mP9A – Circles: Arcs/Sectors

Directions: Read the explanation for each problem type mentioned below. Pay special attention to the methods and techniques used to solve the sample problems. Then, do the practice exercises that follow; use the appropriate method to solve each problem.

Topic Overview: Arcs/Sectors

Before we begin talking about arcs and sectors, we need to learn a new way to measure angles. The standard measure of a complete circle is $360°$ – a number that was set up a long time ago because of its multitude of factors. But it's an artificial way to measure angle, like 12 inches in a foot. A more natural way to measure angles is by using *radians*. Using radian measure, a complete circle measures 2π radians, a straight angle measures π radians, and a right angle $\frac{\pi}{2}$ radians. We can convert between degrees and radians by multiplying an angle by one of the following conversion factors:

radians to degrees: $\frac{180°}{\pi \text{ radians}}$

degrees to radians: $\frac{\pi \text{ radians}}{180°}$

Sample Problem: Radian Measure

Evaluate the following functions:

(a) $\sin\frac{\pi}{6}$

(b) $\tan\frac{\pi}{4}$

(c) $\cos\frac{\pi}{2}$

(a) Use the conversion factor to change this to degrees:

$\frac{\pi}{6}\text{radians} \times \frac{180°}{\pi \text{ radians}} = \frac{180°}{6} = 30°$

$\sin 30° = \mathbf{\frac{1}{2}}$

(b) $\frac{\pi}{4}\text{radians} \times \frac{180°}{\pi \text{ radians}} = \frac{180°}{4} = 45°$

$\tan 45° = \mathbf{1}$

(c) $\frac{\pi}{2}\text{radians} \times \frac{180°}{\pi \text{ radians}} = \frac{180°}{2} = 90°$

$\cos 90° = \mathbf{0}$

Eventually we may get used to the measures of angles in radians, but if we don't, we can always use the conversion. Notice that because π radians $= 180°$, in each of those conversions we simply divided 180 by the number on the bottom of the fraction!

Sample Problem: Radian Measure

Find the measure in radians of the hands of a clock at $1{:}45$.

(A) $\frac{19\pi}{24}$

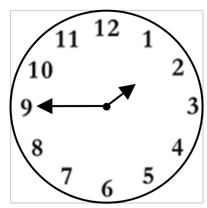
(B) $\frac{15\pi}{24}$

(C) $\frac{17\pi}{12}$

(D) $\frac{3\pi}{4}$

(E) $\frac{4\pi}{7}$

It will be helpful to draw a picture of the clock to figure out what the question is asking for – sometimes drawing a picture can help us immediately eliminate some of the answer choices.



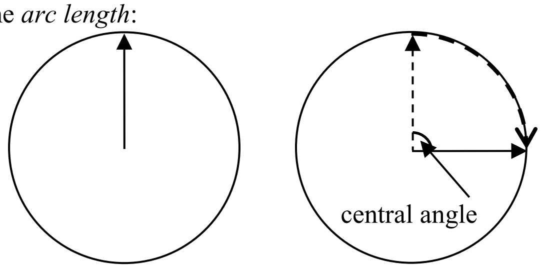
Here's a clock face at $1{:}45$ – notice that the hour hand isn't pointing at the one anymore – it's some distance between 1 and 2! First we need to know the angle between any two numbers on the clock face. Because the entire circle is $360°$, and there are 12 numbers evenly spaced on it, the numbers must be $30°$ apart. But since we're doing this question in radians, we'll say the entire circle is 2π radians, and each number is $\frac{\pi}{6}$ apart. That means that as we move from the 9 to the 1, we'll pass four numbers, so the total angle there is $4 \times \frac{\pi}{6} = \frac{2\pi}{3}$ radians. But how far past the 1 has the hour hand moved so far?

Because there are 60 minutes in an hour, after 45 minutes have pass we are ¾ of the way through the hour – therefore the hand has moved ¾ of the way between the 1 and the 2. $\frac{3}{4}$ of $\frac{\pi}{6}$ is $\frac{\pi}{8}$, so the total angle passed by the hand is $\frac{2\pi}{3}+\frac{\pi}{8}=\frac{19\pi}{24}$. The answer is (A).

We already know that given any circle, we can find the circumference of the circle by using the formula $C = 2\pi r$, and we can find the area by using the formula $A = \pi r^2$. Some problems may ask us to find fractions of these quantities, called *arc length* and *sector area*, respectively. In this lesson we'll look at the formulas for these and the types of problems that we may encounter.

