mP4B – LOGARITHMS

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

TOPIC OVERVIEW: LOGARITHMS

Now that we've learned the ins and outs of exponents, we're going to look at a related topic: logarithms. A logarithm is simply the inverse of an exponent. There is one major relationship between logarithms and exponents:

If $a^b = c$, then $\log_a c = b$.

In the exponential expression, a is called the "base", and b is called the "exponent". We learned that in the last lesson. In the logarithmic expression, a is called the "base", and c is called the "argument". The logarithmic expression is read "log base a of c is equal to b."

Logarithms (often called simply "logs") have a set of rules that's very similar to the set of rules for exponents.

(1) $\log_x ab = \log_x a + \log_x b$

(2) $\log_x \frac{a}{b} = \log_x a - \log_x b$

(3) $\log_x a^b = b\log_x a$

(4) $\log_x x = 1$

(5) $\log_x 1 = 0$

(6) $x^{\log_x a} = a \;\leftrightarrow\; \log_x x^a = a$

Look at the first two rules, and compare them with the first two rules for exponents:

$$\log_x ab = \log_x a + \log_x b \quad \leftrightarrow \quad x^a x^b = x^{a+b}$$

$$\log_x \frac{a}{b} = \log_x a - \log_x b \quad \leftrightarrow \quad \frac{x^a}{x^b} = x^{a-b}$$

They look pretty much the same, right? Multiplying is like adding, and dividing is like subtracting. Because logs and exponents are inverses of each other, the same basic rules apply. The third rule above is a very important one, because it's what allows us to use logs to solve exponential equations:

$\log_x a^b = b\log_x a$

In this equation notice that the b (which was in the exponent initially) moves down to the front of the logarithm. That means that if we can figure out how to take the log, we can find b.

The fourth and fifth rules are like identities – anytime the base and the argument are the same, the log is equal to 1, and the log of <u>any</u> base of 1 is equal to 0.

Finally, the sixth rule shows us how we can use the inverse relationship between logarithms and exponents to simplify complicated expressions.

Occasionally we may see a log expression that has no base written next to it, like so:

$\log 100 = 2$

When there is no base, the base is assumed to be 10. This is called the *common log*, and it is quite…common.

Those are all the basic rules of logarithms. Any question you'll find on the ACT will use one or more of these basic rules. Don't forget the very first thing we discussed – the relationship between logs and exponents.

SAMPLE PROBLEM: LOGARITHMS

Evaluate the following expressions:

a) $\log_5 5$
b) $\log_4 64$
c) $3^{\log_3 x}$
d) $\log_7 7^x$
e) $\log_{10} 1$
f) $\log_{10} 0$

Solutions:

a) The 4th rule tells us that when the base of the log and the argument of the log match, the expression is equal to **<u>1</u>**.

b) This one is a little trickier – the base and the argument don't match. But we can use the same logic we used in the lesson on exponents: we know that 64 is a power of 4 – it's equal to 4^3:

   $\log_4 64 = \log_4 4^3 = \underline{\mathbf{3}}$

c) Here we used the second part of the sixth rule – this is the reason that logs and exponents are inverses of each other. When you do both of them together it's like not doing either of them. Think of squares and square roots.

d) Use the first part of the sixth rule – again with the inverses. This expression is equal to $\underline{\boldsymbol{x}}$.

e) Just like in parts (b) and (c), $\log_7$ and 7^x are inverses. This expression is also equal to $\underline{\boldsymbol{x}}$. It doesn't matter what the base is.

f) The fifth rule tells us that the log of any base of 1 is equal to $\underline{\mathbf{0}}$.

This is sort of a trick question. Set the log expression equal to x and use the definition of the logarithm to rewrite it in exponential form:

$$\log_{10} 0 = x \quad \rightarrow \quad 10^x = 0$$

No matter how small (or negative) we make x, we can never raise 10 to a power and get zero, which means there's no value of x that makes the expression true. That means the log expression is <u>undefined</u>. As a rule, the number that goes into the log (which is called the *argument*) must always be positive.

SAMPLE PROBLEM: LOGARITHMS

If $\log_5 x = 2$, what is the value of x?

The quickest way to solve this is to use the relationship between logs and exponents:

$$a^b = c \quad \leftrightarrow \quad \log_a c = b$$

In this example, the expression we're given is like the one on the right. All we need to do is plug the values into the expression on the left:

$$\log_5 x = 2 \;\rightarrow\; 5^2 = x \;\rightarrow\; x = 25$$

The more you use this rule, the easier it will be. A problem similar to this on the ACT will take you only seconds to solve!

