## Copulae Made Easy

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### 1 Basics of Copula Theory

Lemma 1.1 (Frechet bounds). Copula has the following bounds:

$$\max\left(0, \sum_{i=1}^{d} u_i + 1 - d\right) \le C(u_1, ..., u_d) \le \min(u_1, ..., u_d). \tag{1}$$

*Proof.* Note that  $C(u_1,...,u_d) = \mathbb{P}(\bigcap_{i=1}^d \{U_i \leq u_i\})$  and

$$\bigcap_{i=1}^{d} \{ U_i \le u_i \subset \{ U_i \le u_i \}, \text{ for } i = 1, ..., d.$$

The second inequality (upper bound) follows then by combining these two facts. For the lower bound, re-write the joint CDF as follows:

$$C(u_1, ..., u_d) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{d} \{U_i > u_i\}\right).$$

One has

$$\mathbb{P}\left(\bigcup_{i=1}^d \{U_i > u_i\}\right) \le \sum_{i=1}^d \mathbb{P}(U_i > u_i) = d - \sum_{i=1}^d u_i.$$

Combining ... and the fact that  $C(u) \geq 0$ , yields:  $\max(0, \sum_{i=1}^{d} u_i + 1 - d) \leq C(u_1, ..., u_d)$ .

Theorem 1.2 (Invariance via strict monotonic transformation). Let  $X = (X_1, ..., X_d)$  be a multivariate random variable with copula C and  $T_i$ , i = 1, ...d be strictly increasing functions. Then

$$Y = (Y_1, ..., Y_d) := (T_1(X_1), ..., T_d(X_d))$$

has the same Copula as X, i.e. Copula is invariant under strict monotonic transformation.

*Proof.* Denote by  $G, G_1, ..., G_d$  and  $F, F_1, ..., F_d$  joint- and marginal CDFs of X and Y, respectively. We need to prove that

$$G(y_1, ..., y_d) = C(G_1(y_1), ..., G_d(y_d)).$$
(2)

It follows from the definition of  $Y_i$ , i = 1, ..., d that

$$G_i(y_i) = \mathbb{P}(T_i(X_i) \le y_i) = \mathbb{P}(X_i \le T_i^{-1}(y_i)) = F_i(T_i^{-1}(y_i)). \tag{3}$$

Let's work out the joint CDF of Y:

$$G(y_{1},...,y_{d}) = \mathbb{P}(Y_{1} \leq y_{1},...,Y_{d} \leq y_{d})$$

$$= \mathbb{P}(T_{1}(X_{1}) \leq y_{1},...,T_{d}(X_{d}) \leq y_{d})$$

$$= \mathbb{P}(X_{1} \leq T_{1}^{-1}(y_{1}),...,X_{d} \leq T_{d}^{-1}(y_{d}))$$

$$= F(T_{1}^{-1}(y_{1}),...,T_{d}^{-1}(y_{d}))$$

$$= C(F_{1}(T_{1}^{-1}(y_{1})),...,F_{d}(T_{d}^{-1}(y_{d}))). \tag{4}$$

Combining (3) and (4) yields (2). Note that the last equality in (4) is due to the property of Copula.  $\Box$ 

TODO: provide a generic description or pseudocode how to construct a multivariate rv with a certain dependence structure and given marginal cdfs from scratch.

**Algorithm 1:** Construction of a multivariate rv with given marginal CDFs and dependence structure.

**Input:** Marginal CDFs,  $F_1, ..., F_d$  and a dependence structure, e.g. via an explicit Copula.

**Output:** A multivariate rv X with marginal CDFs given by  $F_i$ , i = 1, ..., d and the given dependence structure.

**Step 1:** Construct  $U = (U_1, ..., U_d)$  where  $U_i \sim \mathcal{U}(0, 1), i = 1, ..., d$  that has joint CDF C(u).

**Step 2:** Work out the inverse functions of the marginal CDFs:

 $F_1, ..., F_d$ . Sometimes a numerical approximation is required when an analytical form is not available or tractable to work with.

Step 3: Obtain  $X = (X_1, ..., X_d)$  where

$$X_i = F_i^{-1}(U_i), \ i = 1, ..., d.$$
 (5)

The rv X has the given marginal cdfs  $F_i$ , i = 1, ..., d and Copula C.

**Remark.** Some preliminary results are necessary here to ease understanding, e.g.  $F_X^{-1}(U) \sim \mathcal{F}_X$ ,  $F_X(X) \sim \mathcal{U}(0,1)$ .

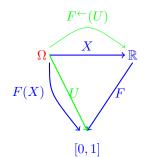
Step 1 in Algorithm 1 involves simulating the multivariate uniform rv U with a given joint CDF.  $\diamond$ 

Let X be a random variable with cumulative distribution function (cdf)  $F_X$ . Then

1. The following holds true

$$F^{\leftarrow}(U) \stackrel{D}{=} X \tag{6}$$

2. Moreover, if F is continuous then  $F(X) \sim \mathcal{U}(0,1)$ .



# 2 Examples of Copulae

#### 2.1 Fundamental Copulae

Independence Copula The independence copula is

$$\Pi(u_1, ..., u_d) = \prod_{i=1}^d u_i.$$
 (7)

Comonotonicity Copula: is the Frechet upper bound from (1):

$$M(u_1, ..., u_d) = \min\{u_1, ..., u_d\}.$$
(8)

This corresponds to the case of perfectly positively dependent rvs, i.e. they are (almost surely) strictly increasing functions of each other so that  $X_i = T_i(X_1)$  for all i = 2, ..., d. More details to come.

**Countermonotonicity copula** is the two-dimensional Frechet lower bound copula from (1) given by:

$$W(u, v) = \max(u + v - 1, 0). \tag{9}$$

#### 2.2 Implicit Copulae

Object is to construct joint CDF C(u). This can be done by postulating an explicit function or by implicitly constructing a multivariate rv and infer the CDF of its rank.

#### 2.3 Explicit Copulae