Programming in Tabled Prolog (very) DRAFT [1]

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Contents

Chapter 1

Background and Motivation: Why Tabling?

There is a flaw in the very foundations of Logic Programming: Prolog is nondeclarative. Of course, everyone knows that real Prolog as it is used is nondeclarative. Prolog programs abound with cuts, var tests, asserts, and all kinds of ugly warts. Everyone agrees that Prolog should be more declarative and much research time has been spent, and spent productively, trying to remedy these problems by providing more declarative alternatives. But I believe that there is a more serious flaw closer to the heart of Prolog, right at its very center, and this flaw has caused a number of problems in logic programming. I'm convinced that we must address this flaw, before we can proceed together productively with logic programming. Of course, whether this is a "flaw" is in the eye of the beholder; we all have arguments over whether a particular behavior is a bug or a feature. As is clear from my presentation, I think I've found a bug; some of you will think it's a feature. Wherever we stand on this issue, however, I believe we must be clear about its implications. Only then will we be able to communicate effectively about where LP is and where it should go.

Let me also mention that I am by no means pointing out something new, unknown to the community; rather I'm trying to raise our consciousness about an inconsistency and its implications.

The problem is the very semantics of logic programming. The foundational results of our field show that for pure horn clauses, the following are equivalent: logical implication, SLD resolution, and the least fixpoint of the T_p operator. And logic programming basically "works" because SLD corresponds to computation (i.e. Prolog) *and* to logical implication (truth in models).

But there is a fly in the ointment, the dirty secret that we Prolog aficionados try to keep in the closet but Prolog programmers are all too aware of. Let's look more carefully at the full SLD tree, the foundation for us Prolog hackers. There are three kinds of paths in an SLD tree: 1) those ending in success, 2) those ending in failure, and 3) those not ending. The foundational results tell us that those ending in success correspond exactly to those instances of the query that logically follow from the program. This is fine and dandy and certainly something we want and need. But as programmers, that's not enough. We programmers also want, need actually, to distinguish between paths that end in failure and paths that don't end at all. When we run a program, and it responds 'no', we may accept that response as a good answer to our query. But if it goes into an infinite loop never coming back with an answer at all, we decide we have a bug in our program and set out to fix it.

So the problem with SLD is not that it doesn't do the right things for success, but that it doesn't do the right thing with the other two possible outcomes. Note that this has nothing to do with how we search the SLD tree, e.g. unfairly with depth first search or fairly with breadth-first search; it's a property of the tree itself. And it has nothing to do with the fact that our theory deals best with ground programs and Prolog programs deal with variables. The problem concerns which paths in the SLD tree are finite and which are infinite.

In Prolog, as compared with other languages, it seems easier to write partially correct programs, programs that, if they halt, they give the right answers. This comes from the power of declarative programming. But it seems much harder to write and reason about totally correct programs, programs that halt in all (the right) cases. So it may be the case that what we gain from declarativeness in our ability to reason about the partial correctness of Prolog programs, we lose in the difficulty of reasoning about total correctness. Theoreticians may accept partial correctness as good enough; but we users have to know precisely how partial.

And sadly, the problems show up for very simple programs. The issue is clearly seen in transitive closure. Consider the definition:

```
tca(X,Y) :- a(X,Y).
tca(X,Y) :- a(X,Z), tca(Z,Y).
```

Prolog programmers know that this definition is fine when the predicate "a" stands for "parent", and then "tca" is "ancestor". But what happens if "a" is defined by the following facts?

```
a(1,1).
a(2,1).
```

Consider what happens when we ask the query: :- tca(1,2). Prolog (that is SLD resolution) goes into an infinite loop. Are we as programmers happy? Do we say that since it hasn't returned, it must mean that you can't get to 2 from 1 in this graph? Not at all. We turn back to fix the program, probably modifying it by adding a loop-check. That is, we add a list of nodes to the predicate "tca", which contains the nodes already visited. Then before continuing to search from a node, we check that it has not already been visited. We're not happy until our program responds 'no' to our query. So clearly the semantics of our programming language is going to have to account for this. And it does. We can't depend on the previously referenced foundational results to give a semantics for pure Prolog as a programming language, where we want to know what both 'yes' and 'no' answers mean. The theory must somehow be extended. This was done in [] by Kenneth Clark who defined the completion of a program. The semantics of (pure) Prolog programs is normally taken to be the logical implications of the completion (plus some standard axioms such as the unique names axiom.) To find the completion of our tca/2 program, we first convert the two implication rules for tca/2 into the equivalent one implication with a disjunction in the antecedent, and then we turn the implication into a biconditional. We do a similar transformation for a/2. Then we take the meaning of the program to be the logical implications of these biconditionals (including some standard axioms). And it is the case that $\sim tca(1,2)$ is not logically implied by this theory (and neither is tca(1,2), of course.) So the completion semantics correctly tells us that SLD will not terminate for this query. And indeed Prolog theoreticians (as well as the programmers) know that tca/2 as defined above is NOT a general definition for transitive closure. (But note that it *is* transitive closure if the graph is acyclic. Maybe this is OK, since it is often the case that algorithms for constrained data are simpler, but here it could be rather confusing, I think.)

But in any case the Prolog community was (mostly) happy with this situation, until the database people showed up and asked the obvious question: Why doesn't the obvious two-line definition of tca/2 indeed mean transitive closure? (To some of us, it felt somewhat as though they were pointing out that the emperor had no clothes.) They pointed out that if one uses the fixed point characterization from the original foundation trilogy, instead of the SLD characterization, it does. Indeed, transitive closure is only the beginning; there are many advantages in using this semantics. *All* programs without function symbols are terminating, and many simple programs now have a natural programmers' semantics. Indeed DCG's now *do* mean the corresponding context free languages, whereas under the completion semantics they don't always. But even more important is what this fixpoint semantics does to the theory of programs. The theory is much simpler, and accords with theory already developed. E.g., automata theory (e.g. CFL's) is now correctly reflected in LP. And also interestingly, this semantics leads the way for a theory of default negation, which has blossomed. This is because if we have a better idea of when things are known false, the theory describing such negative knowledge is simpler, more interesting, and perhaps even more useful.

Actually, the situation for the completion is somewhat worse than implied by the transitive closure example. There are programs for which Prolog goes into an infinite loop even though the completion of the program determines all goals. And this is *not* because Prolog is depth-first; it's because its selection rule is left-to-right. To get Prolog actually to display the behavior predicted by the completion, one would have to interleave computations of all goal orderings, and fail a goal if any of the interleavings failed. This is hardly an attractive alternative for Prolog implementors or programmers.

So I personally think minimal model semantics (i.e., the fixpoint semantics) is better than the completion semantics for logic programming. I think the completion semantics is a bug, which should be fixed, and it should be fixed by taking the fixpoint semantics. There are those who either don't think this is a bug, or don't think that this is a reasonable fix. I certainly accept that the point is arguable. Going to the fixpoint semantics is a big step for at least two reasons:

Objection 1: It violates a principle dear to the hearts of some theoretically inclined logic programmers, that the meaning of a program consists of the logical implications of the theory of the formulas making up the program (or a simple transformation of them).

Objection 2: It violates a principle dear to the hearts of some practically inclined logic programmers (like me), that a program should have the well-known procedural semantics, i.e., be executable by SLD resolution (and WAM engines).

The first objection is real and in some sense is not satisfactorily solvable, I believe. We know that there is no first-order theory that characterizes transitive closure (the usual Prolog loop-check program requires default negation.) I will discuss some implications of this later, and try to claim that things aren't so bad, but I can't say they aren't true. Basically, the purpose of this entire book is to claim that the second objection can be overcome. It is overcome by using OLDT resolution [] (or a generalizaiton SLG Resolution []) in place of SLD resolution in the foundational theorems. OLDT is a resolution strategy very similar to SLD, but one that avoids redundant computation by remembering subcomputations and reusing their results to respond to later requests. This remembering and reuse has been called memoization, tabling, lemmatization, caching, well-formed substring tables, and probably a number of other names. With OLDT instead of SLD, the foundational theorem can be even stronger, including a theorem relating OLDT failure leaves and the fixpoint definition.

OLDT terminates (i.e., has only success and failure leaf nodes) for all queries to programs that have finite minimal models, i.e. that have a least fixpoint of finite size. This guarantees total correctness for the transitive closure definition (over finite graphs). Actually, even more can be said: OLDT terminates for all queries to programs for which only a finite portion of the minimal model is looked at; that is, for all programs with the Bounded Term Size (BTS) property. BTS is slightly circular, and undecidable, but to me intuitively says that any finitary, constructive definition will terminate. (Admittedly, this is also somewhat circular.)

One thing that happens when we do not have first-order theories to explain our semantics, i.e. the first objection above, is that we may leave the realm of recursively enumerable sets. That is, if we did have a first-order theory to explain the semantics, then we'd know that a theorem-prover could enumerate the theorems and the semantics would be r.e. However, by going to the minimal model semantics, we may (and do, when we add negation) get non-r.e. sets. That is, we leave the realm of the computable sets. This might give pause to someone interested in computation, because then we can write programs that aren't in principle computable. Note that this isn't a problem for totally correct programs (which will be recursive programs), which are those we usually want to write. I guess that just as we rely on the programmer to write programs that are terminating, we must rely on the programmer to write programs that are computable.

SLDNF was the obvious (and perhaps only reasonable) extension of SLD to include default negation. If we are to base our new logic programming on OLDT, the question arises as to how OLDT should be extended to handle negation. Given the theory of default negation developed over the last several years, it seems that the well-founded semantics is the appropriate semantics to compute for logic programs with unconstrained negation. (The other possible semantics is the partial stable model semantics, but this is NP complete for propositional programs, so may be not particularly appropriate for a basic computational mechanism.) For OLDT there have been several proposals for computing the well-founded semantics, in particular WELL! and XOLDTNF, which are quite similar. The one I'm proposing here is SLG, which I claim is the (or a) right one.

The simplest reason that SLG is better than the previous two is that they are exponential for general propositional normal programs, whereas SLG is polynomial. Also SLG produces a residual program when there are undefined atoms in the well-founded models, and this residual program can be further processed to find certain partial stable models. So in many cases SLG could be used as a preprocessor for a partial stable model evaluator.

SLG implements the procedural interpretation of Horn clauses, with the addition of tabling. This is a more complex procedural interpretation than SLDNF (and the procedural interpretation of negation makes it even more complex), but I think it is indeed a procedural interpretation and I think it is acceptable. But this is not enough to respond fully to objection 2 above. The Prolog programmers want to know if SLG can be implemented so that it is efficient enough to compete with Prolog, (or actually efficient enough to compete with C++, perhaps our *true* competitor.) I think the answer is 'yes'. Actually, even more than that, I think it will be the case that we will be able to construct an engine that for many useful problems will be significantly faster than Prolog. And these problems will be in what has historically been logic programming's core applications: natural language, AI search, knowledge bases, and databases. The examples in this book will indicate in detail where some of the problems lie. There is much work to do, but I will claim that results we already have in the XSB project indicate that it will definitely be possible.

The approach is to take SLG as the basic computation strategy, and to use SLDNF as an optimization. So when SLDNF can be proved to have the same behavior as SLG, then we can use it and get current Prolog speeds. To do this we need an implementation that fully and closely integrates SLG and SLD, allowing them to be used interchangably and intermixedly. And that is what SLG resolution supports and its implementation in XSB achieves.

Chapter 2

Introduction to Prolog

This chapter introduces the Prolog programming language. Here we will explain a bit of how Prolog works. It is not intended to be a full description of how to become an expert Prolog programmer. For that, after reading this chapter, you should refer to another book, such as *Programming in Prolog* by Clocksin and Mellish, or if you are already very familiar with Prolog, you might look at *The Craft of Prolog* by Richard O'Keefe. While this is an introduction to Prolog, even experts may find something of interest in this chapter, since I explain Prolog in a somewhat unusual way, that may give new insights to old Prolog programmers.

Prolog's name is short for "Programming in Logic" (or really for Programmation Logique?") As its name suggests, Prolog is firmly based on logic, and Prolog programs can be understood as statements in a formal logic. I.e., a Prolog program can be thought of as a set of statements in first-order logic, and the meaning of the program is the set of true implications of those logical statements. This is the approach that is usually taken to describe Prolog programming to novices. However, the amazing thing about logic programming to me is not that it is logic, but that it is programming. These Prolog programs are not only statements in a logic but they are also statements in a programming language. This is referred to in the biz by saying that Prolog programs have a procedural interpretation (i.e., as programs) as well as a declarative interpretation (i.e., as statements in a logic.) The introduction to Prolog that I give here will emphasize its procedurality. This may be anathema to some Prolog purists (and it certainly would have been to me a while ago) but I now feel that this is the best way to introduce logic programming to computer scientists who already know about programming. We will build on your understanding of programming, and use that to lead to logic.

2.1 Prolog as a Procedural Programming Language

Prolog, as a programming language, is a little unusual. It can be understood as a standard procedural language with two unusual properties. It is a procedural language like C or Algol. One programs in a procedural language by writing procedures that carry out particular operations. One specifies what a procedure is to do by using primitive statements and by invoking other procedures. Prolog procedures have only local variables and all the information that a procedure can use or produce must be passed through its arguments.

C can be viewed as a procedural language by thinking of using it without functions; i.e., all functions return void, and information is passed to and from functions through their arguments only.

In Prolog, procedures are called predicates. The two unusual aspects of Prolog are:

1. Prolog has assign-once variables, and
2. Prolog is nondeterministic.

2.1.1 Assign-once Variables

By saying that Prolog has assign-once variables, I mean that any particular variable in a Prolog procedure can only ever get one value assigned to it. A Prolog variable at any point in execution either has a value, which can thereafter never be changed, or it has not yet been given a value. This may seem to be an incredibly strong restriction. In C, for instance, one usually programs by setting a variable's value and then changing it during the execution of the program. For example, to sum up an array of numbers, one sets the accumulator variable to 0 and the loop variable to 1, and then increments the loop variable to step through the array modifying the accumulator at each step. How in the world could this be done with assign-once variables? The secret is that it can be done easily through the use of recursion.

So let's write a simple Prolog program to see how we can do something interesting with only assign-once variables. Let's consider the problem of adding up the numbers in a list. Prolog is a list-processing language, similar to Lisp. Its primary data structure is the tree (called a term), a very common form of which is the list. There are three basic data types in Prolog: 1) integers, 2) floating point numbers, and 3) atoms. The first two should be self-explanatory. Atoms are simply symbols that represent themselves. Prolog terms (trees) are constituted of integers, floats, atoms and other terms.

A list in Prolog is written with square brackets and with its elements separated by commas. For example the following are lists:

```
    [1,2,3]   [aa,bbb,d]   []   [[2,b],or,not,[2,b]]
```

The first is a list of integers, the second a list of atoms, the third is the empty list consisting of no elements, and the fourth is a list containing four elements, the first and last being themselves lists.

So let's now write a program in a made-up procedural language (that is *not* Prolog, but somewhat similar) and see if we can sum up the elements of an integer list with assign-once variables.

```
sum(List,Sum) :-
    if List = []
      then Sum := 0
      else Head := head(List)
           Tail := tail(List)
           sum(Tail,TailSum)
           Sum := TailSum + Head
```

The first line (preceding the `:-`) declares the name of the procedure and its formal parameters. The remaining lines (following the `:-`) make up the body of the procedure definition. We have assumed the existence of two functions, head and tail, to extract the first element of a list, and the remainder of the list after the head is removed, respectively. Sum is a recursive procedure. It takes a list of numbers as its first argument and returns the sum of the numbers in its second. It first checks to see if the list is empty. If so, it sets the sum to 0 and returns directly. If not, it saves the first element of the list in a local variable, `Head`, and then calls `sum` recursively on the tail of the list, getting the sum of the rest of the list in the local variable `TailSum`. It then adds `Head` to `TailSum` to get the value for `Sum`, which it sets and returns. Notice that no single variable gets two different values. The variable `Sum` is not set and then changed; each recursive invocation has a different `Sum` variable and each gets set only once, to its appropriate partial sum. Note also that the loop variable in the iterative version of summing an array is here replaced by the variables containing each sublist. So here too there is no need to have multiply assigned variables. Instead of one variable getting many values, we can instead uses many variables, each getting one value. (Let me point out for those of you who may be worried about efficiency that this is a conceptual point; it may well be that the underlying implementation of such a program would actually use just one location for all the `Sum` variables.)

So we see that we are able get by in this case with assign-once variables. It turns out that this idea of using recursion and the multiple variables at the different recursion levels is very general. This is not just a trick that works in this case only, but is an example of a very general technique.

Now having assign-once variables gives rise to a very interesting phenomenon: assignment can be symmetrical in Prolog. That is, Prolog doesn't have to treat the left and the right sides of an assignment differently, as must be done in normal procedural languages such as C or Java. As a matter of fact, Prolog doesn't have to treat tests and assignments differently either. I.e., Prolog doesn't need two operators, say == for testing and = for assignment as C does; it needs only one.

Let's first consider assignment. Consider the following assignments:

```
X := 5
Y := X
```

We'll assume that neither X nor Y have been assigned a value before this sequence is executed. So X gets the value 5 by the first statement, and then Y is assigned the value of X, so Y gets the value 5 as well. Now consider the following statements:

```
X := 5
X := Y
```

The first statement again assigns 5 to X. Now consider the second. X has the value 5 and Y has no value. Since Prolog is an assign-once language, X can get only one value and it already has it, so we know we can't change it. But Y doesn't yet have a value. So the only reasonable thing to do is to set Y to be 5, i.e., to X's value. Note that this sequence of assignments has the same net effect that the previous sequence had.

This suggests how we can treat both sides of an assignment in the same way. If one of the variables has a value and the other doesn't, then assign the value that the one has to the other. If neither variable has a value yet, then make it so that whenever one of them gets a value, the other gets that same value. If they both have a value, then if it's the same value, then the assignment is a no-op. If the two values are different, then there is a problem since neither can get a new value. In this case we say the computation fails. (We will talk more about failure later.)

Notice that this definition of "assignment" means that any ordering of the same (or symmetric) assignments gives the same result. For example, consider the different ordering of our assignments above:

```
X := Y
X := 5
```

Again assuming that X and Y start with no values, the first statement causes Prolog to bind X and Y together, so that whenever one of them gets a value, the other will also get that same value. Then the second statement causes X to get the value 5, and so Y gets that value, too. So after these two assignments are executed, both X and Y have the value 5, exactly as they do after the previous two versions of these assignments. This is also a very general phenonemon: with this meaning of assignment, any ordering of any set of assignments gives the same result.

So let's rewrite our sum program with these ideas in mind. We will use = for our symmetric assignment statement. (From now on, all our programs will be syntactically correct Prolog, and XSB, programs, so you can type them into XSB and try them out. [sidebar] to explain how to create files, consult them, and run defined predicates).

```
sum(List,Sum) :-
    List = []
     ->   Sum = 0
     ;    List = [Head|Tail],
            sum(Tail,TailSum),
            Sum is TailSum + Head.
```

I've changed the syntax for if-then-else to Prolog's syntax, using `->` and `;`. Here we've said that `Sum = 0`; using the properties of symmetric assignment, we could just as well have said that `0 = Sum`. Consider the symmetric assignment: `List = [Head|Tail]`. The structure on the right is how one constructs a list from a head element and a tail list. (In Lisp it is known as `cons`.) So our symmetric assignment here is even more powerful. We know that the variable `List` has a list as its value. So this assignment assigns both variables `Head` and `Tail` so that they get the values of the first element of `List` and the tail of `List`, respectively. We can see that symmetric assignment is here extended to matching. We match the value in the variable `List`, which is a list, to the structure `[Head|Tail]`. `Head` and `Tail` are the only variables without values, so the symmetric assignment will fill them in with the appropriate values to make the `List` and the `[Head|Tail]` structure the same. This matching process, which we have been referring to as "symmetric assignment", is called *unification.*

Notice that we've used the same operation of unification, `List = []`, in the test of the if-then-else. Here we see a use of failure. Recall that we said that if a symmetric assignment cannot be made because the two sides have values that are different, then the assignment (or unification) fails. The if-then-else construct does the unification and if it succeeds, then it executes the then statement (which follows the `->`); if it fails, it executes the else statement (which follows the `;`.) So even the boolean test of an if-then-else can use our universal unification operation.

Notice, however, that we have not used unification for the last statement that adds Head to the partial sum. This is because here we don't want to match the two sides, since Prolog considers the right side to be a tree (with + at the root and with two leaves of TailSum and Head.) So here we must use an operation that explicitly asks for the tree on the right to be evaluated as a numeric expression. In Prolog that operation is named `is`.

As another example, consider the append procedure:

```
append(L1,L2,L3) :-
    L1 = []
     ->  L3 = L2
     ;   L1 = [X|L1t],
         append(L1t,L2,L3t),
         L3 = [X|L3t].
```

This is a procedure that takes two lists and concatenates them together, returning the resulting list in its third argument. This definition says that if the first list is empty, then the result of concatenating `L1` and `L2` is just `L2`. Otherwise, we let `X` be the head of the first list and `L1t` be its tail. Then we concatenate `L1t` and `L2`, using append recursively, to get `L3t`. Finally we add `X` to the beginning of `L3t` to construct the final result, `L3`.

Consider the following version of append:

```
append(L1,L2,L3) :-
    L1 = [X|L1t]
     ->  L3 = [X|L3t],
         append(L1t,L2,L3t)
     ;   L3 = L2
```

This one looks rather strange, but it also works. We've used the boolean test unification also to deconstruct the list. (This is probably a poor idea in real Prolog programming.) The other perhaps stranger (but less bad) difference is that we've moved the construction of the output list `L3` to before the recursive call to `append`. You might wonder how we can construct a list before we have its components. But with unification, that works just fine. The intuition is that if a variable can get only one value, it doesn't really matter when it gets it. So it is often the case that unifications can be moved earlier. What happens here is that the list cell is constructed before the call to `append`, and then the recursive call to `append` will fill in the tail of that cell with the appropriate value.

We've looked at assign-once variables and seen how they lead to symmetric assignment, which leads to unification. Next let's consider the other unusal property of Prolog programs, the fact that they can be nondeterministic.

2.1.2 Nondeterminism

C is a deterministic programming language (as are Java and most any other language you ar familiar with); at any point in the execution of a C program there is exactly one next step, which seems reasonable since the machines we execute these programs on are (essentially) deterministic. Prolog, however, is nondeterministic. There are points in the execution of a Prolog program when there are multiple legal next steps. The way this is specified in Prolog is to give multiple definitions of the same procedure. For example, we could write a procedure to find both square roots of a positive real number by:

```
a_sqrt(X,Y) :-
    X > 0,
    Y is sqrt(X).
a_sqrt(X,Y) :-
    X > 0,
    Y is -sqrt(X).
```

`a_sqrt` takes a number and returns its square root. Here we want it to return both square roots, one positive and one negative. We can do that by giving two definitions of a procedure `a_sqrt`. A C compiler would complain that the procedure is multiply defined, but Prolog accepts such multiple procedure definitions happily. The first definition checks that the input argument is greater than 0, and if so uses a Prolog primitive builtin to calculate the positive square root. The second definition does the same, but returns the negation of the positive square root. In Prolog terminology, each definition is called a "clause", so `a_sqrt` is defined by two clauses.

Prolog execution as the execution of multiple machines

The way to understand how Prolog handles multiple procedure definitions is first to think of how a deterministic procedural machine executes a procedural program. It maintains a state (usually a stack of activation records) and executes instructions which update that state, calling subprocedures, performing the indicated operations, and returning from subprocedures. To introduce nondeterminism into this picture, we consider what happens when a machine encounters a procedure that has multiple definitions. At this point it duplicates itself, creating a copy for each definition, and each copy continues by executing the definition it is assigned to. Recall that an execution may fail when it does a unification that discloses an inconsistency. When this happens, we can think of the machine as simply disappearing. So we can think of a Prolog execution as a set of executing deterministic machines; whenever any one of them encounters a procedure call of a multiply defined procedure, it forks into multiple machines; whenever a machine fails, it disappears out of existence. The answer to a Prolog program is the set of answers returned by all the individual machines that make it to the final instruction of the program, i.e., that return successfully from the initial procedure call.

So if we invoke the `a_sqrt` procedure defined above with the following procedure call statement:

```
:- a_sqrt(13,Y).
```

we will get two answers:

```
X = 3.6055;
X = -3.6055;
```

Let's revisit the append program we wrote above. Instead of using an if-then-else construct there, we can now use nondeterminism. Consider:

```
append(L1,L2,L3) :-
    L1 = [],
    L3 = L2.
append(L1,L2,L3) :-
    L1 = [X|L1t],
    append(L1t,L2,L3t),
    L3 = [X|L3t].
```

Notice that for whatever the list that is assigned to the variable `L1`, exactly one of the two procedure definitions will fail, and the other will succeed. The first will succeed only if `L1` is the empty list, and the second will succeed only if `L1` is a nonempty list. This program is essentially equivalent to the one above with the if-then-else.

Actually we can now improve this new append program. Consider how a normal procedural programming language passes parameters to procedures. One way is to do it by assignment: local variables are allocated for each formal parameter and the actual parameters are assigned to local variables on invocation. So assignment is used for passing parameters. Prolog can pass parameters this way as well, but instead of using assignment, it uses its symmetric assignment operation, unification (or matching.) So rather than doing a unification in the body of a procedure definition, we can simply put the values to be unified in the place of the formal parameters. So, for example, in the first procedure definition for `append`, rather than assigning the actual parameter to a local variable `L1` and then checking it against the empty list, we can directly check the first argument against the empty list as follows:

```
append([],L2,L3) :-
    L3 = L2.
```

This looks very odd for a conventional procedural language, having a constant in the place of a formal parameter, but with Prolog's symmetric assignment, it works fine. It simply unifies the empty list with the first actual parameter at the time of invocation.

As a matter of fact, whenever we have an explicit unification of a variable with another term, we can replace all occurrences of the variable with the term and eliminate the explicit unification. So we can replace L3 by `L2` in the above clause and we get simply:

```
append([],L2,L2).
```

(When a definition has no body operations, we don't even write the :-.) This procedure definition has no operations in its body. In a traditional procedural language, it would be a no-op, but in Prolog it actually does some work through the unification of its arguments when it is invoked.

The same idea for eliminating explicit unifications can be used on the second clause for `append`, and we obtain the usual Prolog definition of `append`:

```
append([],L2,L2).
append([X|L1t],L2,[X|L3t]) :-
    append(L1t,L2,L3t).
```

2.1.3 Executing Programs in XSB

Now, we can load this definition into XSB and then call it in various ways to experiment with how it works. So we put this definition into a file called, say, appendfile.P. (The '.P' suffix indicates to XSB that this is a file containing source code.) We run XSB and then compile the file and load it into XSB by:

```
 % xsb
```

```
XSB Version 1.4.1 (94/11/21)
[sequential, single word, optimal mode]
| ?- [appendfile].
[Compiling ./appendfile]
[appendfile compiled, cpu time used: 0.901 seconds]
[appendfile loaded]

yes
| ?-
```

The XSB system top-level prompt is '`| ?- `', which is printed when XSB is waiting for the user to enter something. Here we've entered the filename in a list. This requests the system to compile and load the indicated file, which the system then does. The compilation creates an object file, in this case named appendfile.xwam. Then the XSB loader is called to load that file into XSB's space. (If the last-change date of the object file is more recent than the last-change date of the source file, then the compiler is not called, but the object file is loaded.) So now we have the append program in XSB memory and we can ask XSB to execute it. We do this by entering a call at the top-level prompt, as follows:

```
| ?- append([a,b,c],[d,e],X).

X = [a,b,c,d,e]
```

XSB calls the append procedure and executes it, passing the two lists in and when append returns, X has been assigned the answer, which XSB prints. It's possible that there is more than one answer (as would be the case with `a_sqrt` above), so XSB waits to let the user ask for another answer, if desired. To request another answer, the user enters a ';' (and then `<ret>`), to which XSB responds with the next answer, if any. Here the result is as follows:

```
| ?- append([a,b,c],[d,e],X).

X = [a,b,c,d,e];

no
| ?-
```

XSB has responded with 'no' and then returned with the top level prompt. This is because, in this case, there is just one answer so asking for another results in the response of 'no', meaning "no more answers'..

We could, of course, ask for different invocations of append, giving it different lists, but we can also give different forms of invocations. The unification of Prolog allows us to call some procedures in perhaps a surprising variety of different ways.

For example, we can enter the following query (i.e., procedure invocation) and will get the indicated result from XSB:

```
| ?- append([a,b,c],[d,e],[a,b,c,d,e]).

yes
| ?-
```

Here we've given the answer. XSB simply verifies that the answer is correct, and indicates it is by responding 'yes'. In this execution, unifications that set variable values in the previous execution simply verify that the variables already have the correct values. If the values don't check out, as they won't in this following case:

```
| ?- append([a,b,c],[d,e],[a,b,c,d]).

no
| ?-
```

XSB gives a response of 'no' indicating that the first two arguments do *not* concatenate to form the third.

Actually, Prolog can respond to even stranger invocations of our append procedure. Consider the following invocation:

```
| ?- append(X,Y,[a,b,c]).
```

Here we are asking for what two values will concatenate together to form the list [a,b,c]. The tokens beginning with capital letters, `X` and `Y`, are variables, and we are asking the system to fill them in with correct values. (A variable starts with an upper-case letter, and an atom starts with a lower-case letter. We've been using this convention all along, and it is important to know.)

Prolog can answer this query reasonably. There are four possible pairs of lists that do concatenate together to produce `[a,b,c]`, and Prolog will produce them:

```
| ?- append(X,Y,[a,b,c]).

X = []
Y = [a,b,c];

X = [a]
Y = [b,c];
```

```
     X = [a,b]
     Y = [c];

     X = [a,b,c]
     Y = [];

     no
     | ?-
```

Here XSB produced the first answer and then waited for my response. I responded with a `;`, and it responded by producing the next answer. We continued until all the answers were produced. Since Prolog is nondeterministic, queries that have multiple correct answers are reasonable to ask. In this case Prolog answers the query correctly and reasonably.

Let's consider another simple (and well known) Prolog program known as `member`. `Member` is a binary predicate, i.e., it is a procedure with two arguments. It is given an element and a list, and it checks to see whether the element is a member of the list:

```
     member(X,[X|L]).
     member(X,[Y|L]) :-
         member(X,L).
```

The first clause says that `X` is a member of a list whose head is `X`, an unexceptional statement. The second clause `X` is a member of a list if `X` is a member of the tail of the list.

Example executions of `member` are:

```
| ?- member(2,[1,2,3]).

yes
| ?- member(2,[1,3,4]).

no
| ?- member(X,[1,2,3]).

X = 1;

X = 2;

X = 3;

no
| ?-
```

Notice that we can use member to generate all the elements of a list.

(Aside: If you tried to compile this member program exactly as it is written here, you noticed that the XSB compiler issued some warning messages. The first message says that the variable L in the first clause appears only once in that clause. Such a variable is called an *anonymous variable.* An anonymous variable is just a placeholder, since the value that it might get is never used anywhere else, because the variable appears nowhere else. In such cases you are encouraged to use a slightly different notation: instead of starting anonymous variables with upper-case letters, start them with underscores (_), or simply use an underscore alone. Each occurrence of the underscore symbol is a distinct (unnamed) variable. The compiler will not complain if you begin the name of an anonymous variable with an underscore. I strongly suggest following this convention; it will save untold grief that you'll otherwise suffer when you mistype variable names. So an equivalent, but syntactically improved, version of member is:

```
    member(X,[X|_L]).
    member(X,[_Y|L]) :-
        member(X,L).
```

End of aside.)

As a final example of a simple Prolog list-processing predicate, consider the procedure reverse, which is given a list and returns a list that contains the elements of the input list, but in the reverse order.

```
    reverse([],[]).
    reverse([X|L],R) :-
        reverse(L,RL),
        append(RL,[X],R).
```

The first clause says that if the input list is empty, then the resulting list is also the empty list. The second clause says that if the input list has head X and tail L, then first reverse the tail L of the input list obtaining RL, and then add X to the end of RL. The predicate append is used to add this element to the end of the reversed sublist. Notice that we must use [X] as the second argument of append, not just X, because append expects a list there, not an element.

An example of executing reverse is:

```
| ?- reverse([1,2,3],R).

R = [3,2,1];

no
| ?-
```

exactly as expected. You might reasonably think that we should also be able to ask the following query:

```
| ?- reverse(X,[1,2,3]).

X = [3,2,1];
```

And it looks as though everything works fine. However, what has really happened is that after the system produced the expected answer, I asked it for another answer. It should have simply said "no", but instead it went into an infinite loop and didn't respond at all. To understand why Prolog behaves like this, we have to understand more clearly exactly how it goes about evaluating queries.

[add an example to introduce ; and disjunction.]

2.1.4 The Scheduling of Machine Execution in Prolog

Recall that above we described a conceptual model of how Prolog executes nondeterministic programs by talking of a growing and shrinking set of deterministic procedural machines. In order to completely understand how Prolog executes these nondeterministic programs, we must look more carefully at how this growing and shrinking set of machines is actually executed. Clearly Prolog is not executed by actually have a set of hardware machines that grows and shrinks. (While it would be nice, the physics of such machines has not yet been worked out.) Instead these multiple machines must somehow be simulated, or emulated, by a single hardware machine having just one processor.

The Prolog engine keeps track of the states of these multiple machines and uses them to simulate the machine executions. Let's first consider the way Prolog keeps track of the state of a single machine. We can model execution of a single machine by "expanding" procedures. When a procedure is called, the actual parameters are matched with the formal parameters. All variables that get values in the matching process, occurrences in the body of the procedure being called and variables in the calling procedure, are replaced by those values. And the procedure call is replaced by the body of the procedure. The explanation is complex but the idea is simple: procedure calls are replaced by procedure bodies, with the variables appropriately set. For example.....

Now that we have seen how a single machine executes, the real question is in what order does it emulate the multiple machines. Clearly when a query first starts, there is just one machine to execute it. What happens when that machine encounters a procedure defined by multiple clauses? At that point there are several machines to be executed. In what order does Prolog execute them? I.e., how are they scheduled?

===================

The formal counterpart of Prolog execution is the SLD search tree. Each node in the search tree corresponds to a state of one of the machines. Each path through the search tree corresponds to the execution sequence of one of the machines. A branching node in the tree corresponds to a choice point, when a machine is duplicated to create instances of itself that will explore the various alternatives.

Let's look at a simple example to see how this works for the append program when it is called with a final list and it is asked to find all pairs of lists that concatenate to form the given list.

Consider the query:

```
| ?- append(X,Y,[a,b]).
```

First we have to determine a way to represent the states of the individual procedural machines. The state of execution of a procedural program is normally kept by a current instruction pointer and a stack of activation records associated with active procedures, which indicate values of local variables and also where to return when the procedure exits. For our Prolog programs we will use a very abstract representation that will enable us to understand the machine's operations without getting lost in encoding details. We will keep an instance of the variables of the original query and the sequence of procedure calls that remain to be done to complete the machine's computation. So the initial state of the machine is:

```
answer(X,Y) :- append(X,Y,[a,b]).
```

We use the `:-` to separate the query variables, which are grouped by `answer`, from the sequence of procedure calls that remain to be done. Initially the only thing to be done is the initial procedure call. We move a machine to its next state by taking the first procedure call after the `:-` in the machine state, matching it against the head of the chosen clause that defines the procedure, and replacing that call by the body of the clause, with the variables appropriately updated. Thus one step of computation replaces a procedure call by the sequence of procedure calls that make up the body of its definition, with the variables appropriately updated according to the parameter passing method.

So let's now consider the execution of the above query. The execution will be a tree of machine states, with the above machine state at the root. This tree is shown in Figure 2.1.

Prolog generates this tree in a top-down left-to-right order. The left-to-right order of children of a node corresponds to the order in which clauses for the called procedure appear in the text of the program. This is implemented in the Prolog engine by maintaining a stack of alternatives; whenever new alternative computation states are generated, it pushes them onto the stack, and whenever it needs another alternative, it takes the top one from the stack. So the Prolog engine begins by taking the first (and only) state off the stack and matching the first procedure, `append(X,Y,[a,b])`, with the heads of the appropriate procedure definitions (i.e., clauses). For each one that matches, it pushes a new state on the stack, replacing the procedure call by the body of the procedure, updating the variables accordingly. Now this first procedure call matches both clauses, so we generate two new states as children of the root state:

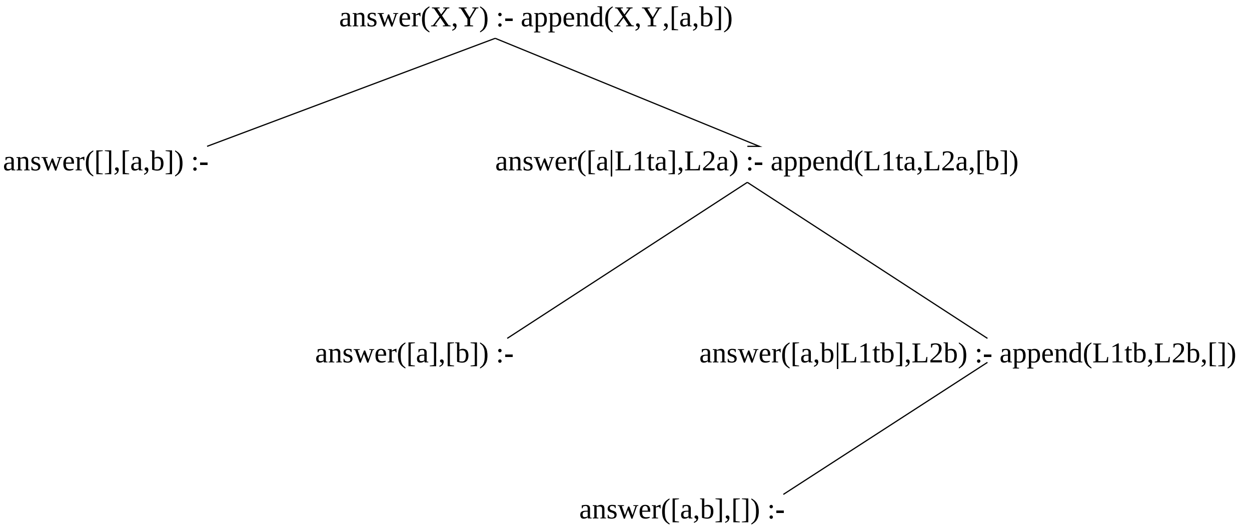

answer(X,Y) :- append(X,Y,[a,b])
answer([],[a,b]) :-
answer([a|L1ta],L2a) :- append(L1ta,L2a,[b])
answer([a],[b]) :-
answer([a,b|L1tb],L2b) :- append(L1tb,L2b,[])
answer([a,b],[]) :-

Figure 2.1: Computation tree for the query: append(X,Y,[a,b]).

```
answer([],[a,b]) :- .
answer([a|L1ta],L2a) :- append(L1ta,L2a,[b]).
```

The second state comes from the second clause, and the procedure call is replaced by the single procedure call in the body of the second clause defining `append`. The first state comes from the first clause, which has no procedure call in its body, so this state has no procedure call to do, and thus is a successful *final state* of the Prolog program. The arguments of the answer template contain the values of the variables of the original query, and so constitute the final answer of this machine. Here they are `[]` and `[a,b]`, which do indeed concatenate to generate `[a,b]`, as expected. Prolog will print this answer out, remove this state from the stack and continue expanding the next state on the top of the stack, here the second child of the root node.

Now consider that state:

```
answer([a|L1ta],L2a) :- append(L1ta,L2a,[b]).
```

It was generated by matching the original procedure call with the second clause for `append`. In a procedural language, whenever a procedure is called, the procedure gets a *new* set of local variables, and in Prolog it is the same. I've indicated that here by giving the variables in the clause new names, by adding 'a' to the end of their original names. Each time I take a clause, I'll have to rename the variables to new ones, so we don't have unintended collisions, and I'll do this by adding a new letter suffix.

Again Prolog expands this state by replacing the first procedure call by the bodies of matching clauses. Again both clauses for `append` match this one so we get two new states on the stack:

```
answer([a],[b]) :- .
answer([a,b|L1tb],L2b) :- append(L1tb,L2b,[]).
```

The top one is again an answer, since it has no procedures left to call, and its values for the result variables are: `[a]` and `[b]`, which again do concatenate to form the desired `[a,b]`.

After the answer is printed and the state removed from the stack, the next state:

```
answer([a,b|L1tb],L2b) :- append(L1tb,L2b,[]).
```

is expanded. Now this procedure call matches only the first of the `append` clauses; the second fails to match because the third argument in the call is `[]` but in the procedure head is `[X|L3t]`. So the new stack is just:

```
answer([a,b],[]) :- .
```

The top state is an answer, which gets popped and displayed, and then the stack is empty, indicating that Prolog has completely finished traversing the computation tree and thus with evaluating the query. It has simulated all the deterministic procedural machines to completion.

Stepping back a bit and thinking about the computation tree, we can quite easily describe the tree by giving an operation that can be applied to a subtree to extend it. Then we can define the tree as the result of applying this operation to an initial (trivial) tree, and all resulting trees until no operation is applicable. This operation is called PROGRAM CLAUSE RESOLUTION.

Definition 2.1.1. (Program Clause Resolution) Given a tree with a node labeled $A : -A_1, A_2, \ldots, A_n$, and a rule in the program of the form $H : -B_1, B_2, \ldots, B_k$, and given that H and B_1 match with matching variable assignment θ, then add a new node as a child of this one and label it with $(A : -B_1, B_2, \ldots, B_k, A_2, \ldots, A_n)\theta$, if it does not already have a child so labeled. Note that the matching variable assignment is applied to all the goals in the new label.

Notice that the entire tree of Figure 2.1 is developed by applying this rule to the trivial tree consisting of the single node `answer(X,Y) :- append(X,Y,[a,b])`. So we can think of Prolog as applying this PROGRAM CLAUSE RESOLUTION rule over and over again (in a top-down backtracking manner) to the initial query to trace out the computation tree.

The example we have used here has a relatively simple execution and Prolog executions can get considerably more complex, but all the basics have been illustrated.

2.2 Grammars in Prolog

Next we turn to more complex examples of Prolog programming. Prolog was originally invented as a programming language in which to write natural language applications, and thus Prolog is a very elegant language for expressing grammars. Prolog even has a builtin syntax especially created for writing grammars. It is often said that with Prolog one gets a builtin parser for free. In this section we will see why this claim is made (and sometimes contested).

Consider the following simple context-free grammar for a small fragment of English.

$S \longrightarrow NP\ \ VP$
$NP \longrightarrow Det\ N$
$VP \longrightarrow TV\ \ NP$
$VP \longrightarrow V$
$Det \longrightarrow the$
$Det \longrightarrow a$
$Det \longrightarrow every$
$N \longrightarrow man$
$N \longrightarrow woman$
$N \longrightarrow park$
$TV \longrightarrow loves$
$TV \longrightarrow likes$
$V \longrightarrow walks$

In this grammar we can derive such simple sentences as:

a man loves the woman
every woman walks
a woman likes the park

We can write a simple Prolog program to recognize this language, by writing a recursive descent parser. We first must decide how to handle the input string. We will use a list in Prolog. For each nonterminal we will construct a Prolog procedure to recognize strings generated by that nonterminal. Each procedure will have two arguments. The first will be an input parameter consisting of the list representing the input string. The second will be an output argument, and will be set by the procedure to the remainder of the input string after an initial segment matched by the nonterminal has been removed. An example will help clarify how this works. The procedure for `np` would, for example, take as its first argument a list `[a,woman,loves,a,man]` and would return in its second argument the list `[loves,a,man]`. The segment removed by the procedure, `[a,woman]`, is an NP. The Prolog program is:

```
s(S0,S) :- np(S0,S1), vp(S1,S).
np(S0,S) :- det(S0,S1), n(S1,S).
vp(S0,S) :- tv(S0,S1), np(S1,S).
vp(S0,S) :- v(S0,S).
det(S0,S) :- S0=[the|S].
det(S0,S) :- S0=[a|S].
```

```
    det(S0,S) :- S0=[every|S].
    n(S0,S) :- S0=[man|S].
    n(S0,S) :- S0=[woman|S].
    n(S0,S) :- S0=[park|S].
    tv(S0,S) :- S0=[loves|S].
    tv(S0,S) :- S0=[likes|S].
    v(S0,S) :- S0=[walks|S].
```

The first clause defines procedure `s`, for recognizing sentences. An input list `S0` is passed into procedure `s`, and it must set `S` to be the remainder of the list `S0` after a sentence has been removed from the beginning. To do that, it uses two subprocedures: it first calls `np` to remove an NP, and then it calls `vp` to remove a VP from that. Since the grammar says that an S is an NP followed by a VP, this will do the right thing. The other rules are exactly analogous.

Having put this program in a file called `grammar.P`, we can load and execute it on our example sentences as follows:

```
% xsb
XSB Version 1.4.1 (94/11/21)
[sequential, single word, optimal mode]
| ?- [grammar].
[Compiling ./grammar]
[grammar compiled, cpu time used: 1.14 seconds]
[grammar loaded]

yes
| ?- s([a,man,loves,the,woman],[]).

yes
| ?- s([every,woman,walks],[]).

yes
| ?- s([a,woman,likes,the,park],[]).

yes
| ?- s([a,woman,likes,the,prak],[]).

no
| ?-
```

When the string is in the language of the grammar, the program will execute successfully through to the end, and the system produces ‘yes’. If the string is not in the language, as in the final example where ‘park’ was misspelled, the system answers ‘no’. We called the `s` procedure with the input string as first argument and we gave it the empty list as second argument, because we want it to match the entire input string, with nothing left over after seeing an `s`.

The grammar above is called a *Definite Clause Grammar* (DCG) and Prolog supports a special rule syntax for writing DCGs. The syntax is simpler, much closer to the syntax one uses in writing context-free grammar rules. When using the DCG syntax, the programmer doesn't have to write all the string variables threaded through the nonterminal procedure calls; the compiler will do it. Here following is the same Prolog program as above, but witten as a DCG:

```
    s --> np, vp.
    np --> det, n.
    vp --> tv, np.
    vp --> v.
    det --> [the].
    det --> [a].
    det --> [every].
    n --> [man].
    n --> [woman].
    n --> [park].
    tv --> [loves].
    tv --> [likes].
    v --> [walks].
```

Notice that these "procedure definitions" use the symbol `-->` instead of `:-` to separate the procedure head from the procedure body. The Prolog compiler converts such rules to (almost) exactly the program above, by adding the extra arguments to the predicate symbols and treating the lists as terminals. The "almost" is because it really translates, for example, a single word list `[loves]` above to the procedure call `'C'(S0,loves,S)`, and includes the definition of this new predicate as:

```
    'C'([Word|String],Word,String).
```

This gives exactly the same effect as the Prolog program for the grammar given above.

Consider another example grammar, this one for simple arithmetic expressions over integers with operators `+` and `*`:

```
expr --> term, addterm.
addterm --> [].
addterm --> [+], expr.
term --> factor, multfactor.
multfactor --> [].
multfactor --> [*], term.
factor --> [I], {integer(I)}.
factor --> ['('], expr, [')'].
```

There are several things to note about this DCG. Notice that the list entries, representing terminals, need not appear alone on right-hand-sides of DCG rules, but may accompany nonterminals. Also notice the first rule for `factor`; it has a variable (`I`) in a list, which will cause it to be matched with, and thus set to, the next input symbol. The following procedure call is enclosed in braces. This means that it matches no input symbols and so its translation to Prolog does NOT result in the string variables being added. It remains just a call to the Prolog procedure with one argument: `integer(I)`. The `integer` procedure is a Prolog builtin which tests whether its argument is an integer. Note also that we must quote the parentheses in the final rule. Otherwise, Prolog's reader would not be able to parse them correctly as atoms.

Consider some example executions of this grammar:

```
% xsb
XSB Version 1.4.1 (94/11/21)
[sequential, single word, optimal mode]
| ?- [grammar].
[Compiling ./grammar]
[grammar compiled, cpu time used: 1.309 seconds]
[grammar loaded]

yes
| ?- expr([4,*,5,+,1],[]).

yes
| ?- expr([1,+,3,*,'(',2,+,4,')'],[]).

yes
| ?- expr([4,5,*],[]).

no
| ?-
```

This grammar is not the most obvious one to write down for this expression language. It is specially constructed to avoid being left recursive. We mentioned above that we were writing a recursive descent parser for the grammar, and that is what one gets for a DCG from Prolog's execution strategy. Prolog execution of the underlying deterministic machines and its use of a stack to schedule them naturally yields a recursive descent parser. And it is well known that a recursive descent parser cannot handle left-recursive grammars; it will go into an infinite loop on them. So in Prolog we must avoid left-recursive grammars.

Also a recursive descent parser can be quite inefficient on some grammars, because it may re-parse the same substring many times. In fact, there are grammars for which recursive descent parsers take time exponential in the length of the input string. When using DCGs in Prolog, the programmer must be aware of these limitations and program around them. It is for this reason that some people respond to the claim that "You get a parser for free with Prolog" with "Maybe, but it's not a parser I want to use."

(Another example for adding arguments and using word/3 instead of strings?).

2.3 Prolog as a Database Query Langauge

Prolog is an elegant language for database queries. In fact if one constrains Prolog programs to use only atoms, integers and reals (no lists or complex terms) and disallows recursive definitions, one gets a database language that is equivalent to a powerful subset of SQL. In this section we will see how this is so.

A relation in relational database theory is a set of tuples. A common example of a database relation is the `employee` relation, which contains a tuple for each employee in a company. Each tuple might contain fields for: employee number, last name, first name, street address, city, state, zipcode, department number, hire date, and salary. It could, of course, contain many more. We can represent a set of tuples in Prolog as a highly nondeterministic procedure, the procedure returning every one of the tuples in the relation.

```
employee(193,'Jones','John','173 Elm St.','Hoboken','NJ',
                                          12345,1,'25 Jun 93',25500).
employee(181,'Doe','Betty','11 Spring St.','Paterson','NJ',
                                          12354,3,'12 May 91',28500).
employee(198,'Smith','Al','2 Ace Ave.','Paterson','NJ',
                                          12354,3,'12 Sep 93',27000).

(and more...)
```

And we might have a department relation which contains for each department, a tuple that gives its number, name, and employee number of its manager.

```
department(1,'Grocery',181).
department(3,'Deli',193).
department(5,'Produce',199).
...
```

Given these basic relations (also called extensional relations), we can define other relations using Prolog procedure definitions to give us answers to questions we might have about the data. For example, we can define a new relation containing the names of all employees making more than $28,000:

```
well_paid_emp(First,Last) :-
    employee(_Num,Last,First,_Addr,_City,_St,_Zip,_Dept,_Date,Sal),
    Sal > 28000.
```

As another example, we could ask for the name of the manager of the Deli department:

```
deli_manager(First,Last) :-
    department(_Deptno,'Deli',MgrID),
    employee(MgrID,Last,First,_Addr,_City,_St,_Zip,_Dept,_Date,_Sal).
```

Here we first call the department relation to find the employee number of the manager of the Deli department; then we call the employee relation to find the first and last names of the employee with that number.

(Should we introduce negation here? Without recursion, it is pretty easy. Would have to talk about safety.)

2.4 Deductive Databases

By staying with the simple data types, but adding recursion to this database language, one gets a language called (positive?) Datalog, which is the language underlying deductive databases. Deductive databases are an extension of relational databases which support more complex data modeling. In this section we will see how simple examples of deductive databases can be represented in Prolog, and we will see more of the limitations of Prolog.

A standard example in Prolog is a geneology database. An extensional relation stores the `parent` relation: `parent(X,Y)` succeeds with `X` and `Y` if `X` has parent `Y`. (Maybe do an example consisting of some English monarchs?) Given this `parent` relation, we can define the ancestor relation as follows:

```
    ancestor(X,Y) :- parent(X,Y).
    ancestor(X,Y) :- parent(X,Z), ancestor(Z,Y).
```

This says that X has ancestor Y if X has parent Y; and X has ancestor Y if there is a Z such that X has parent Z and Z has ancestor Y. Given a definition for the parent relation as follows:

```
parent(elizabeth_II, charles_??).
etc.
```

we can query about ancestors, such as:

```
:- ancestor(elizabeth_II,X).
```

and find all of Queen Elizabeth's ancestors.

This works very nicely in Prolog, since the parent graph is (essentially) a tree. However, if we try the same definition of transitive closure for a graph that is not acyclic like a tree, we can be in trouble. Say we have a relation `owes(X,Y)` which indicates that `X` owes money to `Y`, and we want to define a predicate `avoids(X,Y)`, meaning that `X` tries to avoid running into `Y`. The definition is that people avoid someone to whom they owe money, and they avoid anyone that someone to whom they owe money avoids:

```
avoids(X,Y) :- owes(X,Y).
avoids(X,Y) :- owes(X,Z), avoids(Z,Y).
```

This definition has the same form as the ancestor definition. The problem here is that the `owes` relation may be cyclic. It is possible for Andy to owe money to Bill, Bill to owe money to Carl and Carl to owe money to Bill:

```
owes(andy,bill).
owes(bill,carl).
owes(carl,bill).
```

and if we ask who Andy avoids:

```
| ?- avoids(andy,X).
```

we get:

```
| ?- avoids(andy,X).

X = bill;

X = carl;

X = bill;

X = carl;

X = bill;

X = carl;

X = bill;
....
```

an infinite loop.

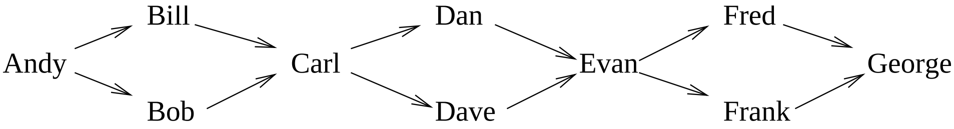


Figure 2.2: Graph on which Prolog is exponential

If we would like to use Prolog as an engine for deductive databases, this shows up a serious problem: that a user can write a simple specification (using only atoms and variablse) and yet Prolog won't give an answer, but will wander off to (or toward) infinity. One couldn't afford to give such a system to a naive database user. There it is important that any query come back with some answer. Think of an SQL system that sometimes went into an infinite loop. It wouldn't be much used, and certainly not by naive users.

This problem of infinite looping is a well-known problem in Prolog and Prolog programmers learn to program around it. The usual fix is to add an extra argument to the `avoids/2` predicate that keeps the list of people encountered in the process of finding the avoidees. Then if one of them is encountered again, the search is made to fail at that point, since it is known that all avoidees from that one have already been found. I.e.,

```
avoids(X,Y,L) :- owes(X,Y), \+ member(Y,L).
avoids(X,Y,L) :- owes(X,Z), \+ member(Z,L), avoids(Z,Y,[Z|L]).
```

Here we've used the Prolog predicate `member/2`, and the Prolog builtin, `\+`, which implements *not*. `\+ member(Y,L)` succeeds just in case `member(Y,L)` fails, and fails if it succeeds. Now with this program and the corresponding query, we get:

```
| ?- avoids(andy,X,[]).

X = bill;

X = carl;

no
| ?-
```

This fix works to avoid the infinite loop, but it sometimes has some undesirable properties. There are graphs for which the computation will be exponential in the number of arcs. Consider the case in which Andy owes money to Bill and Bob, and Bill and Bob owe money to Carl; Carl owes money to Dan and Dave, and Dan and Dave owe money to Evan; Evan owes money to Fred and Frank, and Fred and Frank owe money to George; and so on. The graph of the owes relation can be pictured as in Figure 2.2. On this graph, this query will be exponential. This graph is acyclic, so the original, simpler Prolog program for transitive closure would terminate. But it (and the `member`-enhanced version, too) would recompute the same answers again and again. It would in effect turn this (essentially) linear list into a tree.

