Assignment 2

Basic Techniques in Computer Graphics
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Exercise 1 Linear Basis Transformations

a) In order to express the point p_B with respect to the standard-basis, we can transform it via $p_I = Bp_B$. If we have the point p_I in regards to the standard-basis, we can express it with respect to the basis C as: $p_C = C^{-1}p_I$. That leaves us with the matrix $M_1 = C^{-1}B$ with $p_C = M_1 p_B = C^{-1}Bp_B$.

b) We need to find M_2, such that $C = M_2 B$. Thus $M_2 = CB^{-1}$.

c) We can see:

$$B^T B = \begin{pmatrix} b_1^T \\ b_2^T \\ b_3^T \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} b_1^T b_1 & b_1^T b_2 & b_1^T b_3 \\ b_2^T b_1 & b_2^T b_2 & b_2^T b_3 \\ b_3^T b_1 & b_3^T b_2 & b_3^T b_3 \end{pmatrix}$$

Because b_1, b_2, b_3 are perpendicular this equates to:

$$B^T B = \begin{pmatrix} ||b_1||^2 & 0 & 0 \\ 0 & ||b_2||^2 & 0 \\ 0 & 0 & ||b_3||^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As for BB^T we note: $B^T B = I \Rightarrow BB^T B = B \Rightarrow BB^T BB^{-1} = BB^{-1} \Rightarrow BB^T = I$
Thus we conclude $B^T B = BB^T = I$, which means $B^{-1} = B^T$.

d) $M_2 = CB^{-1}$. By applying c) and because $B^{-1} = B^T$, $C^{-1} = C^T$ we see:

$$M_2^T = (CB^T)^T = (B^T)^T C^T = BC^T = BC^{-1} = (B^{-1})^{-1} C^{-1} = (CB^{-1})^{-1} = M_2^{-1}$$

e) We calculate the matrix M like shown in a) and use the fact from c) for orthonormal bases:

$$M = C^{-1}B = C^T B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Then we change the basis for p_B:

$$p_C = Mp_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 2 Meaningful Geometric Operations

a)

$$p_1 - 2p_2 + p_3 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ y_1 - 2y_2 + y_3 \\ z_1 - 2z_2 + z_3 \\ 1 - 2 + 1 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ y_1 - 2y_2 + y_3 \\ z_1 - 2z_2 + z_3 \\ 0 \end{pmatrix}$$

Thus $p_1 - 2p_2 + p_3$ yields a **vector**.

b)

As offset between two points represents a vector, each of $(p_i - p_0), i = 1, \ldots, n$ represents a vector. So, we have

$$p_0 + \sum_{i=1}^{n}(p_i - p_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^n x_i - x_0 \\ \sum_{i=1}^n y_i - y_0 \\ \sum_{i=1}^n z_i - z_0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 + \sum_{i=1}^n x_i - x_0 \\ y_0 + \sum_{i=1}^n y_i - y_0 \\ z_0 + \sum_{i=1}^n z_i - z_0 \\ 1 \end{pmatrix}$$

Thus $p_0 + \sum_{i=1}^n (p_i - p_0)$ yields a **point**.

c)

$$\alpha p_0 + \beta p_1 = \begin{pmatrix} \alpha x_0 \\ \alpha y_0 \\ \alpha z_0 \\ \alpha \end{pmatrix} + \begin{pmatrix} \beta x_1 \\ \beta y_1 \\ \beta z_1 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha x_0 + \beta x_1 \\ \alpha y_0 + \beta y_1 \\ \alpha z_0 + \beta z_1 \\ \alpha + \beta \end{pmatrix}$$

In general $\alpha + \beta \notin \{0, 1\}$, so this operation is generally **not geometrically meaningful**. But if $\alpha + \beta = 1$ it represents a point or if $\alpha = -\beta$, it yields a vector.

