mP8C – Advanced Algebra Word Problems

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Advanced Algebra WP's

One of the most important skills we have for solving math problems is our ability to turn a problem statement into an equation we can solve – translating the English into algebra. Some words translate almost directly into mathematical operations: "of" usually means "multiply," and "is" means "equals," for example. On the other hand, some statements don't translate as easily. In this lesson we'll hone our skills at this by looking at more complicated question than we've seen before.

Sample Problem: Advanced Algebra

Jassy buys a new pick-up truck for $24,000. The value of the truck decreases at a constant rate of 3% each month for the first two years, and then decreases at a rate of 2% each month for the next five years. If she buys the truck in January of 2013, in what month will the value of the truck fall below $10,000?

(A) August 2013
(B) September 2013
(C) December 2014
(D) August 2015
(E) December 2015

To solve this question, we need to write an equation that represents the value of the truck after some number of months. One good strategy for any algebra questions is to first assign a variable to the quantity we're looking for – in this case, we're looking for the number of months. Let's call it a. We need to subtract 3% of the truck's value for each month that passes – we can represent this with an equation:

$$24{,}000 - 24{,}000(0.03) = 23{,}280$$

This is the value of the truck after 1 month. It would take a long time to repeat this over and over until we reach a number that's less than $10,000, so let's try and find a faster way to do it.

First of all, we can simplify the above equation by factoring 24,000 out of each term on the left:

$$24{,}000 - 24{,}000(.03) = 24{,}000(1 - .03) = \underline{24{,}000(0.97)}$$

Think of the underlined expression above a way to write 97% of the value of the truck – if we take away 3%, then we have 97% left. To find the value of the truck a months after its purchase, we'd have to do that calculation a times. To avoid punching this into our calculator a bunch of times, we can use an exponent to do it much more quickly:

$$24{,}000(0.97)^a = \text{value after } a \text{ months}$$

For the first 24 months, this expression will be:

$$24{,}000(0.97)^{24} = 11{,}554.01$$

Now we know that after 24 months, the value of the truck is still greater than $10,000, which means we need to keep going. The problem statement tells us that after 24 months, the truck's value decreases at a slightly lower rate – 2% each month. Using the same logic as above, we can write an expression for the value of the truck b months *after the 24th month*:

$$\text{Value of the truck} = 11{,}554.01(0.98)^b$$

We want this expression to be less than $10,000, but how do we figure out what b is? One way is to use logarithms, but the ACT won't test your ability to solve equations that way. Another way is to try a couple values of b and "home in" on the right answer. We can use a systematic approach. Let's start with $b = 10$ months, and then increase or decrease it by 2 months depending on the answer we get:

$$11554.01(.98)^{10} = 9440.47$$

This number is smaller than $10,000, but it may not be the first month when this happens. So let's decrease b by 2:

$$11{,}554.01(0.98)^8 = 9829.72$$

We're still lower than $10,000, but we're closer than before. Let's try $b = 7$:

$$11{,}554.01(.98)^7 = 10{,}030.33$$

Now we're above the $10,000 mark. This means that the first month in which the value of the truck is less than $10,000 is the 8th month. But wait! It's really not the 8th month, because we accounted for the first 24 months earlier. So it's really the $(24+8)^{\text{th}}$ month, or the 32^{nd} month. Therefore the value of the truck falls below $\$10{,}000$ 32 months after January of 2013, which will be August of 2015. The answer is (D).

In order to solve this problem in a reasonable amount of time, we needed to use our knowledge of exponents, and write an equation based on the words in the problem statement. Another common type of word problem involves systems of equations. We'll be able to recognize these questions because they'll have more than one thing that we don't know.

SAMPLE PROBLEM: ADVANCED ALGEBRA

Jillian's coin purse contains only nickels, dimes and quarters. The number of quarters is seven less than twice the number of dimes, and the number of quarters is ten less than the number of nickels. She takes her purse to the bank, and they give her $\$11.70$ back. How many nickels did she have?

(A) 19
(B) 29
(C) 31
(D) 41
(E) 59

Because we're being asked to find more than one quantity, we can guess that we will need to write and solve a system of equations. Luckily, the problem statement gives us all the information we need to do this. First, let's assign some variables – we'll call the number of quarters q, the number of dimes d, and the number of nickels n. Now, look at the second sentence:

"The number of quarters is seven less than twice the number of dimes, and the number of quarters is ten less than the number of nickels."

The first part of this sentence can be translated into an equation. "The number of quarters" is q, "is" means "equals," "seven less than" means we'll be subtracting seven from a number, and "twice the number of dimes" is $2d$. Here's how to put it all together:

$q = 2d - 7$

(Notice that the phrase "seven less than $2d$" is translated into "$2d - 7$," not "$7 - 2d$")

The second part of that sentence can also be made into an equation – "the number of quarters" is still q, "ten less than" means we'll be subtracting ten from something, and "the number of nickels" is n:

$q = n - 10$

Now we have two equations, but we have three variables, which means we need a third equation. We can use the total amount of money to write the third equation. Because each quarter is worth 25 cents, we can write the value of the quarters in Jillian's purse as $25q$. The value of the dimes is $10d$, and the value of the nickels is $5n$. Add these together and we get the total amount of money in her purse. This will be our third equation:

$25q + 10d + 5n = 1170$

Notice that we used 25 as the value of a quarter, 10 as the value of a dime and 5 the value of a nickel – which means the numbers are in cents, not dollars. That's why we changed $\$11.70$ into 1170 cents.

Here are all three equations together:

$q = 2d - 7$

$q = n - 10$

$25q + 10d + 5n = 1170$

We can use substitution to solve this system. Because the first two equations both have q in them, we'll solve them *d* and *n* and substitute those expressions into the third equation:

$q = 2d - 7 \quad \rightarrow \quad q + 7 = 2d \quad \rightarrow \quad \boldsymbol{d = \frac{q+7}{2}}$

$q = n - 10 \quad \rightarrow \quad \boldsymbol{n = q + 10}$

$25q + 10\boldsymbol{d} + 5\boldsymbol{n} = 1170$

$25q + 10\left(\boldsymbol{\frac{q+7}{2}}\right) + 5(\boldsymbol{q + 10}) = 1170$

$25q + 5q + 35 + 5q + 50 = 1170$

$35q + 85 = 1170 \quad \rightarrow \quad 35q = 1085 \quad \rightarrow \quad q = 31$

There were 31 quarters in her purse. Now we can use that to find the value of n:

$n = q + 10 = 31 + 10 = 41$

There were 41 nickels. The answer is (D).

SAMPLE PROBLEM: ADVANCED ALGEBRA

The formula for finding the distance d an object moving at speed v_0 travels while undergoing an acceleration a for a time t is

$$d = v_o t + \frac{1}{2}at^2$$

Which of the following is an expression for a in terms of d, v_0, and t?

(A) $\frac{2(d-v_0 t)}{t^2}$

(B) $\sqrt{\frac{2(d-v_0 t)}{t}}$

(C) $2(d - v_0 t)$

(D) $2(d - v_0 t)t^2$

(E) $\frac{2(d-v_0)}{t}$

A problem like this has a pretty straightforward approach – all we're doing is solving for one of the variables. First, move all the terms that don't have an a to one side, and everything else to the other side. In this example, we subtract $v_0 t$ from both sides:

$$d = v_o t + \frac{1}{2}at^2 \quad \rightarrow \quad d - v_0 t = \frac{1}{2}at^2$$

Now we need to isolate the a. First multiply both sides by 2 to eliminate the fraction:

$$d - v_0 t = \frac{1}{2}at^2 \quad \rightarrow \quad 2(d - v_0 t) = at^2$$

Now we divide by t^2 and the problem is finished!

$$2(d - v_0 t) = at^2 \quad \rightarrow \quad a = \frac{2(d-v_0 t)}{t^2}$$

The answer is (A).

Another way to solve a problem like this is to pick numbers for the unknown quantities and plug them in. Be careful using this strategy, though. It can only be used when the problem statement and answer choices contain only variables – if they give you any numbers, it won't work.

Because we're trying to find an expression for a, we pick numbers for all the other quantities and plug them into the given expression. This will give us a numerical value for a. Then we'll plug those same numbers into the answer choices and see which one gives us the same answer. Let's pick some numbers:

$$d = 15, v_o = 6, t = 10$$

It's a good idea to chose numbers that aren't multiples or factors of each other. Now we'll plug them into the equation we were given:

$$d = v_o t + \frac{1}{2}at^2 \quad \rightarrow \quad 15 = 6 \cdot 10 + \frac{1}{2}a \cdot 10^2$$

$$15 = 60 + 50a \quad \rightarrow \quad a = -0.9$$

Now we plug the same numbers into the answer choices and see which one matches:

