2. (a)

// $(x>0) \mid (x=0)$ & $(x>y) \mid (x<0)$ & $(x+1>y)$

// $(x>0) \mid (x \leq 0)$ & $((x=0)$ & $(x>y) \mid (x \neq 0)$ & $(x+1>y))$

```
if (x>0) {
    // true
    // y+1 > y
    x := y+1
    // x > y
} else {
    // (x=0) & (x>y) | (x≠0) & (x+1>y)
    if (x==0) {
        // x > y
        skip
        // x > y
    } else {
        // x+1 > y
        x := x+1
        // x > y
    }
    // x > y
}
// x > y
```

(b) $\Phi_f(f)$ where $f = (a = x!)$

$\Phi_f(f) = \big( (x \geq 1) \wedge wp(a = a x, wp(x = x-1, f)) \big) \vee \big( (x < 1) \wedge f \big)$

X wrong question

1

a) $x \sqsubseteq y \iff \forall \sigma \in \Sigma_i : x(\sigma) \leq y(\sigma)$

$\Rightarrow (\Sigma_i \to [0,1], \sqsubseteq)$ is complete lattice.

f monotonic:

Let $x \sqsubseteq y$ (have to show $f(x) \sqsubseteq Y(x)$)

$x \sqsubseteq y$

$\Rightarrow Ax \sqsubseteq Ay$

$\Rightarrow Ax+b \sqsubseteq Ay+b$

Knaster-Tarski Thm: $f$ has a lfp.

b) $\text{lfp}\, F := \bigsqcup_{n \in \mathbb{N}} F^n(\bot)$.

Algorithm: Compute $f^k(0)$ for increasing $k$

$0, f(0), f^2(0) = f(f(0)), \ldots, \}$ 

Let $(x_i)_{i \geq 0}$ ascending chain in $\Sigma_? \to [0,1]$

$$\bigsqcup_i f(x_i) \overset{!}{=} f\left(\bigsqcup x_i\right)$$

$$\bigsqcup_{i \geq 0} x_i = \lim_{i \to \infty} x_i$$ [Monotonic Convergence Theorem] Boundedness $\Rightarrow$ convergence

By monotonicity $f(x_i)_{i \geq 0}$ is ascending chain.

$$\lim_{i \to \infty} f(x_i) = \lim_{i \to \infty} (Ax_i + b) = A\left(\lim_{i \to \infty} x_i\right) + b$$

$$= f\left(\lim_{i \to \infty} x_i\right).$$

3 (a) $f(\mathrm{lfp}\, g) \overset{\text{assume}}{\sqsubseteq} g(\mathrm{lfp}\, g) = \mathrm{lfp}\, g$

$\Rightarrow \quad \mathrm{lfp}\, g \in S.$

$\mathrm{lfp}\, f = \sqcap S \sqsubseteq \mathrm{lfp}\, g.$

(b) Proof by induction over the structure of $P$.

Let $F \subseteq G$ be predicates.

- $P = skip$ $\quad wp(skip, F) = F \subseteq G = wp(skip, G)$

- $P = (x := E)$

$wp(x := E, F) = F[x := E]$

$F[x := E] = \{ s \in S \mid \exists s' \in F : s[x \mapsto E(s)] = s' \}$

$F \subseteq G$

$\Rightarrow \forall s \in S \left( \exists s' \in F : s[x \mapsto E(s)] = s' \Rightarrow \exists s' \in G : s[x \mapsto E(s)] = s' \right\}$

$\Rightarrow \quad F[x := E] \subseteq G[x := E].$

$\Rightarrow \quad wp(x := E, F) \subseteq wp(x := E, G).$

- $P = \underset{\sim}{P_1} ; \underset{\sim}{P_2}$

wps are monotonic for individual programs.

$F \subseteq G$

$\Rightarrow \quad wp(P_2, F) \subseteq wp(P_2, G)$ (Induction hypothesis)

$\Rightarrow \quad wp(P_1, wp(P_2, F)) \subseteq wp(P_1, wp(P_2, G))$ ( " )

$\Rightarrow \quad wp(P_1; P_2, F) \subseteq wp(P_1; P_2, G).$ (By defn.)

- $P = \text{if } (\varphi) \{P_1\} \text{ else } \{P_2\}$

$F \subseteq G$

$\Rightarrow \quad wp(P_1, F) \subseteq wp(P_1, G)$

$\qquad \wedge\ wp(P_2, F) \subseteq wp(P_2, G)$

$\Rightarrow \quad \varphi \wedge wp(P_1, F) \subseteq \varphi \wedge wp(P_1, G)$

$\qquad \wedge\ \neg\varphi \wedge wp(P_2, F) \subseteq \neg\varphi \wedge wp(P_2, G)$

$\Rightarrow \quad (\varphi \wedge wp(P_1, F)) \vee (\neg\varphi \wedge wp(P_2, F)) \subseteq (\varphi \wedge wp(P_1, G)) \vee (\neg\varphi \wedge wp(P_2, G))$

$\Rightarrow \quad wp(\text{if} \ldots, F) \subseteq wp(\text{if} \ldots, G).$

- $P = \text{while } (\varphi)\ \{P_1\}$

  $wp(\text{while}(\varphi)\{P_1\}, F) = \text{lfp}\ \Phi_F$ , $\Phi_F(x) = (\varphi \wedge wp(P_1, x)) \vee \underbrace{(\neg\varphi \wedge F)}_{\subseteq (\neg\varphi \wedge G)}$.

  $F \subseteq G \Rightarrow \forall x : \Phi_F(x) \subseteq \Phi_G(x)$.

  and, $\Phi_F, \Phi_G$ are monotonic! ??
  (Induction hypothesis).

3. ⓐ $\Rightarrow$ $\boxed{\text{lfp}\ \Phi_F \subseteq \text{lfp}\ \Phi_G}$

   $\Rightarrow wp(\text{while } (\varphi)\{P_1\}, F) \subseteq wp(\text{while } (\varphi)\ \{P_1\}, G)$.

4. (a) P: while (true) {skip} = diverge ;

$wp(P, F) = false$

$wlp(P, F) = true.$

(b)

$$wlp\left(\text{while } (x \neq 0)\ \{x : x = 2\},\ false\right)$$

$$= \inf_{n \geq 0} \Psi^n_{false}(true)$$

$$\Psi(X) = (x \neq 0 \wedge wlp(x := x - 2,\ X)) \vee \cancel{(x = 0 \wedge false)}\ false.$$

$$\Psi(true) = (x \neq 0 \wedge wlp(x := x - 2,\ true))$$

$$= x \neq 0 \wedge true \quad = x \neq 0$$

$$\Psi^2(true)$$

$$= \Psi(x \neq 0) = x \neq 0 \wedge x \neq 2$$

$$\Psi^3(true) = x \neq 0 \wedge x \neq 2 \wedge x \neq 4 \ \ldots$$

$$\Psi^n(true) = \bigwedge_{i=0}^{n-1} x \neq 2i$$

$$\inf_n \Psi^n(true) = x < 0 \vee x \bmod 2 \neq 0.$$