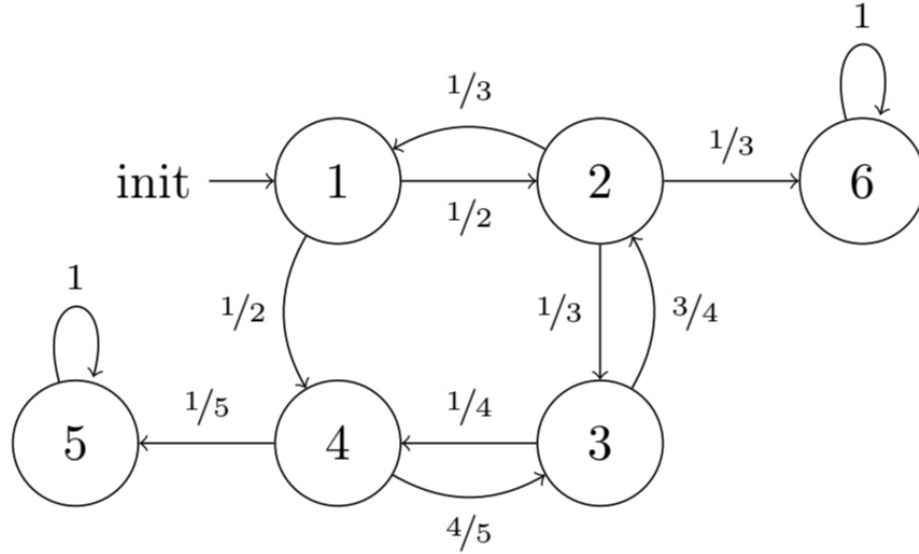


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## Probabilistic Programming Exercise Sheet 2 Solutions Group 7

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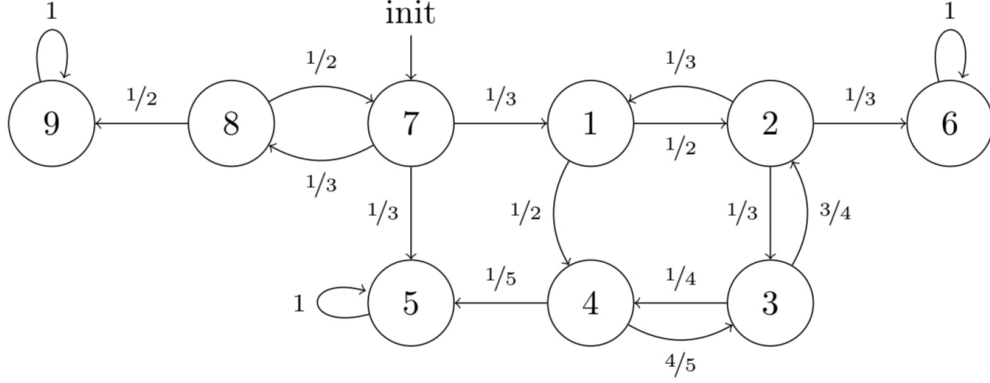
1. We consider the following Markov Chain with state space  $\Sigma = \{1, 2, 3, 4, 5, 6\}$  and goal states  $G = \{6\}$



- (a) From the given Markov Chain above, we formulate  $\Sigma_?$  as follows :  
 $\Sigma_?$  is the set of states that can reach  $G$  by  $> 0$  steps and we can reach goal state 6 from states 1, 2, 3 and 4 in at least 2, 1, 2 and 3 steps respectively. Also, 5 is an absorbing state from which we cannot leave. Therefore, we have,  $Pre^*(G) = \{1, 2, 3, 4, 6\}$  and  $\Sigma_? = Pre^*(G) \setminus G = \{1, 2, 3, 4\}$  for  $G = \{6\}$
- (b) As per definition, we have -  
 $A = (P(\sigma, \tau))_{\sigma, \tau \in \Sigma_?}$  which are the transition probabilities in  $\Sigma_?$   
 $b = (b_\sigma)_{\sigma \in \Sigma_?}$  which are the probabilities to reach  $G$  in 1 step, that is  $b_\sigma = \sum_{\gamma \in G} P(\sigma, \gamma)$   
 For  $G = \{6\}$ , we therefore have

