Exercise Sheet 6

General remarks:

- **Due date:** December 9th 12:30 (before the exercise class). Note that this sheet is worth 150 points but you have two weeks to work on it.
- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.
- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.

Exercise 1 (Probabilistic invariants) $\boxed{30P}$

Consider the following `pGCL` program P:

$$\texttt{while}\ (x > 0)\ \{\ \{\ x := x - 1\ \}\ [p]\ \{\ x := x + 1\ \}\ \}\ .$$

For $p = 1/3$ you have already seen in the lecture that $I = [x > 0] \cdot 2^{-x} + [x \leq 0]$ is a wp-superinvariant[1] of P wrt. the constant post-expectation $f = 1$.

(a) [2P] Give an non-trivial[2] upper bound on the probability that the program terminates when started with $x = 3$. Assume that $p = 1/3$.

(b) [25P] Now let $p \in [0, 1]$. Find a *wp*-superinvariant $I_p \neq 1$ (depending on p) of P wrt. to post-expectation $f = 1$ and prove that your answer is correct.

(c) [3P] Use your result from (b) to compute a non-trivial upper bound on the probability that the program terminates when started in $x = 100$, assuming $p = 0.49$.

Exercise 2 (Exact *wp* via conditional difference boundedness) $\boxed{30P}$

Consider the following `pGCL` program P:

$$\begin{array}{l}\texttt{while}\ (x > 0)\ \{\\ \quad c := c + 1\ ;\\ \quad \{\ x := x - 1\ \}\ [1/2]\ \{\ \texttt{skip}\ \}\\ \}\end{array}$$

Determine $wp[\![P]\!](c)$ exactly. Proceed as follows:

- Guess a fixed point I of Φ_c and verify it.
- Verify that $I \sqsubseteq \mathrm{lfp}\ \Phi_c$ by applying the rule from Slide 24 of Lecture 11 (this will only work if your guessed I is actually the least fixed point). You may assume without proof that P terminates in finite expected time from any initial state.

[1] In fact, I is the *least* fixed point of Φ_f.

[2] A *non-trivial* upper bound on a probability is a bound that is strictly smaller than 1.

Hint: $wp⟦P⟧(c)$ is an expectation that describes the expected value of variable c after program termination given the initial values of x and c. Use this intuition to guess the correct I.

Exercise 3 (Conditioning in pGCL programs) $\boxed{\textbf{30P}}$

Consider the following scenario: A telephone operator has forgotten what day of the week it is. However, she knows that she receives on average ten calls per hour in the week and three calls per hour at the weekend. She observes that she receives four calls in a given hour.

(a) [10P] Write a cpGCL program P modeling the above scenario.
Hint: You may use a sampling statement like $x :\approx \mathcal{D}(p)$, where $\mathcal{D}$ is some discrete distribution and p a parameter.

(b) [5P] Give an expectation f stating (for the program P) that it is a week day.

(c) [15P] Use the *cwp*-calculus to help the telephone operator to decide whether it is a week day. To this end, determine the probability that is a week day by computing $cwp(P, f)$.

Hint: We assume that probability of receiving k calls in an hour is given by a (discrete) poisson distribution. That is, if we know that we receive on average r calls per hour, then the probability of receiving k calls in a given hour is $\frac{r^k}{k!} \cdot e^{-r}$.

Exercise 4 (From pGCL to conditional reward Markov chains) $\boxed{\textbf{30P}}$

Consider the following pGCL program P:

```
1 :   a, b, c := 0, 0, 0;
2 :   {c := c + 1; } [0.5] {skip};
3 :   {a := 1} [0.6] {skip};
4 :   if(a = 1){
5 :       {c := c + 1; b := 1} [0.2] {c := c − 1};
6 :   } else {
7 :       c := 0
8 :   };
9 :   observe (a ≠ 0 ∨ b ≠ 0)
```

(a) [20P] Construct the reward Markov chain corresponding to P (you may use the line numbers to refer to sub-programs).

(b) [10P] Compute the expected value of c after termination of P.

Exercise 5 (Guard strengthening for lower bounds) $\boxed{\textbf{30P}}$

Prove the "guard strengthening for lower bounds" proof rule from lecture 11, slide 26:

Let $P_{loop} = \texttt{while}\ (\varphi)\ \{\ P\ \}$ and $P'_{loop} = \texttt{while}\ (\varphi')\ \{\ P\ \}$, and expectations f and I. Then it holds:

$$\left(\varphi' \Rightarrow \varphi \quad \wedge \quad I \sqsubseteq wp⟦P'_{loop}⟧([\neg\varphi] \cdot f)\right) \quad \text{implies} \quad I \sqsubseteq wp⟦P_{loop}⟧(f)\ .$$

Hint 1: Prove $wp⟦P'_{loop}⟧([\neg\varphi] \cdot f) \sqsubseteq wp⟦P_{loop}⟧(f)$ where $\varphi' \Rightarrow \varphi$ first.

Hint 2: In exercise sheet 4, exercise 3 (a), you have shown the following useful lemma:

Let $(D, \sqsubseteq)$ be a complete lattice and $f, g \colon D \to D$ be monotonic such that for all $d \in D$, $f(d) \sqsubseteq g(d)$. Then, lfp $f \sqsubseteq$ lfp g holds.