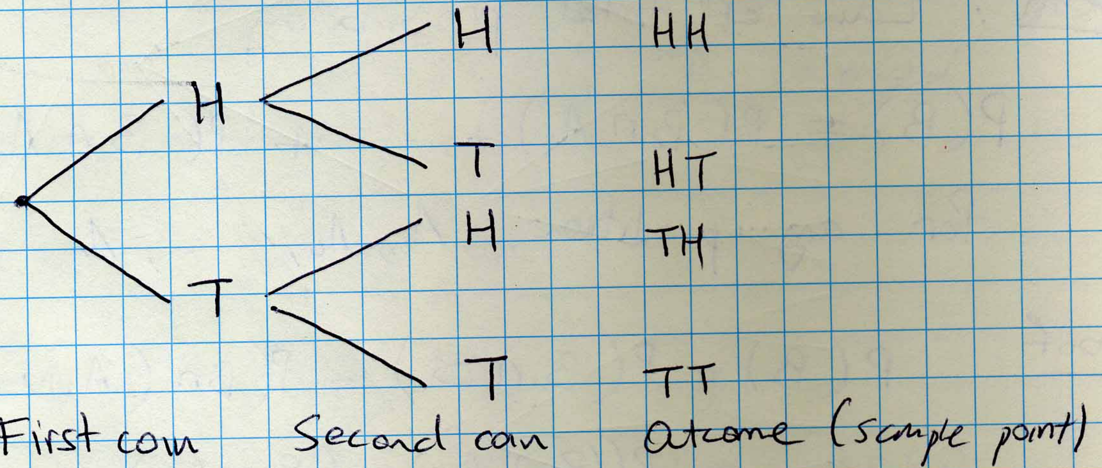
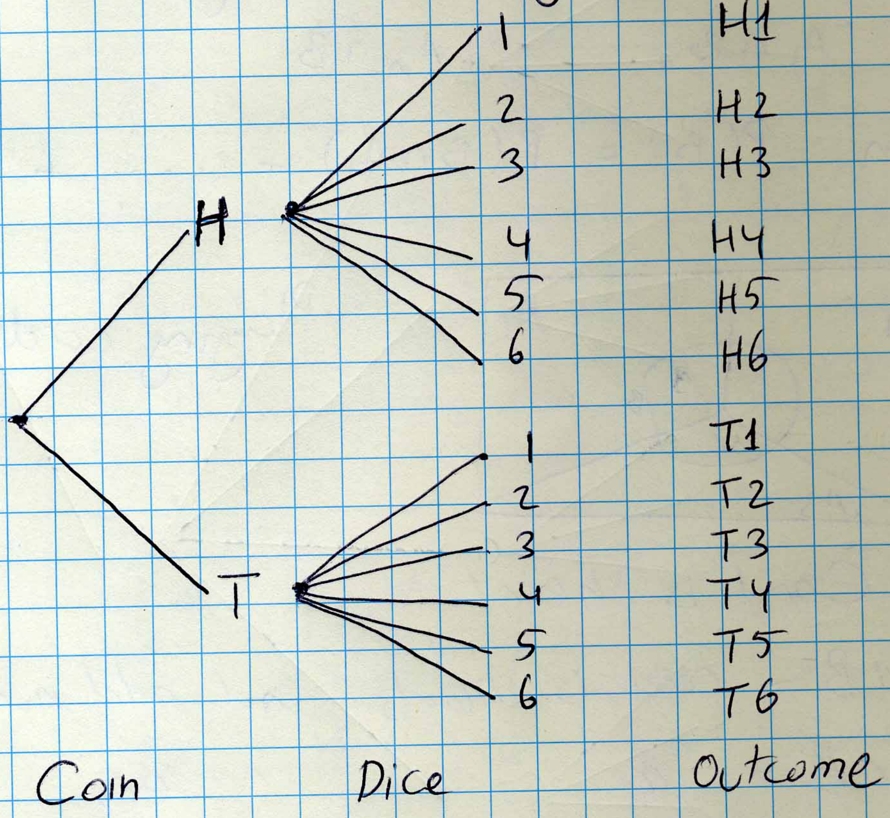
COUNTING SAMPLE POINTS