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{4}{5} & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$

(c) We now have an extension of the previous Markov Chain as follows,



In this new MC, we are given  $G$  as  $G = \{5, 6\}$ . As such, for this new MC and new set of goal states, we have  $\Sigma_\gamma = \{1, 2, 3, 4, 7, 8\}$

In turn, we construct  $A$  and  $b$  as follows (where  $A$  and  $b$  are as defined above in Part (b)).

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{4}{5} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{5} \\ \frac{1}{3} \\ 0 \end{bmatrix} \quad \checkmark$$

We now solve the following system of linear equations to obtain our desired probabilities

$(I - A)\mathbf{x} = \mathbf{b}$ , where  $\mathbf{x} = (x_\sigma)_{\sigma \in \Sigma_\gamma}$  with  $x_\sigma = Pr(\sigma | = \diamond G)$  as the unique solution, where,  $I$  is the identity matrix of order 6. So, we have -

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \\ \frac{1}{5} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

We solve the above system of linear equations and obtain the following solution,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 0.8 \\ 0.4 \end{bmatrix} \quad \checkmark$$

Therefore, the probability of reaching  $G$  from initial position 7 is 0.8.

- (d) For any Markov Chain  $D = (\Sigma, \sigma_I, P)$  with goal set  $\emptyset \neq G \subseteq \Sigma$  such that  $\Sigma_? \neq \emptyset$ , we know that  $G$  is reachable from any state in  $\Sigma_?$ . Further, the transition probability matrix of a finite Markov Chain is stochastic, with every row summing up to 1. Now, to construct the matrix  $A$  for non-null  $\Sigma_?$ , we eliminate the corresponding row(s) and column(s) from  $P$  for which we cannot reach  $G$  in  $> 0$  steps. In fact, we also eliminate the row(s) and column(s) corresponding to the goal state(s). Since a state in  $G$  is reachable from all the states in  $\Sigma_?$ , there has to be a connection or one step path from a state in  $\Sigma_?$  to a state in  $G$  with non-zero probability. Removing the states in  $G$  also removes the corresponding entry in the row for that state in  $\Sigma_?$ . As such, there will exist atleast one row which has a sum strictly less than 1. Hence, the matrix  $A$  is always proper sub-stochastic.

Your idea is correct, but this is not really a formal proof of the claim.

Argue with the transition probabilities of state  $G \in \Sigma_?$  to states in  $G$ . -2P

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2. (a) Check sheet02\_ex2a-b.wppl for the code. ✓  
 (b) Check sheet02\_ex2a-b.wppl for the code. After comparing the two different sampling techniques, the MCMC with Metropolis-Hasting ran in 62 ms, while the rejection sampling took 5658 ms, significantly slower than MCMC! ✓  
 (c)

2. Co As stated in the hint the volume of the  $d$ -dimensional ball is: ?

The acceptance rate of a sample is proportional to the ratio of volumes of the  $d$ -dimensional unit box ( $\pm 1^d = 1$ )

and ball  $\left( \frac{2\pi^{\frac{d}{2}}}{d \cdot (\frac{d}{2} - 1)!} \right)$

The volume is  $2^d$  as you take the box from  $-1$  to  $1$  (to contain the ball / all possible points you can sample)  
-2p

$\Rightarrow$  meaning The acceptance probability is:

$$p = \frac{V_0}{V_B} = \frac{2\pi^{\frac{d}{2}}}{d \cdot (\frac{d}{2} - 1)!}$$

• For  $d = 10$ :

$$p = \frac{2\pi^{\underline{2.5}}}{10 \cdot (4!)} = \frac{34.98}{240} = 0.145$$

$\frac{d}{2} = 5$ , not 2.5 - 1p

$\Rightarrow$  as seen in the lecture, following the geometric distribution Expectation, the Expected number of samples to obtain 1 accepted one is:  $E = \frac{1}{p}$

