CONDITIONAL PROBABILITY

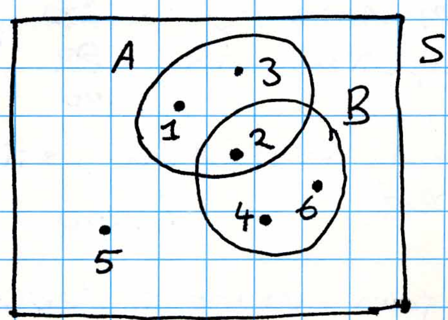
notation: $P(A|B)$ is the probability of event A given that event B occured.

example: Dice Roll $S = \{1,2,3,4,5,6\}$

$A = \{1,2,3\}$ $\quad P(A) = 1/2$

$B =$ dice roll is even $= \{2,4,6\}$

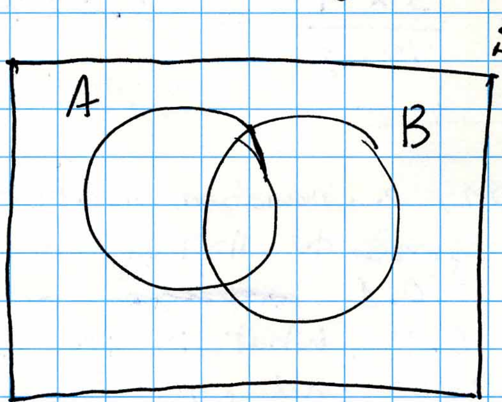
$P(A|B) = ?$



Since we know for sure that B occured, i.e. the outcome must be either 2, 4 or 6, we are working with a reduced sample space. To find $P(A|B)$ we have to count how many out of the new reduced possible set of outcomes is in A. In this example, this corresponds to the outcome 2. Hence $P(A|B) = 1/3$

Notice $P(A|B) < P(A)$ in this case. In other words knowing that the outcome was even (event B) reduced the probability of A occuring.

Note: for $P(A|B)$ to exist, we must have $P(B) \neq 0$



$\Longrightarrow$ if we know B occured

(*)

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Definition

Sample space becomes B

Probabilities are scaled accordingly.

$\boxed{\text{Note: } P(A|B) \neq P(B|A)}$

Theorem: $P(A|B) + P(A'|B) = 1$

using defn. of conditional probability we have:

$$\frac{P(A\cap B)}{P(B)} + \frac{P(A'\cap B)}{P(B)} = \frac{P(A\cap B)+P(A'\cap B)}{P(B)} \quad (**)$$

but since A and A' form a partition of S we have:

$$P(B) = P(A\cap B) + P(A'\cap B)$$

then substituting into $(**)$ above we have $\frac{P(B)}{P(B)} = 1$

There is no similar simple relationship between ~~P(A|B) and P(A|B')~~ $P(A|B)$ and $P(A|B')$.

$$P(A|B') = \frac{P(A\cap B')}{P(B')} = \frac{P(A) - P(A\cap B)}{1-P(B)}$$

Example: The probability of a flight departing on time is $P(D) = 0.83$. The probability of a flight arriving on time is $P(A) = 0.82$. We are also told $P(D\cap A) = 0.78$ (The probability that a flight both departs and arrives on time)

* What is the probability of a flight arriving on time if we know it departed on time?

$$P(A|D) = \frac{P(A\cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

* What is the probability that a flight departed on time if we know it arrived on time?

$$P(D|A) = \frac{P(A\cap D)}{P(D)} = \frac{0.78}{0.82} = 0.95$$

INDEPENDENT EVENTS

Defn. Two events A and B are <u>independent</u> if and only if

$$P(A|B) = P(A) \quad \text{OR} \quad P(B|A) = P(B).$$ Otherwise A and B are dependent.

Intuition If knowing that event B occured doesn't change the probability that A will occur, then B must carry no information about A. In other words, A & B are independent.

