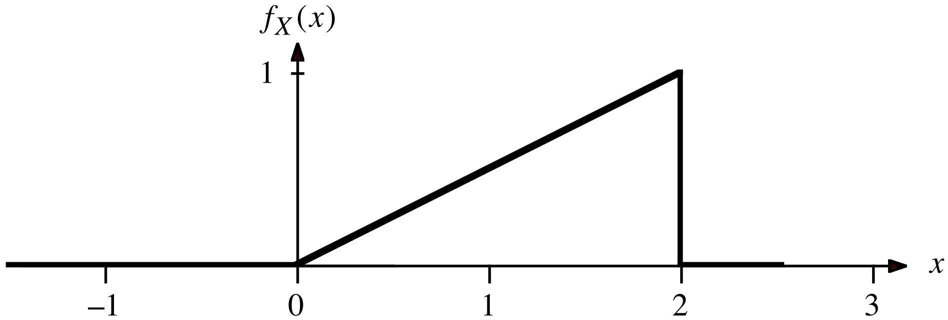
HOMEWORK #5 solution

1. Linear transformations of random variables are useful for generating new probability density functions (pdf's) from standard pdf's such as a uniform distribution. This problem explores linear transformations.

a) Assuming the cumulative distribution, $F_X(x)$, of X is known, find an expression for the cumulative distribution, $F_Y(y)$, of Y where $Y = aX + b$ and $a > 0$.

b) Use the result in (a) to find an expression for pdf $f_Y(y)$ in terms of pdf $f_X(x)$.

c) For the $f_X(x)$ shown below, find $f_Y(y)$ where $Y = 3X - 1$.



SOL'N: a) The cumulative distribution $F_X(x)$ is, by definition, a probability:

$$F_X(x) \equiv P(X \le x)$$

Likewise, the cumulative distribution $F_Y(y)$ is a probability:

$$F_Y(y) \equiv P(Y \le y)$$

Given $Y = aX + b$, we may write this probability for Y in terms of X:

$$P(Y \le y) = P(aX + b \le y)$$

Performing algebraic operations on the inequality inside the parentheses, we may reduce this to a probability for X by itself:

$$P(aX + b \le y) = P(aX \le y - b) = P\left(X \le \frac{y-b}{a}\right)$$

This last expression corresponds to a value of $F_X(x)$

$$P\left(X \le \frac{y-b}{a}\right) = F_X\left(x = \frac{y-b}{a}\right)$$

From the chain of equalities, we have the desired relationship:

$$F_Y(y) = F_X\left(x = \frac{y-b}{a}\right)$$

b) The probability density function $f_Y(y)$ is the derivative *with respect to y* of $F_Y(y)$. We wish to express this in terms of $f_X(x)$, the derivative of $F_X(x)$ *with respect to x*. To resolve the incompatibility of the variables of differentiation, we use the chain rule from calculus:

$$\frac{dF_Y(y)}{dy} = \left.\frac{dF_X(x)}{dx}\right|_{x=\frac{y-b}{a}} \cdot \frac{dx}{dy}$$

or

$$\frac{dF_Y(y)}{dy} = f_X\left(x = \frac{y-b}{a}\right) \cdot \frac{dx}{dy}$$

or

$$\frac{dF_Y(y)}{dy} = f_X\left(x = \frac{y-b}{a}\right) \cdot \frac{d}{dy}\frac{y-b}{a}$$

or

$$\frac{dF_Y(y)}{dy} = f_X\left(x = \frac{y-b}{a}\right) \cdot \frac{1}{a}$$

or

$$\frac{dF_Y(y)}{dy} \equiv f_Y(y) = \frac{1}{a} f_X\left(x = \frac{y-b}{a}\right)$$

c) The following equation describes $f_X(x)$:

$$f_X(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le 2 \\ 0 & otherwise \end{cases}$$

Since $Y = 3X - 1$, we have $a = 3$ and $b = -1$.

Using the result from (b) means we multiply $f_X(x)$ by $1/a$ and substitute $\frac{y-b}{a}$ for x everywhere, including the condition equation, $0 \le x \le 2$:

$$f_Y(y) = \frac{1}{a}\begin{cases} \frac{1}{2}\frac{y-b}{a} & 0 \le \frac{y-b}{a} \le 2 \\ 0 & otherwise \end{cases}$$

or

$$f_Y(y) = \frac{1}{3}\begin{cases} \frac{1}{2}\frac{y+1}{3} & 0 \le \frac{y+1}{3} \le 2 \\ 0 & otherwise \end{cases}$$

or

$$f_Y(y) = \begin{cases} \frac{y+1}{18} & -1 \le y+1 \le 5 \\ 0 & otherwise \end{cases}$$

This pdf is still triangular, as we might have suspected: