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Probabilistic Programming
Exercise Sheet 5 Solutions
Group 7

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1. **Weakest pre-expectation calculus**

   (a) Program $P : \{x := 2\}[1/3]\{$ diverge $\}$. P diverges with probability $\frac{2}{3}$. However, we only consider the post-expectation f on termination to calculate the weakest pre-expectation. Therefore,

$$
\begin{aligned}
& wp(\{x := 2\}[1/3]\{\text{ diverge }\}, x^2) \\
& \qquad = \frac{1}{3} \cdot wp[\![x := 2]\!](x^2) + \frac{2}{3} \cdot wp[\![\text{diverge}]\!](x^2) \\
& \qquad = \frac{1}{3} \times 2^2 + 0 \\
& \qquad = \frac{4}{3}
\end{aligned}
$$

   (b) We need to prove or disprove whether the programs P_1 and P_2 are equivalent w.r.t. the post-expectation $f = x$:
   $P_1 :$ $y := 5;$ if $(y < 0)\{$skip$\}$ else $\{\{x := 1\}[1/2]\{$ skip $\}\}$
   $P_2 :$ $\{\{x := x + 3\}[1/3]\{x := x\}\}[1/2]\{x := 0\}$

$$
\begin{aligned}
wp(P_1, x) &= wp(y := 5;\ \text{if } (y < 0)\{\text{skip}\} \text{ else } \{\{x := 1\}[1/2]\{skip\}\}, x) \\
&= [y < 0]_s \cdot wp[\![skip]\!](x) + [y \geq 0]_s \cdot wp[\![\{x := 1\}[\tfrac{1}{2}]\{skip\}]\!](x) \\
&= 0 \cdot x + 1 \cdot wp[\![\{x := 1\}[\tfrac{1}{2}]\{skip\}]\!](x) \text{ [since s(y):=5]} \\
&= 0 + wp[\![\{x := 1\}[\tfrac{1}{2}]\{skip\}]\!](x) \\
&= \frac{1}{2} wp[\![x := 1]\!](x) + \frac{1}{2} wp[\![skip]\!](x) \\
&= \frac{1}{2} \cdot 1 + 0 \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
wp(P_2, x) &= wp(\{\{x := x+3\}[1/3]\{x := x\}\}[1/2]\{x := 0\}, x) \\
&= \frac{1}{2} \cdot wp[\![\{x := x+3\}[1/3]\{x := x\}\}]\!](x) + \frac{1}{2} \cdot wp[\![\{x := 0\}]\!](x) \\
&= \frac{1}{2} \cdot (\frac{1}{3} \cdot wp[\![x := x+3]\!](x) + \frac{2}{3} \cdot wp[\![x := x]\!](x)) + \frac{1}{2} \cdot 0 \\
&= \frac{1}{2} \cdot (\frac{1}{3} \cdot (x+3) + \frac{2}{3} \cdot x) + 0 \\
&= \frac{1}{6} \cdot (x+3) + \frac{1}{3} \cdot x \\
&= \frac{x}{2} + \frac{1}{2} \\
&= \frac{x+1}{2}
\end{aligned}
$$

From the weakest pre-expectations of P_1 and P_2 w.r.t. post-expectation $f = x$, we can see that P_1 and P_2 are not equivalent in general except for the case where s(x) = 0 for P_2.

(c) P_1 : while $(x \neq x)\{\{x := y+1\}[1/2]\{x := y-1\}\}$
P_2 : while (true) { skip }

$$
wp(P_1, x) = wp(while(x \neq x)\{\{x := y+1\}[1/2]\{x := y-1\}\}, x)
$$

For the while loop, we need to compute the characteristic function.

$$
\begin{aligned}
\Phi_f(X) &= [x \neq x] \cdot wp[\![\{x := y+1\}[1/2]\{x := y-1\}]\!](X) + [x := x] \cdot x \\
&= (false) \cdot wp[\![\{x := y+1\}[1/2]\{x := y-1\}]\!](X) + (true) \cdot x \\
&= 0 + x = x.
\end{aligned}
$$

