mP4C – Complex Rate/Ratio Questions

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Complex Rate/Ratio Questions

We've already looked at some basic concepts with rates, ratios and proportions. In this lesson we'll investigate some of the more difficult problems that the ACT offers on this topic. The first problem type deals with distances, speeds, and times.

The Distance Formula:

$\text{distance} = \text{rate} \times \text{time}$

This formula can be rearranged to be an equation for time or an equation for rate:

$\text{time} = \frac{\text{distance}}{\text{rate}}$ $\qquad$ $\text{rate} = \frac{\text{distance}}{\text{time}}$

Sample Problem: Rates

Sara drives the 650 miles from Washington, D.C. to Atlanta in 10 hours. If she increased her average speed by 10 miles per hour, how much time could she save on the trip?

(A) 45 minutes
(B) 1 hour
(C) 1 hour, 20 minutes
(D) 1 hour, 40 minutes
(E) 2 hours

To solve this problem, we first need to find out how fast Sara drove on her way from Washington to Atlanta. Then we add 10 mph to that speed, and calculate the time it would take her at that rate. Finally, we'll find the difference between the old time and the new time.

To find her first speed, we use the *rate* formula above.

$\text{rate} = \frac{\text{distance}}{\text{time}} = \frac{650 \text{ miles}}{10 \text{ hours}} = 65 \text{ mph}$

According to the problem, Sara is considering raising her speed by 10 mph to 75 mph. How long will it take her to drive 650 miles at his speed? We can use the *time* formula to find out:

$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{650 \text{ miles}}{75 \text{ mph}} = 8\frac{2}{3} \text{ hours}$

To find the answer to the question, we subtract the two times:

$10 \text{ hours} - 8\frac{2}{3} \text{ hour} = 1\frac{1}{3} \text{ hours} = 1 \text{ hour}, 20 \text{ min.}$

The answer is (C).

Sample Problem: Rates

The Voyager spacecraft sends a signal (which travels at the speed of light, 3×10^5 km/sec) from Jupiter back to Earth at a distance of 7.8×10^8 km. The signal is received on the ground by engineers at the Johnson Space Center in Houston, who generate new instructions for the spacecraft (a process that takes 10 minutes), and send a signal back. How many seconds pass from the time that Voyager sends its signal until it receives its new instructions?

(A) 1300 seconds
(B) 2600 seconds
(C) 3200 seconds
(D) 5200 seconds
(E) 5800 seconds

This problem has three parts: the time it takes the signal to return to Earth, the processing time in Houston, and the time it takes for the signal to return to Voyager. For the travel time from Jupiter to Earth, we can use the distance formula:

$\text{time} = \frac{\text{distance}}{\text{rate}} = \frac{7.8 \times 10^8 \text{km}}{3 \times 10^5 \text{km/s}} = 2.6 \times 10^3 \text{s}$

That's 2600 seconds (or 43 minutes, 20 seconds). This will be the transit time for the signal both directions. We need to make sure to add in the processing time – 10 minutes. That's the same as 600 seconds, so we can find the total time by adding the three times together:

$2600 + 600 + 2600 = 5800$ seconds. The answer is (E).

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Notice that the answer choices contain choices that reflect all the possible mistakes we could make in this question. Choice (B) is the time it takes for the signal to go from Earth to Jupiter. (C) represents the time to make a one-way trip and the processing time. And (D) is the round-trip travel time without taking into account the processing time. This tells us that we need to make sure we're answering the question exactly as asked!

Remember that when a problem statement gives you information about rates, distances, and times, it's very likely that you'll end up using the distance formula (in one of its forms) at some point to solve it.

Questions about ratios give many students fits on the ACT. There are a few techniques that make these questions much easier. We'll look at one of these in the next example:

SAMPLE PROBLEM: RATIOS

A quadrilateral has angles in the ratio of $2:5:6:7$. What is the measure of the smallest angle?

(A) 18º
(B) 36º
(C) 48º
(D) 54º
(E) 90º

This problem lends itself perfectly to a strategy that works well for ratio problems where you know the sum of the quantities in the ratio. In this question, we know that the sum of the angles in a quadrilateral is 360º.

The given ratio tells us that the measures of the angles in the quadrilateral could be 2°, 5°, 6°, and 7°. But we know that that's wrong, because $2+5+6+7=20$, and we need the numbers to add up to 360.

Alternatively, the angles could measure $20°$, $50°$, $60°$, and $70°$, because it's clear that those angles are in the same $2:5:6:7$ ratio (we just multiplied each number by 10). But those numbers add up to 200, not 360. Or they could be 10, 25, 30, and 35 (multiplying them all by 5), but that doesn't work either.

