mP8B – SEQUENCES

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

TOPIC OVERVIEW: SEQUENCES

A sequence is a list of numbers that are somehow related to each other – there's a rule which tells us how to get from one number to the next. A simple series is a list of the integers in order:

$$1, 2, 3, 4, 5 \ldots$$

This series can be generated by adding 1 to each number to find the next number. All we need is the starting point. We call the numbers in a sequence *terms*, and in this lesson we'll use the variable a to denote a term in a sequence. The first term in a sequence will be called a_1, the second a_2, and so on. A general term will usually be denoted a_n, and is called the n^{th} term in the sequence.

There are two major categories of sequences: arithmetic and geometric. An arithmetic sequence has a common *difference* (usually called d) between the terms, which means we add a number to each term to generate the sequence. A geometric sequence has a common *ratio* (usually called r), which means we multiply a number by each term to generate the sequence. There are formulas for the *n*th term of each type of sequence:

Arithmetic sequence: $a_n = a_1 + (n-1)d$
Geometric sequence: $a_n = a_1 \cdot r^{n-1}$

SAMPLE PROBLEM: SEQUENCES

What is the 11th term of the following sequence?
$2, 5, 8, 11 \ \ldots$

(A) 26
(B) 28
(C) 32
(D) 35
(E) 38

The first thing to find is the type of sequence we're looking at. This sequence is generated by adding 3 to each term, so it's an arithmetic sequence with a common difference of 3. In order to use the formula for the n^{th} term of a sequence, we'll need to know the first term. The first term, a_1, is 2. Because we're looking for the 11th term, n is 11. Now we can plug the values into the formula:

$$a_n = a_1 + (n-1)d \ \rightarrow a_{11} = 2 + (11-1)3 = 32.$$

The answer is (C).

SAMPLE PROBLEM: SEQUENCES

Find the 6th term of the geometric sequence with $a_2 = 6$ and $a_3 = 18$.

(A) 18
(B) 54
(C) 108
(D) 243
(E) 486

This problem tells us what type of sequence we have, but it doesn't tell us the first term, or the common ratio. But we can figure it out from what we're given – the second and third terms. Just from looking at the terms, we can see that the common ratio is 3: $6 \times 3 = 18$. A more systematic way to figure it out would be to divide the second term into the third:

$$\frac{a_3}{a_2} = \frac{18}{6} = 3 = r$$

Now we can find the first term by *dividing* the common ratio into the second term:

$$a_1 = \frac{a_2}{r} = \frac{6}{3} = 2$$

To answer the question, we use the formula with $a_1 = 2$, $n = 6$, and $r = 3$:

$$a_n = a_1 \cdot r^{n-1} = 2 \cdot 3^{6-1} = 2 \cdot 3^5 = 486.$$

The answer is (E).

Note: In the last examples, the problems could also be solved by simply adding (or multiplying) the common difference (or ratio) to the terms until you get to the one you want. In the first example, you could have done $2 + 3 + 3 + 3 + 3$ ... until you got to 32. In the second example, you could have done $6 \times 3 \times 3 \times 3 \times 3 = 486$. We showed you the formulas because they will work for *any* example, but for questions that ask for terms early in the sequence, just adding or multiplying the numbers may be faster. On the other hand, if the question asks for the 20th or 100th term, definitely use the formulas!

SAMPLE PROBLEM: SEQUENCES

Find the eighth term of the following sequence:

$$2, 5, 10, 17 \dots$$

(A) 75
(B) 65
(C) 50
(D) 45
(E) 37

This sequence has no common difference ($5 - 2 = 3$ but $10 - 5 = 5$), and no common ratio ($5 \div 2 = 2.5$ but $10 \div 5 = 2$). This sequence is neither arithmetic nor geometric, which means we have to find some other pattern.

This sequence contains a fairly common pattern – the squares of the integers. The first four squares are 1, 4, 9, and 16, which we can see are all 1 less than the numbers in the sequence. So each term is actually $n^2 + 1$, where n is the term number. Therefore the 8th term is $8^2 + 1 = 65$.