A tree diagram is a useful visual tool to systematically list the elements of a sample space

ex. Tossing a coin twice.



ex. Coin toss followed by dice roll



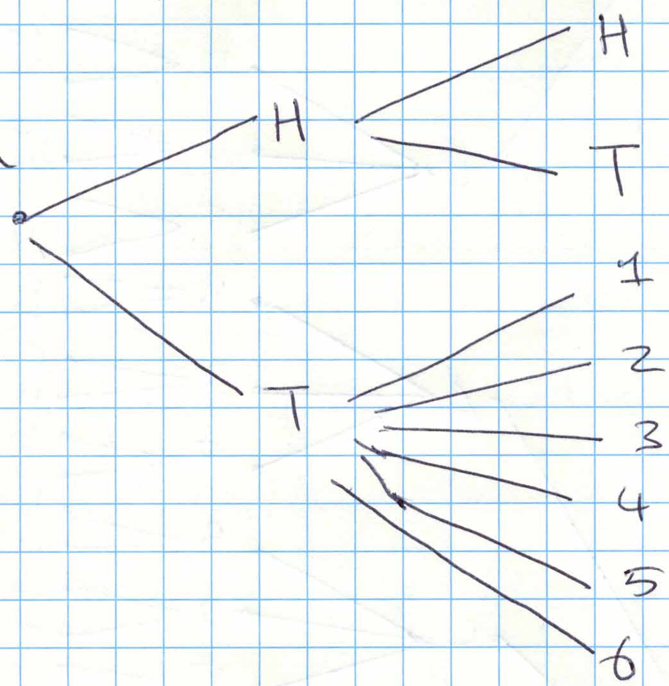
Defn Multiplication rule: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

ex. Tossing coin twice $n_1 = 2, n_2 = 2$ $n_1 n_2 = 4$

ex. Coin toss followed by dice roll $n_1 = 2, n_2 = 6$ $n_1 n_2 = 12$

ex. A coin is flipped and then flipped a second time if a head occured the first time. If a tail occured on the first flip a dice is rolled.

