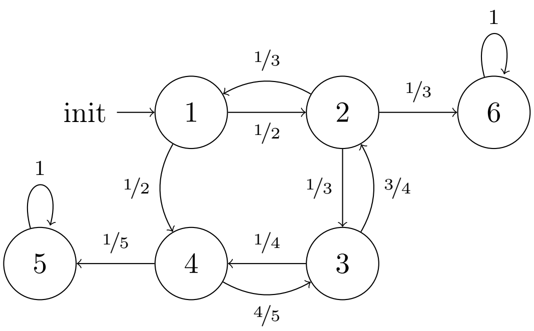
Exercise Sheet 2

General remarks:

- **Due date:** November 4th 12:30 (before the exercise class).
- Please submit your solutions via MOODLE. Remember to provide your matriculation number. It is necessary to hand in your solutions in groups of **three**. You may use the MOODLE forum to form groups.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.
- If you have any questions regarding the lecture or the exercise, please use the forum in MOODLE.
- Provide solutions to programming exercises in a separate plain text file. Please name the files according to the exercise they belong to (e.g. 'sheet01_ex1a.wppl'). All your programs are required to run without errors in the online interpreter available at `http://webppl.org`.

Exercise 1 (Reachability in Markov chains) $\boxed{\textbf{35P}}$

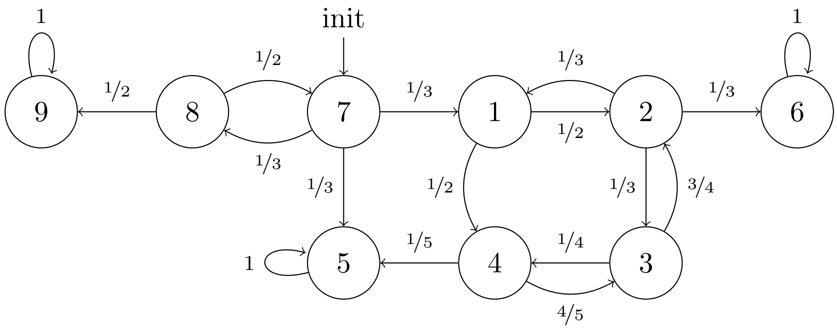
Consider the following Markov chain with state space $\Sigma = \{1, 2, 3, 4, 5, 6\}$ and goal states $G = \{6\}$:



(a) [5P] Compute the set $\Sigma_?$ as defined in the lecture.

(b) [10P] Set up the equation system $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{b}$ defined in the lecture by giving the matrix $\mathbf{A}$ and the vector $\mathbf{b}$. You do not need to solve the system.

(c) [10P] We now extend the Markov chain from above as follows:



This time, let $G = \{5, 6\}$. Compute $Pr(\Diamond G)$ and justify your answer!

(d) [10P] A *sub-stochastic matrix* $\mathbf{M}$ is a square matrix with coefficients in $[0, 1]$ such that each row sum is *at most* 1. M is called *proper* if at least one row sum is *strictly smaller than 1*. Now let $D = (\Sigma, \sigma_I, \mathbf{P})$ be any Markov chain with goal set $\emptyset \neq G \subseteq \Sigma$ such that $\Sigma_? \neq \emptyset$. Prove or disprove: The matrix $\mathbf{A}$ is a proper sub-stochastic matrix.

Exercise 2 (Rejection sampling) $\boxed{\textbf{25P}}$

(a) [10P] Write a WebPPL program that samples uniformly from a d-dimensional ball with radius 1 centered at the origin.

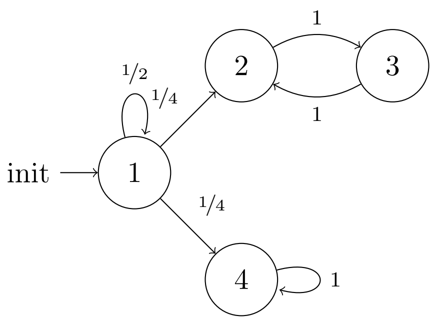
(b) [5P] Compare (qualitatively) the time taken for inferring the posterior distribution using MCMC with Metropolis-Hastings kernel vs. rejection sampling both with parameters $d = 10$ dimensions and $n = 1000$ samples.

(c) [10P] Compute the probability that a sample gets accepted when using a uniform proposal distribution (on $[-1, 1]^d$) for $d \in \{10, 20, 40\}$ dimensions. Also give the expected total number of samples that have to be done for obtaining $n = 1000$ accepted samples. **Hints:**

- The acceptance rate of a sample is proportional to the ratio of volumes of the d-dimensional unit box and ball.
- The volume of a d-dimensional unit ball is $\frac{2\pi^{\frac{d}{2}}}{d \cdot \Gamma(\frac{d}{2})}$ where $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Exercise 3 (Properties of Markov chains) $\boxed{\textbf{25P}}$

Consider the following Markov chain with state space $\Sigma = \{1, 2, 3, 4\}$:



(a) [4P] Classify each state of the Markov chains as either transient, positive recurrent or null recurrent!

(b) [2P] Is the Markov chain irreducible? Justify your answer!

(c) [6P] Give all the irreducible components of the above Markov chain (Hint: There are 2). For each component say whether it is aperiodic and justify your claim.

(d) [8P] Show that the Markov chain has infinitely many stationary distributions.

(e) [5P] Give an example of an irreducible and aperiodic Markov chain with exactly 2 states and compute all its stationary distributions.