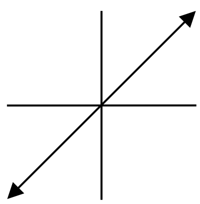
mP6C - Transformations

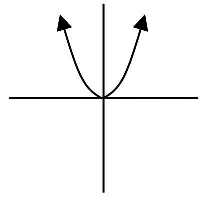
Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Transformations

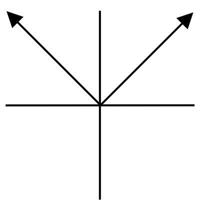
We've seen many different types of functions – linear, quadratic, absolute value, and cubic functions are just a few examples. Each of these functions can be represented graphically, and each has its own *parent function*, which is the simplest form of the graph:



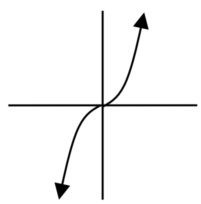
linear: $y = x$



quadratic: $y = x^2$



absolute value: $y = |x|$



cubic: $y = x^3$

We can *transform* these graphs by adding, subtracting, or multiplying numbers to the equations of the functions. There are four transformations that the ACT will expect us to be able to do: translation, rotation, reflection, and shrinks/stretches.

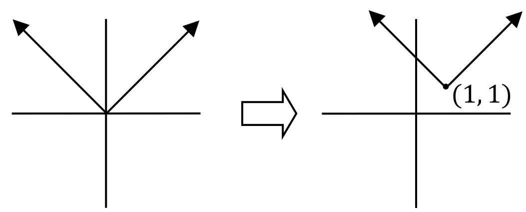
Translations are slides – moving the graph up, down, left, or right. To accomplish this, we just add or subtract a number to the equation of the graph. For a vertical translation, the number is added to the *entire function*; for a horizontal translation, we add or subtract the number to the *x*-value only. Here are some examples of graphs shifted vertically and horizontally by 2 units:

Vertical shift of 2 units up (Adding 2 to the y)	Horizontal shift of 2 units left (Adding 2 to the x)
$y = x^2 \rightarrow y = x^2 + \mathbf{2}$	$y = x^2 \rightarrow y = (x + \mathbf{2})^2$
$y = x^3 \rightarrow y = x^3 + \mathbf{2}$	$y = x^3 \rightarrow y = (x + \mathbf{2})^3$
$y = \lvert x\rvert \rightarrow y = \lvert x\rvert + \mathbf{2}$	$y = \lvert x\rvert \rightarrow y = \lvert x + \mathbf{2}\rvert$

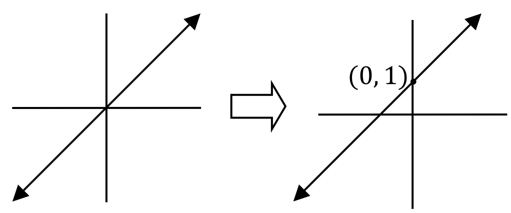
We can see that for vertical shifts the number is *outside* the function, but for horizontal shifts it's *inside* the function. Also pay attention to the directions of the shifts: Adding to the *y* shifts the graph *up,* adding to the x shifts it *left*. The opposites of these statements are also true: subtracting from the y shifts *down*, subtracting from the x shifts *right*.

Below are some graphs of translations (with their parent graphs):

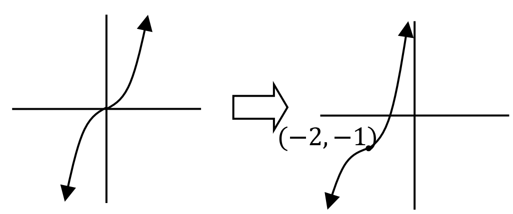
$y = |x - 1| + 1$ (one unit up and one unit right):



$y = x + 1$ (one unit up):



$y = (x + 2)^3 - 1$ (one unit down and two units left):



Sample Problem: Translation

Which of the following functions represents the translation of the graph of $y = x^3$ two units down and one unit to the left?