(A) $\frac{2(d-v_0 t)}{t^2} = \frac{2(15-60)}{10^2} = \frac{-90}{100} = -0.9$

It looks like (A) was the right answer after all! But be careful – when you choose this method, you need to plug your choices into the other answers as well – if two answers give you the same number you need to pick new numbers and try again:

(B) $\sqrt{\frac{2(d-v_0 t)}{t}} = \sqrt{\frac{2(15-60)}{10}} = \sqrt{\frac{-90}{10}} =$ undefined

(C) $2(d - v_0 t) = 2(15 - 60) = -90$

(D) $2(d - v_0 t)t^2 = 2(15 - 60)10^2 = -900$

(E) $\frac{2(d-v_0)}{t} = \frac{2(15-6)}{10} = \frac{18}{10} = 1.8$

Now it's clear that (A) is the only answer choice which matches what we got the first time. This is a method which works best if you don't know how to proceed from reading the question statement. Remember: this only works if all the quantities you're given are variables.

SAMPLE PROBLEM: ADVANCED ALGEBRA

The function $a \leftrightarrow b$ is defined as $\frac{a^2}{2b^2}$ for all a and b.
What is the value of $(2x-2) \leftrightarrow (x-1)$?

(A) $\frac{(2x-2)^2}{x-1}$
(B) $\sqrt{x-2}$
(C) $(2x-2)^2$
(D) 4
(E) 2

The symbol that appears in this question ($\leftrightarrow$) isn't really a special function – it's only defined for this one problem. There's a good chance you'll never see it again. All it does is tell us the rule for this particular question. So we'll just plug in the values we're given and simplify (the simplifying is often the most difficult part of these problems):

$a \leftrightarrow b = \frac{a^2}{2b^2} \quad \rightarrow$

$(2x-2) \leftrightarrow (x-1) = \frac{(2x-2)^2}{2(x-1)^2}$

$\frac{(2x-2)^2}{2(x-1)^2} = \frac{4x^2-8x+4}{2(x^2-2x+1)} = \frac{4(x^2-2x+1)}{(x^2-2x+1)} = \frac{4}{2} = 2$

The answer is (E)

A shortcut for the last example would be to notice that the first expression was exactly twice the second expression and to factor it before plugging it in:

$2(x-1) \leftrightarrow (x-1) = \frac{[2(x-1)]^2}{2(x-1)^2} = \frac{4(x-1)^2}{2(x-1)^2} = 2$

This saves us the trouble of multiplying out the top and bottom; while you might not find that to be difficult, any time we can avoid the potential to make mistakes we should take it!

PRACTICE EXERCISES

1. Josh is 5 years older than Sarah was when Josh was born. Write an equation for Sarah's age in terms of Josh's age.

2. Hui has three nickels for every four dimes. Write an equation for the number of nickels Hui has in terms of the number of dimes.

3. A parking garage costs \$4 for the first hour and \$2 for every hour thereafter. Write an equation for the cost of parking in terms of the total number of hours parked.

4. The larger of two numbers exceeds the smaller by three less than half the larger. Write an equation for the larger number in terms of the smaller one.

5. The population of a bacterial colony was P_0 at the start of an experiment and has doubled every 3 hours since then. Write an equation for the population now in terms of the number of hours since the experiment began.

6. A savings account earns 0.2% interest each month. Write an equation for the amount of money in the account after t years if the starting balance is $\$500$.

7. The Celsius temperature scale is based on the freezing point of water, which is defined as 32° Fahrenheit or 0° Celsius. One degree on the Celsius scale is the same temperature increment as 1.8 degrees on the Fahrenheit scale – thus, if the air temperature increases by x°C, it increases by $1.8x$°F. Write a formula for the outdoor air temperature in Celsius in terms of the same temperature in Fahrenheit.

8. The length of time, t (in seconds), for a complete swing of a simple pendulum can be modeled by the equation $t = 2\pi\sqrt{\frac{L}{40}}$, where L is the length, in feet, of the string. Write an equation for the length of the string in terms of t.

9. The length of a hallway, measured in inches, is i. Measured in feet, the length of the same hallway is f. Tom thus concludes that $f = 12i$. Is Tom correct? Explain.

10. Felicia has pennies, dimes, and quarters in the ratio $2:3:5$. She writes the equation $2p = 3d = 5q$ to represent this situation. Is Felicia correct? Explain.

11. Iris buys a pounds of coffee priced at x dollars per pound and b pounds of coffee priced at y dollars per pound. Write an equation for the total amount she spends on coffee in terms of a, b, x, and y.