The answer is (B).

The last example shows us that not all sequences we'll see will be arithmetic or geometric.

SAMPLE PROBLEM: SEQUENCES

Find the sum of the first five terms of the geometric sequence with $a_2 = 30$ and $a_4 = 7.5$.

(A) 37.5
(B) 58.125
(C) 105
(D) 116.25
(E) 125

Because we're not given the first term or the ratio here, we need to deduce them from what we're given. We can think of the terms in a geometric series with first term a_1 and ratio r:

$a_1 = a_1,\ a_2 = a_1 r,\ a_3 = a_1 r^2,\ a_4 = a_1 r^3,$

$a_5 = a_1 r^4, \dots$

We can see from this pattern that any terms two apart will have a ratio of r^2. The terms we're given are separated by two (they are the 2nd and 4th terms) so their ratio must be r^2:

$$\frac{a_4}{a_2} = r^2 = \frac{7.5}{30} = \frac{1}{4} \quad \rightarrow \quad r = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Now we can find the first term:

$$a_1 = \frac{a_2}{r} = \frac{30}{\frac{1}{2}} = 60$$

To find the sum of the first five terms, we'll find each term and add them all together:

$a_1 = 60 \quad a_2 = 30 \quad a_3 = 15 \quad a_4 = 7.5 \quad a_5 = 3.75$

$60 + 30 + 15 + 7.5 + 3.75 = 116.25$

The answer is (D).

SAMPLE PROBLEM: SEQUENCES

In a certain arithmetic sequence, the third term is b and the common difference is c. What is the sum of the first six terms?

(A) $6b$
(B) $6b + 3c$
(C) $5b + 3c$
(D) $5b$
(E) $3b$

This example doesn't give us any numbers! Not to worry, we can still use the same principles we used on the previous questions. We're given the third term and the difference, which means we can find the second term by *subtracting* c from b and getting $b - c$. We can find the first term by subtracting c from that and getting $b - 2c$. We can find terms after the third by adding c to the term:

1st term: $b - 2c$ 2nd term: $b - c$

3rd term: b 4th term: $b + c$

5th term: $b + 2c$ 6th term: $b + 3c$

We add them all up, combine like terms, and we have our answer:

$(b - 2c) + (b - c) + b + (b + c) + (b + 2c) + (b + 3c) \rightarrow$

$6b + 3c$

The answer is (B).

PRACTICE EXERCISES

For Problems 1 through 10, write the first five terms (a_1 through a_5) of the sequence described.

1. Arithmetic sequence, $a_1 = 30, d = 70$

2. Geometric sequence, $a_1 = \frac{3}{16}, r = 2$

3. Arithmetic sequence, $a_1 = 0, a_5 = 20$

4. Arithmetic sequence, $a_{10} = 17, d = -3$

5. Geometric sequence, $a_1 = 8, a_4 = \frac{125}{8}$

6. Sequence in which $a_1 = 1$, and $a_{n+1} = 3a_n - 1$ for integers $n \geq 1$.

7. Geometric sequence, $a_7 = 1458, r = 3$

8. Arithmetic sequence in which $d = 8$ and $a_1 + a_2 + a_3 + a_4 + a_5 = 120$

9. Geometric sequence, $a_1 = 36, a_5 = 4$

10. Sequence in which $a_6 = 59$ and $a_{n+1} = a_n + n^2$ for integers $n \geq 1$.

11. Fill in the blanks in the below sequence:

    -1, 0, 7, 26, 63, ____, ____

12. Fill in the blanks in the below sequence:

    $\frac{16}{81}, \frac{8}{27}, \frac{4}{9}$, ____, ____, ____, $\frac{9}{4}$, ____, $\frac{81}{16}$