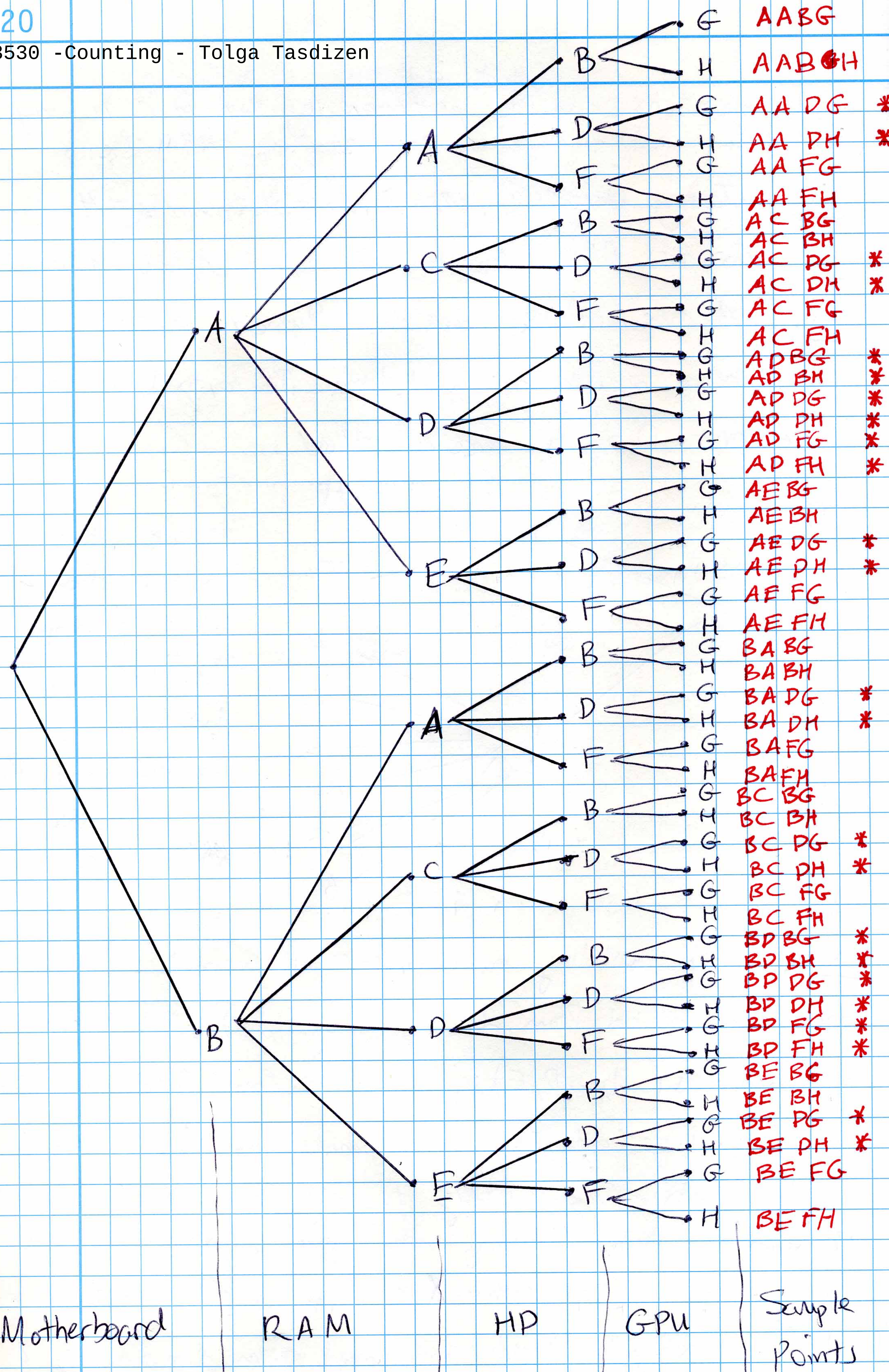
Tree diagram



Notice, the multiplication rule doesn't apply in this case because n_2 is different for the possible outcomes of the first operation.

Ex: I am building a computer from parts. I can buy the motherboard from 2 companies (A, B), RAM from 4 companies (A, C, D, E), hard drive from 3 companies (B, D, F) and graphics card from 2 companies (G, H). For each part, I am equally likely to buy from any one of the companies manufacturing that part. What is the probability of the event X that I build a computer that has at least one part from company D?

ECE 3530 -Counting - Tolga Tasdizen



(1) From the tree we count 48 possible outcomes. In other words, the sample space has $N=48$ points.

(2) From the tree, we count 24 outcomes included in event X. $n=24$

Theorem: If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event X, then $P(X) = n/N$.

(3) Therefore, in our example

$$P(X) = \frac{24}{48} = \frac{1}{2}$$

Defn: Generalized Multiplication Rule: If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of operations can be performed in $n_1 n_2 n_3 \cdots n_k$ ways.

An easier solution to the computer example (No need to draw tree)

