ECE 3530/CS 3130: Engineering Probability & Stats
Lecture Notes Spring 2015

Lecture #8

1 Poker

Poker provides many examples of counting and probability. In these notes, we study a variation of poker we play with "Phase 10" cards in order to have more practice with permutations, combinations, and probability. These are faceless cards, 48 cards in a deck: 12 each of yellow (Y), red (R), green (G), and blue (B). In each color, the 12 cards are numbered 1 through 12. The deck also comes with Wild cards, Skip cards, and an instruction card, which we will remove.

We will play the following game. The dealer will shuffle the cards and then deal you five cards, and the order they are dealt to you does not matter. You may discard any five of the cards — you can discard all five, or none, or any number in between. The discarded cards do not go back into the dealer's deck. The dealer will deal you new cards to take the place of those discarded. After this, the person with the "best hand" out of the players in the group is declared the winner.

The "best" hand is the one least likely to be dealt to you (at the start, before discarding and dealing more cards). The types of hands are as follows, in order of decreasing probability:

1. One pair: the same number appears twice in your hand. Example: { B2, R2, G12, Y5, Y9}.

2. Two pairs: one number appears twice, and a second number appears twice in your hand. Example: { B2, R2, G5, Y5, Y9}.

3. Three of a kind: the same number appears three times in your hand. Example: { B2, R2, G2, Y5, Y9}.

4. Straight: five cards of any color have sequential numbers. Example: { B2, R3, G4, G5, B6}.

5. Full house: one number appears three times, and a second number appears twice in your hand. Example: { B2, R2, G2, B9, Y9}.

6. Flush: the five cards are all of the same color. Example: { B2, B4, B8, B9, B12}.

7. Four of a kind: the same number appears four times in your hand. Example: { B2, R2, G2, Y2, Y9}.

8. Straight flush: five cards of a single color have sequential numbers. Example: { B2, B3, B4, B5, B6}.

9. Royal flush: Five cards of the same color numbered 12, 11, 10, 9, and 8 appear in your hand. Example: { G8, G9, G10, G11, G12}.

Note that some hands fit multiple types; you always take the best (least likely) type. Thus when we count the ways to get "one pair", we exclude two pair, full house, and four of a kind.

If two people have the same hand (or none of the above), then the person with the highest numbered card wins.

1.1 First Deal

At first deal, how big is the sample space? That is, how many ways are there to get a hand of five cards using this deck? There are 48 distinct cards, and order doesn't matter. Thus there are $\binom{48}{5}$ or 1,712,304 ways.

How many ways are there to get a royal flush? There is only one for each color, so there are 4 ways.

How many ways are there to get a straight flush? First you pick one of four colors. Then you pick a starting number, from 1 to 7. Thus there are $(4)(7) = 28$.

How many ways are there to get four of a kind? First, pick a number that will be repeated 4 times in the hand from among 12. Then, pick one additional card from among the remaining 44. Thus $12(44) = 528$ ways.

How many ways are there to get a flush? First, pick one color from the four. Then, select 5 cards from the 12 available in that color. Thus $4\binom{12}{5} = 3168$. However, these include the straight and royal flush. Subtracting them, there are $3168 - 28 - 4 = 3{,}136$ ways.

How many ways are there to get a full house? First, pick a number for the three-peat from among 12. Then, pick three out of four cards that have that number. Then, pick a second number from among the 11 remaining. Then, pick two out of the four cards that have that number. Thus there are $(12)\binom{4}{3}(11)\binom{4}{2} = 3{,}168$.

How many ways are there to get a straight? First pick a starting number, from 1 to 8. For each of the five cards, you may independently pick one of four colors. So there are $(8)(4)^5 = 8192$ ways. However, these include the straight flush. Subtracting them out, there are 8,164 ways.

How many ways are there to get three of a kind? First, pick a number that will be repeated 3 times in the hand from among 12. Then, pick two additional cards from among the remaining 44 (of other numbers). Thus $12\binom{44}{2} = 11352$ ways. These include the full house. Subtracting, there are then 8,184 ways.

How many ways are there to get two pairs? First, pick a number that will be repeated 2 times in the hand from among 12. Choose two out of the 4 with this number. Then, pick a different number that will be repeated 2 times in the hand from among the 11 left. Choose two out of the 4 with this number. Then, pick one of the remaining 40 cards. Thus $12\binom{4}{2}11\binom{4}{2}(40) = 190{,}080$ ways.

How many ways are there to get one pair? First, pick a number that will be repeated 2 times in the hand from among 12. Choose 2 out of the 4 with this number. Then, pick three of the remaining 11 numbers, and each of these three cards can be one of the four colors. Thus $12\binom{4}{2}\binom{11}{3}4^3 =$ 760,320.

1.2 Selection of Discards

Given any hand, how many ways are there to discard? There are five cards, each one can either be discarded or not. Thus there are $2^5 = 32$ ways to discard. One may evaluate the probability of any particular first deal and selection of discards. (This is what is done in the reading for today, an excerpt from Prof. Stewart N. Ethier's *The Doctrine of Chances: Probabilistic Aspects of Gambling* for two types of video poker.) This probability is essentially a conditional probability, because the probability is conditioned on the cards dealt in the first round (which you cannot receive in the 2nd round), and the cards selected to be discarded. The sample space is limited by these conditions, as well as the number of possibilities in each type of hand.