mP6A – Piecewise and Composite Functions

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Piecewise Functions

Most of the functions we've seen up to this point are defined by a single rule, like these:

$y = x^2 - 2x + 4 \quad y = \sqrt{x-9} \quad y = \frac{3x}{x^2+5}$

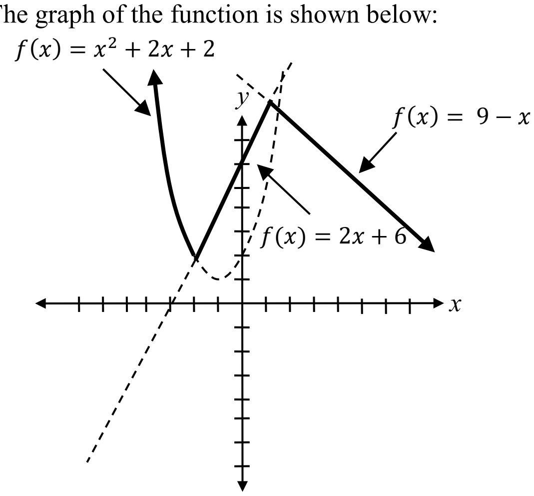
If we're given an x, to find y, we just plug in the x-value and evaluate the function. There is another class of functions called *piecewise* functions that are defined by multiple rules. To evaluate these functions, you need to pay close attention to both the function definition and the x-value you're given. Let's look at an example:

Sample Problem: Piecewise Functions

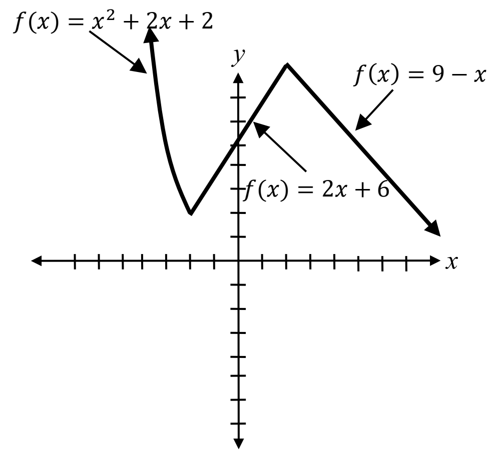
Given the function defined below, evaluate at $x = -5$, -2, and 3:

$$f(x) = \begin{cases} x^2 + 2x + 2 & x < -2 \\ 2x + 6 & -2 \le x \le 1 \\ 9 - x & x > 1 \end{cases}$$

The graph of the function is shown below:



The entirety of each of the three parts of the piecewise function is graphed here – the dotted segments are the parts we don't need, and the solid segments are the parts that are defined for this function:



Piecewise functions are defined with multiple equations, and with limits that tell us when to use each equation. For example, the first row of this function tells us that $f(x) = x^2 + 2x + 2$ when $x < -2$. The next row tells us the equation to use when x is between -2 and 1, and the last row tells us the equation to use when x is greater than 1. Now we can answer the question:

When $x = -5$, we use the first equation:

$f(-5) = (-5)^2 + 2(-5) + 2 = 25 - 10 + 2 = 17$

For $x = -2$, notice that the number -2 appears in the limits for the first and the second equations – it's the right-hand limit in the first one and the left-hand limit in the second. Which equation should we use? Because the second equation says x is greater than *or equal to* -2, we'll use that one. The first equation is used when x is less than -2, and we know that -2 is not less than itself.

$f(-2) = 2(-2) + 6 = -4 + 6 = 2$

For the third value, $x = 3$, we'll use the last equation:

$f(3) = 9 - 3 = 6$

So the answers are 17, 2, and 6.

In the preceding example, f was defined for every value of x: all the numbers less than -2, all the numbers between -2 and 1 (including -2 and 1), and all the numbers greater than 1. Occasionally we're given a function with "holes" in it, that is, places where the function is not defined.

SAMPLE PROBLEM: PIECEWISE FUNCTIONS

Find the point(s) where the following function is undefined:

$$f(x) = \begin{cases} 4x^3 - 8 & x < 4 \\ \log_5 x & 4 < x \leq 6 \\ |6x + 5| & x > 6 \end{cases}$$

(A) $-4, 5$
(B) $-3, 5$
(C) $4, 6$
(D) 4 only
(E) 6 only

Because this question is only asking about the values of x we can use, we don't even really need to look at the functions – which is good, because they're more complicated this time. We can just look at the limits.

The first function is defined when x is less than 4, and the second is defined when x is between 4 and 6. But while that second limit includes 6 (there is a less than or equal to sign there), it doesn't include 4 (the inequality is strictly greater than). This means that the function doesn't include 4 – this means that if we were asked to find $f(4)$, we wouldn't have anything to plug the number into.