(A) $y = (x + 2)^3 - 1$
(B) $y = (x + 1)^3 - 2$
(C) $y = (x - 1)^3 - 2$
(D) $y = (x - 2)^3 + 1$
(E) $y = (x - 2)^3 - 1$

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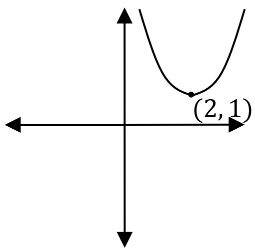
To shift a graph two units down, we need to *subtract* 2 from the entire function. This means we can eliminate choices A, D, and E right away. To shift a graph one unit to the left, we need to add 1 to the x-value. That means that (B) is the correct answer.

Rotations are “spins”. The ACT will only test rotations in a more general sense – the test won’t ask us to find equations for graphs once they’re rotated. However, we may be asked to identify graphs once they have been rotated, or identify the type of rotation that has occurred. Let’s look at an example to make this clearer.

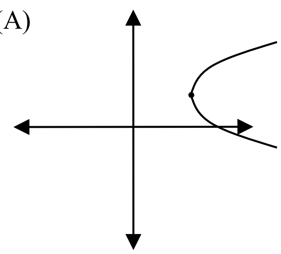
SAMPLE PROBLEM: ROTATION

A function $f(x)$ is shown below:

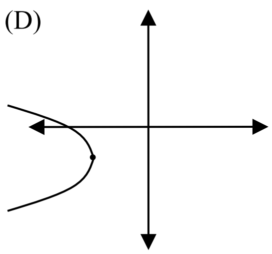


Which of the graphs represents the rotation of this graph $270°$ clockwise about the point $(2, 1)$?

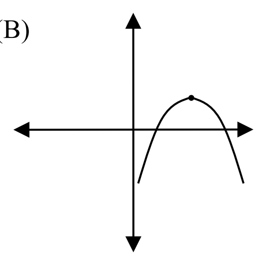
(A)



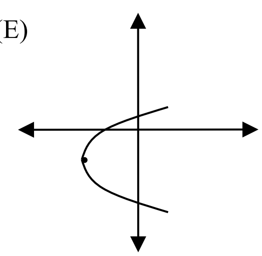
(D)



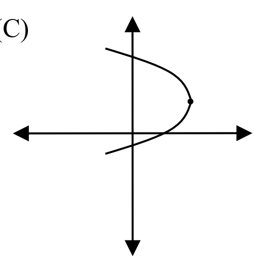
(B)



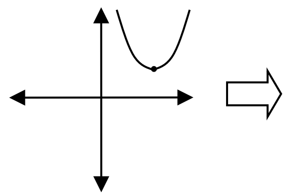
(E)



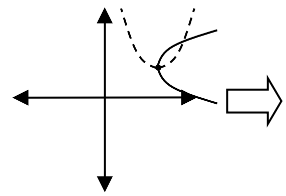
(C)



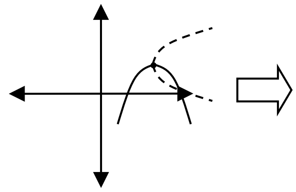
A rotation here is just a spin – the words “about the point $(2, 1)$” tell us where the center of the rotation should be. Remember that $90°$ is a quarter-turn. Because $270°$ is three times $90°$, we need to turn our graph a quarter turn three times, making sure that the point $(2, 1)$ stays in the same spot each time:



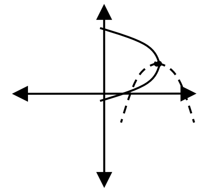
original graph



¼ turn clockwise



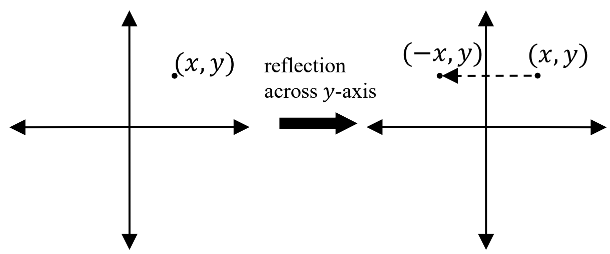
¼ turn clockwise

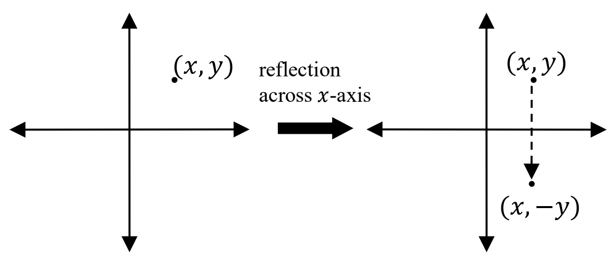


¼ turn clockwise

The final graph is the one we want – it matches choice (C).

