## mP9B – Conic Sections

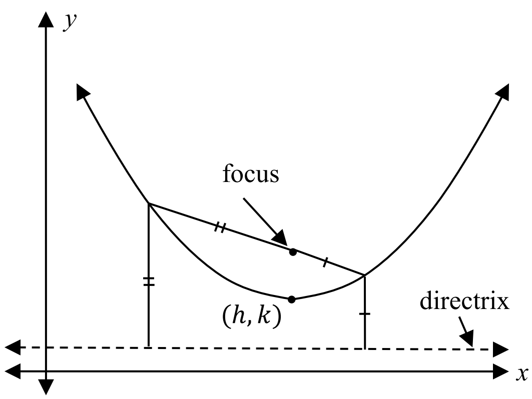
**Directions:** Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

### Topic Overview: Conic Sections

The term *conic section* refers to any one of a group of plane shapes that can be made from cutting a double cone with a plane. The only conic sections the ACT is concerned with are parabolas, circles, and ellipses. We'll talk about the general forms of these, and then look at the types of questions we might encounter.

A parabola is defined as the collection of points that is the same distance from a given point and a given line. That point is called the *focus* and the line is called the *directrix*. Both are depicted in the picture below. The focus is always inside the parabola, and the directrix is on the opposite side. A parabola with vertex $(h, k)$, has this equation:

$$y = \frac{1}{4p}(x-h)^2 + k$$




In this equation, $p$ is the distance from the vertex to both the focus and the directrix.

The parabola shown above has a vertical axis of symmetry. If a parabola has a horizontal axis of symmetry, the equation of that parabola is:

$$x = \frac{1}{4p}(y-k)^2 + h$$

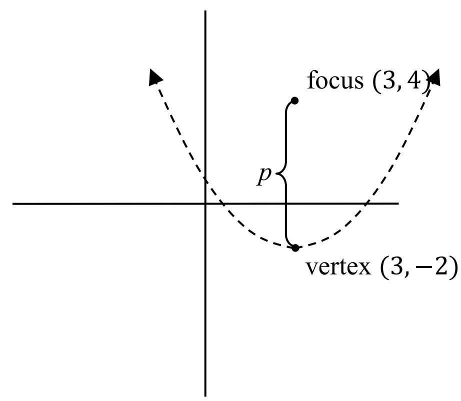
The number in front of the squared term in each equation determines the direction that the parabola opens. For a vertical axis of symmetry, a positive number means that it opens upward, and a negative number means that it opens downward. If the axis of symmetry is horizontal, positive means it opens to the right, and negative means it opens to the left.

### Sample Problem: Parabolas

A parabola has its vertex at $(3, -2)$ and focus at $(3, 4)$. Find the equation of the parabola.

(A) $x = 24(y-3)^2 + 2$
(B) $y = \frac{1}{24}(x-3)^2 - 2$
(C) $y = \frac{1}{24}(x+2)^2 + 3$
(D) $y = 24(x-3) + 2$
(E) $y = 24(x-2)^2 - 3$

To figure out the distances we'll need to solve this problem, it will be helpful to sketch a graph of the information we're given:




From this graph we can see that the parabola will have a vertical axis of symmetry, and will open upwards. The dotted line shows a sketch of what the parabola might look like. We'll use the equation we were given earlier:

$$y = \frac{1}{4p}(x-h)^2 + k$$

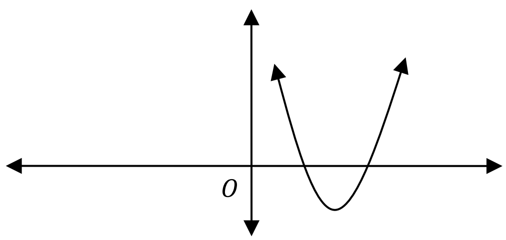
We know that $h = 3$ and $k = -2$. We also can find $p$, the distance from the focus to the vertex. In this example $p = 6$. Therefore the equation of the parabola is:

$$y = \frac{1}{4 \cdot 6}(x-3)^2 + (-2) = \frac{1}{24}(x-3)^2 - 2$$

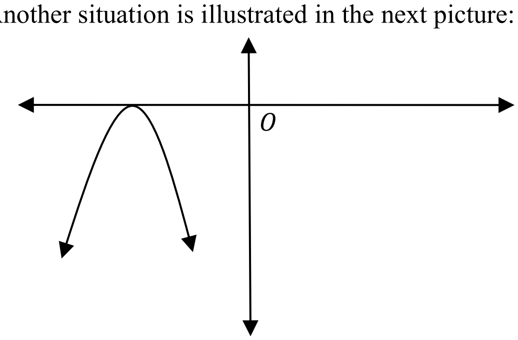
The answer is (B).

In this question, if the focus and vertex had been switched, then the parabola would have opened downward, and the equation would have had a negative coefficient, rather than a positive one.

One more note about parabolas: A parabola is the graph of a quadratic equation, which we can write $y = ax^2 + bx + c = 0$. If we think about this graphically, it means that the solutions of the quadratic equation are the places where $y = 0$, also known as the $x$-intercepts. This can tell us something about the *nature* of the solutions, that is, whether they are real numbers or not. For example, look at the picture below:

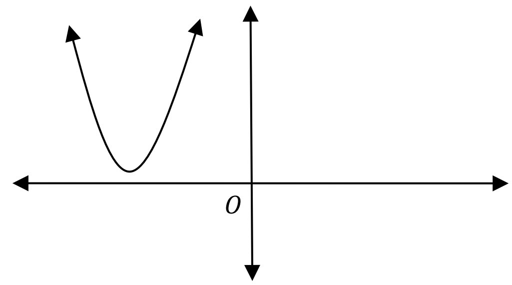



Because this parabola intersects the $x$-axis in two distinct places, we say that the equation that defines the parabola has <u>two real solutions</u>. We can write these parabolas as $y = (x - a)(x - b)$, where $a$ and $b$ are both the intercepts and the solutions to the equation. Another situation is illustrated in the next picture:



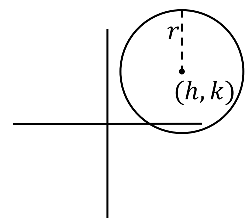

This parabola is tangent to the $x$-axis, which means that its equation has only <u>one real solution</u>. These are parabolas that can be written as $y = (x - a)^2$, where $a$ is the single solution.

The last possibility is shown below. Here, the parabola doesn't intersect the $x$-axis anywhere, and therefore has <u>no real solutions</u>. If we were to factor the equation for a parabola like this, it would be $y = (x - ai)(x - bi)$, where $i^2 = -1$.

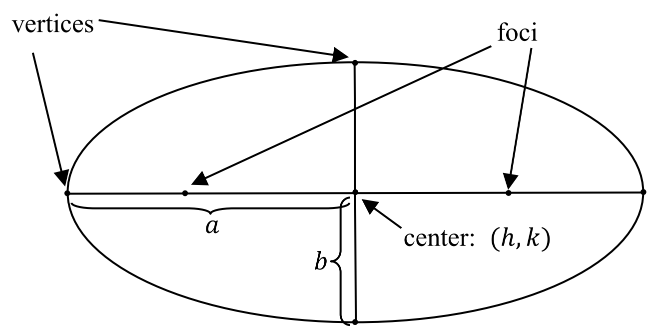



A <u>circle</u> (which we're pretty familiar with already) is the collection of points that's the same distance from a given point. We call that point the *center* of the circle, and that distance is the *radius*. A circle with center $(h, k)$ and radius $r$, has the equation

$$(x - h)^2 + (y - k)^2 = r^2$$




<u>Ellipses</u> are defined as the set of points for which the sum of the distances from two given points (each called a focus) is the same. An ellipse has a lot of different numbers and terms associated with it. The picture below shows an ellipse with all the important points labeled:




The long axis (half of which is labeled $a$) is called the *major axis*, and the shorter is called the *minor axis* (half of which is labeled $b$). The endpoints of both axes are called *vertices* (sometimes the endpoints of the minor axis are called co-vertices). The foci (plural of *focus*) are on the major axis. The above ellipse has equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

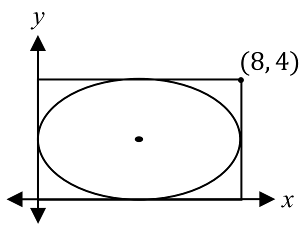
If the major axis were vertical, the equation would be:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

To find the distance we travel from the center to the focus, we find a quantity $c$, by using the equation $c^2 = a^2 - b^2$. Then we move $c$ units in both directions from the center on the major axis to find the foci.

**SAMPLE PROBLEM: ELLIPSES**

In the figure below, the ellipse is inscribed in the rectangle. Which of the following is the equation of the ellipse?




(A) $\frac{(x-4)^2}{16} + \frac{(y-2)^2}{4} = 1$
(B) $\frac{(x+4)^2}{4} + \frac{(y+2)^2}{2} = 1$
(C) $\frac{(x+4)^2}{16} + \frac{(y+2)^2}{4} = 16$
(D) $\frac{(x-4)^2}{2} + \frac{(y-2)^2}{4} = 1$
(E) $\frac{(x+4)^2}{4} + \frac{(y+2)^2}{16} = 1$

To find the equation of the ellipse, we need to find a lot of different numbers: $a$, $b$, $h$, and $k$. The center of the ellipse is at the point $(h, k)$. Because the ellipse is inscribed in the rectangle, we know that the centers of the two shapes are the same. We find the center of the rectangle by moving over 4 units and up 2 units, so the center must be $(4, 2)$. Let's fill in the equation with that information:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-4)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

We're halfway home. Referring back to the picture we drew of a generic ellipse, $a$ is the distance from the center to the edge of the ellipse in the $x$-direction. In this problem, that's the distance from the center to the $y$-axis, which is 4 units. Using the same logic, we find $b$ by moving from the center to the $x$-axis, or 2 units down. Now we can plug those values in and we'll have the equation:

$$\frac{(x-4)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1 \rightarrow$$

$$\frac{(x-4)^2}{4^2} + \frac{(y-2)^2}{2^2} = 1$$

$$\frac{(x-4)^2}{16} + \frac{(y-2)^2}{4} = 1$$

The answer is (A).

If we were asked to find the foci of the ellipse in this problem, we would need to find $c$:

$c^2 = a^2 - b^2 = 16 - 4 = 12.$

So $c^2 = 12$, and $c = 2\sqrt{3}$

Then we would move that far left and right along the major axis from the center, $(4, 2)$:

Foci: $\left(4 + 2\sqrt{3}, 2\right)$ and $(4 - 2\sqrt{3})$

**PRACTICE EXERCISES**

1. Given the function $f(x) = x^2 + x - 2$ and a restricted domain of $-3 \leq x \leq 3$, what is the maximum value of $f(x)$?

2. Find the equation of an ellipse in the standard $(x, y)$ coordinate plane given that the center of the ellipse is at the origin, the vertical major axis is 12 units in length, and the $x$-intercepts are $-4$ and 4.

3. Given a parabola modeled by the function $f(x) = 3x^2 + 3x - 3$, what is the value of $f(-9)$ ?

4. Find the equation of a circle in the standard $(x, y)$ coordinate plane, in center-radius form, given that the center of the circle is $(8, -3)$ and the radius measures 11 units.

5. Find the equation of a circle in the standard $(x, y)$ coordinate plane, in center-radius form, if the end points of the diameter are $(-3, 6)$ and $(3, -2)$.

6. What is the center of the circle in the standard $(x, y)$ coordinate plane with the equation $(x - 4)^2 + (y + 4)^2 = 3$?

