mP6B – Rational Functions

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Rational Functions

One group of functions with its own special set of rules is *rational functions*, which are just functions that are written as fractions – one expression in the numerator and one expression in the denominator. Here are some examples of rational functions:

$f(x) = \frac{x^2+3x+9}{x^4-16}$ $\quad g(x) = \frac{1}{3x+7}$ $\quad h(x) = \frac{4x^3-8x^2+4x-8}{\sqrt{3x^2+9}}$

The most commonly tested concept related to rational functions concerns the *domain* of the functions. Remember that the domain is the set of all x values that will give a valid value of y. For example, the domain of a polynomial is all real numbers – any number x will give us a number for y. For a rational function, however, there is often a number or set of numbers that is not allowed. This is called a *domain restriction*. The most common domain restriction is division by zero.

Sample Problem: Rational Functions

If $\frac{x^2+2}{x^2-3x-28}$, which of the following values CANNOT be x?

(A) -2
(B) 2
(C) 2 and 14
(D) -4 and 7
(E) 4 and 7

Because this is a rational function, to find the possible values for x we should always start by looking at the denominator. Because the denominator of a fraction can never be zero, we can set the denominator equal to zero and find the x's that make this true. Those values are the ones that we *don't* want:

When we set the denominator of our fraction equal to zero, we have a quadratic equation we can solve. The solutions are the values in the domain restriction:

$x^2 - 3x - 28 = 0 \quad \rightarrow$
$(x-7)(x+4) \quad \rightarrow$
$x = 7, -4$

So the two values of x that make the denominator equal to zero are 7 and -4, which means those are the values that we don't want. Therefore the answer is (D).

Sample Problem – Rational Functions

What is the domain of the function $\frac{4x^3-8x^2+4x-8}{\sqrt{3x^2+9}}$?

(A) All real numbers
(B) All real numbers except $x = 2$
(C) All integers greater than 3
(D) $(-\infty, \sqrt{3})$
(E) $(-\sqrt{3}, \sqrt{3})$

Like the previous example, this is a rational function. Therefore we check and see if there are any restriction to both the numerator and denominator of the fraction:

The numerator is a polynomial, and the domain of a polynomial is all real numbers (often written as $\mathbb{R}$). No restrictions there.

We need to check the denominator to see if there are any values of x that make it equal to zero. In this question, notice that there is also a radical in the denominator – this introduces another possible restriction: the expression underneath the radical cannot be less than zero (see the note about imaginary numbers at the end of this example). Looking at the expression under the radical, we can see $3x^2 + 9$ is always at least 9, because $3x^2$ is always greater than or equal to zero. If the expression under the radical is always at least nine, then the radical expression itself is always at least 3. So there are no values of x that make the denominator equal to zero, or that give us a negative number under the radical. Therefore the domain of the function is all real numbers, (A).

Note: The ACT will test your knowledge of imaginary and complex numbers. These topics will be covered in a later section. However, questions that test your knowledge of the domain of a function restrict their answers to the real numbers. You won't have to worry about imaginary numbers in answering these questions.

We said earlier that a rational function is a function written as a fraction. That means we can use the same set of rules we have for fractions to manipulate rational functions. For example, when we multiply fractions together, we multiply the numerators and the denominators:

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$$\frac{2}{7} \times \frac{5}{9} = \frac{10}{63}$$

The same goes for rational functions:

$$\frac{x-1}{y^2+4} \times \frac{x+7}{3y} = \frac{(x-1)(x+7)}{(y^2+4)(3y)} = \frac{x^2+6x-7}{3y^3+12y}$$

When we divide fractions, we multiply by the reciprocal of the second fraction – the same rule applies for rational functions.

$$\frac{\frac{3x+2}{5y}}{\frac{4y-11}{10x}} = \frac{3x+2}{5y} \times \frac{10x}{4y-11} = \frac{2x(3x+2)}{y(4y-11)}$$

The expression above is left in *factored form*. Notice that we canceled a factor of 5 out of the numerator and denominator of the fraction before we performed the multiplication. Just like we did with fractions, we can simplify rational expressions by removing factors that appear in both:

Sample Problem: Rational Functions

Which of the following is equal to $\frac{(3x-6)(x^2-8x-20)}{(6x-60)(x^2-4)}$?