(1) Number of points in sample space

$$N = n_1 n_2 n_3 n_4 = 2 \cdot 4 \cdot 3 \cdot 2 = 48$$

(2) Define events

$Y =$ RAM chip from company D, $P(Y) = ?$

$Z =$ HD from company D, $P(Z) = ?$

(3) # of otcomes with RAM chip from company D

Motherboard		RAM		HD		GPU		
?		D		?		?		
2	x	1	x	3	x	2	=	12

$P(Y) = 12/48 = 1/4$

# of outcomes with HD from company D

Motherboard		RAM		HD		GPU		
?		?		D		?		
2	x	4	x	1	x	2	=	16

$P(Z) = \frac{16}{48} = \frac{1}{3}$

(4) Notice $X = Y \cup Z$

$P(X) = P(Y \cup Z) = P(Y) + P(Z) - P(Y \cap Z)$

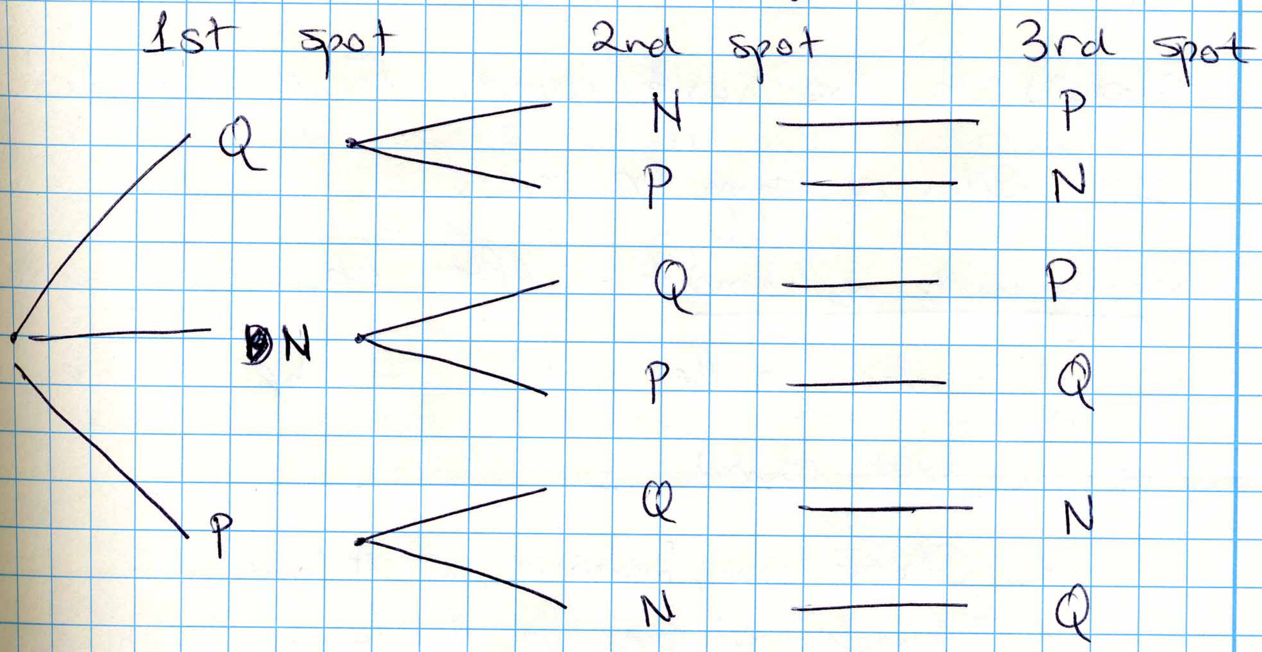
$Y \cap Z$ is the event that both RAM and HD are from company D

Motherboard		RAM		HD		GPU		
?		D		D		?		
2	x	1	x	1	x	2	=	4

$P(Y \cap Z) = 4/48 \Rightarrow P(X) = \frac{12+16-4}{48} = \frac{1}{2}$

Permutation is an arrangement of all or part of a set of objects. (order matters)

Ex: In how many ways can you arrange a quarter, a ~~dime~~ nickel and a penny?



$3 \times 2 \times 1 = \underline{\underline{6 \text{ ways}}}$

Defn: The number of permutations of n objects is $n!$ (read: n factorial)

Defn: $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

Also we define $\boxed{0! = 1}$

Ex: How many ways can you arrange a quarter, nickel, penny and dime?

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = \underline{\underline{24}}$ ways

