Assignment 3

Basic Techniques in Computer Graphics
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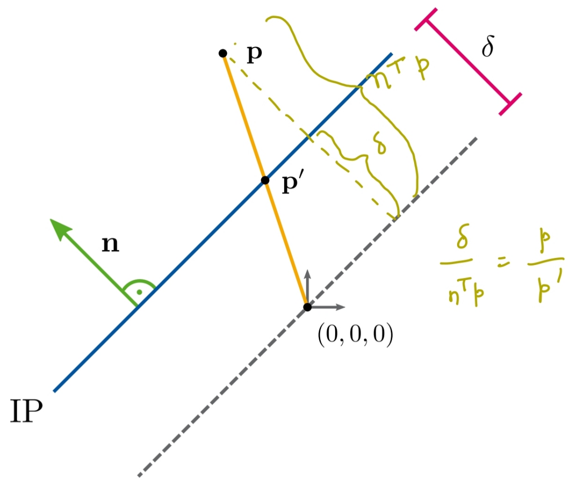
Exercise 1 Projective Geometry

(a) For a 3D scene with the camera located at the origin $(0,0,0)^T$ and image plane (IP) defined by the normal vector n with $||n|| = 1$ and the focal distance δ. The point p' is the projection of point p onto IP.



From the above image, we can see that for a given point $p = (x, y, z)^T$ and normal $n = (n_x, n_y, n_z)^T$ with $||n|| = 1$, the projected point p' can be computed as: $p' = \frac{\delta}{n^T p} \cdot p$ and so the corresponding projection matrix is: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{n_x}{\delta} & \frac{n_y}{\delta} & \frac{n_z}{\delta} & 0 \end{pmatrix}$. Therefore, for the normal vector $n = (0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^T$, we have the following projection matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-\sqrt{2}}{2\delta} & \frac{-\sqrt{2}}{2\delta} & 0 \end{pmatrix}$$

(b) Our assumption is that we picked up a value of δ such that the projection matrix becomes

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

For any point on the line $L(\lambda) = o + \lambda d = (o_x, o_y, o_z) + \lambda(d_x, d_y, d_z)$, we can compute the projection as:

$$P(L(\lambda)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} o_x + \lambda d_x \\ o_y + \lambda d_y \\ o_z + \lambda d_z \\ 1 \end{pmatrix} = \begin{pmatrix} o_x + \lambda d_x \\ o_y + \lambda d_y \\ o_z + \lambda d_z \\ o_y + \lambda d_y - (o_z + \lambda d_z) \end{pmatrix}$$

After de-homogenization we get the point in 3D as: $\begin{pmatrix} \frac{o_x+\lambda d_x}{o_y+\lambda d_y-(o_z+\lambda d_z)} \\ \frac{o_y+\lambda d_y}{o_y+\lambda d_y-(o_z+\lambda d_z)} \\ \frac{o_z+\lambda d_z}{o_y+\lambda d_y-(o_z+\lambda d_z)} \end{pmatrix} = \begin{pmatrix} \frac{o_x+\lambda d_x}{(o_y-o_z)+\lambda(d_y-d_z)} \\ \frac{o_y+\lambda d_y}{(o_y-o_z)+\lambda(d_y-d_z)} \\ \frac{o_z+\lambda d_z}{(o_y-o_z)+\lambda(d_y-d_z)} \end{pmatrix}$

Given $d_y = d_z$, which essentially boils down to $\begin{pmatrix} \frac{o_x+\lambda d_x}{(o_y-o_z)} \\ \frac{o_y+\lambda d_y}{(o_y-o_z)} \\ \frac{o_z+\lambda d_z}{(o_y-o_z)} \end{pmatrix} \underset{\lambda\to\infty}{\longrightarrow} \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}$, otherwise we have,

$\begin{pmatrix} \frac{o_x/\lambda+d_x}{o_y/\lambda+d_y-(o_z/\lambda+d_z)} \\ \frac{o_y/\lambda+d_y}{o_y/\lambda+d_y-(o_z/\lambda+d_z)} \\ \frac{o_z/\lambda+d_z}{o_y/\lambda+d_y-(o_z/\lambda+d_z)} \end{pmatrix} \underset{\lambda\to\infty}{\longrightarrow} \begin{pmatrix} \frac{d_x}{d_y-d_z} \\ \frac{d_y}{d_y-d_z} \\ \frac{d_z}{d_y-d_z} \end{pmatrix}$.