Reflections are just what they sound like – if an object is in front of a mirror, then its image in the mirror is the reflection. To find a reflection, we’ll imagine that there are mirrors lying on certain lines and find the reflection (the image) of a point or a graph in the “mirror”. The pictures below show the reflections of points across the coordinate axes:





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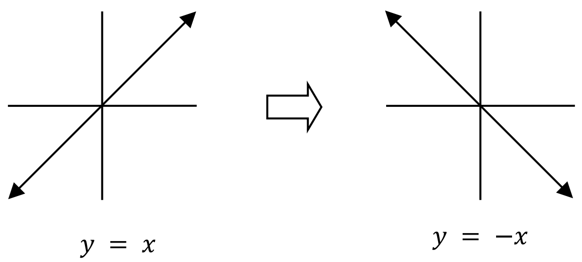
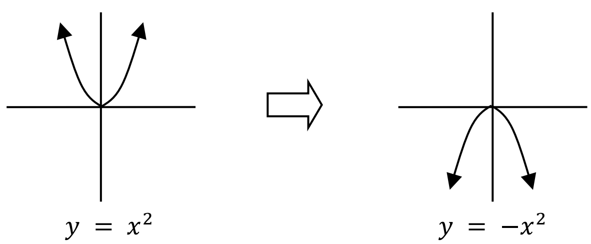
These are the most common types of reflections that will appear on the ACT. The coordinates listed on the graphs give us a kind of rule we can use for these transformations:

Reflection across the *y*-axis: $(x, y) \rightarrow (-x, y)$

Reflection across the *x*-axis: $(x, y) \rightarrow (x, -y)$

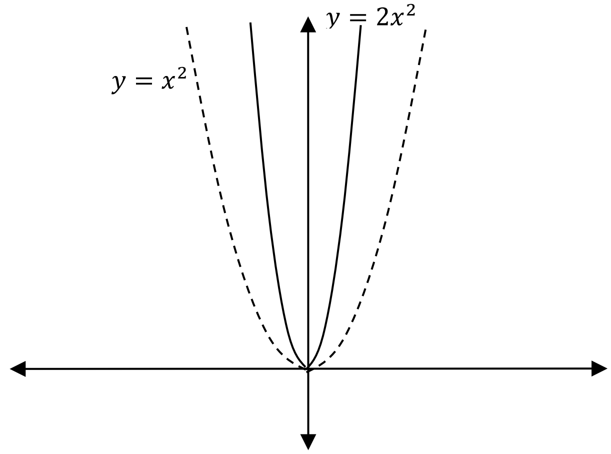
There is another way to write a reflection in the *x*-axis – by making the entire function negative. This is in some ways equivalent to the reflection we saw above, which makes the *y*-coordinate negative, but it's often easier just to remember that if there is a negative sign in front of the leading term, we'll flip the graph over the *x*-axis:



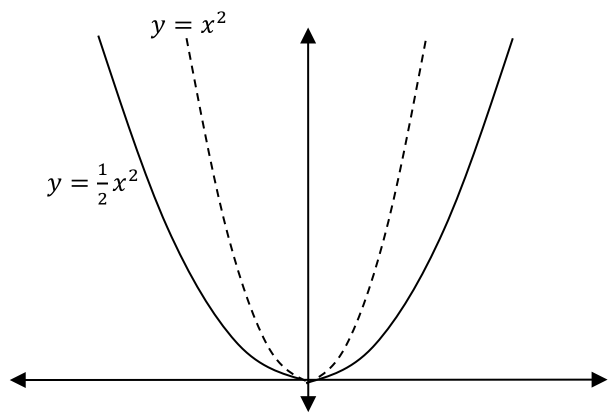


The last transformations we'll need to know to ace this topic on the ACT are called **stretches and shrinks**. These occur when there is a number (besides 1 or -1) multiplied by the leading term of the function (remember that the leading term is the term with the largest power of x in it). If the number is greater than one, we call that a *vertical stretch* (or a *horizontal shrink*). If the number is less than one, the transformation is a *vertical shrink* (or a *horizontal stretch*). The pictures to the right show these, with the parent graph drawn with a dotted line.