13. If the pattern below is continued, how many dots will there be in Figure 5?

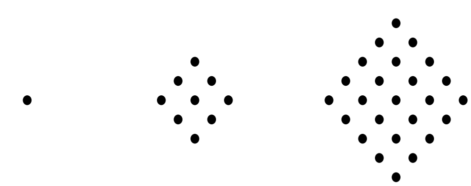

Figure 1 **Figure 2** **Figure 3**

14. If the pattern below is continued, how many dots will there be in Figure 5?

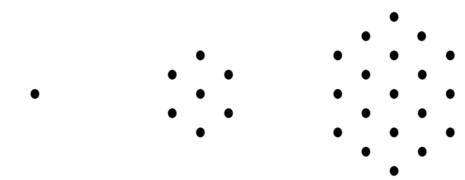

Figure 1 **Figure 2** **Figure 3**

15. Pascal's Triangle is a triangle of integers that can be continued indefinitely in which each row has one more element than the row above it, each row begins and ends with a 1, and otherwise each number is the sum of the numbers above and to the left and right of it. The first few rows of Pascal's Triangle are printed here:

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
…	….....

Following this pattern, complete Rows 5 and 6 of Pascal's Triangle.

TEST EXERCISES (TIME: 10 MINUTES)

16. The first term of an arithmetic sequence is p and the common difference is d. What is the sum of the first seven terms, in terms of p and d?

(A) $p + 6d$
(B) $p + 7d$
(C) $7p + 6d$
(D) $7p + 7d$
(E) $7p + 21d$

17. Which of the following could be part of a geometric sequence?

(A) $1, 4, 9, 16, \ldots$
(B) $1, -1, -3, -5, \ldots$
(C) $2, 3, 6, 18, \ldots$
(D) $16, 8, 4, 2, \ldots$
(E) $1, 2, 4, 6, \ldots$

18. A sequence of numbers is defined such that the first term is 3, and each successive term exceeds twice its preceding term by 1. What is the fifth term?

(A) 11
(B) 33
(C) 48
(D) 49
(E) 63

19. In the sequence defined in Problem 18, where the nth term is denoted a_n, $a_n = ?$

(A) $2^n - 1$
(B) $2^{n+1} - 1$
(C) 2^{n+1}
(D) $2^{n+1} + 1$
(E) $2^{n-1} + 1$

20. Which of the following is true about the arithmetic sequence $4, 0, -4, \ldots$?

(A) The common difference of consecutive terms is 4.
(B) The average of the first five terms is -4.
(C) The sum of the first four terms is 0.
(D) The fifth term is twice the third term.
(E) The sum of the first term and the common difference is 8.

21. Each day in March, Greg's Games sold the same number of copies of Super Mario Kart. At the start of business on March 1st, Greg's Games had 196 copies in stock; by closing time on March 11th, it had 53 copies left. How many copies were sold per day?

(A) 11
(B) 12
(C) 13
(D) 16
(E) 143

22. The fifth term of a geometric sequence is $\frac{27}{64}$, and the common ratio of consecutive terms is $\frac{3}{4}$. What is the first term?

(A) $\frac{6561}{65536}$
(B) $\frac{243}{1024}$
(C) 1
(D) $\frac{4}{3}$
(E) $\frac{16}{9}$

23. The first term of a geometric sequence is eight times the seventh term. What is the ratio of the sixth term to the third term?

(A) $\frac{\sqrt{2}}{4}$
(B) $\frac{1}{4}$
(C) $\frac{1}{2}$
(D) $\frac{\sqrt{2}}{2}$
(E) $\sqrt{2}$

24. Which of the following is true about the sum of the first n terms of the following arithmetic sequence?

1, 3, 5, 7, 9, …

(A) It is always odd.
(B) It is always even.
(C) These sums, evaluated for $n = 1, 2, 3, \ldots$, form an arithmetic sequence.
(D) It is always equal to $2n - 1$.
(E) It is always equal to n^2.

25. The first two terms of a geometric sequence are $a_1 = \frac{1}{25}$, $a_2 = \frac{1}{10}$. The n^{th} term of the sequence, a_n, is $\frac{625}{64}$. What is the value of n?

(A) 5
(B) 6
(C) 7
(D) 8
(E) 9