Now, we use Kleene's fixed point theorem:
$\Phi_x^0(\underline{0}) = \underline{0}$
$\Phi_x^1(\underline{0}) = x$
$\Phi_x^2(\underline{0}) = \Phi_x(\Phi_x^1(\underline{0})) = \Phi_x(x) = x$
.
.
. $\Phi_x^k(\underline{0}) = \Phi_x(\Phi_x^{k-1}(\underline{0})) = \Phi_x(x) = x$
$\implies \lim_{k\to\infty} \Phi_x^k(\underline{0}) = x \implies wp(P_1, x) = x.$

$$
wp(P_2, x) = wp(while(true)\{skip\}, x)
$$

For the while loop, we again compute the characteristic function.

$$
\begin{aligned}
\Phi_f(X) &= [true] \cdot wp[\![\{skip\}]\!](X) + [false] \cdot x \\
&= (true) \cdot X + (false) \cdot x \\
&= X + 0 = X.
\end{aligned}
$$

Now, we use Kleene's fixed point theorem:
$\Phi_x^0(\underline{0}) = \underline{0}$
$\Phi_x^1(\underline{0}) = \underline{0}$
$\Phi_x^2(\underline{0}) = \Phi_x(\Phi_x^1(\underline{0})) = \Phi_x(\underline{0}) = \underline{0}$
.
.
. $\Phi_x^k(\underline{0}) = \Phi_x(\Phi_x^{k-1}(\underline{0})) = \Phi_x(\underline{0}) = \underline{0}$
$\implies \lim_{k\to\infty} \Phi_x^k(\underline{0}) = \underline{0} \implies wp(P_1, x) = \underline{0}$.
So P_2 diverges and hence is not equivalent to P_1.

2. **Continuity of weakest pre-expectations** Fix a pGCL loop $P = \text{while}\,(G)\,\{P'\}$. For all and expectations $X, f \in \mathbb{E}$ we define

$$\Psi_{P,f}(X) = [G] \cdot wp\,(P', X) + [\neg G] \cdot f.$$

(a) If we consider $\Psi_{P,f}(X)$ as a function in f, the first portion of the additive formulation of $\Psi_{P,f}(X)$ i.e. $[G] \cdot wp\,(P', X)$ doesn't depend on f which is essentially continuous in f as any constant function is continuous in its domain. Similarly, in the second part, the negative guard predicate $[\neg G]$ doesn't depend on f, while f is an identity in f both of which are continuous in f. Now, we can use the hints. Firstly, since the product of two continuous functions in the same domain ($\mathbb{E}$) is also continuous, $[\neg G] \cdot f$ is continuous in $f \in \mathbb{E}$. Then, using the fact that the sum of two continuous functions is again continuous, we prove that sum of $[G] \cdot wp\,(P', X)$ and $[\neg G] \cdot f$ i.e. $\Psi_{P,f}(X)$ is a continuous function in $f \in \mathbb{E}$.

(b) Since $(\mathbb{E}, \leq)$ forms a complete lattice, to prove that for fixed post-expectation f, $\Psi_{P,f}$ is continuous in X, we need to show for a chain $\{X_n\}_{n\in\mathbb{N}} \subseteq \mathbb{E}, \sup_{n\in\mathbb{N}} \Psi_{P,f}(X_n) = \Psi_{P,f}(\sup_{n\in\mathbb{N}} X_n)$. We can show this by induction on the structure of our program P.
We can use the fact that for $P = \text{skip}$, $P = \text{diverge}$, $P = x := E$, $wp(P, f)$ is continuous. Also, if P_1, P_2 are such that $wp\,(P_1, f)$ and $wp\,(P_2, f)$ are continuous, then $wp(P, f)$ is continuous for $P = P_1; P_2, P = \text{if}\,(H)\,\{P_1\}\text{ else }\{P_2\}$ and $P = \{P_1\}\,[p]\,\{P_2\}$ for all guards H and probabilities $p \in [0, 1]$. We can use these base cases for $wp(P', X)$, given P' is the body of the while loop and $X \in \mathbb{E}$, which gives us:
We have the following formulations using the induction hypothesis on the structure of P:

