Assignment 4

Basic Techniques in Computer Graphics
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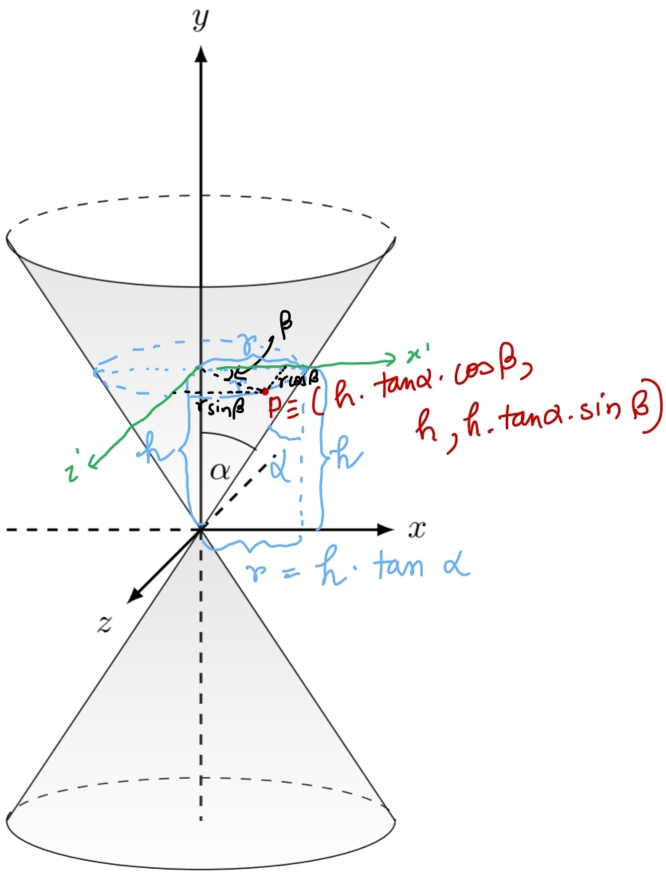
Exercise 1 Explicit and Implicit Representations of Surfaces

We consider a double cone with (half) opening angle α. The cone is centered at the origin and extends infinitely in the y and $-y$ directions.

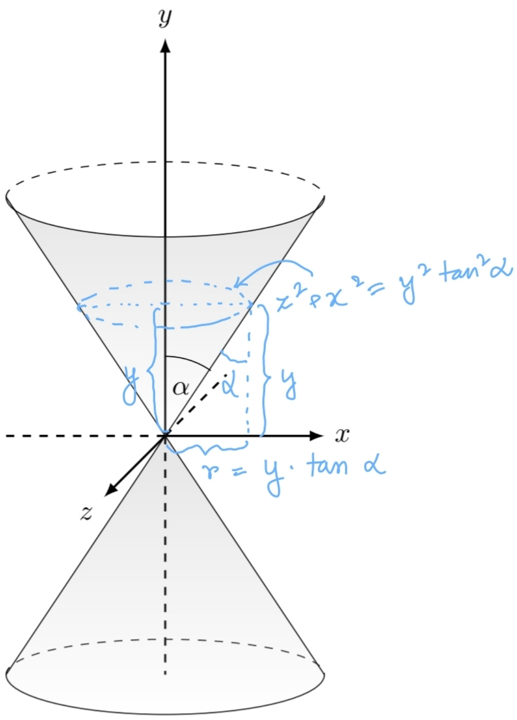
(a) Explicit Representation



To find an explicit or parametric representation of the double cone, we consider any point at distance h from the origin along $y-$axis where h can range from $-\infty$ to ∞, i.e. $h \in \mathbb{R}$. We can see from the above figure that each point on the surface of the double cone at signed distance h lies on the circle with radius $h \cdot \tan\alpha$ and any point on that circle can be represented using the angle β it makes with respect to $x-$axis, where $0 \leq \beta \leq 2\pi$, as $(h \cdot \tan\alpha \cdot \cos\beta, h, h \cdot \tan\alpha \cdot \sin\beta)$. So, we have the function $f_c(\dots) \in \mathbb{R}^3$ as follows:

$$f_c(h, \beta) = \begin{pmatrix} h \cdot \tan\alpha \cdot \cos\beta \\ h \\ h \cdot \tan\alpha \cdot \sin\beta \end{pmatrix}, h \in \mathbb{R}, 0 \leq \beta \leq 2\pi.$$

(b)



Using similar logic as to part (a) of the solution, we can say that any point with y-coordinate y will be on the circle with radius $y \cdot \tan \alpha$ at distance y from the origin. Any point $(x, y, z)^T$ on the circle then can be represented as: $x^2 + z^2 = (y \cdot \tan \alpha)^2$. So, we can derive an implicit representation of the form $F_c(x, y, z) \in \mathbb{R}$ such that $F_c(x, y, z) = 0$ as:

$$x^2 + z^2 - (y \cdot \tan \alpha)^2 = 0.$$

(c) From part (b) we have the quadric form,

$$\begin{aligned} F_c(x, y, z) &= x^2 + z^2 - (y \cdot \tan \alpha)^2 \\ &= (x, -y \cdot (\tan \alpha)^2, z, 0) \cdot (x, y, z, 1)^T \\ &= (x, y, z, 1) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tan^2 \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot (x, y, z, 1)^T \end{aligned}$$

