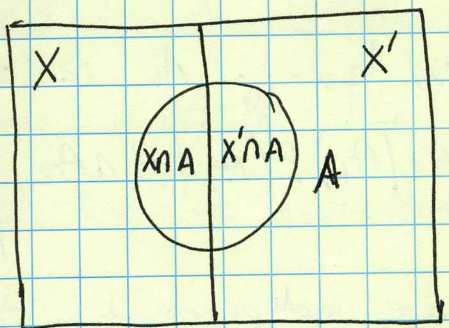
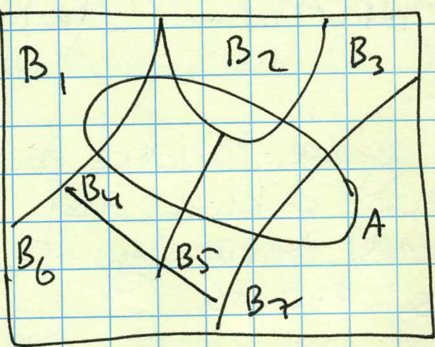
# BAYES RULE




**Total Probability X and X'**

$$P(A) = P(X \cap A) + P(X' \cap A)$$

$$= P(A|X)P(X) + P(A|X')P(X')$$




**Total probability generalization**

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i)$$

defn. of conditional prob

$$= \sum_{i=1}^{k} P(A|B_i)P(B_i)$$

defn. of conditional prob.

substitute

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(A|B_r)P(B_r)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

$$\boxed{P(B_r|A) = \frac{P(A|B_r)P(B_r)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}}$$ **Bayes rule**

Some terminology
- $P(B_i)$ priors
- $P(A|B_i)$ likelihoods
- $P(B_i|A)$ posteriors

Useful in problems where $P(B_i|A)$ are not known but $P(A|B_i)$ and $P(B_i)$ are.

Example: In a manufacturing plant, three machines, $B_1$, $B_2$ and $B_3$ make 30%, 45% and 25% respectively of the products. It is known that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Now suppose that a product is found defective, what is the probability that it was made by machine $B_3$?

soln: $P(B_1) = 3/10$ $\quad P(B_2) = 0.45$ $\quad P(B_3) = 0.25$

A: product is defective

$P(A|B_1) = 0.02$ $\quad P(A|B_2) = 0.03$ $\quad P(A|B_3) = 0.02$

$$P(B_3 | A) = \frac{P(A|B_3)P(B_3)}{\sum_{i=1}^{3} P(A|B_i)P(B_i)} = \frac{0.02 \times 0.25}{0.02\times 0.3 + 0.03 \times 0.45 + 0.02 \times 0.25}$$

$$= \frac{0.005}{0.0245} = \frac{10}{49}$$

Example: Astronauts on the space shuttle realize that oxygen level is dropping. There are 3 possible problems that can cause oxygen levels to drop: a leak in the fuselage (L), malfunctioning oxygen pump (M) and a $CO_2$ filter in need of replacement (F). The astronauts know that

$P(L) = 0.02$, $P(M) = 0.49$ and $P(F) = 0.49$

Ground crew runs simulations to find:

O : event oxygen level dropping

$P(O|L) = 1$

$P(O|M) = 0.4$

$P(O/F) = 0.6$

What should the astronauts try to fix first?

$$P(L|O) = \frac{P(O|L)P(L)}{P(O|L)P(L) + P(O|M)P(M) + P(O|F)P(F)}$$

$$= \frac{1 \times 0.02}{1 \times 0.02 + 0.4 \times 0.49 + 0.6 \times 0.49} = \frac{0.02}{0.51} = 0.039$$

$$P(M/O) = \frac{0.4 \times 0.49}{0.51} = 0.384$$

same denominator as above

$$P(F|O) = \frac{0.6 \times 0.49}{0.51} = 0.576$$

They should check the filter first.

* Notice $P(L/O) + P(M/O) + P(F/O) = 1$
* In this case, there was no way for the astronauts and the ground crew to directly measure probabilities like $P(L/O)$. BUT they could simulate a faulty pump and experimentally estimate probabilities like $P(O/F)$. They also knew the prior probability of each part failing.

## Example

In a manufacturing plant, three machines, $B_1$, $B_2$ and $B_3$ make 30%, 30% and 40% of the products respectively. It is also known that some of these are defective products (event D) we know that

$$P(D|B_1) = 0.1, \; P(D|B_2) = 0.04, \; P(D|B_3) = 0.07$$

Lets look at a few things:

$$P(B_1|D) = \frac{P(D|B_1)P(B_1)}{\sum_{k=1}^{3} P(D|B_k)P(B_k)}$$

Bayes Rule

This is $P(D \cap B_1)$

This is also $P(D)$

$$P(B_1|D) = \frac{0.1 \times 0.3}{0.1 \times 0.3 + 0.04 \times 0.3 + 0.07 \times 0.4}$$

$$= \frac{0.03}{0.07} = 0.428$$

This is $P(D)$

- Notice $P(B_1|D) \neq P(B_1)$ hence events $B_1$ and $D$ are ~~independent~~. NOT INDEPENDENT
- We could have arrived at the same conclusion from $P(B_1 \cap D) = 0.1 \times 0.3 \neq P(B_1)P(D)$

$0.3 \times 0.07$

Lets do the same for $P(B_3|D)$.

$$P(B_3|D) = \frac{0.07\times0.4}{0.1\times0.3+0.04\times0.3+0.07\times0.4}$$

$$= \frac{0.028}{0.07} = 0.4$$

But notice $P(B_3|D) = P(B_3)$ hence $B_3$ and $D$ are independent.