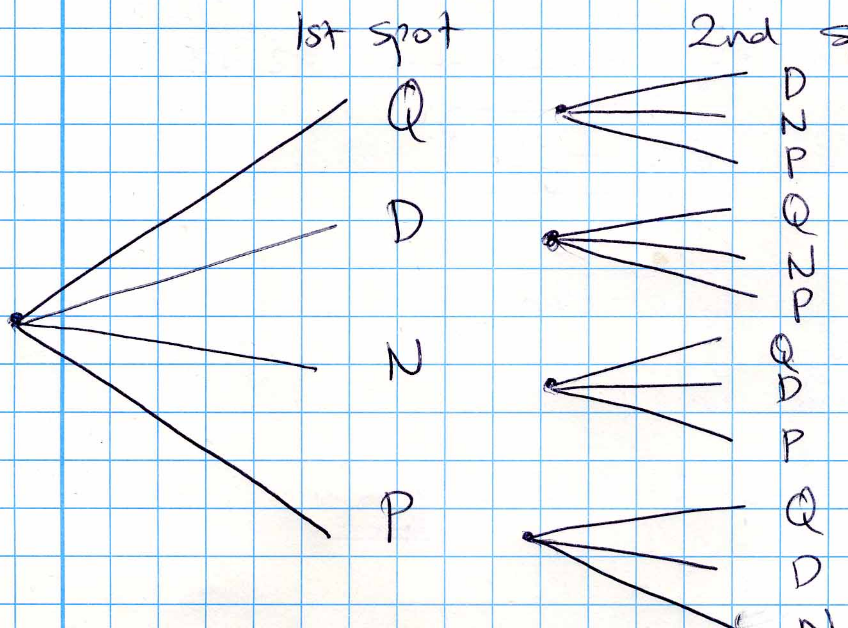
* **Note:** Permutations are used when we are sampling without replacement and order matters.

Def <u>Sampling without replacement</u>: After choosing an object it is not put back in the set of possible choices for the next round. For example, the case with arranging the coins.

Def <u>Sampling with replacement</u>: After choosing an object, we put it back in the set of possible outcomes for the next round.

ex. Choose coin, record it, put it back in bag. If I choose 4 coins in this way the number of possible ways is $4 \cdot 4 \cdot 4 \cdot 4 = 256$.

Ex: In how many ways can I arrange 2 coins from the 4 coins: Quarter, Dime, Nickel and Penny



$4 \times 3 = 12$ ways

<u>Defn</u>: The number of permutations of n distinct objects taken r at a time is $_nP_r = \dfrac{n!}{(n-r)!}$

Back to our example: $_4P_2 = \dfrac{4!}{(4-2)!} = \dfrac{4!}{2!} = \dfrac{4\times3\times\cancel{2\times1}}{\cancel{2\times1}}$

$= \underline{\underline{12}}$

<u>Ex</u>: Drawing names out of a hat for a lottery. Don't put names back in if they get drawn. Suppose we have 60 students and we draw 3 names. How many possible outcomes if we keep track of the order of names.

1st draw: 60 possible names
2nd draw: 59 possible names
3rd draw 58 possible names

Total number of possible outcomes: $60 \cdot 59 \cdot 58 = 205320$

Notice that this is also $_{60}P_3$.

$$_{60}P_3 = \frac{60!}{(60-3)!} = \frac{60!}{57!} = \frac{60\cdot59\cdot58\cdot\cancel{57\cdot56\cdots1}}{\cancel{57\cdot56\cdots1}}$$

Another way to think about it

$$60! = (60\cdot59\cdot58)\cdot(57!)$$

Hence $\dfrac{60!}{57!} = 60\cdot59\cdot58$

How many sets of 3 names are possible if the order they are drawn is ignored?