The conclusion we reach is that there's some number we multiply 2, 5, 6, and 7 by so that when we add them up, we'll get 360. We could keep guessing, but there's a more systematic way to do it. Let's call the angles in the quadrilateral $2x$, $5x$, $6x$, and $7x$. Then we add them up and get 360º, and we can solve for x!

$$2x+5x+6x+7x=360$$

$$20x=360$$

$$x=18$$

Now we need to make sure we plug the value of x back into the ratio to find the measures of the angles themselves:

$$(2\times 18):(5\times 18):(6\times 18):(7\times 18)=$$

$$36:90:108:126$$

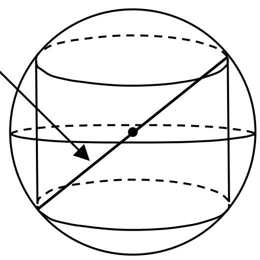
You can check and see that these four angles add up to 360. The smallest angle is 36, so the answer is (B).

Notice that one of the answer choices in the last question was 18º, which was the value of x, but NOT the measure of any of the angles. A mistake some students make is to stop once they find x. Remember that this is just a "multiplier," which helps us find the actual numbers from the ratio.

Sample Problem: Ratios

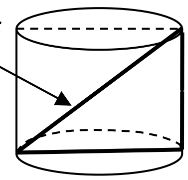
A cylinder is inscribed in a sphere of radius x, as indicated in the figure below. If the height and diameter of the cylinder are equal, what is the ratio of the volume of the sphere to the volume of the cylinder? (Note: the volume of a cylinder is $\pi r^2 h$, and the volume of a sphere is $\frac{4}{3}\pi r^3$.)

(A) 1
(B) $\sqrt{2}$
(C) $\frac{4}{3}$
(D) $\frac{4\sqrt{2}}{3}$
(E) 2



To solve this problem, we'll need a way to write the volumes of each shape so that the variables will all cancel out – notice that none of the answer choices have any variables in them.

If the radius of the sphere is x, then the volume of the sphere is $\frac{4}{3}\pi x^3$. For the cylinder, we'll need to write the radius and height in terms of x. Remember that it's inscribed in the sphere – the two shapes share the same center. That means the diameter of the sphere is the same length as the diagonal distance across the cylinder – that length is $2x$. If we redraw the cylinder with that diagonal we can see there's a triangle in there that will help us write its volume:



This is an isosceles right triangle, which means it's a 45-45-90, and we knew the ratios of the sides of a triangle like that. If the hypotenuse is $2x$, then we can find each leg by dividing by $\sqrt{2}$:

$$\text{Leg} = \frac{2x}{\sqrt{2}} = x\sqrt{2}$$

The height of the triangle is the height of the cylinder, and the base of the triangle is the diameter of the cylinder, or twice the radius of the cylinder. Therefore the volume of the cylinder is:

$$V = \pi r^2 h = \pi \left(\frac{x\sqrt{2}}{2}\right)^2 \cdot x\sqrt{2} = \frac{\pi x^3\sqrt{2}}{2}$$

The ratio we're looking for is the volume of the sphere divided by the volume of the cylinder, which simplifies as follows:

$$\frac{\frac{4}{3}\pi x^3}{\frac{\pi x^3\sqrt{2}}{2}} = \frac{\frac{4}{3}}{\frac{\sqrt{2}}{2}} = \frac{4}{3}\cdot\frac{2}{\sqrt{2}} = \frac{8}{3\sqrt{2}} = \frac{4\sqrt{2}}{3}$$

The answer is (D).

PRACTICE EXERCISES

1. On the Stephen Douglas High School Debate Team, the ratio of boys to girls is $5:6$. What fraction of the debaters are boys?

2. A coffee shop sells only coffee and tea. It sold x cups of coffee and y cups of tea on Friday. What fraction of the drinks sold were cups of tea, in terms of x and y?

3. P people were asked to contribute D dollars each for the cost of pizza at an office party. However, the price of the pizza turned out to be \$1 more per person than had been anticipated. Furthermore, x people canceled after the pizza had already been ordered, forcing the attendees to cover those x people's share of the cost. Write a formula for the amount of *additional* money that had to be collected from each person who attended the party, in terms of P, D, and/or x.

4. ΔABC and ΔDEF are similar. What is the ratio of the area of ΔABC to that of ΔDEF if the ratio of corresponding sides $AB:DE =$

   (A) $1:2$
   (B) $4:9$
   (C) $2:\sqrt{2}$
   (D) $x:y$

5. What is the ratio of the volumes of two cubes if their side lengths are in the ratio:

   (A) $1:2$
   (B) $27:3$
   (C) $3:\sqrt{3}$
   (D) $x:y$

6. Tim runs two laps around a one mile track, the first at a rate of 10 mph, the second at a rate of 8 mph. He infers that his average rate for the whole run was 9 mph. Is Tim correct? Show work to support your answer.

