mP5C - MATRICES

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

TOPIC OVERVIEW: MATRICES

A matrix is a way to organize or list data in rows and columns. Each piece of data in a matrix is called an *entry*, and we name entries by stating the row and column that we find it in. Below is an example of a matrix, and a list of some of the entries in it:

$$\begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 9 \\ 7 & 5 & 8 \end{bmatrix}$$

$$a_{1,1,} = 1 \quad a_{2,1} = 6 \quad a_{3,2} = 5 \quad a_{3,3} = 8$$

This matrix is called a 3×3 matrix, which means that it has 3 rows and 3 columns. This is called the *dimension* of the matrix. If a matrix has the same number of columns and row, it is called a *square* matrix.

The ACT only tests a few operations that can be done with matrices: addition and subtraction, multiplication by a constant, and multiplication of matrices.

To add or subtract matrices, they must have the same dimension. For example, we can subtract a 3×3 from a 3×3 or add a 4×6 to a 4×6, but we can't add a 2×2 and a 2×3. To add (or subtract) matrices, just add (or subtract) the items in the corresponding locations.

SAMPLE PROBLEM: MATRIX ADDITION/SUBTRACTION

$$\begin{bmatrix} 3 & 7 & 1 \\ 3 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 3 \\ 4 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 3+2 & 7+8 & 1+3 \\ 3+4 & 6+9 & 4+1 \end{bmatrix} =$$

$$\begin{bmatrix} 5 & 15 & 4 \\ 7 & 15 & 5 \end{bmatrix}$$

SAMPLE PROBLEM: MATRIX ADDITION/SUBTRACTION

$$\begin{bmatrix} 4x & 17 & y \\ z^2 & y & 4x \end{bmatrix} - \begin{bmatrix} 2x & 8 & y \\ 5 & 2y & -5 \end{bmatrix} =$$

$$\begin{bmatrix} 4x-2x & 17-8 & y-y \\ z^2-5 & y-2y & 4x-(-5) \end{bmatrix} =$$

$$\begin{bmatrix} 2x & 9 & 0 \\ z^2-5 & -y & 4x+5 \end{bmatrix}$$

We can also multiply a matrix by a number or expression. To perform this operation, multiply each entry in the matrix by the expression.

SAMPLE PROBLEM: MATRIX MULTIPLICATION (CONSTANT)

$$3x\begin{bmatrix} 4 & 5x & 4y \\ 3x & 7z & 5 \\ 4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3x \times 4 & 3x \times 5x & 3x \times 4y \\ 3x \times 3x & 3x \times 7z & 3x \times 5 \\ 3x \times 4 & 3x \times 6 & 3x \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 12x & 15x^2 & 12xy \\ 9x^2 & 21xz & 15x \\ 12x & 18x & 15x \end{bmatrix}$$

The last operation, matrix multiplication, is more complicated than the other three. Like with addition and subtraction, there are only certain situations in which we can perform this operation. In order to multiply two matrices, the number of columns of the first matrix must be equal to the number of rows of the second matrix. Let's say we have a 3×3 and a 3×2:

$$[3 \quad 3] \times [3 \quad 2]$$

Because the middle two numbers match, the multiplication is possible.

The resulting matrix will have the dimensions of the outer numbers:

$$[3 \quad 3] \times [3 \quad 2]$$

The resulting matrix will have dimension $[3 \times 2]$

An interesting fact is that, unlike all multiplication that we've encountered so far, matrix multiplication is NOT commutative. Recall that if an operation is commutative, the order in which we apply the operation doesn't matter. For example, integer multiplication is commutative, so $4 \times 2 = 2 \times 4 = 8$. With numbers, addition and multiplication are commutative, while subtraction and division are not. With matrices, addition is commutative, but multiplication and subtraction are not. If A and B are two matrices, AB may or may not be equal to BA.

SAMPLE PROBLEM: MATRIX MULTIPLICATION

$\begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 1 \\ 7 & 3 & 9 \end{bmatrix} = ?$

Before we start, we need to make sure the multiplication is possible. Look at the dimensions of the matrices:

$$[2 \times 2] \ \times \ [2 \times 3]$$

Because the inner numbers match (they're both 2), the multiplication is possible. The resulting matrix will have dimension 2×3 (the outer numbers). To perform the multiplication, first draw an empty answer matrix with the correct dimension:

$$\begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 1 \\ 7 & 3 & 9 \end{bmatrix} = \begin{bmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}$$

To find the entry in the first row and first column of the product, multiply the entries in the first row of the first matrix by the entries in the first column of the second matrix:

$\begin{bmatrix} \mathbf{4} & \mathbf{1} \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} \mathbf{5} & 2 & 1 \\ \mathbf{7} & 3 & 9 \end{bmatrix} =$

$\begin{bmatrix} 4 \times 5 + 1 \times 7 & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix} = \begin{bmatrix} 27 & \_ & \_ \\ \_ & \_ & \_ \end{bmatrix}$

Notice that we multiplied the numbers in order, and added the products together. This is why the numbers need to match – so we have the same number of entries to multiply. Let's do the entry in the first row and the second column:

$\begin{bmatrix} \mathbf{4} & \mathbf{1} \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & \mathbf{2} & 1 \\ 7 & \mathbf{3} & 9 \end{bmatrix} =$

$\begin{bmatrix} 27 & 4 \times 2 + 1 \times 3 & \_ \\ \_ & \_ & \_ \end{bmatrix} = \begin{bmatrix} 27 & 11 & \_ \\ \_ & \_ & \_ \end{bmatrix}$

Continuing in the same manner, we can fill in the answer matrix:

$\begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 1 \\ 7 & 3 & 9 \end{bmatrix} =$

$\begin{bmatrix} 27 & 11 & 4 \times 1 + 1 \times 9 \\ 2 \times 5 + 6 \times 7 & 2 \times 2 + 6 \times 3 & 2 \times 1 + 6 \times 9 \end{bmatrix} =$

$\begin{bmatrix} 27 & 11 & 13 \\ 52 & 22 & 56 \end{bmatrix}$

Use the exercises in this section to practice this technique. It's a little confusing at first, but with a little experience it can be done quickly.

One last concept that is mentioned occasionally on the ACT is the *determinant* of a matrix. The determinant is a number that describes certain characteristics of a matrix, including the existence of an inverse. On the ACTs you will only be asked to find the determinant of 2×2 matrices, and you will be given this formula if asked to do it:

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
the determinant of A is equal to $ad - bc$

To find the determinant, simply plug the numbers from the matrix into the equation and solve. The determinant of a matrix A is written as $det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (note that the brackets become just straight lines in the notation for the determinant).

SAMPLE PROBLEM

Find the determinant of $A = \begin{bmatrix} 3 & 2 \\ 5 & 9 \end{bmatrix}$

Using the formula from above,
$det(A) = 3 \times 9 - 2 \times 5 = 17.$

Practice Exercises

For Questions 1 through 7, perform the indicated operation on the matrices, or write "Not Valid" if the operation cannot be performed:

1. $\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix} = ?$

2. $\begin{bmatrix} 0 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix} = ?$

3. $\begin{bmatrix} x & y & 1 \\ y & 1 & z \\ 1 & z & x \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = ?$

4. $2x\begin{bmatrix} 3x & 5 & y \\ y^2 & -2 & 0 \\ 0.5 & \frac{1}{x} & x^2 \end{bmatrix} = ?$

5. $\begin{bmatrix} -4 & 3 \\ 0 & 1 \\ 2 & -6 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 3 & -2 \\ 4 & 0 \end{bmatrix} = ?$

6. $\begin{bmatrix} 0 & 1 & -3 \end{bmatrix} \times \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix} = ?$

7. $\begin{bmatrix} x & y \\ 2x & 1 \\ -2 & -y \end{bmatrix} + \begin{bmatrix} 1 & 4x & -y \\ y & 0 & 0 \end{bmatrix} = ?$

8. Find the determinant of the matrix $\begin{bmatrix} 14 & -15 \\ 15 & 16 \end{bmatrix}$.

9. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the 3×3 *identity matrix*, and is labeled I_3 for shorthand. Find the product $I_3 \times I_3$.

10. Find the determinant of the matrix

$$\begin{bmatrix} x^2 - 1 & x^3 \\ x & x^2 + 1 \end{bmatrix}$$

TEST EXERCISES (TIME: 10 MINUTES)

11. Which of the following operations will yield a 3×3 matrix as the result?

(A) $\begin{bmatrix} 2 \\ 12 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 9 \end{bmatrix}$

(B) $\begin{bmatrix} n \\ n^2 \\ n^3 \end{bmatrix} - \begin{bmatrix} m^3 \\ m^2 \\ m \end{bmatrix}$

(C) $\begin{bmatrix} 4 & 7 \\ 13 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 20 \\ -2 & 0 & -9 \end{bmatrix}$

(D) $7\begin{bmatrix} x & 4 & 2x \\ -3x & -x & 91 \\ 0 & 0 & -23 \end{bmatrix}$

(E) $\begin{bmatrix} a & 2a & 3 \\ -a & -5 & a^2 \end{bmatrix} \times \begin{bmatrix} -1 & -7 \\ a^3 & 4a \\ a+1 & 2 \end{bmatrix}$

12. If $x\begin{bmatrix} 3 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -17 \\ 2 \end{bmatrix}$, then (x, y) = ?

(A) $(-3, 0)$
(B) $(-3, 1)$
(C) $(-5, -2)$
(D) $(-5, 2)$
(E) $(-3.4, 0.4)$

13. Evaluate $\begin{bmatrix} 2x & 3y \\ 3x & 6y \end{bmatrix} \times \begin{bmatrix} -x & -2y \\ -3y & 4x \end{bmatrix}$ when $x = 1$ and $y = -2$.