2.5 Summary

So we have seen that Prolog is an interesting language that combines logic and computation. Some programs are very simple and elegant and execute very reasonably, such as the `append` program or the `member` program. Other programs, such as those derived from context-free grammars, are very simple and elegant, but sometimes have undesirable computational properties. For example, some context-free grammars have recognition times that are exponential in the length of the input string. This is clearly undesirable, since we know that there are recognition algorithms that work for all context-free grammars in at worst cubic time. And even worse, for some grammars, such as left-recursive grammars, Prolog will go into an infinite loop, not returning an answer at all. We also saw the same problem with the Datalog language. Perfectly good specifications might end up with (very) bad computational behavior. Before we tackle this problem directly, we first develop in the next chapter the theory underlying Prolog and its semantics: the theory of first-order logic.

2.6 Exercises

1. Write and TEST Prolog definitions for the following predicates. You will be graded on the simplicity and elegance of your solutions, AND how well you test them. You can download XSB from xsb.sourceforge.net for unix (for which you must configure and compile, according to instructions) or for Windows (as a zip file that unzips to a set of directories with an executable in XSB/config/i686-pc-cygwin/bin/xsb.exe). Contact me if you need help.

   (a) member(Element,List) iff Element is a member of the list List.
   Give a query that uses it to produce all members of a list that are greater than some integer.

   (b) last(Element,List) iff Element is the last element of list List.

   (c) consecutive(X,Y,L) iff X and Y are consecutive elements (in that order) of list L.

   (d) delete_first(X,Li,Lo) iff list Lo is list Li with the first occurrence of X removed.

   (e) delete_all(X,Li,Lo) iff list Lo is list Li with all occurrences of X removed.

   (f) append(L1,L2,L3) iff list L3 is the result of concatenating list L1 and L2. ***Use it to split a list into all pairs of sublists that concatenate to produce the given list.

   (g) sublist(Sl,L) iff list Sl is a (contiguous) sublist of list L.

   (h) i. intersect(L1,L2,I) iff I is the list that is the intersection of the lists L1 and L2. (Assume L1 and L2 do not have duplicates within themselves but may be in any order. Produce in I one list that contains all the elements in any order that are in both L1 and L2.)

   ii. intersect(L1,L2,I) iff I is the list that is the intersection of the lists L1 and L2. (Assume L1 and L2 are in increasing order (using @¡). The list representing the intersection should also be in order.)

   (i) reverse(L,R) iff list R is the reverse of the list L. What is the complexity of your solution for a list of length n? Explain.

(j) palindrome(L) iff the list L is a palindrome, i.e. reads the same backwards and forwards.

(k) permutation(L1,L2) iff list L2 is a permutation of list L1. (It should be able to generate all permutations.)

(l) split(E,L,P1,P2) iff lists P1 and P2 are a partition of list L of integers, such that all elements of P1 are less than or equal E and all elements of P2 are greater than E.

(m) quick_sort(L,S) iff S is the sorted version of the list L of integers. The idea of quicksort is 1) to choose an element from the list; 2) split the list into two lists putting all elements smaller than (or equal to) the chosen element into the first list and the others into the second (Hint: Use split/4 for this); 3) recursively sort the two sublists; and 4) concatenate the two sorted sublists (putting the chosen element back in the middle).

2. Consider the very simple "English" grammar we discussed in the chapter:

```
s --> np,vp.
np --> det,n.
vp --> tv,np.
vp --> v.
det --> [the].
det --> [a].
det --> [every].
n --> [man].
n --> [woman].
n --> [park].
tv --> [loves].
tv --> [likes].
v --> [walks].
```

Extend this grammar to include some prepositional phrases, as "in the park". Also include several proper names, such as john, mary, and rohit. You should add them to the grammar in a natural way so that they follow the (simple syntactic) rules of English grammar. For example, prepositional phrases can modify nouns and they can modify verb phrases.

Example sentences that you should be able to process include:

(a) a man walks to the park

(b) the man in the park likes mary

(c) the man saw a woman in the park with a telescope

(d) the man in the park with a telescope walks

(e) the man walks in the park with a telescope

(f) many, many more, some of which may not make much sense but are "syntactically" OK.

(Add the necessary words to their appropriate categories.)

Test your grammar on these examples and others of your own choosing.

3. Extend your grammar of the previous problem to construct a parse tree for each sentence it recognizes.

   Run your parser to construct parse trees for each of the example sentences of problem 1, and for several others of your own choosing. Does your parser produce multiple different parses for some of your examples? Should it? Discuss.

2.7 Exercise Discussion

The usual definition of reverse, using append (and called naive reverse) does not work for all calling modes. It only uses unification so why doesn't it work in all calling modes as append/3 does? Consider the following definition:

```
reverse(L,R) :- same_length_list(L,R), reverse1(L,R).

same_length_list([],[]).
same_length_list([_|L1],[_|L2]) :- same_length_list(L1,L2).

reverse1([],[]).
reverse1([X|L],R) :- reverse1(L,Rhs), append(Rhs,[X],R).
```

reverse1/2 is the standard definition of naive reverse. This reverse works in all directions. Even:

```
| ?- reverse(L,R).
```

gives correct answers. We could also use the linear version of reverse1, with an accumulator:

```
reverse1([],R,R).
reverse1([X|L],S,R) :- reverse1(L,[X|S],R).
```

and calling reverse1(L,[],R) in the body of reverse. (Note that same_length_list is done only once and is linear, so this definition of reverse has the complexity of the reverse1 definition.)

To come up with this definition, I was thinking of the definition of naive reverse and why we couldn't interchange the two calls in the body of the recursive rule and put the append first. And the problem is that with the direct call (with first argument bound and second a variable) we get a call of the form

```
   append(-,[+],-)
```

(where - indicate variables and + indicates a bound value.) But there are infinitely many pairs of lists that stand in this relationship, so Prolog is necessarily in an infinite loop. So why do we need to look at only finitely many of these lists? Because we know the length of the resulting list. So by setting the resulting list to be of the right length (the same as the input list), we can make that call have only finitely many (in fact, only 1) solutions. So that's how I came to this definition. Which also points out that for the naive definition, we might as well interchange the body calls and actually do the append first, which will make the definition tail recursive and save space. Of course its time complexity is still quadratic

Chapter 3

Introduction to First-Order Logic

This chapter introduces the basics of logic needed to understand the foundations of Prolog and logic programming. We start with Propositional Logic and then move to first-order logic.

Figure 11 shows in cartoon form what logic tries to do. On the left is a real-world situation of a person looking at a table with blocks on it and saying that “a red block is on the table”. There ia another person, our erstwhile logician, looking at this situation and trying to understand what’s going on with the use of language. The logician separates this mixed-up real-world situation into several parts to study them. First she pulls herself out of the situation so she can study it “from outside”. Second she separates the statement into one component and the part of the real world that the statement is about into another component. Her idea is that a statement is true or false when looked at in the context of the real-world component. For example, when we look at the table with the blocks, we will see that either the statement “a red block is on the table” is a true statement about that situation, i.e., there is indeed a block on the table whose color is red; or it is a false statement about that situation, i.e., looking at the blocks and table, we see that none of the blocks are red. So the meaning of a statement with respect to a real-world situation is either *true* or

```
 Real World Situation                 Statements
----------------------              -------------------
|Pic of person       |--------------|''a red block is |
|looking at a table  |              | on the table''  |
|with blocks on it   |         /    -------------------
|with balloon saying |   person
|``a red block is on |    with
|the table'' and     |     L    \   What statements are about
|another person (with|              ----------------
|L on chest) looking |--------------|  Table with  |
|at them             |              |    blocks    |
----------------------              ---------------
```

false, depending on whether the statement makes a true or false assection about that world. This is called a correspondence theory of meaning, since statements "mean" true or false depending on how they "correspond" with a world situation.

As an aside, note that there are other possible theories of language meaning. As of this writing, there is a US presidential nominating process going on. It is rather clear to me that the language used in that process does not depend on a correspondence theory of meaning. A suprisingly large number of utterances and clearly not true with respect to the world situation, so claiming that politicians make statements that are true is certainly not an adequate account of language in that context. Another theory of language, going by the name of "speech-acts", models statements as efforts by one person to cause another person to construct in his head certain beliefs. I suspect political discourse is much better modeled by such a theory of language. Of course there is the question of what a belief is, and the formaliztion of that may require some correspondence theory of a belief and the world. End of aside.

Returning to our logician (whose role we now play), the problem is how to formalize the language and the world situation in such a way that we can precisely say when a sentence is true (or false) in a world situation. To do this, we need to precisely formalize the language component, which is called the syntax of the system, and the real world component, which is call the semantics of the system. And then we have to define a function that takes a sentence in the syntax and a world situation in the semantics, and produces either true or false. This is the meaning (or semantic) function of the framework.

After we do this, another question arises: Can we deduce the truth values of some sentences given knowledge of truth values of other sentences? This is a question of deduction. Knowing that some sentences are true an a given situation, can we deduce that some other sentence is true in that same sitution? And can we do this without knowing what the situation is? A very simple (yet powerful) example of this is the following: Say we know that we are in a situation in which the sentence "it is raining" is true; furthermore we know that in all situations the sentence "If it is raining, then the grass is wet" is true. Then we should be able to conclude that we are in a situation in which the sentence "the grass is wet" is true. And we know this entirely from the form of the syntax of the sentences, and their meanings with respect to their truth values in situations. This is an example of deduction, and we want the logical system of meaning we construct to support such reasoning.

To summarize, a logic (or logical theory of meaning) has three components: 1) a syntax that defines the well-formed sentences of the language of the logic, 2) a model-theoretic semantics that defines what models of real-world situations are appropriate for the logic language and how those models make the well-formed sentences true or false, and 3) a theory of deduction that gives rules that apply to sentences that preserve some aspects of their truth. We will look first at propositional logic and its components, and then at first-order logic.

3.1 Propositional Logic

Propositional logic takes simple declarative sentences as basic and combines them with sentential operators: “and”, “or”, and “not” (and perhaps others). For example, we might have basic sentences like “John is going to the store”, “It is raining”, and “It is daytime”. Propositional logic has basic connective words: “and”, “or”, and “not”. We can combine the basic sentences to make larger sentences using the basic connective words. For example, we could say: “John is going to the store and it is daytime”, or “It is not raining or it is daytime”.

3.1.1 Syntax

In propositional logic we are interested is the connective words, and not particularly in how meaning is given to the basic declarative sentences. So we will simply represent the basic sentences by letters: p,q,r,s,.... We construct more complex sentences using the connectives to get sentences like: not(p), p or q, p or (q and not(p)), etc.

So the syntax of propositional logic is very simple, and can be given by the following grammar:

```
Sent --> Conjsent
Sent --> Sent or Conjsent
Conjsent --> Primesent
Conjsent --> Conjsent and Primesent
Primesent --> Basicsent
Primesent --> not Primesent
Basicsent --> Prop_sym
Basicsent --> ( Sent )
```

3.1.2 Semantics

The meaning of a sentence in a world is a truth value: true or false. I.e., given a world, each sentence is either true or false in that world. So we will think of a world as assigning to each basic sentence a truth value, i.e., true or false. I.e., a world determines a truth assignment to the propositional symbols (which represent the basic sentences). For example for the basic sentences (propositions) given above, “It is raining” and “It is daytime”, one world might assign “true” to the first and “false” to the second. This would be the world in which it is a wet night. A given set of n propositional symbols can distinguish 2^n different worlds, one for each truth assignment to that set of propositional symbols.

Given the meanings of the propositional symbols, we want to give meanings to the compound sentences, those made up from basics propositional symbols using the propositional connectives. So let P be the set of proposition symbols, and $M : P \rightarrow \{true, false\}$ be a truth assignment (associated with some possible world; i.e., we think of M as representing a possible world.) We can extend M to the set of all sentences, i.e., the compound sentences involving the connectives, as follows:

```
M(and(S1,S2)) = true, if M(S1)=true and M(S2)=true,
              = false, otherwise

M(or(S1,S2)) = true, if M(S1)=true or M(S2)=true, or both
              = false, otherwise

M(not(S)) = true, if M(S)=false
          = false, otherwise.
```

Given a propositional sentence, we canconsider its truth value in each possible world, i.e., for each truth assignment to the propositional symbols in the sentence. For example, consider (not p or q).

```
p     | q     || not p or q
=======================
true  | true  || true
true  | false || false
false | true  || true
false | false || true
```

Here we have set out all four of the truth assignments (worlds) for the basic propositional symbols, p and q, one in each row. And then in the third column, we have entered the corresponding truth value of the compound sentence (not p or q) in that world. This is known as a truth table for (not p or q), and we can make a truth table for any (complex) sentence in the propositional logic. If the sentence has n different propositional variables, its truth table will have $n+1$ columns and 2^n rows.

A sentence whose truth table has a last column of all *true* is called a **tautology**. One whose last column is all *false* is called a **contradiction**. A sentence that is a contradiction is said to be **unsatisfiable** since no world can satisfy it, i.e., make it true. For example, (b or not b) is a tautology. And (p and not p) is a contradiction. Notice that if an arbitrary sentence Q is a tautology, then (not (Q)) is a contradiction.

Tautologies are of particular interest. Say we know that (not (A) or B) is a tautology, for some particular sentences A and B. Say further that we don't know everything about our current world, but we do know that it makes A true. Then we also know that our current world, whatever it is, makes B true. (Exercise, why?) We often write a formula of the form (not(A) or B) as ($A \rightarrow B$), read A implies B, or if A then B. We can extend the logic with this implication operator as a new connective.

3.1.3 Deduction

So to be able to reason about what is true in our current world, without know everything about the world, it is useful to know the tautologies. Now we can always determine if a sentence is a tautology by building its truth table. But a truth table is exponentially large. Is there a way to determine whether a sentence is a tautology without having to construct the entire truth table, just by looking at the sentence itself and maybe related sentences?

There are many different ways to tackle this problem. We will look at one particular approach. First we will consider sentences that have identical truth tables; such sentences are called **equivalent**, indicated by $\Longleftrightarrow$. There are a number of important equivalences; some useful ones are listed here:

1. $A \iff not\ not\ A$
2. $A\ or(B\ and\ C) \iff (A\ or\ B)\ and\ (A\ or\ C)$
3. $not(A\ and\ B) \iff not(A)\ or\ not(B)$
4. $not(A\ or\ B) \iff not(A)\ and\ not(B)$

Now given any sentence, we can use these equivalences, iteratively, to transform it into an equivalent sentence of the following form (called conjuctive normal form): all "not" operators are immediately over propositional symbols, and no "or" is inside the scope of an "and. For example, not(A and (B or not(C))), can be transformed as follows:

$$\begin{array}{l} \quad not(A\ and\ (B\ or\ not(C))) \\ \Longleftrightarrow not(A)\ or\ not(B\ or\ not(C)) \\ \Longleftrightarrow not(A)\ or\ (not(B)\ and\ not(not(C))) \\ \Longleftrightarrow not(A)\ or\ (not(B)\ and\ C) \\ \Longleftrightarrow (not(A)\ or\ not(B))\ and\ (not(A)\ or\ C) \end{array}$$

We call a propositional symbol or the negation of a propositional symbol a "literal". We can see that, for *any* sentence, we can find an equivalent sentence that is a conjunction of disjunctions of literals. We can think of a sentence in this conjunctive normal form as a set of "clauses", one clause for each conjunct, and each clause consisting of a set of literals. So for the example above, not(A and (B or not(C))) is equivalent to the set of clauses $\{\{not(A), not(B)\}, \{not(A), not(C)\}\}$. A set of clauses is true if all the clauses are true (corresponding to the conjunctions in the conjunctive normal form) and a single clause is true if any one (or more) of the literals in it is true (corresponding to the disjunction in the equivalent sentence.) A literal is true in the expected cases: if it is a propositional symbol then it is true just in case the propositional symbol is true; if it is the not of a propositional symbol, then it is true just in case the propositional symbol is false.

Now we will present a method (called **resolution**) to determine whether a set of clauses is a contradiction, i.e. is unsatisfiable. That means that every row of its truth table is false. Given such a method, we can use it to determine whether an arbitrary sentence is a tautology, as follows. We take the formula, put a *not* around it, convert it to an equivalent sentence in conjunctive normal form (using the equivalences above) and then apply resolution, which will tell us whether it is a contradiction. If it is, then the original sentence is a tautology.

Note that any clause that has both a literal and its complement is satisfied by every truth assignment. Thus we can delete such a clause (a tautology) from any set of clauses without affecting that set's unsatisfiability. From now on we will assume no clause in a clause set contains both a literal and its complement.

As mentioned resolution works on clauses. The idea is, given a set of clauses, to choose two from the set that can be resolved, form their **resolvant**, which is a new clause, and add the resolvant to the original set of clauses. And continue doing this until the empty clause is generated, or until no new clauses can be generated. If the empty clause is generated, then the original set of clauses is a contradiction.

So we need to define the "resolvant" of two clauses. Consider two distinct clauses, one of which contains a proposition symbol P and the other of which contains the literal $not(P)$. These clauses can be resolved and the resolvant is all the literals in either of the two clauses excluding the literals P and $not(P)$. For example, the two clauses: $\{p, not(q)\}$ and $\{not(p), not(q), r\}$, can be resolved on p in the first clause and $not(p)$ in the second, producing the resolvant: $\{not(q), r\}$. Notice that we treat the clauses as sets and thus don't have multiple copies of a literal in a clause.

Do a larger example:

Soundness

If we are given a set of clauses that is satisfiable, i.e. for which there exists a truth assignment that makes them all true, then if we form a resolvant from that set and add it back to the set, that new set of clauses is also satisfiable. This follows because consider the truth assignment that makes the entire set of clauses true; it must make both the two clauses that we resolved true. And it must make the propositional symbol we resolved on either true or false. If it makes it true, then it makes its negation false. Then for the clause containing the negation to be true, some literal in it other than that negation must be true. That literal is in the resolvant and so the resolvant is true. Alternatively the satisfying truth assignment might make the proposition symbol we resolve on false. If so, then some other literal in its clause must be true (since the clause is true.) And that literal will be in the resolvant making the resolvant true. So adding resolvants preserves satisfiability of a set of clauses. If we eventually can get the empty clause, which is unsatisfiable, then the original set of clauses must be unsatisfiable. (If you're not clear on why the empty clause must be unsatisfiable, you can back up one step, and see that the only way we can generate the empty clause through resolution is by resolving two singleton clauses of the form $\{p\}$ and $\{not(p)\}$. An it's easy to see that no truth assignment can make both these clauses true.) So this argument shows that our tautology testing procedure is **sound**, i.e., if it says a sentence is a tautology, then it is indeed a tautology.

Completeness

But we would also like for our procedure to be **complete**, i.e., if indeed the original sentence is a tautology, then our procedure will indeed tell us so. It turns out that resolution is indeed complete. The proof is not difficult so we will look at it.

Consider an arbitrary sentence. Let $p1, p2, p3, ..., pn$ be all the proposition symbols that appear in it. Consider the following tree built from these symbols:

```
                     []
                /            \
          ~p                       p
        /     \                  /    \
    ~q           q            ~q          q
   /   \        /  \        /   \       /  \
 ~r     r    ~r    r      ~r     r    ~r    r
 /\     /\   /\    /\     /\    /\    /\    /\
~s s  ~s s  ~s s  ~s s   ~s s  ~s s  ~s s  ~s s
```

(We note that this tree is *not* an execution tree of any kind. It is not used for computation in any way.)

Each path from the root to a leaf corresponds to one truth assignment to the proposition symbols $\{p, q, r, s\}$, the truth assignment that makes all the literal on the path true.

Now consider the following example clauses:

1. $\{p, r\}$
2. $\{\sim p, r, s\}$
3. $\{\sim q, \sim r\}$
4. $\{q, r\}$
5. $\{\sim p, \sim s\}$
6. $\{q, \sim r, s\}$
7. $\{p, q, \sim r\}$

We can use each clause to eliminate various truth assignments represented in the tree as not satisfying the clause (and thus not satisfying the set of clauses.)

For each clause, find all paths from the root down to a node such that the complement of every literal in the clause is in the path (and this is not true for any shorter path.) Mark all such nodes with an 'X'. Do that for every clause in the clause set. Note that any root to leaf path (i.e. truth assignment) that has an X on some node cannot satisfy the set of clauses, because it will make the clause that generated the X false. And note conversely that any root-to-leaf path that does *not* have a node marked with an X will make all the clauses true and so is a satisfying truth assignment. So an unsatisfiable set of clauses will have a tree in which every root-to-leaf path contains a marked node. Consider the subtree from the root down to the first marked node on each path, i.e., a subtree with all its leaves marked and no internal nodes marked. In this tree, there must be a node with both children being leaves. Let the children correspond to not(P) and P for some P. These child nodes were caused to be marked by two clauses, one of which contains not(P) and one of which contains P. These two clauses are resolvable, and will produce a new resolvant clause that would cause some ancestor of these two nodes to be marked.

These observations show that the resolution closure of a set of unsatisfiable clauses must contain the empty clause, the only clause that marks the root of the tree. If it didn't, there would be a resolution that could be done, and thus the set wouldn't be closed.

So the reolution strategy for determining unsatisfiability is sound and complete.

Resolution is highly explosive, there being many different choices for what clauses to resolve and on what literals to resolve them at every step. It turns out that there are a variety of restrictions one can place on the order of generating resolvants that reduce the combinatorial explosion and yet retain the soundness and completeness of the method. Much research has gone into finding such constraints. Another direction for finding efficient deduction strategies has been to restrict the form of the allowable clauses in some way. With a restricted set of clause forms, and restricted ways to apply resolution, efficient strategies do sometimes exist.

3.1.4 Horn Clauses

In this section we consider clauses of a particular form, called Horn clauses. A clause is a Horn clause if it contains at most one positive literal. So the following are examples of Horn clauses:

```
{~p,q,~r,~s}
{~p}
{s}
{~p,q}
{~p,~q,~r,~s}
{}
```

and the following are *not* Horn clauses:

```
{~p,q,~r,s}
{s,t}
{p,~q,~r,~s,u}
```

A Horn clause that does contain a positive literal is called a *Definite Horn Clause* or just a *Definite Clause.* So the first, third and fourth example Horn clauses above are definite clauses.

Now consider resolution among a set of Horn clauses. For two clauses to resolve, one must contain a positive literal, i.e. one must be a definite clause. Also note that the resolvant of two Horn clauses is itself a Horn clause, so Horn clauses are closed under resolution. Also note that resolving two definite clauses together can never directly produce the empty clause, since one (and exactly one) of the positive literals will appear in the resolvant. One way to do resolution with Horn clauses is to start with a nondefinite clause and resolve it with a definite clause to generate another nondefinite clause. Then we use that just-generated nondefinite clause and resolve it against some definite clause to get a new nondefinite clause, and we continue in this way, always using the just-generated resolvant as one clause used to generate the next resolvant. This is called a linear strategy. It turns out that this is also a complete strategy for Horn clauses.

Consider the example, where for convenience we put the positive literal of definite clauses first:

```
1. {~p,~r}
2. {p, ~t}
3. {p, ~q, ~r, ~s}
4. {q, ~r}
5. {r}
6. {s, ~q}.
```

Consider the following resolution tree:

```
     1. {~p,~r}
w/2. /    \ w/3.
{~t,~r}   {~q,~r,~s}
            |w/4.
          {~r,~s}
            |w/5.
           {~s}
            |w/6.
           {~q}
            |w/4.
           {~r}
            |w/5.
            {}
```

This is a tree in which the root is a nondefinite clause and all the nodes are labeled with resolvants and each edge is labeled with the number of the clause that the parent resolved with to generate the child.

Notice that we kept the order of the clauses in generating the resolvant and put the negative literals of the definite clause before the remaining negative literals of the nondefinite clause (eliminating subsequent duplicates.) And we always chose the first literal in the nondefinite clause to resolve on. This is called SLD resolution for Horn clauses, and it turns out to be sound and complete. (A proof is beyond what I want to do here, but can be found in the literature, or reconstructed by looking an all possible resolution trees and determining that any general resolution sequence that leads to the empty clause, can be transformed into one that is in the form here.)

There is another way to write Horn clauses; we can write them as implications. We can define implication (->), read "implies", by:

```
P -> Q == ~P \/ Q
```

and reverse implication (->), read "if", by:

```
P <- Q == P \/ ~Q
```

Note that the propositional sentence `p <- (q/\r/\s)` is logically equivalent to `(p\/~q\/~r\/~s`, and is equivalent to the Horn clause:

```
{p,~q,~r,~s}
```

So definite Horn clauses are implications in which a conjunction of positive literals imply a single positive literal.

Explain (and give example) how SLD resolution (not eliminating redundant literals) is what Prolog does (for predicates without any arguments). Prolog searches the tree of SLD resolution derivations in depth-first order.

Horn Clauses, Unit Resolution

Another strategy for resolution with Horn clauses is called unit resolution. In unit resolution every resolution step involves one clause that has only one positive literal.

So for example:

```
0. {~p,~s}
1. {p,~q,~r}
2. {p,~s}
3. {q,~s,~t}
4. {r}
5. {s}
```

and the following unit resolutions:

```
6. 1 and 4 --> {p,~q}
7. 3 and 5 --> {q,~t}
8. 2 and 5 --> {p}
9. 0 and 5 --> {~p}
10. 8 and 9 --> {}
```

It turns out that unit resolution is complete for Horn clauses.

We think of the indefinite clause as the Goal: it is (the negation of) what we want to prove is logically implied by the truth of all the definite clauses. SLD resolution is called a backward chaining strategy and Unit resolution is called a forward chaining strategy. If we think of our Horn clauses as implications, we can see why: SLD resolution starts from the Goal and reasons backwards using the implications in a backward direction, from consequence to antecedent, to determine what we have yet to establish in order to establish our goal. In Unit resolution we reason starting from facts we know (the unit clauses) and reason forward to derive everything we can conclude, eventually hoping to prove the Goal.

3.2 First Order Logic (FOL)

Propositional logic is very limited in what it can represent and reason about. Taking simple declarative sentences as primitive, and thus not having the ability to look inside and analyze components of simple sentences is very restrictive. First order logic remedies this by allowing primitive declarative sentences to be further decomposed into statements about objects and relationships among objects. For (a classic) example, the sentence "Socrates is a man" in propositional logic is primitive, but in first order logic we can further analyze it and say, essentially, that Socrates is a thing in the set of things called "man", written man(Socrates). And we can analyze the sentence "All men are mortal" as saying that if something is in the set of men, then it is the set of mortal things, written `all(X,man(X) -> mortal(X))`. The advantage of this is that now we can reason about objects and their relationships. For example from these two first-order sentences, we can logically conclude that Socrates is mortal, written mortal(Socrates). In propositional logic, this kind of reasoning can not be modeled.

3.2.1 Syntax

The syntax of FOL is:

```
term_list --> term, term_tail.

term_tail --> [].
term_tail --> [’,’], term_tail.
```

```
term --> [Atom], {fun_sym(Atom}}, opt_par_term_list.
term --> [X],{var_sym(X)}.

opt_par_term_list --> [].
opt_par_term_list --> ['('], term_list, [')'].

atomic_formula --> [Pred], {pred_sym(Pred)}, opt_par_term_list.

formula --> atomic_formula.
formula --> [not],formula.
formula --> formula, ['\/'], formula.
formula --> formula, ['/\'], formula.
formula --> [for_all],[X],{var_sym(X)},formula.
formula --> [there_exists],[X],{var_sym(X)},formula.
formula --> ['('], formula, [')'].
```

In this prolog-like representation, we assume that the function symbols (used in terms) come from some set of symbols, represented in Prolog with the predicate `fun_sym`. Similarly predicate symbols come from some fixed set as do variables, and the variable symbols (called variables) and the function symbols are disjoint.

Assuming that the appropriate symbols are in the appropriate sets (and variables are symbols starting with upper-case letters), examples of terms are:

```
father(michael)
sum(1,3)
sum(5,prod(6,succ(3)))
```

Examples of atomic formulas are:

```
man(socrates)
mortal(socrates)
employee(john,deere,93,124000,'101 Elm Street','Stony Brook','NY',11794)
loves(X,madonna)
```

Examples formulas are:

```
mortal(socrates)
man(socrates) /\ mortal(socrates)
for_all X (not(man(X)) \/ mortal(X))
```

We can define various syntactic properties of formulas that come in handy.

A variable occurrence is *free* in a formula if it does not appear in the scope of a quantifier binding that same variable. It is *bound* otherwise.

A formula is *closed* if it contains no free occurrence of any variable. It is *open* otherwise.

3.2.2 Semantics

The semantics of FOL is given using FOL structures (sometimes called models.) A structure provides a meaning for each function symbol and predicate symbol in the language. Each function symbol and predicate symbol comes with an "arity", a natural number that corresponds to the number of terms that follow them in a parenthesized list in the syntax. (We assume, for simplicity now, each symbol has a fixed arity and appears only in the context of that many following terms.) A structure comes with a domain of objects, say D. A structure gives an n-ary function on D as the meaning of a function symbol of arity n. A 0-ary function symbol is called a constant and appears without any following parenthesized list. For example, "socrates" is a constant. So in particular, a structure assigns a constant to an element of D. A structure gives an n-ary relation on D as the meaning of a predicate symbol of arity n.

We use structures to give meaning to first-order formulas. But first-order formulas contain variables, and we need a way to give meanings to variables. We do this with a "variable assignment", which is an assumption of what values the variables take on. A variable assignment is a function from variables into the domain D of objects of the structure, i.e., `VA:Vars->D`. Now given a structure and a variable assigment VA, we can assign meanings to formulas of FOL.

We'll write a simple Prolog specification to show how meaning is defined. A model M determines a domain, a function mapping and a predicate mapping. We represent that as the following Prolog facts:

```
%% given by the model
model_domain(Obj).                          % for every object Obj in the domain
model(FunctionSym,ArgList,Obj).             % to define every function symbol
model(PredicateSym,ArgList,TruthValue).     % to define every predicate symbol
```

So for example, for a model of arithmetic, we would have facts like: `model_domain(0)`, `model_domain(1)`, `model_domain(2)`, and so on for the domain elements. For functions we would have `model(+,[0,0],0)`, `model(+,[1,0],1)`, `model(+,[1,1],2)` and so on. For the less-than predicate symbol, we would have `model(<,[0,0],false)`, `model(<,[0,1],true)`, `model(<,[1,0],false)`, and so on. Of course, these sets are infinite, so we cannot actually write them down as Prolog programs. But the Prolog specification should be clear; it's just that there would be infinitely many answers to some queries.

Consider constant symbols in our arithmetic language. Let's say that we use boolean notation. So our model must also give a meaning to these 0-ary function symbols making up boolean representation of numbers. That would be done with the facts such as: `model('11',[],3)` and `model('101',[],5)`. We would also have the facts `model('0',[],0)` and `model('1',[],1)` and it is important to note that the first entry in these facts is a symbol, often called a numeral, here a sequence of the 0 and 1 symbols. The last element in these facts is a number, the abstract concept regardless of what notation we use to refer to it.

Now given that representation of a model, we can define the concept of the meaning of a formula in a model under a variable assignment. We will assume that a variable assignment is represented as a list of Var=Obj records. First we have to define the meaning of a term in a model under a variable assignment. Such a meaning will be an object in the model. The definition is as follows:

```
%% meaning of a term in a model under a variable assignment
term_means(Term,VA,Obj) :-
   var_sym(Term),
   member(Term=Obj,VA).
term_means(Term,VA,Obj) :-
   Term =.. [FunSym|Arguments], % e.g. f(a,X,g(b)) =.. [f,a,X,g(b)]
   term_means_list(Arguments,VA,Objs),
   model(FunSym,[Objs],Obj).

%% meaning of a list of terms
term_means_list([],_,[]).
term_means_list([Term|Terms],VA,[Obj|Objs]) :-
    term_means(Term,VA,Obj),
    term_means_list(Terms,VA,Objs).
```

So the meaning of a term is found by using the model to get the meaning of constant symbols, using the variable assignment to get the meanings of variables, and using the model's function definitions to find the result of applying a function to its arguments.

Then given the meaning of terms, we can define the meaning of a formula in a model under a truth assignment as follows:

```
%% meaning of a formula in a model under a variable assignment
formula_means(AFmla,VA,TVal) :-   % atomic formula
    Fmla =.. [PredSym|Args],      % e.g. psym(a,X,g(b)) =.. [psym,a,X,g(b)]
    term_means_list(Args,VA,Objs),
    model(PredSym,Objs,TVal).
formula_means(A /\ B,VA,TVal) :-
    formula_means(A,VA,TVal1),
    formula_means(B,VA,TVal2),
    /\(TVal1,TVal2,TVal).
formula_means(not(A),VA,TVal) :-
    formula_means(A,VA,TVal1),
    not(TVal1,TVal).
```

```
formula_means(exists(V,Fmla),VA,TVal) :-
    (model_domain(Obj),                   % try any domain object
     replace_assignment(V=Obj,VA,NewVA),  % for this variable
     formula_means(Fmla,NewvA,true)       % and if it makes subformula true
     ->TVal = true
     ;  TVal = false
    ).

%% replace a single assignment with another
replace_assignment(V=Obj,VA,NewVA) :-
    append(Prefix,[V=_|Suffix],VA),
    append(Prefix,[V=Obj|Suffix],NewVA).

%% and truth table
/\(true,true,true).
/\(true,false,false).
/\(false,true,false).
/\(false,false,false).

%% or truth table
not(true,false).
not(false,true).
```

To find the meaning of an atomic formula, we find the meanings of its arguments using the term meaning function, and then use the model to see whether that tuple of objects is in the relation that the model associates with that predicate symbol. The meaning of *and* formulas is clear. The meaning of an exists formula is true if we can find some domain object in the model, that when we assign it to the correspoding variable in a new variable assignment (leaving all other individual assignments the same), that variable assignment makes the subformula true.

(Do some examples.)

Properties of FOL formulas:

Two FOL formulas are *logically equivalent* if in every structure and for every variable assignment, they have the same meaning.

The meaning of a closed formula (one with no free variables) is independent of the variable assignment. We normally work with closed formulas, sometimes called sentences.

FOL equivalences (or definitions):

```
not(forall(X,Fmla)) == exists(X,not(Fmla))
not(exists(X,Fmla)) == forall(X,not(Fmla))
forall(X,Fmla1) /\ forall(X,Fmla2) == forall(X,Fmla1/\Fmla2)
```

```
   if X is not free in Fmla2 then
      forall(X,Fmla1) \/ Fmla2 == forall(X,Fmla1\/Fmla2)
   exists(X,Fmla1) \/ forall(X,Fmla2) == exists(X,Fmla1\/Fmla2)
   if X is not free in Fmla2 then
      exists(X,Fmla1) /\ Fmla2 == exists(X,Fmla1/\Fmla2)
```

Given a quantified formula such as `forall(X,Fmla)`, we can choose a new variable not appearing in `Fmla`, say `Y`, and replace every free occurrence of `X` in `Fmla` with `Y` getting `Fmla|X<-Y`, and then `forall(Y,Fmla|X<-Y)` is logically equvalent to `forall(X,Fmla)`.

3.2.3 Deduction

We will again use resolution for our deduction method. Recall that reslution works on formulas in clausal form. In FOL we will call an atomic formula or a negation of an atomic formula a literal. And again a clause will be a set of literals. Now a set of clauses is true if they are true under all variable assignments. A single clause is true under a variable assignment if some literal in it is true under that variable assignment. This definition allows us to think of the variables in clauses as being universally quantified.

So we would like to find, given an arbitrary FOL formula, an equivalent clausal form. This turns out not to be possible, but we can find a clausal form that is satisfiable just in case the original formula is satisfiable, and that is sufficient for our purpose of deduction.

So given an arbitrary FOL formula, we first find a form in prenex normal form (PNF), meaning all the quantifiers have outermost scope, and the nonquantifier "matrix" is in conjunctive normal form. For example the formula:

```
forall(X,exists(Y,p(X) /\ (not(q(X) \/ r(X,Y)))))
```

is in PNF. Now given any FOL formula, we can find an equivalent one in PNF. First we "standardize the variables apart", which means that we change bound variables to ensure that no two distinct quantifiers in the formula bind the same variable. This can always be done by choosing new variables and changing one of the bound variables to the new variable (including all free occurrences in the scope of that binding quantifier). Then we can first push all negations over quantifiers, using the equivalences above, and over "and" and "or" using DeMorgans laws (as for propositional logic), eliminating double negations as necessary. Next we have to pull quantifiers out of conjunctions and disjunctions. Given a formula of the form: `(P /\ exists(X,Q))`, we can transform it to `exists(X,P /\ Q)`, since X does not appear in P (since all variables are standardized apart.) And similarly we can move all quantifiers out over "and" and "or". So for any FOL formula, we can find an equivalent formula in PNF.

For example, consider the formula for "every man loves a woman":
`forall(X,man(X)->exists(Y,woman(Y)/\loves(X,Y)))`. We can find the following chain of equivalences:

```
forall(X,man(X)->exists(Y,woman(Y)/\loves(X,Y)))
==> forall(X,not(man(X))\/exists(Y,woman(Y)/\loves(X,Y)))
==> forall(X,exists(Y,not(man(X)) \/ (woman(Y)/\loves(X,Y))))
==> forall(X,exists(Y,(not(man(X))\/woman(Y)) /\ (not(man(X))\/loves(X,Y))))
```

and the final formula is in PNF.

Now if the formula in PNF has only universal quantifiers, then we are done. The "matrix" of the prenex formula directly provides the set of clauses, since the assumption on the meaning of variables in a set of clauses is that they are universally quantified over the entire set. But what happens if we have existential quantifiers in the quantified prefix, as in the above example?

Here, we are not able to find a logically equivalent formula, in general. However we can find another formula that will be satisfiable if and only if the PNF formula is satisfiable. And for use in resolution theorem proving that is just what we need.

So consider the PNF formula we have:

```
forall(X,exists(Y,(not(man(X))\/woman(Y)) /\ (not(man(X))\/loves(X,Y))))
```

which says, "for every X there is a Y such that some property holds," in this case that Y is a woman that the man X loves. So let's consider a function "lovee" that for every man chooses a woman that he loves. So perhaps lovee(john)=mary where "john loves mary". Now consider the formula:

```
forall(X,(not(man(X))\/woman(lovee(X))) /\ (not(man(X))\/loves(X,lovee(X)))))
```

We have eliminated the existential quantifier on Y, and replaced every occurrence of Y in the quantified formula with "lovee(X)". The claim is: there is a structure that satisfies the first formula if and only if there is a structure that satisfies the second. Consider the "if" direction and assume that there is a structure satisfying the second formula. Then that structure will satisfy the first formula, since the witness for the truth of the exists subformula is given by looking at where the "lovee" function takes the object that the variable assignment assigns to X. Since "lovee" is a function, there will always be such an object, and the second formula ensures that the subformula will be true using that object. Now for the "only if" direction, the truth of the first formula guarantees that there is a total function that takes any X to some Y satisfying the subformula. So there is a function, which can be assigned to the "lovee" function symbol in a new structure that will cause the second formula to be satisfied.

This is an example of a general transformation, called "Skolemization". For any formula in PNF with an existential quantifier in the prefix, we can eliminate the outermost existential quantifier by introducing a new "Skolem" function symbol, with arity equal to the number of universal quantifiers that include the existential formula in their scope (i.e. to the left of the existential.) We then replace every occurrence of the existentially quantified variable with the term consisting of the new function symbol applied to all the enclosing universally quantified variables. This gives us another formula that is satisfiable if and only if the original one is satisfiable. By iterating this process, we can eliminate all existential quantifiers and find a formula with only universal quantifiers. And that formula can now be used to generate the equivalent set of clauses.

For example for our "every man loves a woman" formula, the set of clauses is:

```
{(not(man(X)),woman(lovee(X))},
 {not(man(X)),loves(X,lovee(X))}
}
```

Resolution in FOL

Resolution in FOL works with first-order clauses, i.e., sets of first-order atom formulas and their negations. So the clauses contain variables. We can informally think of a clause with variables as being shorthand for the whole set of clauses one for each way to substitute values for (i.e., instantiate) the variables in the clause consistently. In this way, we can think of the "real" clauses as actually being variable-free in which case they are just propositional clauses, (of course with the complication that there may be infinitely many clauses.) But our intuition should be that when we manipulate these first-order clauses, we are intuitively manipulating their (infinitely many) ground instantiations, i.e., instances with no variables.

So as an example, consider the first-order clauses:

```
{loves(john,X),~woman(X)}
{~loves(Y,pat)}
```

Thinking of these as representing their ground instances, we can see that the first clause includes the instance `{loves(john,pat),~woman(pat)}` (since X can be instantiated to `pat`, and the second clause includes the instance `{~loves(john,pat)}` since Y can be instantiated to `john`. These (instantiated) clauses contain complementary literals and so we can resolve them (as propositional clauses) to obtain a new clause: `{~woman(pat)}`. And note that this fact does indeed follow from the two first-order clauses.

Now consider a more complicated case:

```
{loves(Z,X),likes(Z,X)}
{~loves(Y,pat)}
```

Now the first clause has an instance: `{loves(Z,pat),likes(Z,pat)}` which stands for many ground clauses, one for each value substituted for Z. And the second clause `{~loves(Y,pat)}` stands for many ground clauses, one for each value substituted for Y. Note that for any value, if we substitute it for Z in the first and X in the second, we get two resolvable clauses, and their resolvants are all the clauses we can get by substituting something for Z in `{likes(Z,pat)}`. And note that we can find this clause directly by matching loves(Y,pat) of the second clause with loves(Z,X) of the first, to find the minimal instantiations necessary to make them the same. Then we can apply those instantiations to all the literals in both clauses. Then these clauses can resolve directly generating a new first-order clause representing all the ground resolvants. This is the idea behind first-order resolution.

There a a few things we must be careful about. The matching operation is called unification, and given two atomic formulas, it finds a most general term that is an instance of both formulas. This term is called a most general unifier (mgu). It turns out that given two terms if an mgu exists, it is unique.

There are several algorithms for finding the mgu. One approach is to work with a set of pairs of terms, which we try to unify. If all pairs are identical, we are done and the terms are unifiable. If there is a pair neither of whose components is a variable and whose function symbols are not the same, then we are done and the terms are not unifiable. If one component of some pair is a variable and the variable does not occur in the term it is paired with, but does appear in some term in some other pair, then replace that variable by its paired term in all terms in all other pairs in the set. (Note that this will eliminate all other occurrences of that variable.) If there are two nonvariable terms that are paired and have the same main function symbol, then pair all the corresponding arguments of the two terms and replace the original pair by that set of pairs. Repeat until done. The most general unifier is determined by the remaining set of pairs. At least one of each final pair is a variable.

(More precise definition and give several examples)

Examples...

Two clauses are called **variants** of each other if they are identical up to renaming of variables. Notice that the result of replacing a clause in a set of clauses by a variant preserves its meaning.

Now we can define the first-order resolvant of two first-order clauses. Two clauses resolve if they contain complementary literals, and the atomic formulas of those literals unify (after standardizing their variables apart.) The resolvent is obtained by taking the union of the resolved clauses minus the resolved-upon literals and applying the mgu substitution to that result.

So the resolution refutation algorithm for FOL is now the "same" as the propositional algorithm. Close the set of clauses under this definition of resolution. If the closure contains the empty clause, then the original set is unsatisfiable.

Example:

Consider the example clauses, which we number for future reference:

```
1. { ap(n,X,X) }
```

```
2. { ap(c(X,Y),Z,c(X,W)), ~ap(Y,Z,W) }
3. { ~ap(c(a,c(b,n)),c(d,n),A) }
```

Consider the following sequence of resolution steps to determine that these clauses are inconsistent.

```
3. { ~ap(c(a,c(b,n)),c(d,n),A) }
resolve this with 2 using mgu:
    X = a, Y = c(b,n), Z = c(d,n), A = c(a,W)
obtaining:
4. { ~ap(c(b,n),c(d,n),W) }
resolve this with 2 again (with all variables primed) using mgu:
    X' = b, Y' = n, Z' = c(d,n), W = c(b,W')
obtaining:
5. { ~ap(n,c(d,n),W') }
resolve this with 1 using mgu:
    X = c(d,n), W' = c(d,n)
obtaining:
6. { }
```

Since we derived the empty clause, these clauses are inconsistent. Notice that we can look to see what instantiation of the variable A it was that resulted in the inconsistency, by looking at the mgu's involved:

```
A = c(a,W) from the first mgu
W = c(b,W') from the second, and
W' = c(d,n) from the last
yielding
A = c(a,c(b,c(d,n)))
```

Now note that we can write clauses 1 and 2 in implicational form:

```
ap(Y,Z,W) -> ap(c(X,Y),Z,c(X,W))
ap(n,X,X)
```

or in reverse implicational form, and expanding the predicate name of *ap*:

```
append(c(X,Y),Z,c(X,W)) :- append(Y,Z,W).
append(nil,X,X).
```

This is actually the append program, where we are representing the empty list with the constant nil, and using c/2 as the list constructor.

Horn Clauses, SLD

The definition of Horn clauses applies to first-order clauses: A clause is a Horn clause if it contains at most one positive literal. Resolution can be specialized to Horn clauses just as in the propositional case, and for Horn clauses we specialize general resolution to SLD resolution.

Examples:

Prolog as FO Horn Clauses

Prolog is depth-first search through the tree of SLD-resolution proofs.

Consider:

```
{anc(X,Y), ~parent(X,Y)}
{anc(X,Y), ~parent(X,Z), ~anc(Z,Y)}
{parent(e,t)}
{parent(t,d)}
{parent(t,p)}
{parent(d,h)}
{parent(h,m)}
```

This is the clause version of our old friend, the Prolog program for defining ancestor.

Show the SLD tree, and Prolog evaluation.

Limitations of Prolog/SLD and what can be done

Prolog has infinite loops, as we've seen. There are trivial Horn clause programs that make perfect sense logically, but put SLD (and thus Prolog) into an infinite loop. E.g. p :- p. But maybe more importantly

```
{anc(X,Y), ~parent(X,Y)}
{anc(X,Y), ~parent(X,Z), ~anc(Z,Y)}
```

if parent is cyclic. Or

```
{anc(X,Y), ~parent(X,Y)}
{anc(X,Y), ~anc(X,Z), ~parent(Z,Y)}
```

which logically is just as good as the previous definition.

Other natural examples from the previous chapter are: a) the expression grammar and b) the simple English grammar as extended in the exercises. The expression grammar we gave has the wrong associativity for the + and * operators; they are normally considered to be left associative (e.g., 1+2+3 is considered as (1+2)+3 and not 1+(2+3)), but the grammar used right-associativity. Had we used the natural grammar for left associative operators, we would have had a left-recursive grammar, and Prolog goes into an infinite loop for such grammars. Similarly for the simple English grammer in which we wanted to say that a noun phrase is a noun phrase followed by a prepositional phrase. The natural rule to do this, verb—np –¿ np, pp.—, again left-recursive. So adding prepositional phrases in a way that Prolog could handle them required some rather complicated, and perhaps non-intuitive, machinations.

(Read intro now.)

Let's consider only Datalog programs, which are Horn clauses with only constants and variables (i.e., no function symbols of arity greater than 0.)

We can define an operator, called the T_P operator, on sets of ground atomic formulas as follows:

```
Tp(S) = {H : there is a ground instance, H :- B1,B2,...,Bn, of a rule,
             and for each Bi, Bi is in S}
```

This is one-step of inference: if we assume all the facts of set S are true, and the rules are true, then T_P(S) is what we can infer to be true in one step. We can infer what is true in any number of steps beginning from no assumptions as follows:

```
S_0 = Tp([]).
S_1 = Tp(S_0).
S_2 = Tp(S_1).
...
S_n = Tp(S_n-1).
```

And then we take the union of all the S_i. This will give us all the atomic formulas that must be true in every structure that makes all the rules true. This process essentially produces the "least Model" of the program. For Datalog programs, this process is finite. $S_i \subset S_{i+1}$ for every i, so the sequence is monotonically increasing. Since there are only finitely many possible ground atomic formulas for a finite Datalog program, this sequence must eventually reach a fixed point where $S_{k+1} = S_k$.

(Do ancestor example.)

Bottom-up computation starts with the facts and derives new facts using the rules. The idea is that you start assuming that you don't know anything: i.e., that all the relations (corresponding to the rule heads and facts) are empty. Then you add tuples to those relations as is required by the rules. The simple example is:

Program:

```
e(a,b).
e(a,c).
e(c,d).
e(d,f).
p(X,Y) :- e(X,Y).
p(X,Y) :- p(X,Z),e(Z,Y).
```

So there are two relations (sets of ordered pairs) defined by this program: e/2 and p/2. We start assuming they are empty:

```
e/2 = { }
p/2 = { }
```

Now we use the current tuples in e/2 and p/2 (none, at the moment) to see if these tuples and the rules and facts force us to add more tuples. Well, the 4 facts for e in the program force us to add those pairs to the relation e/2. So we get

```
e/2 = {<a,b>,<a,c>,<c,d>,<d,f>}
p/2 = { }
```

Now we do it again, to see if we need to add more tuples, using these current sets. So since we have `<a,b>` in e/2 and the first p-rule says that p(X,Y) if e(X,Y), we must add `<a,b>` to p/2. And similarly for the other pairs in e/2, so now we get:

```
e/2 = {<a,b>,<a,c>,<c,d>,<d,f>}
p/2 = {<a,b>,<a,c>,<c,d>,<d,f>}
```

OK, do it again. Now we have `<a,c>` in p/2 and `<c,d>` in e/2, and the second p/2 rule says that p(X,Y) if p(X,Z) and e(Z,Y). so with X=a,Z=c and Y=d, we have to add `<a,d>` to p/2. And similarly for `<c,d>` in p/2 and `<d,f>` in e/2, that same rule requires that we add `<c,f>` to p/2. So at this point we get:

```
e/2 = {<a,b>,<a,c>,<c,d>,<d,f>}
p/2 = {<a,b>,<a,c>,<c,d>,<d,f>,<a,d>,<c,f>}
```

And then again: now we have `<a,d>` in p/2 and `<d,f>` in e/2, so the second p/2 rule requires that we add `<a,f>` to p/2, and we get:

```
e/2 = {<a,b>,<a,c>,<c,d>,<d,f>}
p/2 = {<a,b>,<a,c>,<c,d>,<d,f>,<a,d>,<c,f>,<a,f>}
```

and now we can't get anything new. We just get pairs we already have, so we have hit the fixed point, and are done.

The T_P operator takes a current set of relations for the predicates in the program, and uses the rules of the program P and the tuples in the set of relations, and generates a new set of relations. It does this the way we did in each step above: Look for instantiated rules where we have facts for all the atomic formulas on the right-hand side of a rule, and then add the left-hand side. So for example, what we did above, in terms of the T_P operator:

```
Tp(e/2={ },p/2={ })  =  T1  =  (e/2= {<a,b>,<a,c>,<c,d>,<d,f>}, p/2={ })
Tp(T1)  =  T2  =  (e/2= {<a,b>,<a,c>,<c,d>,<d,f>},
                   p/2= {<a,b>,<a,c>,<c,d>,<d,f>})
Tp(T2)  =  T3  =  (e/2= {<a,b>,<a,c>,<c,d>,<d,f>},
                   p/2= {<a,b>,<a,c>,<c,d>,<d,f>,<a,d>,<c,f>})
Tp(T3)  =  T4  =  (e/2= {<a,b>,<a,c>,<c,d>,<d,f>},
                   p/2= {<a,b>,<a,c>,<c,d>,<d,f>,<a,d>,<c,f>,<a,f>})
Tp(T4)  =  T4
```

and we reached the fixed point. Notice that we started with the empty relations and at each step applied the the previous step. This will always generate a non-decreasing sequence of sets. If there are only finitely many possible tuples, it will always stop at some point, and we will have reached a fixed point.

Now if want to determine if a query is implied by a program, find the LFP of the program using the T_P operator, and see if the query is in the LFP. So this is another proof technique for Datalog programs. (It is directly extensible to all Horn clause programs, but then the LFP may be infinite.)

So this shows that for Datalog programs we don't *have* to live with the infinite loops of Prolog. There is an alternative.

But there are problems with the T_P approach: it is not goal-directed. E.g., to find my ancestors, we have to find everybody's ancestors. That can be a lot of unnecessary work.

We can see a similar (the same) phenonomenon with inductively defined functions. Recall from high school how we defined the factorial function, n!

```
0! = 1
n! = n*(n-1)! if n>0
```

There are two ways to think about evaluating, say 5!: We can think "bottom-up":

```
0! = 1, 1! = 1*0!=1*1=1, 2! = 2*1!=2*1=2,
3! = 3*2!=3*2=6, 4! = 4*3!=4*6=24, 5! = 5*4!=5*24=120.
```

Or we can compute it “top-down” as a functional programming language would by:

```
5! = 5*4! = 5*(4*3!) = 5*(4*(3*2!)) = 5*(4*(3*(2*1!))) = 5*(4*(3*(2*(1*0!)))) =
5*(4*(3*(2*(1*1)))) = 5*(4*(3*(2*1))) = 5*(4*(3*2)) = 5*(4*6) = 5*24 = 120.
```

For factorial, these two ways of computing it do essentially the same amount of work. But consider a definition of fibonacci:

```
fib(0) = 1
fib(1) = 1
fib(n) = fib(n-1) + fib(n-2) if n>1
```

Computing this top-down takes exponential work. (Do example)

Computing it bottom-up takes linear work. (Do example)

So it seems that bottom-up can be much better than top-down. But consider the definition:

```
lg(1) = 0
lg(n) = 1 + lg(n//2) if n>1. (where // is integer division)
```

Computing this bottom up takes linear work, but computing it top-down takes only log work. So in this case top-down is much better than bottom-up. So it’s clear that neither bottom up nor top-down is uniformly better than the other. The question is whether we can come up with a single evaluation strategy that gets the best of both strategies, for example would be linear on the given definition for the fibonacci function and log on the definition of the lg function.

In the case of evaluting Datalog programs, there are two approaches (which turn out to be essentially the same approach): Tabling, and magic sets. Tabling modifies Prolog backward chaining evaluation to remember goals and answers to avoid rederiving redundant goals. Magic sets modifies the T_P approach to add filtering predicates so that only essential goals are derived.

We will look at the Tabling approach.

3.3 Exercises

1. Prolog has infix operations that allow you to write propositional expressions, such as `(p /\ q \/ not(p /` We will use this syntax to represent propositional expressions, with `/\` meaning ”and”, `\/` meaning ”or” and not meaning ”not”.

We will represent a truth assignment as a list, such as:

```
[p=true,q=false,r=false]
```

where each element in the list defines the mapping of a propositional symbol to true or false.

Write a Prolog predicate eval_tv(PropExp,TruthAsgn,TV) which takes a propositional expression, PropExp, and a truth assignment in TruthAsgn, and returns the truth value of that expression under that truth assignment in TV.

For example

```
| ?- eval_tv((p /\ q \/ not(q \/ p)),[p=false,q=false],TV).

TV = true ;

no
| ?-
```

2. Extra Credit: (Not for the squeamish. Only for those with time and interest.) Write a predicate that, given a propositional formula, will print out its truth table.