SAMPLE PROBLEM: LOGARITHMS

If $\log_3 s = 0.031$ and $\log_3 t = 0.139$, evaluate $\log_3 \frac{s^4}{t^3}$

This problem requires us to *expand* the log expression we're given – this is a common question type. We generally use the first three rules to do this. First, we can use the rule for division to separate the s's and the t's:

$$\log_3 \frac{s^4}{t^3} = \log_3 s^4 - \log_3 t^3$$

Now we can use the third rule to move the exponents down to the front of the logs:

$$\log_3 s^4 - \log_3 t^3 = 4\log_3 s - 3\log_3 t$$

Finally, we substitute the values we were given in the problem statement:

$$4\log_3 s - 3\log_3 t = 4(0.031) - 3(0.139) = -0.293$$

SAMPLE PROBLEM: LOGARITHMS

What is the value of $\log_u \frac{u^8}{u^2}$?

(A) 2
(B) 4
(C) 6
(D) u^2
(E) u^4

There are two ways to approach this question: First, we can use the second rule to expand the expression, and then the last rule to evaluate:

$$\log_u \frac{u^8}{u^2} = \log_u u^8 - \log_u u^2 \qquad \text{(Rule 2)}$$

$$\log_u u^8 - \log_u u^2 = 8 - 2 = 6 \qquad \text{(Rule 6)}$$

Another way to solve this is to simplify the exponential expression inside the logarithm first, and then use the last rule to evaluate:

$$\log_u \frac{u^8}{u^2} = \log_u u^{8-2} = \log_u u^6 = 6$$

To recap: Logarithms are closely related to exponents, and there is a set of rules we apply when dealing with them:

- When given an equation with logarithms, use the relationship between logs and exponents to solve

- When given an expression to evaluate, use the rules of logs to expand the expression and then evaluate it.

- Logs and exponents are inverses of each other, so when both are applied to an expression they cancel each other out.

PRACTICE EXERCISES

For questions 1-12, solve for x:

1. $x = \log_{17} 1$
2. $x = \log_{27} 3$
3. $x = \log_4 8$
4. $x = \log_{\sqrt{2}} 4$
5. $\log_x 1000 = 3$
6. $\log_x 347 = 1$
7. $\log_x 0.0225 = 2$
8. $\log_x \left(\frac{1}{64}\right) = -12$
9. $\log_{16} x = \frac{3}{4}$
10. $\log_3 x = 0$
11. $\log_{0.1} x = -8$
12. $\log_{\sqrt{2}} x = -7$

For questions 13-20, find the value of the expression in exact form if possible. If not, write the expression as a single logarithm of a single argument:

13. $\log_3 2 + \log_3 3$
14. $\log_4 75 - \log_4 15$
15. $\log_6 12 + \log_6 3$
16. $7\log_{15} 3 + 7\log_{15} 5$
17. $x^{2\log_x \sqrt{23}}$
18. $1 - \log_{10} 5$
19. $\left(10^{\log_{10} 2}\right)^{\log_2 10}$
20. $\log_4 6 \cdot \log_6 4$

TEST EXERCISES (TIME: 10 MINUTES)

21. If $\log_2 x + \log_2(2x) = \log_2 14x$, then $x =$?

(A) 1
(B) $\frac{14}{3}$
(C) 7
(D) 14
(E) 42

22. If $x = 3^{\log_3 2}$, then $2^{\log_2 x} =$?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 6

23. If $\log_n x = 16$, what is $\log_{n^2} x$?

(A) 2
(B) 4
(C) 8
(D) 16
(E) 64

24. What is the value of $\frac{\log_2 8}{\log_8 2}$?

(A) $\frac{8}{3}$
(B) $\frac{1}{9}$
(C) 1
(D) 3
(E) 9

25. If $\log_{10} y = \log_{10} x + 3$, what is y expressed as a function of x ?

(A) $y = x + 3$
(B) $y = 3x$
(C) $y = x + 1000$
(D) $y = 1000x$
(E) $y = x^3$

26. If $\log_5(A^5) = 2.25$, then $\log_5(A^2) =$?

(A) 0.15
(B) 0.45
(C) 0.81
(D) 0.90
(E) 1.50

27. If $a = \log 100$, then which of the following expressions is equal to $\log 100{,}000$?

(A) $1 - 2a$
(B) $5a$
(C) $a + 1$
(D) $2a$
(E) $\frac{5}{2}a$

28. Given that $x = \log_2 3$, what is $\log_2 72$ expressed in terms of x ?

(A) $9x$
(B) $24x$
(C) $2x + 3$
(D) $3x^2$
(E) $8x^2$

29. If $6^{x+2} - 4 \cdot 6^x = 2^{x+6}$, then $x =$?

(A) $\log_2 3$
(B) $\log_3 2$
(C) $\log_2 6$
(D) $\log_3 6$
(E) $\log_6 2$

30. If $\log_{12} 2 = r$ and $\log_{12} 3 = s$, which of the following choices correctly expresses r in terms of s ?

(A) $r = \frac{6}{s}$
(B) $r = 1 - 2s$
(C) $r = \frac{1-s}{2}$
(D) $r = \sqrt{\frac{1}{s}}$
(E) $r = \sqrt{1 - s}$