Assignment 6

Basic Techniques in Computer Graphics
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Exercise 1 Barycentric Coordinates

(a) Linear System For a given triangle defined by $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ and a point $\mathbf{p} \in \mathbb{R}^2$, we can represent the point $\mathbf{p}$ using barycentric coordinates via the following equation:

$$\mathbf{p} = \alpha \cdot \mathbf{a} + \beta \cdot \mathbf{b} + \gamma \cdot \mathbf{c}, \text{ where } \alpha + \beta + \gamma = 1.$$

So, we can write γ as: $\gamma = 1 - \alpha - \beta$, which gives us:

$$\begin{aligned} &\mathbf{p} = \alpha \cdot \mathbf{a} + \beta \cdot \mathbf{b} + (1 - \alpha - \beta) \cdot \mathbf{c} \\ \implies &\alpha \cdot (\mathbf{a} - \mathbf{c}) + \beta \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{p} - \mathbf{c} \end{aligned}$$

$$\implies \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \mathbf{p} - \mathbf{c}$$

which gives us the 2-by-2 linear system in the form $\mathbf{A}\mathbf{x} = \mathbf{r}$ such that $\mathbf{A} = \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix} = \begin{bmatrix} a_x - c_x & b_x - c_x \\ a_y - c_y & b_y - c_y \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ (barycentric coordiates in 2D), and $\mathbf{r} = \mathbf{p} - \mathbf{c} = \begin{bmatrix} p_x - c_x \\ p_y - c_y \end{bmatrix}$

(b) Closed-Form Solution Using Cramer's rule we can derive the closed-form solutions for barycentric coordinates in 2D as:

$$\alpha = \frac{\det \begin{bmatrix} \mathbf{p} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}}{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}} = \frac{(p_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(p_y - c_y)}{(a_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(a_y - c_y)}$$

$$\beta = \frac{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{p} - \mathbf{c} \end{bmatrix}}{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}} = \frac{(a_x - c_x)\cdot(p_y - c_y) - (p_x - c_x)\cdot(a_y - c_y)}{(a_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(a_y - c_y)}$$

(c) Example Given, $\mathbf{a} = (2, 2)^\top, \mathbf{b} = (10, 2)^\top, \mathbf{c} = (2, 4)^\top, \mathbf{p} = (3, 3)^\top$, we can use the closed-form solution derived in (b) to get:

$$\begin{aligned} \alpha = \frac{\det \begin{bmatrix} \mathbf{p} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}}{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}} &= \frac{(p_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(p_y - c_y)}{(a_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(a_y - c_y)} \\ &= \frac{(3-2)(2-4) - (10-2)(3-4)}{(2-2)(2-4) - (10-2)(2-4)} = \frac{-2+8}{0+16} = \frac{6}{16} = \frac{3}{8}, \\ \beta = \frac{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{p} - \mathbf{c} \end{bmatrix}}{\det \begin{bmatrix} \mathbf{a} - \mathbf{c} & \mathbf{b} - \mathbf{c} \end{bmatrix}} &= \frac{(a_x - c_x)\cdot(p_y - c_y) - (p_x - c_x)\cdot(a_y - c_y)}{(a_x - c_x)\cdot(b_y - c_y) - (b_x - c_x)\cdot(a_y - c_y)} \\ &= \frac{(2-2)(3-4) - (3-2)(2-4)}{(2-2)(2-4) - (10-2)(2-4)} = \frac{0+2}{0+16} = \frac{2}{16} = \frac{1}{8}. \end{aligned}$$

So, we have $\gamma = 1 - \alpha - \beta = 1 - \frac{3}{8} - \frac{1}{8} = 1 - \frac{4}{8} = 1 - \frac{1}{2} = \frac{1}{2}$. Therefore, the barycentric coordinates of $\mathbf{p}$ in the triangle $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are $(\frac{3}{8}, \frac{1}{8}, \frac{1}{2})$.