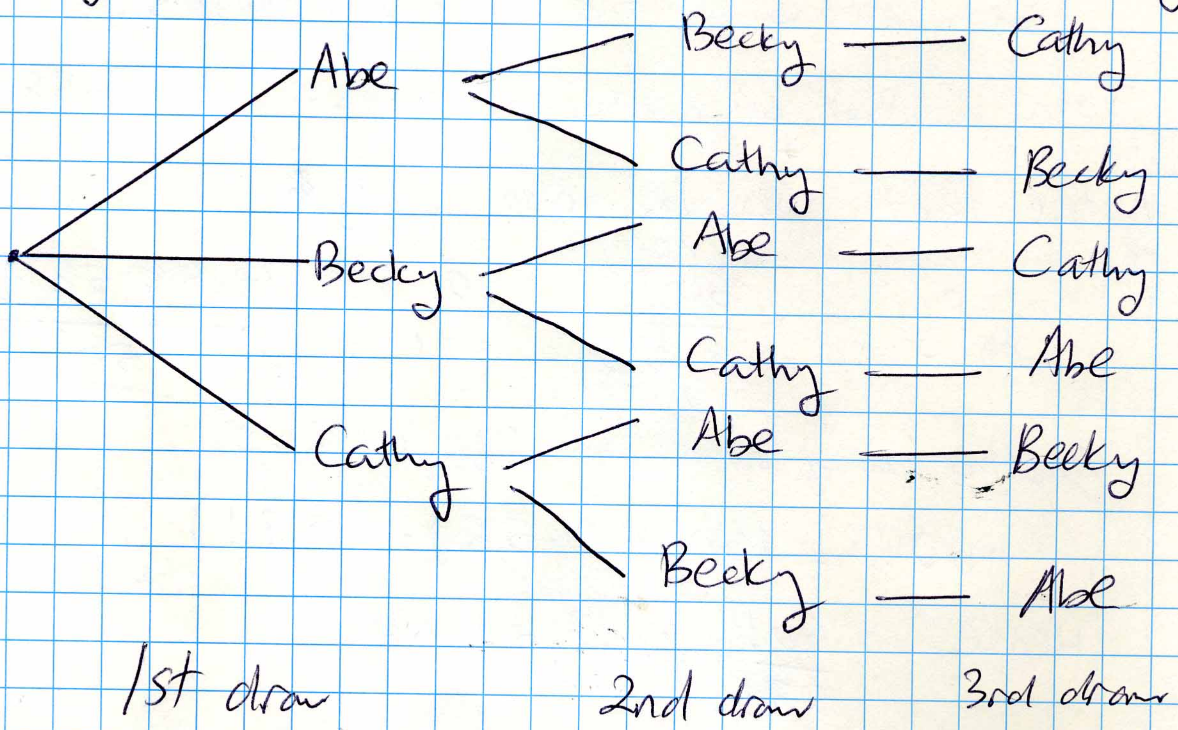
Defn The # of possible <u>combinations</u> (order doesn't matter) of n distinct objects taken r at a time is

$$ {}_nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!} $$

Note 1 Combinations are used when we are sampling without replacement and order does NOT matter.

Note 2 $$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{\text{\# permutations of } r \text{ out of } n \text{ objects}}{\text{\# permutations of } r \text{ objects (taken } r \text{ at a time)}} $$

Suppose Abe, Becky and Cathy won the lottery. They could have been drawn in $3! = 3 \cdot 2 \cdot 1 = 6$ ways



The same goes for any combination of 3 winning names.

Therefore, to answer how many sets of 3 names are possible if the order is ignored:

$$ {}_nC_r = \frac{\#\text{ permutations of 3 out of 60}}{\#\text{ permutations of 3 (out of 3)}} = \frac{{}_{60}P_3}{{}_3P_3} $$

$$ {}_nC_r = \frac{60!}{(60-3)!\,3!} = \frac{60\cdot 59\cdot 58}{3\cdot 2\cdot 1} = 34{,}220 $$

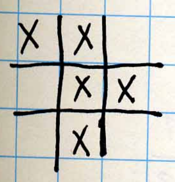
Question: What is the probability that Abe, Becky and Cathy win?

P(Abe, Becky, Cathy drawn from hat in any order)

$$ = \frac{1}{{}_nC_r} = \frac{1}{34{,}220} $$

Ex: How many ways are there to put 5 x's and 4 o's on a tic-tac-toe board?

soln. Find # ways to place 5 x's in 9 squares leaving 4 squares blank. Since all x's are the same, order doesn't matter



$$ \#\text{ ways to place 5 x's in 9 squares} = \binom{9}{5} $$

$$ = \frac{9!}{(9-5)!\,5!} = 126 $$

Now find # ways to put 4 o's in the remaining 4 empty spots.

$$ \binom{4}{4} = \frac{\cancel{4!}}{\cancel{4!}(4-4)!} = \frac{1}{0!} = \frac{1}{1} = 1 $$

Final answer $126 \cdot 1 = 126$