A check of the other endpoint of the inequality, 6, tells us that it is defined – we use the second function for it. Therefore the only place the function is undefined is at $x = 4$. The answer is (D).

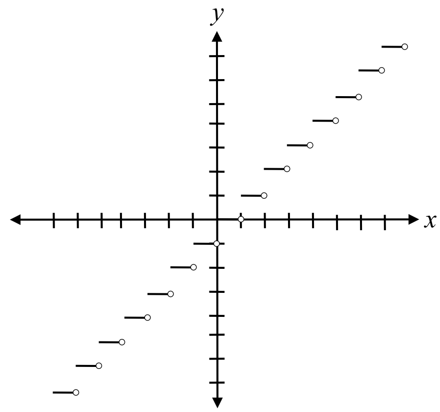
These two concepts – evaluation and domain – constitute the majority of questions about piecewise functions

One special function that can be thought of as a piecewise function is called the *greatest integer function*, usually written as $[x]$. When we're given a function, usually we plug a value in, do some calculations, and out pops the value of the function. The greatest integer function takes a number and returns the *greatest integer less than or equal to the number*. For example, $[3.5] = 3$, because 3 is the largest integer that's still less than 3.5. Here are a few more examples:

$[e] = 2$ $\qquad$ $[-3.9] = -4$

$[\pi] = 3$ $\qquad$ $[4] = 4$

For reference, the graph of the greatest integer function looks like this:



The circles on the right hand edges of the lines indicate that as x increases, when you reach the next integer the value of the function increases.

Let's look at an example of how this concept might be incorporated into an ACT question:

SAMPLE PROBLEM: PIECEWISE FUNCTIONS

If $[x]$ is the greatest integer function, defined as the greatest integer less than or equal to x, what is the value of $[x] - [x + 1]$?

(A) $-x$
(B) -1
(C) 0
(D) 1
(E) cannot be determined

For a question like this, we should plug in a few values and see if a pattern emerges. It's good to use a variety of different types of numbers – positive, negative, zero, integers, decimals, and so on. Let's try -2.5, 0, and 3.6:

$[-2.5] - [-2.5 + 1] = [-2.5] - [-1.5] =$
$-3 - (-2) = -1$

$[0] - [0 + 1] = [0] - [1] = 0 - 1 = -1$

$[3.6] - [3.6 + 1] = [3] - [4.6] = 3 - 4 = -1$

Because all these values are -1, we can be fairly certain that the answer is (B). Thinking about it logically, the greatest integer function basically rounds a number down the integer below it. Since integers are exactly 1 unit apart, adding 1 will automatically take the value of the greatest integer function up 1. Therefore $[x] - [x+1] = x - (x+1) = -1$.

TOPIC OVERVIEW: COMPOSITE FUNCTIONS

We know that to evaluate a function $f(x)$ at a given value, we simply plug the value in for x everywhere we see it in the function. Sometimes the "value" we're given is another function. When this happens, we call the evaluation *composition of functions*. While it may look a little intimidating, the same principle applies – we plug the value in based on the rule of the function. There are two common ways to denote composition of functions:

(1) $f(g(x))$

This represents f composed with g and it means you evaluate function f at g – that is, you plug g into f. We'll use this notation in this lesson.

This is read "f of g of x"

(2) $(f \circ g)(x)$

This means the same thing as the previous notation, but is a little less common. Many high school algebra classes introduce composition this way, so it's good to know that it means the same thing.

SAMPLE PROBLEM: COMPOSITE FUNCTIONS

If $f(x) = \sqrt{2x+2}$ and $g(x) = \frac{1}{2}x^2 - 1$, what is $f(g(x))$?

Here we need to evaluate f at g. Just plug the expression for g into f anywhere you see an x:

$$f(g(x)) = f\left(\frac{1}{2}x^2 - 1\right) = \sqrt{2\left(\frac{1}{2}x^2 - 1\right) + 2} =$$
$$\sqrt{x^2 - 2 + 2} = \sqrt{x^2} = |x|$$

You can see we rewrote $f(g(x))$ with the expression for g inside the parentheses. This can help us remember exactly what we're doing and avoid making careless mistakes.

We can also switch the order around, and find $g(f(x))$. Here we'll substitute the function f into the function g:

$$g(f(x)) = g(\sqrt{2x+2}) = \frac{1}{2}\left(\sqrt{2x+2}\right)^2 - 1 =$$
$$\frac{1}{2}(2x+2) - 1 = x + 1 - 1 = x$$

When $f(g(x)) = g(f(x)) = x$, we say that the functions are inverses of each other. Note that in general, for two random functions f and g, $f(g(x))$ is usually <u>not</u> equal to $g(f(x))$. The next exercise gives you a chance to investigate this further.

