ECE 3530/CS 3130: Engineering Probability & Stats
Lecture Notes Spring 2015

Lecture #3

1 Total Probability

We mentioned in passing the very important rule that for *any* two events A and B,

$$P[A] = P[A \cap B] + P[A \cap B^c]. \tag{1}$$

This is a simple example of the law of total probability applied to the *partition*, B and B^c. Now, let's present the more general law of total probability. First, the definition of partition:

Def'n: *Partition*
A countable collection of mutually exclusive events $C_1, C_2, \ldots$ is a partition if $\bigcup_{i=1}^{\infty} C_i = S$.

Examples:

1. For any set C, the collection C, C^c.
2. The collection of all simple events for countable sample spaces. Eg., $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$.

Def'n: *Law of Total Probability*
For a partition $C_1, C_2, \ldots$, and event A,

$$P[A] = P[A \cap C_1] + P[A \cap C_2] + P[A \cap C_3] + \cdots \tag{2}$$

Note: You must use a partition!!!, but A can be any event.

Example: Die and Odd
Let's consider a fair 6-sided die again, but define the following sets: $E_1 = \{1, 2, 3\}$, $E_2 = \{4, 5\}$, and $E_3 = \{6\}$. Also let F be the event that the number rolled is odd. First, find:

1. $P[E_i \cap F]$ for $i = 1, 2, 3$.
2. Do E_1, E_2, and E_3 form a partition?
3. Use (1.) and (2.) to calculate $P[F]$.

Solution:

1. Using the discrete uniform probability law, $P[E_1 \cap F] = 2/6$, $P[E_2 \cap F] = 1/6$, and $P[E_3 \cap F] = 0/6$.

2. Well, $E_1 \cup E_2 \cup E_3 = \{1, 2, 3, 4, 5, 6\} = S$ and the intersection of any two of the E_i sets is the null set. So they do form a partition.

3. By the law of total probability, $P[F] = P[E_1 \cap F] + P[E_2 \cap F] + P[E_3 \cap F] = \frac{2+1+0}{6} = 0.5$.

Example: Traits
People may inherit genetic trait A or not; similarly, they may inherit genetic trait B or not. People with both traits A and B are at higher risk for heart disease. For the general population, $P[A] = 0.50$, and $P[B] = 0.10$. We know that $P[A \cap B^c]$, that is, the probability of having trait A but not trait B, is 0.48. What is the probability of having both trait A and B?
Solution: Use the law of total probability with partition B and B^c:

$$P[A] = P[A \cap B] + P[A \cap B^c].$$

We can rearrange to: $P[A \cap B] = P[A] - P[A \cap B^c]$. Using the given information:

$$P[A \cap B] = 0.50 - 0.48 = 0.02.$$

A Venn diagram is particularly helpful here.

1.1 Trees

A graphical method for organizing information about how a multiple-stage experiment can occur. See Figure 1.

Example: Customers Satisfied by Operator
A tech support phone bank has exactly three employees. A caller is randomly served by one of the three. After a customer is served, he or she is surveyed. TData is kept as to the employee (by number, 1, 2, or 3) and whether the caller was satisfied (S). The data shows:

- $P[S \cap 1] = 0.40$
- $P[S \cap 2] = 0.20$
- $P[S \cap 3] = 0.18$

Questions:

1. What is the probability that a caller is satisfied?

2. Are the probabilities of a caller being served by each of the three employees equal?

Solution:

1. Since 1, 2, and 3 are a partition, $P[S] = P[S \cap 1] + P[S \cap 2] + P[S \cap 3] = 0.4 + 0.2 + 0.18 = 0.78$.

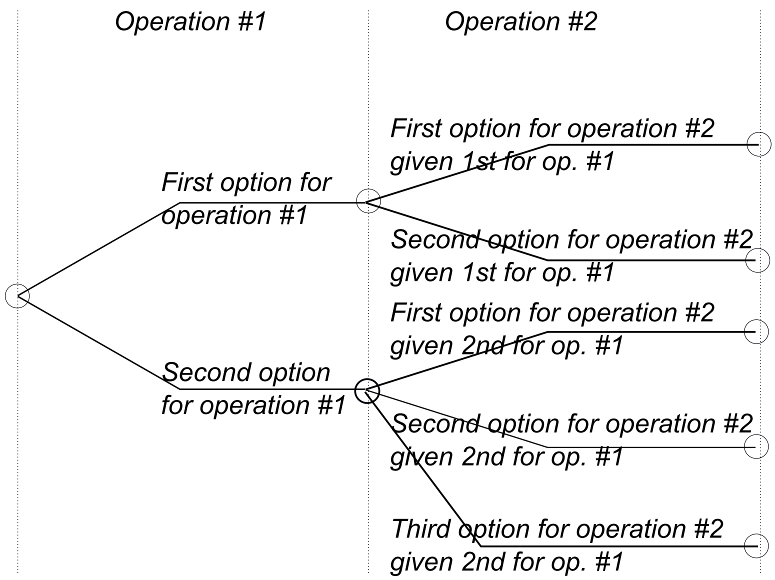

Figure 1: A tree shows the options of a first operation, and then for each of those options, what options exist for the second operation. The number of branches at each node does not need to be the same.

2. The three probabilities, $P[1]$, $P[2]$, and $P[3]$ can't be equal. Employee 1 has too high of a probability. By the conjunction bound, $P[1] \geq P[S \cap 1] = 0.40$. Note you can get the conjunction bound by considering that

$$\begin{aligned} P[1] &= P[1 \cap S] + P[1 \cap S^c] \\ P[1] &\geq P[1 \cap S]. \end{aligned} \tag{3}$$

   The latter line because the probability $P[1 \cap S^c] \geq 0$ by probability Axiom 1.

   In any case, the probability a caller is served by 1 is at least 40%. If the three employees were equally likely, they'd each have probability of 0.333.