Vertical Stretch (Horizontal Shrink):



Vertical Shrink (Horizontal Stretch):



It doesn't matter whether we think of these as vertical or horizontal transformations, but it's easier to pick one direction and remember it that way. For example, we can stick with the vertical direction and remember that a number greater than one means stretch vertically and a number between zero and one means shrink vertically.

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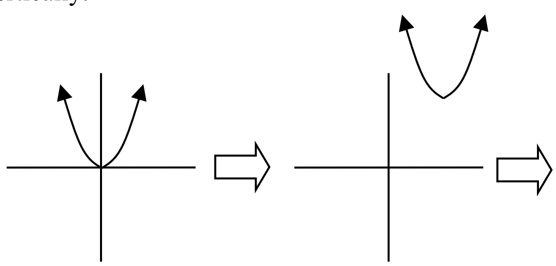
SAMPLE PROBLEM: TRANSFORMATIONS

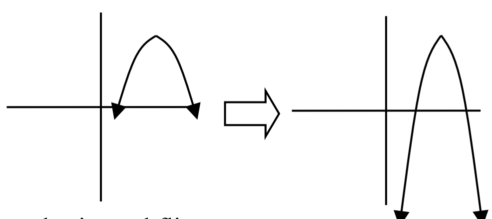
Describe the transformations of the graph of $y = -2(x-2)^2 + 3$ from its parent graph.

This function has many different transformations attached to it. First, let's notice that the parent graph is a quadratic graph. Then look at the translations: The -2 on the inside of the parentheses tells us to move the graph 2 units right, and the $+3$ on the outside tells us to move it 3 units up. The -2 multiplied by the $(x-2)^2$ term tells us two things: we'll need to flip the graph over the x–axis and we'll need to stretch it vertically:



parent graph: $y = x^2$

shift 2 units right and 3 units up



horizontal flip

vertical stretch

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PRACTICE EXERCISES

1. List the four types of transformations that are covered in ACT problems.

For Questions 2 through 5, graph the indicated function and its parent function on the same set of axes. (Examples of parent functions: $y = x$,$y = x^2$, $y = |x|$, etc.) Also list the transformations that must be applied to the parent function to produce the graph of the indicated function.

2. $y = 3x - 4$

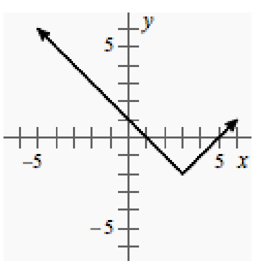
3. $y = |x + 1| + 2$

4. $y = \frac{1}{4}(x + 1)^2$

5. $y = -\sqrt{x + 2} - 2$

For Questions 6 through 9, write the equation of the function in each graph.

6.



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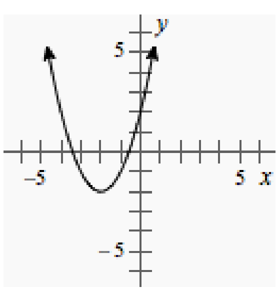

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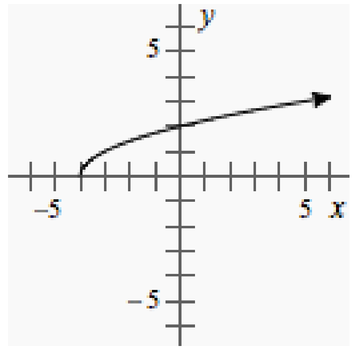
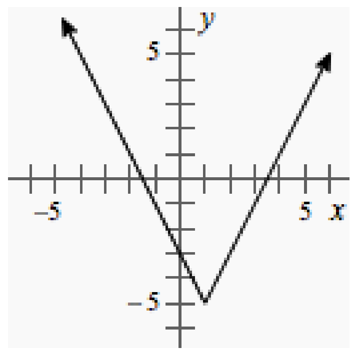


9.



10. Sketch the graph of the function $f(x) = 2x - 3$ rotated 90° clockwise about the point $(2, 1)$.

11. Sketch the graph of the function $f(x) = x^2$ rotated 90° counterclockwise about the point $(-1, 1)$.

12. The graph of the function $f(x) = (x - 3)^2$ is reflected across the y-axis, then across the x-axis. What is an equation of the resulting function?

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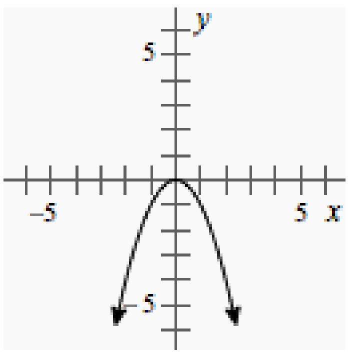
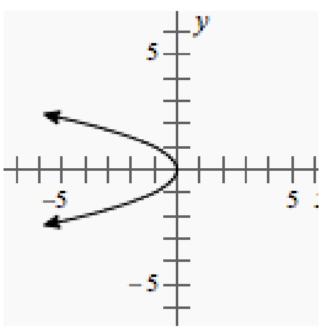
TEST EXERCISES (TIME: 10 MINUTES)

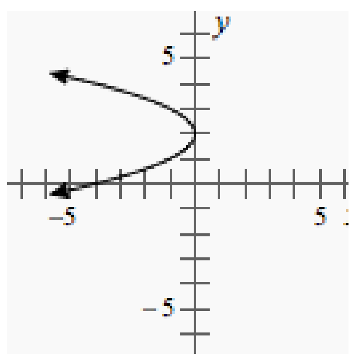
13. If the graph of $y = 2x - 5$ is reflected across the x-axis, the result is the graph of:

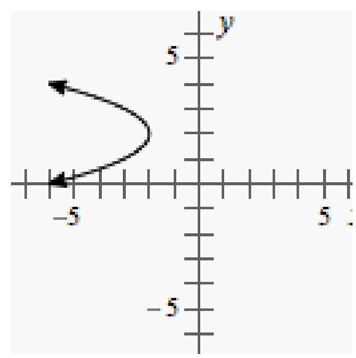
    (A) $y = 2x + 5$
    (B) $y = -2x + 5$
    (C) $y = -2x - 5$
    (D) $y = 0.5x + 5$
    (E) $|y| = 2x - 5$

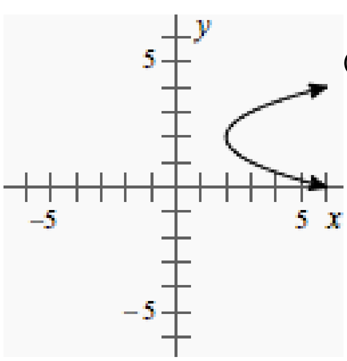
14. If the graph of $y = -x^2$ (depicted below) is rotated 90° clockwise about the point $(0, 2)$, the result is which of the following graphs?



(A)


(B)


(C)


(D)


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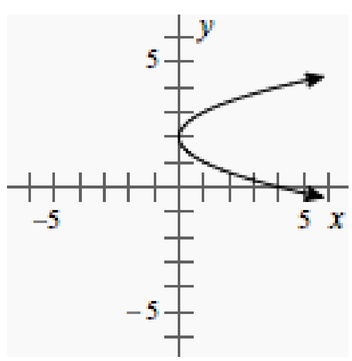

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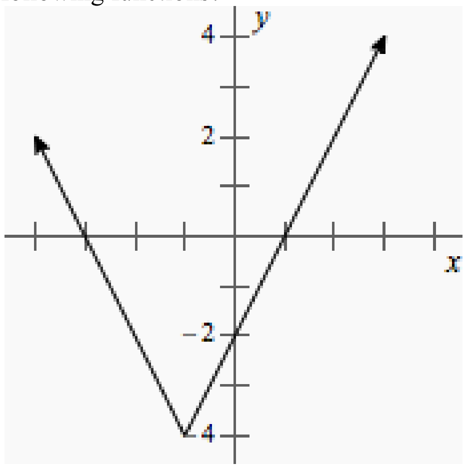
15. The graph below is the graph of which of the following functions?



(A) $f(x) = 2|x| - 3$
(B) $f(x) = 2|x - 1| - 4$
(C) $f(x) = 2|x - 5|$
(D) $f(x) = 2|x - 4| + 1$
(E) $f(x) = 2|x + 1| - 4$

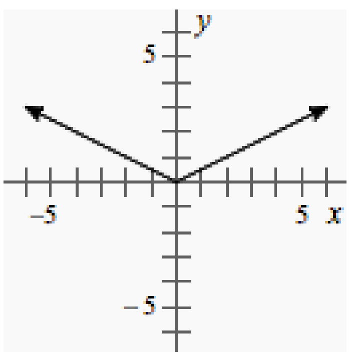
16. If $a \neq 0$ and the graph of $y = ax^2$ contains the point $(4, 5)$, then the graph of $y = a(x - 3)^2$ MUST contain the point:

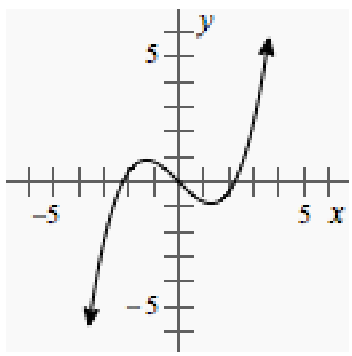
(A) $(4, 2)$
(B) $(4, 8)$
(C) $(1, 5)$
(D) $(7, 5)$
(E) $(7, 2)$

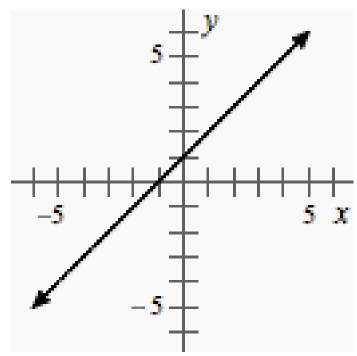
17. To obtain the graph of $y = x^2 - 2x + 1$, the graph of $y = x^2$ must be shifted:

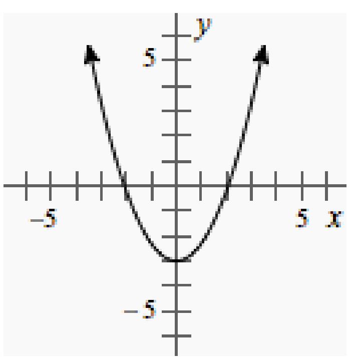
(A) one unit to the right.
(B) one unit to the left.
(C) one unit up.
(D) one unit to the left and one unit up.
(E) two units to the right and one unit up.

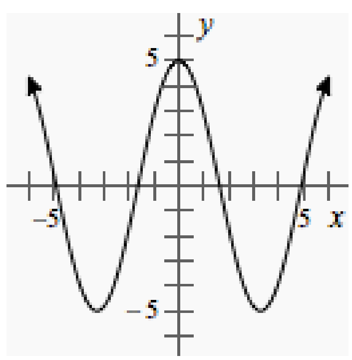
18. A function f is an odd function if and only if $f(x) = -f(-x)$ for every value of x in the domain of f. The functions below are graphed in the standard xy-coordinate plane. Which graph shows an odd function?

(A)


(B)


(C)


(D)


(E)


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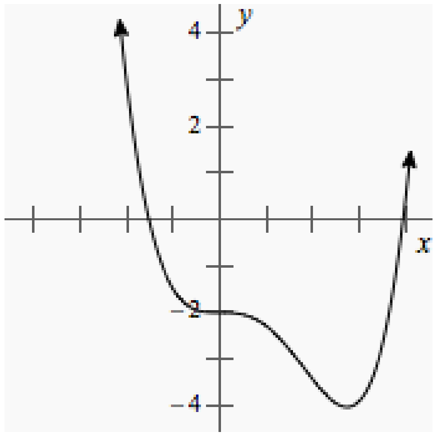
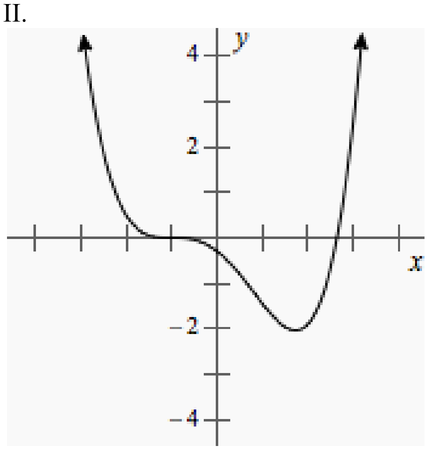
19. Consider the two graphs below:

I:



II.