Therefore, we get $Q_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tan^2 \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(d) Here we consider the general quadric

$$\mathbf{Q} = \begin{pmatrix} 2a & b & c & d \\ b & 2e & f & g \\ c & f & 2h & i \\ d & g & i & 2j \end{pmatrix}$$

which defines the surface $(x,y,z,1)\cdot\mathbf{Q}\cdot(x,y,z,1)^T = F(x,y,z) = 0$.
Therefore, we derive:

$$F(x,y,z) = (x,y,z,1)\cdot\begin{pmatrix}2a & b & c & d\\ b & 2e & f & g\\ c & f & 2h & i\\ d & g & i & 2j\end{pmatrix}\cdot(x,y,z,1)^T$$
$$= (x,y,z,1)\cdot\begin{pmatrix}2ax+by+cz+d\\ bx+2ey+fz+g\\ cx+fy+2hz+i\\ dx+gy+iz+2j\end{pmatrix}$$
$$= 2ax^2+bxy+czx+dx+bxy+2ey^2+fyz+gy+czx+fyz+2hz^2+iz+dx+gy+iz+2j$$
$$= 2(ax^2+ey^2+hz^2+bxy+fyz+czx+dx+gy+iz+j)$$

Now, we can compute the gradient of $F(x,y,z)$ as:

$$\nabla F(x,y,z) = \begin{pmatrix}\frac{\partial F(x,y,z)}{\partial x}\\ \frac{\partial F(x,y,z)}{\partial y}\\ \frac{\partial F(x,y,z)}{\partial z}\end{pmatrix}$$
$$= \begin{pmatrix}4ax+2by+2cz+2d\\ 2bx+4ey+2fz+2g\\ 2cx+2fy+4hz+2i\end{pmatrix}$$
$$= \begin{pmatrix}4a & 2b & 2c & 2d\\ 2b & 4e & 2f & 2g\\ 2c & 2f & 4h & 2i\end{pmatrix}\cdot(x,y,z,1)^T$$
$$= 2\begin{pmatrix}2a & b & c & d\\ b & 2e & f & g\\ c & f & 2h & i\end{pmatrix}\cdot(x,y,z,1)^T$$

which gives us, $\mathbf{G} = \begin{pmatrix}4a & 2b & 2c & 2d\\ 2b & 4e & 2f & 2g\\ 2c & 2f & 4h & 2i\end{pmatrix} = 2\begin{pmatrix}2a & b & c & d\\ b & 2e & f & g\\ c & f & 2h & i\end{pmatrix}$.

(e)

$$F_e(x,y,z) = x(x-2)+y(y-4)+3z(2\sqrt{3}-z)-4 = x^2+y^2-3z^2-2x-4y+6\sqrt{3}z-4\ldots \quad (1)$$

Comparing the expanded form of $F(x,y,z)$ we got in part (d) for the general quadric with (1), we get:

$$2a = 1, 2e = 1, 2h = -3, b = f = c = 0, 2d = -2, 2g = -4, 2i = 6\sqrt{3}, 2j = -4.$$

Now by replacing by the corresponding values in general quadric $\mathbf{Q}$, we have:

$$\mathbf{Q_e} = \begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -2\\ 0 & 0 & -3 & 3\sqrt{3}\\ -1 & -2 & 3\sqrt{3} & -4\end{pmatrix}$$

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(f) Using the expression derived in part (d) for matrix $\mathbf{G}$ and replacing the values calculated from (1), we get:

$$\mathbf{G} = \begin{pmatrix} 4a & 2b & 2c & 2d \\ 2b & 4e & 2f & 2g \\ 2c & 2f & 4h & 2i \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -6 & 6\sqrt{3} \end{pmatrix}$$

which gives us,

$$\begin{aligned} \nabla F(1,5,2\sqrt{3}) &= \mathbf{G} \cdot (1,5,2\sqrt{3},1)^T \\ &= \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -6 & 6\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 2\sqrt{3} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 6 \\ -6\sqrt{3} \end{pmatrix} \end{aligned}$$

$$||\nabla F(1,5,2\sqrt{3})|| = || \begin{pmatrix} 0 \\ 6 \\ -6\sqrt{3} \end{pmatrix} || = \sqrt{0+36+108} = 12$$

So, the normal at point $p = (1,5,2\sqrt{3},1)^T$ of the surface defined by $\mathbf{Q}_e$, $\mathbf{n}(1,5,2\sqrt{3}) = \frac{\nabla F(1,5,2\sqrt{3})}{||\nabla F(1,5,2\sqrt{3})||} = \frac{(0,6,-6\sqrt{3})^T}{12} = (0, \frac{1}{2}, -\frac{-\sqrt{3}}{2})^T$.

Exercise 2 Frustum Transformation for Orthogonal Projections

(a) The viewing frustum becomes cuboid as each of the points on the near plane gets projected along the normal vector of the near plane onto the far plane. This includes the corner-points of the viewing frustum, thus each edge between the near and far plane is parallel to each other. The near plane and the far plane are also parallel to each other and each of the edges of out frustum is orthogonal to the near as well as the far plane, so our viewing frustum forms a cube.

(b) The translation T can be defined as follows:

$$T := \begin{pmatrix} 1 & 0 & 0 & -l - \frac{r-l}{2} \\ 0 & 1 & 0 & -b - \frac{t-b}{2} \\ 0 & 0 & 1 & n - \frac{n-f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix maps $\begin{pmatrix} l + \frac{r-l}{2} \\ b + \frac{t-b}{2} \\ -n + \frac{n-f}{2} \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then we need to scale the matrix:

$$S := \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In total we get the final transformation matrix M:

$$M := S \cdot T = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -l - \frac{r-l}{2} \\ 0 & 1 & 0 & -b - \frac{t-b}{2} \\ 0 & 0 & 1 & n - \frac{n-f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{2 \cdot l}{r-l} - 1 \\ 0 & \frac{2}{t-b} & 0 & -\frac{2 \cdot b}{t-b} - 1 \\ 0 & 0 & \frac{2}{n-f} & \frac{2 \cdot n}{n-f} - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$