# Exercise Sheet 4

**General remarks:**

- **Due date:** November 18nd 12:30 (before the exercise class).
- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.
- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.

**Exercise 1 (Markov Chains and Fixed Points)** **25P**

Consider a Markov chain $D = (\Sigma, \sigma_I, \mathbf{P})$ with goal set $G \subseteq \Sigma$. Let[1] $\Sigma_? = \Sigma \setminus G$. Further, define the matrix $\mathbf{A}$ via $\mathbf{A} = (\mathbf{P}(\sigma,\tau))_{\sigma,\tau\in\Sigma_?}$, and the vector $\mathbf{b} = (b_\sigma)_{\sigma\in\Sigma_?}$ where $b_\sigma = \sum_{\gamma\in G}\mathbf{P}(\sigma,\gamma)$.

In the rest of this exercise we identify vectors whose entries are indexed by the states in $\Sigma_?$ with functions of type $\Sigma_? \to [0,1]$. For instance, we have $\mathbf{b}\colon \Sigma_? \to [0,1], \mathbf{b}(\sigma) = b_\sigma$.

Now consider the function

$$f\colon (\Sigma_? \to [0,1]) \to (\Sigma_? \to [0,1]),\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}\ .$$

(a) [15P] Define a partial order on $\Sigma_? \to [0,1]$ and prove that $f$ has a least fixed point wrt. this order.

(b) [10P] Use the results of lecture #7 to formulate an iterative algorithm that approximates the least fixed point of $f$ *from below*.

**Exercise 2 (Program verification with weakest pre-conditions)** **25P**

(a) [10P] Compute $wp(P, f)$ for the program $P$ and post-condition $P$ below:

$P$: $\texttt{if}\ (x > 0)\ \{\ x := y + 1\ \}\ \texttt{else}\ \{\ \texttt{if}\ (x = 0)\ \{\ \texttt{skip}\ \}\ \texttt{else}\ \{\ x := x + 1\ \}\ \}$

$f$: $x > y$

(b) [15P] For the following loop $P$, compute the loop-characteristic functional $\Phi_f(f)$ with post-condition $f$ and parameter $f$. What does the result tell you about lfp $\Phi_f$ and $wp(P, f)$? Hint: $(\mathbb{P}, \Rightarrow)$ is a complete lattice.

$P$: $\texttt{while}\ (x \geq 1)\ \{\ a := a * x;\ x := x - 1\ \}$

$f$: $a = x!$

[1] Note that this definition of $\Sigma_?$ slightly differs from the one given in lecture #3

**Exercise 3 (Monotonicity of weakest pre-expectations)** $\boxed{\textbf{25P}}$

The goal of this exercise is to prove that $wp(P)$ is a *monotonic* predicate transformer.

(a) [8P] Let $(D, \sqsubseteq)$ be a complete lattice and $f, g\colon D \to D$ be monotonic such that for all $d \in D$, $f(d) \sqsubseteq g(d)$. Show that $\text{lfp}\ f \sqsubseteq \text{lfp}\ g$.
*Hint:* You may use without proof that $\text{lfp}\ f = \bigsqcap S$, where $S = \{d \in D \mid f(d) \sqsubseteq d\}$.

(b) [17P] Prove that for all `GCL` programs $P$, the function $wp(P)\colon \mathbb{P} \to \mathbb{P}$ is monotonic.

**Exercise 4 (Weakest liberal pre-conditions)** $\boxed{\textbf{25P}}$

(a) [5P] Give a `GCL` program $P$ along with a post-condition $f$ such that $wp(P, f) \neq wlp(P, f)$. You may not use the `diverge` statement in your program!

(b) [10P] Determine the inputs for which the following program $P$ does *not* terminate by computing the *weakest liberal pre-condition* with respect to a suitable post-condition:

$$\texttt{while}\ (x \neq 0)\ \{\ x := x - 2\ \}$$

(c) [10P] Prove by induction on the program structure that for any `GCL` program $P$ and post-condition $f$ it holds that $wp(P, f) \sqsubseteq wlp(P, f)$.
*Hint:* Use Exercise 3 (b). You may use that $wlp(P)$ is monotonic without proof.