Monty Hall proof with Bayes rule

C_k : car is behind door k. $k = 1, 2$ or 3
H_{ij} : host opens door j after player picks door i

$P(C_k) = 1/3$ for $k = 1, 2,$ or 3

$$P(H_{ij} \mid C_k) = \begin{cases} 0 & \text{if } i = j \quad \text{can't open door picked by player} \\ 0 & \text{if } j = k \quad \text{can't reveal car} \\ 1/2 & \text{if } i = k \quad \text{player was correct} \\ 1 & \text{if } i \neq k \text{ and } j \neq k. \end{cases}$$

More specifically for $k = 1$

i	j	k	$P(H_{ij} \mid C_k)$	
1	1	1	0	violates both
1	2	1	1/2	
1	3	1	1/2	
2	1	1	0	can't reveal
2	2	1	0	can't open
2	3	1	1	
3	1	1	0	can't reveal
3	2	1	1	
3	3	1	0	can't open

You picked door 1, host revealed 3

What should you do now, switch to 2 or stay with 1?

$$P(C_1 \mid H_{13}) = \frac{P(H_{1,3} \mid C_1)\, P(C_1)}{\sum_{k=1}^{3} P(H_{13} \mid C_k)\, P(C_k)}$$

$$= \frac{1/2 \times 1/3}{1/2 \times 1/3 + 1 \times 1/3 + 0 \times 1/3} = \frac{1/6}{1/2} = 1/3$$

$$P(C_2 \mid H_{13}) = \frac{1 \times 1/3}{1/2} = 2/3 \quad !!$$

$$P(C_3 \mid H_{13}) = \frac{0 \times 1/3}{1/2} = 0.$$

SWITCH!!!

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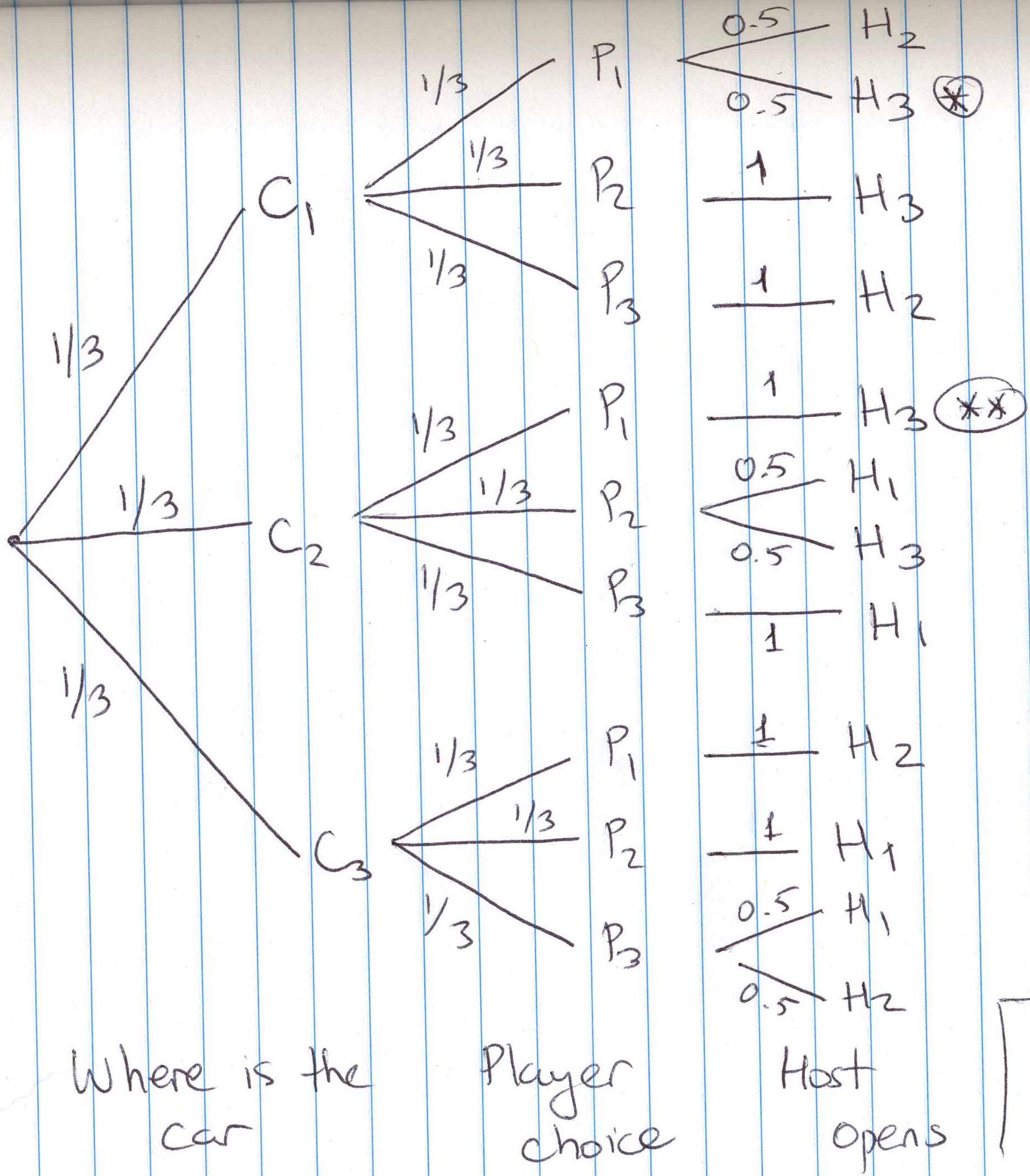
i	j	k	$P(H_{ij} \mid C_k)$	
1	1	1	0	violates both
1	1	2	0	can't open
1	1	3	0	can't open
1	2	1	1/2	
1	2	2	0	can't reveal
1	2	3	1	
1	3	1	1/2	
1	3	2	1	
1	3	3	0	can't reveal



Player Picked Door I (P_1)
Host opened Door 3 (H_3)
Switch or stay?

$$P(C_1 \mid P_1 \cap H_3)$$

$$= \frac{P(C_1 \cap P_1 \cap H_3)}{P(P_1 \cap H_3)}$$

C_1, C_2, C_3 form a partition

$$= \frac{P(C_1 \cap P_1 \cap H_3)}{\sum_{k=1}^{3} P(C_k \cap P_1 \cap H_3)}$$

$$= \frac{*}{* + **}$$ $\longrightarrow$ Notice $P(C_3 \cap P_1 \cap H_3) = 0$. It is not in the tree

$$= \frac{1/3 \times 1/3 \times 0.5}{1/3 \times 1/3 \times 0.5 + 1/3 \times 1/3 \times 1}$$

$$= 1/3$$

Similarly can compute

$$P(C_2 \mid P_1 \cap H_3) = \frac{2}{3}$$