   Prolog has builtin predicates write(Term) that writes the term Term, and nl, which writes out a newline. Try for example:

   ```
   | ?- write(hello),nl,write(world).
   ```

3. Write a DCG to recognize and parse propositional sentences.

4. Write a Prolog predicate prop_clause/2 that takes a propositional formula (using `/\`, `\/`, and `not`, as in the previous problems and produces an equivalent list of clauses. A clause is represented as a list of literals. This predicate should in effect convert the propositional formula into conjunctive normal form and then produce the resulting list of clauses. (You do not have to create the CNF form if you find a more direct way to get the same result.)

5. Write a Prolog predicate resolve/1 that takes a list of clauses, such as those generated in problem 1, applies the resolution strategy, succeeding if the clauses are unsatisfiable, and failing otherwise.

   One way to write resolve is as a recursive predicate that chooses a pair of resolvable clauses, generates the resolvent, adds it to the set of clauses if it's new, and recurses, until the empty clause is generated, or no new clauses can be generated. Now you need to choose only one resolvable pair at each step; you don't want to backtrack and choose all pairs nondeterministically. To do this in Prolog, you can use a metapredicate, once/1, which when applied to a predicate call, will return only the first answer the call generates.

   So for example, we get:

   ```
   | ?- once(member(X,[a,b,c])).
   X = a;
   no
   | ?-
   ```

Using once/1, you can choose only one of the many ways resolvants might be generated at each step.

(a) Add writeln/1 calls at appropriate places in your code so that you can see, for each the resolvant, what clauses were used to generate it.

(b) Test your program with reasonable clause sets. Come up with an interesting example with some semantics for the propositions, and explain what your proof system does.

6. This problem involves representing a first-order structure in a Prolog program and evaluating the meaning of formulas in such a structure.

   (a) Consider the representation of a first-order structure in Prolog. For our problem, the structure will be an abstraction of a simple world with several people in it and several relations relating the people. We use the following representation:

      i. To represent the domain, we will use a unary Prolog predicate, dom/1, which will return each element of the domain. So, for example, the fact:

         `dom(1).`

         would indicate that 1 is in the domain of the structure.

      ii. To represent the relations, we will use a binary predicate, relation/2, which returns pairs in which the first is a predicate symbol (of arity n) in the language, and the second is a list of length n representing a tuple in the corresponding relation. So, for example, the fact:

         `relation(<,[3,17]).`

         would indicate that the predicate symbol `<` is mapped to a relation that contains the ordered pair `<3,17>`.

      iii. To represent the functions, we will use a binary predicate, function/2, which returns pairs in which the first is a function symbol (of arity n) in the language, and the second is a list of length n+1 representing the function arguments and its value for those arguments in the graph of the corresponding function. So, for example, the fact:

         `function(+,[2,3,5]).`

         would indicate that `+` is a function symbol in the language that is mapped by the structure to a binary function that, given arguments 2 and 3, returns 5.

   Task: Give definitions of these three Prolog predicates for a first-order structure representing the following situation:

      i. There are 4 people: john_jones, mary_smith, bill_rogers, jane_doe.

      ii. john_jones and bill_rogers are men, where men are represented by a unary relation symbol 'man'. mary_smith and jane_doe are women, where women are represented by a unary relation symbol 'woman'.

      iii. Some people love other people, represented by a binary relation symbol 'loves'. In particular john_jones loves mary_smith and she loves him back. And bill_rogers loves jane_doe, but jane_doe loves john_jones.

   iv. Every person has a spouse, represented by a unary function symbol 'spouse'. In particular, john_jones and mary_smith are married (and thus are each other's spouses) and bill_rogers and jane_doe are also married.

(b) Write first-order formulas (in the language above) to represent the meanings of the following English sentences. (You may use implication (`->`) as a logical connective, if you wish.) The constants we have in the logic are 'John', 'Mary', 'Jane', and 'Bill' (which refer to the obvious objects in the structure), the unary predicate symbols are 'man' and 'woman', the binary predicate symbol 'loves', the unary function symbol 'spouse'.

   i. John loves Mary.
   ii. Bill loves Jane but Jane loves John.
   iii. Mary loves a man.
   iv. Jane does not love her spouse.
   v. Every man loves his spouse.
   vi. Not every woman loves her spouse.
   vii. Every man loves a woman. (i.e., for every man there is a woman that he loves)
   viii. There is a man that every woman loves.

(c) We will represent first-order formulas in Prolog as shown in the following examples.

```
forall(x1,exists(y2,loves(x1,y2)))
forall(x1,not(man(x1))\/exists(y,woman(y)/\loves(x1,y)))
forall(x,loves(x,spouse(x)))
```

So variables are represented by Prolog atoms that begin with the letters x, y, or z followed by a (possibly empty) sequence of digits.

Universal quantification is represented by the Prolog term with forall at the root and its left child a variable, and its right child a formula. And existential quantification is similar.

The representation of terms and the other form of formulas should be clear from the examples.

Task: Write a Prolog predicate:

```
mean_fmla(+Formula,+VariableAsgn,-TV)
```

that takes a first-order formula, a variable assignment, and returns true if the formula is true in the structure represented by the Prolog predicates dom/1, relation/2 and function/2. A variable assignment is represented as a list of terms Varable=Object. For example, a variable assignment for the structure of problem 1 might look like: [x=john_jones,y1=jane_doe]. We will assume that VariableAsgn has terms for every variable that appears free in Formula.

We will need an auxiliary predicate, call it mean_term, where mean_term(+Term,+VariableAsgn,-Obj) takes a first-order logical term, a variable assignment, and returns the object that term means in the structure defined by dom/1 and friends.

We will need a Prolog predicate to break apart a term that is an atomic formula (or a function application) into its components. We use the builtin (infix) predicate =.. as the following example shows:

```
| ?- loves(john,spouse(x2)) =.. Term.
Term = [loves,john,spouse(x2)];
no
| ?-
```

So given a left-hand argument of a term (tree), =.. breaks it apart into a list, where the first element of the list is the atom at the root of the term tree, and the tail of the list is a list of the terms that are the immediate children of the root. =.. works in either direction, i.e., it will also construct a term tree from a list representing the root atom followed by its children (but we don't need that direction in this problem.)

Task: Define the necessary predicates in Prolog, run mean_fmla/3 on all the example formulas you came up with in problem 2 just above. Make up some other formulas, including some that are false in the structure of problem 1, and run mean_fmla on them, explaining why they get the answers that they do.

Chapter 4

Tabling and Datalog Programming

In the previous chapter we saw several limitations of Prolog. When we considered grammars in Prolog, we found that the parser provided by Prolog "for free" is a recursive descent parser and not one of the better ones that we'd really like to have. When looking at deductive databases, we found that some perfectly reasonable programs go into an infinite loop, for example transitive closure on a cyclic graph. We had to go to some lengths to program around these limitations, and even then the results were not completely satisfying.

XSB implements a feature not (yet) found in many other Prolog systems. It is the notion of tabling, also sometimes called memoization or lemmatization. The idea is very simple: never make the same procedure call twice: the first time a call is made, remember all the answers it returns, and if it's ever made again, use those previously computed answers to satisfy the later request. In XSB the programmer indicates what calls should be tabled by using a compiler directive, such as:

```
:- table np/2.
```

This example requests that all calls to the procedure `np` that has two arguments should be tabled. Predicates that have such declarations in a given program are called tabled predicates.

A simple example of a use of tabling is in the case of a definition of transitive closure in a graph. Assume that we have a set of facts that define a predicate `owes`. The fact `owes(andy,bill)` means that Andy owes Bill some money. Then we use `owes` to define a predicate `avoids` as we did in the previous chapter. A person avoids anyone he or she owes money to, as well as avoiding anyone they avoid.

```
:- table avoids/2.
avoids(Source,Target) :- owes(Source,Target).
avoids(Source,Target) :-
    owes(Source,Intermediate),
    avoids(Intermediate,Target).
```

Here we are assuming that the edges of a directed graph are stored in a predicate `owes/2`. The rules in this program are the same as those used in Prolog to define ancestor. The difference is that in XSB we can make the table declaration, and this declaration guarantees that this predicate will be correctly computed, even if the graph in `owes/2` *is* cyclic. Intuitively it's clear that any call to `avoids` will terminate because there are only finitely many possible calls for any finite graph, and since tabling guarantees that no call is ever evaluated more than once, eventually all the necessary calls will be made and the computation will terminate. The problem with Prolog was that in a cyclic graph the same call was made and evaluated infinitely many times.

Indeed, executing this program on the graph:

```
owes(andy,bill).
owes(bill,carl).
owes(carl,bill).
```

for the query `avoids(andy,X)`, which we saw go into an infinite loop without the table declaration, yields the following under XSB:

```
warren% xsb
XSB Version 1.4.2 (95/4/6)
[sequential, single word, optimal mode]
| ?- [graph].
[Compiling ./graph]
[graph compiled, cpu time used: 0.589 seconds]
[graph loaded]

yes
| ?- avoids(andy,Y).

Y = bill;

Y = carl;

no
| ?-
```

XSB tabled execution as the execution of concurrent machines

We understood a Prolog evaluation as a set of executing deterministic procedural machines, increasing in number as one of them executes a multiply-defined procedure, and decreasing in number as one of them encounters failure. Then we saw how it was implemented by means of a depth-first backtracking search through the tree of SLD computations, or procedure evaluations. To add the

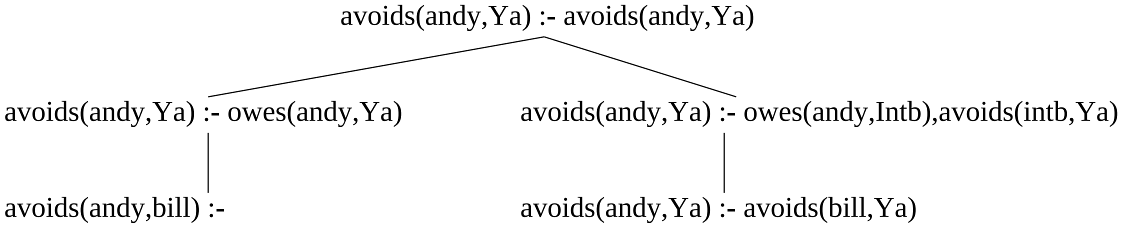

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- owes(andy,Intb),avoids(intb,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- avoids(bill,Ya)

Figure 4.1: Tree for avoid(andy,Ya) goal server

concept of tabling, we have to extend our computational model. Tabling execution is best understood as computation in a concurrent programming language. Nontabled predicates are evaluated exactly as in SLD, with the intuition of a procedure call. Evaluating a tabled predicate is understood as sending the goal to a cacheing goal server and then waiting for it to send back the answers. If no server for the goal exists, then one is created and begins (concurrently) to compute and save answers. If a server for this goal already exists, none needs to be started. When answers become available, they can be (and eventually will be) sent back to all waiting requesters. So tabled predicates are processed "asynchronously", by servers that are created on demand and then stay around forever. On creation, they compute and save their answers (eliminating duplicates), which they then will send to anyone who requests them (including, of course, the initiating requester.)

Now we can see tabled execution as organized around a set of servers. Each server evaluates a nondeterministic procedural program (by a depth-first backtracking search through the alternatives) and interacts with other servers asynchronously by requesting answers and waiting for them to be returned. For each answer returned from a server, computation continues with that alternative.

The abstraction of Prolog computation was the SLD tree, a tree that showed the alternative procedural machines. We can extend that abstraction to tabled Prolog execution by using multiple SLD trees, one for each goal server.

Let's trace the execution of the program for reachability on the simple graph in `owes/2`, given the query `:- avoids(andy,Ya)`. Again, we start with a query and develop the SLD tree for it until we hit a call to a tabled predicate. This is shown in Figure 4.1. Whereas for SLD trees, we used a pseudo predicate `ans` to collect the answers, for SLD trees for goal servers, we will use the entire goal to save the answer, so the left-hand-side of the root of the SLD tree is the same as the right-hand-side. Computation traces this tree in a left-to-right depth-first manner.

So the initial global machine state, or configuration, is:

```
avoids(andy,Ya) :- avoids(andy,Ya).
```

Then rules are found which match the first goal on the right-hand-side of the rule. In this case there are two, which when used to replace the expanded literal, yield two children nodes:

```
avoids(andy,Ya) :- owes(andy,Ya).
```

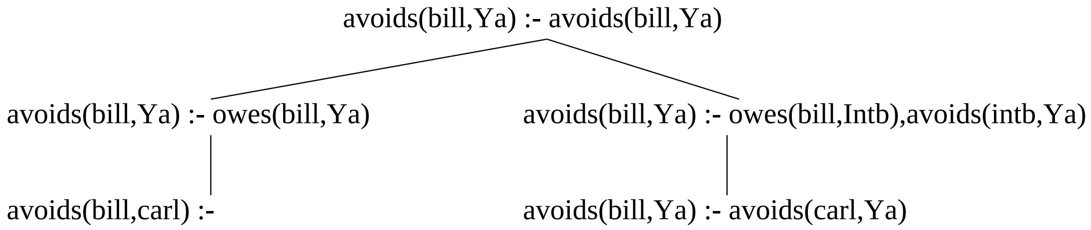

avoids(bill,Ya) :- avoids(bill,Ya)
avoids(bill,Ya) :- owes(bill,Ya)
avoids(bill,Ya) :- owes(bill,Intb),avoids(intb,Ya)
avoids(bill,carl) :-
avoids(bill,Ya) :- avoids(carl,Ya)

Figure 4.2: Tree for avoid(bill,Ya) goal server

```
  avoids(andy,Ya) :- owes(andy,Intb),avoids(Intb,Ya).
```

Computation continues by taking the first one and expanding it. Its selected goal (the first on the right-hand side) matches one rule (in this case a fact) which, after replacing the selected goal with the (empty) rule body, yields:

```
  avoids(andy,bill) :-
```

And since the body is empty, this is an answer to the original query, and the system could print out `Y=bill` as an answer.

Then computation continues by using the stack to find that the second child of the root node:

```
  avoids(andy,Ya) :- owes(andy,Intb),avoids(Intb,Ya).
```

should be expanded next. The selected goal matches with the fact `owes(andy,bill)` and expanding with this results in:

```
  avoids(andy,Ya) :- avoids(bill,Ya).
```

Now the selected goal for this node is `avoids(bill,Ya)`, and `avoids` is a tabled predicate. Therefore this goal is to be solved by communicating with its server. Since the server for that goal does not exist, the system creates it, and schedules it to compute its answers. This computation is shown in Figure 4.2.

This computation sequence for the goal `avoid(bill,Y)` is very similar to the previous one for `avoid(andy,Y)`. The first clause for `avoids` is matched, followed by the one fact for `owes` that has `bill` as its first field, which generates the left-most leaf of the tree of the figure. This is an answer which this (concurrently executing) server could immediately send back to the requesting node in Figure 4.1. Alternatively, computation could continue in this server to finish the tree pictured in Figure 4.2, finally generating the right-most leaf:

```
avoids(bill,Ya) :- avoids(carl,Ya)
```

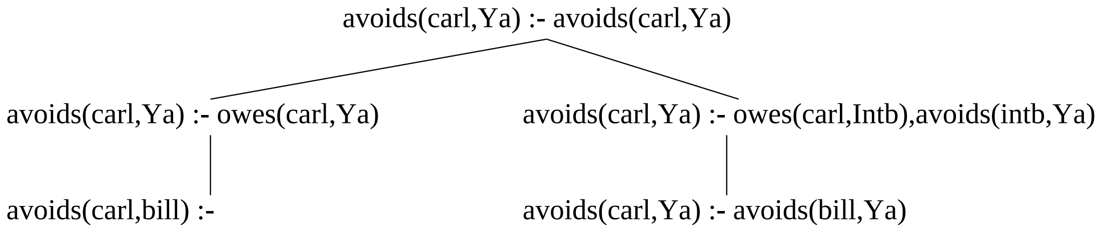

avoids(carl,Ya) :- avoids(carl,Ya)
avoids(carl,Ya) :- owes(carl,Ya)
avoids(carl,Ya) :- owes(carl,Intb),avoids(intb,Ya)
avoids(carl,bill) :-
avoids(carl,Ya) :- avoids(bill,Ya)

Figure 4.3: Tree for avoids(carl,Ya) goal server

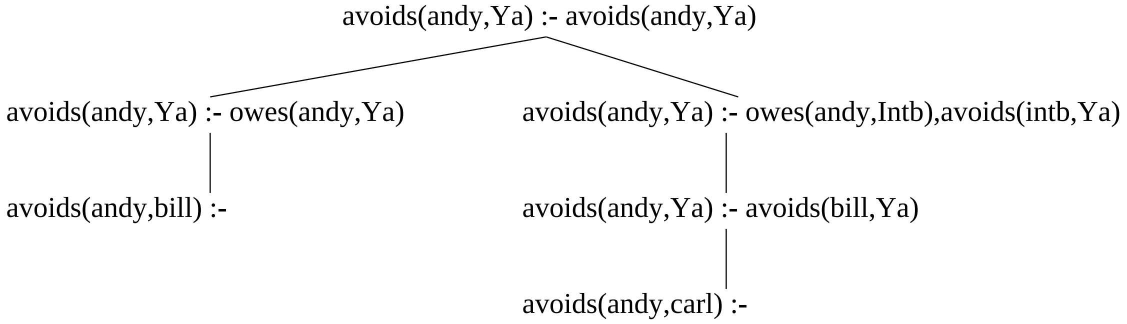

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- owes(andy,Intb),avoids(intb,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- avoids(bill,Ya)
avoids(andy,carl) :-

Figure 4.4: Updated tree for avoid(andy,Ya) goal server

Now since the selected goal here is tabled, the server for it is queried and its response is awaited. Again, there is no server for this goal yet, so the system creates one and has it compute its answers. This computation is shown in Figure 4.3.

This computaton is beginning to look familiar; again the form of the computation tree is the same (because only one clause matches `owes(carl,Y)`). Again an answer, `avoids(carl,bill)`, is produced (and is scheduled to be returned to its requester) and computation continues to the right-most leaf of the tree with the selected goal of `avoids(bill,Ya)`. This is a tabled goal and so will be processed by its server. But now the server *does* exist; it is the one Figure 4.2. Now we can continue and see what happens when answers are returned from servers to requesters. Note that exactly *when* these answers are returned is determined by the scheduling strategy of our underlying concurrent language. We have thus far assumed that the scheduler schedules work for new servers before scheduling the returning of answers. Other alternatives are certainly possible.

Now in our computation there are answers computed by servers that need to be sent back to their requesters. The server for `avoids(bill,Ya)` (in Figure 4.2) has computed an answer `avoids(bill,carl)`, which it sends back to the server for `avoids(andy,Ya)` (in Figure 4.1). That adds a child to the rightmost leaf of the server's tree, producing the new tree shown in Figure 4.4. Here the answer (`avoids(bill,carl)`) has been matched with the selected goal (`avoids(bill,Ya)`) giving a value to `Ya`, and generating the child `avoids(andy,carl) :-`. Note that this child is a new answer for this server.

Computation continues with answers being returned from servers to requesters until all answers have been sent back. Then there is nothing left to do, and computation terminates. The trees of the three servers in the final state are shown in Figure 4.5. Duplicate answers may be generated (as we see in each server) but each answer is sent only once to each requester. So duplicate answers are eliminated by the servers.

Let's be more precise and look at the operations that are used to construct these server trees. We saw that the SLD trees of Prolog execution could be described by giving a single rule, Program Clause Resolution, and applying it over and over again to an initial node derived from the query. A similar thing can be done to generate sets of server trees that represent the computation of tabled evaluation. For this we need three rules:

Definition 4.0.1. (Program Clause Resolution) Given a tree with a node labeled $A : -A_1, A_2, \ldots, A_n$, which is either a root node of a server tree or A_1 is not indicated as tabled. Also given a rule in the program of the form $H : -B_1, B_2, \ldots, B_k$, (with all new variables) and given that H and B_1 match with matching variable assignment θ, then add a new node as a child of this one and label it with $(A : -B_1, B_2, \ldots, B_k, A_2, \ldots, A_n)\theta$, if it does not already have a child so labeled. Note that the matching variable assignment is applied to all the goals in the new label.

Definition 4.0.2. (Subgoal Call) Given a nonroot node with label $A : -A_1, A_2, \ldots, A_n$, where A_1 is indicated as tabled, and there is no tree with root $A_1 : -A_1$, create a new tree with root $A_1 : -A_1$.

Definition 4.0.3. (Answer Clause Resolution) Given a non-root node with label $A : -A_1, A_2, \ldots, A_n$, and an answer of the form $B : -$ in the tree for A_1, then add a new node as child of this node labeled by $(A : -A_2, \ldots, A_n)\theta$, where θ is the variable assignments obtained from matching B and A_1 (if there is not already a child with that label.)

So for example the trees in Figure 4.5 are constructed by applying these rules to the initial tree (root) for the starting goal. XSB can be understood as efficiently constructing this forest of trees. We have seen that XSB with tabling will terminate on a query and program for which Prolog will loop infinitely. It turns out that this is not just an accident, but happens for many, many programs. For example, here we've written the transitive closure of owes using a right recursive rule, i.e., the recursive call to `avoids` follows the call to `owes` in the second rule defining `avoids`. We could also define `avoids` with a rule that has a call to `avoids` before a call to `owes`. That definition would not terminate in Prolog for any graph, but with tabling, it is easily evaluated correctly.

4.1 More on Transitive Closure

We saw in the previous section how XSB with tabling will correctly and finitely execute a transitive closure definition even in the presence of cyclic data. Actually, this is only a simple example of the power of tabled evaluation.

We can write another version of transitive closure:

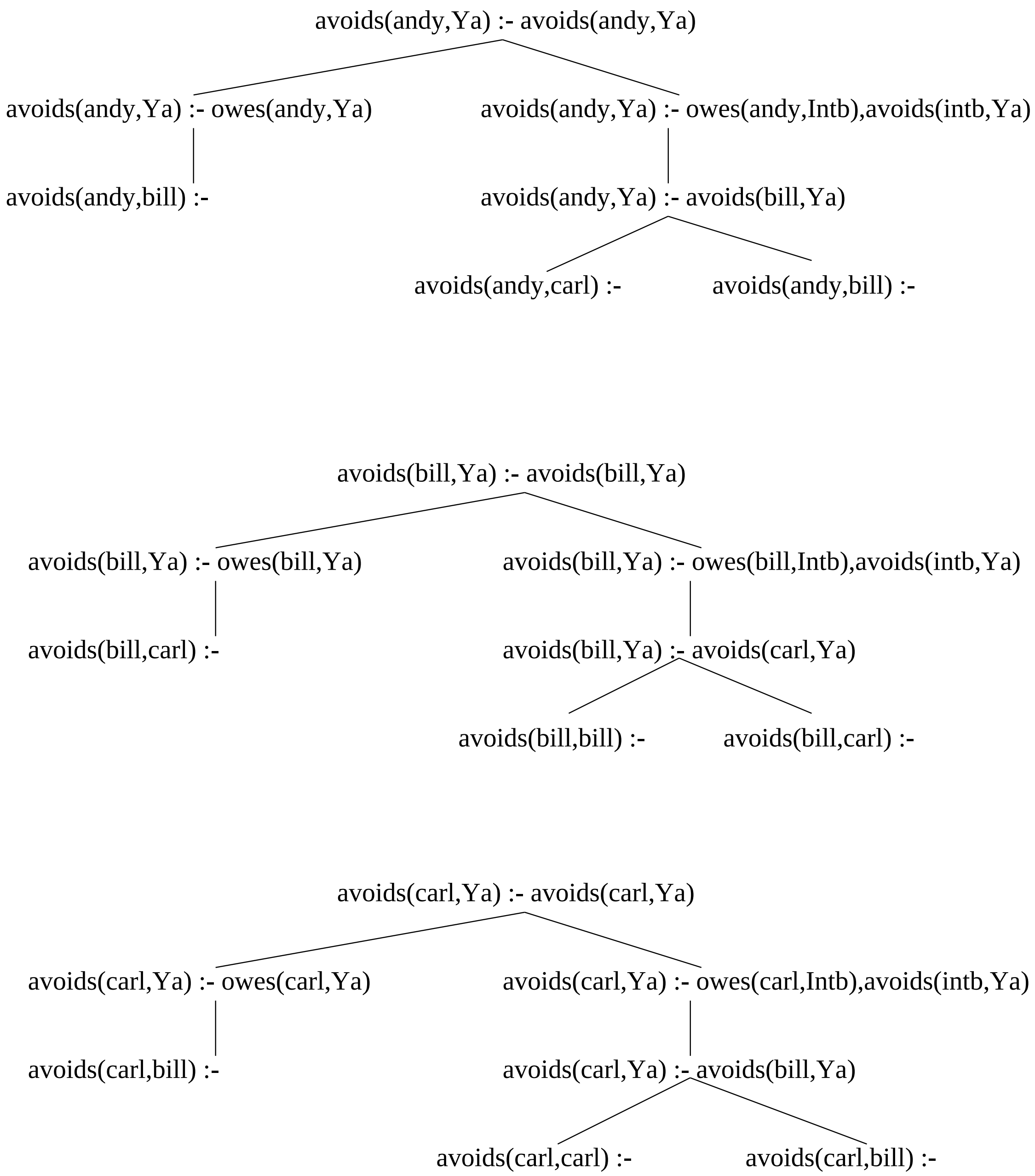

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- owes(andy,Intb),avoids(intb,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- avoids(bill,Ya)
avoids(andy,carl) :-
avoids(andy,bill) :-
avoids(bill,Ya) :- avoids(bill,Ya)
avoids(bill,Ya) :- owes(bill,Ya)
avoids(bill,Ya) :- owes(bill,Intb),avoids(intb,Ya)
avoids(bill,carl) :-
avoids(bill,Ya) :- avoids(carl,Ya)
avoids(bill,bill) :-
avoids(bill,carl) :-
avoids(carl,Ya) :- avoids(carl,Ya)
avoids(carl,Ya) :- owes(carl,Ya)
avoids(carl,Ya) :- owes(carl,Intb),avoids(intb,Ya)
avoids(carl,bill) :-
avoids(carl,Ya) :- avoids(bill,Ya)
avoids(carl,carl) :-
avoids(carl,bill) :-

Figure 4.5: Final state for all goal servers for query avoids(andy,Ya)

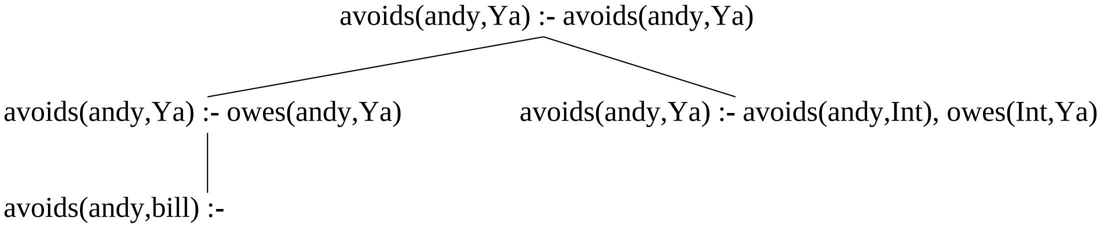

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- avoids(andy,Int), owes(Int,Ya)
avoids(andy,bill) :-

Figure 4.6: Beginning of evaluation of avoids(andy,Ya) for left-recursive transitive closure definition

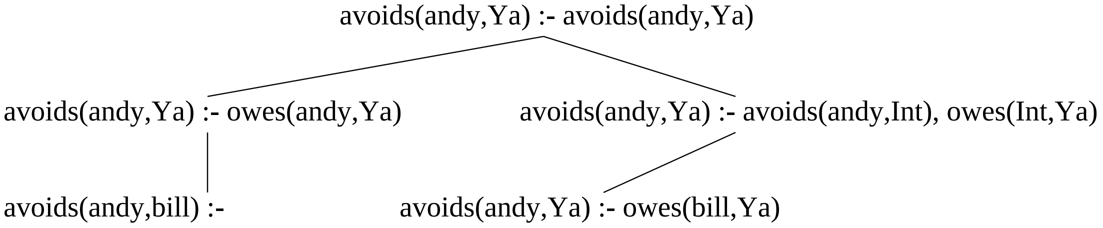

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- avoids(andy,Int), owes(Int,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- owes(bill,Ya)

Figure 4.7: More of the evaluation of avoids(andy,Ya) for left-recursive transitive closure definition

```
:- table avoids/2.
avoids(Source,Target) :- owes(Source,Target).
avoids(Source,Target) :-
    avoids(Source,Intermediate),
    owes(Intermediate,Target).
```

This one is left recursive. A Prolog programmer would not consider writing such a definition, since in Prolog it is guaranteed to be nonfinite. But with tabling, this definition works fine. As a matter of fact, it is generally a more efficient way to express transitive closure than is right recursion. In this section we will look at various versions of transitive closure and compare their efficiency.

Let's consider the evaluation of the same `avoids` query on the same `owes` data as above, but using the left-recursive definition of avoids.

Figure 4.6 shows the state of the initial server when it first encounters a request to a server. Note that this time the request to a server is to the server for `avoids(andy,Ya)`, and this is the server itself. (The names of the variables don't matter when finding a server; they are just "placeholders", so any server with the same arguments with the same pattern of variables works.) The server does have an answer already computed, so it can send it back to the requester (itself), and that results in the tree of Figure 4.7. Now the new leaf, created by the returned answer, can be expanded (by Program Clause Resolution) yielding a new answer, `avoids(andy,carl)`. This answer can be returned to the (only) requester for this server, and that generates a second child for the requester node; this state is shown in Figure 4.8.

Now this node can be expanded (by Program Clause Resolution) to obtain the tree of Figure 4.9 Here we have generated another answer, but it is the same as one we've already generated, so returning it to the requester node will *not* generate any new children. All operations have been

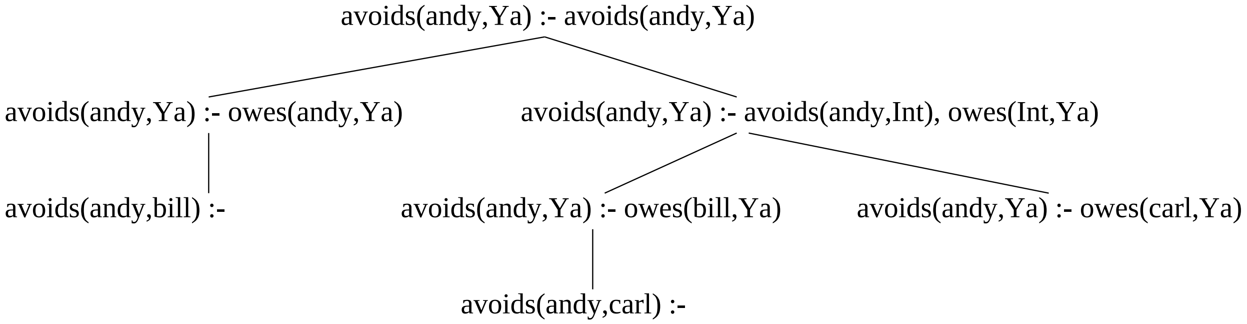

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- avoids(andy,Int), owes(Int,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- owes(bill,Ya)
avoids(andy,Ya) :- owes(carl,Ya)
avoids(andy,carl) :-

Figure 4.8: More of the evaluation of avoids(andy,Ya) for left-recursive transitive closure definition

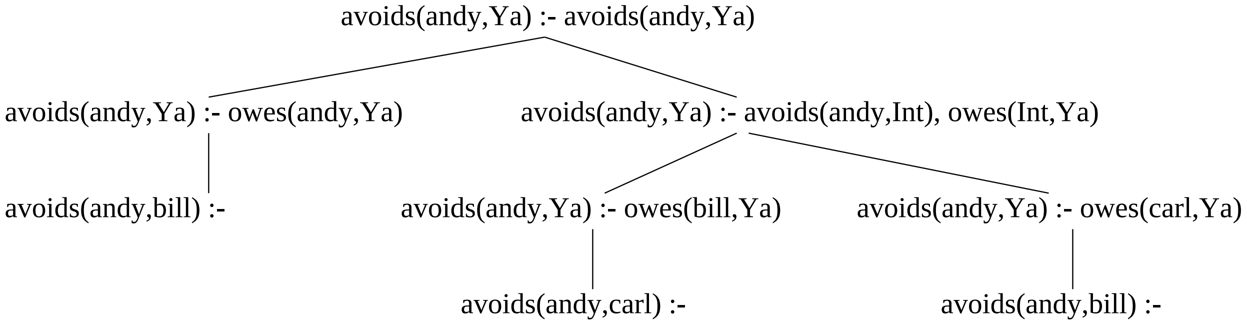

avoids(andy,Ya) :- avoids(andy,Ya)
avoids(andy,Ya) :- owes(andy,Ya)
avoids(andy,Ya) :- avoids(andy,Int), owes(Int,Ya)
avoids(andy,bill) :-
avoids(andy,Ya) :- owes(bill,Ya)
avoids(andy,Ya) :- owes(carl,Ya)
avoids(andy,carl) :-
avoids(andy,bill) :-

Figure 4.9: Final forest for avoids(andy,Ya) for left-recursive transitive closure definition

applied and no more are applicable, so we have reached the final forest of trees, a forest consisting of only one tree. Note that we have the correct two (distinct) answers: that andy avoids bill and andy avoids carl.

The right-recursive definition and the left-recursive definition of avoids both give us the correct answers, but the left-recursive definition (for this query) generates only one tree, whereas the right recursive definition generates several. It seems as though the left-recrsive definition would compute such queries more efficiently, and this is indeed the case.

Consider transitive closure over an `owes` relation that defines a cycle. E.g.,

```
owes(1,2).
owes(2,3).
owes(3,4).
...
owes(99,100).
owes(100,1).
```

defines a graph with a cycle of length 100. How would the trees in the forest look after evaluation of a query to `avoids(1,X)` using the right-recursive transitive closure definition? For each n between 1 and 100, there is a tree with root: avoids(n,Y). And each such tree will have 100 leaf answer nodes. So the forest will have at least 100^2 nodes, and for a cycle of length n the forest would be of size $O(n^2)$.

What does the forest look like if we use the left-recursive definition? It has one tree with root, `avoids(1,Y)`, and that tree has 100 answer leaves. Generalizing from the tree of Figure 4.9, we see that it is a very flat tree, and so for a cycle of length n, the tree (forest) would be of size $O(n)$. The left-recursive definition is indeed the more efficient to compute with. Indeed the complexity of a single-source query to the left-recursive version of transitive closure is linear in the number of edges in the graph reachable from the source node.

4.2 Other Datalog Examples

In the previous section we saw how tabling can finitely process certain programs and queries for which Prolog would go into an infinite loop. Tabling can also drastically improve the efficiency of some terminating Prolog programs. Consider reachability in a DAG. Prolog will terminate, but it may retraverse the same subgraph over and over again.

Let's reconsider the mostly linear `owes` graph at the end of the previous chapter (shown in Figure 2.2) on which Prolog had exponential complexity. Consider evaluating the query `avoids(andy,X)` with the left-recursive tabled definition of transitive closure. The forest for this evaluation will again consist of a single tree, and that tree will be very flat, similar in form to the one of Figure 4.9. Thus tabled evaluation will take linear time. So this is an example in which Prolog (with its right recursive definition) will terminate, but take exponential time; XSB with the left-recursive definition and tabling will terminate in linear time.

The "doubly-connected linear" graph used here may seem unusual and specially chosen, but the characteristics of the graph that cause Prolog to be exponential are not that unusual. Many naturally occurring directed graphs have multiple paths to the same node, and this is what casues the problem for Prolog. [For example, consider a graph (generated by graph-base [**?**]) that places 5-letter English words in a graph with an edge between two words if one can be obtained from the other by changing a single letter.... (get example from Juliana, and see how it works.)

Transitive closure is perhaps the most common example of a recursive query in Datalog, but other query forms can be encountered. Consider the definition of `same_generation`. Given binary relations `up` and `down` on nodes, define a binary relation on nodes that associates two nodes if one can be reached from the other by going n steps up and then n steps down, for some n. The program is:

```
same_generation(X,X).
same_generation(X,Y) :-
    up(X,Z1),
    same_generation(Z1,Z2),
    down(Z2,Y).
```

The name of the predicate arises from the fact that if we let up be defined by a "parent_of" relation and down be defined by the "child_of" relation, then same_generation/2 defines people in the same generation.

[to be continued...]

4.3 Some Simple Graph Problems

4.3.1 Stongly Connected Components in a DAG

Consider the problem of finding connected components in a directed graph. Assume we have a node and we want to find all the nodes that are in the same connected component as the given node.

The first thought that comes to mind is: given a node X, find those nodes Y that are reachable from X and from which you can get back to X.

```
sameSCC(X,Y) :- reach(X,Y), reach(Y,X).
```

So we will assume that edges are given by an edge/2 relation:

```
:- table reach/2.
reach(X,X).
reach(X,Y) :- reach(X,Z), edge(Z,Y).
```

Indeed given a node X, this will find all nodes in the same strongly connected component as X, however it will in general take $O(n * e)$ time, where n is the number of nodes and e is the number of edges. The reason is that given an X, there are n possible Z values and for each of them, we will find everything reachable from them, and each search can take $O(e)$ time.

However, we can do better. It is known that this problem can be solved in $O(e)$ time. The idea is, given a node X, to find all nodes reachable from X following edges forward. Then find all nodes reachable from X following edges backward (i.e., follow edges against the arrow.) Then intersect the two sets. That will be the set of nodes in X's SCC, because if Y is in both these sets, you can follow the edges forward from X to Y and then since there is also a backwards path from X to Y, there is forward path from Y to X, so you can get from X to Y and back to X following edges forward. So the program that does this is:

```
% sameSCC(+X,-Y)
sameSCC(X,Y) :- reachfor(X,Y), reachback(X,Y).
```

```
:- table reachfor/2, reachback/2.
reachfor(X,X).
reachfor(X,Y) :- reachfor(X,Z),edge(Z,Y).

reachback(X,X).
reachback(X,Y) :- reachback(X,Z),edge(Y,Z).
```

Let's now consider its complexity to see why it is $O(e)$. For a fixed value X, the computation of the query `reachfor(X,Y)` takes $O(e)$ time. Then we may have $O(n)$ calls to `reachback(X,Y)` (one for each Y) but they all use one underlying call to `reachback(X,_)` which takes $O(e)$ and is done only once. So when we add all that up (assuming a connected graph), we get $O(e)$ time.

(NOTE:DSW expand with ideas for when back-edge needs to be relative to nodes reachable from a source; as when edge is a state transition function. Need subsumption, but can use a "poor man's" subsumption... It only uses a subsuming call if that call is already completed.)

4.3.2 Connected Components in an Undirected Graph

Another problem is to find connected components in an undirected graph. The usual procedural algorithm is linear in the number of edges. One starts by ordering the nodes. Then proceed by taking the next unmarked node, calling it a leader, and marking it and all nodes reachable from it. This is continued until all nodes are marked.

This seems to be a difficult problem to solve in linear time with a pure datalog program. The following program solves this by using an "inflationary not" operator, which is definable using XSB primitives (as is shown.)

(NOTE:DSW–fix, expand,or delete.)

```
:- table leader/2.
leader(N,T) :-
for(I,1,31),    % total number of nodes = 31
inot(leader(I,_)),  % not yet determined whether leader or not
(N=I, T=true    % so it must be a leader
         ;
 reachable(I,N), T=false  % and then mark all reachable as nonleaders.
        ).

inot(Q) :-
excess_vars(Q,[],[],Vars),
get_calls(Q,S,R),
is_most_general_term(Vars),
get_returns(S,R),
!,
fail.
inot(_).

for(I,I,H) :- I =< H.
for(I,L,H) :- L < H, L1 is L+1, for(I,L1,H).


:- table reachable/2.
reachable(X,Y) :- edge(X,Y).
reachable(X,Y) :- reachable(X,Z),edge(Z,Y).
```

4.4 Genome Examples

[We need to get the semantics of the genome queries from Tony. Does anybody remember?]

4.5 Subsumptive Tabling

[Maybe introduce subsumptive tabling here? Then later chapter on applications of it. Maybe Anderson(?) pointer analysis. And meta-interpreter to see it's doing "linear" bottom-up evaluation of propositional horn clauses, when called with open call.]

4.6 Inferring When to Table

Up to now whenever we wanted calls to a predicate to be tabled, we explicitly coded a table directive to indicate the specific predicate to table. There is a facility in XSB for the programmer to direct the system to choose what predicates to table, in which case the system will generate table directives automatically. There are two directives that control this process: `auto_table/0` and `suppl_table/1`. When such a directive is placed in a source file, it applies to all predicates in that file when it is compiled.

`auto_table/0` causes the compiler to table enough predicates to avoid infinite loops due to redundant procedure calls. The current implementation of `auto_table` uses the call graph of the program. There is a node in the call graph of a program for each predicate, `P/N`, that appears in the program. There is an edge from node for predicate `P/N` to the node for predicate `Q/M` if there is a rule in the program with an atom with predicate `P/N` in the head and a literal with predicate `Q/M` in the body. The algorithm constructs the call graph and then chooses enough predicates to table to ensure that all loops in the call graph are broken. The algorithm, as currently implemented in XSB, finds a minimal set of nodes that breaks all cycles. The algorithm can be exponential in the number of predicates in the worst case[1]. If the program is a Datalog program, i.e., it has no recursive data structures, then `auto_table` *is* guaranteed to make all queries to it terminate finitely. Termination of general programs is, of course, undecidable, and `auto_table` may or may not improve their termination characteristics.

The goal of `auto_table` is to guarantee termination of Datalog programs, but there are other uses tabling. Tabling can have a great effect on the efficiency of terminating programs. Example 4.6.1 illustrates how a multiway join predicate can use tabling to eliminate redundant subcomputations.

Example 4.6.1. *Consider the following set of relations describing a student database for a college:*

1. *student(StdId,StdName,Yr): Student with ID* `StdId` *and name* `StdName` *is in year* `Yr`*, where year is 1 for freshman, 2 for sophomores, etc.*
2. *enroll(StdId,CrsId): Student with ID* `StdId` *is enrolled in the course with number* `CrsId`*.*
3. *course(CrsId,CrsName): Course with number* `CrsId` *has name* `CrsName`*.*

We define a predicate `yrCourse/2`*, which, given a year, finds all the courses taken by some student who is in that year:*

```
yrCourse(Yr,CrsName) :-
    student(StdId,_,Yr), enroll(StdId,CrsId), course(CrsId,CrsName).
```

Note that it will most likely be the case that if one student of a given year takes a course then many students in the same year will take that same course. Evaluated directly, this definition will result in the course name being looked up for every student that takes a given course, not just for every course taken by some student. By introducing an intermediate predicate, and tabling it, we can elminate this redundancy:

```
yrCourse(Yr,CrsName) :-
    yrCrsId(Yr,CrsId), course(CrsId,CrsName).

:- table yrCrsId/2.
yrCrsId(Yr,CrsId) :-
    student(StdId,_,Yr), enroll(StdId,CrsId).
```

The intermediate predicate `yrCrsId` *is tabled and so will eliminate duplicates. Thus* `course` *will only be accessed once for each course, instead of once for each student. This can make a very large difference in evaluation time.*

[1] The algorithm to find such a minimal set of predicates corresponds to the *feedback vertex set* problem and is NP-Complete [2].

In this example a table has been used to eliminate duplicates that arise from the database operations of a join and a projection. Tables may also be used to eliminate duplicates arising from unions.

The `suppl_table/1` directive is a means by which the programmer can ask the XSB system to perform such factoring automatically. The program:

```
:- edb student/3, enroll/2, course/2.
:- suppl_table(2).
yrCourse(Yr,CrsName) :-
    student(StdId,_,Yr), enroll(StdId,CrsId), course(CrsId,CrsName).
```

will automatically generate a program equivalent to the one above with the new intermediate predicate and the table declaration.

To understand precisely how `suppl_table` works, we need to understand some distinctions and definitions of deductive databases. Predicates that are defined by sets of ground facts can be designated as *extensional predicates.* The extensional predicates make up the extensional database (EDB). The remaining predicates are called *intensional predicates*, which make up the intensional database (IDB), and they usually have definitions that depend on the extensional predicates. In XSB the declaration:

```
          :- edb student/3, enroll/2, course/2.
```

declares three predicates to be extensional predicates. (Their definitions will have to be given elsewhere.) We define the data dependency count of an IDB clause to be the number of tabled IDB predicate it depends on plus the number of EDB predicates it depends on (*not* through a tabled IDB predicate.) The command:

```
      :- suppl_table(2).
```

instructs XSB to factor any clause which has a data dependency count of greater than two. In Example 4.6.1 the data dependency count of the original form of `join/2` is three, while after undergoing supplementary tabling, its count is two. Choosing a higher number for `suppl_table` results in less factoring and fewer implied table declarations.

The next subsection describes somewhat more formally how these transformations affect the worst-case complexity of query evaluation.

On the Complexity of Tabled Datalog Programs

The worst-case complexity of a Datalog program (with every predicate tabled) is:

$$\sum_{clause} (len(clause) + k^{num\_of\_vars(body(clause))})$$

where k is the number of constants in the Herbrand base (i.e., in the program). One can see how this can be achieved by making all base relations to be cross products of the set of constants in the program. Assume the call is completely open. Then if there are v_1 variables in the first subgoal, there will be k^{v_1} tuples. Each of theses tuples will be extended through the second subgoal, and consider how many tuples from the second subgoal there can be: k^{v_2} where v_2 is the number of variables appearing in the second subgoal and not appearing in the first. So to get through the second subgoal will take time $k^{v_1} * k^{v_2}$. And similarly through the entire body of the clause. Each subgoal multiplies by a factor k^v where v is the number of new variables. And every variable in the body of the clause is new once and only once. This is the reason for the second component in the summation above. The first component is just in case there are no variables in the clause. For an entire program one can see that the complexity (for a nonpropositional) datalog program is $O(k^v)$ where v is the maximum number of variables in the body of any clause.

We can use folding to try to improve the worst-case efficiency of a Datalog program. Consider the query:

```
(7) :- p(A,B,C,D),q(B,F,G,A),r(A,C,F,D),s(D,G,A,E),t(A,D,F,G).
```

It has 7 variables (as indicated by the number in parentheses that precedes the query), so its worst-case efficiency is $O(n^7)$. However, we can fold the first two subgoals by introducing a new predicate, obtaining the following program:

```
(6) :- f1(A,C,D,F,G),r(A,C,F,D),s(D,G,A,E),t(A,D,F,G).
(6) f1(A,C,D,F,G) :- p(A,B,C,D),q(B,F,G,A).
```

This one has a maximum of 6 variables in the query or in the right-hand-side of any rule, and so has a worst-case complexity of $O(n^6)$.

We can do a couple of more folding operations as follows:

```
(5) :- f2(A,D,F,G),s(D,G,A,E),t(A,D,F,G).
(5) f2(A,D,F,G) :- f1(A,C,D,F,G),r(A,C,F,D).
(6) f1(A,C,D,F,G) :- p(A,B,C,D),q(B,F,G,A).

(4) :- f2(A,D,F,G),f3(D,G,A),t(A,D,F,G).
(4) f3(D,G,A) :- s(D,G,A,E).
```

```
(5) f2(A,D,F,G) :- f1(A,C,D,F,G),r(A,C,F,D).
(6) f1(A,C,D,F,G) :- p(A,B,C,D),q(B,F,G,A).
```

Thus far, we have maintained the order of the subgoals. If we allow re-ordering, we could do the following. For each variable, find all the variables that appear in some subgoal that it appears in. Choose the variable so associated with the fewest number of other variables. Factor those subgoals, which removes that variable (at least). Continue until all variables have the same number of associated variables.

Let's apply this algorithm to the initial query above. First we give each variable and the variables that appear in subgoals it appears in.

```
A:BCDEFG
B:ACDFG
C:ABDF
D:BCDEFG
E:ADG
F:ABGCD
G:ABFDE
```

Now E is the variable associated with the fewest number of other variables, so we fold all the literals (here only one) containing E, and obtain the program:

```
(6) :- p(A,B,C,D),q(B,F,G,A),r(A,C,F,D),f1(D,G,A),t(A,D,F,G).
(4) f1(D,G,A) :- s(D,G,A,E).
```

Now computing the new associated variables for the first clause, and then choosing to eliminate C, we get:

```
A:BCDFG
B:ACDFG
C:ABDF
D:ABCFG
F:ABGCD
G:ABFD

(5) :- f2(A,B,D,F),q(B,F,G,A),f1(D,G,A),t(A,D,F,G).
(4) f1(D,G,A) :- s(D,G,A,E).
(5) f2(A,B,D,F) :- p(A,B,C,D),r(A,C,F,D).
```

Now computing the associated variables for the query, we get:

```
a:bdfg
b:adfg
d:abfg
f:abdg
g:abfd
```

All variables are associated with all other variables, so no factoring can help the worst-case complexity, and the complexity is $O(k^5)$.

However, there is still some factoring that will eliminate variables, and so might improve some queries, even though it doesn't guarantee to reduce the worst-case complexity.

```
(4) :- f3(A,D,F,G),f1(D,G,A),t(A,D,F,G).
(5) f3(A,D,F,G) :- f2(A,B,D,F),q(B,F,G,A).
(4) f1(D,G,A) :- s(D,G,A,E).
(5) f2(A,B,D,F) :- p(A,B,C,D),r(A,C,F,D),

(3) :- f4(A,D,G),f1(D,G,A).
(4) f4(A,D,G) :- f3(A,D,F,G),t(A,D,F,G).
(5) f3(A,D,F,G) :- f2(A,B,D,F),q(B,F,G,A).
(4) f1(D,G,A) :- s(D,G,A,E).
(5) f2(A,B,D,F) :- p(A,B,C,D),r(A,C,F,D).
```

The general problem of finding an optimal factoring is conjectured to be NP hard. (Steve Skiena has the sketch of a proof.)

4.7 Datalog Optimization in XSB

[Do we want to do it at all, and if so, here?] I think we do want it, but I don't know about here.

4.8 Table Builtins

table builtins: get_calls, get_returns, abolish_all_tables, ...

trie_assert, trie_retract (or maybe in section on large files, or maybe in a separate chapter on indexing.)

Do example to extract parses from a table created by recognition of a string in a CF grammar. (How to do, maybe interpreter.)

As examples, how about suspend / resume (which Rui has working at least partly) and the cursors / server example?

4.9 Exercises

1. In this problem you will explore how XSB handles transitive closure for binary relations defining various graphs. The first task is to define predicates that generate those graphs. In each case, we will assume that the graph predicate will always be called with the first argument bound. Each graph will be parameterized by a number M, which will be defined by a Prolog fact: size/1. So if M is 10000, there will be a fact: size(10000). defined in your Prolog program.

   (a) Define a binary predicate cycle/2 such that cycle(i,i+1) for

   ```
   0 =< i < M, and cycle(M,0).
   ```

   Draw a picture of this graph (with integers as nodes) for M = 8.

   (b) Define a binary predicate tree/2 such that

   ```
   tree(i,2*i+1) for 0 =< i < M, and
   tree(i,2*i+2) for 0 =< i < M.
   ```

   Draw a picture of this graph (with integers as nodes) for M = 4.

   (c) Define a binary predicate dline/2 such that

   ```
   dline(i,i+1) if 0 =< i < M and i mod 4 = 0  (i.e. i is a multiple of 4)
   dline(i,i+2) if 0 =< i < M and i mod 4 = 0
   dline(i+1,i+4) if 0 =< i < M and i mod 4 = 0
   dline(i+2,i+4) if 0 =< i < M and i mod 4 = 0
   ```

   Draw a picture of this graph (with integers as nodes) for M = 12.

2. For each of the predicates define in problem 1, give the rules for transitive closure of that graph, possible in three ways: the right-recursive definition, the left-recursive definition, and the doubly-recursive definition. They should be named:

   ```
   tc_cycle_rr/2 (for the right recursive transitive closure of cycle)
   tc_cycle_lr/2 (for the left recursive..)
   tc_cycle_dr/2 (for the doubly recursive..)

   tc_tree_rr/2  (right recursive tree)
   tc_tree_lr/2  (left recursive tree)

   tc_dline_rr/2 (right recursive)
   tc_dline_lr/2 (left recursive)
   ```

   You are to populate the following benchmarking table:

graph	tabled	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10
tc_cycle_rr/2	Y										
tc_cycle_lr/2	Y										
tc_cycle_dr/2	Y										
tc_tree_rr/2	N										
tc_tree_rr/2	Y										
tc_tree_lr/2	Y										
tc_dline_rr/2	N										
tc_dline_rr/2	Y										
tc_dline_lr/2	Y										

where M1 - M10 are integers, in increasing order that are chosen to show reasonable ranges of cpu times for all the benchmarks. These benchmarks can vary significantly in speed, so to get a reasonable time for the fastest you'll need a reasonably large M10; any time greater than 30 seconds should be represented as TO (for time out). (I.e., cancel any query that takes longer, using ctrl-c.) The "tabled" column having "Y" means that the corresponding transitive closure definition is declared as tabled. If it is "N", it is not tabled, but simple Prolog is used.

The values in the cells are the cpu-times for running XSB on the 0-source query for the row. E.g., the number in the ¡tc_cycle_rr/2, M1¿ cell should be the cputime to run the query tc_cycle_rr(0,X) (when tc_cycle_rr/2 is tabled) to find all X's reachable from 0.

This can be done by the following query:

```
| ?- cputime(T0),(tc_cycle_rr(0,_),fail ; true), cputime(T1),
  Time is T1-T0, writeln(Time).
```

You might find the following predicate helpful:

```
:- dynamic size/1.          % needed at beginning of code!
:- import for/3 from basics.
test(Goal,M) :-
    N is M//10,             % 10 equal intervals
    for(I,1,10),            % for I=0 to 10
    NM is I*N,              % the next decile
    abolish_all_tables,     % clear all tables
    retractall(size(_)),    % clear old size
    assert(size(NM)),       % set new size for base predicate
    cputime(T0a),           % find cputime from start
    (call(Goal),fail;true), % do the query, all answers
    cputime(T1a),
    Timea is T1a-T0a,
    writeln([Goal,NM,Timea]), % write out goal, size, and cputime
    fail.
```

And example call:

```
| ?- test(tc_cycle_lr(0,_),1000000).
```

Chapter 5

Prolog Programs as Inductive Definitions

5.1 Introduction

The evaluation of pure Prolog programs is usually formalized as SLD resolution applied to first-order Horn clauses. This is how we described it in our chapter 3 on First Order Logic. However, there is another, perhaps more basic and fundamental way to understand logic programs by understanding them as providing *Inductive Definitions* of sets of terms. In this formulation, the first order logic of Horn clauses is just one particular interpretation of such inductively defined sets of terms.

This chapter is a slight diversion to explore this alternative way to understand (and formalize) logic programs. [DSW: It may be that this chapter and chapter should be interchanged, with ID being fundamental and FOL being derived.]

We present a new formulation of the foundations of Logic Programming. We emphasize that logic programming is fundamentally programming with inductive definitions. The form of inductive definitions then have a natural interpretation as defining least models of sets of Horn clauses in first-order logic. In this formulation logic is not fundamental; the inductive definition is fundamental, and logic is a derived aspect of the form of the inductive definition.

In this formulation the evaluation of logic programs is computation of the set defined by an inductive definition of a particular form. We will explore various strategies for evaluating these inductive definitions, and will see that these correspond to Horn clause proof strategies in first-order logic.

The contribution of this chapter does not include any new deep technical results, but rather is a re-orientation of how one thinks about logic programming. We provide insights into how logic programming is not simply first-order logic, but whose semantics fundamentally includes the notion of minimality, and whose computation is productively thought of as evaluation of inductively defined sets rather than first-order theorem proving. This presentation builds on and elaborates the foundational contributions of Marc Denecker and his colleagues [**?**].

5.2 Motivation

Logic Programming has from the beginning been understood (and taught) as being based fundamentally on the Horn Clause subset of first-order logic; its semantics based on minimal models; and its evaluation strategy as based on the theorem proof method of SLD resolution. This is clearly good, but it does require various restrictions on the logic, such as the restriction to Herbrand models, and the restriction to minimal models that is required to handle negation. These restrictions complicate the simple elegance of first-order logic. And they can (and have) led to confusions among logic programmers between properties of fixed point logic and traditional first-order logic.

The approach to the foundations of logic programming that is proposed in this chapter is based not on first-order logic, but on inductive definitions, with the relation to logic as arising from a particular correspondence between the particular form of inductive definitions and first-order models. In this formulation the minimality of models is a fundamental property of inductive definitions and not what might seem as a somewhat ad hoc restriction to minimal models in first-order logic. And the correspondence to Herbrand models of first-order logic is natural and obvious.

We do not avoid any of the difficulties/complications of Logic Programming - e.g., all the difficulties of handling negation remain - but we feel that this new approach makes them easier to motivate and understand.

5.3 Informal Review of Inductive Definitions

We are interested in inductively defined *sets*. An inductively defined set is a subset of a carrier set generated by a set of operators.

An operator on a carrier set D is an element of $\mathcal{F}(D) \times D$, where $\mathcal{F}(D)$ denotes the set of finite subsets of D. Elements of the first component of an operator are called *antecedents* of the operator, and the second component is called the *consequent*.

An operator $Op = \langle A, d \rangle$, for $A \subseteq D$, applied to a set $S \subseteq D$, denoted $Op(S)$, is $S \cup \{d\}$ if $A \subseteq S$, and is S otherwise.

A set of operators Ops applied to a set S, denoted $Ops(S)$, is $\bigcup_{Op \in Ops} Op(S)$.

A set $S \subseteq D$ is closed under an operator, Op, if $Op(S) = S$. A set is closed under a set of operators if it is closed under every operator in the set, i.e., $Ops(S) = S$.

Definition 5.3.1. (Inductively defined set) The set inductively defined by a set of operators is the smallest set closed under the operators.

A very simple example is the set of even natural numbers. The carrier set is the natural numbers N. The operators are $\{\langle\{n\}, n+2\rangle : n \in N\} \cup \{\langle\emptyset, 0\rangle\}$. Then the even natural numbers are the smallest set containing the number 0 and closed under the operators. (How could we inductively define the set of *odd* natural numbers?)

We next define the carrier set for Prolog and describe how pure Prolog rules determine operators. The carrier set of interest is the set of finite, labeled, ordered trees.

5.4 Finite, Labeled, Ordered Trees

The objects of interest in logic programming are finite, labeled, ordered trees with leaves of atoms, integers, or floats. (We refer to them in the following simply as *trees*.) Each internal node is labeled by an atom, has a finite number of children, and the order of the children of a node is significant.

We use *terms* as a sequential way to represent certain sets of trees. A single tree is represented by using a pre-order traversal, enclosing the sequence of children in parentheses with the children separated by commas. Labels are (singly) quoted ASCII sequences. Atomic leaves are also quoted ASCII sequences; numeric leaves are represented as usual in programming languages.

Definition 5.4.1. (Ground Term) A *ground term* is a string in the following grammar, and represents a finite, labeled, ordered tree:

```
term --> integer | float | atom | atom ( term-sequence )
term-sequence --> term | term , term-sequence
atom --> ' ascii-seq '
atom --> unq-atom
ascii-seq --> ascii | ascii ascii-seq
unq-atom --> ascii-lc | unq-atom asci-alphanum
```

The integer and float terms are written as usual in programming languages, as optionally signed sequences of digits, with a decimal point for floats. If an atom starts with a lower-case alpha character, and contains only alphanumeric characters (and underscores), then the atom need not be quoted. The usual convention is to choose, when convenient, the names of atoms to be unquoted atoms to make them easier to write and avoid the clutter of extraneous quotes.

Example 5.4.1. *The following are examples of ground terms that represent distinct trees:*

```
193
'David S. Warren'
f(a_bc,12,13.5)
f(12,13.5,a_bc)
'+'(5,'*'(7,14))
cons(a,cons(b,cons(c,nil)))
employee(george,sales,1988,50000)
```

We extend the definition of terms to include some that represent (certain) *sets* of trees by introducing "variables."

Definition 5.4.2. (Term) A *term* is a string in the language of the grammar above for *ground terms* augmented by the following rules:

```
term --> variable
variable --> ascii-uc | variable ascii-alphanum
```

A leaf node of a (non-ground) term tree can be a variable. So a variable can appear in the position of any subterm in a term. (Note an internal atomic label cannot be replaced by a variable since it is not a full subterm, but just a piece of one.) A variable is indicated by an unquoted sequence of alphanumeric characters (plus the underscore) that starts with an uppercase alpha character (or underscore).

Example 5.4.2. *Examples of non-ground terms:*

```
X
_variable_1
equal(X,X)
f(Atom,12,13.5)
cons(Head,Tail)
employee(Name,Dept,Birth_Date,Salary)
```

A term that does *not* contain a variable is said to be "ground." A ground term corresponds to a (single) tree. A term with variables corresponds to the set of trees that can be obtained from the term by replacing each variable uniformly with any tree (i.e., ground term). "Uniformly" means that all occurrences of the same variable must be replaced with the same tree. We make this more precise.

Definition 5.4.3. (Substitution) A *substitution* is a function from some set of variables to terms. A *ground substitution* is a function from variables to ground terms.

Example 5.4.3. *Examples of substitutions are as follows, written as sets of pairs, called replacements:*

```
{X->'David', Y->f(aaa,ZZ)}
{Name->george, Dept->sales, Birth_Date->1988, Salary->50000}
{Head->a, Tail->cons(b,cons(c,nil))}
```

The second and third examples are ground substitutions; the first is not.

Definition 5.4.4. (Substitution Instance) The substitution instance of a term T under substitution θ, written $T\theta$, is the result of replacing every variable V that occurs in T with the term that is the image of V under θ.

Definition 5.4.5. (Ground Instance) A *ground instance of term T* is $T\theta$ for some ground substitution θ such that $T\theta$ is a ground term.

Examples:

Example 5.4.4.

```
cons(Head,Tail){Head->a, Tail->cons(b,cons(c,nil))} =
    cons(a,cons(b,cons(c,nil)))

employee(Name,Dept,Birth_Date,Salary)
     {Name->george, Dept->sales, Birth_Date->1988, Salary->50000} =
   employee(george,sales,1988,50000)

equal(X,X){X->bill,Y->george} = equal(bill,bill)
```

5.5 Inductive Definitions of Sets of Trees

We next describe pure Prolog rules (including facts) and use them to define operators on sets of terms. Such a set of operators will inductively define a set of terms in the framework described above.

Definition 5.5.1. (Rule) A *rule* is a string in the language of the following grammar:

```
rule --> term .
rule --> term :- body-terms .
body-terms -> term | term , body-terms
```

The first term of a rule is called the *head* of the rule; the comma-separated sequence of terms after the `:-` symbol is called the *body* of the rule.

Definition 5.5.2. (Inductive Program) An *inductive program* (or simply *program* for short) is a sequence of rules, i.e., a string in the language of the following grammar:

```
program --> rule | rule program
```

Example 5.5.1. *A simple example of a program is:*

```
nil.
cons(aa,Tail) :- Tail.
```

This is a program with two rules (or clauses). The first does not have a body list (and is called a fact*); the second has a body consisting of a single term.*

Definition 5.5.3. (Substitution applied to a rule) (Needed??) The result of applying substitution θ to rule R, denoted by $R\theta$, is the rule obtained by replacing every variable V in R by its image under θ.

Definition 5.5.4. (Rule Operator) A rule $R = t$:- $t_1, t_2, ..., t_n$ determines a set of operators: for every ground instance of R under a substitution θ, there is an operator $\langle\{t_1\theta, t_2\theta, ..., t_n\theta\}, t\theta\rangle$.

Definition 5.5.5. (Rule operators from program P) Let Ops^P be the set of all operators determined by all rules of program P.

Definition 5.5.6. (Meaning of program P, $\mathbf{M^{least}(P)}$) The meaning of a program P is the set inductively defined by its rule operators. I.e., it is the smallest set of trees closed under all the operators of Ops^P.

Here we should establish that there exists such a unique set of terms. This follows from the fact that the intersection of two sets that are closed under a set of operators is closed under those operators. Then we can take the intersection of all such closed sets, which is unique, and is the smallest closed set. Note that the set of all trees is closed under any set of operators, so we have at least one set in the intersction.

This is a fine definition of the meaning of a program, but it doesn't suggest a way to compute it. There will almost certainly be infinitely many sets closed under the operators of a program, so generating and intersecting them would not be reasonable. Fortunately there are other ways to define this same set.

Definition 5.5.7. (Meaning of a program P, $\mathbf{M^{bu}(P)}$) Let
$M_0(P) = Ops^P(\emptyset)$ and
$M_n(P) = Ops^P(M_{n-1}(P))$.
Then let $M^{bu}(P) = \bigcup_{i=0}^{\infty} M_i(P)$

This definition can be used to compute the meaning of a program that generates a finite number of operators. We note that $M_i(P) \subseteq M_{i+1}(P)$ for all i. We first apply all the operators to the empty set, the apply all the operators to that set, and continue in this way applying all operators to the previously generated set obtaining a new set until no new elements are added to the set. This means that all the rest of the sets in the infinite sequence are the same set and we are finished. Since the number of operators is finite, we can only add finitely many elements, at most the consequents of all the operators. The *bu* in the name of the meaning function stands for “bottom-up”. (Do an example.)

We need to prove that $M^{bu}(P) \iff M^{least}(P)$ for all programs P. (This is generally true for any set of operators on any carrier set.)

However, before we prove this, we give yet another definition of this same set, one that uses proof sequences.

Definition 5.5.8. (Proof Sequence) Given a set of operators *Ops* on a carrier set D, a *proof sequence for* $d_n \in D$ *from Ops* is a finite sequence of elements of D, $\langle d_1, d_2, ..., d_n\rangle$, such that for every d_i in the sequence, there is an operator $\langle A, d_i\rangle$ such that $A \subseteq \{d_1, d_2, ..., d_{i-1}\}$. I.e., all elements in the antecedent of the operator appear in the sequence before d_i. Note that the consequent of an operator with an empty antecedent may appear anywhere in the sequence. (The first element must, of course, be such an antecedent.)

Definition 5.5.9. (Meaning of a program P, $\mathbf{M^{prf}(P)}$)
$M^{prf}(P)$ is the set containing exactly every element $d \in D$ for which there exists a proof sequence for d from the operations of P.

Now we should prove that $M^{prf}(P) = M^{bu}(P) = M^{least}(P)$. We will leave that to the reader. It is not very difficult and is a good exercise to become familiar with these definitions.

================

Definition 5.5.10. (T_P Operator) For program P, T_P is an operator from sets of trees to sets of trees, as follows:

$$T_P(S) = \bigcup_{R \in P} T_R(S)$$

Definition 5.5.11. (Semantics of Inductive Programs, M_P) The meaning of a program P is the least fixed point of the operator T_P.

This is well defined, since T_P is a monotonic operator and so is known to have a least fixed point.

Exercise to prove if T_P has a fixed point, it has a (unique) least fixed point.

Example 5.5.2. *Given the program P:*

```
nil.
cons(aa,Tail) :- Tail.
```

the set

```
  {nil, cons(aa,nil), cons(aa,cons(aa,nil)),
    cons(aa,cons(aa,cons(aa,nil))), ...}
```

is a fixed point of T_P, since the first rule generates `nil`*, and the second rule generates for each element in the set, the element following it. Thus all the elements of the set are generated and so this set is a fixed point of T_P. One can see that there is no smaller set that is also a fixed point.*

5.6 Evaluation of Inductive Programs

We provide two different definitions of the least fixed point of the operator T_P, which can be used to compute elements of the fixed point: a) the union of iterations of the program operator, T_P, and b) a proof that a element is in the set. We leave the proof that these are equivalent to the least fixed point definition as an exercise for the reader.

Definition 5.6.1. (Iterated Operator Application) Given a set of trees S, the iterated application of T_P to S, $T_P^i(S)$, is defined as follows:
$T_P^1(S) \;=\; T_P(S)$
$T_P^{i+1}(S) \;=\; T_P(T_P^i(S))$

Definition 5.6.2. (Iterated fixed point of T_P, M_P^{Iter}) For program P, the iterated fixed point of T_P, M_P^{Iter}, is defined as follows:

$$M_P^{Iter} \;=\; \bigcup_{i=1}^{\infty} T_P^i(\emptyset)$$

Example 5.6.1. *Consider the program P:*

```
nil.
cons(aa,Tail) :- Tail.
```

Then $T_P^1(\emptyset) \;=\; \{nil\}$
$T_P^2(\emptyset) \;=\; T_P(\{nil\}) \;=\; \{nil, cons(aa, nil)\}$
$T_P^3(\emptyset) \;=\; T_P(\{nil, cons(aa, nil)\}) \;=\; \{nil, cons(aa, nil), cons(aa, cons(aa, nil))\}$
...
So we can see that M_P^{Iter}, being the union of these sets, is the set

```
{nil, cons(aa,nil), cons(aa,cons(aa,nil)),
  cons(aa,cons(aa,cons(aa,nil))), ...}
```

Next we define a proof of a tree from a program, P.

Definition 5.6.3. (Proof from program P) A *proof of tree t_n from program P* is a sequence of trees, $t_0, t_1, t_2, ..., t_n$ such that for every tree t_i in the sequence, there are trees t_{i_1}, t_{i_2}, ..., t_{i_k} in the sequence preceding t_i such that t_i :- t_{i_1}, t_{i_2}, ..., t_{i_k} is a ground instance of a rule $R \in P$.

Definition 5.6.4. (Trees provable from program P, M_P^{Prf}) $M_P^{Prf} \;=\; \{t :$ There is a proof of t from program $P\}$, i.e., the set of all trees for which there is a proof from P.

Example 5.6.2. *Again consider the program P:*

```
a. nil.
b. cons(aa,Tail) :- Tail.
```

(with rules labeled for reference) and a proof of `cons(aa,cons(aa,cons(aa,nil)))` *from P:*

```
1. nil
2. cons(aa,nil)
3. cons(aa,cons(aa,nil))
4. cons(aa,cons(aa,cons(aa,nil)))
```

with the trees of the proof sequence numbered for reference. Tree `1.` *is justified by rule* `a.`*, which has no body and so does not require any trees earlier in the proof. Tree* `2.` *is justified by* `cons(aa,nil) :- nil`*, a ground instance of rule* `b.`*, whose body,* `nil`*, is tree* `1.` *in the proof. Tree* `3.` *is justified by* `cons(aa,cons(aa,nil)) :- cons(aa,nil)`*, a ground instance of rule* `b.`*, whose body,* `cons(aa,nil)`*, is tree* `2.` *in the proof. Tree* `4.` *is justified by* `cons(aa,cons(aa,cons(aa,nil))) :- cons(aa,cons(aa,nil))`*, a ground instance of rule* `b.`*, whose body,* `cons(aa,cons(aa,nil))`*, is tree* `3.` *in the proof. Thus every tree in the sequence is justified by an instance of a program rule, so the sequence is a proof.*

Note that from any proof, we can easily extract a proof for any tree that appears in the proof simply by deleting the trees that follow it.

Theorem 1. *For every program P,* $M_P = M_P^{Iter} = M_P^{Prf}$.

Proof. Exercise for the reader. □

A few examples:

Example 5.6.3. *We can define the natural numbers with the following program:*

```
  0.
  s(X) :- X.
```

where s(X) is the successor of X. So the natural number n is represented by a linear tree of n internal nodes labeled with s and the single leaf of 0.

Then we can define addition of natural numbers. We'll represent an addition fact of, say, 1+2=3, with the tree `plus(s(0),s(s(0)),s(s(s(0))))`*. We want the subtrees of a plus tree to be restricted to trees that represent natural numbers. So we will define a set of (linear) trees each of whose roots is labeled by* `nat`*, whose child is a natural number tree (as defined above.)*

Then the `plus` *program is:*

```
n1.  nat(0).
n2.  nat(s(X)) :- nat(X).

p1.  plus(0,M,M) :- nat(M).
p2.  plus(s(N),M,s(NM)) :- plus(N,M,NM).
```

Here we have defined the `nat` *trees with the first two rules, and then* `plus` *with the last two rules. The third rule, which builds a* `plus` *tree for adding* 0*, constrains the second (and third) subtree to be a natural number subtree. (We label the rules to allow us to easily refer them them.)*

Example 5.6.4. *We can use this program to define another set of trees, of the form* `mult(N1,N2,N3)`*, where N1, N2, and N3 stand for subtrees representing natural numbers n_1, n_2, and n_3 such that $n_1 * n_2 = n_3$.*

```
m1.  mult(0,M,0) :- nat(M).
m2.  mult(s(N),M,NM) :- mult(N,M,NMProd), plus(M,NMProd,NM).
```

5.6.1 Top-Down Computation

Let's try to determine whether `plus(s(s(0)),s(s(0)),s(s(s(s(0)))))` is in M_P, where P is the program including rules `p1`, `p2`, `n1`, and `n2`. How might we go about automatically determining this? We have two possible approaches: use the iterative fixed point definition or use the definition of proofs. We can try to use either; (Maybe do iterative first? But then want a different example maybe.) Let's start with the proof approach.

Say we're given the `plus` tree above; how might be go about trying to construct a proof for this tree? We can try to construct a proof by starting with the last element of the proof, which we know must be this tree, and then trying to add enough elements earlier in the sequence to construct a valid proof. So for this example, a proof we desire must have the following form:

```
...
...
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

I,e., it must be some sequence of trees whose last element is the tree whose membership in M_P we are trying to determine. This final tree must be "justified" in the proof by a rule and previous trees. The only rule that could justify this term is rule `p2`. And the required instance of that rule must be:

```
plus(s(s(0)),s(s(0)),s(s(s(s(0))))) :- plus(s(0),s(s(0)),s(s(s(0)))).
```

The substitution that makes the head of the rule match the tree to be justified determines this instance. So we can extend our potential proof to look like:

```
...
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

Now the last tree is justified, and since *all* trees in a proof must be justified, we now have to turn to justifying the new tree we just added. Again, the only rule that can justify the second-to-last tree is rule p2 with instance:

```
plus(s(0),s(s(0)),s(s(s(0)))) :- plus(0,s(s(0)),s(s(0))).
```

which allows us to extend our potential partial proof to be:

```
...
plus(0,s(s(0)),s(s(0)))
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

Next we need to justify that third-to-last tree we just added. The only rule that could have this term as an instance of its head is rule p1, and it has instance:

```
plus(0,s(s(0)),s(s(0))) :- nat(s(s(0))).
```

so we extend our potential partial proof to:

```
...
nat(s(s(0)))
plus(0,s(s(0)),s(s(0)))
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

Now we need to justify nat(s(s(0)). This can be done only with rule n2, giving us:

```
...
nat(s(0))
nat(s(s(0)))
plus(0,s(s(0)),s(s(0)))
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

and using the n2 rule again, we get:

```
...
nat(0)
nat(s(0))
nat(s(s(0)))
plus(0,s(s(0)),s(s(0)))
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

And now we can justify `nat(0)` using rule (or actually fact) `n1`. But this fact doesn't require any additional tree to be added to our potential proof in order to be justified. Now since every tree is justified by a rule instance whose body trees are earlier in the sequence, this sequence itself, with no additional trees, is a proof. Therefore we've succeeded in constructing a proof of `plus(s(s(0)),s(s(0)),s(s(s(s(0)))))`; it is:

```
nat(0)
nat(s(0))
nat(s(s(0)))
plus(0,s(s(0)),s(s(0)))
plus(s(0),s(s(0)),s(s(s(0))))
plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
```

And thus we've established that the tree `plus(s(s(0)),s(s(0)),s(s(s(s(0)))))` $\in P$.

This example has many nice properties that a more complex program and query tree might not enjoy. We now ask whether we extend this computational idea of "backward proof construction" to handle those more complicated situations.

The first generalization requires us to handle rules with more than one term in their bodies. In that case we must add all the body trees to our potential proof and justify all of them in turn. This just requires that we keep track of the set of trees that remain to be justified. For example, consider the definition:

```
1.  mathReq :- discreteMath, calc.
2.  discreteMath :- setTheory, logic.
3.  calc :- calc1, calc2, calc3.

4.  setTheory.
5.  logic.
6.  calc1.
7.  linearAlg.
8.  calc2.
9.  calc3.
```

whose rules 1 through 3 define the courses required to satisfy a set of math requirements, and whose facts 6 through 9 define the courses taken by a particular student. If `mathReq` is in M_P for this definition P, then the student has satisfied the math requirements for the major.

We try to generate a proof of `mathReq` from these rules. Note that the rules have multiple body terms (basic trees), so we will need to keep track of which we ones we have started processing to find a justification. We will mark the trees in our developing proof with a "*" if a potential justification for it has not yet been started, but will need to be found. We start with:

```
...
*mathReq
```

The first step is to posit a proof that ends with the tree `mathReq`, which needs to be justified and so is marked with an "*". So we find a rule (rule 1) that might justify `mathReq` and extend the potential proof:

```
...
*discreteMath
*calc
mathReq
```

We remove the "*" from `mathReq` since its justification has now been started, and we added the body trees of rule 1 to the potential proof, with "*"'s indicating they need to be justified. Next we choose the earliest tree in the proof with an "*" to justify. Since we want to try to justify body trees in left-to-right order, we added them to the potential proof in reverse order to their appearance in the body of the rule. So accordingly, we next choose `discreteMath` to expand for justification, getting:

```
...
*setTheory
*logic
discreteMath
*calc
mathReq
```

Next, choosing `setTheory`, we have that as a fact 4, so that adds no new trees to justify, and we get:

```
...
setTheory
*logic
discreteMath
*calc
mathReq
```

Now `logic` is the first non-justified term, so we choose it to justify. Again we have the fact 5, and so our partial proof becomes:

```
...
setTheory
logic
discreteMath
*calc
mathReq
```

Next `calc` requires justification, so we use rule 3, and obtain:

```
...
*calc1
*calc2
*calc3
setTheory
logic
discreteMath
calc
mathReq
```

Now `calc1` is the next to justify, and with the fact 6, we get:

```
...
calc1
*calc2
*calc3
setTheory
logic
discreteMath
calc
mathReq
```

And then choosing
calc2 and then `calc3`, with the facts 8 and 9 we eventually get:

```
calc1
calc2
calc3
setTheory
logic
```

```
discreteMath
calc
mathReq
```

Since no tree has an "*", we have fully justified all of them, and thus we have a proof of `mathReq` in *P*.

The other program complication we have to account for is when we have more than one rule that might be used to justify a particular tree. For example, consider the following extension of our requirements program:

```
1.  mathReq :- discreteMath, calc.
2.  discreteMath :- setTheory, logic.
3.  discreteMath :- setTheory, linearAlg.
4.  calc : honorsCalc1, honorsCalc2.
5.  calc :- calc1, calc2, calc3.

6.  setTheory.
7.  honorsCalc1.
8.  calc1.
9.  linearAlg.
10. calc2.
11. calc3.
```

We have added alternative ways to satisfy the `discreteMath` and `calc` requirements, and have provided facts for a different student, who satisfies the requirements differently. Now we want to justify `mathReq` for this student.

At each point that we choose a rule to use to justify tree, we might have multiple rules that could be chosen. We will try them in order, choosing the first (or next) and remembering the number of the next rule to try if this one fails to produce a justification.

We begin with:

```
...
*mathReq
```

We use rule 1 to get:

```
...
*discreteMath
*calc
mathReq
```

Since there is only the one rule that might justify `mathReq`, we don't have to remember any alternative rule. Next we try to justify `discreteMath`, and we use its first rule to move to:

```
...
*setTheory
*logic
(3)discreteMath
*calc
mathReq
```

Here `discreteMath` is in the process of being justified, so we have removed its "*", but there is another rule that might be applied to justify it, so we record this next rule number with it. Then we proceed as usual to justify `setTheory`, getting:

```
...
setTheory
*logic
(3)discreteMath
*calc
mathReq
```

And now we want to justify `logic`. But there is no fact (or rule) for this tree, and so there is no justification for `logic`. So this is called "failure" and we must try another alternative. So we will find the first tree (in our potential partial proof) that has a unused alternative, i.e., a rule number associated with it. Here it is `discreteMath`, and the rule number is 3. So we remove every tree back to this numbered tree, change the number to the next alternative (if there is one, and deleting it if none). So here we get:

```
...
*setTheory
*LinearAlg
discreteMath
*calc
mathReq
```

Then we can justify `setTheory` with fact 6, and `LinearAlg` with fact 9, yielding:

```
...
setTheory
LinearAlg
discreteMath
*calc
mathReq
```

Then next we need to justify `calc`, which has two rules, so using the first rule, 4, and recording the second, we get:

```
...
*honorsCalc1
*honorsCalc2
setTheory
LinearAlg
discreteMath
(5)calc
mathReq
```

Using the fact 7, we get:

```
...
honorsCalc1
*honorsCalc2
setTheory
LinearAlg
discreteMath
(5)calc
mathReq
```

Now again, we have no rule or fact that could justify `honorsCalc2`, so we must fail and try an alternative, now using the top numbered tree, here `(5)calc`, getting:

```
...
honorsCalc1
*honorsCalc2
setTheory
LinearAlg
discreteMath
(5)calc
mathReq
```

OOPS. When go back to remove star to begin justification, must move formerly starred tree to top of stack...

A nice property of our example is that at each point in trying to construct our proof, there was exactly one rule that could generate an instance whose head matched the tree we were trying to justify. More generally, it may be that there may be any (finite) number of rules (and rule instances) whose heads might match the tree to be justified. If there are *no* rule instances with a matching head tree, then that tree cannot be justified, and the current potential partial proof cannot be extended to a proof. If there are more than one instance that can match the tree to be justified, we must try all of them in turn if we want to find a proof (or all proofs). Since some partial proofs may not be extensible to a full proof, as we just noted, we need to search through all the possible proofs to be sure to find one if one exists.

(DSW: Do an example that shows these complications, and how to manage them...)

multiple rule bodies; non-determinism.

5.6.2 Lazy Grounding in Top-Down Computation

[Comment DSW: Make proofs in traditional order, forward. To introduce computation, start with propositional program and motivate backwards chaining, goal-directed computation. Then do plus example (no existential variables) with ground call. Then do plus example asking for output; now must "guess" the grounding, so introduce lazy grounding. Then do example with mult, showing even with ground call that existential variables requires lazy grounding. Then think about generalizing lazy grounding to full unification...]

We can use either M_P^{Iter} or M_P^{Prf} as the basis for a computation mechanism for determining membership of trees in a set determined by a program P, that is answers for queries to P.

Using M_P^{Iter} leads to a computation strategy known as "bottom-up evaluation", whereas M_P^{Prf} leads to a strategy known as "top-down evaluation".

We will develop the top-down strategy. The idea is to start with a query, consisting of a sequence of trees. We try to construct a proof for every tree in that sequence. We start constructing a proof by first taking the query sequence as the initial sequence of our prospective proof. We take another copy of the query sequence to actively process as follows as we add to our proof. We process the active query sequence by taking the first tree and finding a rule that has a ground instance with that tree as its head. We add the trees of body of the rule instance to our growing proof, and we replace the tree in the active query sequence with the sequence of trees that make up the body of the grounded rule, producing a new active query sequence. Then we continue in this same way to process every tree in the active query sequence, and continue until its list of trees becomes empty. If we reach the empty list of terms, we have accumulated a proof of the first tree in the initial query sequence. In fact we have a proof of every tree in the proof sequence, obtainable by deleting all trees preceding that tree in the larger sequence.

Example 5.6.5. *Given P:*

```
1.  nat(0).
2.  nat(s(X)) :- nat(X).

3.  plus(0,M,M) :- nat(M).
4.  plus(s(N),M,s(NM)) :- plus(N,M,NM).
```

```
5.  mult(0,M,0) :- nat(M).
6.  mult(s(N),M,NM) :- mult(N,M,NMProd), plus(M,NMProd,NM).
```

and we construct the following proof of `mult(s(s(0)),s(s(0)),s(s(s(s(0)))))` *from P:*

```
a) mult(s(s(0)),s(s(0)),s(s(s(s(0)))))
b) mult(s(0),s(s(0)),s(s(0))),
c) plus(s(s(0)),s(s(0)),s(s(s(s(0)))))
d) mult(0,s(s(0)),0),
e) plus(s(s(0)),0,s(s(0)))
f) nat(s(s(0)))
g) nat(s(0))
h) nat(0)
i) plus(s(0),0,s(0))
j) plus(0,0,0)
k) nat(0)
l) plus(s(0),s(s(0)),s(s(s(0))))
m) plus(0,s(s(0)),s(s(0)))
n) nat(s(s(0)))
o) nat(s(0))
p) nat(0)
```

Consider how this sequence can be built. Our query is `mult(s(s(0)),s(s(0)),s(s(s(s(0)))))`*. So a proof of this tree must use rule 6 since that's the only one that has a ground instance with the term as head. So we choose the substitution of* `{N->s(0), M->s(s(0)), NM->s(s(s(0))), NMProd->s(s(0))}` *which when applied to rule 6 gives a ground instance of that rule with the query as head. So we add the two ground body terms, b) and c), to our (potential) proof and then must extend the proof to prove each of them. To prove b) we again take an instance of rule 6 using substitution* `{N->0, M->s(s(0)), NM->0, NMProd->0}`*. This adds the trees d) and e) to our potential proof. Then we choose tree d) to prove, and use substitution* `{M->s(s(0))}` *applied to rule 5 to get a ground instance with tree d) as head. So we add its body to our growing proof as tree f). Then rule 2 with substitution* `{X->s(0)}` *leads to proof element g), and the same rule with substitution* `{X->0}` *generates a rule instance with head g) and body h), which is added to the proof. h) is a fact in the program: rule 1, so it doesn't generate anything further that needs proof. So we next go back to prove e):* `plus(s(s(0)),0,s(s(0))}`*. We generate a ground instance of rule 4, and generate proof element i). Then i) leads to j) using rule 4 again, and j) to k) using rule 2. Now we can go back to item c) that still needs a proof, and an instance of rule 4 leads to adding item l) to the proof, which leads to m), which leads to n), which leads to o), which leads to p). At this point every item in the proof has been used to generated a rule instance, with it as head and whose body trees have been added to (and thus appear in) the proof following the head tree. Thus we have a complete proof of the initial query tree.*

We see that we have constructed a proof for this query in this case, but we have done this by hand, choosing rules and choosing ground substitutions that generate a proof. Is this a general algorithm that could be programmed to automatically find a proof of an arbitrary query from an arbitrary program, whenever one exists?

We must organize the search for a proof, choosing the right rules to instantiate and the right substitutions that generate desired ground instances of the rules. In our example above, we always choose the right rule and the right instantiation that led to a proof. Choosing the right rule was easy in this case since we can see that in every case, there was exactly one rule whose head could be instantiated to match the desired tree. But this won't always be the case for arbitrary rules and queries. Sometimes there will be no rules and sometimes there will be several. If there are no rules, then we cannot extend the current potential proof into a full complete proof. So we must fail the current proof attempt. If there are several rules, we must try them all in turn, since any one (or more) of them might lead to a proof. And we will want to find all the proofs for a query. It may, of course, be the case that multiple rules will be tried, but after more computation we find that no rule matches a subsequent tree, in which case that proof attempt fails, but one that used a different rule, or substitution, might lead to a full roof. So we must search them all. With careful organization, this can be done.

However, there is a more difficult problem when we look at the problem of choosing "the right" ground substitution after we've chosen the rule to match. In many cases, we can see that only one substitution will work, and there is no problem. For example when we wanted to prove `nat(s(0))` in our example, the only choice for a substitution when we use rule 2 is `{X=0}` and applying that substitution to the rule generates a ground instance. However, if we look at trying to prove the original query, `mult(s(s(0)),s(s(0)),s(s(s(s(0)))))`, we see that only rule 6 can be used and by matching the query to the head of the rule, we can determine the partial substitution `{N->s(0), M->s(s(0)), NM->s(s(s(0)))}`, but we don't know what the value for the variable `NMProd` should be; it is not determined by the matching rule head. For the proof above, since we know what the trees rooted by `mult` and `add` are intended to mean, we can, with that knowledge, guess the right instance, but in general, for arbitrary programs, we can't assume an algorithm will have our insight. We might just guess every possibility, trying each in turn. But there are infinitely many possible trees that could work, so that is not feasible.

In fact, this problem might arise even for queries to `plus` trees. Our example query above was ground, but in general we will want to be able to evaluate a query such as `plus(s(0),s(s(0)),X)`, where we ask for instances of the query, i.e., for values of the variable `X`, in this case to be told the value of $1 + 2$. Note that in this case we aren't given a value for `X` in the query; we want the evaluation strategy to determine it. Can this be done?

The answer is yes and the idea is to do "lazy grounding". That is, if we can't determine what the value for a variable is, we leave it as a variable and wait until later in the computation, when value is determined. So rather than computing with trees (i.e., ground terms), we generalize the computation to work with general terms, which may contain variables. Then rather than require ground instances at each step in our proof construction, we allow general terms. We can think of the computation as maintaining the *set* of trees that are the possible ground instances of the?????

WORK ON PROOF FORM TO SHOW SUBSTITUTIONS AND RULES AS ANNOTATIONS all possible substitutions Each step in our computation can be thought of as further

(Then ask how do we know what instances to choose. Do matching of heads to query, but this doesn't ground all of them. What to do with them? Do lazy grounding; only ground as much as necessary, and leave the others as variables to be grounded later when required. Do the example and show. See how much of unification is needed. Try to see if can get away with just matching here (I hope.) Then introduce case where full unification is needed....

Definition 5.6.5. (Query and Answer) A *query* is a non-empty sequence of terms. An *answer for a query Q to program P* is a substitution *theta* for the variables in Q such that for every tree $t \in Q$, $t\theta \in M_P$. (DSW: this is only for *an* answer; and what about ground queries? Put this defn later, after example of ground query and answer of true or false. And after introducing variables in queries.)

5.7 Conclusion

if any...

Chapter 6

Grammars

In this chapter we will explore how tabling can be used when writing DCG grammars in XSB. Tabling eliminates redundancy and handles grammars that would infinitely loop in Prolog. This makes the "parser you get for free" in XSB one that you might well want to use.

6.1 An Expression Grammar

Consider the following expression grammar, expressed as a DCG. This is the "natural" grammar one would like to write for this langauge.

```
% file grammar.P
:- table expr/2, term/2.

expr --> expr, [+], term.
expr --> term.
term --> term, [*], primary.
term --> primary.
primary --> ['('], expr, [')'].
primary --> [Int], {integer(Int)}.
```

This grammar is left-recursive and would cause a problem for Prolog, but with the table declarations, XSB handles it correctly. Notice that we must table `expr/2` because the XSB parser adds 2 arguments to the nonterminal symbol `expr`. (An alternative would be to use the `auto_table` directive.) After compiling and loading this program, we can execute it:

```
| ?- [grammar].
[Compiling ./grammar]
```

```
[grammar compiled, cpu time used: 0.419 seconds]
[grammar loaded]

yes
| ?- expr([1,+,2,*,3,*,’(’,4,+,5,’)’],[]).
Removing open tables.

yes
| ?-
```

Here the system answers “yes” indicating that the string is recognized. (The message “Removing open tables” is given when a query with no variables evaluates to true. It indicates that once the first successful execution path is found, computation terminates, and any tables that are not fully evaluated have been deleted.) This grammar is not only more elegant than the one we wrote in Prolog for the same langauge, it is more “correct”. What I mean by that is that this grammar associates “+” and “*” to the left, as is usual, rather than to the right as the did the Prolog grammar that we gave in an earlier chapter.

So far we’ve only seen DCG’s that represent simple context-free grammars. The DCG representation is actually much more powerful and can be used to represent very complicated systems. As a first example, let’s say we want to implement an evaluator of these integer expressions. We can add semantic arguments to the nonterminals above to contain the value of the subexpression recognized. The following grammar does this:

```
% file grammar.P
:- table expr/3, term/3.

expr(Val) --> expr(Eval), [+], term(Tval), {Val is Eval+Tval}.
expr(Val) --> term(Val).
term(Val) --> term(Tval), [*], primary(Fval), {Val is Tval*Fval}.
term(Val) --> primary(Val).
primary(Val) --> [’(’], expr(Val), [’)’].
primary(Int) --> [Int], {integer(Int)}.
```

Recall that the braces are used to indicate that the enclosed calls are calls to Prolog predicates, not to nonterminals which recognize parts of the input string.

We can compile and run this program to evaluate the expression provided and have it return its integer value as follows:

```
| ?- [grammar].
[Compiling ./grammar]
[grammar compiled, cpu time used: 0.75 seconds]
[grammar loaded]
```

```
yes
| ?- expr(Val,[1,+,2,*,3,*,'(',4,+,5,')'],[]).

Val = 55;

no
| ?-
```

Notice that the string arguments are added *after* any explicit arguments. So we wrote what looked like a procedure definition for `expr` that had only one argument, but we get a definition that has 3 arguments, and that's the way we called it at the top-level prompt.

As mentioned, this grammar treats the operators as left associative, which is the usual convention for arithmetic expressions. While it doesn't matter semantically for the operations of "+" and "*", which are associative operators anyway, were we to extend this evaluator to handle "−" or "/" (as is very easy), then this correct associativity would be critical.

6.2 Representing the Input String as Facts

In actuality, this way to process grammars with tabling is not as efficient as it might be. The reason is that the arguments to the tabled nonterminals consist of lists, and so each time a call is copied into the table, a long list may have to be copied. Also answers consist of tails of the list corresponding to the input string, and these may be long as well. We can use a slightly different representation for the input string to avoid this inefficiency.

Instead of representing the input string as a list, we will store it in the database, representing it by a set of facts. We will think of each word in the input sentence as being numbered, starting from 1. Then we will store the string as a set of facts of the form, `word(`$n, word$`)`, where $word$ is the n^{th} word in the input string. For example, the string used in the example above, `[1,+,2,*,3,*,'(',4,+,5,')']`, would be represented by the following facts:

```
word(1,1).
word(2,+).
word(3,2).
word(4,*).
word(5,3).
word(6,*).
word(7,'(').
word(8,4).
word(9,+).
word(10,5).
word(11,')').
```

Recall that we said that the DCG translation translates lists in the body of DCG rules to calls to the predicate 'C'/3, which is defined to process the list input. But we can redefine this predicate to look at the word/2 predicate as follows:

```
'C'(I,W,I1) :- word(I,W), I1 is I+1.
```

(We could alternatively use word/3 facts and explicitly store the two consecutive integers, so no computation would be involved.) With this definition of 'C'/3, we can use the same DCG for parsing but now, rather than using lists to represent positions in the input, the system uses integers.

```
% grammar.P
:- table expr/3, term/3.
'C'(I,W,I1) :- word(I,W), I1 is I+1.

eval_string(String,Val) :-
    retractall(word(_,_)),
    assert_words(String,1,N),
    abolish_all_tables,
    expr(Val,1,N).

assert_words([],N,N).
assert_words([Word|Words],N,M) :-
        assert(word(N,Word)), N1 is N+1, assert_words(Words,N1,M).

expr(Val) --> expr(Eval), [+], term(Tval), {Val is Eval+Tval}.
expr(Val) --> term(Val).
term(Val) --> term(Tval), [*], primary(Fval), {Val is Tval*Fval}.
term(Val) --> primary(Val).
primary(Val) --> ['('], expr(Val), [')'].
primary(Int) --> [Int], {integer(Int)}.
```

Here we've defined a predicate eval_string to take an input string as a list, assert it into the database as a set of word/2 facts and then to call expr to parse and evaluate it. Notice that we need both to retract any facts previously stored for word/2 and to abolish any tables that were created during previous evaluations of strings. This is because old tables are no longer valid, since the new input string changes the meanings of the integers that represent positions in the input string.

We can compile and call this predicate as follows:

```
| ?- [grammar].
[Compiling ./grammar]
++Warning: Redefining the standard predicate: C / 3
```

```
[grammar compiled, cpu time used: 1.069 seconds]
[grammar loaded]

yes
| ?- eval_string([1,+,2,*,3,*,’(’,4,+,5,’)’],V).

V = 55;

no
| ?-
```

The warning is to alert the user to the fact that a standard predicate is being redefined. In this case, that is exactly what we want to do, and so we can safely ignore the warning.

6.3 Mixing Tabled and Prolog Evaluation

We can extend this evaluator in the following interesting way. Say we want to add exponentiation. We introduce a new nonterminal, `factor`, for handling exponentiation and make it right recursive, since exponentiation is right associative.

```
:- table expr/3, term/3.

expr(Val) --> expr(Eval), [+], term(Tval), {Val is Eval+Tval}.
expr(Val) --> term(Val).
term(Val) --> term(Tval), [*], factor(Fval), {Val is Tval*Fval}.
term(Val) --> factor(Val).
factor(Val) --> primary(Num), [^], factor(Exp),
    {Val is floor(exp(log(Num)*Exp)+0.5)}.
factor(Val) --> primary(Val).
primary(Val) --> [’(’], expr(Val), [’)’].
primary(Int) --> [Int], {integer(Int)}.
```

However, we don’t table the new nonterminal. Prolog’s evaluation strategy handles right recursion in grammars finitely and efficiently. In fact, Prolog has linear complexity for a simple right-recursive grammar, but with tabling it would be quadratic. Thus an advantage of XSB is that it allows tabled and nontabled predicates to be freely intermixed, so that the programmer can choose the strategy that is most efficient for the situation at hand.

6.4 So What Kind of Parser is it?

A pure DCG, one without extra arguments (and without look-ahead symbols which we haven't discussed at all), represents a context-free grammar, and the Prolog and XSB engines provide recognizers for it. A context-free recognizer is a program that, given a context-free grammar and an input string, responds "yes" or "no" according to whether the input string is in or is not in the language of the given grammar.

We noted that the recognizer that "you get for free" with Prolog is a recursive descent recognizer. The recognizer "you get for free" with XSB and tabling is a variant of Earley's algorithm, or an active chart recognition algorithm (ref Peter Norvig and B. Sheil.)

The worst-case complexity of the recognizer under XSB (with all recursive nonterminals tabled) is $O(n^{k+1})$ where n is the length of the input string and k is the maximum number of nonterminals and terminals on the right-hand-side of any rule. A little thought shows that this is consistent with the discussion of the complexity of Datalog programs under XSB in Chapter ??. This is an example of a situation in which tabling turns an otherwise exponential algorithm (recursive descent) into a polynomial one (active chart recognition.)

Any grammar can be changed to another grammar that represents the same language but has two (or fewer) nonterminal symbols on the right-hand-side of every rule. This is the so-called Chomsky normal form. So if we transform a grammar into this form, then its worst-case complexity will be $O(n^3)$.

In fact, the folding and tabling that is done automatically by the XSB compiler when the `:- suppl_table.` and `:- edb word/2.` directives are given results in exactly the transformation necessary to transform a grammar to Chomsky normal form. So giving those directives guarantee the best worst-case complexity.

For unambiguous grammars, the complexity is actually $O(n^2)$. I find this an intriguing situation, which is particularly pleasant, since it is undecidable whether a context-free grammar is or is not ambiguous. So it will be undecidable to determine what the actual complexity of the algorithm is, given a grammar. But, no problem; the algorithm will automatically tune itself to the data (grammar in this case) to be more efficient in the unambiguous case.

These complexity results are very good and make the parser that "you get for free" with XSB quite a desirable parser.

(DSW: Describe the CKY algorithm, giving an example, and then connect that to evaluating DCG's with tabling. Tabling fills the same entries in the CKY table, but does it in a top-down way, on demand. Try to explain it.)

6.5 Building Parse Trees

The desirable complexity results of the previous section hold for *recognition* of context-free languages. But often it is the case that one wants, given a string in the language, to construct the parse tree(s) for it. An easy way to do this is to add an argument to each nonterminal to contain the parse tree, and then add the necessary code to each rule to construct the appropriate tree. For example, the following rule from our expression example:

```
    expr(Val) --> expr(Eval), [+], term(Tval), {Val is Eval+Tval}.
```

could become:

```
expr(Val,+(E1,T1)) --> expr(Eval,E1), [+], term(Tval,T1), {Val is Eval+Tval}.
```

We've added a second argument and in it constructed the parse tree for the entire expression phrase given the parse trees for the component expression and term phrases. All the other rules would be extended accordingly.

This is very easy to do and in almost all cases is the best way to construct the parse tree. However, from a complexity standpoint, it has certain drawbacks. It may cause us to lose the polynomial complexity we had for the recognition problem. For example, consider the following grammar:

```
    :- auto_table.
    s --> b,[c]

    b --> b,b.
    b --> [a].
```

Here nonterminal `s` generates a list of "a"s followed by a single "c". If we compile this grammar with XSB, using the `auto_table` directive, we get a program that will recognize strings correctly in $O(n^3)$ time. But if we add parameters to construct the parse tree, thusly:

```
    :- auto_table.
    s(s1(B,c)) --> b(B), [c].

    b(b1(B1,B2)) --> b(B1), b(B2).
    b(b2(a)) --> [a].
```

it may take exponential time. Now the string $a^n c$ has exponentially many parses (all bracketings of the length n string of "a"s), so it will clearly take exponential time to produce them all. This is not so much of a problem; if there are exponentially many, there's no way to produce them all without taking exponential time. However, if we try to recognize the string a^n, this will take exponential time to report failure. This is because the system will construct all the (exponentially many) initial parses from the nonterminal "b" and then each time fail when it doesn't find a terminating "c" in the input string.

One might think that we could easily just maintain two versions of the grammar: one with no parameters to do recognition, and one with a parse-tree parameter. Then we'd first recognize and if there were no parses, we'd simply report failure, and if there were parses, we'd reprocess the input string, this time using the parsing version of the grammar. But this doesn't always work either. For example, say we added a few rules to our grammar above to obtain:

```
    :- auto_table.
    s --> b,[c].
    s --> g,[d].

    b --> b,b.
    b --> [a].

    g --> g, [a].
    g --> [a].
```

Here, the input string of `[a,a,a,a,a,a,d]` has only one parse, but naively parsing it with the parse-annotated grammar will construct exponentially many initial segments of parses, which come from the first rule and the rules for the nonterminal "b". So if we are serious about this problem, we must be a little more sophisticated.

We will use a representation of parse trees that uses an abstraction. An abstraction will be a term of the form `{X:p(...,X,...)}` and is intended to mean that X can take on values that would be assigned to `X` by calling `p(...,X,...)`. We will construct such abstraction terms to represent our parse trees. Consider the following example that adds a parse tree to the grammar above.

```
:- table b/3,g/3,s/3,b_abs/3,g_abs/3.
    s(s(P,c)) --> b_abs(P), [c].
    s(s(P,d)) --> g_abs(P), [d].

    b(b(P1,P2)) --> b_abs(P1), b_abs(P2).
    b(a) -->  [a].

    g(g(P),S0,S) :-
           g_abs(P,S0,S1), 'C'(S1,a,S).
    g(a,S0,S) :-
           'C'(S0,a,S).
    b_abs({X:b(X,S0,S)},S0,S) :- b(_,S0,S).

    g_abs({X:g(X,S0,S)},S0,S) :- g(_,S0,S).
```

For each nonterminal symbol that appears in the body of a rule, such as `b`, we add a new nonterminal symbol, such as `b_abs`, that calls `b` but returns an abstraction for the parse argument. And every nonterminal in the body of a rule is replaced by a call to its abstraction version. We here have tabled all the predicates to minimize redundancy. Now when this program is evalated, each nonterminal constructs a representation for its parse term using abstractions for its subterms. So now multiple different parses for a substring do not cause multiple returned answers; only one answer is returned, which is an abstraction that represents (and can be used to generate) the multiply answers when desired.

For example, consider the following call, after loading the above grammar:

```
| ?- s(P,[a,a,a,a,c],[]).

P = s({_h324 : b(_h324,[a,a,a,a,c],[c])},c);

no
| ?-
```

The parse is represented as a term whose main functor symbol is `s/2` with first argument an abstraction and second argument as `c`. We can get the actual terms for that abstraction by unfolding the abstraction by calling the indicated subgoal.

The following predicate, `ext_term/2`, can be used to do this unfolding and generate the parses:

```
%% ext_term(+AbsTerm,-ConcreteTerm)

ext_term(AT,AT) :- var(AT), !.
ext_term({X:R},T) :- !,
call(R),
ext_term(X,T).
ext_term(AT,T) :-
AT =.. [F|As],
ext_term_list(As,Ts),
T =.. [F|Ts].

ext_term_list([],[]).
ext_term_list([A|As],[T|Ts]) :-
ext_term(A,T),
ext_term_list(As,Ts).
```

With the definition, we get the following results:

```
| ?- s(P,[a,a,a,a,c],[]),ext_term(P,T).

P = ...
T = s(b(a,b(a,b(a,a))),c);

P = ...
T = s(b(a,b(b(a,a),a)),c);

P = ...
T = s(b(b(a,a),b(a,a)),c);

P = ...
T = s(b(b(a,b(a,a)),a),c);

P = ...
T = s(b(b(b(a,a),a),a),c);

no
| ?-
```

(I've elided the answers for the variable P, since they are large and not of interest here. Also, this example goes back to the standard DCG input representation as a list. We would normally want to use the word/3 representation for efficiency.)

To summarize, this abstraction representation allows us to delay the full construction of each parse tree until the end of the string parsing, instead representing the parses by "pointers" into the tables. The abstractions can be thought of as "pointers" to terms in the tables that are later retrieved. This avoids the exponential explosion of parse trees since a single "pointer" can point to a set of subtrees.

Thinking a little more generally, we can think of this as a way to represent the set of proofs of a program, which avoids the exponential explosion of multiple proofs.

6.6 Computing First Sets of Grammars

The previous examples show that XSB can process grammars efficiently, using them to determine language membership and the structure of strings. But XSB can also do other kinds of grammar processing. In the following, we use XSB to compute the FIRST sets of a grammar.

$FIRST(\alpha)$, for any string of terminals and nonterminals α, is defined to be the set of terminals that begin strings derived from α, and if α derives the empty string then the empty string is also in $FIRST(\alpha)$. $FIRST_k(\alpha)$ is the generalization to length k strings of terminals that are prefixes of strings derived from α.

We will assume that a grammar is stored in a predicate `==>/2`, with the head of a rule as an atomic symbol and the body of a rule as a list of symbols. Nonterminals are assumed to be those symbols for which there is at least one rule with it as its head. (`==>/2` is declared as an infix operator.)

The predicate `first(SF,K,L)` is true if the list SF of grammar symbols derives a string whose first K terminal symbols are L.

```
% The definition of FIRST:
% first(SentForm,K,FirstList) computes firsts for a context-free grammar.
:- table first/3.
first(_,0,[]).
first([],K,[]) :- K>0.
first([S|R],K,L) :- K>0,
    (S ==> B),
    first(B,K,L1),
    length(L1,K1),
    Kr is K - K1,
    first(R,Kr,L2),
    append(L1,L2,L).
first([S|R],K,[S|L]) :- K>0,
    \+ (S ==> _),    % S is a terminal
    K1 is K-1,
    first(R,K1,L).
```

The first rule says that the empty string is in $FIRST_0(\alpha)$ for any α. The second rule says that the empty string is in $FIRST_k(\alpha)$ for α being the empty sequence of grammar symbols. The third rule handles the case in which the sequence of grammar rules begins with a nonterminal, `S`. It takes a rule beginning with S and gets a string `L1` (of length `K1` $\leq$ `K`) generated by the body of that rule. It gets the remaining `K`$-$`K1` symbols from the rest of the list of input grammar symbols. And the fourth rule handles terminal symbols.

Consider the following example of computing the first sets for the expression grammar:

```
/** %Grammar:
e ==> [e,+,t].
e ==> [e,-,t].
e ==> [t].
t ==> [t,*,f].
t ==> [t,/,f].
t ==> [f].
```

```
f ==> [int].
f ==> ['(',e,')'].
***/

| ?- [firsts].
[Compiling .\firsts]
[firsts compiled, cpu time used: 0.0150 seconds]
[firsts loaded]
yes
| ?- first([e],1,F).
F = [(];
F = [int];
no
| ?- first([e],2,F).
F = [(,(];
F = [(,int];
F = [int];
F = [int,+];
F = [int,*];
F = [int,/];
F = [int,-];
no
| ?-
```

This is a relatively simple declarative (and constructive) definition of $FIRST_k$. But without the table declaration it would not run for many grammars. Any left-recursive grammar would cause this definition to loop in Prolog.

6.7 Linear Parsing of LL(k) and LR(k) Grammars

(This section involves a more advanced topic in XSB programming, metainterpretation. It may be helpful to read the later section on metainterpreters if the going gets tough here.)

As discussed above, parsing context-free grammars with tabling results in an Earley-type parser. This is the parser we get when we write DCGs. We can also write an XSB program that takes a grammar as input (defined in a database predicate as in the example of `first/3`) and a string (defined by the database `word/3` predicate) and succeeds if the grammar accepts the string. With such processing we can compute and use the $FIRST$ sets to make the grammar processing more deterministic. This is similar to what is done in LL(k) and LR(k) parsing, but there the emphasis is on compile-time analysis and complete determinacy. Here the approach is more interpretive and supports nondeterminacy. However, if the grammars are indeed of the appropriate form (LL(k) or LR(k)), the corresponding interpreters presented here will have linear complexity.

It is very easy to write a simple context-free grammar parser in XSB. Again, we assume that the grammar is stored in facts of the form `NT ==> Body` where `NT` is a nonterminal symbol and `Body` is a list of terminals and nonterminals.

```
:- table parse/3.

% parse(Sym,S0,S) if symbol Sym generates the string from S0 to S.
parse(Sym,S0,S) :-
    word(S0,Sym,S).
parse(Sym,S0,S) :-
    (Sym ==> Body),
    parseSF(Body,S0,S).

% parseSF(SF,S0,S) if sentential form SF generates the string from S0 to S.
parseSF([],S,S).
parseSF([Sym|Syms],S0,S) :-
    parse(Sym,S0,S1),
    parseSF(Syms,S1,S).
```

The predicate `parse/3` recognizes strings generated by a single grammar symbol; the first clause recognizes terminal symbols directly and the second clause uses `parseSF/3` to recognize strings generated by the sentential form that makes up the body of a rule for a nonterminal. `parseSF/3` simply maps `parse/3` across the sequence of grammar symbols in its sentential form argument.

Were we not to add the table declaration, we would get a recursive descent recognizer. But with the table declaration, we get the Earley-type recognizer of XSB as described above. The tabling reflects right through the programmed recognizer.

Next we add look-ahead to this recognizer by computing `FIRST` sets and making calls only when the next symbols are in the first set of the sentential form to be processed. This will give us an LL(k)-like recognizer. Hovever, the form of `FIRST` we need here is slightly different from the one above. Here we want to include the context in which a `First` set is computed. For example, we may want to compute the `FIRST_2` set of a symbol `N`, but `N` only generates one symbol. The definition of `first` above would return a list of length one for that symbol. Here we want to take into account the context in which `N` is used. For example, we may know that in the current context `N` is immediately followed by another nonterminal `M`, and we know the `FIRST` of `M`. Then we can compute the `FIRST_2` of `N` in the following context of `M` by extending out the one symbol in `first` of `N` with symbols in `FIRST` of `M`.

The following definition of `firstK/3` computes such first sets.

```
:- table firstK/3.

% firstK(+SF,+Follow,-First), where K = length(Follow)
```

```
firstK([],Follow,Follow).
firstK([Sym|SF],Follow,First) :-
    firstK(SF,Follow,SymFollow),
    ((Sym ==> Body)
     ->   firstK(Body,SymFollow,First)
     ;    First = [Sym|FirstTail]
          append(FirstTail,[_],SymFollow),
    ).
```

The predicate firstK/3 takes a sentential form SF, a follow string Follow, and returns in First a first string of SF in the context of the follow string. The value of k is taken to be the length of the list Follow; this will be the length of the look-ahead.

We can now extend our parser to have a top-down look-ahead component, similar to an LL(k) recognizer:

```
TEST THIS SUCKER!
:- table parseLL/4.    % without this, it is an LL(k)-like parser,
                        % but with it, what is it???

% parseLL(Sym,Follow,S0,S) if symbol Sym generates the string from S0 to S.
parseLL(Sym,Follow,S0,S) :-
    (Sym ==> Body)
     -> firstK(Body,Follow,First),
        next_str(First,S0),        % do the look-ahead, continuing only if OK
        parseLLSF(Body,Follow,S0,S)
     ;  word(S0,Sym,S).

% parseLLSF(SF,Follow,S0,S) if sentential form SF generates the string from S0 to S.
parseLLSF([],_Follow,S,S).
parseLLSF([Sym|Syms],Follow,S0,S) :-
    firstK(Syms,Follow,SymFollow),
    parseLL(Sym,SymFollow,S0,S1),
    parseLLSF(Syms,Follow,S1,S).

next_str([],_).
next_str(['$'|_],S) :- \+ word(S,_,_).  % $end of string
next_str([Sym|Syms],S) :- word(S,Sym,S1),next_str(Syms,S1).
```

The predicate parseLL/4 recognizes a string generated by a grammar symbol, using lookahead. The condition tests whether Sym is a nonterminal, and if it is and can be rewritten as Body, it checks to see whether the next symbols in the input string belong to the first set of the body of that rule. If not, it needn't process that rule because it cannot succeed. For an LL(k) grammar, only one rule will ever apply; the others all will be filtered out by this lookahead. So in this case a rule is never tried unless it is the only rule that might lead to a parse. If the symbol is a terminal, it simply checks to see whether the symbol matches the input.

`parseLLSF/4` maps `parseLL/4` across a sequence of grammar symbols in a sentential form. It uses `firstK/3` to compute the follow strings of a symbol, which are needed in `parseLL` to compute the first strings in the correct context.

In this LL(k)-like parser, we tested that the next symbols were in the first set just before we called `parseLLSF`. We could also check the lookahead just before returning. This gives us an LR(k)-like parser, as follows:

```
% parseLR with tabling to get an lr(k)-like algorithm.

:- table parse/4.
parseLR(Sym,Follow,Str0,Str) :-
    word(Str0,Sym,Str),
    next_str(Follow,Str).
parseLR(Sym,Follow,Str0,Str) :-
    (Sym ==> RB),
    parseLRSF(RB,Follow,Str0,Str).

parseLRSF([],Follow,Str,Str) :-
    next_str(Follow,Str).
parseLRSF([Sym|SF],Follow,Str0,Str) :-
    firstK(SF,Follow,SymFollow),
    parseLR(Sym,SymFollow,Str0,Str1),
    parseLRSF(SF,Follow,Str1,Str).
```

For this parser, we compute a follow string for a sentential form, and only return from parsing that sentential form if the next input symbols match that follow string. For an LR(k) grammar, the parser will not fail back over a successful return (unless the entire string is rejected.) This allows the parser to work in linear time.

The above program was written as a Prolog program, but we can write the identical program as a DCG and have the DCG transformation put in the string variables, as follows:

```
:- table parseLR/4.
parseLR(Sym,Follow) --> [Sym], look(Follow).
parseLR(Sym,Follow) -->
    {(Sym ==> RHS)},
    parseLRSF(RHS,Follow).

parseLRSF([],Follow) --> look(Follow).
parseLRSF([Sym|SF],Follow) -->
    {firstK(SF,Follow,SymFollow)},
    parseLR(Sym,SymFollow),
    parseLRSF(SF,Follow).

look([]) --> [].
look([Word|Words]), [Word] --> [Word], look(Words).
```

This recognizer differs from an LR(k) recognizer in that it computes the look-ahead strings as needed. Also it processes each look-ahead string separately; i.e., `Follow` is a single string, not a set of strings. In an LR(k) recognizer, the lookahead tables are computed once and stored.

6.8 Parsing of Context Sensitive Grammars

Another more powerful form of grammars is the class of context sensitive grammars. They contain rules that have strings on both the left-hand-side and the right-hand-side, as opposed to context-free rules which require a single symbol on the left-hand-side. A constraint on context sensitive rules is that the length of the string of the left-hand side is at least one, and is less than or equal to the length of the string on the right-hand side. (Without this constraint, one gets full Turing computability and the recognition problem is undecidable.) As a simple example, consider the following context sensitive grammar:

```
1. S --> aSBC
2. S --> aBC
3. CB --> BC
4. aB --> ab
5. bB --> bb
6. bC --> bc
7. cC --> cc
```

This grammar generates all strings consisting of a nonempty sequence of a's followed by the same number of b's followed by the same number of c's. Consider the following derivation:

```
S
aSBC      rule 1
aaBCBC    rule 2
aaBBCC    rule 3
aabBCC    rule 4
aabbCC    rule 5
aabbcC    rule 6
aabbcc    rule 7
```

Even though in this simple example the rules fire in order, it's not difficult to see that rule 3 will have to fire enough times to move all the C's to the right over the B's, and then rules 4-7 will fire enough times to turn all the nonterminals into terminals.

The question now is how to represent this in XSB. We can think of the XSB DCG rules as running on a graph that starts as a linear chain representing the input string. Then each DCG context-free rule tells how we can add edges to that linear graph. For example, a DCG rule `a --> b,c.` tells us that if there is an arc from node X to node Y labeled by b and also an arc from node Y to node Z labeled by c, then we should add an arc from node X to node Z labeled by a. So the DCG rule in Prolog, `a(S0,S) :- b(S0,S1),c(S1,S).`, read right-to-left, says explicitly and directly that if there is an arc from S0 to S1 labeled b and an arc from S1 to S labeled c, then there is an arc from S0 to S labeled by a. We can think of DCG rules as rules that add labeled arcs to graphs. This is exactly the way that chart-parsing is understood [ref].

Now we can extend this way of understanding logic grammars to context-sensitive rules. A context-sensitive rule, with say two symbols on the left-hand-side, can be seen also as a graph-generating rule, but in this case it must introduce a new node as well as new arcs. So for example, a rule such as `AB --> CD`, when it sees two adjacent edges labeled C and D, should introduce a new node and connect it with the first node of the C-arc labeling it A, and also connect it to the final node of the D-arc, labeling that new arc with B. So we add two new XSB rules for a context sensitive rule such as `AB --> CD`, as follows:

```
    a(S0,p1(S0,S)) :- c(S0,S1), d(S1,S).
    b(p1(S0,S),S) :- c(S0,S1), d(S1,S).
```

which explicitly add the arcs and nodes. We have to introduce a new name for the new node. We choose to identify the new nodes by using a functor symbol that uniquely determines the rule and left-hand internal position, and pairing it with the names of the end points of the base arc. So in this case, p1 uniquely identifies the (only) internal position in the left-hand-side of this rule. Other rules, and positions, would have different functors to identify them uniquely.

Now we can represent the context sensitive grammar above using the following XSB rules:

```
:- auto_table.

s(S0,S) :- word(S0,a,S1),s(S1,S2),b(S2,S3),c(S3,S).
s(S0,S) :- word(S0,a,S1),b(S1,S2),c(S2,S).

c(S0,p0(S0,S)) :- b(S0,S1),c(S1,S).
b(p0(S0,S),S) :- b(S0,S1),c(S1,S).

word(S0,a,p1(S0,S)) :- word(S0,a,S1),word(S1,b,S).
b(p1(S0,S),S) :- word(S0,a,S1),word(S1,b,S).
```

```
word(S0,b,p2(S0,S)) :- word(S0,b,S1),word(S1,b,S).
b(p2(S0,S),S) :- word(S0,b,S1),word(S1,b,S).

word(S0,b,p3(S0,S)) :- word(S0,b,S1),word(S1,c,S).
c(p3(S0,S),S) :- word(S0,b,S1),word(S1,c,S).

word(S0,c,p4(S0,S)) :- word(S0,c,S1),word(S1,c,S).
c(p4(S0,S),S) :- word(S0,c,S1),word(S1,c,S).

% define word/3 using base word (separation necessary)
word(X,Y,Z) :- base_word(X,Y,Z).

% parse a string... assert words first, then call sentence symbol
parse(String) :-
    abolish_all_tables,
    retractall(base_word(_,_,_)),
    assertWordList(String,0,Len),
    s(0,Len).

% assert the list of words.
assertWordList([],N,N).
assertWordList([Sym|Syms],N,M) :-
    N1 is N+1,
    assert(base_word(N,Sym,N1)),
    assertWordList(Syms,N1,M).
```

We can run this grammar to parse input strings as follows:

```
warren% xsb
XSB Version 1.7.2 (7/10/97)
[Sun, optimal mode]
| ?- [csgram].
[csgram loaded]

yes
| ?- parse([a,a,b,b,c,c]).

yes
| ?- parse([a,a,a,b,b,c,c,c]).

no
| ?- parse([a,a,a,a,b,b,b,b,c,c,c,c]).

yes
| ?-
```

We factored the above program, added more tabling declarations and introduced some write statements to allow us to see how this context-sensitive recognizer actually processes strings. Here follows the log for processing the string 'aabbcc':

```
| ?- parse([a,a,b,b,c,c]).
1. [cC-->cc,4,p4,6]
2. [bC-->bc,3,p3,p4(4,6)]
3. [bC-->bc,3,p3,5]
4. [bB-->bb,2,p2,p3(3,5)]
5. [bB-->bb,2,p2,p3(3,p4(4,6))]
6. [bB-->bb,2,p2,4]
7. [aB-->ab,1,p1,p2(2,4)]
8. [aB-->ab,1,p1,p2(2,p3(3,p4(4,6)))]
9. [aB-->ab,1,p1,p2(2,p3(3,5))]
10. [aB-->ab,1,p1,3]
11. [aB-->ab,1,p1,3]
12. [bC-->bc,3,p3,5]
13. [bB-->bb,2,p2,p3(3,5)]
14. [aB-->ab,1,p1,p2(2,p3(3,5))]
15. [CB-->BC,p2(2,p3(3,5)),p0,5]
16. [S-->aBC,1,p1(1,p2(2,p3(3,5))),p2(2,p3(3,5)),p0(p2(2,p3(3,5)),5)]
17. [CB-->BC,p2(2,p3(3,5)),p0,5]
18. [cC-->cc,4,p4,6]
19. [bC-->bc,3,p3,p4(4,6)]
20. [bB-->bb,2,p2,p3(3,p4(4,6))]
21. [aB-->ab,1,p1,p2(2,p3(3,p4(4,6)))]
22. [CB-->BC,p2(2,p3(3,p4(4,6))),p0,p4(4,6)]
23. [S-->aBC,1,p1(1,p2(2,p3(3,p4(4,6)))),p2(2,p3(3,p4(4,6))),p0(p2(2,p3(3,p4(4,6))),p4(4,6)
24. [CB-->BC,p2(2,p3(3,p4(4,6))),p0,p4(4,6)]
25. [S-->aSBC,0,1,p0(p2(2,p3(3,p4(4,6))),p4(4,6)),p4(4,6),6]

yes
| ?-
```

The writes were added where answers are returned, the order is essentially bottom-up. Each log item includes first the grammar rule that applies, then the inital node it starts from, the structure symbols of the new nodes it generates, and then the node it terminates on. While this is rather complicated, we can extract a successful derivation that uses items: 25, 23, 21, 22, 20, 19, 18.

So now we have generalized DCG's to include processing of context-sensitive grammars and languages. The builtin DCG notation doesn't support context sensitive languages, but we can write the necessary rules directly as XSB rules, as we did above. It is interesting to note that the XSB rules we generate for a single context-sensitive rule all have the same body, and that the logical implications

```
p <- r & s.
q <- r & s.
```

are logically equivalent to the single implication:

```
p & q <- r & s.
```

So it would be very natural to extend the Prolog notation to support "multi-headed" rules, which would be compiled to a set of "single-headed", i.e., regular Prolog, rules. Were we to do this, we could write the context-sensitive rule:

```
AB --> CD
```

as the single (multi-headed) XSB rule:

```
a(S0,p1(S0,S)), b(p1(S0,S),S) :- c(S0,S1), d(S1,S).
```

which looks very much like the original context sensitive rule. In fact, we can see rule as the formula:

```
for-all([S0,S],
        (there-is X (a(S0,X),b(X,S))) ->
           there-is(S1,c(S0,S1),d(S1,S)))
```

where the term `p1(S0,S)` is a Skolem term introduced to eliminate the existential quantifier over the head of the rule.

This suggests how we might want to extend the DCG notation to support context-sensitive rules through the support of multi-headed rules.

6.9 Substring Matching

(NOTE: Redo this as follows:

match(Pattern,LeftToSee,StringLeft) :-

end)

Consider the problem of finding whether a given string appears as a contiguous substring in another given string. This is not quite a grammar problem but it does involve strings. We will assume that the strings (called Pat and Str, respectively) are represented as Prolog lists. It is very easy in Prolog to write a simple predicate to succeed if Pat appears as a (contiguous) substring in Str and fail if not:

```
:- import append/3 from basics.

match(Pat,Str) :-
    append(_Pre,StrTail,Str),
    append(Pat,_Suff,StrTail)
```

This simply breaks Str into an ignored prefix and tail, and then sees if Pat is a prefix of that tail.

This straightforward algorithm has complexity of $O(n * m)$ where n is the length of Str and m is the length of Pat. It is known that there is an algorithm for this problem that is $O(n)$ (when n is much larger than m, as would be the usual case.) We can derive that algorithm by creating a slightly more complicated version of match and then using tabling appropriately.

So instead of writing the predicate as match(Pat,Str), let's write it as match(Pat,StrPrefix,StrSuffix), where we split the subject string into two pieces: a prefix and a suffix. We will assume that the length of StrPrefix is less than the length of Pat.

The idea is that we will match Pat against the longest tail of StrPrefix, returning the prefix of Pat that matched and the suffix that was left over. Then we'll see if the suffix that was left over is a prefix of StrSuffix, returning the sequence that matches, plus the symbol that causes the mismatch if it doesn't match the entire pattern suffix. Now in case it does not match the pattern suffix entirely, we need to delete the first symbol of the entire subject string (StrPrefix concatenated with StrSuffix.) We can construct a representation for that string using the pieces we've collected in our previous matching. The program is:

```
:- import append/3 from basics.

match(Pat,StrPre,StrSuff) :-
    match_suffix_with_remainder(Pat,StrPre,PatMatched,PatLeft),
    match_to_mismatch(PatLeft,StrSuff,PatWithMismatch,StrAfterMismatch,Matched),
    (Matched == match
     ->  append(PatMatched,PatWithMismatch,[_|NewStrPrefix]),
         match(Pat,NewStrPrefix,StrAfterMismatch)
     ;   true
    ).
```

So match_suffix_with_remainder/4 matches the (longer) Pat with some suffix of StrPre and returns the prefix of Pat that matched and the suffix that was left over. Then match_to_mismatch takes the unmatched tail of the Pat and matches it against StrSuff and returns in PatWithMismatch the part of the pattern suffix that matched including the mismatching symbol as its last symbol, and returns the string after that mismatch in StrAfterMismatch. If the entire PatLeft matched, then the flag Matched is returned as match, and we're done, having found a match. Otherwise, we can append the PatMatched and PatWithMismatch and delete its head to get the prefix of the subject string we should next try to match. And StrAfterMismatch is the suffix of that string. So that's what we call match/3 with in the recursive call. Notice that the length of NewStrPrefix is less than the length of Pat, since PatMatched concatenated with PatWithMismatch is at most the length of Pat, and NewStrPrefix is one symbol shorter.

The helper predicates are defined as:

```
match_suffix_with_remainder(Pat,SubPat,PatPrefix,PatSuffix) :-
    (append(SubPat,PatSuffix,Pat)
     ->  PatPrefix = SubPat
     ;   SubPat = [_|SubPat1],
         match_suffix_with_remainder(Pat,SubPat1,PatPrefix,PatSuffix)
    ).

match_to_mismatch([],Str,[],Str,match).
match_to_mismatch([X|Pat],[Y|Str],[Y|PatWithMismatch],StrAfterMismatch,M) :-
    (X == Y
     ->  match_to_mismatch(Pat,Str,PatWithMismatch,StrAfterMismatch,M)
     ;   PatWithMismatch = [],
         StrAfterMismatch = Str,
         M = mismatch
    ).

match([],_) :- !.
match(Pat,[X|Str]) :-
    match(Pat,[X],Str).
```

The goal here was to come up with an algorithm that was $O(n)$ for subject string of length n. The main work is done in match_to_mismatch/5 since that is where new symbols of the subject string are initially seen. If we delay the append until we've entered match, we can get the append and match_suffix_with_remainder together. So we change match/3 to match/4 and pass in the two pieces that will used to make up the string prefix:

```
:- import append/3 from basics.

match(Pat,StrPre1,StrPre2,,StrSuff) :-
    append(StrPre1,StrPre2,[_|StrPre]),
    match_suffix_with_remainder(Pat,StrPre,PatMatched,PatLeft),
    match_to_mismatch(PatLeft,StrSuff,PatWithMismatch,StrAfterMismatch,Matched),
    (Matched == match
     ->  match(Pat,PatMatched,PatWithMismatch,StrAfterMismatch)
     ;   true
    ).
```

And now we can factor out the append and `match_suffix_with_remainder` to form a single call:

```
:- import append/3 from basics.

match(Pat,StrPre1,StrPre2,,StrSuff) :-
    app_match_suffix_with_remainder(Pat,StrPre1,StrPre2,PatMatched,PatLeft),
    match_to_mismatch(PatLeft,StrSuff,PatWithMismatch,StrAfterMismatch,Matched),
    (Matched == match
     ->  match(Pat,PatMatched,PatWithMismatch,StrAfterMismatch)
     ;   true
    ).

:- table app_match_suffix_with_remainder/5.
app_match_suffix_with_remainder(Pat,StrPre1,StrPre2,PatMatched,PatLeft) :-
    append(StrPre1,StrPre2,[_|StrPre]),
    match_suffix_with_remainder(Pat,StrPre,PatMatched,PatLeft).
```

And we table that new predicate. Consider the possible arguments passed to `app_match_suffix_with_rema` `Pat` is always the same, and the concatenation of `StrPre1` with `StrPre2` is a proper prefix of `Pat` followed by a single symbol. So the number of calls depends only on m, the length of the pattern, and on the number of different symbols in the alphabet, but not on n, the length of the subject string. So if the table lookup can be done in constant time, then we have an $O(n)$ algorithm.

If this actual program is run with XSB, the table lookup time will be $O(m)$ since XSB will compare the entire strings. But by changing the representation of the strings so that the arguments are indexes into a string stored in the database, as we did for grammars, this can be made constant. Exercise for the reader....

6.10 Exercises

1. (Is this exercise appropriate here? If I write a chapter on Abstract Interpretation, maybe move there and add an exercise on AI.).

   Write a XSB program to parse a simple programming procedural language and construct an abstract syntax tree for it. The syntax is taken from the Modula-3 programming language, but greatly simplified.

The grammar is following. (The empty string is indicated by e, for epsilon; — indicates alternative rules (or), and parentheses indicate grouping except when quoted in which case they indicate tokens):

```
module ::= MODULE identifier ; block identifier .
block ::= declaration_sequence block_body
block_body ::= END | BEGIN statement_sequence END
declaration_sequence ::= declaration_sequence declaration | e
declaration ::= VAR variable_declaration_list ;
variable_declaration_list ::= variable_declaration
          (e | ; variable_declaration_list)
variable_declaration ::= identifier_list : type
identifier_list ::= identifier (e | , identifier_list)
statement_sequence ::= statement statement_sequence | e
statement ::= assignment_statement ; | if_statement ; |
              while_statement ; | print_statement ;
assignment_statement ::= identifier := expression
if_statement ::= IF expression THEN statement_sequence
                   ELSE statement_sequence END
while_statement ::= WHILE expression DO statement_sequence END
print_statement ::= PRINT '(' expression ')'
type ::= INT
expression ::= simple_expression relop simple_expression | simple_expression
relop ::= < | > | =
simple_expression ::= simple_expression addop term | term
addop ::= + | -
term ::= term mulop primary | primary
mulop ::= * | /
primary ::= identifier | integer | '(' simple_expression ')'
```

You will be provided a scanner in file scanner.P, which will read a source file containing a program, and produce a set of token facts that can be processed by your DCG grammar. The scanner will return keywords directly as tokens; identifiers as terms ident(IdentName); integers as int(IntegerVal); and special characters as themselves as single tokens. The tokens will be asserted as word/3 facts, and you can use your DCG to process them, as indicated in parse_file/2 below.

Your grammar must recognize all programs according to this grammar and, for each recognized program, produce an abstract syntax tree (AST) for that program. The AST must have the following form:

(a) Expressions should have operators as interior nodes (named by the operator symbol) and have as children the ASTs for their operands. Identifiers should be ident(Identifier) and integers int(Integer).

(b) An assignment statement AST should have the form assign(Var,ExprAST); an if-then-else statement AST should have the form if(ExprAST,ThenStmtAST,ElseStmtAST);

a while statement AST should have the form while(ExprAST,DoStmtAST); a print statement AST should have the form print(ExprAST).

(c) A statement sequence AST should be a list of statement ASTs.

(d) An identifier list AST should be a list of identifiers.

(e) A variable declaration is an IdentifierListAST, as is a variable declaration list; as is a declaration; as is a declaration sequence. (Since the langauge has only INT variables, type indications are unnecessary; we need only the names of the variables.)

(f) A block AST is of the form block(IdentifierListAST,StmtListAST).

(g) A module AST is of the form module(Name,BlockAST).

You should write a predicate parse_file/2, which takes the name of a file containing a program, and if the program is in the language of the grammar, returns its Abstract Syntax Tree.

You may import the scanner by putting the scanner.P file in your directory and then adding the following declaration to your grammar program:

```
:- import scan_file/2 from scanner.
```

Then you can call it with the following code:

```
parse_file(ProgramFile,AST) :-
    scan_file(ProgramFile,Length),
    module(AST,0,Length).
```

Chapter 7

Automata Theory in XSB

In this chapter we explore how we can use XSB to understand and implement some of the formal systems of automata theory. We will begin by defining finite state machines (FSM), and exploring executable specifications in XSB of string acceptance, epsilon-free machines, deterministic machines and other interesting notions.

7.1 Finite State Machines

We represent a finite state machine using three relations:

1. m(MachineName,State,Symbol,TargetState) which describes the transition relation for a machine, with MachineName being the name of the machine (to allow us to represent many machines using the same relation), State is a state of the FSM, Symbol is an input symbol, and TargetState is a state to which the machine transitions from State on seeing Symbol.
2. mis(MachineName,InitialState) where InitialState is the initial state of the FSM named MachineName.
3. mfs(MachineName,FinalState) where FinalState is a final state of the FSM named MachineName.

By including a MachineName in each tuple, we can use these relations to represent a number of different FSM's. This will be convenient later.

We will use the symbol `''` (the atom with the empty name) as a pseudo-symbol to represent epsilon transitions, transitions a machine can make without consuming any input symbol.

For example, the following relations represent the machine pictured in Figure 7.1, which we will call m0s1s2s, since it recognizes strings made up of a string of 0's followed by a string of 1's

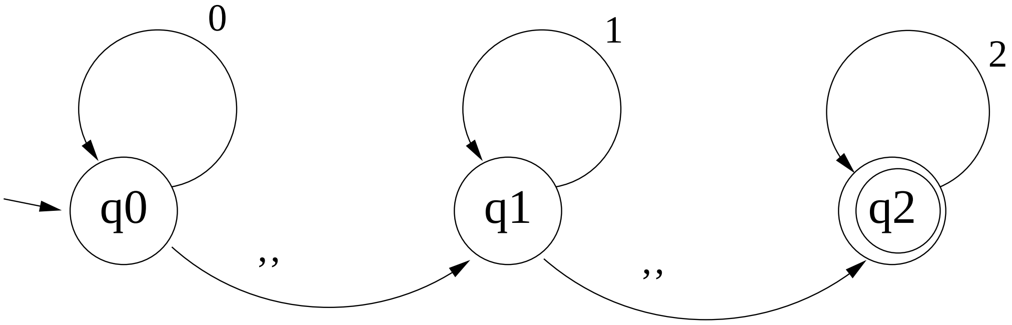


Figure 7.1: The finite state machine: m0s1s2s

followed by a string of 2's:

```
m(m0s1s2s,q0,0,q0).
m(m0s1s2s,q0,'',q1).
m(m0s1s2s,q1,1,q1).
m(m0s1s2s,q1,'',q2).
m(m0s1s2s,q2,2,q2).

mis(m0s1s2s,q0).

mfs(m0s1s2s,q2).
```

We represent strings with two relations. Again we will name strings for convenience.

1. string(StringName,Index,Symbol,Index1) where StringName is the name of the string, Symbol is the Index1-th symbol in the string and Index is Index1-1. For example, the string "001112", which we will call s1, would be represented by:

   ```
   string(s1,0,0,1).
   string(s1,1,0,2).
   string(s1,2,1,3).
   string(s1,3,1,4).
   string(s1,4,1,5).
   string(s1,5,2,6).
   ```

   and string "021", which we'll name s2, would be represented by:

   ```
   string(s2,0,0,1).
   string(s2,1,2,2).
   string(s2,2,1,3).
   ```

2. stringlen(StringName,Length) where Length is the length of the string named StringName. For example, for the previous examples, we would have the facts:

```
    stringlen(s1,6).
    stringlen(s2,3).
```

A FSM is said to accept a string if it executes in the following way: it starts in the initial state, and makes a transition to the next state along a path labeled by the symbol it is looking at. It consumes that symbol and then makes another transition based on the next symbol in the string. It continues in this way until all symbols are consumed, and if the machine is then in a final state, the string is accepted; otherwise it is rejected. If there is an epsilon-transition from one state to another, the machine can make such a transition without consuming a symbol of the input.

Now we can easily write a specification in XSB that defines when a machine accepts a string, as follows:

```
:- auto_table.

% A machine accepts a string if the machine starts in the initial state,
%    recognizes the string, ending in a final state and has consumed the
%    entire string.
accept(MachineName,StringName) :- mis(MachineName,StateStart),
        recognize(MachineName,StringName,StateStart,StateFinal,0,StringFinal),
        mfs(MachineName,StateFinal),
        stringlen(StringName,StringFinal).

% recognize(MachineName,StringName,MState0,MState,SLoc0,SLoc) is true
% if machine MachineName started in state MState0 can transition to
% state MState by recognizing the substring from location SLoc0 to SLoc
% of the string named StringName.

% The empty input string
recognize(_,_,MState,MState,SLoc,SLoc).
% regular transitions
recognize(MachineName,StringName,MState0,MState,SLoc0,SLoc) :-
        string(StringName,SLoc0,Symbol,SLoc1),
        m(MachineName,MState0,Symbol,MState1),
        recognize(MachineName,StringName,MState1,MState,SLoc1,SLoc).
% Epsilon transitions
recognize(MachineName,StringName,MState0,MState,SLoc0,SLoc) :-
        m(MachineName,MState0,'',MState1),
        recognize(MachineName,StringName,MState1,MState,SLoc0,SLoc).
```

The definition of `accept` says that a machine accepts a string if StateStart is the initial state of the indicated machine, and the machine transits from StateStart to StateFinal while recognizing the string starting from 0 and ending at StringFinal, and StateFinal is a final state of the machine, and StringFinal is the length of the string.

The definition of `recognize/6` describes how a machine moves through its states while recognizing (or generating) a sequence of symbols. The first clause says that for any machine and any string, when the machine stays in the same state, no symbols of the string are processed. The second clause says that a machine moves from MState0 to MState recognizing a substring if the next symbol in the substring is Symbol, and there is a transition of the current machine on that Symbol that takes the machine from MState0 to MState1, and the machine recognizes the rest of the substring from that state MState1 getting to MState. The third clause handles epsilon transitions; it says that a machine moves from MState0 to MState recognizing a substring if there is an epsilon transition from MState0 to a MState1 and the machine recognizes the entire string from MState1 to MState.

For example, accept(m0s1s2s,s1) succeeds, but accept(m0s1s2s,s2) fails:

```
warren% xsb
XSB Version 1.6.0 (96/6/15)
[sequential, single word, optimal mode]
| ?- [automata].
[automata loaded]

yes
| ?- accept(m0s1s2s,s1).
++Warning: Removing incomplete tables...

yes
| ?- accept(m0s1s2s,s2).

no
| ?-
```

This is a right-recursive definition of `recognize/6`, so it might seem that this should not need tabling. And indeed for this particular machine and string, Prolog would evaluate this definition just fine. However, there are machines for which tabling is required. Can you give an example of such a machine?

Also, it is possible to give this specification in a left-recursive manner. That is also an exercise for the reader.

7.1.1 Intersection of FSM's

Given two FSM's, one can ask the question as to whether there is a string that both machines accept, that is, whether the intersection of the languages accepted by the two machines is non-empty.

This turns to be possible, and not very difficult. As a matter of fact, we've already essentially written a program that does this. You might have noticed that our representations of strings and machines are actually very similar. In fact, a string can be understood as a FSM, simply by viewing the string/4 predicate as a machine transition predicate. In this case the string's states are integers, the initial state is 0 and the final state is the string length. Viewed this way, a string is simply a FSM that recognizes exactly that one string.

Viewed in this way, the `accept/2` predicate above, which we wrote as determining whether a FSM accepts a string, can be trivially modified to determine whether two machines accept languages with a non-empty intersection. We leave it to the reader to modify the definition of `accept/2` to check intersection, and to test it with several examples.

7.1.2 Epsilon-free FSM's

Two FSM's are said to be *equivalent* if they accept exactly the same set of strings. Given any nondeterministc FSM, it is always possible to find an equivalent one that has *no* epsilon transitions. In fact, given a machine as defined and represented above, we can easily define such an equivalent epsilon-free machine. So given a machine named *mach* and defined in m/4, mis/2 and mfs/2, we will define the transitions, initial state and final state for its epsilon-free version named *efree(mach)* as follows:

```
% epsilon-free machines

% first define emoves as any sequence of epsilon transitions
emoves(_,State,State).
emoves(Mach,State0,State) :-
        emoves(Mach,State0,State1),
        m(Mach,State1,'',State).


% define the transition relation of the efree machine
m(efree(Mach),State,Symbol,TargState) :-
        emoves(Mach,State,State1),
        m(Mach,State1,Symbol,State2),
        Symbol \== '',
        emoves(Mach,State2,TargState).

% define the initial and final states of the efree machine
mis(efree(Mach),IS) :- mis(Mach,IS).
mfs(efree(Mach),FS) :- mfs(Mach,FS1),emoves(Mach,FS,FS1).
mfs(efree(Mach),FS) :- mfs(Mach,FS1),emoves(Mach,FS1,FS).
```

The predicate `emoves/3` defines for any machine the set of pairs of states such that the machine can move from the first state to the second state without consuming any symbols in the input string. Then with this definition, the rule defining transitions says that an epsilon-free machine can move from State to TargState on Symbol if it can move from State to State1 using only epsilon moves, can move from State1 to State2 on seeing Symbol (which is *not* epsilon) and can make epsilon moves from State2 to TargState.

The initial state of the epsilon-free machine is exactly the initial state of the original machine. The final state of the epsilon-free machine is any state from which you can get to a final state of the original machine using only epsilon transitions, or any state you can get to from a final state using only epsilon transitions.

For example:

```
warren% xsb
XSB Version 1.6.0 (96/6/15)
[sequential, single word, optimal mode]
| ?- [automata].
[automata loaded]

yes
| ?- m(efree(m0s1s2s),So,Sym,Ta),writeln(m(efree(m0s1s2s),So,Sym,Ta)),fail.
m(efree(m0s1s2s),q0,0,q0)
m(efree(m0s1s2s),q0,0,q1)
m(efree(m0s1s2s),q0,0,q2)
m(efree(m0s1s2s),q1,1,q1)
m(efree(m0s1s2s),q1,1,q2)
m(efree(m0s1s2s),q2,2,q2)
m(efree(m0s1s2s),q0,1,q2)
m(efree(m0s1s2s),q0,1,q1)
m(efree(m0s1s2s),q1,2,q2)
m(efree(m0s1s2s),q0,2,q2)

no
| ?- mis(efree(m0s1s2s),IS),writeln(mis(efree(m0s1s2s),IS)),fail.
mis(efree(m0s1s2s),q0)

no
| ?- mfs(efree(m0s1s2s),FS),writeln(mfs(efree(m0s1s2s),FS)),fail.
mfs(efree(m0s1s2s),q2)
mfs(efree(m0s1s2s),q0)
mfs(efree(m0s1s2s),q1)

no
| ?-
```

The diagram for `efree(m0s1s2s)` is shown in Figure 7.2.

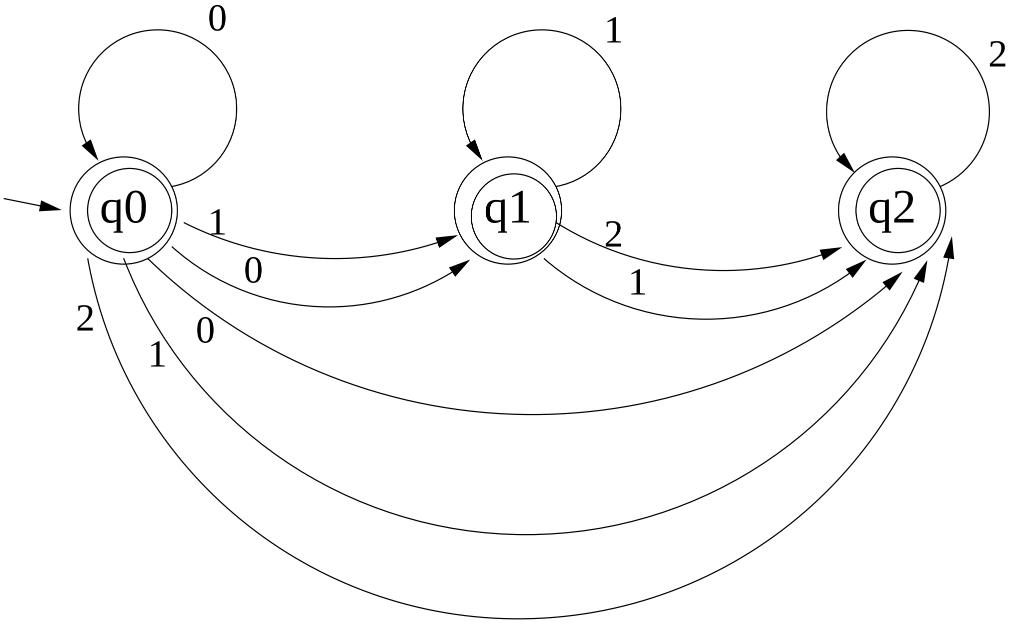


Figure 7.2: The finite state machine: efree(m0s1s2s)

7.1.3 Deterministic FSM's

A deterministic FSM is a machine such that for every state for any symbol there is at most one transition from that state labeled with that symbol (and there are no epsilon transitions.) This means that there is never a choice in how the machine is to proceed when it sees a symbol: there will be at most one state to move to, given that symbol. The question arises as to whether given an arbitrary FSM there is always an equivalent deterministic FSM, i.e., a deterministic FSM that accepts the same language, i.e., exactly the same set of strings.

The answer turns out to be "yes", and it is not difficult to see why. Given a nondeterministic (ND) machine, we can construct a deterministic machine each of whose states corresponds to a set of states in the nondeterministic machine. The idea is that, after seeing a string, the deterministic machine will be in a state corresponding to a set of ND states just in case the ND FSM could be in any one of the ND states after seeing the same string. As a very trivial example, say we had a ND machine with three states: $q1$, $q2$, $q3$, with $q1$ the initial state and transitions from $q1$ to $q2$ on symbol a and from $q1$ to $q3$ also on a. Then the deterministic machine would have two states, $\{q1\}$ and $\{q2, q3\}$ (each being a set of the original machine's states), and a transition from the first to the second on symbol a.

The following specification describes this construction. Rather than constucting *all* the states (which would necessarily be exponential in the number of states in the nondeterministic machine), we will only construct those that are reachable from the initial state. This may be a much smaller number. Also, for this particular specification to be constructive, we need to constrain the set of possible deterministic states in some way, and this seems a good way.

We will assume that the machine is an epsilon-free machine. If `efreemach` is the name of an epsilon-free machine then `det(efreemach)` is the name of an equivalent deterministic machine.

```
:- import member/2 from basics.
:- import tsetof/3 from setof.

% Assume Mach is an epsilon-free machine.
% A state is reachable if it is the initial state or if it can be
%   reached by one step from a reachable state.
reachable(Mach,S) :- mis(Mach,S).
reachable(Mach,S) :- reachable(Mach,S1),m(Mach,S1,_,S).

% The next state of the deterministic machine given a state and symbol
%   is the set of states of the nondeterministic machine which are a
%   next state starting from some element of the current state of the
%   deterministic machine. (Mach is assumed to be epsilon-free.)
m(det(Mach),State0,Sym,State) :-
    reachable(det(Mach),State0),
    tsetof(NDS, a_next(Mach,State0,Sym,NDS), State).

% A state is a next state if it is a next state reachable in one step
%   from some member of the current state of the deterministic machine.
a_next(Mach,DState,Sym,NDState) :-
    member(S1,DState),
    m(Mach,S1,Sym,NDState).

% The initial state is the singleton set consisting of the initial
%   state of the nondeterministic machine.
mis(det(Mach),[IS]) :- mis(Mach,IS).

% A final state is a reachable deterministic state that contains some
%   final state.of the nondeterministic machine.
mfs(det(Mach),FS) :- mfs(Mach,NFS), reachable(det(Mach),FS),member(NFS,FS).
```

Now we can use this specification to find a deterministic machine that is equivalent to the nondeterministic machine *m*0*s*1*s*2*s*:

```
| ?- m(det(efree(m0s1s2s)),S,Sy,T),writeln(m(det(m0s1s2s),S,Sy,T)),fail.
m(det(m0s1s2s),[q0],0,[q0,q1,q2])
m(det(m0s1s2s),[q0],1,[q1,q2])
m(det(m0s1s2s),[q0],2,[q2])
m(det(m0s1s2s),[q0,q1,q2],0,[q0,q1,q2])
m(det(m0s1s2s),[q0,q1,q2],1,[q1,q2])
m(det(m0s1s2s),[q0,q1,q2],2,[q2])
```

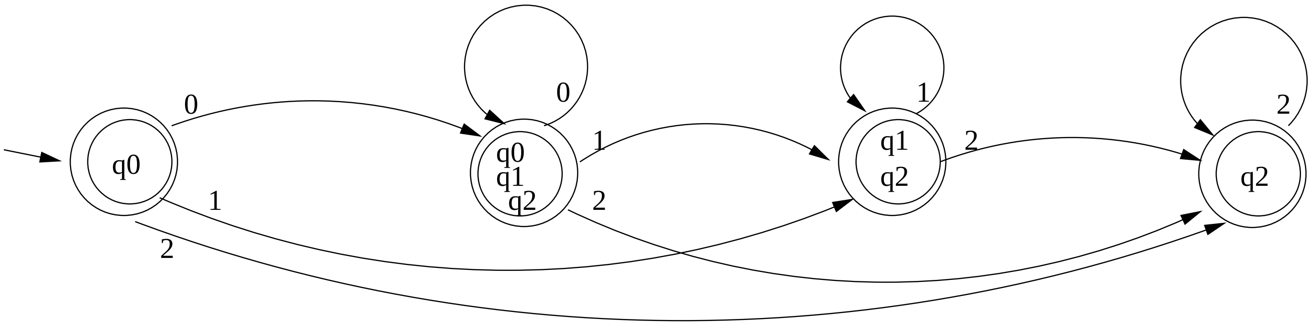


Figure 7.3: The finite state machine: det(efree(m0s1s2s))

```
m(det(m0s1s2s),[q1,q2],1,[q1,q2])
m(det(m0s1s2s),[q1,q2],2,[q2])
m(det(m0s1s2s),[q2],2,[q2])

no
| ?- mis(det(efree(m0s1s2s)),S),writeln(mis(det(m0s1s2s),S)),fail.
mis(det(m0s1s2s),[q0])

no
| ?- mfs(det(efree(m0s1s2s)),S),writeln(mfs(det(m0s1s2s),S)),fail.
mfs(det(m0s1s2s),[q1,q2])
mfs(det(m0s1s2s),[q0,q1,q2])
mfs(det(m0s1s2s),[q2])
mfs(det(m0s1s2s),[q0])

no
| ?-
```

The diagram for `det(efree(m0s1s2s))` is shown in Figure 7.3.

7.1.4 Complements of FSM's

One may also want to construct a machine that accepts the complement of the set of strings a given machine accepts. This turns out to be possible and reasonably easy to do, once we straighten out a minor issue. Up to now we have been mostly ignoring the alphabet of symbols that make up the strings. This has been fine for accepting strings, since if a symbol never appears in any transition, the machine can never accept any string containing that symbol. But when we talk about strings that a machine rejects, we have to give the alphabet with respect to which to take the complement. For example, what is the complement of the language that consists of all strings consisting of only a's and b's? It is the empty set if the alphabet is $\{a, b\}$, but if the alphabet is $\{a, b, c\}$, then it is the set of all strings over $\{a, b, c\}$ that contain at least one c.

Using our current representation, we will assume that the alphabet of a given machine is the set of symbols that appear on *some* transition of that machine. While this seems to be a reasonable assumption, note that it could be incorrect for the second example of the complement of all strings of a's and b's given above. (If we were committed to being very precise, we could add an XSB relation which for each machine defined the set of symbols in its alphabet.)

The basic idea for generating the complement machine is simply to take the original machine but to take the complement of its final states to the be final states of the complement-accepting machine. But there are a couple of things we have to guarantee before this works. First the original machine must be deterministic, since otherwise for a given string it might end in both a final and a nonfinal state. In this case it should be rejected by the machine accepting the complement language, but simply inverting the final and nonfinal states would result in it being accepted. So we will always start with a deterministic machine. Second, we have to be sure that the machine gets to some state on *every* input. That is, there must always be a transition that can be taken, regardless of the symbol being scanned. That is, every state must have an outgoing transition for every symbol in the alphabet. And here is where the importance of the alphabet is clear.

So we will separate our construction into two parts: first we will complete the machine (assuming it is deterministic) by adding transitions to a new state, called "sink," when there are no transitions on some symbols; and second we will complement a completed machine.

```
% completed machine
% A symbol is in the alphabet of a machine if it appears in a non-epsilon
%    transition.  (Note that this is our convention and for some machines
%    could be wrong.)
alphabet(Mach,Sym) :-
    m(Mach,_,Sym,_),
    Sym \== ''.

% S is a (possibly reachable) state in machine if it's initial or has an
%    incoming edge.
is_state(Mach,S) :- m(Mach,_,_,S).
is_state(Mach,S) :- mis(Mach,S).

% The initial states and final states of the completed machine are the
%   same as the original machine.
mis(completed(Mach),IS) :- mis(Mach,IS).
mfs(completed(Mach),FS) :- mfs(Mach,FS).

% Assume Mach is deterministic
% There is a transition to ``sink'' if there is no other transition on
%    this symbol from this state.
m(completed(Mach),So,Sy,sink) :-
    is_state(Mach,So),
    alphabet(Mach,Sy),
    tnot(isatransition(Mach,So,Sy)).
% Machine transitions from sink to sink on every symbol
m(completed(Mach),sink,Sy,sink) :-
    alphabet(Mach,Sy).
% Otherwise the same as underlying machine
m(completed(Mach),So,Sy,Ta) :-
    m(Mach,So,Sy,Ta).

% There is a transition if there's a state it transits to.
isatransition(Mach,So,Sy) :-
    m(Mach,So,Sy,_).
```

Now given a completed machine, we can easily generate the complement machine simply by interchanging final and nonfinal states:

```
% complement machine
% Asume machine is completed and deterministic.
% The transitions of the complement machine are the same.
m(complement(Mach),So,Sy,Ta) :- m(Mach,So,Sy,Ta).

% The initial state of the complement machine is the same.
mis(complement(Mach),S) :-  mis(Mach,S).

% A state is a final state of the complement if it is NOT the final state
%    of the underlying machine.
mfs(complement(Mach),S) :-
    is_state(Mach,S),
    tnot(mfs(Mach,S)).
```

With these definitions, we can compute the complement of our simple machine m0s1s2s:

```
| ?- [automata].
[automata loaded]

yes
| ?- m(complement(completed(det(efree(m0s1s2s)))),S,Sy,T),
     writeln(m(complement(m0s1s2s),S,Sy,T)),fail.
m(complement(m0s1s2s),[q2],1,sink)
m(complement(m0s1s2s),[q2],0,sink)
m(complement(m0s1s2s),[q1,q2],0,sink)
m(complement(m0s1s2s),sink,2,sink)
m(complement(m0s1s2s),sink,1,sink)
m(complement(m0s1s2s),sink,0,sink)
```

```
m(complement(m0s1s2s),[q1,q2],2,[q2])
m(complement(m0s1s2s),[q1,q2],1,[q1,q2])
m(complement(m0s1s2s),[q0,q1,q2],2,[q2])
m(complement(m0s1s2s),[q0,q1,q2],1,[q1,q2])
m(complement(m0s1s2s),[q0,q1,q2],0,[q0,q1,q2])
m(complement(m0s1s2s),[q2],2,[q2])
m(complement(m0s1s2s),[q0],2,[q2])
m(complement(m0s1s2s),[q0],1,[q1,q2])
m(complement(m0s1s2s),[q0],0,[q0,q1,q2])

no
| ?- mis(complement(completed(det(efree(m0s1s2s)))),S),
     writeln(mis(complement(m0s1s2s),S)),fail.
mis(complement(m0s1s2s),[q0])

no
| ?- mfs(complement(completed(det(efree(m0s1s2s)))),S),
     writeln(mfs(complement(m0s1s2s),S)),fail.
mfs(complement(m0s1s2s),sink)

no
| ?-
```

Given these definitions, we can now write a specification that determines when two machines accept the same language. With complement and intersection, we can define subset: $A \subseteq B \iff A \cap \overline{B} = \emptyset$. We leave it as an exercise for the reader to write and test such a specification.

7.1.5 Minimization of FSM's

Another question of interest is whether a given FSM has "redundant" states. That is, is it as small as it can be or is there a smaller machine, i.e., one with fewer states, that can recognize the same language.

So the idea is, given a machine, to see whether it has redundant states. The first step is to determine whether two states in the machine are distinguishable, i.e., whether there is some string such that when the machine is started in the respective states, one computation will lead to an accepting state and the other won't. The following specification defines (and computes) distinguishable states.

```
% Assume Mach is a deterministic machine
% S1 and S2 are distinquishable if S1 is final and S2 is not.
distinguishable(Mach,S1,S2) :-
        mfs(Mach,S1),
        is_state(Mach,S2),
        tnot(mfs(Mach,S2)).
% S1 and S2 are distinquishable if S2 is final and S1 is not.
distinguishable(Mach,S1,S2) :-
        mfs(Mach,S2),
        is_state(Mach,S1),
        tnot(mfs(Mach,S1)).
% S1 and S2 are distinguishable if some symbol Sy takes them to states that
%    are distinguishable.
distinguishable(Mach,S1,S2) :-
        m(Mach,S1,Sy,T1),
        m(Mach,S2,Sy,T2),
        distinguishable(Mach,T1,T2).
```

The first two rules say that states are distinguishable if one is final and the other is not. For this we need the constraint that the initial machine be deterministic. The third rule says that states are distinguishible if there is a symbol on which they make transitions to distinguishable states.

As an example of finding distinguishable states, we can use the following machine:

```
m(dfa,a,0,b).
m(dfa,a,1,f).
m(dfa,b,0,g).
m(dfa,b,1,c).
m(dfa,c,0,a).
m(dfa,c,1,c).
m(dfa,d,0,c).
m(dfa,d,1,g).
m(dfa,e,0,h).
m(dfa,e,1,f).
m(dfa,f,0,c).
m(dfa,f,1,g).
m(dfa,g,0,g).
m(dfa,g,1,e).
m(dfa,h,0,g).
m(dfa,h,1,c).

mis(dfa,a).
mfs(dfa,c).
```

(draw a picture. Could also use the determistic version of ms0s1s2, since it has an extra state q0, I think.)

And with this machine we get the following evaluation:

```
| ?- distinguishable(dfa,S1,S2),S1@<S2,writeln(d(S1,S2)),fail.
d(d,e)
d(d,g)
d(d,h)
d(b,d)
d(b,e)
d(b,f)
d(b,g)
d(b,c)
d(a,b)
d(a,d)
d(a,f)
d(a,g)
d(a,h)
d(a,c)
d(f,g)
d(f,h)
d(e,f)
d(e,g)
d(e,h)
d(g,h)
d(c,d)
d(c,e)
d(c,h)
d(c,g)
d(c,f)

no
| ?-
```

In the query we filtered to keep only the cases in which the first state has a smaller name than the second. This was just to avoid printing out all the commutative and reflexive pairs.

What is more interesting than finding two states that are distinguishable is finding two states that are **not** distinguishable. In this case one of the states is unnecessary and can be eliminated from the machine. So using this definition of `distinguishable`, we can construct a minimal DFSM that accepts the language given by a machine by merging indistinguishable states in that machine.

```
% min (assuming the machine is deterministic), reduce by
%   indistinguishability.
m(min(Mach),So,Sy,Ta) :-
    reachable(min(Mach),So),  % So is a set of indistinguishable states
    member(Ss,So),            % Ss is one of them..
    m(Mach,Ss,Sy,T),          % ..that can transit to T on Symbol Sy
    tsetof(S,indistinguishable(Mach,T,S),Ta).  % target is those indistinguishable from T
```

```
% The initial (final) state is the set of states indistinguishable
%   from the initial (final) state of the base machine.
mis(min(Mach),IS) :-
    mis(Mach,Bis),
    tsetof(S,indistinguishable(Mach,Bis,S),IS).
mfs(min(Mach),FS) :-
    mfs(Mach,Bfs),
    tsetof(S,indistinguishable(Mach,Bfs,S),FS).

indistinguishable(Mach,S1,S2) :-
    is_state(Mach,S1),
    is_state(Mach,S2),
    tnot(distinguishable(Mach,S1,S2)).
```

And executing this with the previous example, we get:

```
| ?- m(min(dfa),S,Sy,T),writeln(m(min(dfa),S,Sy,T)),fail.
m(min(dfa),[a,e],1,[f])
m(min(dfa),[a,e],0,[b,h])
m(min(dfa),[f],1,[g])
m(min(dfa),[f],0,[c])
m(min(dfa),[b,h],1,[c])
m(min(dfa),[b,h],0,[g])
m(min(dfa),[g],1,[a,e])
m(min(dfa),[g],0,[g])
m(min(dfa),[c],1,[c])
m(min(dfa),[c],0,[a,e])

no
| ?- mfs(min(dfa),S),writeln(mfs(min(dfa),S)),fail.
mfs(min(dfa),[c])

no
| ?- mis(min(dfa),S),writeln(mis(min(dfa),S)),fail.
mis(min(dfa),[a,e])

no
| ?-
```

(Draw state diagram) Note that the state “d” does not appear in this collapsed machine. This is because it has no in-transitions and is not the initial state, so it doesn’t appear in our reduction. It is actually equivalent to state “f”, and could be merged with it.

7.1.6 Regular Expressions

Regular expressions are another way of specifying finite state languages, that is sets of strings of symbols. A regular expression over an alphabet Σ is:

1. a symbol from Σ, or
2. an expression $(RE1 * RE2)$, where $RE1$ and $RE2$ are regular expressions, or
3. an expression $(RE1 + RE2)$, where $RE1$ and $RE2$ are regular expressions, or
4. an expression $@(RE)$, where RE is a regular expression.

We associate with each regular expression (RE) a set of strings over Σ (i.e., a language.) We will use XSB (Prolog) rules to define the set of strings associated with a RE. An RE will be represented as a Prolog term, and the definition below shows that we are using $*$ for concatentation, $+$ for alternation, and @ for iteration.

The following program, when given a regular expression, accepts strings that are in the language represented by that expression:

```
% Is StringName in the language represented by Exp?
reacc(Exp,StringName) :-
    reacc(Exp,StringName,0,F),
    stringlen(StringName,F).

% An atom represents itself
reacc(A,S,From,To) :- atomic(A),string(S,From,A,To).
% Concatenation of E1 and E2
reacc((E1*E2),S,From,To) :-
    reacc(E1,S,From,M),
    reacc(E2,S,M,To).
% Alternation
reacc((E1+_E2),S,From,To) :- reacc(E1,S,From,To).
reacc((_E1+E2),S,From,To) :- reacc(E2,S,From,To).
% Iteration if 0 or more occurrences
reacc(@(_E),_S,From,From).
reacc(@(E),S,From,To) :-
    reacc(@(E),S,From,Mid),
    reacc(E,S,Mid,To).
```

Now we can test whether string s1 (00112) is in the language represented by the regular expression $@(0) * @(1) * @(2)$:

```
| ?- reacc(@(0)* @(1)* @(2),s1).
++Warning: Removing incomplete tables...

yes
| ?-
```

and it is.

THIS FOLLOWING DEVELOPMENT OF FSMs FROM REs IS INCORRECT. IT CONFLATES MULTIPLE OCCURRENCES OF THE SAME RE SUBEXPRESSION. FIX OR DELETE!!!! -DSW

It turns out that regular expressions represent exactly the same languages that finite state machines do. Given a regular expression, we can construct a finite state machine that recognizes the same language (and vice versa.) So first we will construct a machine given a regular expression. To make the construction easy, we will represent machine names and states in an interesting way. Given a regular expression RE, we will use m(RE) to name the machine for recognizing the same langauge as RE. And the initial state for the constructed machine that recognizes the language of RE will be named i(RE) and the final state (there will always be exactly one in our construction) will be f(RE). Note that we are free to name the machines and states anything we want, so this choice is simply a convenience.

The following rules define the FSM given a regular expression:

```
% The machine for an atomic RE, simply transits from its initial state
%    to its final state on the given atomic symbol.
%    (All the others will be epsilon transitions.)
m(re(RE),i(RE),RE,f(RE)) :- atomic(RE).

% To recognize concatenated expressions:
% Connect the initial state of the compound expr to the initial state of
%    the first subexpr.
m(re(RE1*RE2),i(RE1*RE2),'',i(RE1)).

% Connect the final state of the first subexpr to the initial state of
%     the second subexpr.
m(re(RE1*RE2),f(RE1),'',i(RE2)).

% Connect the final state of the second subexpr to the final state of
%     the compound expr.
m(re(RE1*RE2),f(RE2),'',f(RE1*RE2)).

% And finally must add the transitions of the machines for the
%    subexpressions.
m(re(RE1*_RE2),S,Sy,T) :- m(re(RE1),S,Sy,T).
```

```
m(re(_RE1*RE2),S,Sy,T) :- m(re(RE2),S,Sy,T).

% The process is analogous for alternation.
m(re(RE1+RE2),i(RE1+RE2),'',i(RE1)).
m(re(RE1+RE2),i(RE1+RE2),'',i(RE2)).
m(re(RE1+RE2),f(RE1),'',f(RE1+RE2)).
m(re(RE1+RE2),f(RE2),'',f(RE1+RE2)).
m(re(RE1+_RE2),S,Sy,T) :- m(re(RE1),S,Sy,T).
m(re(_RE1+RE2),S,Sy,T) :- m(re(RE2),S,Sy,T).

% and for iteration
m(re(@(RE)),i(@(RE)),'',f(@(RE))).
m(re(@(RE)),i(@(RE)),'',i(RE)).
m(re(@(RE)),f(RE),'',f(@(RE))).
m(re(@(RE)),f(@(RE)),'',i(@(RE))).
m(re(@(RE)),S,Sy,T) :- m(re(RE),S,Sy,T).

% and the initial and final states are just those named i() and f().
mis(re(RE),i(RE)).
mfs(re(RE),f(RE)).
```

As an example, consider the following execution

```
| ?- m(re(a*b*c),S,Sy,T),writeln(m(re(a*b*c),S,Sy,T)),fail.
m(re(a * b * c),i(a * b * c),,i(a * b))
m(re(a * b * c),f(a * b),,i(c))
m(re(a * b * c),f(c),,f(a * b * c))
m(re(a * b * c),i(a * b),,i(a))
m(re(a * b * c),f(a),,i(b))
m(re(a * b * c),f(b),,f(a * b))
m(re(a * b * c),i(a),a,f(a))
m(re(a * b * c),i(b),b,f(b))
m(re(a * b * c),i(c),c,f(c))

no
| ?- m(det(efree(re(a*b*c))),S,Sy,T),writeln(m(re(a*b*c),S,Sy,T)),fail.
m(re(a * b * c),[i(a * b * c)],a,[f(a),i(b)])
m(re(a * b * c),[f(a),i(b)],b,[f(b),f(a * b),i(c)])
m(re(a * b * c),[f(b),f(a * b),i(c)],c,[f(c),f(a * b * c)])

no
| ?-
```

Here we first constructed the FSM that recognizes the single string *abc*, which has 9 transitions, most of them epsilon transitions. In the second query, we found the deterministic version of that machine. Notice that this dropped the number of transitions to the minimal three.

The final problem concerning FSM's and regular expressions that we will consider is that of, given a machine, constructing an equivalent regular expression.

```
re(S,T,0,Mach,RE) :- is_state(S),is_state(T),S\==T,tsetof(Sy,m(Mach,S,Sy,T),RE).
re(S,S,0,Mach,[''|RE]) :- is_state(S),tsetof(Sy,m(Mach,S,Sy,T),RE).
re(I,J,K,Mach,[RE1* @(RE2) * RE3,RE4]) :- K>0,
    K1 is K-1,
    re(I,K,K1,Mach,RE1),
    re(K,K,K1,Mach,RE2),
    re(K,J,K1,Mach,RE3),
    re(I,J,K1,Mach,RE4).
```

7.2 Grammars Revisited

In a previous chapter we explored how we could represent grammars in tabled Prolog and obtain efficient recognition algorithms. Recall the Prolog program from the DCG for the expression context-free grammer:

```
% file grammar.P
:- table expr/3, term/3.

expr(S0,S) --> expr(S0,S1), word(S1,+,S2), term(S2,S).
expr(S0,S) --> term(S0,S).
term(S0,S) --> term(S0,S1), word(S1,*,S2), primary(S2,S).
term(S0,S) --> primary(S0,S).
primary(S0,S) --> word(S0,'(',S1), expr(S1,S2), word(S2,')'S).
primary(S0,S) --> word(S0,Int,S1), integer(Int).
```

We understood a grammar rule, for example the first one, as saying that an expr generated a string from position S0 to S in the input string if an expr generated a string from position S0 to S1 and the word '+' spanned from positions S1 to S2, and a term spanned from position S2 to S.

There is another way to understand DCG rules declaratively. Recall that we can consider the input string word/3 representation as a simple FSA, that recognizes exactly the input string. The first string position is the initial state, the last string position is the accepting state, and the word facts define the state transitions. We can think of the DCG rules as add new transitions, on new symbols (nonterminal symbols) to that basic (linear) FSA. An atomic formula, such as expr(S0,S) defines a transition from state S0 to S on the symbol 'expr'. Then we can understand a rule, for example the first one, as saying that if there is a transition from state S0 to S1 on the symbol 'expr', and from state S1 to S2 on '+' and from state S2 to S on 'term', then there is a transition from state S0 to S on 'expr'. So the grammar rules (along with the facts of the initial string) can be understood as defining a finite state automoton. The initial set of word/3 facts defines the initial transitions of the FSA (on terminal symbols) and the DCG rules define further transitions (on nonterminal symbols). The first state of string is the initial state of the FSA and the last state of the string is the final, accepting state of the FSA.

So the question is: What strings does the defined FSA accept? The answer is all sentential forms (i.e., strings of terminals and nonterminals) that can eventually generate the given initial string of terminals. So the query expr(InitialState,FinalState) is true just in case the sentential form consisting of the single nonterminal 'expr' can generate the initial input string. I.e., if the initial string is in the language generated by expr.

So all of this is just a slightly different way to understand what a DCG specifies, i.e., how a DCG specifies a language.

Looked at this way, we can specify the facts specifying the input string using existential variables, for example the string $1 + 2$, as follows:

```
exists(S1,exists(S2,exists(S3,exists(S4,
  initial_state(S1)/\word(S1,1,S2)/\word(S2,+,S3)
       /\word(S3,2,S4)/\final_state(S4)))))
```

and a rule, such as the first one is:

```
forall(S0,forall(S,exists(S1,exists(S2,
    expr(S0,S1)/\word(S1,+,S2)/\term(S2,S)))))
```

Then the DCG rules define further transitions in this FSA, and the goal:

```
| ?- initial_state(S0),final_state(S),expr(S0,S).
```

is true just in case there is a transition from the initial state to the final state labeled by 'expr'.

Understood this way, it is easy to see how to extend DCG's to express context-sensitive grammars. Recall that grammar rules define transitions in a FSA that recognizes sentential forms that generate the input string. Given a context sensitive rule such as:

```
AB --> CD.
```

we want to define a new pair of transitions on A and then on B, if there already are transitions on C and D. We can say this logically with the formula:

```
forall(S0,forall(S,thereis(S1,'A'(S0,S1)/\'B'(S1,S)) -->
              thereis(S3,'C'(S0,S3)/\'D'(S3,S))))
```

So this rule asserts the (conditional) existence of a state (with variable S1) and two transitions. In our extended Prolog, this is the rule:

```
S3 ^ (c(S0,S3),D(S3,S)) :- a(S0,S1),B(S1,S).
```

7.3 Push-Down Automata

7.4 Exercises

1. Represent a FSA using the following three relations:

   (a) m(MachineName,State,Symbol,TargetState) which describes the transition relation for a machine, with MachineName being the name of the machine (to allow us to represent many machines using the same relation), State is a state of the FSM, Symbol is an input symbol, and TargetState is a state to which the machine transitions from State on seeing Symbol. An epsilon transition is indicated by a Symbol of ".

   (b) mis(MachineName,InitialState) where InitialState is the initial state of the FSM named MachineName.

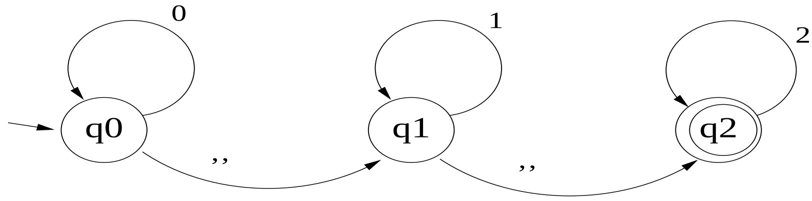
   (c) mfs(MachineName,FinalState) where FinalState is a final state of the FSM named MachineName.

   Represent strings with the following two relations:

   (a) string(SName,I1,Sym,I2), where SName is the name of the string (to permit representing multiple strings); Sym is the symbol at position I2, and I1 is I2-1.

   (b) stringlen(SName,Length), where SName is the name of the string, which has length Length.

   Do the following exercises concerning FSA's.

   (a) 

   Give the facts of m/4, mis/2, and mfs/2 that represent this machine. Give the facts that define strings 0011112, 00021110

(b) Write a predicate accept(Mach,String) which is true if machine Mach accepts the string String.

(c) Given the definition of the machines M1 and M2, write a predicate m(intersect(M1,M2),State,Symbol that defines the transition relation of the machine that accepts the intersection of the two languages accepted by M1 and M2.

Similarly write mis(intersect(M1,M2),IS) and mfs(intersect(M1,M2),FS).

(d) Given the definition of a machine Mach, write a predicate m(efree(Mach),State,Symbol,TargetState) that defines the transition relation of the (or a) epsilon-free machine that recognizes the same language as Mach. [An epsilon-free machine is one that has not epsilon transitons.]

Similarly write mis(efree(Mach),IS) and mfs(efree(Mach),FS).

(e) Similarly write definitions of m(det(Mach),State,Symbol,TargetState), mis(det(Mach),IS) and mfs(det(Mach),FS), where det(Mach) is a deterministic FSA that recognizes the same language as Mach.

2. Write an XSB program to interpret the simple procedural programming language you parsed in assignment 6, and produce the appropriate output, as generated by the PRINT statements.

The semantics of the language should be clear from its syntax. The basic predicate you will need to write should be:

```
interpM3(AST,StateIn,StateOut)
```

where AST is the Abstract Syntax Tree of the portion of the program you are interpreting; StateIn is the state of the program variables at the beginning of that execution; and StateOut is the state of the program variables after the program segment has been executed. You may produce the output, generated by interpreted PRINT statements as a side-effect of the interpretation (by using writeln/1 in XSB.)

One simple and easy representation of the program state is a list of terms of the form: VariableName=VariableValue.

As a very simple example, your predicate should do the following:

```
| ?- interpM3(assign(x,+(ident(x),int(1))),[x=3],StateOut).

StateOut = [x=4];

no
```

You should use the AST's you generated in Exercise ?? of Chapter 5. Note that the / operand means integer division, and so must produce an integer.

Chapter 8

Dynamic Programming in XSB

Dynamic Programming is the name for a general strategy used in algorithms when one organizes the computation to be done in such a way that subproblems are evaluated only once instead of many times. With this description of the dynamic programming strategy, one can see that the tabling strategy of XSB is a *dynamic* dynamic programming strategy. That is, regardless of how the computation is structured at compile time (by the programmer), tabling ensures that subproblems are evaluated only once. So this suggests that problems amenable to dynamic programming solutions might be particularly appropriate for evaluating with XSB. This is indeed the case, and in this chapter we will see a number of examples.

These problems have a common characteristic. They all can be solved by writing down a simple specification of the problem. However, if one thinks as a Prolog programmer about the execution of the specification, it seems horrendously redundant and inefficient. But executing it with tabling declarations eliminates the redundancy and actually turns the specification into an efficient algorithm.

8.1 The Knap-Sack Problem

The first problem we will consider is the knap-sack problem. The idea is that we have a knap-sack and a bunch of things of various sizes to put in it. The question is whether there is a subset of the things that will fit exactly into the knap-sack. The problem can be formally stated as follows:

> Given n items, each of integer size k_i ($1 \leq i \leq n$), and a knap-sack of size K. 1) determine whether there is a subset of the items that sums to K. 2) Find such a subset.

We will represent the items and their sizes by using a set of facts `item/2`, where `item(3,5)` would mean that the third item is of size 5.

To determine whether there is a subset of items that exactly fill the knap-sack, we can just nondeterministicaly try all alternatives.

```
% ks(+I,+K) if there is a subset of items 1,...,I that sums to K.
ks(0,0).                % the empty set sums to 0
ks(I,K) :- I>0,         % don't include this Ith element in the knapsack
    I1 is I-1, ks(I1,K).
ks(I,K) :- I>0,         % do include this Ith element in the knapsack
    item(I,Ki), K1 is K-Ki, K1 >= 0, I1 is I-1, ks(I1,K1).
```

The first clause says that the empty set takes no space in the sack. The second clause covers the case in which the Ith item is *not* included in the sack. The third clause handles the case in which the Ith item *is* included in the sack.

This program could be exponential in the number of items, since it tries all subsets of items. However, there are only I^2 possible distinct calls to ks/2, so tabling will make this polynomial.

This program just finds whether a packing of the knapsack exists; it doesn't return the exact set of items that fit. We could simply add a third argument to this definition of ks/2 which would be the list of items added to the knap-sack. But that might then build an exponential-sized table. For example with every item of size one, there are exponentially many items to include to make a sum. So instead of simply adding another parameter and tabling that predicate, we will use ks/2 to avoid constructing a table unnecessarily. Note that this is similar to how we constructed a parse tree for a grammar by using the recognizer. Notice that ksp/3 uses ks/2 in its definition.

```
ksp(0,0,[]).
ksp(I,K,P) :- I>0,
    I1 is I-1, ks(I1,K),
    ksp(I1,K,P).
ksp(I,K,[I|P]) :- I>0,
    item(I,Ki), K1 is K-Ki, K1 >= 0, I1 is I-1, ks(I1,K1),
    ksp(I1,K1,P).
```

(There is something going on here. Can we figure out a syntax or conventions to make this uniform?)

```
% ks(+I,+K) if there is a subset of items 1,...,I that sums to K.
ks(0,0).
ks(I,K) :- ks1(I,K,_).
  ks1(I,K,I1) :- I>0,I1 is I-1, ks(I1,K).
ks(I,K) :- ks2(I,K,_).
  ks2(I,K,I1) :- I>0, item(I,Ki), K1 is K-Ki, K1 >= 0, I1 is I-1, ks(I1,K1).
```

```
ksp(0,0,[]).
ksp(I,K,P) :- ks1(I,K,I1), ksp(I1,K,P).
ksp(I,K,[I|P]) :- ks2(I,K,I1), ksp(I1,K1,P).
```

8.2 Sequence Comparisons

Another problem where dynamic programming is applicable is in the comparison of sequences. Given two sequences A and B, what is the miniml number of operations to turn A into B? The allowable operations are: insert a new symbol, delete a symbol, and replace a symbol. Each operation costs one unit.

A program to do this is:

```
/* sequence comparisons.  How to change one sequence into another.
A=a_1 a_2 ... a_n
B=b_1 b_2 b_3 ... b_m
Change A into B using 3 operations:
    insert, delete, replace: each operation costs 1.
*/

% c(N,M,C) if C is minimum cost of changing a_1...a_N into b_1...b_M
:- table c/3.
c(0,0,0).
c(0,M,M) :- M > 0.             % must insert M items
c(N,0,N) :- N > 0.              % must delete N items
c(N,M,C) :- N > 0, M > 0,
        N1 is N-1, M1 is M-1,
        c(N1,M,C1), C1a is C1+1,        % insert into A
        c(N,M1,C2), C2a is C2+1,        % delete from B
        c(N1,M1,C3),                    % replace
                a(N,A), b(M,B), (A==B -> C3a=C3; C3a is C3+1),
        min(C1a,C2a,Cm1), min(Cm1,C3a,C).       % take best of 3 ways

min(X,Y,Z) :- X =< Y -> Z=X ; Z=Y.

% example data
a(1,a). a(2,b). a(3,b). a(4,c). a(5,b). a(6,a). a(7,b).
b(1,b). b(2,a). b(3,b). b(4,b). b(5,a). b(6,b). b(7,b).
```

The first three clauses for c/3 are clear; most of the work is done in the last clause. It reduces the problem to a smaller problem in three different ways, one for each of the operations of insert, delete, and replace. Each reduction costs one unit, except that “replacement” of a symbol by itself costs nothing. It then takes the minimum of the costs of these ways of turning string A into string B.

In Prolog this would be exponential. With tabling it is polynomial.

8.3 ??

Dynamic Programming, e.g. optimal association for matrix multiplication

Searching, games (see Bratko)

Pruning (alpha-beta search)

8.4 Exercises

1. Given two sequences A and B, what is the minimal number of operations to turn A into B? The allowable operations are: insert a new symbol, delete a symbol, and replace a symbol. Each operation costs one unit.

   Hint: Represent string A as facts a(N,Sym), meaning that the symbol at position N in string A is Sym. Similarly for B. Then write a predicate cost(N,M,C) which is true if the minimum cost of turning a1...aN into b1...bM is C.

Chapter 9

Tabling for Initialization and Indexing

[This is essentially my paper “Top-down and Bottom-up Evaluation Procedurally Integrated,” which appeared in ICLP2018, and TPLP. I will do major reworking to make it appropriate as a chapter in this book.]

9.1 abstract

This paper describes how XSB combines top-down and bottom-up computation through the mechanisms of variant tabling and subsumptive tabling with abstraction, respectively.

It is well known that top-down evaluation of logical rules in Prolog has a procedural interpretation as recursive procedure invocation [?]. Tabling adds the intuition of short-circuiting redundant computations [?]. This paper shows how to introduce into tabled logic program evaluation a bottom-up component, whose procedural intuition is the initialization of a data structure, in which a relation is initially computed and filled, on first demand, and then used throughout the remainder of a larger computation for efficient lookup. This allows many Prolog programs to be expressed fully declaratively, programs which formerly required procedural features, such as assert, to be made efficient.

9.2 Introduction

Top-down and bottom-up algorithms have long been seen as alternative, even competitive, ways to evaluate Datalog (and Prolog) programs [?, ?], with proponents of each algorithm arguing for its particular advantages [?, ?, ?]. Systems implement either top-down evaluation, e.g., XSB (by default) [?] and other Prolog systems that include tabling, such as YAP [?], Caio Prolog [?], BProlog [?] and as a library in SWI Prolog [?], or bottom-up evaluation, e.g., LDL++ [?], Coral [?] and LogicBlox [?], among many others. In this paper we consider the properties and advantages of each strategy and propose that they be integrated in a single system, with each having a distinct procedural interpretation. Then each can be used when its evaluation strategy is most appropriate and efficient. This paper describes how they are integrated in the XSB tabled logic programming system.

9.3 Brief Review of Top-Down and Bottom-up Evaluation

To briefly review tabled top-down and bottom-up evaluation of Datalog rule sets, we consider the following well-known example of transitive closure for a simple graph:

```
:- table p/2.
                                    e(a,b).     e(b,c).
p(X,Y) :- e(X,Y).                   e(e,a).     e(c,b).
p(X,Y) :- p(X,Z),e(Z,Y).            e(d,e).
```

Example 9.3.1. *For query* `p(a,A)`*, tabled top-down evaluation proceeds as follows:*

```
1 p(a,A)           add query p(a,?) to table
2   e(a,A).        resolve 1 with 1st rule
3     () A=b       resolve 2 with fact e(a,b), add answer p(a,b) to table
4   p(a,Z),e(Z,A)  resolve 1 with 2nd rule
5     e(b,A)       resolve 4 with answer p(a,b) from table
6       () A=c     resolve 5 with fact e(b,c), add answer p(a,c) to table
7     e(c,A)       resolve 4 with answer p(a,c) from table
8       () A=b     resolve 7 with fact e(c,b), p(a,b) in table, don't add
```

Example 9.3.2. *Bottom-up evaluation for the same query proceeds in iterations. At each iteration it uses the program rules and the facts currently known to infer new facts.*

```
Iteration 0: infer the program facts:
        e(a,b), e(e,a), e(d,e), e(b,c), e(c,b)
Iteration 1: Use those facts and program rule 1 to infer:
  new:  p(a,b), p(e,a), p(d,e), p(b,c), p(c,b)
Iteration 2: Use rule 2 and previous inferred facts to infer:
  new:  p(a,c) [from p(a,b) and e(b,c)]
        p(e,b) [from p(e,a) and e(a,b)]
        p(d,a) [from p(d,e) and e(e,a)]
        p(b,b) [from p(b,c) and e(c,b)]
        p(c,c) [from p(c,b) and e(b,c)]
Iteration 3: Again use rule 2 with previously inferred facts to infer:
  new:  p(e,c) [from p(e,b) and e(b,c)
        p(d,b) [from p(d,a) and e(a,b)
          (We get duplicates, e.g., p(a,b) from p(a,c) and e(c,b),
           but they aren't added.)
Iteration 4: Again use rule 2
        p(d,c) [from p(d,b) and e(b,c)]
Iteration 5: Nothing new to infer, so stop
```

These logs show the difference: Top-down uses the binding on the first argument of the query to guide its search, so here it never has to look at nodes `d` and `e` and their successors. Bottom-up computes the entire transitive closure relation, and would then select from it those p-tuples with value `a` in the first position. This might be more efficient if we were to ask many queries to the `p/2` relation, with different first arguments, since the `p` relation would be computed only once, and then every subsequent query would require only a simple lookup in that table.

In this particular case, there is not much to gain from computing the entire relation first, because the top-down strategy (saving tables across queries) would efficiently compute only the new tuples required for the current query, using the previously computed answers as needed. But there are (many) cases in which that efficient incremental computation is not what happens with multiple top-down queries to the same relation.

9.4 Example of Need for Bottom-Up Evaluation

Example 9.4.1. *Consider a simple system with a corpus of sentences, which allows a user to input a sentence and find all the sentences in the corpus that share a word with the input sentence. The system uses a relation of strings, called* `corpus(CorSent)`*, which contains the sentences of the corpus, and defines a predicate* `share(+InSent,-CorSent,-Word)`*, that, given an input sentence, returns each corpus sentence that shares a word with it, and the shared word. It can be defined using a helper tokenizing predicate,* `scan/2`*, as follows:*

```
share(InSent,CorSent,Word) :-
    scan(InSent,InWordList),
    corpus(CorSent),
    scan(CorSent,CorWordList),
    member(Word,InWordList),
    member(Word,CorWordList).
```

Top-down evaluation by Prolog proceeds as follows: Given a string as input, `scan/2` *tokenizes it, providing the list of words it contains. Then for each sentence in the corpus, we tokenize it, and return the corpus sentence for each word in the input sentence token list that is also in the corpus sentence token list.*

This is a reasonable approach for a single query, but if we want to answer multiple similar queries to the same corpus, this will re-tokenize every string in the corpus for each query, which would be extremely redundant and inefficient: Answering a single query for one input sentence essentially requires almost all the work (i.e., tokenizing every corpus sentence) that is necessary to answer a query for any input sentence. In the presence of multiple queries, it would be much better to construct a table of the corpus sentences and their tokens only once, and then use this table to answer each input query. We can simply reorder and fold the above program clause to create and use that `corpus_word/2` relation as follows:

Example 9.4.2.

```
share(InSent,CorSent,Word) :-
    scan(InSent,InWordList),
    member(Word,InWordList),
    corpus_word(CorSent,Word).

corpus_word(CorSent,Word) :-
    corpus(CorSent),
    scan(CorSent,CorWordList),
    member(Word,CorWordList).
```

This separates out the relation we want to generate and reuse, but notice that variant tabling does not solve our redundant computation problem. The predicate `corpus_word/2` will be called with its second argument, `Word`, bound. So for each different word in the input sentence, `corpus_word/2` will be entered, thus re-scanning every corpus sentence for each new input word. This is even worse than the previous unfolded program.

However, if we can ensure that the `corpus_word/2` definition is evaluated bottom-up, generating a complete table that is indexed on its second argument and that table is used for every call from `share/3`, then the redundant computation will be eliminated.

So this is an example of a situation in which we want to build (i.e., initialize) a table containing more answers than we immediately need: we want answers computed for all corpus words, even though we initially need only those for a single word.

In procedural terms, we understand the bottom-up evaluation of `corpus_word/2` as building an intermediate data structure to more efficiently evaluate various subqueries that will be needed later. This is a common requirement in algorithm development. Experienced Prolog programmers normally implement such requirements non-declaratively by initially computing the necessary tuples and adding them to a dynamic predicate using the Prolog builtin `assert/1`. But with bottom-up evaluation, such programs can be purely declarative and yet have the equivalent computational properties.

9.5 Specifying Bottom-Up Evaluation in XSB

The question now is how to indicate to the evaluation system that the first call to `corpus_word/2` is to cause it to be evaluated bottom-up to fill the table that is to be indexed on its second argument, and how the evaluation system will efficiently carry out that evaluation.

XSB uses Subsumptive Tabling With Abstraction (STWA) to carry out the bottom-up evaluation. Normally tables in XSB are variant tables, i.e. an entry in a variant table is used to satisfy a subsequent goal if that goal is a variant of the goal that generated the table entry, i.e., equal up to change of variable. In subsumptive tabling [?, ?, ?, ?, ?], a table entry is used to satisfy a subsequent goal if the subsequent goal is subsumed by the generating goal, i.e., is an instance of the generating goal. Note that in such a case the answers to the subsumed goal will (eventually) be in the table of answers for the previously encountered subsuming goal.[1] The programmer may specify, using a `table_index/2` directive, that when a subsumptively tabled goal is called, the calling goal should be abstracted before it is actually invoked, so that a table for the more general call is constructed, and then only the answers unifying with the original unabstracted goal are returned to the initial call. In this case, since a general table is constructed, a later subgoal that is subsumed by that more general goal will use this general table to satisfy the later subgoal and not make the specific subgoal call to do clause resolution.

The `table_index/2` directive was recently added to XSB to support the creation of subsumptive tables with multiple arbitrary indexes [?]. An index[2] on a single argument of the table is specified by just giving an integer that indicates the position in the tabled predicate of the argument to be indexed. Joint indexes, i.e. indexes on multiple arguments, are specified by terms made by separating the multiple position integers by the symbol +. Multiple different indexes are provided in a list. For example, say we want four indexes on a subsumptive table for `p/4`, so that a call to `p/4` (after its table is built) first checks if the call has arguments 1 and 2 bound and if so uses a joint index on those arguments; and if not, then checks if it has just argument 1 bound and if so uses an index on that; and if not then checks if arguments 2, 3 and 4 are bound and if so uses a joint index on those postions; and if not then checks if argument 4 is bound and if so uses an index on that position; and if not, then throws an error (because that would be an unindexed call and we should be notified). So this would require the directive:

```
:- table_index(p/4,[1+2,1,2+3+4,4]).
```

The `table_index/2` directive not only specifies the indexes on the resulting subsumptive table but also what abstraction should be done to the initial call to the subsumptively tabled predicate. The assumption is that every index will have some calls that use it. Since we want to compute the tuples for only one abstracted call to the tabled predicate, every position that might be unbound on some particular call must be abstracted in the initial subsumptive call, so that the one table will contain all answers for a call not bound on that position. This implies that the arguments that should be bound in the abstracted call to the subsumptively tabled predicate are those that appear in every specified index; all others must be variables. For the `p/4` example, there are no such arguments, so the first call will be fully abstracted, i.e., all arguments will be variables.[3]

There are situations where we want the initial call to a subsumptively tabled predicate to be abstracted to the fully open call, even if there are positions appearing in all index specifications. For example, we may need only one index but want full abstraction, as in our example for `corpus_word/2`. In such a case, we must use 0 as an index specifier, meaning "no index", and it is placed last.

Thus for the `corpus_word/2` subsumptive table, we want:

[1] Not all predicates defined in Prolog can be correctly tabled. For example, variant tabled predicates should not use builtins that change the global state, such as `assert/1` or `retract/1`; subsumptively tabled predicates, in addition, should not use builtins that depend on the instantiation state of variables, such as `var/1` and many uses of the cut (!) operation.

[2] These index specifications are the same as those used for dynamic predicates in XSB.

[3] In this case we call it "subsumptive tabling with *full* abstraction".

```
:- table_index(corpus_word/2,[2,0]).
```

which declares that the predicate corpus_word/2 should be computed bottom-up. We want an index on the second argument and the first call should be abstracted to the call with all variables. Since every corpus_word/2 subgoal is subsumed by this most general goal, it will never be called after the first time, but the table that is constructed at the first call will be used to satisfy all those subsequent calls. The first call to corpus_word/2 will build (or initialize) the table (or data structure), and from then on every call will directly access that table; and in our case use the index for constant time lookup.

The directive table_index/2 is implemented in XSB by a program transformation, which we briefly describe by example. The following code (slightly modified for clarity) is generated by the compiler for the p/4 example above,
:- table_index(p/4,[1+2,1,2+3+4,4]):

```
:- table p1234/4, p4231/4 as subsumptive.
p(A,B,C,D) :-
    nonvar(A) -> p1234(A,B,C,D)
    ; nonvar(D) -> p4231(D,B,C,A)
    ; table_error('Illegal Mode in call to p/4').
p1234(A,B,C,D) :-
    var(A),var(B),var(C),var(D) -> p_base(A,B,C,D)
    ; p1234(E,F,G,H),E = A,F = B,G = C,H = D.
p4231(A,B,C,D) :-
    var(D),var(B),var(C),var(A) -> p1234(D,B,C,A)
    ; p4231(E,F,G,H),E = A,F = B,G = C,H = D.
p_base(A,B,C,D) :- ... original p rules ...
```

p1234/4 and p4231/4 are versions of p/4 with its arguments permuted. Subsumptive tables implement trie indexing, thus providing indexing on any initial sequence of arguments. Here two tables are needed: p1234/4 for indexes 1 and $1 + 2$, and p4231/4 for indexes 4 and $2 + 3 + 4$.[4] The first clause tests the calling mode to select which tabled permutation to call. Each permutation predicate will normally be entered twice, once with some arguments bound, which will result in a (second) call to itself with all arguments unbound. From then on every other call will be subsumed by that most general call and so its answers will be served from that generated table. Notice that p4231/4 calls p1234/4 and not p_base/4 and so will be filled by getting its answers from the table of p1234/4. Thus the original clauses that define p/4 (now in p_base/4) will be called only once.

9.6 Intermixing Top-down and Bottom-up evaluation

It is possible to combine top-down filtering with bottom-up evaluation.

[4] The complier determines the minimum number of tables required to cover all the declared indexes.

Example 9.6.1. *Say our corpus of sentences in Example 9.4.1 is partitioned to indicate those from a single book. We would have another argument to* `corpus/1` *now:*
`corpus(BookISBN,Corent)`*. And top-level predicate:*
`share(InSent,BookISBN,CorSent)`*, and sub-predicate:*
`corpus_word(BookISBN,CorSent,Word)`*, defined in terms of* `corpus/2`*.*

```
share(InSent,ISBN,CorSent) :-
    scan(InSent,InWordList),
    member(Word,InWordList),
    corpus_word(ISBN,CorSent,Word).

corpus_word(ISBN,CorSent,Word) :-
    corpus(ISBN,CorSent),
    scan(CorSent,CorWordList),
    member(Word,CorWordList).
```

We assume that `corpus/2` *is indexed on its first field, BookISBN, and require that ISBN be bound on a call to* `share/3`*. Then we would invoke partial bottom-up evaluation with the declaration:*

```
:- table_index(corpus_word/3,[1+3,1]).
```

This will abstract a call to `corpus_word/3` *to* `corpus_word(+,-,-)`*, where* `+` *indicates a bound argument and* `-` *a variable argument. Positions that appear in* all *indexes in the* `table_index` *declaration, here* `1`*, will not be abstracted; all others will be. Now the sentences of each book will be processed bottom-up independently. When a new book is queried for the first time, the table for words in sentences for that book will be computed bottom-up and stored in the table. Subsequent queries to the same book will directly use the constructed table. So this is an example in which demand-driven computation (here with respect to ISBN) can be combined with data-driven computation (here for the set of corresponding $< CorSent, Word >$ pairs) to evaluate a query (here to* `share/3`*.)*

That a data-driven computation can use a demand-driven subquery has already been seen in the original Example 9.4.1. While we haven't shown the code for the `scan/2` predicate, it is naturally defined using demand-driven computation. E.g., a DCG defining the scanner would use top-down evaluation of a subquery. So the original example shows how the bottom-up evaluation of `corpus_word(-,-)` uses the top-down evaluation of `scan(+,-)`.

9.7 Procedural Interpretation of Bottom-up Evaluation

To understand top-down evaluation of rules, one intuitively thinks of subgoal solving as nondeterministic procedure invocation, with backtracking to explore alternate computation paths. To understand top-down evaluation with tabling, one thinks of procedure invocation but with short-circuiting for previously encountered procedures (with the same parameters). To understand bottom-up evaluation as provided by Subsumptive Tabling With Abstraction (STWA) through the `table_index/2` directive, one thinks of indexed table construction (or initialization) from which future retrieval will be done.

A bottom-up computation is appropriate when there is a predicate that is efficiently computable in one mode and yet it will be needed multiple times in another mode. For example, `corpus_word(+.-)` of Example 9.4.1 can be efficiently evaluated in that direction, i.e., given a sentence find the words in it. But the application requires multiple calls of the mode `corpus_word(-,+)`, i.e., given a word find the sentences that contain it. In this situation, bottom-up evaluation is called for.

The most basic example of this phenomenon is indexing itself. For example, if one wants to find the value associated with a particular key in an unordered sequence of key-value pairs, one must search the pairs sequentially, which on average takes time linear in the number of pairs. With the same time complexity to search for one item, one can (with perfect hashing) build a hash table which allows *all* such key lookups to be done in constant time.

9.8 STWA for File Input

Part of the attractiveness of logic programming is that it treats programs (usually mostly rules) and data (usually facts) the same, but in most large systems there is an important pragmatic difference between them: Programs are rules (and facts) that are known statically at compile time; data are facts (and maybe rules) that are not known until runtime when they are retrieved from a file. Facts are normally accessed through a filename passed to an executing program, where the records of the file are initially read and asserted into a predicate in memory, which is used by the program rules. It is not clear how this could be made fully declarative. But consider the following use of STWA:

Example 9.8.1. `:- table_index(emp_data/4,[1+2,1]).`

```
emp_data(FileName,EmpId,Name,Addr) :-
    open(FileName,read,InpStream),
    repeat,
    read(InpStream,Term),
    (Term == end_of_file
     -> !, close(InpStream), fail
     ;  Term = emp(EmpId,Name,Addr)
    ).
```

This definition uses low-level procedural Prolog operations to read the records of a file containing `emp/3` *facts and return them as answers to a call to* `emp_data/4`*, where the first argument of the call is the name of the file to read. Because of the* `table_index/2` *directive, the first call will be abstracted to* `emp_data(+,-,-,-)`*. (Recall that arguments in the intersection of all the indexed arguments, here argument 1, must be bound in any call, and will not be abstracted for the initial table-filling call.) So with this definition, a programmer can get access to the tuples in any employee file (one that contains* `emp/3` *facts) by invoking a simple subgoal, with the first argument providing* *the name of the file. There is no need for the programmer to make an explicit call (e.g., Prolog's* `ensure_loaded/1`*) to initialize the* `emp_data/3` *predicate using Prolog's nonlogical* `assert/1`*. The first call will automatically read the file in, building an index on the first and on the first and second arguments. Since XSB uses tries to represent tables [?], this is here just a single trie, and the* `Filename` *is stored only once in the table, so the overhead of using the 4-ary predicate is essentially zero.*

Of course a programmer could read different sets of employee facts from different files simply by invoking the emp_data goal with different file names.

Generalizing this approach we can make the entire file system a part of the data-space of a Prolog program. Definitions such as in Example 9.8.1 simply map (portions of) the file system into Prolog fact-defined predicates. We can make this easier for the programmer by providing a single "file-system" predicate that views the entire file system as a single relation. For example, consider a system predicate [?]: `data_records(+FileName,+FileFormat,?RecordTerm)`, which, given a filename and a file format, reads the file, parses each line/record to a Prolog term as specified by `FileFormat`, and returns each such term nondeterministically in `RecordTerm`.

Now a Prolog programmer can use `table_index/2` and more easily define `emp_data/4` as:

```
:- table_index(emp_data/4,[1+2,1]).
emp_data(FileName,EmpId,Name,Addr) :-
    data_records(FileName,read,emp(EmpId,Name,Addr)).
```

to cause the `emp_data/4` predicate to be loaded from the named datafile on its first call. With this framework, the Prolog programmer has declrative access to all data in the file system, without having to worry explicitly about how and when to load it into memory.

9.9 Does Subsumptive Tabling with Abstraction (STWA) Really Do Bottom-Up Evaluation?

Subsumptive tabling with abstraction has the properties of bottom-up evaluation, described above, of generating a full table once and then using the resulting table for subsequent queries. But does it actually build that table using the well-understood bottom-up, forward-chaining, data-directed algorithm, as described in Section 9.3? The short answer is yes (mostly), which we now demonstrate.

Example 9.9.1. *Consider how XSB would compute the transitive closure of Example 9.3.2 under STWA. The program is:*

```
:- table_index(p/2,[1,0]).
                                  e(a,b).     e(b,c).
p(X,Y) :- e(X,Y).                 e(e,a).     e(c,b).
p(X,Y) :- p(X,Z),e(Z,Y).          e(d,e).
```

Instrumented to produce output showing the engine's progress (see 9.13), we get (with annotations added for clarity):

```
| ?- p(a,A).
(Iteration 1)
enter rule 1: p(X,Y)
from fact: e(a,b) infer p(a,b)
from fact: e(b,c) infer p(b,c)
from fact: e(e,a) infer p(e,a)
from fact: e(c,b) infer p(c,b)
from fact: e(d,e) infer p(d,e)
(Iteration 2)
enter rule 2 p(X,Y)
from table: p(a,b) and fact e(b,c) infer p(a,c)
from table: p(b,c) and fact e(c,b) infer p(b,b)
from table: p(e,a) and fact e(a,b) infer p(e,b)
from table: p(c,b) and fact e(b,c) infer p(c,c)
from table: p(d,e) and fact e(e,a) infer p(d,a)
from table: p(a,c) and fact e(c,b) infer p(a,b)
from table: p(b,b) and fact e(b,c) infer p(b,c)
from table: p(e,b) and fact e(b,c) infer p(e,c)
from table: p(c,c) and fact e(c,b) infer p(c,b)
from table: p(d,a) and fact e(a,b) infer p(d,b)
(Iteration 3)
from table: p(e,c) and fact e(c,b) infer p(e,b)
from table: p(d,b) and fact e(b,c) infer p(d,c)
(Iteration 4)
from table: p(d,c) and fact e(c,b) infer p(d,b)
A = b;
A = c;
no
| ?-
```

This log closely follows that of Example 9.3.2: the "enter rule" lines show that each rule (and thus the predicate) is entered only once. From then on everything is returned through the table for `p(X,Y)`*. The only differences are the repeated answers in Iteration 2, where duplicate answers are computed but not added to the table. (Were we more detailed in our description of Example 9.3.2, these would have appeared there, too.)*

Intuitively, what happens here is that the initial call to `p(a,A)` is fully abstracted to `p(X,Y)` and that call is made. The first rule returns all its answers to the table, which we have labeled "Iteration 1". Then the second rule is entered and the open call to `p(X,Z)` (the first subgoal of that second rule) is made, found to have previously been made, and suspended, to return answers from the table. The rest of the log, of iterations 2 through 4, consists of answers from the table being returned to that call, joined with facts e/2 and the resulting answers added (if new) to the table. So computation across the bodies of rules may be suspended to wait for new answers to show up in subsumptive tables. This, in effect, uses XSB's scheduling queues to get the equivalent of semi-naive bottom up computation using pointers. This computation directly mirrors the bottom-up evaluation of the query `p(X,Y)`.

Theorem 2. *Assume H is a Horn program and P a predicate in H such that all predicates in H are reachable from P in the predicate call graph of H. (I.e., the goal predicate potentially depends on all predicates.) Assume that for every rule body in H, there is some instance of it that is true in the least model of H. Then evaluation of predicate P in H under Subsumptive Tabling With* Full *Abstraction (STWFA) is equivalent to a bottom-up computation of the least model of H.*

Proof. See 9.14 □

9.10 Bottom-up Proofs for Propositional Horn Clauses in Linear Time

For a deeper understanding of XSB's STWA and bottom-up evaluation, we consider the well-known three-line meta-interpreter, applied to propositional programs.

Example 9.10.1. *Consider the (folded variation of the) classic meta-interpreter of Prolog programs:*

```
interp(true).
interp((A,B)) :- interp(A), interp(B).
interp(G) :- interpAtom(G).

interpAtom(G) :- (G <- Body), interp(Body).
```

It is well-known that if we add `<-/2` facts to define a (propositional) Horn clause program and evaluate `interp(Goal)` for some `Goal` in Prolog, this will carry out the top-down Prolog evaluation of `Goal` with respect to the program. Similarly, if we add the declaration:

```
:- table interpAtom/1.
```

this will carry out top-down evaluation with (variant) tabling.

Now consider instead adding the declaration:

```
:- table_index(interpAtom/1,[1,0]).
```

and posing the query: `interpAtom(Prop)`, with `Prop` a variable. Recall that this declaration causes XSB to use subsumptive tabling. There is no abstraction necessary for this query since it is already fully abstract, so all subsequent calls to `interpAtom/1` will be served from the table constructed by that initial call. The claim is that this will cause XSB to evaluate the propositional program in `(<-)/2` using a bottom-up algorithm. To see this, consider how this query will be processed. Note that since there is only the one call to `interpAtom(_)`, there can be no propagation of demand through (partially) instantiated arguments. The initial goal is `interpAtom(_)`, the open call. So this initial call, being subsumptively tabled, will add the call to the table and invoke the predicate code to do clause resolution. Now every other call to `interpAtom(P)` for any particular proposition `P` (here only the one in the third clause of `interp/1`) will not invoke the code for `interpAtom/1`, but will return answers that have previously been added to the table and will suspend waiting for a new answer to show up in the table to then be returned. The first answers returned cannot require a recursive call to `interpAtom/1`, so they can come from only the first clause of `interp/1`, and those will be the facts of the program in `(<-)/2`. So the first answers to show up in the table for `interpAtom/1` will be the program facts, just as in bottom-up evaluation. Then those facts can be returned to the recursive call to `interpAtom/1` (in the third clause of `interp/1`.) So rules all of whose body atoms are facts will have their recursive calls to `interpAtom/1` satisfied (since they will be in the `interpAtom(_)` table on which they are waiting), and will succeed returning the propositions in their heads, to be added to the table for `interpAtom(_)`. And so forth.

We can see the order that XSB generates the propositional answers to our program above by executing the above meta-interpreter (with logging operations, as shown in 9.13):

```
p <- q,v,r,s.          s <- true.
p <- q,s,t.            u <- s,p,v,r.
q <- u,r.              u <- r,t.
q <- q,t,v.            t <- true.
r <- s.
```

where we get (lightly edited):

```
| ?- interpAtom(p).
1.  Var demanded
2.  p <- q,v,r,s   initial clause
3.  p <- q,s,t     initial clause
4.  q <- u,r       initial clause
5.  q <- q,t,v     initial clause
6.  r <- s         initial clause
7.  s <- true      initial clause
8.  s              from true and s <- true
9.  u <- s,p,v,r   initial clause
10. u <- p,v,r     from s and u <- s,p,v,r
11. u <- r,t       initial clause
12. t <- true      initial clause
13. t              from true and t <- true
14. r              from s and r <- s
15. u <- t         from r and u <- r,t
16. u              from t and u <- t
17. q <- r         from u and q <- u,r
18. q              from r and q <- r
19. q <- t,v       from q and q <- q,t,v
20. q <- v         from t and q <- t,v
21. p <- s,t       from q and p <- q,s,t
22. p <- t         from s and p <- s,t
```

```
23. p             from t and p <- t
24. p <- v,r,s    from q and p <- q,v,r,s
25. u <- v,r      from p and u <- p,v,r
```

Each line (from 2 on) shows a derived clause that is either an initial program clause, or derived from a previous clause by removing the first proposition in its body if it is proved (or is the constant 'true'). A proposition is proved if it is the head of a derived clause with an empty body. Each step is the result of a unit resolution, whose participating clauses are indicated on the line. Unit resolution generates a form of bottom-up propositional reasoning for Horn clauses.

Performing iterative bottom-up evaluation of this program, we get:

```
  Iteration 0:
    s, t: program facts
  Iteration 1:
    r: from r <- s
  Iteration 2:
    u: from u <- r,t
  Iteration 3:
    q: from q <- u,r
  Iteration 4:
    p: from p <- q,s,t
```

And we see these propositions showing up in exactly this order in the XSB STWA log above.

9.11 Performance of XSB STWA for Bottom-Up Evaluation

In this section we show XSB's performance on bottom-up evaluation of a propositional program using a meta-interpreter. The system used is XSB Version 3.8.0 on a Mac Pro with an i7-4870HQ CPU running at 2.5-GHz with 16 GB of RAM and the 64-bit Windows 10 Operating System. The program we use is a "triangular" program, of the form:

```
p1 :- p2,p3,p4,p5.
p2 :- p3,p4,p5.
p3 :- p4,p5.
p4 :- p5.
p5.
```

This program is an example with 5 rules (and facts) and 15 proposition occurrences. The meta-interpreter is:

```
interp_goal(true) :- !.
interp_goal((G1,G2)) :- !, interp_atom(G1), interp_atom(G2).
```

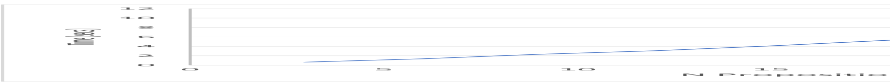


Figure 9.1: Times for Bottom-up Evaluation of Triangular Programs

```
interp_goal(G) :- interp_atom(G).

:- table interp_atom/1.
interp_atom(G) :- interp_atoms(G).

:- table_index(interp_atoms/1,[0]).
interp_atoms(G) :- (G <- Gs), interp_goal(Gs).
```

Notice that we use *both* a variant table (on `interp_atom/1`) *and* a subsumptive table (on the equivalent `interp_atoms/1`.) This is to improve efficiency, since lookup and retrieval from a variant table in XSB is significantly faster than for a subsumptive table.[5] And in these triangular programs there are many lookups of the same variant call; each proposition is looked up once per occurrence in the body of a rule, so on average there are approximately `NumRules/2` lookups per proposition, so lookups completely dominate the computation time.

Figure 9.1 shows the cpu time for bottom-up evaluation of various-sized triangular programs. N corresponds to the number ($*10^6$) of proposition occurrences in the program, or more precisely, the number in the largest triangular program that has $N * 10^6$ or fewer proposition occurrences.

Times are just for the meta-interpretation of the triangular programs; the time for generating the `(<-)/2` facts for the programs is not included. The graph is essentially a straight line, showing the linearity of the algorithm. The propositional Horn clause satisfaction problem is known to be

[5] Without the additional variant table, the evaluation is about 25 times slower, showing significant room for improvement in XSB's implementation of subsumptive table goal lookup. Evaluation of this program with only variant tabling is about twice as fast; not surprising since that evaluation is also linear, it doesn't duplicate tables, and XSB is optimized for variant tables.

linear in the number of proposition occurrences [?], and this meta-interpreter is indeed linear on all propositional logic programs.

Triangular programs have no loops so pure top-down evaluation without tabling will terminate, but will be very slow due to many redundant computations. For example to evaluate without tabling the single query, `p1`, to a triangular program with 325 proposition occurrences takes c. 4.1 seconds, and with 465 occurrences c. 126 seconds. This contrasts with the 15,000,000 occurrences in the program that is computed by tabled bottom-up evaluation in 4.1 seconds (as shown in the figure).

9.12 Conclusion

In this paper we have described the integration of bottom-up evaluation with top-down evaluation in tabled Prolog by means of Subsumptive Tabling With Abstraction (STWA). We provided a procedural interpretation for bottom-up evaluation, in the context of Prolog programming, as efficient indexed data structure initialization. This allows many formerly non-declarative Prolog programs that contain assert/1 to be naturally expressed as declarative programs with no sacrifice in efficiency.

We have shown how XSB provides a natural model for evaluating Horn programs, based on nondeterministic procedural interpretation using subsumptive tabling with abstraction, that incorporates both top-down and bottom-up computation in a single uniform framework. And this model provides linear evaluation of all queries to proposition Horn programs for all intermixings of demand and data driven computation.

9.13 Programs to Generate Traces

Program that generates trace of bottom-up evaluation of `?- p(a,X).`

```
:- table_index(p/2,[1,0]).
p(X,Y) :-
    writeln('enter rule 1: p'(X,Y)),
    e(X,Y),
    write('from fact: e'(X,Y)),
    write(' infer p'(X,Y)),
p(X,Y) :-
    writeln('enter rule 2 p'(X,Y)),
    p(X,Z),
    write('from table: p'(X,Z)),
    e(Z,Y),
    write(' and fact e'(Z,Y)),
    write(' infer p'(X,Y)).

e(a,b).     e(b,c).
```

```
e(e,a).     e(c,b).
e(d,e).
```

Program that generates the annotated trace of bottom-up meta-interpretation of proposition clauses.

```
:- op(1200,xfx,('<-')).
:- import conset/2, coninc/2 from gensym.

:- table_index(interpAtom/1,[0]).
interpAtom(G) :-
    conset('_ctr',0),
    log('%S demanded',args(G)),
    (G <- Body),
    log('%S :- %S initial clause',args(G,Body)),
    interp(Body,G).

interp(true,H) :- !, log('%S from true and %S :- true',args(H,H)).
interp((A,B),H) :- !,
    interpAtom(A),
    log('%S :- %S from %S and %S :- %S',args(H,B,A,H,(A,B))),
    interp(B,H).
interp(G,H) :-
    interpAtom(G),
    log('%S from %S and %S :- %S',args(H,G,H,G)).

p <- q,v,r,s.          s <- true.
p <- q,s,t.            u <- s,p,v,r.
q <- u,r.              u <- r,t.
q <- q,t,v.            t <- true.
r <- s.
```

9.14 Subsumptive Tabling with Full Abstraction (STWFA) is Bottom-up

This section proves a close relationship between the evaluation of Horn programs using subsumptive tabling with full abstraction and their bottom-up evaluation.

9.14.1 Multiple Machine Model of Tabled Horn Clause Evaluation

We describe a model for STWFA evaluation as a set of (virtual) machines. Given a Horn clause program and a goal, each machine carries out the evaluation of the goal in the program along a different deterministic path. Without tabling there would be a machine corresponding to each root-to-leaf path of the SLD tree for the query. With subsumptive tabling with full abstraction, the model is as described next.

A global table of subgoals is maintained. Associated with each subgoal is a set of answers corresponding to proven instances of the subgoal. This table is maintained and used by all the executing machines. It is monotonic in that goals and answers are only added to the table and never deleted.

Computation starts with a single machine that is given the initial goal to evaluate. When a machine encounters a goal, it looks to see if the corresponding most-general goal is in the global table. If it is *not* in the table, a) it adds it (with an empty list of associated answers), b) for each clause for that most-general goal, it forks off a duplicate of itself to process it, and c) remembering the goal it encountered, it suspends on the table entry it just added (to later process answers that are associated with that most-general goal). If the most-general goal already *is* in the table, the machine (remembering its encountered goal) suspends on that table entry.

When a machine encounters a failure, it simply disappears. When a machine returns an answer to a goal (i.e., completes execution to the end of a clause for that goal), it adds the computed goal instance to the end of the list of answers associated with that goal in the table, and then disappears.

Whenever there is a new answer for a table entry for a goal, each machine that is suspended on that entry, whose associated goal unifies with that new answer, forks off a copy of itself. This new machine uses that answer to update its state and then continues executing. The suspended machine remains suspended having marked in the table that that answer has been returned.

This set of machines is normally simulated by a sequential emulator, which includes a scheduler that determines which of the next possible machine operations will be performed. At any point in the execution, there may be a machine ready to a) evaluate the next goal of some clause (or return an answer if there are no more goals for that clause) or b) fork a machine that is suspended on a table entry to process a new (unifying) answer. For specificity and simplicity, we will assume that answers associated with a table entry are maintained in the order they are generated, and suspended machines return them in that same order.

9.14.2 Example of STWFA Evaluation

To better understand this model, we trace its evaluation of the graph reachability example, with the edge relation defined by a simple join:

```
:- table_index(p/2,[0]).
                              q(a,b).          r(a).
p(X,Y) :- e(X,Y).             q(a,d).          r(b).
p(X,Y) :- p(X,Z),e(Z,Y).      q(e,a).          r(c).
                              q(d,e). q(b,d).  r(e).
e(X,Y) :- q(X,Y),r(Y).        q(b,c). q(c,b).
```

We trace the evaluation of the query p(a,X). A state of a machine is represented by a rule form: $H \leftarrow B_1, B_2, ..., B_k$, where $k >= 0$. This represents a machine about to call subgoal B_1, which on its return will be followed by calls to B_2 through B_k, and then a return to H. When $k = 0$, it is about to return the answer H to its call.

The table is represented by a set of forms: $Goal : [AnsList], [H \leftarrow B_1, B_2, ..., B_k : N, ...]$ where $Goal$ is a most-general goal, $AnsList$ is a list of instances of Goal (a.k.a. answers). The indexed rule form $H \leftarrow B_1, B_2, ..., B_k : N$ represents a suspension: B_1 is an instance of $Goal$, and N is an integer between 0 and the length of the answer list, and is the index of the last answer from B_1 that has been returned, i.e., for which the suspension has been cloned and continued. This pair of a rule and index represents a machine that is suspended on the goal of this table entry; there may be multiple such suspensions.

