow$ as seen in the lecture, following the geometric distribution Expectation, the Expected number of samples to obtain 1 accepted one is: $E = \frac{1}{p}$

$\Rightarrow$ meaning to obtain 1000 accepted samples we will need: $1000E = \frac{1000}{p} = \frac{1000}{0.145} = 6859.7$

$= \boxed{6860 \quad \text{samples}}$

$\Rightarrow$ similarly, for $\underline{d = 20}$

$$\boxed{P = \frac{2\pi^{10}}{20 \cdot 9!} = 0.0258}$$

$$E = 38.749$$

$$1000E = 38{,}749{,}34 \approx \boxed{38{,}750 \quad \text{samples}}$$

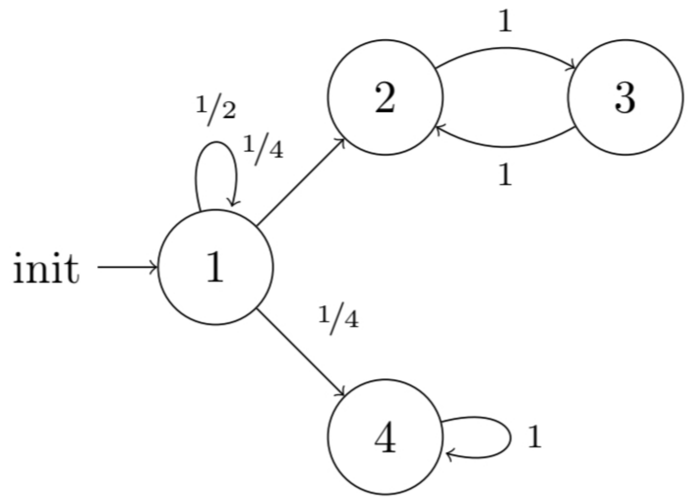
$\Rightarrow$ and for $\underline{d = 40}$:

$$\boxed{P = \frac{2\pi^{20}}{40 \cdot 19!} = 3.6 \cdot 10^{-9}}$$

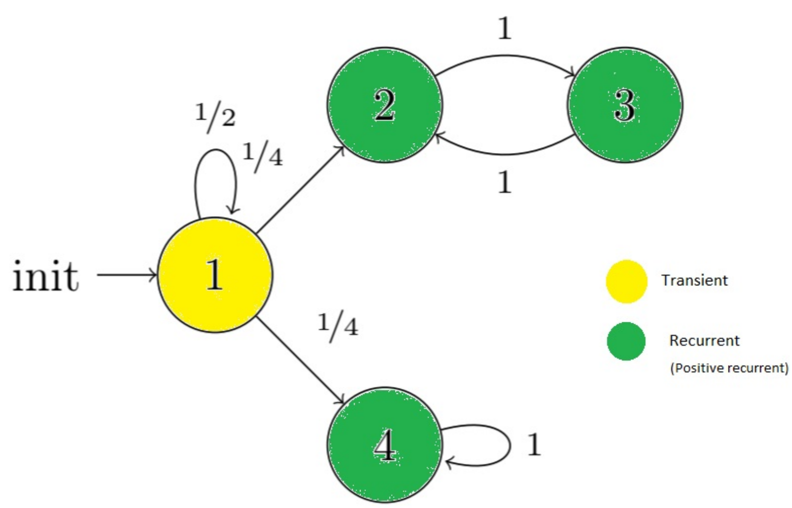
$$E = 277{,}413{,}226.2$$

$$1000E = \boxed{2.77 \cdot 10^{11} \quad \text{samples}}$$

3. We consider the following Markov Chain with a finite state space $\Sigma = \{1, 2, 3, 4\}$



(a) To classify each state of the above Markov Chain as either transient, positive recurrent or null recurrent, we first look at strongly connected components such as state 4 which is an absorbing state and returns to the same state after reaching 4 in one step making it a positive recurrent state. Similarly, states 2 and 3 belong to a strongly connected component for which expected number of steps necessary to come back to those states are 2. Therefore, states 2 and 3 are also positive recurrent. From state 1, we could go to either 2 or 4 and never return to 1 which makes it a transient state. Consequently, for the given Markov Chain we have (also see the figure below),
Transient states: {1}
Positive recurrent states: {2, 3, 4}



(b) The Markov Chain is **not irreducible** since it's not strongly connected. We can not reach states 1, 2 or 3 from state 4 since 4 is an absorbing state which makes the Markov chain reducible.

(c) We can use the same arguments given in the solution of 3 (a) to find the irreducible components of the above Markov chain. States 2 and 3 form a strongly connected component which can not be left. Also, the absorbing state 4 (with self-loop) forms its own irreducible component. So the **irreducible components are {2, 3} and {4}**. For the irreducible component {2, 3}, we have the transition probability matrix $P_{2,3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where we have $P_{2,3}^2 = I_2(2 \times 2 \text{ identity matrix})$ and so, $P_{2,3} = P_{2,3}^3 = P_{2,3}^5 = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_{2,3}^2 = P_{2,3}^4 = P_{2,3}^6 = \cdots = I_2$. Therefore the component {2, 3} is periodic with a periodicity of 2. On the other hand, {4} has a period of 1 meaning it's aperiodic.

(d) To calculate the stationary distribution, we can solve the system of equations $x = x \cdot P$, where, P is the transition probability matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and x a probability vector $[x_1, x_2, x_3, x_4]^T$ such that $\sum_{i=1}^{4} x_i = 1$. So, we have

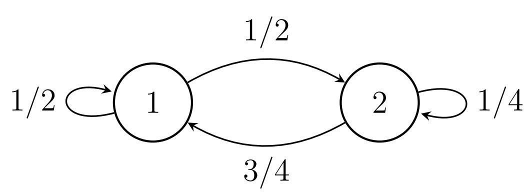
$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$\implies \frac{x_1}{2} = x_1$, $\frac{x_1}{4} + x_3 = x_2$, $x_2 = x_3$ and $\frac{x_1}{4} + x_4 = x_4$

$\implies x_1 = 0, x_2 = x_3$ and $x_4 = 1 - x_2 - x_3$ as $\sum_{i=1}^{4} x_i = 1$.

Therefore we can find infinitely many solutions for x_2 and x_3 for $x_2 = x_3 \in [0, 1]$ which shows that the Markov chain has infinitely many stationary distributions.

(e) An example of an irreducible and aperiodic Markov chain with exactly 2 states can be



So, we have the transition probability matrix

$$\text{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

We can see from the Markov chain that it's strongly connected and hence irreducible. Also, the Markov chain has a period of 1 meaning it's aperiodic.

To calculate the stationary distribution, we can solve the system of equations $x = x \cdot P$, where, P is the transition probability matrix and x a probability vector $[x_1, x_2]^T$ such that $\sum_{i=1}^{2} x_i = 1$.

Therefore, we get $\frac{x_1}{2} + \frac{3x_2}{4} = x_1$ and $\frac{x_1}{2} + \frac{x_2}{4} = x_2$

$\implies x_1 = \frac{3x_2}{2}$ and $x_1 = 1 - x_2$

$\implies 1 - x_2 = \frac{3x_2}{2}$

$\implies x_2 = \frac{2}{5}$ and $x_1 = \frac{3}{5}$.

Hence, the stationary distribution for our example Markov chain is $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$.