(A) $\begin{bmatrix} -38 & -16 \\ -75 & -36 \end{bmatrix}$

(B) $\begin{bmatrix} -10 & -42 \\ 23 & -72 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -2 \\ 9 & -8 \end{bmatrix}$

(D) $\begin{bmatrix} -2 & -24 \\ 18 & -48 \end{bmatrix}$

(E) $\begin{bmatrix} -26 & -51 \\ 16 & -72 \end{bmatrix}$

14. If $x + y = 0$, then $\begin{bmatrix} -6 & 3 & 3y \\ 4 & 12 & -2y \end{bmatrix} \times \begin{bmatrix} -x \\ 1 \\ 2 \end{bmatrix}$ = ?

(A) $\begin{bmatrix} 9 \\ 4 \end{bmatrix}$

(B) $\begin{bmatrix} 3 \\ 12 \end{bmatrix}$

(C) $\begin{bmatrix} -2x \\ 15 \\ 2y \end{bmatrix}$

(D) $\begin{bmatrix} 12x + 3 \\ 12 - 8x \end{bmatrix}$

(E) $\begin{bmatrix} 9y - 3 \\ 48 - 10x \end{bmatrix}$

15. Kenneth buys 10 pounds of Italian sausage for \$43.90. Some of it is sweet sausage, which costs \$4.99 per pound, while some of it is hot sausage, which is on sale for \$3.99 per pound. Which of the following matrix equations is equivalent to a system of linear equations that can be solved to determine how many pounds of each type of sausage Kenneth bought?

(A) $\begin{bmatrix} 1 & 1 \\ 4.99 & 3.99 \end{bmatrix} \times \begin{bmatrix} s \\ h \end{bmatrix} = \begin{bmatrix} 10 \\ 43.90 \end{bmatrix}$

(B) $\begin{bmatrix} s & h \\ 3.99 & 4.99 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 43.90 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 \\ 3.99 & 4.99 \end{bmatrix} \times \begin{bmatrix} 10 \\ 43.90 \end{bmatrix} = \begin{bmatrix} s \\ h \end{bmatrix}$

(D) $\begin{bmatrix} 3.99 & 1 \\ 4.99 & 1 \end{bmatrix} \times \begin{bmatrix} s \\ h \end{bmatrix} = \begin{bmatrix} 10 \\ 43.90 \end{bmatrix}$

(E) $\begin{bmatrix} s & 10-s \\ 3.99 & 4.99 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 43.90 \end{bmatrix}$

16. By definition, the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ equals $ad - bc$. Find the value of $\begin{vmatrix} 2m & -\frac{2}{n} \\ n & \frac{1}{m} \end{vmatrix}$.

(A) $2m + 2n$
(B) -4
(C) 0
(D) 4
(E) $4mn$

17. By definition, the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ equals $ad - bc$. Find the value of $\begin{vmatrix} x-3 & y-7 \\ y & x-2 \end{vmatrix}$ when $x + y = 6$.

(A) $-6x - 7y$
(B) $-6 - 12y$
(C) $4x - 10$
(D) 12
(E) 14

18. If the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & -a \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $abcd = ?$

(A) -2
(B) -1
(C) 0
(D) 1
(E) 2

19. If the product $\begin{bmatrix} 4x & y & -1 \end{bmatrix} \times \begin{bmatrix} 0.25y \\ y^2 \\ xy \end{bmatrix} = -\frac{1}{8}$, then $y = ?$

(A) $-\frac{1}{8}$
(B) $-\frac{1}{2}$
(C) 2
(D) -2
(E) The value of y cannot be determined from the given information.

20. By definition, the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ equals $ad - bc$. If $\begin{vmatrix} 3x & -2y \\ 2x & -y \end{vmatrix} = 4$, then the value of $x + \frac{1}{y} = ?$

(A) 4
(B) $\frac{1}{4}$
(C) $\frac{5}{4}x$
(D) $5x$
(E) $\frac{5}{4}y$