$$\sup_{n\in\mathbb{N}} \Psi_{P,f}(X_n) = \sup_{n\in\mathbb{N}}\{[G] \cdot wp(P', X_n) + [\neg G] \cdot f\} = [G] \cdot \sup_{n\in\mathbb{N}} wp(P', X_n) + [\neg G] \cdot f \quad (1)$$

$$\Psi_{P,f}(\sup_{n\in\mathbb{N}} X_n) = [G] \cdot wp(P', \sup_{n\in\mathbb{N}} X_n) + [\neg G] \cdot f \quad (2)$$

Equating (1) and (2), we can see that to prove Scott-continuity, we just have to show that $\sup_{n\in\mathbb{N}} wp(P', X_n) = wp(P', \sup_{n\in\mathbb{N}} X_n)$.
Now, using the base cases above:

- For $P' = \text{skip}$, $P' = \text{diverge}$, $P' = x := E$, $X_n \in \mathbb{E}$, we have $\sup_{n\in\mathbb{N}} wp(P', X_n) = wp(P', \sup_{n\in\mathbb{N}} X_n)$ [by continuity].
- Also if, $P' = P_1'; P_2', P' = \text{if}\,(H)\,\{P_1'\}\text{ else }\{P_2'\}$ and $P' = \{P_1'\}\,[p]\,\{P_2'\}$ for all guards H and probabilities $p \in [0, 1]$, by induction hypothesis we again have, $\sup_{n\in\mathbb{N}} wp(P', X_n) = wp(P', \sup_{n\in\mathbb{N}} X_n)$.

Here, we can assume P' is loop-free without loss of generality or it can be flattened into one for nested loops, which in turn proves that by induction on the structure of P', $\Psi_{P,f}$ is continuous in X.

3. **Reasoning with invariants** Not solved

4. **A syntax for expectations**

   (a) First we consider: $\reflectbox{S}\ z : [z < y \cdot y] \cdot z \equiv y^2$. We can then construct using this to convert $f = \sqrt[3]{x}$ to a syntactic expectation in `Exp` as: $\reflectbox{S}\ y : [y \cdot\ (\reflectbox{S}\ z : [z < y \cdot y] \cdot z) < x] \cdot y$

   (b) First we consider: $\reflectbox{S}\ z : [z < x \cdot x] \cdot z \equiv x^2$. Then, we can write $f = \frac{2}{2 \cdot x^2 + 7}$ as a syntactic expectation in `Exp` as: $\reflectbox{S}\ y : [y \cdot (2 \cdot (\ \reflectbox{S}\ z : [z < x \cdot x] \cdot z) + 7) = 2] \cdot y$

   (c) We need to use a condition that $a(x)$ is a constant with respect to x as a guard along with a syntactic expectation that evaluates to 1. We can use the guard (φ) $x > y$ or $x < y$ so that x and y are different valuations of free variables. So, the syntactic expectation in the form $[\varphi] \cdot f$ is: $\reflectbox{S}\ z : [x < y] \cdot (z \cdot a(y) = a(x)) + [x > y] \cdot (z \cdot a(y) = a(x))$.

   (d) We use a condition that ensures that a represents a monotonic function in x, as a guard along with a syntactic expectation evaluating to 1. To represent monotonicity, we need that if $x \geq y$ then either $a(x) \geq a(y)$ (if increasing) or $a(x) \leq a(y)$ (if decreasing) $\forall x, y$. Combining the above conditions into guards in our syntactic form we get: $\reflectbox{S}\ z : [x > y] \cdot (z \cdot a(x) < a(y)) + [x < y] \cdot (z \cdot a(x) < a(y)) + [x = y] \cdot (z \cdot a(x) = a(y))$.