(A) $\frac{3(x-2)}{x^2-4}$

(B) $\frac{3x-10}{6x}$

(C) $\frac{3}{10}$

(D) $\frac{1}{2}$

(E) 1

When we're trying to simplify a rational function, the first thing we should do is factor – we need to break the functions we're given down into smaller pieces. In the numerator, we can factor a 3 out of the first expression, and use trinomial factoring to factor the second expression:

$$\frac{(3x-6)(x^2-8x-20)}{(6x-60)(x^2-4)} = \frac{3(x-2)(x+2)(x-10)}{(6x-60)(x^2-4)}$$

In the denominator, we can factor a 6 out of the first term, but what can we do with the second term? It's not a trinomial, but if you pay close attention you'll see that it does express the difference of two squares – a common factoring pattern! We can factor the denominator like so:

$$\frac{3(x-2)(x+2)(x-10)}{(6x-60)(x^2-4)} = \frac{3(x-2)(x+2)(x-10)}{6(x-10)(x-2)(x+2)}$$

This is in simplest form, because all the factors are linear, and none of the linear factors have any common factors in them. Now we can cancel out the terms common to the numerator and denominator:

$$\frac{3(x-2)(x+2)(x-10)}{6(x-10)(x-2)(x+2)} = \frac{3\cancel{(x-2)}\cancel{(x+2)}\cancel{(x-10)}}{6\cancel{(x-10)}\cancel{(x-2)}\cancel{(x+2)}} = \frac{1}{2}$$

The answer is (D)

We also know that if we're given a fraction, we can write it as two fractions, both with the same denominator:

$$\frac{15}{22} = \frac{7}{22} + \frac{8}{22}$$

As long as the new numerators add up to the old one, these two expressions are always equivalent. We may be asked to do this with rational functions, and the same rules apply.

Sample Problem: Rational Functions

$\frac{z^2+9}{6z} = ?$

(A) $\frac{z^2+3}{2}$

(B) $\frac{z+9}{6}$

(C) $\frac{z}{2} + 3$

(D) $z + 3$

(E) $\frac{z}{6} + \frac{3}{2z}$

To simplify this expression, it's important to realize that we can't cancel out any factors unless they are factors of both the numerator and the denominator – this means they have to be a factor of *everything* in the numerator and *everything* in the denominator. In this question, it's tempting to see that 9 and 6 both have a factor of 3 in them and do this:

$\frac{z^2+9}{6z} = \frac{z^2+3}{2z}$ **THIS IS INCORRECT!**

Unless the 3 is a factor of the z^2 term, we can't cancel it out! This is a very common mistake, and one that the answer choices will usually reflect: the test-makers will put those incorrect answers on the test to keep us honest. What we can do is separate this fraction into two fractions first, and then simplify:

$$\frac{z^2+9}{6z} = \frac{z^2}{6z} + \frac{9}{6z} = \frac{z}{6} + \frac{3}{2z}$$

The answer is (E).

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TOPIC OVERVIEW: GRAPHING RATIONAL FUNCTIONS

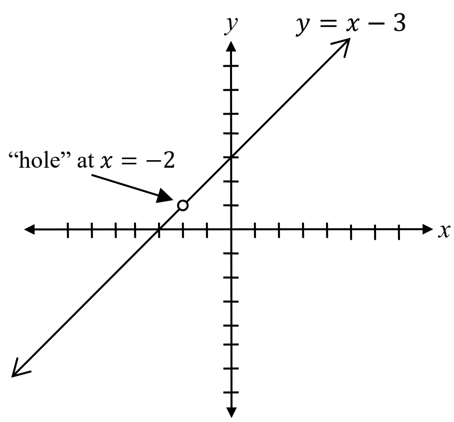
One more topic we will investigate is what the graphs of these rational functions look like. For example, take this function:

$y = \frac{x^2-x-6}{x+2}$

We know that we can factor this and simplify:

$y = \frac{(x+2)(x-3)}{x+2} \quad \rightarrow \quad y = x - 3$

Can we graph this as just this linear function, or is there something else going on? It's clear from the form of the original function that $x = -2$ is a domain restriction, so we don't want to include it in the graph of the function. But we canceled that factor out, so it no longer appears to be a problem! When this happens, we say that the graph has a "hole" at $x = -2$, and we graph the simpler function, leaving out the excluded value:



We indicate that a function has a hole by drawing an open circle at that x-value. Note that a function has a hole only when that value is canceled out of <u>both the numerator and the denominator</u>. We'll look at another example with a more complicated function:

SAMPLE PROBLEM: GRAPHING RATIONAL FUNCTIONS

Graph the following function:

$$y = \frac{2x^4-32}{2x^2-8}$$

Our first step in a problem like this is to see if there is any simplifying we can do. We need to make sure that if we happen to cancel any factors out of the numerator and denominator while we simplify, we note those as the holes in the graph.

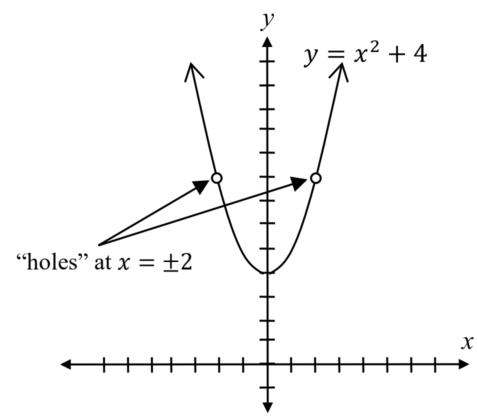
$y = \frac{2x^4-32}{2x^2-8} =$

$\frac{2(x^4-16)}{2(x^2-4)} =$

$\frac{2(x^2-4)(x^2+4)}{2(x^2-4)} =$

$x^2 + 4$

Notice that we canceled the term $x^2 - 4$. This gives us two holes, at $x = 2$ and at $x = -2$. If it's not clear why there are two holes, notice that we could have factored $x^2 - 4$ into two terms (in both the numerator and then denominator) and then canceled those terms. Now we can graph the simplified function, making sure to indicate the holes:



To review:

- The domain of a rational function is usually found by finding values where the denominator is equal to zero and not allowing them. When we're finding domain, we can ignore the possibility of imaginary values.
- We treat rational functions like we treat fractions, using the same rules and operations to manipulate and simplify them
- Only cancel out numbers or expressions if they are factors of <u>both the numerator and the denominator</u>.
- When graphing a rational function, simplify and then graph the expression, graphing any excluded values as "holes"

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PRACTICE EXERCISES

For Questions 1-4, perform the indicated operation and simplify as much as possible. Leave your answer in factored form wherever possible.

1. $\frac{x^2-9}{x^3+2x^2} \div \frac{x-3}{x+2}$

2. $\frac{1}{x+1} + \frac{2}{(x+2)}$

3. $\frac{x}{x-1} + \frac{1}{4x-1}$

4. $\frac{\left(\frac{2}{x}+\frac{3}{x-1}\right)}{\frac{1}{x}+x-2}$

For Questions 5-8, give the domain of the function.

5. $f(x) = \frac{x^2-25}{x^2+5}$

6. $f(x) = \frac{(x-3)\sqrt{x-7}}{(x-7)\sqrt{x-3}}$

7. $f(x) = \frac{\sqrt{16-x^2}}{x^4-81}$

8. $f(x) = \frac{4x^2}{\sqrt{8-x^3}}$

For Questions 9 and 10, graph the function, clearly indicating any domain restrictions.

9. $f(x) = \frac{x^2+9x+18}{x+6}$

10. $f(x) = \frac{x^6-4x^4}{10x^2-40}$

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TEST EXERCISES (TIME: 10 MINUTES)

11. What is the value of $\frac{(x^2-7x+12)}{xy+4y}$ when $x = 4$ and $y = -3$?

(A) 0
(B) 0.125
(C) 1
(D) 3
(E) Undefined

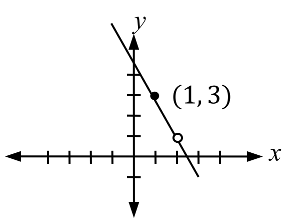
12. The expression $\frac{x}{x-3} - \frac{3}{x+2}$ is equivalent to which of the following?

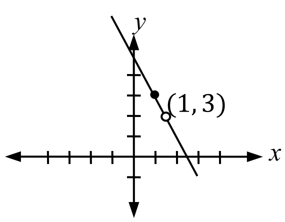
(A) $\frac{x-3}{2x-1}$
(B) $\frac{1}{x+2}$
(C) $\frac{x-3}{x+2}$
(D) $\frac{x^2+5x-9}{(x-3)(x+2)}$
(E) $\frac{x^2-x+9}{(x-3)(x+2)}$

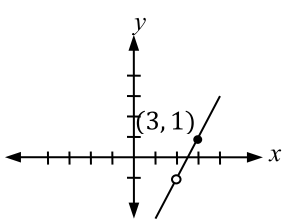
13. What are the values of x for which $\frac{(x-1)(x+1)}{(x+1)(x+2)}$ is undefined?

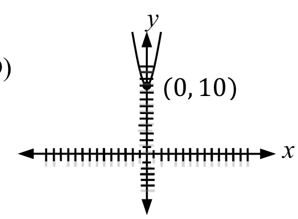
(A) 1 only
(B) -2 only
(C) -1 and -2 only
(D) 1, -1, and -2 only
(E) 1 and 2 only

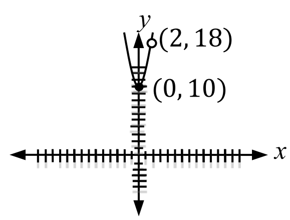
14. Which of the following graphs below represents $y = \frac{2x^2-9x+10}{2-x}$?

(A)


(B)


(C)


(D)


(E)


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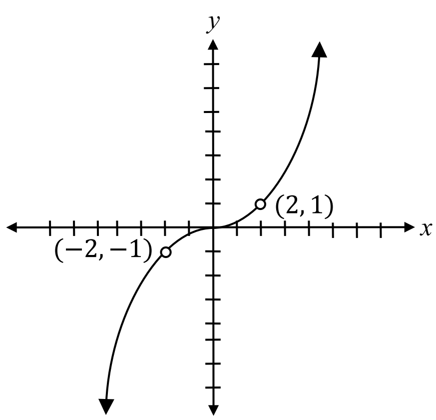
15. What is the sum of all the solutions of the equation $\frac{x+3}{x+4} = \frac{x+3}{x+1}$?

(A) -5.5
(B) -3
(C) -2.5
(D) 0
(E) The equation has no solutions.

16. Which of the following is a simplified expression equal to $\frac{(x+3)^2-25}{x-2}$ for all $x \neq 2$?

(A) $x+1$
(B) $x+8$
(C) $x-2$
(D) $x+7$
(E) $-x-8$

17. The graph below is the graph of which of the following functions?



(A) $f(x) = \frac{x^3}{x^2-4}$
(B) $f(x) = \frac{x^3}{8x^2-32}$
(C) $f(x) = \frac{x^4+2x^3}{8x+16}$
(D) $f(x) = \frac{x^5-4x^3}{8x^2-32}$
(E) $f(x) = \frac{x^4-2x^3}{8x-16}$

18. For which of the following values of x is $\frac{x^2-4x+4}{(x-2)\sqrt{x-2}}$ undefined?

(A) $x < 2$
(B) $x \leq 2$
(C) $x = 2$
(D) $x > 2$
(E) $x \neq 2$

19. If the function $f(x) = \frac{(x-4)(x-2)+k}{(x+1)(x-7)}$ has a domain of all real numbers, then $x =$?

(A) 7
(B) 1
(C) -1
(D) -15
(E) It is not possible for $f(x)$ to have a domain of all real numbers

20. For all x in the domain of the function $f(x) = \frac{1}{x-x^2}$, this function is equivalent to:

(A) $\frac{1}{x} - \frac{1}{x^2}$
(B) $\frac{1}{x} + \frac{1}{x+1}$
(C) $\frac{1}{x} - \frac{1}{x-1}$
(D) $\frac{1}{x-1} - \frac{1}{x+1}$
(E) $\frac{1}{x^2} - \frac{1}{x-1}$

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