mP4A – Exponents/Roots

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Exponents and Roots

You may remember seeing exponents and roots in your studies previously. Recall that exponents are a shorthand way of writing repeated multiplication. The numbers or variables in the upper right of the expression are called *exponents*. The number or variable to the left and below the exponent is called the *base. Powers* are numbers written with exponents. There are some basic rules that you need to remember – we'll show them here for reference:

(1) $x^a x^b = x^{a+b}$ (2) $\frac{x^a}{x^b} = x^{a-b}$

(3) $(x^a)^b = x^{ab}$ (4) $x^a y^a = (xy)^a$

(5) $x^0 = 1$ (6) $x^{-a} = \frac{1}{x^a}$

Sample Problem: Rules of Exponents

Simplify the expression $\frac{(x^2)^{-2}y^3z^{-4}}{x^4y^{-5}z^4}$

To simplify these expressions, our first step should be to get rid of all the parentheses. In this question, there's only one set – around the x^2. We remove it by using rule (3):

$$\frac{(x^2)^{-2}y^3z^{-4}}{x^4y^{-5}z^4} = \frac{x^{-4}y^3z^{-4}}{x^4y^{-5}z^4}$$

Now we need to combine the exponents for each variable. Because we have x, y, and z terms on the top and the bottom, we'll use rule (2) for each:

$$\frac{x^{-4}y^3z^{-4}}{x^4y^{-5}z^4} = x^{-4-4}y^{3-(-5)}z^{-4-4} = x^{-8}y^8z^{-8}$$

The next step is to make all the exponents positive – we do this by putting the negative exponents in the denominator, and the positive exponents in the numerator, using rule (6):

$$x^{-8}y^8z^{-8} = \frac{y^8}{x^8z^8}$$

The last step in this example is to use rule (4) to combine the terms that have the same exponent – here each exponent is 8, so we can combine them into one big term:

$$\frac{y^8}{x^8z^8} = \left(\frac{y}{xz}\right)^8$$

This is the final answer.

Exponential equations are equations in which the variable is in the exponent. Many of these can be solved by using the rules of exponents. The basic idea is expressed by the rule below:

$$\text{If } x^a = x^b, \text{then } a = b$$

This means that if we have two expressions with the same base, then the exponents are equal to each other. Here's an example that will make this clearer:

Sample Problem: Exponential Equation

Solve for x: $3^x 9^{2x+3} = 3^{7x}$

To solve this equation, we need to get each side to be expressions with the same base. Because we recognize 9 as a power of 3, we can re-write it as such:

$3^x 9^{2x+3} = 3^{7x}$
$3^x (3^2)^{2x+3} = 3^{7x}$

Now we can apply the various rules for exponents to get each side to be a simple power of 3:

$3^x (3^2)^{2x+3} = 3^{7x}$
$3^x 3^{4x+6} = 3^{7x}$
$3^{5x+6} = 3^{7x}$

Now we have the situation presented in the rule above – two expressions with the same base. We can set the exponents equal to each other and solve:

$5x + 6 = 7x$
$6 = 2x$
$x = 3$

Remember that to use this rule, you must have the exact situation indicated in the rule above: one term on each side of the equation that is a power of the same base.

We've seen what positive and negative exponents do, but what about exponents that are fractions? These are called *rational* exponents, and they represent the last type of exponent we'll need to study in order to be ready for the ACT. A rational exponent denotes both a *power* and a *root*.

$$x^{\frac{2}{3}} = \sqrt[3]{x^2}$$

The numerator of the fraction becomes the power, and the denominator becomes the root. It can also be written this way:

$$x^{\frac{2}{3}} = \sqrt[3]{x}^2$$

It doesn't matter whether you do the root or the power first, the answer will turn out the same.

SAMPLE PROBLEM: RATIONAL EXPONENTS

Evaluate $9^{\frac{3}{2}}$

Based on our picture above, we can write this one of two ways:

Either $\sqrt{9^3}$ or $\sqrt{9}^3$

Evaluating either of these should give us the same answer:

$\sqrt{9^3} = \sqrt{729} = 27$ or $\sqrt{9}^3 = 3^3 = 27$

And they do.

