# HOMEWORK #5 solution



5. The probability density function for a 2-dimensional gaussian (or normal) distribution is described by the following formula:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho_{XY}^2}} e^{-(x^2 - 2\rho_{XY}\cdot xy + y^2)/2(1-\rho_{XY}^2)}$$

where $\rho_{XY} \equiv$ correlation coefficient for $X$ and $Y$

a) Show that $f(x, y)$, may be written as a product of two 1-dimensional normal (or gaussian) pdf's when $\rho_{XY} = 0$.

b) For $y = 3$ and $\rho_{XY} = 1/3$, write $f(x, 3)$ as a constant, $k$, times a 1-dimensional normal (or gaussian) pdf. Find the values of the following: $k$, $\mu$, and $\sigma^2$.

**SOL'N:** a) We first substitute $\rho_{XY} = 0$ into $f(x, y)$:

$$f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

Because we have a sum in the exponent, we may write $f(x, y)$ as a product:

$$f(x,y) = \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}$$

This splits into two standard normal distributions. One in $x$ and one in $y$.

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Thus, we have the following 1-dimensional normal pdf's:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

**NOTE:** The factor of $1/2\pi$ must be split into two terms that result in each of the individual pdf's having a total area equal to unity.

b) Substituting $y = 3$ and $\rho_{XY} = 1/3$ yields the following expression for $f(x, 3)$:

$$f(x,3) = \frac{1}{2\pi\sqrt{1-\left(\frac{1}{3}\right)^2}} e^{\frac{-(x^2-2x+9)}{2\left(1-\left(\frac{1}{3}\right)^2\right)}}$$

or

$$f(x,3) = \frac{1}{2\pi\sqrt{8/9}} e^{\frac{-(x^2-2x+9)}{2(8/9)}}$$

Our goal is to write $f(x, 3)$ in the following form:

$$f(x,3) = k\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

We achieve this goal by completing the square in the exponent:

$$f(x,3) = \frac{1}{2\pi\sqrt{8/9}} e^{\frac{-(x-1)^2+8}{2(8/9)}}$$

We rewrite the exponential as a product.

$$f(x,3) = \frac{1}{2\pi\sqrt{8/9}} e^{\frac{-(x-1)^2}{2(8/9)}} e^{\frac{-8}{2(8/9)}}$$

We now identify $\mu = -1$ from the $(x - 1)$ in the exponent.

$$\mu = -1$$

We also identify $\sigma^2 = 8/9$ from the denominator in the exponent.

$$\sigma^2 = 8/9$$

Using these values, we have the following expression from which we may identify $k$:

$$f(x,3) = \frac{e^{\frac{-8}{2(8/9)}}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

or

$$k = \frac{e^{\frac{-8}{2(8/9)}}}{\sqrt{2\pi}} = \frac{e^{-4.5}}{\sqrt{2\pi}} = 0.00443$$

**NOTE:** The moral of the story is that the conditional probability $f_{X|Y}(x \mid y = 3)$, which is a scaled version of $f(x,3)$, will be a normal distribution.