## mP8A – Complex Numbers

**Directions:** Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

### Topic Overview: Complex Numbers

When we first learn about square roots, we are told that the number inside the radical must always be greater than or equal to zero. The answer to the question "What is the square root of $-1$?" is answered with *imaginary numbers*. The number $i^2$ is defined to be equal to $-1$:

$$\boldsymbol{i^2 = -1} \qquad \rightarrow \qquad \boldsymbol{i = \sqrt{-1}}$$

$i$ is often referred to as the *imaginary number*, and it allows us to take all kinds of new square root that were off-limits before:

$\sqrt{-16} = \sqrt{16 \times -1} = \sqrt{16} \times \sqrt{-1} = 4i$

$\sqrt{-98} = \sqrt{49 \times -1 \times 2} = \sqrt{49} \times \sqrt{-1} \times \sqrt{2} = 7i\sqrt{2}$

The ACT will test certain basic concepts regarding complex numbers, all of which we will review in this lesson. Don't worry about the practical applications of this idea – they do exist, but the ACT won't expect you to know anything about them.

One idea we do want to be aware of relates to the powers of $i$. Look at the following series of expressions and notice the pattern that emerges:

$i = \sqrt{-1}$ $\qquad$ $i^2 = i \cdot i = -1$

$i^3 = i^2 \cdot i = -i$ $\qquad$ $i^4 = i^3 \cdot i = 1$

$i^5 = \sqrt{-1}$ $\qquad$ $i^6 = -1$ $\qquad$ $i^7 = -i$ $\qquad$ $i^8 = 1$

$i^9 = \sqrt{-1}$ $\qquad$ $i^{10} = -1$ $\qquad$ $i^{11} = -i$ $\qquad$ $i^{12} = 1$

The pattern becomes clear after a couple of repetitions. Every fourth power of $i$ is equal to 1, and then the pattern repeats. That means we can answer questions about large powers of $i$ without having to go through all the smaller powers:

### Sample Problem: Complex numbers

Which of the following is equal to $i^{119}$?

(A) $-2$
(B) $-1$
(C) $1$
(D) $i$
(E) $-i$

From the pattern described above, we know that every fourth power of $i$ is equal to $1$ – that is, each power of $i$ that's divisible by 4 is equal to 1. The power we're given, 119, is not divisible by 4, so we'll find the largest multiple of 4 smaller than $119$ and work our way up. That power of $i$ is 116:

$i^{116} = 1$ $\qquad$ $i^{117} = i$ $\qquad$ $i^{118} = -1$ $\qquad$ $i^{119} = -i$

The answer is (E).

A *complex* number is a number that has an imaginary part and a real part. We can write a complex number as $a + bi$, where $a$ is the real part, and the $bi$ is the imaginary part. In reality, every number is a complex number, and the numbers that we call real numbers have $b = 0$. Here are some examples of complex numbers:

$1 + i \quad (a = 1, b = 1)$
$6\sqrt{3} - 14i \quad (a = 6\sqrt{3}, b = -14)$
$117 \quad (a = 117, b = 0)$
$-12 - 4i \quad (a = -12, b = -4)$

To multiply complex numbers, we use the same rules as we always do – just remember that when we square $i$ it's equal to $-1$.

### Sample Problem: Complex numbers

Which of the following is equal to $(4 - 5i)(5 + 2i)$?

(A) $5$
(B) $5i$
(C) $22 + 33i$
(D) $22 - 17i$
(E) $30 - 17i$

This is a problem we can do using the FOIL method we learned for polynomials – remember to combine like terms at the end:

$(4 - 5i)(5 + 2i) = 20 + 8i - 25i - 10i^2 =$

$20 - 17i - 10(-1) = 30 - 17i$. The answer is (E).

Remember that when we learned about polynomial factoring we learned the factoring pattern for the difference of two squares:

$$a^2 - b^2 = (a + b)(a - b)$$

Now that we know how to use complex numbers, we can write the *sum* of two squares:

$$a^2 + b^2 = (a + bi)(a - bi)$$

Just for fun, let's check this result:

$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 =$

$a^2 - (b^2)(-1) = a^2 + b^2$ ✓

Knowing this pattern allows us to eliminate imaginary numbers that appear in the denominator of fractions – remember that an imaginary number is really a radical, and mathematical convention says that we shouldn't leave radicals in the denominators of fractions.

**SAMPLE PROBLEM: COMPLEX NUMBERS**

Which of the following is equal to $\frac{i}{4+i} \cdot \frac{2i}{4-i}$?

(A) $-\frac{2}{15}$
(B) $-\frac{2}{17}$
(C) $\frac{2}{17}$
(D) $-\frac{2i}{16+8i}$
(E) $\frac{2i}{17-8i}$

As we always do with fractions, we'll multiply the numerators of the fractions to find the numerator of the answer, and multiply the denominators of the fractions to find the denominator of the answer:

$\frac{i}{4+1} \cdot \frac{2i}{4-1} = \frac{i \cdot 2i}{(4+i)(4-i)} = \frac{2i^2}{16-4i+4i-i^2} =$

$\frac{-2}{16-(-1)} = -\frac{2}{17}$

The answer is (B).