<u>Arc Length:</u>

Imagine a clock at $12:00$ noon. 15 minutes later, the minute hand has moved from the 12 to the 3. If we traced the path followed by the tip of the minute hand, it would be an *arc*, and we have a formula that can tell us the *arc length*:



$$\text{Arc Length} = \frac{\text{angle}}{360} \times \boldsymbol{2\pi r}$$

The angle in the formula is the central angle, which is, unsurprisingly, measured at the center of the circle, shown in the diagram above. Notice that part of the formula is the same as the circumference formula – this makes sense because we're looking at a fraction of the circumference.

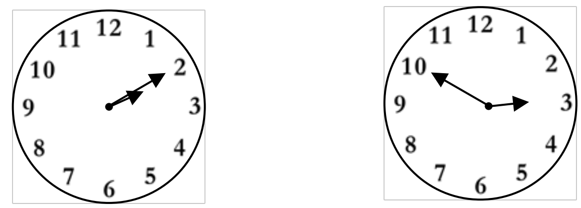
Notice that there are two arcs that can be drawn in the picture above – a shorter one that measures $90°$, and a longer one that measures $270°$. The smaller arc is referred to as a *minor* arc, and the larger is called a *major* arc.

SAMPLE PROBLEM: ARC LENGTH

What is the arc length traced out by the minute hand of a clock from $2:10$ to $2:50$ if the length of the minute hand is 9 inches?

(A) 18π
(B) 16π
(C) 12π
(D) 12
(E) 8π

If we look at the formula, we can see that the only two variables we don't know are the angle and the radius. In this problem, we're given the radius, so we need to find the angle. Think about what a clock face looks like:



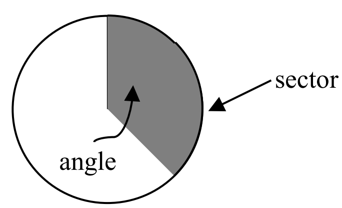
The picture on the left shows the clock face at $2:10$, while the picture on the right shows it at $2:50$. We need to know the measure of the angle between the 2 and the 10. We need to move 8 numbers, and we saw in a previous example that the angle between each number on the clock face is 30º. Therefore the total angle must be $8 \times 30º = 240º$. Now we can use the formula:

$$\text{Arc Length} = \frac{\text{angle}}{360} \times 2\pi r = \frac{240}{360} \times 2\pi \times 9 = \frac{2}{3} \cdot 18\pi = 12\pi$$

The answer is (C).

<u>Sector Area:</u>

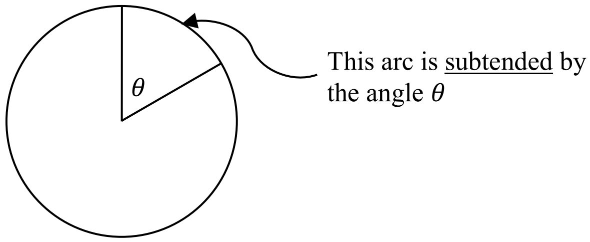
If a circle is measured to be 24 inches across, we can easily find its area by using the formula $A = \pi r^2$. But what if we want to know the area of just a part of the circle? This shape is called a *sector,* and its area is called *sector area*. We have a formula for this that's very reminiscent of the arc length formula:



$$\text{Sector Area} = \frac{\text{angle}}{360} \times \boldsymbol{\pi r^2}$$

Notice that this formula has the formula for area in it, just like the formula for arc length has the formula for circumference.

A word that is often used when discussing arc length and sector area is *subtended*. If an arc is subtended by an angle, it means that the arc is intercepted by that angle:

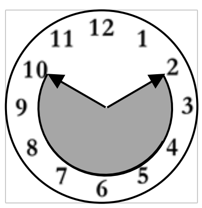


SAMPLE PROBLEM: SECTOR AREA

Find the area swept out by the minute hand in the previous example.

(A) 18
(B) 18π
(C) 24π
(D) 27
(E) 54π

We'll draw the picture of what the question is asking for before we apply the formula:



We already know the angle is 240º. So we can plug our numbers into the formula:

$$\text{Sector Area} = \frac{\text{angle}}{360} \times \pi r^2 = \frac{240}{360} \times \pi \times 81 = 54\pi$$

The answer is (E)

Usually we're not given problems where we're explicitly given the radius or the angle – we have to figure those things out as well. But at the end of the day, any problem that asks us for an arc length or a sector area will use one of the two formulas we just learned.

SAMPLE PROBLEM: ARC LENGTH

What is the length of the arc that the tip of the minute hand of a clock passes through between $3{:}30$ P.M. and $5{:}25$ P.M.?

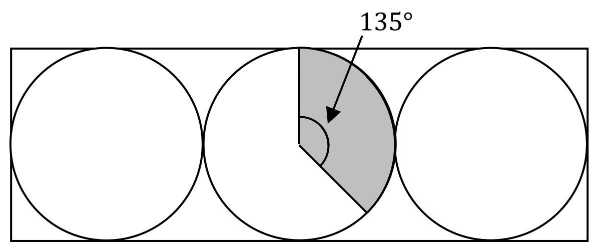
(A) $720°$
(B) $690°$
(C) $640°$
(D) $485°$
(E) $285º$

Notice that more than an hour passes between the two times we're given – really almost two hours. One hour for the minute hand means a full circle, or 360º. Therefore, our answer should be somewhere between $360°$ and $720º$. We can immediately eliminate answers (A) and (E).

At $3{:}30$, the minute hand is pointing directly at the 6. At $5{:}25$, it's pointing at the 5. So the minute hand goes all the way around once, and then goes around 11 numbers, from the 6 to the 5. Using the fact that the angle between two numbers is 30º (which we learned in the first example), we can calculate the total angle as $360° + 11 \times 30° = 690°$. The answer is (B).

SAMPLE PROBLEM: ARC LENGTH/SECTOR AREA

In the figure below, three congruent circles are inscribed in a rectangle. The area of the marked sector is 24π. What is the perimeter of the rectangle?



(A) 48
(B) 48π
(C) 64π
(D) 128
(E) 128π

In this problem, we need to find the radius of the circles to find the perimeter of the rectangle – because the circles are inscribed, the distance from the top of the circle to the bottom (the diameter) is the height of the rectangle. Let's use the formula to find the radius:

$$\text{Sector Area} = \frac{\text{angle}}{360} \times \pi r^2$$

$$24\pi = \frac{135}{360} \times \pi \times r^2 \quad \rightarrow \quad 24\pi = \frac{3}{8} \times \pi \times r^2$$

$$24 = \frac{3}{8} \times r^2 \quad \rightarrow \quad r^2 = 24 \times \frac{8}{3} = 64 \quad \rightarrow \quad r = 8$$

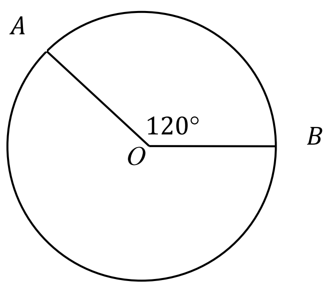
Therefore the diameter of the circle is 16, which is also the height of the rectangle.

Because the width of the rectangle is the same as the "width" of three circles, the width must be three times the diameter, or 48. The perimeter of the rectangle is $16 + 48 + 16 + 48 = 128$. The answer is (D).

PRACTICE EXERCISES

1. In a circle with a circumference of 36π feet, find the length of minor arc, in terms of π, which contains a central angle of 45 degrees.

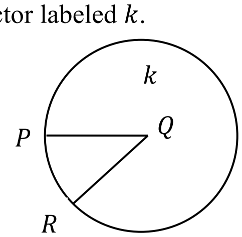
2. Circle O has a central angle that measures $72°$ and a radius that is 7 inches. What is the length, to the nearest tenth, of the minor arc that is subtended by the central angle?



3. In the circle above, $\angle AOB$ measures $120°$ and the area is 196π square centimeters. Find the measure of minor arc AB, in terms of π.

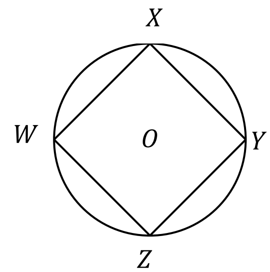
4. If the central angle of a sector of a circle is 108°, what fraction of the area of the circle is the area of the sector?

5. In circle Q shown below, the circumference is equal to 12π units and the measure of central angle $\angle PQR$ is 36°. Find the area, to the nearest square unit, of the sector labeled k.



6. The central angle of a sector of a circle measures $60°$ and the area of the sector is 24π square inches. What is the radius of this circle?

7. Square $WXYZ$ is inscribed in circle O as shown below. If the area of the circle is 81π square meters, find the length of minor arc WX, in terms of π.