7. Tim runs two more laps, the first in 9 minutes, the second in 7 minutes. He concludes that he ran at an average rate of 8 minutes per mile. Is Tim correct? Show work to support your answer.

8. A cell phone company charges \$$1.50$ for the first ten minutes of a call, and \$$0.08$ per minute after that. Write an equation for the cost C of the call in terms of its length in minutes, m, assuming $m \geq 10$.

9. Kevin takes 4 minutes to fold a paper crane. Sharon can do it in 3 minutes, and Cindy takes only 2 minutes and 24 seconds. How many minutes will it take the three working together to fold 1000 paper cranes?

10. Pablo is a poet. He wrote at a rate of 5 poems per hour until his hand cramped, slowing him down to 4 poems per hour. If Pablo finished 55 poems in 12 hours, how many hours after he started did his hand cramp?

TEST EXERCISES (TIME: 10 MINUTES)

11. The cost C of a taxi ride is $\$4.00$ for the first half-mile, plus $\$0.30$ for each $\frac{1}{8}$ mile after that. Which of the following is an equation for C in terms of the number of miles m that a passenger travels?

(A) $C = 4 + 0.3(8m - 0.5)$
(B) $C = 4 + 0.3(8m - 4)$
(C) $C = 4 + 0.3(0.125m - 0.5)$
(D) $C = 4 + 0.0375m$
(E) $C = 4 + 2.4m$

12. A shoe store accidentally ordered 12 extra pairs of men's shoes and 12 too few pairs of women's shoes, resulting in a ratio of men's to women's shoes of $7:8$. If the ratio was meant to be $5:6$, how many pairs of women's shoes was the store supposed to order?

(A) 90
(B) 96
(C) 120
(D) 180
(E) 540

13. A car traveling 30 miles per hour will go how many feet in 3 seconds? (note: 1 mile $= 5280$ feet)

(A) 44
(B) 132
(C) 250
(D) 880
(E) 7920

14. Irene is walking to the park at $4:30$ at a rate of 3 miles per hour. Her sister Iris is walking home from the park at $5:00$ at a rate of 2.5 miles per hour, along the same route. If the two girls meet each other at $5:48$, how many miles is it from the girls' house to the park?

(A) 1.5
(B) 4.4
(C) 5.9
(D) 7.1
(E) 7.4

15. If an equilateral triangle is inscribed in a circle of radius r, what is the ratio of the perimeter of the triangle to the circumference of the circle?

(A) $\frac{3}{2\pi}$
(B) $\frac{3}{\pi}$
(C) $\frac{3\sqrt{3}}{2\pi}$
(D) $\frac{3\sqrt{3}}{4\pi}$
(E) $\frac{3\sqrt{3}}{8\pi}$

16. If, for positive real numbers a and b, the ratio of $135a$ to $14b$ is twice the ratio of $7b$ to $375a$, then what is the ratio of $16a$ to b ?

(A) $16:14$
(B) $32:15$
(C) $25:72$
(D) $72:25$
(E) $224:225$

17. A 100 gallon tub has a faucet and a drain. If the faucet is run with the drain open for 6 minutes and then the drain is closed, it will take 10 more minutes to fill the tub. If the faucet is run with the drain open for 13 minutes before the drain is closed, it will take 5 more minutes to fill the tub. The drain empties the tub at a rate of how many gallons per minute?

(A) 2
(B) 5
(C) 7
(D) 10
(E) 13

18. Consider two triangles, ΔABC and ΔDEF for which the following ratios hold true:

$m\angle A : m\angle D = 1:2$
$m\angle B : m\angle E = 2:1$
$m\angle C : m\angle F = 1:1$
$m\angle A : m\angle F = 1:3$

The ratio of segment lengths $AB:AC$ is:

(A) $2:1$
(B) $1:2$
(C) $2:\sqrt{3}$
(D) $2:3$
(E) Not able to be determined from the given information.

19. A volunteer event is selling commemorative t-shirts. Two-thirds of the shirts are large while one-third are small. Four-fifths of the small shirts are black; the other one-fifth are white. Overall, the ratio of white to black shirts is $1:3$. What fraction of the white shirts are large?

(A) $\frac{1}{4}$
(B) $\frac{3}{40}$
(C) $\frac{9}{10}$
(D) $\frac{4}{5}$
(E) $\frac{11}{15}$

20. Two runners are racing around a circular track. One completes a lap in 84 seconds; the other in 90 seconds. If they start at the same place and time, how many seconds will it take for the faster runner to pass the slower one, having gotten a full lap ahead?

(A) 90
(B) 504
(C) 540
(D) 1260
(E) 1350