Obvious example If we flip two coins A & B and I tell you that B came up heads, what is the probability that A was heads?

$$P(A = \text{heads} \mid B = \text{heads}) = \frac{P(A=\text{heads} \cap B=\text{heads})}{P(B=\text{heads})}$$

Sample space

Fair coins: Each of the 4 outcomes equally likely

A	B	Prob
heads	heads	1/4
heads	tails	1/4
tails	heads	1/4
tails	tails	1/4

$\rightarrow P(A=\text{heads} \cap B=\text{heads})$

$\Sigma \rightarrow P(B=\text{heads}) = 1/2$

$$P(A=\text{heads} \mid B=\text{heads}) = \frac{1/4}{1/2} = 1/2$$

But also $P(A=\text{heads}) = 1/2$

Therefore, these two events are independent.

Dice roll. $S = \{1,2,3,4,5,6\}$

A: dice roll is even $\quad$ B: dice roll is greater than 2

$= \{2,4,6\}$ $\quad P(A) = 1/2$ $\quad = \{3,4,5,6\}$ $\quad P(B) = \frac{4}{6}$

$$* \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad A \cap B = \{4,6\}$$

$$= \frac{2/6}{4/6} = \frac{2}{4} = 1/2$$

so $P(A \mid B) = P(A)$

$$* \text{ Also } P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{1/2} = 4/6 = P(B)$$

In fact, if $P(A \mid B) = P(A)$ then $P(B \mid A) = P(B)$

$$P(A \mid B) \equiv \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)\,P(B)}$$

This is another definition of the independence of A & B

$$\Rightarrow \underbrace{\frac{P(A \cap B)}{P(A)}}_{\text{But this is } P(B \mid A)} = P(B)$$

Given
A: College graduate $\quad$ B: Smoker $\quad$ C: Heart disease

$P(A) = 0.7$ $\quad P(B) = 0.1$ $\quad P(C) = 0.05$

$P(A \cap C) = 0.035$ $\quad P(B \cap C) = 0.03$

$$P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{0.035}{0.7} = 0.05 = P(C)$$

A & C independent

$$P(C \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{0.03}{0.1} = 0.3 \neq P(C)$$

B & C dependent

MULTIPLICATIVE RULES

Theorem. Directly from the definition of conditional probability we also have $P(A \cap B) = P(A|B)\,P(B)$.

Theorem. A & B are independent events if and only if $P(A \cap B) = P(A)\,P(B)$

Example: An electrical engineering lab has 20 probes of which 3 are bad. A student selects 2 probes randomly, what is the probability that both are bad?

events A : First probe is bad

B : Second probe is bad

Question is asking $P(A \cap B)$

Multiplicative rule: $P(A \cap B) = P(A)\,P(B|A)$

$$P(A) = 3/20$$

Now if we know A occurred then there are 19 probes remaining of which 2 are bad, so

$$P(B|A) = 2/19$$

Therefore $P(A \cap B) = 3/20 \cdot 2/19 = 3/190$

Example A rigged coin is twice as likely to come up heads than tails. If the coin is flipped 3 times what is the probability of getting 3 heads?

soln. $P(H) = 2/3 \quad P(T) = 1/3$

Since the three flips are independent

$$P(HHH) = P(H)\,P(H)\,P(H) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

* What is the probability of event A that 2 heads occur?

$$P(A) = P(HHT) + P(HTH) + P(THH)$$

$$= \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = 4/9$$

Generalization of the multiplicative rules

Defn If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \ldots P(A_k|A_1 \cap \ldots \cap A_{k-1})$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\ldots P(A_k) = \prod_{n=1}^{k} P(A_n)$$

Example: In the Yahtze dice game 5 dice are rolled and a winning roll has three of a kind and two of a kind (but not five of a kind) Previously, we computed the probability of winning as 0.0386

* What is the probability of not getting a single winning roll in 5 tries?