(c) As we could see from part (b) of our solution, the condition needed to have a vanishing point of $L(\lambda) = o + \lambda d = (o_x, o_y, o_z) + \lambda(d_x, d_y, d_z)$ can be formulated as: $d_y \neq d_z$.

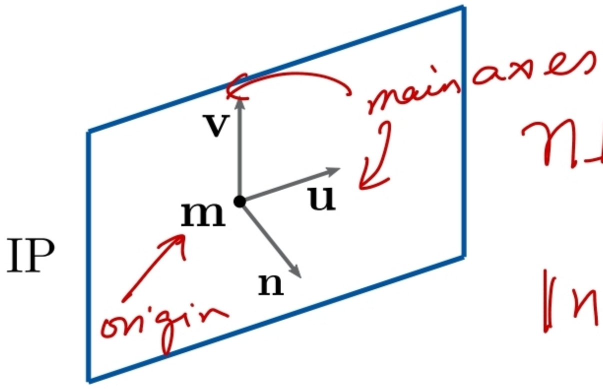
(d) The edge p_0p_1 can be expressed as $(-1, 2, 2) + \lambda(-2+1, 4-2, 0-2) = (-1, 2, 2) + \lambda(-1, 2, -2)$. We can use the condition we got in part (c). Here $d_y = 2 \neq -2 = d_z$, so the vanishing point does exist and it can be calculated as: $\begin{pmatrix} \frac{d_x}{d_y-d_z} \\ \frac{d_y}{d_y-d_z} \\ \frac{d_z}{d_y-d_z} \end{pmatrix} = \begin{pmatrix} \frac{-1}{2-(-2)} \\ \frac{2}{2-(-2)} \\ \frac{-2}{2-(-2)} \end{pmatrix} = \begin{pmatrix} \frac{-1}{4} \\ \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix}$.

The edge p_1p_2 can be expressed as $(-2, 4, 0)+\lambda(2+2, 0-4, -2-0) = (-2, 4, 0)+\lambda(4, -4, -2)$. Here $d_y = -4 \neq -2 = d_z$, so the vanishing point does exist and it can be calculated as: $\begin{pmatrix} \frac{d_x}{d_y-d_z} \\ \frac{d_y}{d_y-d_z} \\ \frac{d_z}{d_y-d_z} \end{pmatrix} = \begin{pmatrix} \frac{4}{-4-(-2)} \\ \frac{-4}{-4-(-2)} \\ \frac{-2}{-4-(-2)} \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$.

The edge p_2p_0 can be expressed as $(2, 0, -2)+\lambda(-1-2, 2-0, 2+2) = (2, 0, -2)+\lambda(-3, 2, 4)$. Here $d_y = 2 \neq 4 = d_z$, so the vanishing point does exist and it can be calculated as: $\begin{pmatrix} \frac{d_x}{d_y-d_z} \\ \frac{d_y}{d_y-d_z} \\ \frac{d_z}{d_y-d_z} \end{pmatrix} = \begin{pmatrix} \frac{-3}{2-4} \\ \frac{2}{2-4} \\ \frac{4}{2-4} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -1 \\ -2 \end{pmatrix}$.