SAMPLE PROBLEM: RATIONAL EXPONENTS

Evaluate $8^{-\frac{4}{3}}$

In this example, we need to use our rule for negative exponents as well as the rules for rational exponents. The negative tells us that we need to put the expression in the denominator of a fraction first:

$8^{-\frac{4}{3}} = \frac{1}{8^{\frac{4}{3}}}$

Now apply the rules for rational exponents: the numerator becomes the power and the denominator becomes the root:

$\frac{1}{8^{\frac{4}{3}}} = \frac{1}{\sqrt[3]{8}^4} = \frac{1}{2^4} = \frac{1}{16}$

The final topic we'll discuss relating to exponents is a more in-depth look at scientific notation. Scientific notation is a way of writing very large (or very small) numbers using a number between 1 and 10 and a power of 10. The number between 1 and 10 is referred to as the *mantissa*, the *significand,* or the *coefficient*. To avoid confusing the different parts of the notation, for the rest of this lesson we'll call this number the coefficient. To multiply (or divide) numbers in scientific notation, multiply (or divide) the coefficients, and then multiply (or divide) the powers of ten. If necessary, adjust the coefficient so that the answer is back in scientific notation.

SAMPLE PROBLEM: SCIENTIFIC NOTATION

$(1.2 \times 10^{-15})(9.6 \times 10^{9}) = ?$

First we'll multiply the coefficients, and then we'll multiply the powers of 10:

$$\begin{aligned}(1.2 \times 9.6)(10^{-15}10^{9}) &= (11.52)(10^{-15+9}) \\ &= 11.52 \times 10^{-6}\end{aligned}$$

Our last step is to convert this number back into scientific notation: the coefficient is larger than 10. We do this by moving the decimal point left or right until the number is the correct size. Here we have to move the decimal point once to the left:

$11.52 \times 10^{-6} \rightarrow 1.152 \times 10^{-5}$

Because changing a number from 11.52 to 1.152 is making it *smaller*, we need to make the power of ten *bigger* to keep the entire expression the same size. That's why the exponent changed from -6 to -5. Another way to think about it is that we divide the coefficient by ten, so we have to multiply the power by ten.

SAMPLE PROBLEM: SCIENTIFIC NOTATION

$(2.4 \times 10^{12}) \div (7.2 \times 10^{-4}) = ?$

As before, we need to divide the coefficients and divide the powers of ten:

$(2.4 \times 10^{12}) \div (7.2 \times 10^{-4}) =$
$(2.4 \div 7.2)(10^{12} \div 10^{-4}) =$
$0.33 \times 10^{12-(-4)} =$
0.33×10^{16}

We need to get the coefficient of this number between 1 and 10. To do this, we have to move the decimal point once to the right. Because this is making the coefficient *bigger*, we need to make the power of ten *smaller*:

$0.33 \times 10^{16} = 3.3 \times 10^{15}$

To add and subtract with scientific notation, we use a different method than we did for multiplication and division. First, we'll change the powers of 10 to be the same power. Then we can add (or subtract) the coefficients. Finally, we'll change the answer back into scientific notation. We'll look at a couple of examples to see this process in action.

SAMPLE PROBLEM: SCIENTIFIC NOTATION

$1.5 \times 10^9 + 7.2 \times 10^8 = ?$

The first step is to change the powers of ten so that they are the same. We do this by changing one of the two coefficients as well as the power of 10 it is attached to – remember that if we make the coefficient *smaller*, we need to make the power of 10 *bigger*, and vice versa. We'll do this problem both ways.

First, we'll get both powers to be 10^8. To do this, we'll need to make the first power of 10 smaller, which means we need to make its coefficient bigger:

$1.5 \times 10^9 = 15 \times 10^8$

Now we can perform the addition by adding the coefficients and keeping the power of ten the same:

$1.5 \times 10^9 + 7.2 \times 10^8 =$
$15 \times 10^8 + 7.2 \times 10^8 =$
$(15 + 7.2) \times 10^8 =$
22.2×10^8

Before we finish, we'll need to change this back into scientific notation – it's no longer in the proper format because the decimal number, 22.2, is not between 1 and 10.

We'll divide 22.2 by 10 and multiply 10^8 by 10:

$22.2 \times 10^8 = 2.22 \times 10^9$

We could also have chosen to make both powers of 10 equal to 10^9 instead:

$7.2 \times 10^8 = 0.72 \times 10^9$

$1.5 \times 10^9 + 7.2 \times 10^8 =$
$1.5 \times 10^9 + 0.72 \times 10^9 =$
$(1.5 + 0.72) \times 10^9 =$
2.22×10^9

We got the same answer both times, which is reassuring.

SAMPLE PROBLEM: SCIENTIFIC NOTATION

$5.2 \times 10^{-7} - 1.4 \times 10^{-9} = ?$

This problem has subtraction instead of addition, and it has negative exponents instead of positive ones. We'll still use the same process to answer the question, though: change the powers of ten to match each other, subtract the coefficients, and convert the answer back into scientific notation.

If we change the first number's power of ten to be equal to 10^{-9}, we'll have to make the coefficient larger (we're making the power of ten *smaller*):

$5.2 \times 10^{-7} = 520 \times 10^{-9}$

Notice that because the power of ten changed by 2, we had to move the decimal point *twice* in the coefficient.