SAMPLE PROBLEM: COMPOSITE FUNCTIONS

If $f(x) = \sqrt[3]{\sqrt{x} + 27}$ and $g(x) = 49x^6$, what is $f(g(3))$?

(A) 3
(B) 6
(C) 18
(D) 27
(E) 81

To answer a question like this, we have two options. First, we can substitute the function for g into f, and then plug the 3 in, or we can start by plugging the 3 into g, and then plug the output of g into f. Depending on how complicated the functions are, one way is often easier than the other. But because you are allowed to use a calculator on the ACT, plugging the number in first is often the safer choice.

If we evaluate g at the given value first, we can see that

$$g(3) = 49 \cdot 3^6 = 49 \cdot 729 = 35{,}721.$$

That doesn't seem like much of a number, but when we plug it into f where we see an x, notice that we're taking a square root:

$$f(g(3)) = f(35721) = \sqrt[3]{\sqrt{35721} + 27} =$$
$$\sqrt[3]{189 + 27} = \sqrt[3]{216} = 6$$

The answer is (B).

PRACTICE EXERCISES

1. If $x > 0$, $f(x) = (x-2)(x+2)$ and $g(x) = \sqrt{x+4}$, what is $g(f(x))$ in terms of x?

2. If $f(x) = 3x - 4$ and $g(x) = x^2 + 3x + 4$, what is $g(f(x))$ in terms of x?

3. A function r is defined as follows:

$$r(x) = \begin{cases} (x+3)^2 - 7 & x < 0 \\ (x-3)^2 + 7 & x \geq 0 \end{cases}$$

What is the value of $r(3) - r(-3)$?

4. Given that f and g are two functions, each with a domain of all real numbers, are $f(g(x))$ and $g(f(x))$ always equivalent? Explain and/or give an example to illustrate your answer.

5. If $p(x) = \sqrt{x}$ and $q(x) = 2^{x-2}$, evaluate $\frac{q(p(x))}{p(q(x))}$ when $x = 4$.

6. If $g(x) = [x]$, the greatest integer function, then for all x, $g(g(x)) - g(x)$ = ?

For Questions 7 through 10, graph each function in the coordinate plane.

7. $f(x) = \begin{cases} 2x - 3 & x < 0 \\ 3 - 2x & x \geq 0 \end{cases}$

8. $f(x) = \begin{cases} x + 5 & x < -2 \\ x^2 - 1 & -2 \leq x < 3 \\ -4 & x \geq 3 \end{cases}$

9. $f(x) = x - [x]$

10. $f(x) = \frac{[2x]}{2}$

TEST EXERCISES (TIME: 10 MINUTES)

11. If $f(x) = 3\sqrt{x} + 2$ and $g(x) = (x-1)^2$, then $g(f(4)) - f(g(3)) = ?$

    (A) -89
    (B) $-17 - 6\sqrt{3}$
    (C) 29
    (D) 41
    (E) $64 - 12\sqrt{3}$

12. If $f(x) = x + 3$ and $g(f(x)) = x^2 + 6x + 13$, what is $g(x)$ in terms of x?

    (A) $x^2 + 6x + 10$
    (B) $2x^2 + 3x + 5$
    (C) $x^2 + 6x + 9$
    (D) $(x+4)^2$
    (E) $x^2 + 4$

13. Consider the function defined as

$$f(x) = \begin{cases} x^2 - kx & x < 2 \\ x^2 + kx & x \geq 2 \end{cases}$$

    If this function has the same value at $x = 1$ and $x = 3$, then $k = ?$

    (A) -3
    (B) -2
    (C) -1
    (D) 1
    (E) 2

14. Let the operation • be defined such that • $x = 2x - 1$. This operation can be performed several consecutive times as denoted by multiple • signs; for example, •• x = • (• x) and so on. What is the value of ••••• 2 = ?

    (A) 15
    (B) 31
    (C) 33
    (D) 57
    (E) 63

15. If the function $[x]$ is the greatest integer function, defined for all real numbers x as the greatest integer less than or equal to x, and if $n = 0.5$, then $[n^2 - n] = ?$

    (A) -2
    (B) -1
    (C) 0
    (D) 1
    (E) 2

16. At which value(s) of x is the following function undefined?

$$f(x) = \begin{cases} |x-3| & x < -4 \\ 2|x-1| - 3 & -4 < x \leq 0 \\ x^2 - 3 & x > 0 \end{cases}$$

