④ a) In order to prove that $(D \to D, \preceq)$ is a partial order we will show:

i) <u>monotonicity</u>:

let f be a monotonic function in D, $f \in D \to D$, for $\forall d \in D$ $f(d) \in D$, since $(D, \sqsubseteq)$ is a complete lattice (thus a partial order) we know $f(d) \sqsubseteq f(d)$ $\{$from monotonicity of $(D, \sqsubseteq)\}$

$\Longleftrightarrow f \preceq f$

by defenition

ii) <u>transitivity</u>:

let $f, g, h \in D \to D$ such that $f \preceq g$, $g \preceq h$, thus, by defenition, $\forall d \in D$ $f(d) \sqsubseteq g(d)$, $g(d) \sqsubseteq h(d)$. by transitivity of $(D, \sqsubseteq)$ we get $f(d) \sqsubseteq h(d)$ for $\forall d \in D$, thus $f \preceq g$.

iii) <u>antisymmetry</u>:

let $f, g \in D \to D$, such that $f \preceq g$, $g \preceq f$. thus, by defenition: $\forall d \in D$ $f(d) \sqsubseteq g(d)$, $g(d) \sqsubseteq f(d)$. from antisymmetry of $(D, \sqsubseteq)$ we get $f(d) = g(d)$. since this is true for $\forall d \in D$ we can conclude $f = g$!

b) we've seen that $(D \to D, \preceq)$ is indeed a partial order, let there B a subset $S \subseteq D \to D$ of monotonic functions in D.

we construct a function f, which for any given $d \in D$ $s(d) \sqsubseteq f(d)$ for $\forall s \in S$, such a function exist since we can just define: $f(d) := \max\{s(d) \mid s \in S\}$. such a function is also still monotonic since $\forall s \in S$ are monotonic as well $\Rightarrow$ if $d_1, d_2 \in D$, $d_1 \sqsubseteq d_2$ let $s_i(d_1)$ B the max value for d_1 out of S and $s_j(d_2)$ B the max value for d_2 out of S, we get $s_j(d_1) \sqsubseteq s_i(d_1) \sqsubseteq s_i(d_1) \sqsubseteq s_j(d_2)$

from transitivity: $s_i(d_1) \sqsubseteq s_j(d_2) \Rightarrow f(d_1) \sqsubseteq f(d_2)$!

f is monotonic as well $\Rightarrow f \in D \to D$.

by defenition f is an upper bound of S.

now lets assume there exist g which is also an upper bound of S and $g \sqsubseteq f$, thus $\exists d \in D$ $\forall s_i \in S$ $s_i(d) \sqsubseteq g(d) \sqsubseteq f[d]$ contradicting the fact that $f(d)$ is defined as the max of $\{s(d) \mid s \in S\}$

$\Longrightarrow$ f is lub \ supremum of S!

$\Rightarrow$ similiraly we can define function h by the minimum values and get that h is the infimum of S thus f, h are $\in D \to D$ and are the supremum and infimum of an arbitrary $S \subseteq D \to D$.

thus $(D \to D, \sqsubseteq)$ is a complete lattice.

$\square$

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c) The statement is correct:

we'll prove it by using the hint.

i) first we'll prove that because D is satisfying the ACC, $D \to D$ also satisfies the same condition. Let there be a function $f \in (D \to D)$ since D is ACC then f's Image and PreImage is also limited! Thus making a chain $F \subseteq (D \to D)$ follow the rules required from ACC.

ii) In our scenario where D satisfies ACC lfp is actually a monotone function: $f_1, f_2 \in (D \to D)$ $f_1 \sqsubseteq f_2$, since D is limited (and basically finite) both functions will have $d_1, d_2 \in D$ where $d_1 = f_1(d_1)$ $d_2 = f_2(d_2)$ (from pigeonhole principle), and since we assumed $F_1 \sqsubseteq f_2$ and the function are monotonic we can derive that $lfs(f_1) \sqsubseteq lfs(f_2)$.

iii) $\Rightarrow$ Now we can use the hint:

$$\phi(\bigsqcup S) = \bigsqcup \phi(S)$$

$$\Downarrow$$

plugging In:

$\phi = lfs$

$S = \tilde{F} \subseteq (D \to D)$

$$\Downarrow$$

$$lfs(\bigsqcup \tilde{F}) = \bigsqcup lfs(f \mid f \in \tilde{F})$$

END!

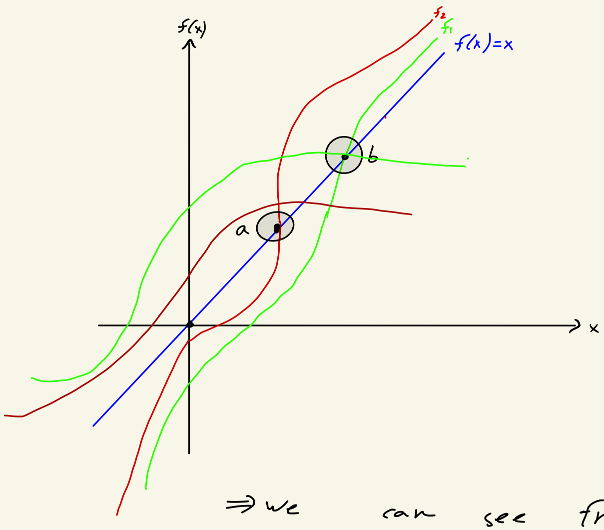
NEXT PAGE IS A SKETCH!

c) we will disprove the statement by a counter example:

- $D := \mathbb{R}$
- $\sqsubseteq := \leq$

$\Rightarrow (\mathbb{R}, \leq) \to$ a complete lattice

- lets look at the following function $f_1, f_2 \in \mathbb{R} \to \mathbb{R}$



$\Rightarrow$ we can see from the graph, that both of the Functions are monotone.

$\Rightarrow$ let's choose $\to \mathcal{F} = \{f_1 \leq f_2\}$.

In this case $\sqcup \mathcal{F} = f_2$, as described in (4.a) thus we can see that the $lfp(\sqcup F) = lfp(f_2) = a$

$\Rightarrow$ now lets look at $\{lfp(f) \mid \in \mathcal{F}\} = \{a, b\}$

$\Rightarrow \sqcup\{lfp(f) \mid \in \mathcal{F}\} = \sqcup\{a, b\} = b$

The equation does not hold $\triangledown$