**SAMPLE PROBLEM: COMPLEX NUMBERS**

What are the zeros of the equation $y = x^2 + 2x + 10$?

(A) $x = 10,\ x = 1$
(B) $x = 5,\ x = -2$
(C) $x = 5i,\ x = -2i$
(D) $x = 1 - 3i,\ x = 1 + 3i$
(E) $x = 1 - i\sqrt{11},\ x = 1 + i\sqrt{11}$

To find the zeros of a quadratic equation, we set the equation equal to zero and solve for $x$. We have a few methods for this: factoring, completing the square, and the quadratic formula. It's always best to try factoring first, because it is the fastest. In this case, though, we can't factor – we can see that there are no factors of 10 whose sum or difference is equal to 2. Let's try the quadratic formula:

$$y = x^2 + 2x + 10 \quad \rightarrow \quad a = 1, b = 2, c = 10$$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4(1)(10)}}{2(1)} =$

$\frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

The answer is (D).

The last example shows us that we can now find the solutions to any quadratic equation, even if the roots aren't real numbers.

**PRACTICE EXERCISES**

1. Veronica was asked to express $\sqrt{-12}$ as a complex number and she wrote $12i$ . In doing this, she committed a common error. What was her error, and what is the correct answer?

For Problems 2 through 4, simplify as much as possible. Make sure your answer is in rationalized form (no complex numbers in the denominator)!

2. $\frac{i}{1+i}$

3. $(2-i)^2$

4. $\frac{4i^4-2i^3}{2i^6+7i^7}$

5. If the roots of $ax^2+bx+c=0$ are not real numbers, what must be true of the quantity $b^2-4ac$ ? Why? (Think of the Quadratic Formula.)

For Problems 6 and 7, solve the given quadratic equation for *x*.

6. $9x^2+4=0$

7. $5x^2+6x+5=0$

8. Evaluate $(-1)^{-\frac{3}{2}}$ as a real or complex number in rationalized form.

9. For all integers *k*, what is the value of $i^k+i^{k+1}+i^{k+2}+i^{k+3}$?

10. If $z=\frac{(-1+\sqrt{3}i)}{2}$, where $i^2=-1$, then $z^3=$? (Hint: Use FOIL and simplify multiple times.)

**TEST EXERCISES (TIME: 10 MINUTES)**

11. For $i^2 = -1$, $i^{33} - i^{35} + i^{37} =$ ?

(A) $-3$
(B) $i^{35}$
(C) $i^{105}$
(D) $3i^{35}$
(E) $3i^{105}$

12. For $i^2 = -1$, what is the value of $i + 2i^2 + 3i^3 + 4i^4 =$ ?

(A) $-6 + 4i$
(B) $-6 - 2i$
(C) $2 - 2i$
(D) $2 + 4i$
(E) $6 + 4i$

13. All of the following numbers (where $i^2 = -1$) are equal EXCEPT:

(A) $i^{71}$
(B) $-i^{23}$
(C) $i^{103}$
(D) $-i^{57}$
(E) $i^{11}$

14. The roots of the equation $x^2 + 2x + 2 = 0$ are:

(A) $-1$ and $1$
(B) $-i$ and $i$
(C) $1 - i$ and $-1 - i$
(D) $-1 - i$ and $-1 + i$
(E) $1 - i$ and $1 + i$

15. For $i^2 = -1$, $\frac{5+i}{-5+i} =$ ?

(A) $\frac{-24-10i}{26}$
(B) $\frac{24+10i}{26}$
(C) $1$
(D) $26$
(E) $-1 + i$

16. For the complex numbers $z_1 = a + bi$ and $z_2 = a - bi$, where $a$ and $b$ are nonzero real numbers and $i^2 = -1$, all of the following must be real numbers EXCEPT:

(A) $z_1 + z_2$
(B) $z_1 z_2$
(C) $\frac{z_1}{z_2}$
(D) $z_1^2 + z_2^2$
(E) $(z_1 - z_2)^2$

17. If $(a + bi)(1 + 2i) = 4 + 13i$, where $a$ and $b$ are real numbers and $i^2 = -1$, then $a + b =$ ?

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

18. The product of two complex numbers, $(a + bi)(c + di)$, is:

(A) always real.
(B) never real.
(C) real if $\frac{c}{a} = \frac{d}{b}$.
(D) real if $\frac{c}{a} = -\frac{d}{b}$.
(E) complex if $bd > ac$.

19. Where $i^2 = -1$, for which of the following values of $k$ does $i^k = i^{k^2} = i^{k^3} \ldots$ ?

(A) 2 only
(B) 4 only
(C) 5 only
(D) 2 and 4
(E) 4 and 5

20. The complex number $a + bi$ (where $i^2 = 1$) is added to the square of itself, and the result is a real number. If $b \neq 0$, then $a =$ ?

(A) $-1$
(B) $-0.5$
(C) $0$
(D) $0.5$
(E) $1$