Now we can subtract the coefficients:

$520 \times 10^{-9} - 1.4 \times 10^{-9} = 518.6 \times 10^{-9}$

The final step is to change the number back to scientific notation – we need to make the coefficient *smaller*, so we'll make the power of ten *larger*:

$518.6 \times 10^{-9} = 5.186 \times 10^{-7}$

Like the previous example, we also could have changed the power of ten attached to the second number given:

$1.4 \times 10^{-9} = 0.014 \times 10^{-7}$

$5.2 \times 10^{-7} - 0.014 \times 10^{-7} =$
$(5.2 - 0.014) \times 10^{-7} =$
5.186×10^{-7}

PRACTICE EXERCISES

For questions 1-4, simplify the expression:

1. $\frac{ab^{-1}}{a^{-2}b}$

2. $(9x^4y^{-2})^{\frac{3}{2}}$

3. $\frac{x^3y^{-4}+y^3x^{-4}}{x^{-4}y^{-4}}$

4. $\left(\frac{p^3q^{-3}r^6}{p^{-1}q^5r^4}\right)^{-\frac{1}{2}}$

For questions 5-8, apply the indicated operation, and leave your answer in scientific notation:

5. $2 \times 10^6 + 8 \times 10^5$

6. $6 \times 10^{-4} - 6 \times 10^{-5}$

7. $3.2 \times 10^{10} \cdot 2.5 \times 10^{-7}$

8. $1.0 \times 10^{-6} \div 5.0 \times 10^3$

For questions 9-12, solve the equation for all possible values of x:

9. $3^x = \left(\frac{1}{9}\right)^{x-9}$

10. $25^x = 125^{2x}$

11. $4^{5x} = 8^{3x-3} + 512^{x-1}$

12. $27^{7x+9} \cdot 4^{6x} = 8^{9x+3} \cdot 9^{3x+9}$

TEST EXERCISES (TIME: 10 MINUTES)

13. $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} = ?$

(A) $\frac{15}{16}$
(B) $\frac{31}{32}$
(C) $\frac{1}{128}$
(D) -14
(E) -31

14. For all $x \neq 0$ and $y \neq 0$, $\frac{x^y}{x^{-y}} = ?$

(A) 1
(B) x
(C) $\frac{1}{x}$
(D) x^{y^2}
(E) x^{2y}

15. $\left(\left(\left(3^{\frac{3}{2}}\right)^{\frac{4}{3}}\right)^{\frac{5}{4}}\right)^{\frac{6}{5}} = ?$

(A) $\frac{1}{27}$
(B) $\frac{1}{3}$
(C) 9
(D) 27
(E) $\frac{317}{60}$

16. For all nonzero x, y, and z values, $\frac{xy^2z^{-3}}{x^{-2}y^3z^{-4}} = ?$

(A) $\frac{z}{xy}$
(B) $\frac{x^3z}{y}$
(C) $\frac{x^3}{yz}$
(D) $\frac{1}{xyz}$
(E) $\frac{x^3z^7}{y}$

17. What is the area, in square meters, of a square whose sides are each 6.0×10^9 meters long?

(A) 6.0×10^{18}
(B) 6.0×10^{81}
(C) 1.2×10^{19}
(D) 3.6×10^{19}
(E) 3.6×10^{82}

18. $\sqrt{2} \times \sqrt[3]{4} \times \sqrt[4]{8} \times \sqrt[12]{2} = ?$

(A) 2
(B) $2 \times \sqrt[12]{2}$
(C) 4
(D) $4 \times \sqrt[12]{2}$
(E) $8\sqrt{2}$

19. If $y = x^{-\frac{1}{2}}$, $z = y^{\frac{1}{3}}$, and $\sqrt[12]{x} = 3$, what is the value of z ?

(A) $\frac{1}{3}$
(B) $\frac{1}{9}$
(C) 3
(D) 9
(E) 729

20. If q is a real number such that $q^{-\frac{5}{3}} = 243$, then $\sqrt[3]{q} - \sqrt{q} = ?$

(A) $-9 - \sqrt{3}$
(B) $\frac{\sqrt{3}}{9}$
(C) $\frac{3-\sqrt{3}}{9}$
(D) $3 - 3\sqrt{3}$
(E) $\sqrt{3} - 1$

21. How many positive integer values of x exist such that $x^{\frac{3}{2}}$ is an integer less than 100?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

22. Which of the following sets includes all possible values of x that satisfy the equation $x^{x+8} = x^{2x+3}$?

(A) $\{-1, 0, 1\}$
(B) $\{-1, 0, 3\}$
(C) $\{1, 3, 5\}$
(D) $\{0, 1, 5\}$
(E) $\{-1, 1, 5\}$