8. If the central angle of a circle measures $40°$, then the area of the circle is how many times greater than the area of the sector?

9. The area of a given circle is 9π square feet. An arc of this circle is subtended by a central angle that measures $20°$. What is the length, in terms of π, of the subtended arc?

10. The central angle of the sector of a circle is $24°$ and the diameter of the circle measures 30 units. What is the area of the sector in terms of π.

TEST EXERCISES (TIME: 10 MINUTES)

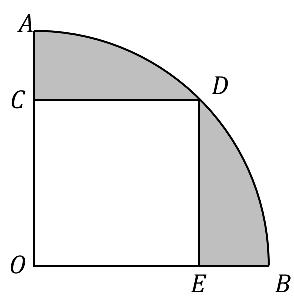
11. What is the length, in terms of π, of an arc intercepted by one side of a regular hexagon that is inscribed in the circle if the circle has a diameter of 16 units?

(A) $\frac{8\pi}{3}$
(B) 3π
(C) 8π
(D) $\frac{32\pi}{3}$
(E) 16π

12. Jarreau looks at the clock on the wall and notes that the time is $12:25$. The minute hand traverses an angle of $846°$ before he looks again. What time is it now?

(A) $2:06$
(B) $2:31$
(C) $2:46$
(D) $3:06$
(E) $4:31$

13. Square $CDEO$ is inscribed in $90°$ circular sector OAB with center O as shown. If $OC = OE = 4$, what is the total area of the shaded regions?



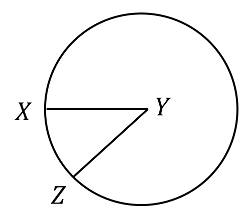
(A) $2\pi - 16$
(B) $4\pi - 16$
(C) $8\pi - 16$
(D) $4\pi - 32$
(E) $8\pi - 32$

14. In a given circle with a central angle that measures $\theta°$ and a radius of r units, what happens to the area of the sector subtended by θ if the length of the radius is tripled?

(A) The area is divided by nine.
(B) The area is divided by three.
(C) There is no change in the area
(D) The area is multiplied by three.
(E) The area is multiplied by nine.

15. In a given circle centered at C, points A and B lie on the circle. If the measure of $\angle ACB = 40°$ and the length of $\overline{BC} = 27$ centimeters, what is the area, in terms of π, of the sector containing $\angle ACB$?

(A) 9π
(B) 27π
(C) 81π
(D) 108π
(E) 144π

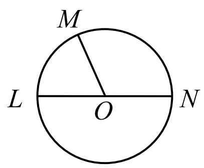


16. If angle XYZ, as shown above, has a measure of $30°$ and subtends an arc of length 12 feet, what is the length of the radius, to the nearest foot, of the circle?

(A) 13
(B) 23
(C) 24
(D) 30
(E) 36

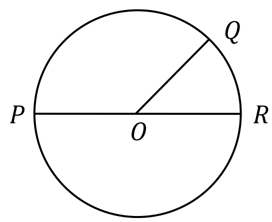
17. Find the measure in radians of $\angle AOC$ on a circle centered at O, given that the area of sector AOC is 15π and the length of minor arc AC is 5π.

(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{3}$
(C) $\frac{\pi}{2}$
(D) $\frac{2\pi}{3}$
(E) $\frac{5\pi}{6}$



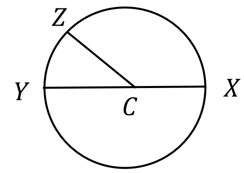
18. In circle O as shown above, with diameter $\overline{LN}$, the ratio of $\angle LOM$ to $\angle NOM$ is $5:7$. If the area of circle O is 225π in^2, what is the length, to the nearest meter, of minor arc MN?

(A) 27
(B) 36
(C) 47
(D) 67
(E) 132



19. In circle O shown above P and R are endpoints of a diameter that passes through center O. Point Q lies on circle O and $\angle QOP$ measures $108°$. The shortest distance traveling along the circle from R to Q is what percent of the distance travelling along the circle from R to P?

(A) 25%
(B) 30%
(C) 35%
(D) 40%
(E) 45%



20. In circle with center C as shown above, the area of sector XCZ is $\frac{5\pi}{2}$ ft^2 and $m\angle XCZ = 144°$. What is the diameter of circle C?

(A) 2.5
(B) 5.0
(C) 7.5
(D) 10.0
(E) 12.5