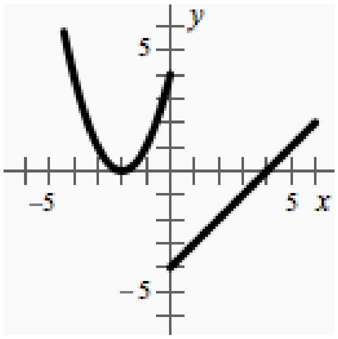
    (A) -4 only
    (B) 0 only
    (C) -4 and 0
    (D) -4, 0, and 3
    (E) None

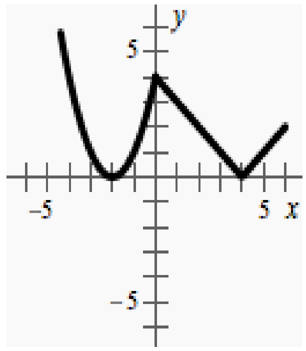
17. Consider the functions $f(x) = x^2 - 2$ and $g(x) = \sqrt{kx}$. If the graph of $f(g(x))$ passes through the point $(3, 13)$, then $k = ?$

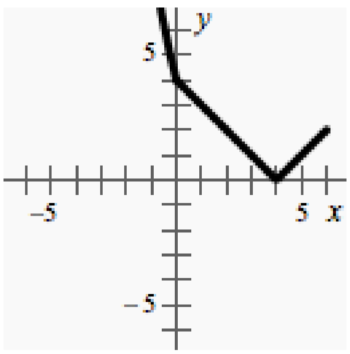
    (A) 5
    (B) 9
    (C) 11
    (D) 13
    (E) 15

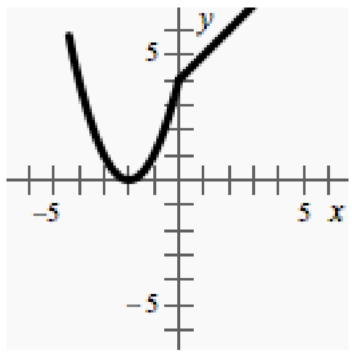
18. Which of the following is the graph of the function $f(x)$ defined below?

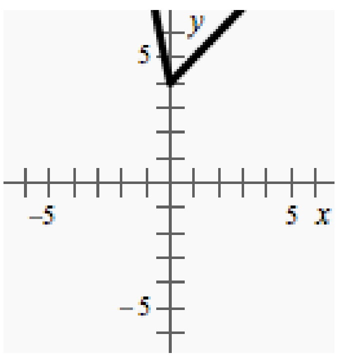
$$f(x) = \begin{cases} (x+2)^2 & x \leq 0 \\ |x-4| & x > 0 \end{cases}$$

(A) 

(D) 

(B) 

(E) 

(C) 

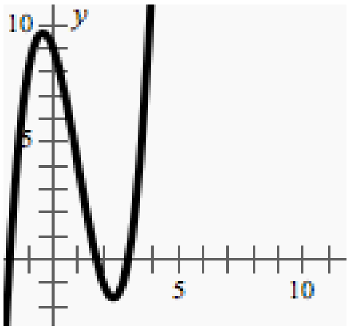
19. Stephen defined the operation $\lozenge$ such that $x \lozenge y = \frac{4}{x-y}$. Given that $3 \lozenge k = 16$, and $k \lozenge (-3.5) = m^2$, which of the following could be the value of m?

(A) -0.8
(B) -0.75
(C) 0.6
(D) 0.2
(E) 2.75

20. Consider the functions f and g defined below:

$f(x) = x^2 - 3$
$g(x) = x - 3$

Which of the following is the graph of $f\big(g(x)\big)$?

(A)


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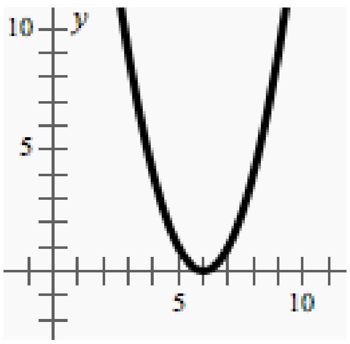

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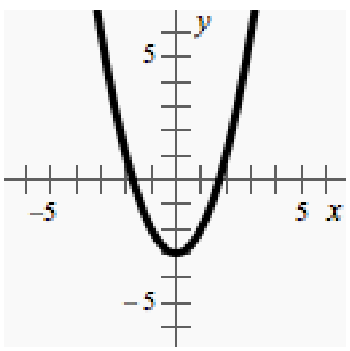

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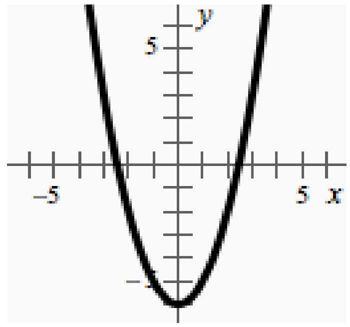

(E)