If graph I is the graph of $y = f(x)$, then graph II could be the graph of:

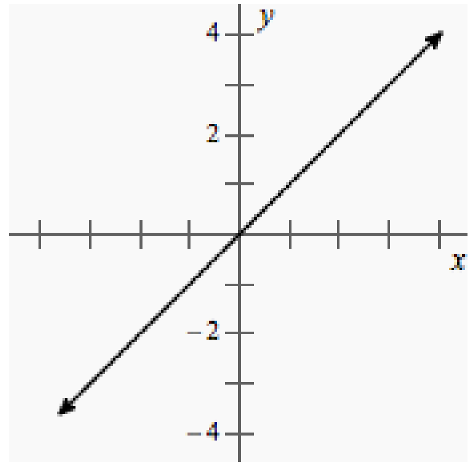
(A) $y = f(x + 1) + 2$
(B) $y = f(x - 1) + 2$
(C) $y = f(x + 2) + 1$
(D) $y = f(x - 2) + 1$
(E) $y = f(x - 2) - 2$

20. If the graph of $y = |x - 3|$ is reflected about the line $x = 1$, the result is the graph of:

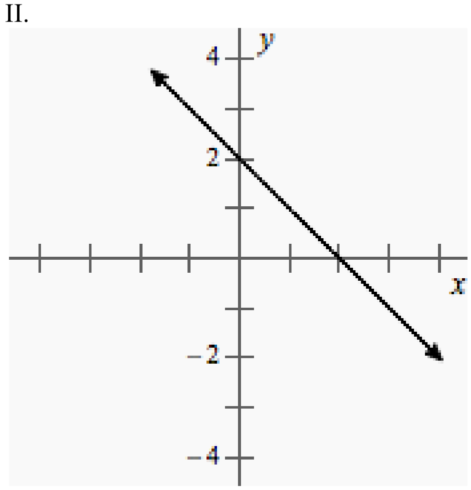
(A) $y = |x - 3|$
(B) $y = |x - 2|$
(C) $y = |x - 1|$
(D) $y = |x + 1|$
(E) $y = |x + 3|$

21. Consider the following two graphs:

I:



II.



Which of the following transformations, if applied to the function in graph I, would produce the function in graph II?

(A) Reflection across the line $y = 1$
(B) $270°$ clockwise rotation about the point $(0, 2)$
(C) $90°$ clockwise rotation about the point $(1, 1)$
(D) Either A or B
(E) Either A or C

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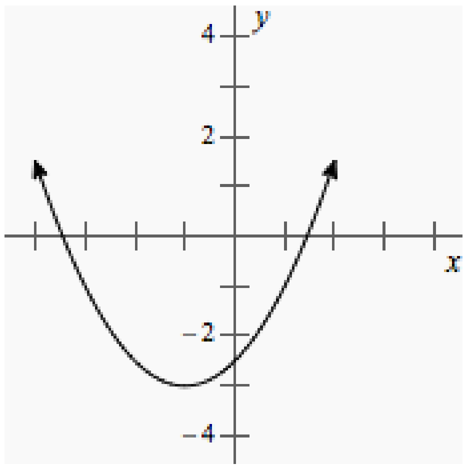
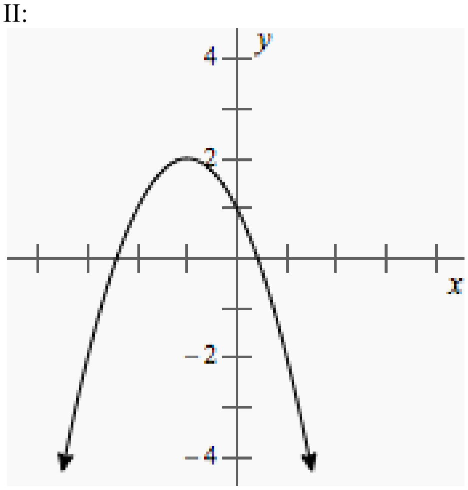
22. Consider the two graphs below:

I:



II:



If graph I is the graph of $y = f(x) - 3$, then graph II could be the graph of:

(A) $y = -2f(x+2)$
(B) $y = 2 - f(x)$
(C) $y = 2 - 2f(x)$
(D) $y = 5 - f(x)$
(E) $y = 5 - f(x+2)$

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