7. The function $h(t) = -4t^2 + 40t$ models the flight of a projectile where $t$ represents time in seconds and $h$ represents the height in meters above the ground. How many meters above the ground is the projectile when $t = 4$?

8. Find the equation of an ellipse that has its center at the origin, a vertex at $(8, 0)$, and a minor axis 10 units in length.

9. Find the equation of the circle in the standard $(x, y)$ coordinate plane that has a radius of 2 and the same center as the circle determined by $x^2 + 2x + y^2 - 8y + 3 = 0$.

10. What happens to the graph of an ellipse when $x$ is replaced with $(x - 3)$?

**TEST EXERCISES (TIME: 10 MINUTES)**

11. Which of the following is the equation of the largest circle that can be inscribed in an ellipse with the equation as follows: $\frac{(x+2)^2}{7} + \frac{(y-3)^2}{9} = 1$?

(A) $x^2 + y^2 = 9$
(B) $x^2 + y^2 = 7$
(C) $(x-2)^2 + (y+3)^2 = 7$
(D) $(x+2)^2 + (y-3)^2 = 7$
(E) $(x+2)^2 + (y-3)^2 = 9$

12. The ordered pair $(k, y)$ lies on a circle in the standard $(x, y)$ coordinate plane defined by the equation $x^2 + y^2 = 144$. What is the largest possible value of $k$ in an ordered pair that lies on this circle?

(A) 6
(B) 12
(C) 24
(D) 48
(E) 144

13. What is the length of the major axis of the ellipse defined by the equation: $\frac{x^2}{25} + \frac{y^2}{100} = 1$ ?

(A) 5
(B) 10
(C) 15
(D) 20
(E) 25

14. What is the standard form of the equation for the parabola with a vertex at the origin in the standard $(x, y)$ coordinate plane, and a focus at $(-2, 0)$? (Note: the standard form of a parabola with a horizontal axis is $(y-k)^2 = 4p(x-h)$, with vertex $(h, k)$, focus $(x+p, k)$, with $p$ representing the distance from the vertex to the focus.)

(A) $x^2 = -\frac{1}{8}y$
(B) $x^2 = -8y$
(C) $x^2 = 2y$
(D) $y^2 = -2x^2$
(E) $y^2 = -8x$

15. A circle in the standard $(x, y)$ coordinate plane is tangent to the $y$-axis at 5 and tangent to the $x$-axis at $-5$. Which of the following could be the equation of this circle?

(A) $x^2 + y^2 = 5$
(B) $x^2 + y^2 = 25$
(C) $(x+5)^2 + (y-5)^2 = 5$
(D) $(x+5)^2 + (y-5)^2 = 25$
(E) $(x-5)^2 + (y+5)^2 = 25$

16. Which of the following intersections cannot form a circle? (Note: A single point is not considered a circle.)

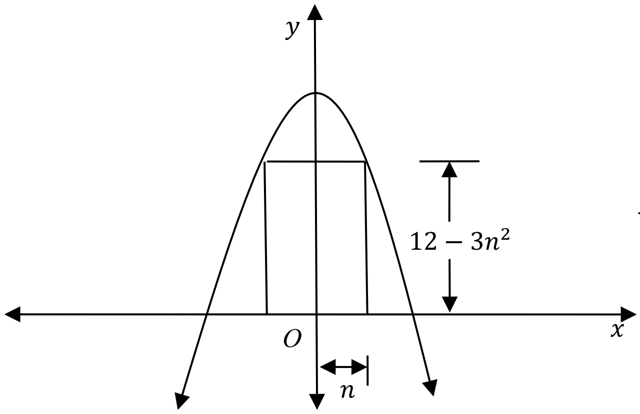
(A) A plane and the surface of a cylinder
(B) A line and the surface of a cylinder
(C) A plane and the surface of a cone
(D) A plane and the surface of a sphere
(E) The surface of a cylinder and the surface of a sphere

17. The graph of $f(x) = x^2 - 4x + 3$ is a parabola and the axis of symmetry is given by the equation $x = 2$. Which of the following ordered pairs represent the point on the parabola that is symmetric to the point $(-1, y)$?