- Could reach the same conclusion from

$$P(D|B_3) = P(D)$$

$0.07 = 0.07$

given — computed in the denominator of Bayes rule

- Still one more way

$$P(D\cap B_3) = P(D|B_3)P(B_3) = 0.07\times0.4 = 0.028$$

$$P(D)P(B_3) = \underbrace{0.07}_{\text{computed}} \times \underbrace{0.4}_{\text{given}} = 0.028$$

If these are equal then D and $B_3$ independent.

ANOTHER EXAMPLE

Coin A : Fair %50 heads %50 tails

$$P(\text{heads}|A) = 0.5 \quad P(\text{tails}|A) = 0.5$$

Coin B : Rigged %75 heads %25 tails

$$P(\text{heads}|B) = 0.75 \quad P(\text{tails}|B) = 0.25$$

You select coin A or coin B with equal probability. You toss the selected coin and get heads. What is the probability that you had selected coin B? In other words, $P(B|\text{heads}) = ?$

Bayes Rule : $P(B|\text{heads}) =$

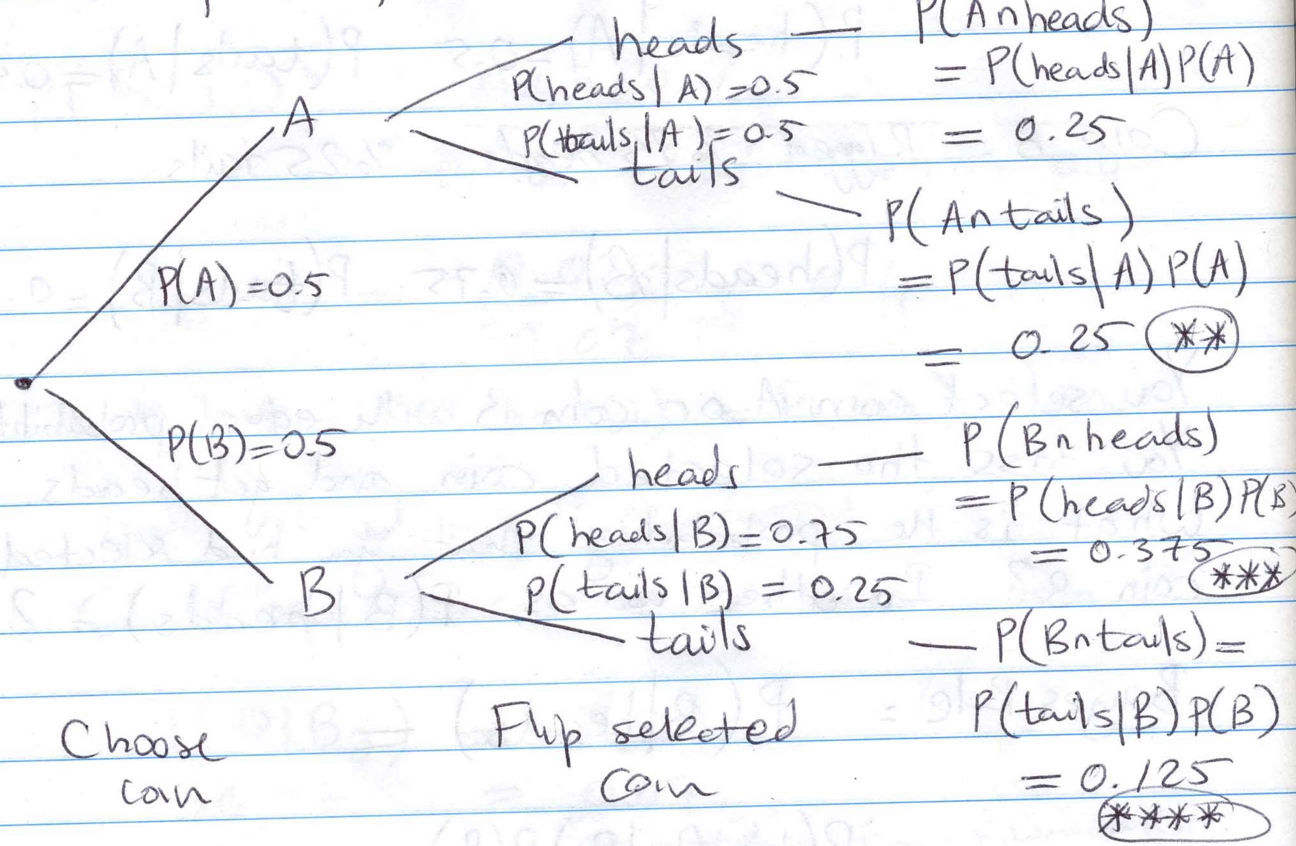
$$\frac{P(\text{heads}|B)P(B)}{P(\text{heads}|A)P(A) + P(\text{heads}|B)P(B)}$$

You selected coin A or B with equal probability so $P(A) = P(B) = 0.5$ plug into eqn. above

$$P(B|\text{heads}) = \frac{0.75 \times 0.5}{0.5 \times 0.5 + 0.75 \times 0.5} = \frac{3}{5}$$

Notice $P(B|\text{heads}) \neq P(B)$. Knowing the outcome was heads made the probability that you had selected coin B larger than 1/2.

Same problem, tree solution:




Now, $P(B|\text{heads}) = \dfrac{P(B \cap \text{heads})}{P(\text{heads})}$

$$= \frac{***}{* + ***}$$

$$= \frac{0.375}{0.25 + 0.375} = 3/5$$