Copulae Made Easy

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1 Basics of Copula Theory

Lemma 1.1 (Frechet bounds). *Copula has the following bounds:*

$$\max\left(0, \sum_{i=1}^{d} u_i + 1 - d\right) \le C(u_1, ..., u_d) \le \min(u_1, ..., u_d). \tag{1}$$

Proof. Note that $C(u1, ..., u_d) = \mathbb{P}(\bigcap_{i=1}^{d}\{U_i \le u_i\})$ and

$$\bigcap_{i=1}^{d}\{U_i \le u_i \subset \{U_i \le u_i\}, \text{ for } i = 1, ..., d.$$

The second inequality (upper bound) follows then by combining these two facts. For the lower bound, re-write the joint CDF as follows:

$$C(u_1, ..., u_d) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{d}\{U_i > u_i\}\right).$$

One has

$$\mathbb{P}\left(\bigcup_{i=1}^{d}\{U_i > u_i\}\right) \le \sum_{i=1}^{d}\mathbb{P}(U_i > u_i) = d - \sum_{i=1}^{d} u_i.$$

Combining ... and the fact that $C(u) \ge 0$, yields: $\max(0, \sum_{i=1}^{d} u_i + 1 - d) \le C(u_1, ..., u_d)$. □

Theorem 1.2 (Invariance via strict monotonic transformation). *Let $X = (X_1, ..., X_d)$ be a multivariate random variable with copula C and T_i, $i = 1, ...d$ be strictly increasing functions. Then*

$$Y = (Y_1, ..., Y_d) := (T_1(X_1), ..., T_d(X_d))$$

has the same Copula as X, i.e. **Copula is invariant under strict monotonic transformation**.

Proof. Denote by $G, G_1, ..., G_d$ and $F, F_1, ..., F_d$ joint- and marginal CDFs of$X$ and Y, respectively. We need to prove that

$$G(y_1, ..., y_d) = C(G_1(y_1), ..., G_d(y_d)). \tag{2}$$

It follows from the definition of $Y_i,\ i = 1, ..., d$ that

$$G_i(y_i) = \mathbb{P}(T_i(X_i) \le y_i) = \mathbb{P}(X_i \le T_i^{-1}(y_i)) = F_i(T_i^{-1}(y_i)). \tag{3}$$

Let's work out the joint CDF of Y:

$$\begin{aligned} G(y_1, ..., y_d) &= \mathbb{P}(Y_1 \le y_1, ..., Y_d \le y_d) \\ &= \mathbb{P}(T_1(X_1) \le y_1, ..., T_d(X_d) \le y_d) \\ &= \mathbb{P}(X_1 \le T_1^{-1}(y_1), ..., X_d \le T_d^{-1}(y_d)) \\ &= F(T_1^{-1}(y_1), ..., T_d^{-1}(y_d)) \\ &= C(F_1(T_1^{-1}(y_1)), ..., F_d(T_d^{-1}(y_d))). \end{aligned} \tag{4}$$

Combining (3) and (4) yields (2). Note that the last equality in (4) is due to the property of Copula. $\square$

TODO: provide a generic description or pseudocode how to construct a multivariate rv with a certain dependence structure and given marginal cdfs from scratch.

Algorithm 1: Construction of a multivariate rv with given marginal CDFs and dependence structure.

Input: Marginal CDFs, $F_1, ..., F_d$ and a dependence structure, e.g. via an explicit Copula.

Output: A multivariate rv X with marginal CDFs given by $F_i,\ i = 1, ..., d$ and the given dependence structure.

Step 1: Construct $U = (U_1, ..., U_d)$ where $U_i \sim \mathcal{U}(0,1),\ i = 1, ..., d$ that has joint CDF $C(u)$.

Step 2: Work out the inverse functions of the marginal CDFs: $F_1, ..., F_d$. Sometimes a numerical approximation is required when an analytical form is not available or tractable to work with.

Step 3: Obtain $X = (X_1, ..., X_d)$ where

$$X_i = F_i^{-1}(U_i),\ i = 1, ..., d. \tag{5}$$

The rv X has the given marginal cdfs $F_i,\ i = 1, ..., d$ and Copula C.

Remark. Some preliminary results are necessary here to ease understanding, e.g. $F_X^{-1}(U) \sim \mathcal{F}_X$, $F_X(X) \sim \mathcal{U}(0,1)$.

Step 1 in Algorithm 1 involves simulating the multivariate uniform rv U with a given joint CDF. $\diamond$

Let X be a random variable with cumulative distribution function (cdf) F_X. Then

1. The following holds true
$$F^{\leftarrow}(U) \overset{D}{=} X \tag{6}$$
2. Moreover, if F is continuous then $F(X) \sim \mathcal{U}(0,1)$.

$$\begin{tikzcd}
\Omega \arrow[rr, "X"] \arrow[rr, bend left, "F^{\leftarrow}(U)"] \arrow[dr, "U"] \arrow[dr, bend right, "F(X)"'] & & \mathbb{R} \arrow[dl, "F"] \\
& {[0,1]} &
\end{tikzcd}$$

2 Examples of Copulae

2.1 Fundamental Copulae

Independence Copula The independence copula is

$$\Pi(u_1, ..., u_d) = \prod_{i=1}^{d} u_i. \tag{7}$$

Comonotonicity Copula: is the Frechet upper bound from (1):

$$M(u_1, ..., u_d) = \min\{u_1, ..., u_d\}. \tag{8}$$

This corresponds to the case of *perfectly positively dependent* rvs, i.e. they are (almost surely) strictly increasing functions of each other so that $X_i = T_i(X_1)$ for all $i = 2, ..., d$. More details to come.

Countermonotonicity copula is the two-dimensional Frechet lower bound copula from (1) given by:

$$W(u, v) = \max(u + v - 1, 0). \tag{9}$$

2.2 Implicit Copulae

Object is to construct joint CDF $C(u)$. This can be done by postulating an explicit function or by implicitly constructing a multivariate rv and infer the CDF of its rank.

2.3 Explicit Copulae