Homework 7

1. The signal amplitude, X, in an undersea cable has an exponential distribution that falls off as amplitude increases. In other words, small-amplitude signals are more likely than large-amplitude signals. The signal amplitude also falls off as the distance, Y, from the signal source increases. A company has been hired to test the cable, and they are interested in the signal amplitudes they will find if they tap into the cable at randomly chosen points. The following probability density function (pdf) describes the probability density of finding signal amplitude, X, and randomly choosing a distance, Y, from the signal source:

$$f(x,y)=\begin{cases} e^{y}e^{-xe^{y}} & 0<x<\infty \text{ and } 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find μ_X. (Remember that μ_X means the same thing as $E(X)$.) Hint: use a change of variables, $z=e^{y}$.

b) Find σ_Y.

2. Consider the joint discrete random variables K and L. You are given that the joint pdf is given as:

$$f_{K,L}(k,l)=\begin{cases} ak^{l}, & k\in\{1,2,3\},\, l\in\{1,2,3\} \\ 0, & \text{Otherwise} \end{cases}$$

for some constant a.

a) Plot the joint pdf or create a table listing its values. (By hand is fine.)

b) Find the value of a that makes $f_{K,L}(k,l)$ a valid joint pdf.

c) Find the marginal pdf's $f_K(k)$ and $f_L(l)$.

d) Find the conditional pdf $f_{L|K}(l\,|\,k=1)$.

e) Are K and L independent?

f) Find the mean and covariance of K and L.

g) Find the correlation coefficient, ρ_{KL}.

3. a) Walpole 3.42. Copied here: Let X and Y denote the lengths in life, in years, of two components in an electrical system. If the joint pdf of these variables is

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{Otherwise} \end{cases}$$

find $P[0 < X < 1 \mid Y = 2]$.

b) For the same joint pdf, are X and Y independent?

4. An engineer is assigned the task of generating random variables with specified means and variances for use in Monte Carlo simulations. (An example of a Monte Carlo simulation is the estimation of an integral by picking random points, (x, y), and determining how often those points lie beneath the curve.) The engineer has been told (erroneously) that only the mean and standard deviation of the random variables is important.

Suppose the engineer is using linear combinations of the following uniform and normally distributed random variables that are typically found in math libraries on computers:

$$X \sim u[0,1] \quad \text{or} \quad f_X(X) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y \sim n(\mu_Y = 0, \sigma_Y = 1) \quad \text{or} \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(y-\mu_Y)^2/2}$$

a) For a desired mean, μ_Z, and variance, σ_Z^2, the engineer decides to use a linear combination of X and Y: $Z = aX + bY$. Find a and b in terms of μ_Z and σ_Z^2.

b) The engineer mistakenly thinks that if $Z = g(X)$, then it is always the case that $\mu_Z = g(\mu_X)$. Despite this erroneous assumption, the engineering finds that this result holds true for $Z = aX + b$. Determine what values of a and b the engineer could have used. Note: X still refers to the uniform random variable defined just above part (a).

5. The engineer described in problem 4 tries to approximate a standard normal distribution by, first, summing two independent samples of X (that we refer to X_1 and X_2), second, shifting to remove the mean, and third, multiplying by six.

$$Z = 6(X_1 + X_2 - 1)$$

As before, X_1 and X_2 are uniformly distributed on [0, 1].

a) Find σ_Z^2. Is this the correct value for a standard normal distribution?

b) Find the correlation between X_1 and Z. That is, find $\rho_{X_1 Z} \equiv \dfrac{\sigma_{X_1 Z}}{\sigma_{X_1}\sigma_Z}$.

ANS:

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2.
3.
4.
5.