$\Rightarrow$  meaning to obtain 1000 accepted samples

we will need:  $1000E = \frac{1000}{p} = \frac{1000}{0.145} = 6859.7$

6860 samples

⇒ similarly, for  $d = 20$

$$P = \frac{2\pi^{10}}{20 \cdot 9!} = 0.0258$$

$$E = 38.749$$

$$1000E = 38,749,34 \approx 38,750 \text{ samples}$$

⇒ and for  $d = 40$ :

$$P = \frac{2\pi^{20}}{40 \cdot 19!} = 3.6 \cdot 10^{-9}$$

$$E = 277,413,226.2$$

$$1000E = 2.77 \cdot 10^{11} \text{ samples}$$

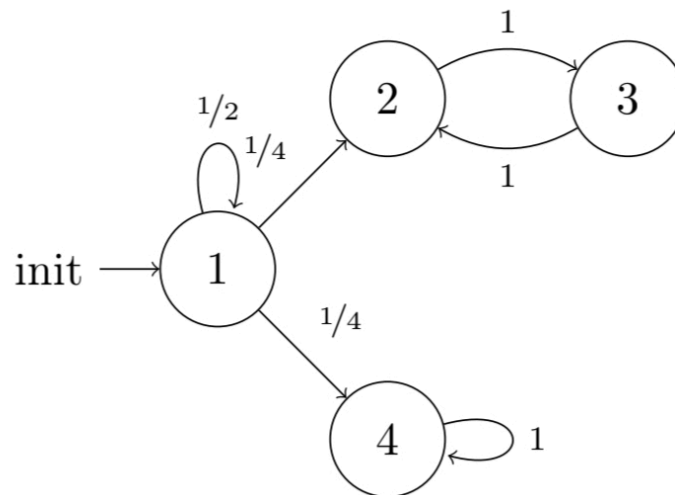
(V)

Your results are wrong due to the wrong computation of  $\text{Vol}(B_n)$ .

However, your calculations themselves are (mostly) correct afterwards.

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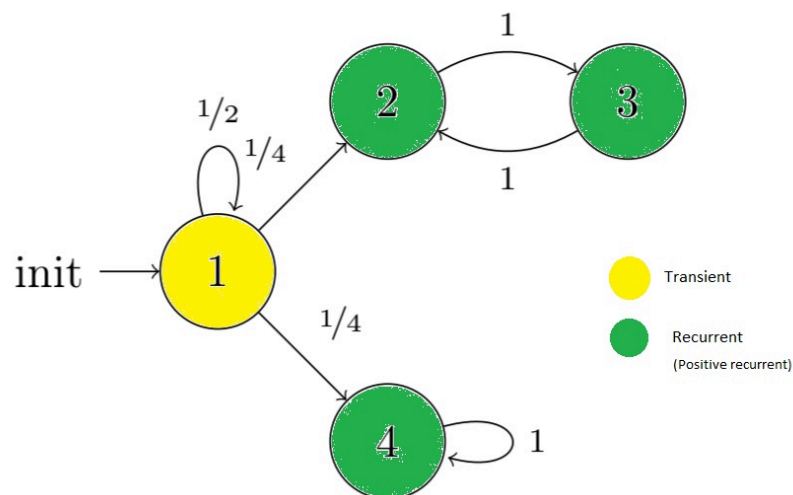
3. We consider the following Markov Chain with a finite state space  $\Sigma = \{1, 2, 3, 4\}$



- (a) To classify each state of the above Markov Chain as either transient, positive recurrent or null recurrent, we first look at strongly connected components such as state 4 which is an absorbing state and returns to the same state after reaching 4 in one step making it a positive recurrent state. Similarly, states 2 and 3 belong to a strongly connected component for which expected number of steps necessary to come back to those states are 2. Therefore, states 2 and 3 are also positive recurrent. From state 1, we could go to either 2 or 4 and never return to 1 which makes it a transient state. Consequently, for the given Markov Chain we have (also see the figure below),

**Transient states: {1}**

**Positive recurrent states: {2, 3, 4}**



- (b) The Markov Chain is **not irreducible** since it's not strongly connected. We can not reach states 1, 2 or 3 from state 4 since 4 is an absorbing state which makes the Markov chain reducible.
- (c) We can use the same arguments given in the solution of 3 (a) to find the irreducible components of the above Markov chain. States 2 and 3 form a strongly connected

component which can not be left. Also, the absorbing state 4 (with self-loop) forms its own irreducible component. So the **irreducible components are {2, 3} and {4}**. ✓  
 For the irreducible component {2, 3}, we have the transition probability matrix  $P_{2,3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , where we have  $P_{2,3}^2 = I_2$  ( $2 \times 2$  identity matrix) and so,  $P_{2,3} = P_{2,3}^3 = P_{2,3}^5 = \dots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $P_{2,3}^2 = P_{2,3}^4 = P_{2,3}^6 = \dots = I_2$ . Therefore the component {2, 3} is periodic with a periodicity of 2. On the other hand, {4} has a period of 1 meaning it's aperiodic. ✓

- (d) To calculate the stationary distribution, we can solve the system of equations  $x = x \cdot P$ ,

where,  $P$  is the transition probability matrix  $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  and  $x$  a probability vector

$[x_1, x_2, x_3, x_4]^T$  such that  $\sum_{i=1}^4 x_i = 1$ . So, we have

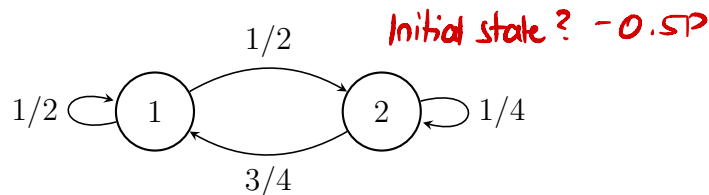
$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{2} = x_1, \frac{x_1}{4} + x_3 = x_2, x_2 = x_3 \text{ and } \frac{x_1}{4} + x_4 = x_4$$

$$\Rightarrow x_1 = 0, x_2 = x_3 \text{ and } x_4 = 1 - x_2 - x_3 \text{ as } \sum_{i=1}^4 x_i = 1. \quad \checkmark$$

Therefore we can find infinitely many solutions for  $x_2$  and  $x_3$  for  $x_2 = x_3 \in [0, 1]$  which shows that the Markov chain has infinitely many stationary distributions.

- (e) An example of an irreducible and aperiodic Markov chain with exactly 2 states can be



So, we have the transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

We can see from the Markov chain that it's strongly connected and hence irreducible. Also, the Markov chain has a period of 1 meaning it's aperiodic.

To calculate the stationary distribution, we can solve the system of equations  $x = x \cdot P$ , where,  $P$  is the transition probability matrix and  $x$  a probability vector  $[x_1, x_2]^T$  such that  $\sum_{i=1}^2 x_i = 1$ .

$$\text{Therefore, we get } \frac{x_1}{2} + \frac{3x_2}{4} = x_1 \text{ and } \frac{x_1}{2} + \frac{x_2}{4} = x_2$$

$$\Rightarrow x_1 = \frac{3x_2}{2} \text{ and } x_1 = 1 - x_2$$

$$\Rightarrow 1 - x_2 = \frac{3x_2}{2}$$

$$\Rightarrow x_2 = \frac{2}{5} \text{ and } x_1 = \frac{3}{5}$$

Hence, the stationary distribution for our example Markov chain is  $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$ .

*Are there other stationary distributions? - 1P*