(e) As part of the definition of vanishing points we know that parallel lines have the same vanishing point. Thus we can shift our line onto a parallel line that intersects with the origin. This implies that the vanishing point of a line
$L(\lambda) = o + \lambda d = (o_x, o_y, o_z) + \lambda(d_x, d_y, d_z)$ is the same as the vanishing point of the line $L'(\lambda) = 0 + \lambda d = \lambda(d_x, d_y, d_z)$. Every point on this line will be projected onto the image plane at the same point, which is the intersection of L' and the image plane. The vanishing point of the line L is this intersection-point.

(f)



$n \perp u, n \perp v,$
$u \perp v$
$\|n\|_2 = \|u\|_2 = \|v\|_2 = 1$

In homogeneous coordinates, we need to express a 3D point $(x, y, z, w)^T$ on the image plane as a 2D point $(\alpha, \beta, w)^T$ within the local coordinate system with the origin m and main axes u and v.

We can first compute the translation matrix that moves m $= (m_x, m_y, m_z)$ to the origin as:

$$T(-m) = \begin{pmatrix} 1 & 0 & 0 & -m_x \\ 0 & 1 & 0 & -m_y \\ 0 & 0 & 1 & -m_z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we can use the following matrix for linear basis transformation from the standard basis to the basis formed by u, v, n:

$$(u|v|n|e_4)^T = \begin{pmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combining the above transformations we get:

$$\begin{aligned} &(u|v|n|e_4)^T \cdot T(-m) \\ &= \begin{pmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ n_x & n_y & n_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -m_x \\ 0 & 1 & 0 & -m_y \\ 0 & 0 & 1 & -m_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_x & u_y & u_z & -(u_x m_x + u_y m_y + u_z m_z) \\ v_x & v_y & v_z & -(v_x m_x + v_y m_y + v_z m_z) \\ n_x & n_y & n_z & -(n_x m_x + n_y m_y + n_z m_z) \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_x & u_y & u_z & -u^T m \\ v_x & v_y & v_z & -v^T m \\ n_x & n_y & n_z & -n^T m \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Now, by dropping the coordinate in the direction of n, we get:

$$M = \begin{pmatrix} u_x & u_y & u_z & -u^T m \\ v_x & v_y & v_z & -v^T m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is the matrix $M \in \mathbb{R}^{3 \times 4}$ we needed.

Exercise 2

(a) The transformation matrix associated with rotation by 45° around the y-axis can be computed as:

$$R_y(45) = \begin{pmatrix} \cos(45°) & 0 & \sin(45°) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45°) & 0 & \cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation matrix associated with scaling by $(1, 4, 16)$ (in x-, y-, and z-direction respectively) is:

$$S(1,4,16) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transformation matrix associated with translation by 10 units along the x-axis is:

$$T(10,0,0) = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, the transformation matrix $M \in \mathbb{R}^{4\times 4}$ that can be used to transform the triangle can be computed by concatenating the above transformations as:

$$M = T(10,0,0) \cdot S(1,4,16) \cdot R_y(45)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 4 & 0 & 0 \\ -\frac{16}{\sqrt{2}} & 0 & \frac{16}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 10 \\ 0 & 4 & 0 & 0 \\ -\frac{16}{\sqrt{2}} & 0 & \frac{16}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) In the lecture we have seen that the matrix used to perform a transformation M on a normal is given by $(M^T)^{-1}$. We can conclude that any transformation-matrix that can be used as-is to transform normals must have the following property:

$$(M^T)^{-1} = M \Rightarrow M^T = M^{-1}$$

This is the case for rotation matrices:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cdot\cos\alpha - \sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha - \sin\alpha\cdot\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Showing this for the 2x2 matrix is sufficient as this shows that the property is given for any rotation around any of the 3 standard-axis. Also this property remains given if you multiply multiple matrices with this property.

However this is not the case for scaling transformations as the inverse of such a matrix is given by:

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Neither is this the case for translations:

$$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus we can use the matrix as-is to transform normals if the transformation performed is a rotation.