In the following trace, `S:` indicates the set of machine states that are available to be scheduled, separated by semi-colons and `T:` indicates the state of the table. We trace the evaluation of the query `p(a,X)`.

```
S: <-p(a,X)
T:

Create new table entry for p/2, suspend on it, fork machines for each rule:
S: p(X,Y)<-e(X,Y); p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]

Create new table entry for e/2, suspend on it, fork machines for each rule:
S: e(X,Y)<-q(X,Y),r(Y); p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]

Create new table entry for q/2, suspend, fork machines for each rule:
S: q(a,b)<-; q(a,d)<-; q(e,a)<-; q(d,e)<-; q(b,c)<-; q(b,d)<-; q(c,b)<-;
   p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[],[e(X,Y)<-q(X,Y),r(Y):0]

7 steps, returning q/2 facts to the q(X,Y), adding them to its table:
S: p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):0]

Fork off 7 copies of the suspended e<-q,r machine, one for each answer:
S: e(a,b)<-r(b); e(a,d)<-r(d); e(e,a)<-r(a); e(d,e)<-r(e); e(b,c)<-r(c);
   e(b,d)<-r(d); e(c,b)<-r(b); p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]

Create new table entry for r/1, suspend on it, fork one for each r/1 rule:
S: r(a)<-; r(b)<-; r(c)<-; r(e)<-;
   e(a,d)<-r(d); e(e,a)<-r(a); e(d,e)<-r(e); e(b,c)<-r(c); e(b,d)<-r(d)
   e(c,b)<-r(b); p(X,Y)<-p(X,Z),e(Z,Y)
```

```
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]a
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[],[e(a,b)<-r(b):0]

4 steps, returning 4 answers to r/1 call:
S: e(a,d)<-r(d); e(e,a)<-r(a); e(d,e)<-r(e); e(b,c)<-r(c); e(b,d)<-r(d);
   e(c,b)<-r(b); p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],[e(a,b)<-r(b):0]

7 steps each calling r/1 and suspending on that call:
S: p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):0, e(a,d)<-r(d):0, e(e,a)<-r(a):0,
         e(d,e)<-r(e):0, e(b,c)<-r(c):0, e(b,d)<-r(d):0,
         e(c,b)<-r(b):0]

7 (*4) steps returning r/1 answers to suspended machine.  5 of the
  suspended machines match on answer and 2 match no answers, generating
  5 new machines:
S: e(a,b)<-; e(e,a)<-; e(d,e)<-; e(b,c)<-; e(c,b)<-;
p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

5 steps returning 5 answers to the e/2 table:
S: p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],[p(X,Y)<-e(X,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

The machine suspended on e/2 forks 5 new machines, one for each new answer:
S: p(a,b)<-; p(e,a)<- ; p(d,e)<-; p(b,c)<-; p(c,b)<-; p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[],[<-p(a,X):0]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],[p(X,Y)<-e(X,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

5 steps, adding 5 new answers to p/2 entry:
S: p(X,Y)<-p(X,Z),e(Z,Y)
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b)],[<-p(a,X):0]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],[p(X,Y)<-e(X,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

Machine suspends on call to p(X,Z):
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b)],[<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):0]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],[p(X,Y)<-e(X,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

5 new machines forked from machine suspended on p/2:
S: p(a,Y)<-e(b,Y); p(e,Y)<-e(a,Y); p(d,Y)<-e(e,Y); p(b,Y)<-e(c,Y); p(c,Y)<-e(b,Y)
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b)],[<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):5]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],[p(X,Y)<-e(X,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

All 5 just generated machines suspend on their calls to e/2:
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b)],[<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):5]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):0, p(e,Y)<-e(a,Y):0,
           p(d,Y)<-e(e,Y):0, p(b,Y)<-e(c,Y):0, p(c,Y)<-e(b,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

The 5 just suspended machines fork of 5 new machines; each machine
  matching exactly one of the 5 new answers:
```

```
S: p(a,c)<-; p(e,b)<-; p(d,a)<-; p(b,b)<-; p(c,c)<-
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b)],[<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):5]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

Each of the 5 new machines add a new answer to the p/2 table entry:
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):5]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

The machine suspended on p(X,Z) in the p/2 table forks off machines
  for each of the 5 just-add p/2 answers:
S: p(a,Y)<-e(c,Y); p(e,Y)<-e(b,Y); p(d,Y)<-e(a,Y); p(b,Y)<-e(b,Y); p(c,Y)<-e(c,Y)
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):10]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

All 5 of those machines suspend on their calls to e/2.
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):10]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):0, p(e,Y)<-e(b,Y):0, p(d,Y)<-e(a,Y):0,
           p(b,Y)<-e(b,Y):0,  p(c,Y)<-e(c,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

All 5 of those suspensions fork off a total of 5 new machines, each
  one matching one of the 5 existing answers for e/2:
S: p(a,b)<-; p(e,c)<-; p(d,b)<-; p(b,c)<-; p(c,b)<-
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):10]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
       [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

Those 5 machines return just 2 new answers to the p/2 entry (3 are duplicates):
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):10]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

The suspend p<-p,e machine forks 2 new machines, one for each just-added answer;
S: p(e,Y)<-e(c,Y); p(d,Y)<-e(b,Y)
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):12]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

Those 2 new machines suspend on their e/2 calls:
S:
```

```
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):12]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):0
           p(d,Y)<-e(b,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

Those 2 suspensions fork 2 new machines from all the e/2 answers,
  each matching exactly one:
S: p(e,b)<-; p(d,c)<-
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):12]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

These two machines add just one new answer to the p/2 table entry:
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b),p(d,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):12]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

The p<-p,e suspension on p/2 forks off a new machine for that new answer:
S: p(d,Y)<-e(c,Y)
```

```
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b),p(d,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):13]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

That machine calls e/2 and suspends:
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b),p(d,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):13]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5, p(d,Y)<-e(c,Y):0]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

That suspension forks one machine for the one matching answer in e/2:
S: p(d,b)<-
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
              p(e,c),p(d,b),p(d,c)],
             [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):13]
   e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5, p(d,Y)<-e(c,Y):5]
   q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
   r(X):[r(a),r(b),r(c),r(e)],
        [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
         e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
         e(c,b)<-r(b):4]

That machine fails since the answer it wants to add is a duplicate:
S:
T: p(X,Y):[p(a,b),p(e,a),p(d,e),p(b,c),p(c,b),p(a,c),p(e,b),p(d,a),p(b,b),p(c,c),
             p(e,c),p(d,b),p(d,c)],
            [<-p(a,X):0,p(X,Y)<-p(X,Z),e(Z,Y):13]
  e(X,Y):[e(a,b),e(e,a),e(d,e),e(b,c),e(c,b)],
          [p(X,Y)<-e(X,Y):5, p(a,Y)<-e(b,Y):5, p(e,Y)<-e(a,Y):5,
           p(d,Y)<-e(e,Y):5, p(b,Y)<-e(c,Y):5, p(c,Y)<-e(b,Y):5,
           p(a,Y)<-e(c,Y):5, p(e,Y)<-e(b,Y):5, p(d,Y)<-e(a,Y):5,
           p(b,Y)<-e(b,Y):5, p(c,Y)<-e(c,Y):5, p(e,Y)<-e(c,Y):5,
           p(d,Y)<-e(b,Y):5, p(d,Y)<-e(c,Y):5]
  q(X,Y):[q(a,b),q(a,d),q(e,a),q(d,e),q(b,c),q(b,d),q(c,b)],
          [e(X,Y)<-q(X,Y),r(Y):7]
  r(X):[r(a),r(b),r(c),r(e)],
       [e(a,b)<-r(b):4, e(a,d)<-r(d):4, e(e,a)<-r(a):4,
        e(d,e)<-r(e):4, e(b,c)<-r(c):4, e(b,d)<-r(d):4,
        e(c,b)<-r(b):4]
```

At this point, the only thing that can be scheduled is the `<-p(a,X)` suspension, which would match the two answers in the table with first field `a` and terminate.

9.14.3 Relationship of STWFA and Bottom-Up

It is clear that, in general, STWFA is not exactly bottom-up least model computation, since STWFA depends on the initial goal. For example, consider the program:

```
p(X) :- q(X).                q(a).
r(X) :- q(X),p(X).           q(b).
```

and the query `p(X)`. In this case, `r(X)` will never be called and so with STWFA its extension will not be evaluated, whereas bottom-up least-model evaluation would generate it.

Consider also the example:

```
p(X) :- q(X),r(X),s(X).          q(a).
                                 r(b).
                                 s(c).
```

and query `p(X)`. In this case, even though `s/1` is reachable from `p/1` in the dependency graph of this program, `s(X)` will never be called under STWFA since there is no `X` such that `q(X)` and `r(X)` is true. So here again is a case where bottom-up, which would, of course, compute the extension of `s/1`, differs from STWFA.

If these possibilities are excluded, we can show a close relationship between STWFA and bottom-up evaluation.

Theorem 1
Assume H is a Horn program and P a predicate in H such that all predicates in H are reachable from

P in the predicate call graph of H. (I.e., the goal predicate potentially depends on all predicates.) Assume that for every rule body in H, there is some instance of it that is true in the least model of H. Then STWFA evaluation of predicate P in H is equivalent to a bottom-up computation of the least model of H.

Proof. (Sketch) We formulate a bottom-up Horn clause evaluator and then compare it to the multiple machine model of STWFA described above.

Consider the following formulation of a bottom-up Horn clause evaluator. For each clause, there is a process to generate facts for instances of its head atom. There is a global set of proven facts that is initialized to empty. The rule processes run as follows: Each rule process looks to find an instance of its clause such that all its body atoms appear in the global fact set. For each such clause instance, it adds the head atom to the global fact set, if it is not already a member. The processes continue to run to a fixpoint, i.e., until no process can add any new fact to the global set.

By synchronizing and scheduling the rule processes so that they iteratively generate a new set of facts to be added to the fact set by using facts only from earlier iterations, one obtains the well-known iterative bottom-up evaluation algorithm.

However, any order of execution of the rule processes is possible, and constitutes an evaluation method appropriately called bottom-up.

In STWFA, a machine that is generated for a rule when its most-general subgoal is first entered corresponds to a rule process in the bottom-up algorithm. The execution of such a machine (and its descendants), which suspend and fork off new machines to process new answers, corresponds to the bottom-up rule process for the clause that finds rule instances whose body atoms are in the global fact set. Finally, a machine returning a new answer to a predicate corresponds to a rule process adding a new fact to the global fact set.

The difference between the two algorithms is how the rule processes/machines get set up in the first place. For the bottom-up algorithm, the rule processes are set up at the same time during an initialization phase. With STWFA, the corresponding machines for the rules of a particular predicate are generated at the first call to a goal for that predicate. STWFA will generate relations only for predicates on which the goal predicate depends, since it will not initialize rule processor machines for other predicates that are never called.

The theorem requires that the goal depends on all predicates in the program, and that every clause has a true instance in the least model of the program. This will guarantee that STWFA will eventually call every predicate and so all rule-processing machines will be initialized.

We can fix a scheduling order for STWFA, and then find an equivalent scheduling order for bottom-up evaluation with rule processes. □

Of course, if the conditions of the theorem are not met, STWFA still computes the correct answers to the query, and computes them in a bottom-up manner; it just does not compute relations that are unnecessary (given its left-to-right selection strategy).

Chapter 10

HiLog Programming

XSB includes a capapbility to process programs which have complex terms in predicate of functor position. This allows programmers to program in a higher-order syntax, and so this extension of Prolog is called HiLog. Programmers can think of programming with parameterized predicates or with predicate variables. HiLog also supports a new way of programming with sets. We will explore these issues in this chapter.

HiLog is actually a very simple extension to Prolog. The definition of a basic term in Prolog is as follows:

- A term is a atomic symbol or a variable, or
- A term is of the form: $s(t_1, t_2, ..., t_n)$ where s is an atomic symbol and the t_i are terms.

Note that the symbol in functor (or predicate) position must be a symbol. HiLog generalizes this to allow an arbitrary term itself. So the definition of a term in HiLog is:

- A term is a atomic symbol or a variable, or
- A term is of the form: $t_0(t_1, t_2, ..., t_n)$ where the t_i are terms.

Computationally these terms are matched just as Prolog terms, so intuitively HiLog programs work very similarly to Prolog programs. However, they encourage different ways of thinking about programs and support different programming paradigms.

10.1 Generic Programs

Because one can use a complex term as a predicate in HiLog, one can program "generic predicates." For example, consider a predicate function, i.e., a function that takes a predicate and returns another predicate. An interesting such predicate function might be `closure`. `closure` takes a binary predicate and returns a predicate for the transitive closure of the corresponding binary relation. So for example, `closure(parent)` would be the transitive closure of the `parent` relation, i.e., the ancestor relation, and `closure(child)` would be the descendent relation. We can define this `closure` predicate function in HiLog as follows:

```
closure(R)(X,Y) :- R(X,Y).
closure(R)(X,Y) :- R(X,Z), closure(R)(Z,Y).
```

Now given any binary relation, one can use use this definition to compute its closure. For example, we can define a binary predicate, `parent` as follows:

```
:- hilog parent.
parent(able,adam).
parent(able,eve).
parent(cain,adam).
parent(cain,eve).
etc
```

and then we can use the generic definition of closure to find anscestors:

```
| ?- closure(parent)(cain,X).
  etc.
```

Notice that we must declare the symbol `parent` to be a hilog symbol using the directive:

```
:- hilog parent.
```

This is necessary because the XSB system allows a mixture of HiLog programming and Prolog programming, and the system distinguishes between HiLog symbols and Prolog symbols in how it represents them. The HiLog term $t_0(t_1, t_2, ..., t_n)$ is represented as the Prolog term $apply(t_0, t_1, t_2, ..., t_n)$. Thus the system must know, for example, that `parent` is a hilog symbol so it knows to represent `parent(cain,adam)` as the Prolog term `apply(parent,cain,adam)`.

Another useful generic predicate is `map`. `map` takes a binary function and returns a function that when given a list, returns the list that results from applying that function to each element of the given list. Again, we can write a natural definition for it:

```
map(F)([],[]).
map(F)([X|Xs],[Y|Ys]) :- F(X,Y), map(F)(Xs,Ys).
```

So, for example, we can use this generic function to add one to every element of a list, double every element of a list, or square every element of a list. Given the definitions:

```
:- hilog successor,double,square.
successor(X,Y) :- Y is X+1.
double(X,Y) :- Y is X+X.
square(X,Y) :- Y is X*X.
```

we can do

```
| ?- [(hilog)].
[Compiling ./hilog]
% Specialising partially instantiated calls to apply/3
[hilog compiled, cpu time used: 0.59 seconds]
[hilog loaded]

yes
| ?- map(successor)([2,4,6,8,10],L).

L = [3,5,7,9,11];

no
| ?- map(double)([2,4,6,8,10],L).

L = [4,8,12,16,20];

no
| ?-
```

This definition of `map` is a bit more general than the one normally found in functional languages, which is not surprising since Prolog is a relational language and this is really a relational definition. For example, `map(successor)` is relation a relation over pairs of lists. If we give to `map` a nonfunctional relation, then the map of that relation is also nonfunctional.

(Think of an interesting example.)

Another interesting example is the generic function `twice`. `twice` takes an input function (or relation) and returns a function that applies the input function twice. (From DHDWarren and MVanEmden.) In standard mathematical notation: $\mathit{twice}(f)(x) = f(f(x))$. By turning `twice` into a relation and essentially writing down this definition, we get:

```
twice(F)(X,R) :- F(X,U), F(U,R).
```

And we can run it:

```
| ?- [twice].
[Compiling ./twice]
[twice compiled, cpu time used: 0.659 seconds]
[twice loaded]

yes
| ?- twice(successor)(1,X).

X = 3;

no
| ?- twice(twice(successor))(1,X).

X = 5;

no
| ?- twice(twice(square))(2,X).

X = 65536;

no
| ?- twice(twice(twice(double)))(1,X).

X = 256;

no
| ?-
```

This interesting thing here is that `twice(f)` for a function `f` produces a function similar to `f`, so we can apply `twice` to a result of `twice` and get a quadrupling (or octupling, ...) effect.

We can add another rule for `twice` (and make it a hilog symbol):

```
:- hilog twice.
twice(X,twice(X)).
```

This rule says that applying `twice` itself to a function argument gives a term representing the resulting function. So now we can even apply twice to itself to produce a function that we can then apply to one of basic functions to produce a function to apply to a number (that lives in the house that Jack built), as follows:

```
| ?- twice(twice)(double,Fun),Fun(1,X).

Fun = twice(twice(double))
```

```
X = 16;

no
| ?- twice(twice(twice))(double,Fun),Fun(1,X).

Fun = twice(twice(twice(twice(double))))
X = 65536;

no
| ?- twice(twice(twice))(successor,Fun),Fun(1,X).

Fun = twice(twice(twice(twice(successor))))
X = 17;

no
| ?-
```

DHDWarren (and a followup paper by Martin vanEmden et al.) explore issues around using Prolog to implement higher-order aspects of functional languages. This example is taken from there, but is expressed in HiLog's syntax, rather than Prolog's. HiLog's syntax makes the development more perspicuous.

(Do we(I) want to develop a bit more of lambda calculus, and show how to do more general higher-order programming?)

10.2 Object Centered Programming in XSB with HiLog

HiLog can also be used to program in an object-centered way.

Object oriented programming: C-logic and variants.

(Dealing with pragmatics of HiLog in XSB? and modules and recompilation? Interactions with tabling.)

Chapter 11

Debugging Tabled Programs

4-port Table Debugger.

The names may not be the best, but they should be clear.

1. Call
   - Call Untabled
   - Call Tabled: New Subgoal
   - Call Tabled: Use Incomplete Table
   - Call Tabled: Use Completed Table
2. Exit
   - Exit Untabled
   - Check/Insert Answer followed by
     - Redundant Answer - fail
     - Insert Answer – succeed
3. Redo
   - Retry Program Clause
   - Retry Answer Clause
4. Fail
   - Fail Untabled
   - Check Complete followed by
     - Completing tables (table list)
     - Rescheduling Answers for (table list)

How does tabling affect the debugger commands

Old commands :

Abort cleans up uncompleted tables.

Skip , Leap should work.

Break allows tables to be partially visible.

New Commands:

Ancestors. At least for tables.

Various Table examination mechanisms built upon Table builtins.

Show incomplete tabled subgoals.

Show returns for a table.

Show ancestors for each suspension of an incomplete tabled subgoal.

Chapter 12

Aggregation

In logic programming it is often the case that one wants to compare the various solutions to a single goal with each other. For example, we may have an employee relation that stores employee names, their departments and salaries, and we want to find, say, the total salary of all the employees. This requires querying the employee relation to retrieve the salaries and then combining all the solutions to find their sum. In an SQL database query, this is done using aggregate functions and the GROUP BY clause.

Prolog provides several general predicates, the so-called all-solutions predicates, to allow a programmer to do such things. The all-solution predicates, such as findall/3, bagof/3 and setof/3, accumulate all the solutions to a particular goal into a list. The programmer can then use normal recursive predicates to compute the desired function over that list, for example, the sum of the elements.

To be concrete, to find the total of the salaries, a Prolog programmer could write:

```
:- bagof(Sal,(Name,Dept)^employee(Name,Dept,Sal),Salaries),
   listsum(Salaries,MaxSal).
```

The predicate `listsum/2` simply takes a list of numbers and returns their sum. The all-solutions predicate `bagof/2` takes a template, a goal, and returns a list of instantiations of the template, one for each solution to the goal. In this case, the template is simply the variable `Sal`, so we will get back a list of salaries. The goal is:

```
(Name,Dept)^employee(Name,Dept,Sal)
```

The variables in the term before the ^ symbol indicate the *existential* variables in the goal. The values of `Sal` in successful solutions to the goal are accumulated into a single list, regardless of the values of the existential variables. In this case, we want all salary values, regardless of the employee's name or department. Another possibly interesting alternative query would be:

```
    :- bagof(Sal,(Name)^employee(Name,Dept,Sal),Salaries),
       maximum(Salaries,TotalSals).
```

This query, without the variable `Dept` being existentially quantified, groups together solutions that have the same department, and returns nondeterministically, for each department, the list of salaries for employees in that department. So this is what one would get using a "group by" query in the database language SQL.

The `bagof/3` all-solutions predicate is used here because we don't want to eliminate duplicate salaries. If two employees have the same salary, we want to add both numbers; we want the sum of salaries, not the sum of *different* salaries.

One computational disadvantage of Prolog's all-solutions predicates is that regardless of the function to be computed over the bag (or set) of solutions, that list must still be created. To accumulate the sum of a set of numbers, it certainly seems inefficient to first construct a list of them and then add them up. Clearly one could just accumulate their sum as they are produced. In XSB if the goal is tabled, the situation is exacerbated, in that the set of solutions is in the table already; building another list of them seems a bit redundant.

XSB, being an extension of Prolog, supports its all-solutions predicates, but it also uses tabling to support several other such predicates, which are described in this chapter.

12.1 Tabled Agggregation and Lattice Operations: Min, Max

In XSB we can use a table declaration to indicate that we want to aggregate the returned values of a particular argument. For example, to find the maximum of all salaries of employees, we could declare:

```
:- table maxEmployeeSal(lattice(max/3)).
maxEmployeeSal(Sal) :- employee(_,_,Sal).
```

and define:

```
max(X,Y,Z) :- (Y > X -> Z = Y ; Z = X).
```

We define the predicate `MaxEmployeeSal/1` to be true of employee salaries, and we declare that it is to be tabled using the lattice operation `max/3` on its only argument. We define `max/3` to be the operation that takes two numbers as input and produces the larger as output. Notice that this operation does define a lattice over numbers, which is why we use the `lattice` indicator in the table declaration. In this situation, `maxEmployeeSal/1` must be called with a variable. XSB computes the first answer and puts it in the table, and then fails back to find other answers. As each answer is found, the max operation is called with the previous answer and the new answer, and the result of that operation replaces the previous answer in the table. So the table keeps only one answer, and in this case it is the maximum of all the numbers seen so far.

12.2 Tabled Agggregation and the Fold Operation: sum

We can use the employee/3 relation to define a predicate total_salaries/1 that gives the total of the salaries of all employees (in any department), as follows:

```
:- table total_salaries(fold(sum/3,0)).
total_salaries(Sal) :- employee(_Name,_Dept,Sal).

sum(X,Y,Z) :- Z is X+Y.
```

Here we use the *fold* indicate how to combine the answers returned to the table. —em Fold takes a binary function (as a ternary predicate), here sum/3, and its identity, here 0, and applies function to the identity for the first element returned to the table. Then for every following element returned, it applies the function to the element currently in the table and the new element, and replaces the current table entry with the resulting value. So in this example, it will produce the sum of all salary values as the value of Sal.

We can also use tabled aggregatio to get the effect of "group by" in SQL. Consider the following definition that defines a predicate that for each department gives the sum of salaries of employees in that department:

```
:- table dept_salaries(_,fold(sum/3,0)).
dept_salaries(Dept,Sal) :- employee(_Name,_Dept,Sal).

sum(X,Y,Z) :- Z is X+Y.
```

Here the variable in the first argument of the predicate dept_salaries in the table declaration indicates that answers should be grouped by that argument. So in this case, dept_salaries will provide, for each department, the sum of the salaries of the employees in that department.

We use HiLog and tables to support aggregation. In HiLog one can manipulate predicates, or the names of sets. So we can construct a set, or bag really, that contains all the salaries in our example of the simple employee relation:

```
    :- hilog salaries.
    salaries(Sal) :- employee(_Name,_Dept,Sal).
```

The symbol `salaries` is the name of a unary predicate that is true of all salaries, or rather is the name of a *bag* of all salaries. It is a bag since it may contain the same salary multiple times. XSB provides a predicate `bagSum` which can be used to sum up the elements in a named bag. So given the definition of the HiLog predicate `salaries/1` above, we can get the sum of all the salaries with:

```
    :- bagSum(salaries,TotalSals).
```

The first argument to `bagSum` is the name of a bag, and the second is bound to the sum of the elements in the bag.

As an interesting example of aggregation and recursion, consider the following problem. Say our university is considering instituting a policy that guarantees that no supervisor shall make less than anyone he or she supervises. Since they do not want to lower anyone's salary, to initiate such a policy, they will have to give raises to supervisors who make less than one of their employees. And this may cascade up the chain of command. We want to write a program that will calculate how much they will have to spend initially to start this new program. The following program does this:

```
% include needed predicates
    maximum(X,Y,Z) :- X>=Y -> Z=X; Z=Y.
    sum(X,Y,Z) :- Z is X+Y.

% The total cost is the sum of the raises.
    :- table totcost(fold(sum/3,0)).
    totcost(Cost) :- raise(Cost).

% A raise is the max of the posible new salaries (own and
% subordinates' salaries) minus the old salary.
    raise(Raise) :-
        possNewSal(Emp,NSal),
        emp(Emp,_,OSal),
        Raise is NSal-OSal.

% A possible new salary is either one's old salary or the max of the
% possible new salaries of one's immediate subordinates.
    :- table(possNewSal(_,lattice(maximum/3))).
    possNewSal(Emp,Sal) :- emp(Emp,_,Sal).
    possNewSal(Emp,Sal) :-
        dept(Dept,Emp), emp(Sub,Dept,_),
        possNewSal(Sub,Sal).

% dept(Dept,Mgr): department data
    dept(univ,provost).
    dept(ceas,deanCEAS).
    dept(cs,chairCS).
    dept(ee,chairEE).

% emp(Name,Dept,Salary):
    emp(provost,univ,87000).
    emp(deanCEAS,univ,91000).
    emp(chairCS,ceas,95000).
    emp(chairEE,ceas,93000).
    emp(prof1CS,cs,65000).
    emp(prof2CS,cs,97000).
    emp(prof1EE,ee,90000).
    emp(prof2EE,ee,94000).
```

Here is the execution of this program for this data:

```
| ?- [raises].
[Compiling ./raises]
% Specialising partially instantiated calls to apply/3
% Specialising partially instantiated calls to apply/2
[raises compiled, cpu time used: 1.669 seconds]
[raises loaded]

yes
| ?- totcost(C).

C = 19000;

no
| ?-
```

And indeed, it would cost $19,000 to upgrade everyone's salary appropriately.

We can combine aggregation with dynamic programming (which is actually what is happening to some extent in the previous example) to nice effect.

A variation on the knapsack problem discussed above in the section on dynamic programming uses aggregation to find an optimal knapsack packing. (This example taken most recently from JLP 12(4) 92, Clocksin, Logic Programming specification and exectuion of dynamic-programming problems. He refers to Sedgewick, Algorithms A-W 88.) Recall that in the earlier knapsack problem we were given a set of integers and we try to see if, given a target integer, we can choose a subset of the given integers that sum to the target integer. The version we consider here is a bit more complicated, and perhaps more realistic. Here we have a set of kinds of items; each item has a value and a size (both are integers.) We are given a knapsack of a given capacity and we want to pack the knapsack with the items that maximize the value of the total. We assume that there is an unlimited supply of items of each kind.

This problem can be formulated as follows:

```
maximum(X,Y,Z) :- X>=Y -> Z=X; Z=Y.

:- table cap/2.
% cap(Size,Cap) if capacity of knapsick of size Size is Cap.
cap(I,Cap) :- I >= 0, small_cap(I,Cap).

% small_cap(BigSize)(BigCap) if there is some item with ISize and IVal
% such that the capacity of a knapsack of size (BigSize-ISize) has
% capacity (BigCap-Ival).
:- table small_cap(_,lattice(maximum/3)).
small_cap(BigSize,BigCap) :-
item(ISize,IVal),
SmallSize is BigSize-ISize,
cap(SmallSize,SmallCap),
BigCap is IVal+SmallCap.
% every knapsack (>=0) has capacity of 0.
small_cap(BigSize)(0) :- BigSize >= 0.
```

Here the tabling of `cap/2` is not necessary, since the call to `bagMax/2` is tabled automatically.

A simple example of executing this program is:

```
%Data:
item(10,18).
item(8,14).
item(6,10).
item(4,6).
item(2,2).

| ?- [aggreg].
[Compiling ./aggreg]
[aggreg compiled, cpu time used: 0.861 seconds]
[aggreg loaded]

yes
| ?- cap(48,C).

C = 86;

no
| ?- cap(49,C).
```

```
C = 86;

no
| ?-
```

And we can see that indeed to fill a knapsack of size 48, one should take four of item 10 (for total value 72), and one of item 8 (with value 14), giving us a total value of 86.

Another problem that combines dynamic programming and aggregation is the problem of finding the way to associate a matrix chain product to minimize the cost of computing the product.

```
minimum(X,Y,Z) :- X=<Y -> Z=X; Z=Y.

% mult_costI,J,C) if C is the cost of the cheapest way to compute the
% product M_I x M_{I+1} x ... x M_J.
mult_cost(I,I,0).
mult_cost(I,J,C) :- I<J, factor(I,J,C).

% factor(I,J) is true of costs obtained by computing the product of
% matrices between I and J by factoring the chain at any point between
% I and J and assuming optimal costs for the two factors.
:- table factor(_,_,lattice(minumum/3)).
factor(I,J,C) :-
J1 is J-1,
between(I,K,J1),
mult_cost(I,K,C1),
K1 is K+1,
mult_cost(K1,J,C2),
        I1 is I-1,
r(I1,Ri1), r(K,Rk), r(J,Rj),
C is C1+C2+Ri1*Rk*Rj.

between(X,X,_).
between(X,Y,Z) :- X < Z, X1 is X+1, between(X1,Y,Z).

% r(I,N) if N is the number of rows in the I-1st matrix.  (The last is
% the number of columns in the last matrix.)
r(0,5).
r(1,3).
r(2,6).
r(3,9).
r(4,7).
r(5,2).
```

An example run is:

```
| ?- [mult_cost].
[Compiling .\aggreg3]
[aggreg3 compiled, cpu time used: 0.125 seconds]
[aggreg3 loaded]

yes
| ?- mult_cost(1,4,C).

C = 456;

no
| ?-
```

12.3 Paths with Maximal Numbers of Edges

Consider the following example in which we try to find nice ways to explore a park while going from one point to another in it. Say the park has various interesting things to see and paths between them, so all nodes are interesting. We'll represent the paths in the park as a directed graph, with the points of interest as the nodes. The goal now is, given a source and a destination node, find all "node maximal" paths from the source to the destination, those that pass through the most intermediate distinct nodes. We can encounter a node more than once, and we can traverse an edge more than once, but we won't iterate around loops. The idea is that we want to take a more-or-less direct route to our target, but we'd like to see as many points of interest as is possible along the way.

The following program will compute the set of such node-maximal paths. We define `supset/2` to have the superset as the first argument and the subset as the second argument. The `po` aggregate indicator causes XSB to find all the minimal answers, with respect to the indicated order predicate, here `supset/2`.

```
    :- import member/2,append/3, reverse/2 from basics.

% stroll(X,Y,Path) if Path is a way to go from X to Y seeing many things.
    stroll(X,Y,Path) :- walk(X,Y,BPath), reverse(BPath,Path).

% superset(L1,L2) if L1 is a superset of L2.
    :- index superset/2-2.
    superset(_,[]).
    superset(L2,[X|L1]) :- member(X,L2), superset(L2,L1).

% L is in walk(X,Y) if L is a (reversed) path from X to Y.
% (must be tabled because of left-recursion.)
    :- table walk(_,_,po(superset/2)).
    walk(X,Y,[Y,X]) :- edge(X,Y).
    walk(X,Y,[Y|P]) :- walk(X,Z,P), edge(Z,Y).
```

Here `walk(X,Y)` is a parameterized predicate name which represents the set of paths that go from node `X` to node `Y`. Each path is represented as a list of nodes (in reverse order of traversal.) The `po` indicator in the table aggregated argument causes XSB to find the greatest minimal values returned under the indicated partial order, (under the assmption that the first argument of the partial order predicate is less than the second argument).. In this case, it will find each path that traverses the maximum number of nodes possible (without repeated loops, maximum because the first argument of the `superset/2` predicate is a superset of the second argument. Here is the execution of the program on the data shown.

```
    edge(a,b).
    edge(b,c).
    edge(b,d).
    edge(c,d).
    edge(d,e).
    edge(a,f).
    edge(f,g).
    edge(g,e).

| ?- [aggreg].
[aggreg loaded]

yes
| ?- stroll(a,e,P).

P = [a,b,c,d,e];

P = [a,f,g,e];

no
| ?-
```

We get two answers since, for example, there is no path from `a` to `e` passing through both `b` and `f`.

Some aggregate operations can be implemented using either `lattice` or `po`. Finding the maximum of a set of answers is a good example. Both of the following definitions are correct for finding the maximum of the only argument of `salary/1`:

```
    maximum(X,Y,Z) :- X >= Y -> Z = X ; Z = Y.
    :- table salary(lattice(max/3)).
```

and

```
maximum(X,Y) :- X >= Y.
:- table salary(po(max/2)).
```

The first treats `maximum/3` as a lattice operation, which any total order such as `maximum` determines, and the second treats(`maximum/2` as a partial order, which any total order is. In such cases it is more efficient to use `lattice` form because it can take advantage of the fact that at any point in time, there will be at most one value in the table.

12.4 Recursive Aggregation

Aggregation interacts in an interesting way with tabling. We've already seen that we can use them both: in the scenic-path example, we needed tabling to compute the paths using left recursion. We also saw an interesting recursive application of aggregation in the salary-raising example. We continue to explore the interaction of tabling, aggregation and recursion in this section by developing a program to compute the shortest paths between nodes in a (positive) weighted graph.

12.4.1 Shortest Path

To compute the shortest path between two points in a graph, we first define a HiLog predicate `short_path(Source,Target)`, that given two nodes returns short paths from the first to the second. There may be several short paths between two nodes, but we will be sure that one of them must be the shortest path:

```
minimum(X,Y,Z) :- X =< Y -> Z = X ; Z = Y.
:- table shortest_path(X,Y,lattice(min/3)).

shortest_path(X,Y,Len) :- edge(X,Y,Len).
shortest_path(X,Y,Len) :-
    shortest_path(X,Z,PLen),
    edge(Z,Y,ELen),
    Len is PLen + ELen.

:- short_path().
% There's a short path if there's an edge,
short_path(X,Y)(D) :- edge(X,Y,D).
% or if there is a short path to a predecessor and then an edge.
short_path(X,Y)(D) :-
    bagMin(short_path(X,Z),D1),
    edge(Z,Y,D2),
    D is D1 + D2.
```

The first clause says that there is a short path from X to Y of length D if there is an edge from X to Y with weight D. The second clause says there is a short path from X to Y of length D if we take the minimum of the short paths from X to a predecessor (Z) of Y and we get D by adding the distance along the edge to Y from the predecessor.

Now to get the shortest path, we simply take the shortest of the short paths:

```
% The shortest path is the minimum of the short paths
shortest_path(X,Y,D) :- bagMin(short_path(X,Y),D).
```

This program in fact works for cyclic graphs, as long as all loops have nonnegative distance. To see why it works, we must look at it more closely. Normally we think of computing an aggregation by creating all the elements in the bag, and then performing the aggregation on the entire set. However, doing that here, with a cyclic graph, would result in a bag with infinitely many elements, since there are infinitely many different paths through a cyclic graph. It is clear that we can't construct and test every element in a bag of infinitely many elements. bagMin must return an answer before it has seen all the elements. Notice that if a graph has a self-loop, say from node a to node a, then a D such that short_path(X,a)(D) is defined in terms of the minimum of a bag that contains D itself. This turns out to be well-defined, because the minimum operator is monotonic. It works computationally because in the case of recursive definitions, the bagMin may return an answer before it has seen all the answers. At any point it returns the best one it has seen so far: if another one comes along that is better, it returns that one; if another comes along that is no better, it just ignores it, and fails back to find another.

So the order in which answers are generated can effect how much computation is done. If poor answers are returned first and many paths are computed using those poor answers, then all that work is unnecessary and will have to be done again with improved answers. Whereas if the best answer is returned first, then much less total computation will have to be done. So the complexity of this routine is dependent on the scheduling strategy of the underlying engine. We will look at these issues more later.

12.4.2 Reasoning with Uncertainty: Annotated Logic

We will look at examples including computing with annotated logic and Fitting's LP over bilattices.

```
:- import bagMin/2 from aggregs.
:- hilog minimum.
minimum(X,Y,Z) :- X =< Y -> Z=X ; Z=Y.

sumlist([],0).
sumlist([X|L],S) :- sumlist(L,S1), S is S1+X.

:- op(500,xfx,@).
```

```
G:D :- orFun(G,D).
orFun(G,D) :- bagMin(andFun(G),D).

andFun(G)(D) :- G@L,sumlist(L,D).

p(X,Y)@[D] :- edge(X,Y):D.
p(X,Y)@[D1,D2] :- p(X,Z):D1,edge(Z,Y):D2.

edge(a,b)@[5].
edge(b,d)@[6].
edge(b,c)@[1].
edge(c,e)@[3].
edge(e,d)@[1].
edge(a,c)@[7].
edge(c,d)@[2].
```

12.4.3 Longest Path

longest path.

```
% longest path (without loops)
:- import bagMax/2 from aggregs.
:- import member/2 from basics.

longpath(X,Y,P)(D) :- edge(X,Y,D), P=[F|R], \+member(F,R).
longpath(X,Y,P)(D) :-
bagMax(longpath(X,Z,[Z|P]),D), edge(Z,Y,D), Y\==Z, \+member(Y,P).

:- hilog maximum.  maximum(X,Y,Z) :- X @< Y -> Z=Y ; Z=X.

edge(a,b,5).
edge(a,b,6).
edge(a,d,4).
edge(d,b,5).
edge(b,c,3).
edge(b,c,4).
```

12.4.4 Markov Decision Processes

See IV's slides, and

```
:- import for/3 from basics.

u(State,Action,Reward) :-
I = 10,
u_i(I,State,_Action0,Reward0),
iterate(State,10,Reward0,Action,Reward).

iterate(State,I,OldReward,Action,Reward) :-
NextI is I+10,
u_i(NextI,State,NewAction,NewReward),
(NewReward - OldReward < 0.000000000001
 ->Action = NewAction,
Reward = NewReward
 ; iterate(State,NextI,NewReward,Action,Reward)
).

%u_i(Iteration,State,Action,Reward)
u_i(1,State,none,Reward) :-
r(State,Reward).
u_i(I,State,Act,Reward) :-
I>1,Iminus1 is I-1,
argmax_sum_rewards_i(Iminus1,State,SumActRew),
SumActRew = argmax(Act,SumRew),
g(Disc),
r(State,Rew0),
Reward is Rew0 + Disc*SumRew.

%argmax_sum_rewards(Iteration,State,argmax(Action,Reward)).
:- table argmax_sum_rewards_i(_,_,lattice(argmax/3)).
argmax_sum_rewards_i(I,State,argmax(Action,Reward)) :-
sum_rewards_i(I,State,Action,Reward).

%sum_rewards_i(Iteration,State,Action,SumReward)
:- table sum_rewards_i(_,_,_,fold(sum/3,0.0)).
sum_rewards_i(I,St0,Act,Rew) :-
u_i(I,St,_PAct,Rew0),
t(St0,Act,St,Prob),
Rew is Prob*Rew0.

sum(X,Y,Z) :- Z is X+Y.
argmax(X,Y,Z) :-
X = argmax(_,R1),
Y = argmax(_,R2),
(R1 >= R2 -> Z = X ; Z = Y).
```

```
%t(Sou,Act,Tar,Prob).
t(a,a1,a,0.5).
t(a,a1,b,0.5).
t(a,a2,c,1.0).
t(b,b1,a,0.25).
t(b,b1,b,0.75).
t(c,c1,c,0.5).
t(c,c1,b,0.5).

%r(State,Val)
r(a,12.0).
r(b,-4.0).
r(c,2.0).

%g(Discount)
g(0.9).
```

12.5 Scheduling Issues

Discuss breadth-first-like scheduling. Give example of graph that has exponential shortest-path with depth-first scheduling. Give benches on both depth-first and breadth-first scheduler. (For benches, maybe just give relative times, so as not to date it.)

(ts) I agree, Ive been using relative times.

12.6 Stratified Aggregation

Issues of stratified findall here, stratified aggregation?

Chapter 13

Incremental Table Maintenance

Describe incremental maintenance.

Chapter 14

Negation in XSB

Negation in the context of logic programming has received significant attention. Prolog implements a negation-as-failure inference rule, succeeding the negation of a goal if the goal itself cannot be finitely successfully proved. This implements a kind of closed-world assumption, in that a proposition is assumed to be false if it cannot be proved to be true. Recall that with pure Horn clauses, one cannot prove a negative fact, since the universally true structure (assigning true to all atoms) is always a model of any set of Horn clauses. The closed world assumption allows us to assume facts are false if they cannot be proved true. This is a useful operator, which can be used to represent (and program) interesting situations.

Consider a Datalog database, defining a single relation that describing the employees of an organization, all defined by simple ground facts. Consider the situation in which we ask a simple query: “Is Bill Gates an employee?” The query processor will answer “no”; what does this mean, logically? It does not mean that it has proved that not(employee(’Bill Gates’)) is true in all models of the database. The database says only that certain facts are true; it says nothing about any facts being false. So a structure in which all the given employee facts are true and another fact saying Bill Gates is an employee is also true is a structure satifying all the database facts and so is a model. So clearly not(employee(’Bill Gates’)) is not logically implied by the database program. So the “no” answer to the initial query means that we cannot prove that Bill Gates is an employee; it does not mean that we have proved that Bill Gates is not an employee.

Consider a simple example of defining the predicate `bachelor` using the predicates `married` and `male`:

```
bachelor(X) :- male(X), \+ married(X).

male(bill).
male(jim).

married(bill).
married(mary).
```

The rule says that an individual is a bachelor if it is male and it is not married. (\+ is the negation-as-failure operator in Prolog.) The facts indicate who is married and who is male. We can interrogate this program with various queries and we get the following:

```
| ?- [negation].
[Compiling ./negation]
[negation compiled, cpu time used: 0.1 seconds]
[negation loaded]

yes
| ?- bachelor(bill).

no
| ?- bachelor(jim).

yes
| ?- bachelor(mary).

no
| ?- bachelor(X).

X = jim;

no
| ?-
```

as expected. The closed-world assumption is applied here: for example, when there is no fact that says that jim is married, we assume that he is not married. Also when there is no fact saying that mary is male, we assume she is not.

It is important to note that with pure Horn clauses, i.e., simple implications with conjunctions of (positive) atomic formulas as antecedents and single atomic formulas as consequents, *nothing* can be proved to be false. I.e., a set of Horn clauses never logically implies the negation of any atomic formula. Given an arbitrary set of Horn clauses, H, and an arbitrary atomic formula, A. Consider the structure ST that makes every atomic formula true. ST makes H true and makes A true. Therefore H does not logically imply not(A). So it is clear that if we are given a Horn clause program, as for `married` above, and we ask a negative goal, such as:

```
| ?- \+ married(jim).
```

a "yes" answer does *not* mean that the program logically implies `not(married(jim))`. It means that using these rules alone, we cannot conclude that married(jim).

Another way to think about the answers to negative queries to a Horn clause program is to think in terms of structures. We saw that every set of Horn clauses has a unique minimal Herbrand model. That's the model constructed by the bottom-up application of the T_P operator. Given this unique model, we can answer both positive and negative queries: a positive query is true if it is in the least Herbrand model; a negative query is true if its subgoal is *not* in the least Herbrand model. So we can use the unique minimal model to characterize answers to both positive and negative queries to Horn programs.

Before we get completely carried away with using negation, we need to look at situations in which there are problems. There are two sources of problems: floundering and nonstratification. Let's first consider a floundering query and program. Say we wrote the bachelor rule as:

```
bachelor(X) :- \+ married(X), male(X).
```

This looks to be an equivalent definition, since after all, the comma is conjunction so the order of literals in the body of a program (as simple as this one) shouldn't matter. But now consider the results of the same queries as above when they are submitted to this program:

```
| ?- bachelor(bill).

no
| ?- bachelor(jim).

yes
| ?- bachelor(mary).

no
| ?- bachelor(X).

no
| ?-
```

The answers are fine for the specific queries concerning bill, jim and mary, but the general query asking for all bachelors fails, whereas we would expect it to generate the answer jim. The reason is that the generated subquery of `\+ married(X)` is able to prove `married(X)` is true (for X=bill (and X=mary as well)), and so `\+ married(X)` fails. The problem is that the implementation of the operator `\+` only works as we want it to when it is applied to an atomic formula containing no variables, i.e., a ground atomic formula. It is not able to generate bindings for variables, but only test whether subgoals succeed or fail. So to guarantee reasonable answers to queries to programs containing negation, the negation operator must be allowed to apply only to ground literals. If it is applied to a nonground literal, the program is said to *flounder*. Prolog systems in general allow the `\+` operator to be applied to nonground literals and so the programmer may get unexpected results. Often another operator, `not`, is provided which acts just like `\+` except that it gives an error message when applied to a nonground literal. (In XSB `not` and `\+` give the same, unsafe, results.)

In fact, when we call the query `\+ married(X)`, with `X` a variable, it succeeds if it is not the case that there exists a value for X for which `married(X)` succeeds. So when a negated goal contains a variable, the negation has wider scope than the existential quantifier which binds the variable. If there are no variables in the subgoal, then there is no quantifier to interfere.

The other problem that may arise in programs with negation, that of nonstratification, is a bit more subtle. While answers to negative queries to a purely Horn clause program are completely characterized by the program's least model, in our bachelor example, we are using negative queries in the definitions of other predicates, not just as top-level queries. Such uses can lead to more complex issues. Consider the following rules:

```
shave(john,john).
shave(bill,bill).
shave(barber,X) :- \+ shaves(X,X).
```

This program says that 'John' shaves himself, and 'Bill' shaves himself, and that the barber shaves someone if that person does not shave himself. The question is: Who shaves the barber? This is known as the "Barber paradox": If the barber does not shave himself, then since he shaves all such people, the rule says that he does shave himself. And if he does shave himself, then since he only shaves people who don't shave themselves, he doesn't shave himself.

The question of whether the barber shaves himself can be posed in Prolog as:

```
| ?- shave(barber,barber).
```

So what should Prolog answer? Actually Prolog will go into an infinite loop here, which seems reasonable, since either a yes or no answer seems incorrect given the argument above.

Consider another example of this kind of phenomenon. Say we want to define a predicate, `normal`, that is true of all reasonable sets. A set is normal if it doesn't contain itself as a member. (A set containing itself is rather weird, when you think about it. So we are using the idea of normality to exclude these weird sets.) To define `normal`, we give the rule:

```
normal(S) :- \+ in(S,S).
```

where the predicate `in` denotes membership. Now we want to have the constant `n` denote the set of all normal sets. So `X` is in `n` just in case `X` is a normal set. The following rule reflects this:

```
in(X,n) :- normal(X).
```

Now consider what happens if we ask this program whether `n` is itself a normal set: `normal(n)`, which reduces to `\+ in(n,n)`, which reduces to `\+ normal(n)`. So to show that `n` is normal, we have to show that `n` is not normal. This is essentially a simple formalization of Russell's paradox.

Clearly there is something a little odd with these examples. The oddity is that in each case, a predicate, (`normal(n)` or `shave(barber,barber)`) is defined in terms of its own negation. Normally we consider rules as defining predicates, and this is an odd kind of cyclicity which we want to avoid in definitions. Programs that avoid such cycles through negation in their definitions are called *stratified* programs. Notice that Prolog would go into an infinite loop when asked queries that involve a cycle through negation.

So we need to have a better idea of what we want these programs that include negation to mean. That is, we need a semantics for these programs. For positive programs (i.e., Horn clause programs) we have a very nice and clean semantics; actually we have several of them, all defining the same thing. A query to a positive program succeeds if 1) an instance is logically implied by the program clauses, or 2) if there is an SLD resolution proof of it, or 3) if an instance is in the least fixed point of the T_P operator for the program. The question is whether we can come up with a simple and clean semantics for programs with negative goals in their bodies.

14.1 Completion Semantics

The first approach to a semantics of programs with negative literals in their bodies was proposed by Keith Clark [], and is known as Clark's Completion. The idea here is to use the first idea above for the semantics: to use the program to come up with a set of formulas which will logically imply the all the appropriate queries, including negative ones. Clearly the implications obtained by treating the negation operator as logical negation won't do, since they cannot imply any negative goal. E.g.

```
forall(X,bachelor(X) <- male(X) /\ not(married(X)))

male(bill).
male(jim).

married(bill).
married(mary).
```

do not imply `not(bachelor(mary))`, which according to our intuitions we require. Those formulas cannot imply `not(bachelor(mary))` since there is a model of these formulas in which bachelor(mary) is true (one in which everyone is a bachelor, which satisfies the first rule true.)

So this set of formulas doesn't work, but maybe another one will. Since we want to think of the first program rule as *defining* the relation bachelor, and we usually use "if and only if" for definitions, we might turn all the "if"s into "if and only if"s as follows:

```
forall(X,bachelor(X) <-> (male(X) /\ not(married(X))))

forall(X,male(X) <-> X=bill \/ X=jim)

forall(X,married(X) <-> X=bill \/ X=mary)
```

Its easy to see that for the first rule, we simply made the quantifier explicit and turned the "if" into an "iff". For the male/1 predicate, we had to create a single rule for all the facts, creating a rule with a single "if", and then we changed that "if" into "iff". The intuitive meaning of the second rule that defines the relation male/1 is that X is a male if and only if X is either bill or jim.

Now this is a set of logical formulas, and we would like the meaning of a ground query to be true if this set of formulas (called the completion of the program) logically implies the query. There is one more issue we need to handle to make this work. Given just the if-and-only-if rules above, we still cannot prove that bill is not a bachelor. The problem is that it may be that both constants bill and jim refer to the same individual in the model. (Intuitively this might be more understandable had I chosen to name the second male as james instead of as bill.) So to be able prove that bill is not married, we have to know that `not(bill=jim)`. So to make the completion semantics work, we add additional inequality axioms to say that no two constants are equal. We also need to add inequality axioms to say that functions are freely interpreted, i.e., that two function applications are equal only if they are the same function symbol applied to equal arguments.

With these additional inequality axioms and the completion of the bachelor program, we can prove not(bachelor(bill)) as follows:

From `forall(X,bachelor(X) <-> (male(X) /\ not(married(X))))`, we can infer `(bachelor(bill) <->` And from this and `married(bill)` and `not(bill=jim)`, we can infer `not(bachelor(bill))`. So for this program we have succeeded in giving a meaning to programs with negation that satisfy our intuitions.

However, there are other programs for which the completion gives somewhat unintuitive results. For example, consider the following transitive closure program:

```
e(a,a).
e(b,a).

p(X,Y) :- e(X,Y).
p(X,Y) :- e(X,Z),p(Z,Y).
```

Its completion is (assuming free variables are universally quantified):

```
e(X,Y) <-> (X=a /\ Y=a) \/ (X=b /\ Y=a)

p(X,Y) <-> e(X,Y) \/ exists(X,e(X,Z) /\ p(Z,Y))
```

Here we again put the two standard rules for transitive closure together into a single rule and then turned the "if" into and "iff".

Now consider the transitive closure of this simple graph; it is {p(a,a), p(b,a)}, since there are no paths other than the length-one paths corresponding to the edges. So since p(a,b) is actually not in the transitive closure, we would like to have the completion of this program *logically imply* that is *not* in the transitive closure. I.e., we would like the completion to imply `not(p(a,b))`. But it doesn't.

Notice that the structure that makes the following atoms true:

```
  {e(a,a),e(b,a),p(a,a),p(b,a),p(a,b),p(b,b)}
```

satisfies all the rules. It clearly satisfies the first rule. To see that is satisfies the second rule, consider all that rule's instances:

```
1. p(a,a) <-> e(a,a) \/ exists(Z,e(a,Z) /\ p(Z,a))
2. p(b,a) <-> e(b,a) \/ exists(Z,e(b,Z) /\ p(Z,a))
3. p(a,b) <-> e(a,b) \/ exists(Z,e(a,Z) /\ p(Z,b))
4. p(b,b) <-> e(b,b) \/ exists(Z,e(b,Z) /\ p(Z,b))
```

Clearly the first and second instances are true since both sides evaluate to true. Consider the third instance: the left-hand side (p(a,b)) is true, e(a,b) is false, but let Z be a, and then we see that e(a,a) / p(a,b) is true in the structure, so the right-hand side is also true, making this entire thrid formula true. And consider the fourth instance: the left-hand side is true, e(b,b) is false, but if we take Z to be a, we get (e(b,a) / p(a,b)) which is true in the structure; so both sides are true and the fourth instance is true as well. So all the instances of the completion of the second rule of the program are true, so there exists a model of the completion (this one) that makes p(a,b) true. So clearly the completion does not logically imply not(p(a,b)). So what this means is that under the completion semantics our definition of "transitive closure" does not define real transitive closure.

For this reason, many have concluded that the completion semantics is not the desired semantics for logic programs. It may handle programs with negation in a reasonable way, but it goes a step backward for positive programs, in particular one as simple as transitive closure.

The problem here can be seen to boil down to rules like:

```
p :- p.
```

In the least model of this program, p is false. But the completion of this rule: `p <-> p` is a tautology so it does not imply that p is false (or that p is true); it imposes no constraint on `p` at all. By looking carefully at our transitive closure example, we can see this "self-supporting" phenomenon going on, with p(a,b), for example.

Recall what we wanted the program completion to do for us for this program. We wanted to construct a set of first-order formulas derived from the transitive closure program that would logically imply exactly the facts in the transitive closure and the negation of facts not in the transitive closure. It turns out that this is not possible; there is *no* such set of first-order formulas. This is a basic theorem of first-order logic. [].

So rather than basing our semantics of logic programs with negation on the notion of logical consequence of a first-order theory, let's turn to look at a possible least-fixed point characterization.

14.2 Negation through Fixed Points

Let's consider how we might understand the bachelor program using least fixed points. Again, the program:

```
bachelor(X) :- male(X), \+ married(X).

male(bill).  male(jim).

married(bill).  married(mary).
```

Here we can first compute the least fixed point of the program consisting just of the two married facts, which (trivially) contains just those two facts. Now having completing that fixed point, we can compute a fixed point for bachelor, which depends on the negation of married.

The paradoxical examples show that definitions of predicates that depend on their own negations are problemmatic. So our first approach to providing a semantics for negation in logic programming will involve prohibiting such programs. A program that doesn't include cyclic definitions that involve negation is called *stratified.*

A Prolog program is *Predicate Stratified* if each predicate, P, can be assigned an integer (a stratum), S(P), such that for every pair of predicates P_1 and P_2 in rule in the program of the form:

```
P_1(..) :- ..., P_2(...), ...
```

`S(P_1) >= S(P_2)`, and for every pair of predicates P_1 and P_2 in rule in the program of the form:

```
P_1(..) :- ..., \+ P_2(...), ...
```

`S(P_1) > S(P_2)`.

For example, consider the bachelor program:

```
bachelor(X) :- male(X), \+ married(X).

male(bill).
male(jim).

married(bill).
married(mary).
```

This program has three predicates: `bachelor`, `male`, and `married`. We can assign strata as follows:

```
S(male) = 1
S(married) = 1
S(bachelor) = 2
```

and this stratification function satisfies the requirements that come from the one rule, i.e., that S(bachelor) >= S(male), and S(bachelor) > S(married). So this program is said to be predicate stratified. There are many possible stratification functions that satisfy the necessary constraints. We can define a "least" one, in which all the strata are as small (positive) integers as possible. The one given here is least.

Consider the program that defines normal sets:

```
in(X,n) :- normal(X).
normal(S) :- \+ in(S,S).
```

For this program S(in) must be >= S(normal), because of the first rule, and S(normal) must be > S(in), because of the second rule. It is clear that this is not possible, so no stratification function exists for this program, and the program is said to be not predicate stratified, or unstratified, or unstratifiable.

There is a simple algorithm for determining whether a stratification function exists for a program and finding one if there does. The algorithm tries to construct a stratification function but will fail if none exists. The algorithm is as follows: First assign every predicate the stratum 1. Then for every pair of predicates P_1 and P_2 that appear in a rule of the program of the form:

```
P_1(..) :- ..., \+ P_2(...), ...
```

If $S(P_1)$ <= $S(P_2)$, then set $S(P_1)$ to be $S(P_2) + 1$. Repeat this until no change needs to be made, in which case the final S is a stratification (and a least one), or until for some predicate P, S(P) > n, where n is the number of distinct predicates in the program, in which case there is no stratification.

Given a stratification for a program, we can define a unique Herbrand model for the program bottom-up using the idea the T_P operator. We define the meanings of the predicates in a stratified program in the order of their stratification numbers. First look at all the predicates at stratum 1. None of these depend negatively on any predicate, so those rules, denoted by $P1$, are a set of Horn clauses. We can construct the least model of these rules, using the T_{P1} operator for the stratum 1 program, which generates the set of ground atomic formulas, M_{P1}, that are logical consequences of $P1$.

Now M_{P1} defines all the predicates at stratum 1. Now consider all predicates at stratum 2, $P2$. They depend on other stratum 2 predicates and stratum 1 predicates, perhaps negatively on stratum 1 predicates. We can now apply the T_{P2} operator iteratively to M_{P1} and now interpreting a negative atomic formula body of a rule in $P2$ as true iff A is not in M_{P1}. We continue in this way through the strata, at each step using the atoms computed at an earlier step to interpret the negative literals in the bodies of rules. In this way we can construct in iterated fixed point, which will be a model of rules in the program. This model is called the *Perfect Model* of the stratified program.

For example:

```
1. p :- q, \+r, s.
2. q :- p.
3. q :- s.
4. r :- \+s, t.
5. s.
```

A stratification can be computed as follows:

```
S(p) = 1
S(q) = 1
S(r) = 1
S(s) = 1
S(t) = 1

S(p) = 2, by rule 1, and S(r) = 1
S(q) = 2, by rule 2, and S(p) = 2
S(r) = 2, by rule 4, and S(s) = 1

S(p) = 3, by rule 1, and S(r) = 2
S(q) = 3, by rule 2, and S(p) = 3
```

So this program is stratified, and its stratification is:

```
S(p) = 3
S(q) = 3
S(r) = 2
```

```
S(s) = 1
S(t) = 1
```

With this stratification we can compute the perfect model. $P1$:

```
5. s.
```

and M_{P1}: `{s}`. Then $P2$ is

```
4. r :- \+s, t.
```

and M_{P2} = `{s}`, since `\+s` is false (since s is in M_{P1}) and then

```
1. p :- q, \+r, s.
2. q :- p.
3. q :- s.
```

and M_{P3} is `{s, q, p}`, since s is in M_{P2} and r is not in M_{P2}.

There are a few things to note about the perfect model. It is an iterated fixed point, and so is defined by transfinite induction. I.e., we may have to iterate to infinity at one stratum before we can move to the next stratum. So this means that the perfect model may not be recursively enumerable.

When a program is predicate stratified, it is generally agreed that the perfect model provides the correct semantics for it. However, there are many programs that are not predicate stratified but for which Prolog computes what seems to be the correct answers. Consider for example the definition of even natural numbers (represented in successor notation):

```
even(0).
even(succ(X)) :- \+ even(X).
```

Prolog computes this correctly. But the program is not predicate stratified, since the predicate `even/1` depends on itself negatively. But notice that when it does, its argument is smaller, so there actually is no cycle through negation in the definition; the definition "grounds out" at 0. Also note that we can transform any predicate stratified propositional program into a nonstratified (predicate logic) program simply by introducing one unary predicate, say, c/1, and then replacing every propositional symbol, P, in the original program by c(P). Clearly the program hasn't changed in any fundamental sense, but if the original program had any negation in it at all, the transformed program will not be predicate stratified; the only predicate is c/1 so it will depend negatively on itself.

To address these problems, more refined definitions of stratification were developed. In particular the notion of "local stratification" was defined. The idea of local stratification is to ground the program, and then to determine if the (usually infinite) set of ground atomic formulas can be placed in strata so that there is no cycle through negation. For example, the even/1 program above is locally stratified, since we can assign the stratum 1 to the ground atomic formula even(0), and n+1 to the formula even(succ(...succ(0))) with n successors. With this stratification, it's easy to see that any atom depends negatively only on atoms with smaller strata.

However, there are disadvantages to local stratification. For example it is undecidable whether a finite predicate logic program is locally stratified, which means that were we to use that as the definition of a meaningful program, it would undecidable whether a program has meaning.

So the search continued for a definition that would give meaning to all programs, stratified (in whatever form) or not.

[How/whether to include this following...]

As an example of stratified negation, consider the situation in which we have a set of terms and a nondeterministic reduction operation over them. Then given a term, we want to reduce it until further operations don't simplify it any more. We will allow there to be cycles in the reduction operation and assume that terms that reduce to each other are equivalently fully reduced.

This situation can be abstractly modeled by considering the terms to be nodes of a directed graph with an edge from N1 to N2 if the term at N1 directly reduces to the term at N2. Now consider the strongly connected components (SCCs) of this graph, i.e. two nodes are in the same SCC if each can be reached from the other. We will call an SCC a final SCC if the only nodes reachable from nodes in that SCC are others in that SCC. Now given a node, we want to find all nodes reachable from that node that are in final SCCs.

So first we define reachable:

```
:- table reachable/2.
reachable(X,Y) :- reduce(X,Y).
reachable(X,Y) :- reachable(X,Z), reduce(Z,Y).
```

Next we can define `reducible` to be true of nodes that can be further reduced, i.e., those nodes from which we can reach other nodes that cannot reach back:

```
reducible(X) :- reachable(X,Y), tnot(reachable(Y,X)).
```

`tnot` is the negation operator for tabled goals. It checks to see that the call doesn't flounder, giving an error message if it does. It can be applied only to a single goal, and that goal must be a tabled predicate. With this predicate we can next define the predicate `fullyReduce` that is true of pairs of nodes such that the first can be reduced to the second and the second is not further reducible:

```
:- table reducible/1.
fullyReduce(X,Y) :- reachable(X,Y),tnot(reducible(Y)).
```

Note that we must table `reducible` because `tnot` can only be applied to predicates that are tabled.

So with these definitions and the following graph for reduce:

```
reduce(a,b).
reduce(b,c).
reduce(c,d).
reduce(d,e).
reduce(e,c).
reduce(a,f).
reduce(f,g).
reduce(g,f).
reduce(g,k).
reduce(f,h).
reduce(h,i).
reduce(i,h).
```

we can ask queries such as:

```
| ?- fullyReduce(a,X).

X = c;

X = h;

X = d;

X = k;

X = i;

X = e;

no
| ?-
```

which returns all nodes in final SCCs reachable from node `a`.

However, we may now wish to generate just one representative from each final SCC, say the smallest. We can do that with the following program:

```
fullyReduceRep(X,Y) :- fullyReduce(X,Y), tnot(smallerequiv(Y)).

smallerequiv(X) :- reachable(X,Y), Y@<X, reachable(Y,X).
```

Now we get:

```
| ?- fullyReduceRep(a,X).

X = c;

X = h;

X = k;

no
| ?-
```

Note that this is an example of a stratified program. The predicate `reachable` is in the lowest stratum; then `reducible` is defined in terms of the negation of `reachable` so it is in the next stratum; then `fullyReduce` is defined in terms of the negation of `reducible`, so it is in the third stratum. `smallerequiv` is in the first stratum with `reachable`; and `fullyReduceRep` is in the same stratum as `fullyReduce`.

14.3 General Negation

There have been many proposals for a semantics for general logical programs, i.e., programs with unrestricted negation. When considering general semantics, there are a few very simple (propositional) programs that we can use to show their particular characteristics. The programs are the following:

```
p :- p.
```

This is a simple program in which p depends (only) on itself. Notice that Prolog goes into an infinite loop on this program, and Tabled Prolog terminates with p being false.

The second program is a direct negative loop.

```
p :- \+ p.
```

The final program has two propositional symbols, each of which depends directly on the other negatively.

```
p :- \+ q.
q :- \+ p.
```

We will use these programs to help understand (and distinguish) the various semantics for general logic programs that we consider in the following sections.

14.3.1 Clark's Completion Semantics

Idea is to turn the implications of rules into biconditionals and then to treat closed world negation as logical negation.

So take program: Put rules with same head together into one rule (with disjuctive bodies). Add existential quantifiers for variables in the body but not the head.

Then turn the if's into iff's.

This is called the completion of the program.

Example:

even(0). even(s(X)) :- \+even(X).

Completion:

```
even(X) <==> exists(Y, (X = 0 \/ (X=s(Y) /\ not(even(Y))))).
```

Also must add "Clark's Equality Theory" (CET), to define equality: not(a=b) for every pair of constants a and b. f(X1,..,Xn) = g(Y1,..,Yn) iff f=g, and Xi=Yi for all Y. not(X=f(..,T,..)) if X appears in term T.

Then the meaning of a program is the set of logical consequences of the completion of the program and CET.

So is even(s(s(0))) a logical consequence of the completion of the program and CET?

```
even(s(s(0))) <==> exists(Y,(s(s(0))=0\/(s(s(0))=s(Y)/\not(even(Y)))))
    <==> not(even(s(0)))
    <==> not(exists(Y,(s(0)=0\/(s(0)=s(Y)/\not(even(Y))))))
    <==> not(not(even(0)))
    <==> not(not(exists(Y,(0=0\/...))))
    <==> not(not(true))
    <==> true.
```

Yes. So Clark's completion gives us what we want for the even program.

What does it do with:

```
p :- p.
```

The completion of this program is simply `p<==>p`, which is a tautology. So its logical consequences are just the tautologies, so it does not imply either p or not(p). So from this program we cannot conclude that p is true or that p is false. Notice that this differs from the perfect model semantics in which we can conclude from this program that p is false.

This does have an interesting effect on the transitive closure definition. Consider the program:

```
e(a,a).
e(b,a).

p(X,Y) :- e(X,Y).
p(X,Y) :- e(X,Z),p(Z,Y).
```

We can ground this program into:

```
e(a,a).
e(b,a).

p(a,a) :- e(a,a).
p(a,b) :- e(a,b).
p(b,a) :- e(b,a).
p(b,b) :- e(b,b).
p(a,a) :- e(a,a),p(a,a).
p(a,a) :- e(a,b),p(b,a).
p(a,b) :- e(a,a),p(a,b).
p(a,b) :- e(a,b),p(b,b).
p(b,a) :- e(b,a),p(a,a).
p(b,a) :- e(b,b),p(b,a).
p(b,b) :- e(b,a),p(a,b).
p(b,b) :- e(b,b),p(b,b).
```

And now the completion is:

```
e(X,Y) <==> (X=a,Y=a) ; (X=b,Y=a).

p(a,a) <==> e(a,a) ; (e(a,a),p(a,a)) ; (e(a,b),p(b,a)).
```

```
p(a,b) <==> e(a,b) ; (e(a,a),p(a,b)) ; (e(a,b),p(b,b)).
p(b,a) <==> e(b,a) ; (e(b,a),p(a,a)) ; (e(b,b),p(b,a)).
p(b,b) <==> e(b,b) ; (e(b,a),p(a,b)) ; (e(b,b),p(b,b)).
```

Now using the first biconditional in the 2nd and 4th rule, we can simplify this to:

```
e(a,a).
e(b,a).
not(e(a,b)).
not(e(b,b)).

p(a,a).
p(a,b) <==> p(a,b)
p(b,a).
p(b,b) <==> p(a,b)
```

And now note that just as in the positive loop, we cannot prove either that p(a,b) is true or that it is false. So this means that the completion semantics does not imply that in this graph, you cannot get from a to b. So this program does not define transitive closure.

So this shows that the completion semantics does not agree with the perfect model semantics. In the perfect model semantics, p(a,b) is definitely false. For this reason, the completion semantics is not normally considered a good semantics for logic programs. People feel that the above transitive closure program ought to indeed define transitive closure. And since under the completion semantics, it doesn't, the completion semantics is not a good semantics.

It is RE, and it is the logical consequences of some set of formulas...

14.3.2 Stable Model Semantics

Another approach to giving semantics to general logic program was proposed by Michael Gelfond and Vladimir Lifschitz [?] and is known as Stable Models. This approach defines a model for a program as a fixed point of a particular operator, the Gelfond-Lifschitz (or GL) operator (not surprisingly.)

We will consider propositional programs. Stable models of predicate programs are defined in terms of the (possibly infinite) ground instantiation of the program. A model is represented as a set of proposition symbols, the model making all symbols in the set true and all those not in the set false. For any general propositional logic program, the GL operator takes a set of propositional symbols and returns a set of proposition symbols. It is defined using a transformation of the given program called the GL-reduct of a program. Given a program P and a set of proposition symbols S, the GL-reduct of P is the program obtained from P by

1. deleting any rule which has a negative literal in its body whose proposition symbol is in S, and
2. deleting from the body of any rule any negative literal whose proposition symbol is not in S.

For example, the GL-reduct of program:

```
p :- q, \+r, s.
q :- \+s.
r := \+q, s.
```

with respect to the set of symbols:

```
{p, q}
```

is

```
p :- q, s.
q.
```

One can think of the GL-reduct of a program with respect to a set S as being the program one would get by assuming the propositions in S are true when interpreting negative body literals of program rules.

Notice that the GL-reduct of any program with respect to any set S is a definite program. In the process of constructing the GL-reduct we delete all negative literals in a program.

Now we can define a Stable Model of a general propositional logic program. A stable model of a program P is a set of proposition symbols M such that the least model of the GL-reduct of P with respect to M is M. I.e., it is a fixed point of the operator obtained by taking the least model of the GL-reduct.

Proposition: If a program P is predicate stratified, then P has exactly one stable model, which is the perfect model.

Examples: For the program:

```
p :- p.
```

the only stable model makes `p` false. This follows from the previous proposition, as well as from direct construction.

For the program:

```
p :- \+p.
```

there are *no* stable models.

For the program:

```
p :- \+q.
q :- \+p.
```

there are two stable models: one that makes p true and q false, and one that makes p false and q true.

Stable models have several drawbacks. Computing the stable model of a propositional program is NP-complete, so the best general algorithm known is exponential. Also, it does not have the "relevance" property. That is, to find whether there is a stable model that makes a particular proposition symbol p true, one has to look at the entire program, not just the propositions that are used to define p. For example, if the program contains a rule `p :- \+p`, then there is no stable model at all, regardless of what the rest of the program is (as long as it doesn't involve p.)

14.3.3 Well-Founded Semantics

Another approach to defining the meaning of logic programs with default negation is the Well-Founded Semantics, which uses a 3-valued logic [**?**]. In a 3-valued logic a proposition may be true or false or *undefined*. By using the truth value undefined judiciously, we can provide a single 3-valued model for each logic program with negation.

We will motivate the definition of the well-founded semantics by first considering what we can infer if we have only partial knowledge of some predicates that are used, but not defined, in an old-fashioned definite logic program.

Consider a situation in which we have some (definite) rules that define some predicates inductively, and use other predicates in their definitions, but don't define them. We will call the used, but not defined, predicates as open predicates. We will assume that we have partial knowledge of the truth values of atoms of the open predicates. As a simple, concrete example, consider our friend *reachability* in a graph:

```
reach(X,Y) :- edge(X,Y).
reach(X,Y) :- reach(X,Z), edge(Z,Y).
```

Here `reach/2` is defined using the `edge/2` predicate, but `edge/2` is not defined. We can say that the `reach` definition is parameterized by the open predicate `edge/2`; a definition of the `edge/2` predicate determines a complete definition of the `reach/2` predicate.

But perhaps we have only partial knowledge of the `edge/2` relation: say we know that there is an edge from `a` to `b`, an edge from `b` to `a`, no edge from `a` to `c` and no edge from `b` to `c`, but we don't know about other possible edges. (We'll assume the graph has only these three nodes.) The question is: What can we conclude about the `reach/2` relation using this incomplete information about the open predicate `edge/2`? Intuitively, we know that `b` is reachable from `a`, `a` is reachable from `b`, `a` is reachable from `a`, `b` is reachable from `b`, `c` is not reachable from `a`, and `c` is not reachable from `b`. But we don't know if `a` is reachable from `c`, if `b` is reachable from `c`, or if `c` is reachable from `c`. Let's be more precise (and general) concerning how we might come to such conclusions.

We assume that the known true `edge` facts are T = {`edge(a,b)`, `edge(b,a)`} and the known false `edge` atoms are F = {`edge(a,c)`, `edge(b,c)`}. How do we conclude what `reach` atoms can be known true and what `reach` atoms can be known false? To find ones that must be true, we can just add the known facts to the (definite) program and find its least model. Anything true in that model must be true even if some (or all) of the unknown open atoms would turn out to be false. For our example, we get the following program:

```
edge(a,b).
edge(b,a).
reach(X,Y) :- edge(X,Y).
reach(X,Y) :- reach(X,Z), edge(Z,Y).
```

Now we see what is implied by this set of rules: here we get {`reach(a,b)`, `reach(b,a)`, `reach(a,a)`, `reach(b,b)`} by taking the least fixed point of this program. These are facts that must be true in any model consistent with the partial knowledge we have of the `edge/2` relation. We have made conservative assumptions and what is still true under these most conservative assumptions must indeed be true. We can say that this program defines the "definitely true" atoms for `reach`.

We have just seen how to conclude what `reach` atoms must be true; now how do we determine what atoms must be false? We similarly create a program using our assumptions of what `edge` atoms are true and false, but this time we assume that all `edge` atoms *not* known to be false are actually true. So the program we get is:

```
edge(a,b).
edge(b,a).
edge(c,b).
edge(c,a).
edge(a,a).
edge(b,b).
edge(c,c).
reach(X,Y) :- edge(X,Y).
reach(X,Y) :- reach(X,Z), edge(Z,Y).
```

The seven `edge` facts are those not known to be false, i.e. those that might be true. We use this program, taking the least model, to see what `reach` atoms "might be true". We can say this program defines the "possibly true" atoms. In this case, we get: {reach(a,b), reach(b,a), reach(c,b), reach(c,a), reach(a,a), reach(b,b), reach(c,c)}. Any reach atom *not* in this set will have to be false in any model consistent with our initial edge assumptions. So we conclude that any reach atom in the complement of this set must be false in any consistent model. So we know that reach(b,c) and reach(a,c) must be false. We have made liberal assumptions about what might be true and anything still not true under these most liberal assumptions must indeed be false.

To review, we have described a way to use partial or incomplete knowledge of facts that are used in the definitions of inductively defined predicates to infer information about those predicates, information including what instances must definitely be true and what are possibly true (and whose complement must definitely be false.) The idea for determining the definitely true defined instances is to assume that all defining facts not known true are false. Then anything that must be true in this situation is definitely true. To determine the definitely false atoms, we assume all defining atoms not known to be false are indeed true; anything that still isn't possibly true under these assumptions must definitely be false.

We can use this idea to give a semantics to programs with negative literals in their bodies; i.e., inductively defined predicates that use negations in their definitions. For simplicity we will consider only propositional programs here. In order to use a negative literal to help infer a fact, we must know that the literal is true, i.e., the atom that is negated is false. But how can we determine such things? We will use the idea we have just developed of reasoning with partial information about open predicates to approach this problem. We start by initially disconnecting the negative literals in the bodies of clauses from their positive forms and just thinking of them as new propositions. E.g., for the literal tnot(p)[1], we introduce a new proposition symbol, say neg_p. We then replace all negative literals in the program with their new positive forms to get a purely positive program with open predicates. The resulting program can be seen as similar to the old program but now parameterized by the newly introduced neg_* predicates.

So let's apply our approach to reasoning about definitions with partial knowledge about open predicates. We begin by assuming that we know nothing about these new open predicates, i.e., none of their atoms are known true or false. So to determine what defined predicates must definitely be true, we interpret all the neg_* predicates as false, and see what is true in the resulting least model. Those defined atoms are now known true. And to find the false atoms, we interpret all the open predicates as true, take the least model to see what could conceivably be true, and then take its complement to find the definitely false atoms.

Let's consider an example; the initial general program with negation is:

```
p :- r, tnot(t).
q :- r, tnot(s), tnot(u).
r :- s.
r.
s :- tnot(q), r.
```

[1] Recall that tnot is the tabled negation operator in XSB.

```
t.
```

We transform it to its open form:

```
p :- r, neg_t.
q :- r, neg_s, neg_u.
r :- s.
r.
s :- neg_q, r.
t.
```

To determine the well-founded model of the original program, we will maintain two programs with the open propositions: one that tells us what atoms must be true (called the *definitely true* program), and a program that tells us what atoms could possibly be true (called the *possibly true* program) and therefore tells us what atoms must be false. Each program will be a conservative approximation and we will iteratively modify the programs to improve their accuracy.

The initial definitely true program and definitely false program are:

```
%  definitely true                %  possibly true

p :- r, neg_t.                    p :- r, neg_t.
q :- r, neg_s, neg_u.             q :- r, neg_s, neg_u.
r :- s.                           r :- s.
r.                                r.
s :- neg_q, r.                    s :- neg_q, r.
t.                                t.
                                  neg_p.  neg_q.  neg_r. neg_s.  neg_t.  neg_u.
```

In the definitely true program we assume that none of the `neg_*` propositions are true; in the possibly true program we assume they are all true. If we take the least model of the definitely true program, we get {`r`, `t`}. These propositions will true regardless of the truth values of the `neg_*` propositions, so we will want them to be true in the well-founded model of the original program. And the least model of the possibly true program (on the defined propositions) is {`p`, `q`, `r`, `s`}. So any defined proposition symbol *not* in this set must be false, and we will want it false in the well-founded model.

Now we can try to use each of these programs to improve the accuracy of the other. There really is a connection between the pair of propositions, say `p` and `neg_p`: if one is true, then the other should be false. So if we deduce that `r` is definitely true in the well-founded model, then we know that `neg_r` is definitely false, i.e., not possibly true. And that allows us to update our possibly true program by deleting the fact for `neg_r`. Similarly, if we know that `u` is definitely false then `neg_u` is definitely true, and we can update our definitely true program by adding `neg_u` to it.

Having changed our definitely true and possibly true programs, we can again find their least fixed points and see if we have learned something new that will allow us to further update the programs. We continue to add neg_* atoms to the definitely true program and remove neg_* atoms from the possibly true program in this way until we learn nothing new. The resulting programs define the well-founded semantics of the original program.

Consider this process for our example; the least model of the true program contains r and t, so these are known true. Since they are known true, neg_r and neg_t must be false, so we can remove them from the possibly true program, improving our estimate of the possibly true atoms and getting an updated possibly true program:

```
%  definitely true               % possibly true

p :- r, neg_t.                   p :- r, neg_t.
q :- r, neg_s, neg_u.            q :- r, neg_s, neg_u.
r :- s.                          r :- s.
r.                               r.
s :- neg_q, r.                   s :- neg_q, r.
t.                               t.
                                 neg_p.  neg_q.  neg_s.  neg_u.
```

Now looking at the least model of this new possibly true program, we see that neither u nor p is possibly true, i.e., not in the least model of the possibly true program and thus must be false. So we can improve our estimate of the true atoms by adding neg_u and neg_p to our definitely true program, obtaining:

```
%  definitely true               % possibly true

p :- r, neg_t.                   p :- r, neg_t.
q :- r, neg_s, neg_u.            q :- r, neg_s, neg_u.
r :- s.                          r :- s.
r.                               r.
s :- neg_q, r.                   s :- neg_q, r.
t.                               t.
neg_p. neg_u.                    neg_p.  neg_q.  neg_s.  neg_u.
```

Now looking at the current versions of the two programs: the neg version of every defined atom in the least model of the definitely true program has been removed from the possibly true program; and the neg version of every defined atom not in the least model of the possibly true program has been added to the definitely true program. So we can no longer improve our estimates of the definitely true and possibly true atoms, which leaves us with the well-founded model of the original program. The atoms true in the well-founded model are the defined atoms in the least model of the final definitely true program; the atoms false in the well-founded model are the defined atoms *not* in the least model of the final possibly true program. Thus for this program `r` and `t` are true in the well-founded model, `p` and `u` are false, and `s` and `q` are undefined.

To review this process: the definitely true program starts with no `neg_*` atoms and gains them as their positive counterparts are found not to be possibly true. The possibly true program starts with all the `neg_*` atoms and loses them as their positive counterparts are found to be definitely true. The process continues until no more improvements can be made. Then we read off the well-founded model from the final programs.

14.4 Other Stuff to think about?

We need examples! Planning? Use for-all type problems, e.g. to find if all nodes reachable from a given node are red, find if it is not the case that there exists a node reachable from the given node that is not red.

****

Issues of safety.

14.5 Approximate Reasoning

Use course prerequisites example. Introduce undefined truth value. Add undetermined facts for courses currently being taken. Then requests for whether have satisfied requirements will give: true if satisfied without any current course false if not satisfied even if all current courses are passed undetermined if satisfaction depends on outcome of current courses.

Categorization examples? Propositional Horn clauses for bird identification. Allow negation as failure, explicit negation.

```
cardinal :- red, crested.
bluejay :- blue, crested.
robin :- red_breasted, ~crested.

Use undef as a kind of null value?:
binarize relation
undef :- ~undef.
%emp(Name,Sal,Age) -> empsal/2 and empage/2
emp(david,50000,_|_) is represented as ‘‘facts’’:

empsal(david,50000).
empage(david,_X) :- undef.
```

```
or empage(david,X) :- between(45,X,55),undef.
(so can fail if not between 45 and 55.)
```

14.6 Representation of Partial Knowledge with Well-Founded Models

We can use the undefined truth value to encode the fact that we don't know whether a particular proposition is true or false. As an example, we will use a (simplification of a) simple game called *Wumpus*. The game is played on a grid, and a hunter starts on some grid square and moves across the board searching for the one other square that contains the pot of gold. The hunter can move left, right, up, or down, and his goal is to move from where he is to the square that has the gold and pick it up. However, there are some squares on the grid that are dangerous. Some squares contain pits, and if the hunter moves to such a square, he falls into the pit and is killed. The hunter can sense when he is adjacent to a pit by feeling a breeze. So if he moves to a square and feels a breeze, he knows that some adjacent (immediately left, right, up, or down) square contains a pit. So the hunter must move from where he is to the gold while avoiding falling into any pit.

The idea will be to represent the state of knowledge of the hunter by a well-founded model. The following are fixed rules of how the game works:

```
%% grid provides the size of the grid.
grid(4,4).

%% A neighbor is a square to either side or above or below the current
%%  square (and still on the grid.)
neighbor(I,J,NI,J) :- NI is I+1, grid(MI,_), NI =< MI.
neighbor(I,J,NI,J) :- NI is I-1, NI >= 1.
neighbor(I,J,I,NJ) :- NJ is J+1, grid(_,MJ), NJ =< MJ.
neighbor(I,J,I,NJ) :- NJ is J-1, NJ >= 1.

%% feel(I,J,Sense) if the hunter has felt "Sense" on square I,J

:- table feel/3.
%% the hunter can feel only one of breeze and no_breeze
feel(I,J,breeze) :- tnot(feel(I,J,no_breeze)).
feel(I,J,no_breeze) :- tnot(feel(I,J,breeze)).

:- table safe/2.
%% A square is safe if it is not a pit
safe(I,J) :- tnot(pit(I,J)).
%% and if the hunter has sensed something there it is also safe.
```

```
safe(I,J) :- visited(I,J).
%% A square is safe if the hunter feels no breeze on some neighbor.
safe(I,J) :-
neighbor(I,J,NI,NJ),
visited(NI,NJ),
feel(NI,NJ,no_breeze).

:- table pit/2.
%% A square is a pit if it is not safe
pit(I,J) :- tnot(safe(I,J)).
%% A square is a pit if some neighbor feels a breeze but no other of
%% its neighbors is a pit.
pit(I,J) :-
neighbor(I,J,NI,NJ),
feel(NI,NJ,breeze),
tnot(another_neighbor_unsafe(NI,NJ,I,J)).

%% Another neighbor (not (NI,NJ) is a pit.
:- table another_neighbor_unsafe/4.
another_neighbor_unsafe(I,J,NI,NJ) :-
neighbor(I,J,PI,PJ),
(PI =\= NI ; PJ =\= NJ),
tnot(safe(PI,PJ)).
```

Consider the following state, in which the hunter has visited squares 1,1 and 1,2, and felt a breeze on 1,1 and no breeze on 1,2:

```
%% a sample state
feel(1,1,breeze).
feel(1,2,no_breeze).
visited(1,1).
visited(1,2).
```

With these rules and this state, we compute the following:

```
| ?- consult(wumpus,[spec_off]).
[Compiling .\wumpus]
[wumpus compiled, cpu time used: 0.1100 seconds]
[wumpus loaded]

yes
| ?- pit(2,1).
```

```
yes
| ?- safe(2,2).

yes
| ?- safe(2,3).

undefined
| ?-
```

We see that the hunter can infer that there is a pit in square 2,1, since it is the neighbor of a cell in which a breeze has been felt (1,1), and all other neighbors of that cell (only 1,2) is known to be safe.

So with this representation, we can develop a strategy for the hunter to safely move around the grid as follows. The hunter uses WFM of the current knowledge base to determine what adjacent cells are safe and moves to one of them. He then adds to the knowledge base that he has visited the new square and what he sensed in that square. And he repeats the process.

If the new information was undefined in the previous state, we can update the previous WFM in XSB by using `force_answer_true/1`. However, in the example above, the hunter needs to add the `visited` fact for the new square he moved to, and this fact was false in the previous WFM, not undefined. I.e., we need to change the knowledge of visited from false to true, not from undefined to true. So in this system the WFM must be computed anew in each new state. (There may be another way to encode this problem so that all knowledge acquisition would be moving from facts with undefined truth values to those with distinct values, but I don't see it now.)

Now in an actual game, a situation may arise in which the hunter has several places he can move but is unable to infer that any of them is safe. In this case, if he wants to find the gold, he must make a potentially dangerous move to a square which may be a pit. The WFM provides no information as to whether some square might more likely be safe than another square. We could extend the WFM with probabilities to try to infer which squares are more likely to be safe. This can be done within the PRISM (Sato) framework...

leftover stuff for now...........

XSB, with its tabling, does not improve over Prolog in handling floundering queries; all calls to negative subgoals must be ground in XSB for the closed-world interpretation of negation to be computed. XSB does extend Prolog in allowing nonstratified programs to be evaluated, and we will discuss how it does that later in the chapter. However, first we will explore how we can use tabling with stratified programs to compute some interesting results. *****end of leftover............

There have been a number of approaches to defining the meaning of logic programs with the negation-as-failure operator. We will describe one general approach that will shed light on several others. The intuitive idea is that Prolog programs are to be understood as rules that define predicates, not as first-order logical formulas. When we are just dealing with positive Horn rules, then these two intuitions coincide in some sense. We saw that the least model of a Horn program defines the set of predicates whose atoms occur in heads of rules in the program, and the facts for those predicates are exactly the logical implications of the Horn clauses. So we can use logical implication to define relations from Horn clauses. However, when we include the negation operator, and want to interpret it in a “negation as failure” way, things are not so simple.

(Aside and Overview)

The story for the following sections on proposed semantics for Prolog with `\+`.

1. We want to “characterize” the negation operator (`\+`) in Prolog. What does it precisely mean? Prolog goals have 3 outcomes: yes, no, and infinite loop. So our characterization of the meaning of goals will need to have 3 values: true, false and undefined (or undetermined, or unknown.) There are several generally accepted approaches to providing such a semantics.

2. The first is the completion semantics. Here a goal is true if it is logically implied by the ”completion” of the program; it is false if its negation is logically implied by the completion, and undefined otherwise. Result, a) doesn’t exactly characterize prolog with `\+`: consider goal p and rule p :- p,f. Prolog gives undefined (infinite loop); completion gives false. and b) completion (and Prolog) give undefined for p:-p, and thus doesn’t give transitive closure as meaning of the program for transitive closure. So this tells us that maybe we don’t want a semantics for `\+` in Prolog, but for `\+` in Tabled Prolog.

3. Define Perfect models, only for predicate stratified programs. Gives nice definition of `\+` in Tabled Prolog, but only for stratified programs. All goals are true or false (no undefined.) Correctly characterizes all terminating Tabled Prolog programs (when `\+` is implemented in the “obvious” way.)

4. Define Stable Model semantics for all (inc nonstratified) programs. Program has 0 or more stable models: goal is true under SM semantics if true in all SMs, false if false in all SMs, and undefined otherwise. Problems: exponentiality, nonrelevance. And for many interesting uses (e.g., combinatorial problems) SM characterizes a solution, e.g. a coloring of a graph. So interesting aspect of an SM is the truth of various goals in the same model (i.e., a coloring) rather than the truth of a goal in all models (e.g., node 1 must be green in all valid colorings.)

5. Define Well-Founded semantics. ...

6. Partial Stable Models, and connections between SM and WFS.

(END Aside and Overview)

(Take 2 Aside and Overview)

1. How to give a semantics for Normal Logic Programs: Prolog programs with unconstrained negation.

2. Define stratified programs, and give their semantics.

3. Define semantics of parameterized programs: i.e., Some predicates given by relations, and take least fixed point using those definitions for those predicates.

4. Define WFS as the limit of an iteration of parameterized programs: For every predicate symbol p, create a symbol p'. Replace in original program every p under a negation by p'. Then start by taking meaning of this modified program parameterized by p'= and p'=Universal. And alternating and iterating...

5. Then define FO(ID): a. A set of defined predicates, and a set of open predicates b. A set of rule with atoms of defined predicates in their heads (and open or defined, pos or neg, literals in their bodies) c. A set of first-order sentences (using defined and open predicates)

A model of an FO(ID) program is any first-order structure (for the defined and open predicates), such that the relations for the defined predicates are the (2-valued) well-founded model parameterized by the relations of the open predicates, and all the FO-sentences are true in the structure.

6. Then Stable Models can be defined in FO(ID).

1. Given a program (with negation). If we know *nothing* about the truth or falsity of the negative subgoals, what could we still infer about true and false propositions?

Goals are true if they can be proved with absolutely NO help from the negative literals. So delete all rules with negative literals and take least model, and all propositions true there must be true.

Goals are true if they can not be proved no matter what help they got from the negative literals. So delete all negative literals from the program rules, take least model, and propositions not in it are false.

2. So given a program with negations, we use the above idea to determine known true propositions and known false propositions, T1 and F1. Now given that we know the truth values of these subgoals, we reduce the program using them. I.e., Delete each rule with a negative literal whose proposition is in T1, and delete from each remaining rule each negative literal whose proposition is in F1. Now this program (along with T1 and F1) should give us the same meaning as the original program. Now repeat the process of step 1 using this program, to get more propositions known true and known false.

3. And continue until there is no change in the true and false sets. The resulting T and F sets determine the well-founded model, with propositions in neither T nor F being undefined.

(END COMMENT)

Chapter 15

Meta-Programming

15.1 Meta-Interpreters in XSB

Meta-interpreters. How one can write meta-interpreters and table them and get tabled evaluation. I.e. tabling “lifts” through meta-interpreters.

Could do XOLDTNF metainterpreter, exponential, but programmable. Probably want to use aggregation.

15.1.1 A Metainterpreter for Disjunctive Logic Programs

Do disjunctive LP metainterpreter

15.1.2 A Metainterpreter for Explicit Negation

Do explicit negation metainterpreter.

15.2 Abstract Interpretation

Abstract Interpretation examples. (partial evaluation, and assert.)

Show low overhead of tabling in meta-interpreter, due to how tables are implemented as tries.

15.2.1 AI of a Simple Nested Procedural Language

(see warren/xsb-tests/procabsint/*)

In this section we will see how to use XSB to construct a simple abstract interpreter for a procedural programming language. Such abstract interpreters can be used to do various kinds of data flow analyses. The abstract interpretator that we develop here is actually quite sophisticated; for example, it does interprocedural analysis. The interesting part here is how easy and straightforward it is to construct one with XSB, and therefore how we can be confident of its correctness.

The idea is to first construct a concrete interpreter for the object language. The Prolog programming language makes this particularly easy. We can run the concrete interpreter on various object programs and make sure that it is working reasonably well. After we have the concrete interpreter, we can easily change the operations to be abstract operations that operate over the abstract domain. Then we could execute programs over the abstract domain. This sounds easy (and it is in XSB) but a couple of issues arise. First, when computing over the abstract domain, the outcome of conditional tests cannot always be determined. This means that what was a deterministic concrete program becomes a nondeterministic abstract program. Since we can't know which branch a specialization of the abstract program might take, we have to try them all. Now in XSB, that is not a problem, since XSB supports nondeterminism. Second, since we can't determine the exact outcome of conditionals, we don't know when to exit from loops. So a simple execution of most any abstract program would loop infinitely. What is really wanted is a least fixed point which will give reachable abstract states, and tabling in XSB gives exactly that. So XSB is ideally suited for this kind of abstract interpretaton problem.

Consider how this approach works in a specific case. The XSB (actually Prolog) program shown below is an interpreter for a simple procedural language that supports nested procedures, static scoping, and call-by-value parameter passing. It is far from trivial and we will discuss how each component works.

First we describe how the execution environment is maintained. When a procedure is executing, it must have access to all variables that are visible to it. With each invocation of a procedure there is an activation record (AR) that stores its local variables. This is maintined in our interpreter as a simple list of (variable-name, variable-value) pairs. So when a procedure is executing, it will have access to its own AR that stores its local variables. But it must also have access to variables global to it, i.e., those in enclosing blocks. These are in ARs for the enclosing procedures. So the state for a procedure is kept as a list of ARs, the first being the procedure's own AR, the second being the AR of the immediately enclosing procedure, etc. We call such a list of AR's a Stack.

The following simple predicates get and set variable values in a Stack. They take a level number, indicating how global the variable is: 0 indicates local, 1 indicates in the immediately enclosing block, etc. They also take a Stack, and a variable name.

```
:- import append/3, memberchk/2 from basics.
```

```
% getVal(+N,+Stack,+Var,-Val)
getVal(0,[AR|_],Var,Val) :- memberchk((Var,Val),AR).
getVal(N,[_AR|Stack],Var,Val) :- N>0, N1 is N-1, getVal(N1,Stack,Var,Val).

% setVal(+N,+StackIn,+Var,+Val,-StackOut)
setVal(0,[AR|StackIn],Var,Val,[NAR|StackIn]) :-
repl_pair(AR,Var,Val,NAR).
setVal(N,[AR|StackIn],Var,Val,[AR|StackOut]) :-
N > 0, N1 is N-1,
setVal(N1,StackIn,Var,Val,StackOut).

repl_pair([(Var,_)|AR],Var,Val,[(Var,Val)|AR]) :- !.
repl_pair([P|AR],Var,Val,[P|NAR]) :- repl_pair(AR,Var,Val,NAR).
```

The interpreter takes as input the abstract syntax tree of an object program.

```
% evaluate a program

eval(module(_Name,Block)) :-
evalBlock(Block,[],0,[],_Stack).

evalBlock(block(Decls,Stmts),Pars,K,Stack0,Stack) :-
remFirst(K,Stack0,Stack1),
append(Pars,Decls,Locals),
evalStmts(Stmts,[Locals|Stack1],[_|Stack2]),
addFirst(K,Stack0,Stack2,Stack).

remFirst(0,L,L).
remFirst(N,[_|L0],L) :- N>0, N1 is N-1, remFirst(N1,L0,L).

addFirst(0,_,Stack2,Stack2).
addFirst(N,[AR|Stack0],Stack2,[AR|Stack]) :-
N>0, N1 is N-1,
addFirst(N1,Stack0,Stack2,Stack).

evalStmts([],Stack,Stack).
evalStmts([Stmt|Stmts],Stack0,Stack) :-
evalStmt(Stmt,Stack0,Stack1),
evalStmts(Stmts,Stack1,Stack).

evalStmt(assign(var(I,Name),Exp),Stack0,Stack) :-
evalExp(Exp,Stack0,Val),
setVal(I,Stack0,Name,Val,Stack).
evalStmt(while(Bool,Stmts),Stack0,Stack) :-
```

```
evalExp(Bool,Stack0,BVal),
(BVal =:= 0
 ->Stack = Stack0
 ; evalStmts(Stmts,Stack0,Stack1),
evalStmt(while(Bool,Stmts),Stack1,Stack)
).
evalStmt(if(Bool,Then,Else),Stack0,Stack) :-
evalExp(Bool,Stack0,BVal),
(BVal =\= 0
 ->evalStmts(Then,Stack0,Stack)
 ; evalStmts(Else,Stack0,Stack)
).
evalStmt(call(I,Name,ActPars),Stack0,Stack) :-
getVal(I,Stack0,Name,proc(Forms,Body)),
evalPars(ActPars,Forms,Stack0,ParLocals),
evalBlock(Body,ParLocals,I,Stack0,Stack).
evalStmt(print(Exps),Stack,Stack) :-
eval_print_exps(Exps,Stack).
evalStmt(dump,Stack,Stack) :-
writeln(Stack),nl.

evalPars([],[],_Stack,[]).
evalPars([Exp|Exps],[(Var,_)|Vars],Stack,[(Var,Val)|Decls]) :-
evalExp(Exp,Stack,Val),
evalPars(Exps,Vars,Stack,Decls).

eval_print_exps([],_).
eval_print_exps([Exp|Exps],Stack) :-
evalExp(Exp,Stack,Val),
writeln(Val),
eval_print_exps(Exps,Stack).

evalExp(int(V),_,V).
evalExp(var(I,Name),Stack,Val) :-
getVal(I,Stack,Name,Val).
evalExp(op(+,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
V is V1+V2.
evalExp(op(*,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
V is V1*V2.
evalExp(op(-,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
V is V1-V2.
evalExp(op(<,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
(V1<V2 -> V=1; V=0).
evalExp(op(>,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
(V1>V2 -> V=1; V=0).
evalExp(op(=,E1,E2),Stack,V) :-
evalExp(E1,Stack,V1),
evalExp(E2,Stack,V2),
(V1=:=V2 -> V=1; V=0).



% compose stmt operations
evalStmts([],Stack,Stack).
evalStmts([Stmt|Stmts],Stack0,Stack) :-
    evalStmt(Stmt,Stack0,Stack1),
    evalStmts(Stmts,Stack1,Stack).

% extract envs for called block and exec body in that context
evalBlock(block(Decls,Stmts),Pars,Level,Stack0,Stack) :-
    keepTail(Level,Stack0,Stack1),
    append(Pars,Decls,Locals),
    evalStmts(Stmts,[Locals|Stack1],[_|Stack2]),
    replTail(Level,Stack0,Stack2,Stack).

% compute value of an expression in a context
evalExp(Exp,Stack,Val) :- ......

% evaluate a statement, generating new Stack
evalStmt(assign(var(I,Name),Exp),Stack0,Stack) :-
    evalExp(Exp,Stack0,Val), setVal(I,Stack0,Name,Val,Stack).
evalStmt(while(Bool,Stmts),Stack0,Stack) :-
    evalExp(Bool,Stack0,BVal),
    (BVal =:= 0 -> Stack = Stack0
               ;  evalStmts(Stmts,Stack0,Stack1),
                  evalStmt(while(Bool,Stmts),Stack1,Stack)).
evalStmt(if(Bool,Then,Else),Stack0,Stack) :-
    evalExp(Bool,Stack0,BVal),
    (BVal =\= 0 -> evalStmts(Then,Stack0,Stack)
               ;  evalStmts(Else,Stack0,Stack)).
evalStmt(call(I,Name,ActPars),Stack0,Stack) :-
```

```
    getVal(I,Stack0,Name,proc(Forms,Body)),
    evalPars(ActPars,Forms,Stack0,ParLocals),
    evalBlock(Body,ParLocals,I,Stack0,Stack).
```

To obtain an abstract interpreter that does uninitialized variable analysis:

Underline evalExp call in assignment clause to point out that its definition is changed to implement abstract operations over the abstract domain of {uninitialized, hasValue}. Constant is mapped to hasValue. Binary ops return hasValue if both their operands are hasValue, otw uninitialized.

Add

```
:- table evalStmt/3.
```

at top.

Cross out the BVal =?= 0 conditions from the while and if-then-else clauses, to get regular disjunctions, instead of Prolog's if-then-else.

Chapter 16

XSB Modules

The XSB module system and how it works.

Does this deserve a chapter of its own, or should it be under System Facilities.

Chapter 17

Handling Large Fact Files

17.1 Indexing

When a call is made to a predicate that has multiple clauses in its definition, it is important for efficiency for the system to be able to quickly determine which defining clauses might match the given call. If it can quickly be determined that many clauses can never match the given clause, then much time can be saved that might otherwise be spent trying each clause in turn. This is known as the clause indexing problem in Prolog implementations.

As a simple example, consider a predicate, say employee/5, that is defined by a large number, say 10000, of ground facts, which might come from a table in a relational database. Then consider a goal:

```
| ?- employee(15663,LastName,FirstName,Dept,Salary).
```

which asks for the name, department, and salary of employee with ID 15663. A naive implementation of Prolog would look at all 10000 employee facts to determine which one (or ones!) have 15663 in its first field. It would do a unification for each of the 10000 clauses, almost all of which fail. To avoid doing all this work, all Prolog implementation support some kind of indexing. Traditionally Prolog implementations build an index on the first argument of every predicate. That means that they build some kind of structure, usually a hash table, to allow direct access to clauses given the value of their first field. Then when a call is made with a value (and not a variable) for the first field, the hash table is accessed to find what clauses might match, and then only for those clauses is full unification done. Of course, if a call is made that has a variable in the first argument, then the hash table (i.e., index) for the first argument could not be used and all clauses would have to be accessed. It is easy to see how this could save immense amounts of time in the evaluation of some queries. Proper indexing is critical for efficient XSB query evaluation; it is not at all unusual to be able to speed up a query by orders of magnitude by finding the best ways to index predicates defined by a large number of clauses.

XSB supports a variety of ways to index clauses. The ways supported differ for predicates whose clauses are compiled (i.e., are stored in a .xwam file and then loaded into memory) and predicates that are dynamic (i.e., their clauses are created by asserting them directly into memory during runtime.)

In XSB compiled predicates can be indexed in only relatively simple ways. One can request only one (or no) index be built; the default is on the first argument, but the programmer can choose any other argument if desired. For example, to specify that the compiled predicate employee/5 is to be indexed on its second argument, add the following directive to the source file containing the facts for employee/5:

```
:- index(employee/5,2).
```

The first argument to the index directive indicates the predicate to be indexed, and the second argument indicates the argument to use to build the index. (If the second argument is 0, then no index is built.) The index is built using the main function symbol of each occurrence of the chosen argument. For example, the default indexing would be very good for the employee/5 example and query above. The first argument of every employee/5 fact is a constant and the index would take advantage of the constant 15663 in the query. However, consider a different situation in which employeeDivID/5 facts had employee IDs of the form [DivNo,EmpNo], i.e., a list of two subfields indicating the division the employee was hired into and their ID number within that division. In this case all the employeeDivID/5 facts have the same main functor symbol in their first argument, the list functor, so an index would be completely useless in this situation, since every first argument main functor symbol would hash to the same bucket. So in this case it would be better to store the two subfields of the ID as separate top-level fields in an employee/6 predicate, and then index on the ID within the division. This would give pretty good indexing, but notice that, since we can only index on one argument, we can't take advantage of knowing both the ID and the division in a query; the system would have to look through all the clauses for employees in different divisions that had the same ID number.

Another limitation of indexing of compiled predicates arises because we can index on only one argument. Say we had employee/5 queries like:

```
| ?- employee(15663,LastName,FirstName,Dept,Salary).
and
| ?- employee(ID,'Warren',FirstName,Dept,Salary).
```

i.e, both queries for data given the employee's ID, and queries for data given the employee's last name. Since for compiled clauses we must choose only one argument for the index, we would have to decide which form of query is more important, choose that index, and suffer poor performance for queries of the other form.

The more flexible indexing that XSB supports for dynamic predicates, those whose clauses are asserted (or dynamically loaded) at runtime, provides solutions to these problems. For dynamic predicates, multiple hash-based indexes, joint indexes, and structure sensitive indexes are supported. Alternatively, the programmer can choose to index a predicate using a new indexing technique, called trie-indexing.

One can specify a joint index of up to three fields. For example, we can create a joint index for (dynamic predicate) employee/6, where the first two fields are Division and ID, respectively, with the following directive:

```
| ?- index(employee/6,1+2).
```

In this case an index is built using (the main functor symols of) both the first and second fields of employee/6 clauses. Then if a call to employee/6 provides values for both of those fields, then this index is used. Note, however, that if only one of the two fields is provided in the call, the index (being a hash-index) cannot be used; so such a call will have to look at *all* clauses defining employee/6.

Indexes for dynamic predicates are built at the time clauses are asserted into the predicate. I.e., the index directive just declares that future asserts into the indexed predicate will have the indicated index built. It does *not* re-index any clauses currently defining the predicate. This means that one (almost) always declares an index for a dynamic predicate when it has no defining clauses, before any asserts to that predicate have been done.

In addition to declaring joint indexes, one can declare multiple indexes for the same dynamic predicate. For example, we might want two indexes on employee/5, one on the first argument (the ID) *and* a joint index on the second and third arguments (the last and first names). This could be done with with directive:

```
| ?- index(employee/5,[1,2+3]).
```

The programmer provides the multiple index specifications in a list. So this directive says to build an index on (the main functor symbol of) the first argument, and to build a joint index on the second and third argument. Then if a call to employee/5 provides a value for the first argument, the first index will be used; if it provides values for both the second and third arguments, the second (joint) index will be used. (If it provides values for all three arguments, the first index will be used, because it was specified first in the list.)

For dynamic predicates the programmer can also request an argument to be indexed using “star indexing”, in which case the index will be a joint index on (up to) the first five symbols of the indicated argument. So, for example, to request a star index the employeeDivID/5 on its first argument (which, recall, is a two-element list):

```
| ?- index(employeeDivID/5,*(1)).
```

Now a query such as:

```
| ?- employeeDivID([3,1433],Last,First,Dept,Sal).
```

would use the star index on the first argument, and use both components of the first argument. Star indexing can be freely combined the the default main-functor symbol indexing. Note that a query such as:

```
| ?- employeeDivID([3,X],Last,First,Dept,Sal).
```

cannot take advantage of the star index at all, since it doesn't have values to find a hash value for the joint index.

The programmer can request that a dynamic predicate be trie indexed by:

```
| ?- index(employeeDivID/5,trie).
```

Trie indexing cannot be combined iwth any other indexing and it provides a slightly different semantics for clauses defining the predicate. First, only fact-defined predicates can be declared as trie-indexed; no rules are allowed as they are for the other indexing forms.) Second, duplicate facts are not supported; i.e., asserting a duplicate clause into a trie-indexed predicate is a no-op. Third, the order of facts is *not* preserved; i.e., a fact may be added

17.2 Compiling Fact Files

(Revise in light of previous (added) section)

Certain applications of XSB require the use of large predicates defined exclusively by ground facts. These can be thought of as "database" relations. Predicates defined by a few hundreds of facts can simply be compiled and used like all other predicates. XSB, by default, indexes all compiled predicates on the first argument, using the main functor symbol. This means that a call to a predicate which is bound on the first argument will quickly select only those facts that match on that first argument. This entirely avoids looking at any clause that doesn't match. This can have a large effect on execution times. For example, assume that p(X,Y) is a predicate defined by facts and true of all pairs ¡X,Y¿ such that $1 <= X <= 20, 1 <= Y <= 20$. Assume it is compiled (using defaults). Then the goal:

```
| ?- p(1,X),p(X,Y).
```

will make 20 indexed lookups (for the second call to p/2). The goal

```
| ?- p(1,X),p(Y,X).
```

will, for each of the 20 values for X, backtrack through all 400 tuples to find the 20 that match. This is because p/2 by default is indexed on the first argument, and not the second. The first query is, in this case, about 5 times faster than the second, and this performance difference is entirely due to indexing.

XSB allows the user to declare that the index is to be constructed for some argument position other than the first. One can add to the program file an index declaration. For example:

```
:- index p/2-2.

p(1,1).
p(1,2).
p(1,3).
p(1,4).
...
```

When this file is compiled, the first line declares that the p/2 predicate should be compiled with its index on the second argument. Compiled data can be indexed on only one argument (unless a more sophisticated indexing strategy is chosen.)

17.3 Dynamically Loaded Fact Files

The above strategy of compiling fact-defined predicates works fine for relations that aren't too large. For predicates defined by thousands of facts, compilation becomes cumbersome (or impossible). Such predicates should be dynamically loaded. This means that the facts defining them are read from a file and asserted into XSB's program space. There are two advantages to dynamically loading a predicate: 1) handling of much larger files, and 2) more flexible indexing. Assume that the file qdata.P contains 10,000 facts defining a predicate q(X,Y), true for $1 <= X <= 100, 1 <= Y <= 100$. It could be loaded with the following command:

```
| ?- load_dyn(qdata).
```

XSB adds the ".P" suffix, and reads the file in, asserting all clauses found there. Asserted clauses are by default indexed on the first argument (just as compiled files are.)

Asserted clauses have more powerful indexing capabilities than do compiled clauses. One can ask for them to be indexed on any argument, just as compiled clauses. For dynamic clauses, one uses the executable predicate *index*/3. The first argument is the predicate to index; the second is the field argument on which to index, and the third is the size of hash table to use. For example,

```
| ?- index(q/2,2,10001).
```

```
yes
| ?- load_dyn(qdata).
[./qdata.P dynamically loaded, cpu time used: 22.869 seconds]

yes
| ?-
```

The index command set it so that the predicate $q/2$ would be indexed on the second argument, and would use a hash table of size 10,001. It's generally a good idea to use a hash table size that is an odd number that is near the expected size of the relation. Then the next command, the $load_dyn$, loads in the data file of 10,000 facts, and indexes them on the second argument.

It is also possible to put the *index* command in the file itself, so that it will be used when the file is dynamically loaded. For example, in this case the file woulstart with:

```
:- index(q/2,2,10001).

q(1,1).
q(1,2).
q(1,3).
...
```

Unlike compiled cclauses, asserted clauses can be indexed on more than one argument. To index on the second argument if it is bound on call, or on the first argument if the second is not bound and the first is, one can use the index command:

```
:- index(q/2,[2,1],10001).
```

This declares that two indexes should be build on $q/2$, and index on the second argument and an index on the first argument. If the first index listed cannot be used (since that argument in a call is not bound), then the next index will be used. Any (reasonable) number of indexes may be specified. (It should be noted that currently an idex takes 16 bytes per clause.)

Managing large extensional relations load_dyn, load_dync, cvt_canonical. Database interface, heterogeneous databases (defining views to merge DB's)

17.4 Indexing Static Program Clauses

For static (or compiled) user predicates, the compiler accepts a directive that performs a variant of $unificationfactoring$ [1].

....

Bibliographic Notes

The idea of using program transformations as a general method to index program clauses was presented in a rough form by [3] [1] extented these ideas to factor unifications ...

Chapter 18

Programming with Sets

18.1 Introduction

Prolog is fundamentally based on sets of tuples of terms, i.e., relations. But sometimes it is useful to manipulate explicitly sets of terms, as values of Prolog variables. For small sets, one often uses sorted lists (without duplicates) as a canonical representation, but since this representation has linear lookup and linear update, it is unsuited for handling large sets. XSB supports a powerful abstract data type for manipulating large sets of ground terms.

The set representation has essentially log time lookup and update and is canonical, and by use of the XSB facility of "interned terms", can efficiently be combined with tabling. It turns out to be surprisingly powerful.

18.2 Interned (or HashCons-ed) Terms

Describe interning, and predicate `intern_termhash/2`.

18.3 The prolog_db Data Structure

Describe this library. Use much from the documentation of lib/prolog_db.tex in the XSB system distribution.

18.4 Conclusion (or Summary?)

if any...

Chapter 19

Managing Large, Updatable State

19.1 Introduction

Prolog (and XSB) supports the dynamic updating of its global database, i.e., its set of clauses that define the predicates that it uses. For a predicate to be updatable, it must be declared to be *dynamic*. Changes to the global clase set (a.k.a. database) are persistent across backtracking. Any Prolog program that uses dynamic predicates is inherently nondeclarative, and thus to be avoided when possible. The question is when is it reasonably possible. This chapter describes how we can often use the `prolog_db` set data structure, described in the previous chapter, along with methodologies for using it to avoid the need for assert and retract, the Prolog builtins to change dynamic predicates in the global clause set.

19.2 The prolog_db Data Structure

Describe this library. Use much from the documentation of lib/prolog_db.tex in the XSB system distribution.

19.3 Examples

Various examples...

19.4 Conclusion (or Summary?)

if any...

Chapter 20

XSB System Facilities

compiler options

Foreign code interface

Calling XSB from C

Bibliography

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