Soln: Each roll of the 5 dice together is one try. Each try is independent of the other tries. In otherwords, the result of try one has no influence on try 2.

Therefore:

$$P(\text{lose}, \text{lose}, \text{lose}, \text{lose}, \text{lose}) = P(\text{lose}) \cdot P(\text{loose}) \cdot \ldots P(\text{loose})$$

$$= P(\text{loose})^5 = (1 - 0.386)^5 = 0.82$$

* What is the probability of getting 3 winners in 44 tries?

Lets consider the sequence win, win, win, followed by $44 - 3 = 41$ ~~losses~~ losses

with order $P(3\text{ win}, 41\text{ loose}) = P(\text{win})^3\, P(\text{loose})^{41} = 0.0386^3 (1-0.0386)^{41}$

$= 1.14 \times 10^{-5}$

But there are ${}_{44}C_3$ ways to choose three winning rolls in 44 tries

without order $P(3\text{ win}, 41\text{ loose without order}) = {}_{44}C_3 \times 1.14 \times 10^{-5}$

≈ 0.15

Example: A bag contains 10 red stones and 90 blue stones. What is the probability of selecting red, blue, red in that order when sampling <u>without</u> replacement? What is the probability of selecting two reds and one blue?

* 1st selection 10 out of 100 red

Probability of red on 1st selection is $\frac{10}{100}$

*

Given first selection red:

- sample space (possible outcomes) changes
- Now we have 9 red 90 blue remaining
- Probability of blue on 2nd selection is $\frac{90}{99}$

Given first = red and second = blue:

- * 9 red 89 blue remaining
- Probability of red on 3rd selection is $\frac{9}{98}$

* Due to ~~independence~~ multiplicative rule

$$P(RBR) = \underset{P(R)}{\frac{10}{100}} * \underset{P(B|R)}{\frac{90}{99}} * \underset{P(R|RB)}{\frac{9}{98}} = \boxed{\frac{9}{11 \times 98}}$$

Note: these are dependent events

* Similarly

$$P(RRB) = \frac{10}{100} * \frac{9}{99} * \frac{90}{98} = \frac{9}{11 \times 98}$$

notice $P(RRB) = P(RBR)$

* Also $P(BRR) = \frac{90}{100} * \frac{10}{99} * \frac{9}{98} = \frac{9}{11 \times 98}$

$$P(\text{two out of 3 red}) = \frac{3 * 9}{11 \times 98} \approx \boxed{0.025}$$

\# ways we can arrange 2 Red and 1 blue

RRB, RBR, BRR.

Ex: A bag contains 10 red stones and 90 blue stones. What is the probability of selecting red, blue, red in that order when sampling with replacement? What is the probability of selecting two reds and one blue?

* 1st selection 10 out of 100 red. So probability of red on first selection is 10/100

* 2nd selection 90 out of 100 red (sample space hasn't changed because we are sampling with replacement). So probability of blue on second selection is $\frac{90}{100}$

* 3rd selection, still 10 out of 100 red. So prob. of red is 10/100

Independent events

* Again using independence of the selections

$$P(RBR) = \frac{10}{100} * \frac{90}{100} * \frac{10}{100} = \boxed{\frac{9}{1000}}$$

* Similarly $P(RRB) = \frac{10}{100} * \frac{10}{100} * \frac{90}{100} = \frac{9}{1000}$

and $P(BRR) = \frac{90}{100} * \frac{10}{100} * \frac{10}{100} = \frac{9}{1000}$

* $P(\text{two out of 3 red}) = \frac{3 \times 9}{1000} = 0.027$

Ex: At the press of a button, a random number generator is used in a computer game to display a red stone with probability 0.1 or a blue stone with probability 0.9. What is the probability of getting two red stones and one blue stone when the button is pressed 3 times?

Soln. This equivalent to the previous example because the scenario described is sampling with replacement