12. The function f is defined such that $f(x) = -\frac{x^2}{4}$. Vincent defines a new operation ϕ such that $x\phi y = \frac{f(x)}{f(y)}$. What is $n\phi\sqrt{n}$ in terms of n?

TEST EXERCISES (TIME: 10 MINUTES)

13. Kallie has nickels, dimes, and quarters in her pocket in a ratio of $5:4:3$. The number of nickels is 6 less than the sum of the number of dimes and the number of quarters. How much money does Kallie have in nickels, dimes, and quarters?

(A) $\$1.40$
(B) $\$2.80$
(C) $\$4.20$
(D) $\$5.60$
(E) $\$7.00$

14. Beth defines a new operation, $\otimes$, as follows: $a \otimes b = ab - \frac{a}{b}$. If $6 \otimes n = 9$, then what is the sum of all possible values of n?

(A) -1.5
(B) 0
(C) 1.5
(D) 2.0
(E) 2.5

15. At Rio Vista High School, the number of seniors is $\frac{4}{5}$ of the number of juniors, the number of sophomores is $\frac{2}{3}$ of the number of juniors, and there are twice as many juniors as freshmen. If there are 356 students at the school, how many are sophomores?

(A) 60
(B) 80
(C) 96
(D) 120
(E) 180

16. The Ideal Gas Law states that the product of the pressure P and volume V of a sample of gas is equal to the product of the amount n of the gas (in moles), the temperature T, and a constant R: this law is thus usually written $PV = nRT$. A sample of hydrogen gas at constant temperature is moved from a spherical container of radius 8 cm to a smaller spherical container, and the pressure of the sample doubles as a result. What is the radius of the smaller sphere to the nearest hundredth of a centimeter?

(A) 2.00
(B) 2.83
(C) 4.00
(D) 5.04
(E) 6.35

17. A taxi driver charges a flat rate of $\$4.00$ for the first half mile plus $\$0.50$ for each $\frac{1}{10}$ of a mile after that. Which of the following is a formula for the number of miles traveled, m, in terms of the total dollar cost C of a ride that is at least half a mile long?

(A) $m = 4 + 0.05C$
(B) $m = 4 + 0.05(C - 4)$
(C) $m = 0.2(C - 1.5)$
(D) $m = 0.2(C - 4)$
(E) $m = 20(C - 4)$

18. At the start of 2001, Grace invested $\$1000$ in a savings bond which pays annual compound interest. At the start of 2002, she invested another $\$1000$ in an identical bond. At the start of 2003, her total investment was worth $\$2106.23$. What was the interest rate of the two bonds?

(A) 2.0%
(B) 2.5%
(C) 3.0%
(D) 3.5%
(E) 4.0%

19. The product of a number and that number divided by 4 is equal to 4 less than the sum of that number and 3. What is that number?

(A) 1
(B) 2
(C) 3
(D) 5
(E) 7

20. The electrical resistance, R, in ohms, of a length of wire can be approximated by the model $R = \frac{10{,}850}{d^2} - 0.34$ for any wire diameter, d, in mils (1 mil = 0.001 inch), such that $5 \leq d \leq 100$. What is the approximate diameter, in *inches*, of a wire with a resistance of 12 ohms?

(A) 2.97×10^{-3}
(B) 2.97×10^{-2}
(C) 7.5×10^{-2}
(D) 8.7×10^{-1}
(E) 75.0

21. Andrew is half as old as Lauren will be 16 years from now. Four years from now, Lauren will be 20% older than Andrew. What is the sum of their current ages?

    (A) 36
    (B) 37
    (C) 42
    (D) 47
    (E) 48

22. *Inflation* is a decrease over time in the value of a currency, leading to a corresponding *increase* in the price of goods and services. An *inflation-adjusted price* is an estimate of what the past price of a product would be measured in today's dollars. It is calculated using the formula:

$$I.A.P. = P \times \left(1 + \frac{r}{100}\right)^t$$

where P is the price of the product in some past year, t is the number of years elapsed since then, and r is the annual percentage rate of inflation. If a loaf of bread cost $\$0.45$ in 1970, an identical loaf cost $\$1.80$ in 2010, and the rate of inflation between 1970 and 2010 was 2% per year, by how much does the price of the 2010 loaf of bread exceed the inflation-adjusted price of the 1970 loaf?

    (A) $\$0.33$
    (B) $\$0.54$
    (C) $\$0.81$
    (D) $\$1.47$
    (E) $\$5.42$