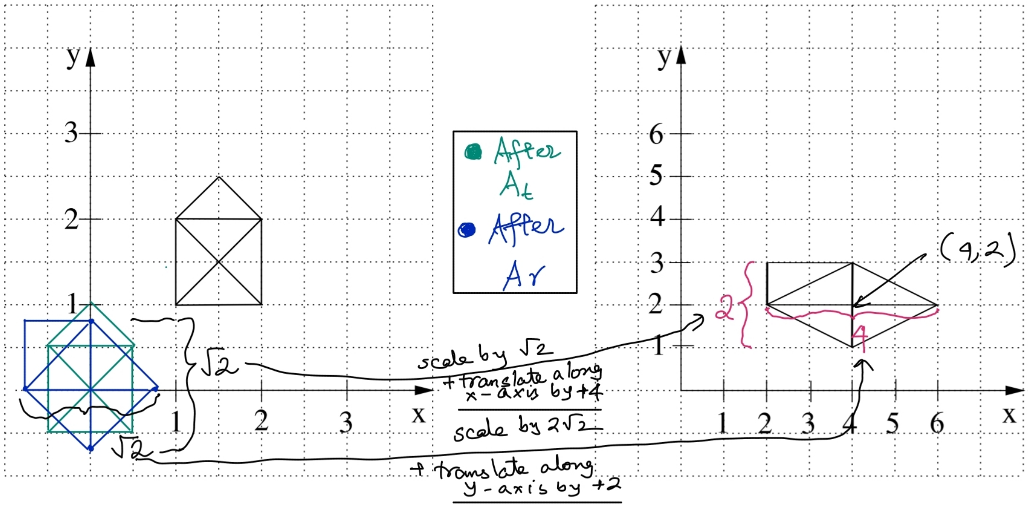
d)

$$\alpha p_0 + \beta p_1 + (1 - \alpha - \beta)p_2 = \begin{pmatrix} \alpha x_0 \\ \alpha y_0 \\ \alpha z_0 \\ \alpha \end{pmatrix} + \begin{pmatrix} \beta x_1 \\ \beta y_1 \\ \beta z_1 \\ \beta \end{pmatrix} + \begin{pmatrix} (1-\alpha-\beta)x_2 \\ (1-\alpha-\beta)y_2 \\ (1-\alpha-\beta)z_2 \\ 1-\alpha-\beta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_0 + \beta x_1 + (1-\alpha-\beta)x_2 \\ \alpha y_0 + \beta y_1 + (1-\alpha-\beta)y_2 \\ \alpha z_0 + \beta z_1 + (1-\alpha-\beta)z_2 \\ \alpha + \beta + 1 - \alpha - \beta \end{pmatrix} = \begin{pmatrix} \alpha x_0 + \beta x_1 + (1-\alpha-\beta)x_2 \\ \alpha y_0 + \beta y_1 + (1-\alpha-\beta)y_2 \\ \alpha z_0 + \beta z_1 + (1-\alpha-\beta)z_2 \\ 1 \end{pmatrix}$$

Thus this operation yields a **point**.

Exercise 3 Linear & Affine Transformations



a) We first use a translation to map the point $\begin{pmatrix} 1.5 \\ 1.5 \\ 1 \end{pmatrix}$ to the origin $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

$$A_t = \begin{pmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we rotate the image by 45°:

$$A_r = \begin{pmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now we need to scale by $2 \cdot \sqrt{2}$ in x-direction and by $\sqrt{2}$ in y-direction.

$$A_s = \begin{pmatrix} 2 \cdot \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lastly we need to use a translation once more:

$$A_{t'} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

In summary we have the transformation matrix

$$A = A_{t'} A_s A_r A_t = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \cdot \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot \sqrt{2} & 0 & 4 \\ 0 & \sqrt{2} & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 4 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

b) The origin is mapped onto the following:

$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2 & -2 & 4\\1 & 1 & -1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}4\\-1\\1\end{pmatrix}$$

As for the standard basis:

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2 & -2 & 4\\1 & 1 & -1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2 & -2 & 4\\1 & 1 & -1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\0\end{pmatrix}$$