(A) $(0, 3)$
(B) $(1, 0)$
(C) $(3, 0)$
(D) $(4, 3)$
(E) $(5, 8)$

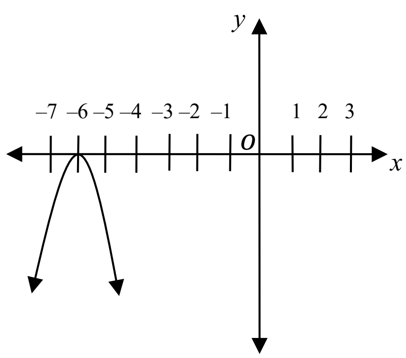
18. A circle in the standard $(x, y)$ coordinate plane is tangent to the $x$-axis and its center is located at the point $(3, -3)$. Where is the point $(0, 3)$ located?

(A) On the circle
(B) On the $x$-axis
(C) Outside the circle
(D) Inside the circle
(E) Cannot be determined from the given information



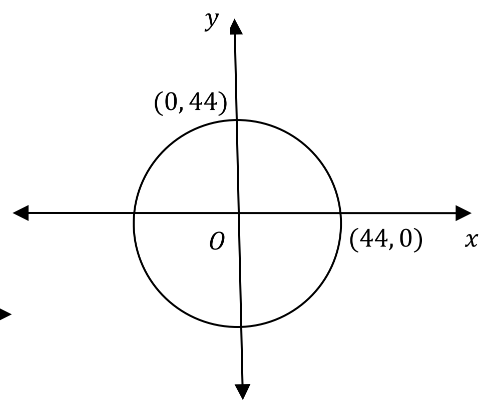

19. In the standard $(x, y)$ coordinate plane as shown above, the base of a rectangle lies on the $x$-axis, and the vertices of the opposite side of the rectangle lie on the parabola modeled by $f(x) = 12 - 3n^2$. Suppose $n$ represents any value of $x$ such that $0 < n < 2$. Which of the following expressions represents the area, in square units, of any such rectangle?

(A) $12n - 3n^3$
(B) $24n - 6n^3$
(C) $9n^2 - 72n + 144$
(D) $-9n^2 - 72n + 144$
(E) $9n^4 - 72n^2 + 144$



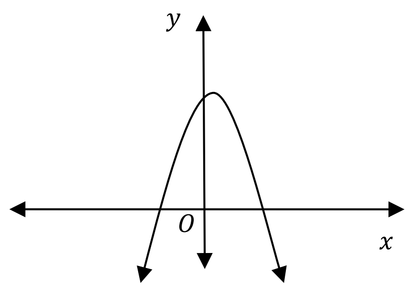

20. The graph of a parabola in the standard $(x, y)$ coordinate plane as shown above is tangent to the $x$-axis at $x = -6$. Which of the following could be the equation for this parabola?

(A) $y = x^2 + 6$
(B) $y = -x^2 - 6$
(C) $y = -(x + 6)^2$
(D) $y = -(x - 6)^2$
(E) $y = x^2 + 6x - 6$




21. What is the equation of the circle in the standard $(x, y)$ coordinate plane as shown above?

(A) $x + y = 44$
(B) $(x + y)^2 = 44$
(C) $(x + y)^2 = 442$
(D) $x^2 + y^2 = 44$
(E) $x^2 + y^2 = 442$




22. The graph of $y = ax^2 + bx + c$ in the standard $(x, y)$ coordinate plane is shown above. When $y = 0$ which of the following best describes the solution set for $x$?

(A) 1 real solutions
(B) 2 real solutions and 1 imaginary solution
(C) 2 real solutions
(D) 1 real solution and 1 imaginary solution
(E) No real solutions