(c) Let $M = T(10,0,0) \cdot S(1,4,16) \cdot R_y(45)$ be the transformation of the triangle as derived in (a). From the lecture we know that the transformation-matrix to transform the the normal of the triangle is given by $(M^{-1})^T$. Thus we need to compute:

$$(M^{-1})^T = ((T(10,0,0) \cdot S(1,4,16) \cdot R_y(45))^{-1})^T = (R_y(45)^{-1} \cdot S(1,4,16)^{-1} \cdot T(10,0,0)^{-1})^T$$

We know:

$$R_y(45)^{-1} = R_y(45)^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S(1,4,16)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T(10,0,0)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which means for $M^{-1} = R_y(45)^{-1} \cdot S(1,4,16)^{-1} \cdot T(10,0,0)^{-1}$:

$$M^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -10 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{16\cdot\sqrt{2}} & -\frac{10}{\sqrt{2}} \\ 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{16\cdot\sqrt{2}} & -\frac{10}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And we get the transformation-matrix for normals:

$$M_n = (M^{-1})^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{16\cdot\sqrt{2}} & -\frac{10}{\sqrt{2}} \\ 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{16\cdot\sqrt{2}} & -\frac{10}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{16\cdot\sqrt{2}} & 0 & \frac{1}{16\cdot\sqrt{2}} & 0 \\ -\frac{10}{\sqrt{2}} & 0 & -\frac{10}{\sqrt{2}} & 1 \end{pmatrix}$$

We now compute the normal of the initial triangle:

Firstly, we define $v_1 = p_3 - p_1 = \begin{pmatrix} 6-8 \\ 4-(-10) \\ 6-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 14 \\ 4 \end{pmatrix}$ and

$v_2 = p_2 - p_1 = \begin{pmatrix} -4-8 \\ -10-(-10) \\ -2-2 \end{pmatrix} = \begin{pmatrix} -12 \\ 0 \\ -4 \end{pmatrix}$. Now we calculate:

$$n' = v_1 \times v_2 = \begin{pmatrix} -2 \\ 14 \\ 4 \end{pmatrix} \times \begin{pmatrix} -12 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 14\cdot(-4) - 4\cdot 0 \\ 4\cdot(-12) - (-4)\cdot(-2) \\ -2\cdot 0 - 14\cdot(-12) \end{pmatrix} = \begin{pmatrix} -56 \\ -56 \\ 168 \end{pmatrix}$$

We normalize the vector and get the normal-vector:

$$n = \frac{n'}{||n'||} = \begin{pmatrix} -\frac{56}{\sqrt{34496}} \\ -\frac{56}{\sqrt{34496}} \\ \frac{168}{\sqrt{34496}} \end{pmatrix} \approx \begin{pmatrix} -\frac{56}{185.731} \\ -\frac{56}{185.731} \\ \frac{168}{185.731} \end{pmatrix} \approx \begin{pmatrix} -0.302 \\ -0.302 \\ 0.905 \end{pmatrix}$$

Now we transform the normal vector n with the transformation-matrix M_n:

$$n'_t = M_n \cdot n \approx \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{16\cdot\sqrt{2}} & 0 & \frac{1}{16\cdot\sqrt{2}} & 0 \\ -\frac{10}{\sqrt{2}} & 0 & -\frac{10}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} -0.302 \\ -0.302 \\ 0.905 \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.426 \\ -0.076 \\ 0.053 \\ 0 \end{pmatrix}$$

We normalize the vector once again and get the normal-vector after the transformation:

$$n_t = \frac{n'_t}{||n'_t||} \approx \begin{pmatrix} -\frac{0.426}{0.436} \\ -\frac{0.076}{0.436} \\ \frac{0.053}{0.436} \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.977 \\ -0.174 \\ 0.122 \\ 0 \end{pmatrix}$$

n_t now is the normal-vector of the triangle after applying the given transformation to it.