ECE 3530/CS 3130: Engineering Probability & Stats
Lecture Notes Spring 2015

Lecture #5

1 Counting

1.1 Birthday Paradox: a Permutation Example

Example: Birthday Paradox
What is the probability that two people in this room will have the same birthday? Assume that "birthday" doesn't include year, and that each day of the year is equally likely (and exclude the leap day), and that each person's birthday is independent.
Solution:

1. How many ways are there for n people to have their birthday? Answer: Each person can have their birthday happen in one of 365 ways, assuming 365 days per year. So: 365^n.

2. How many ways are there to have all n people have **unique** birthdays? The first one can happen in 365 ways, the second has 364 left, and so on: ${}_{365}P_n = 365!/(365-n)!$.

3. Discrete uniform probability law:

$$P\left[\exists \text{no duplicate birthdays}\right] = \frac{365!/(365-n)!}{365^n}$$

See Fig. 1, which ends at $n = 30$. At $n = 50$, there is only a 3% chance. At $n = 60$, there is a 0.6% chance. At 110 students (our entire class) the probability is on the order of 10^{-8}. This is called the birthday paradox because most people would not guess that it would be so likely to have two people with a common birthday in a group as small as 20 or 30.

1.2 Combinations

Permutations require that the *order matters*. For example, in the three pills example where you take one pill at a time until you die or are cured, where K is the pill that will kill you, and C is the pill that will cure you, KC is different than CK (as different as life and death!). Certainly in words or license plates, as well, the order of the chosen letters matter. However, sometimes order doesn't matter, ie, if you are counting how many different sets could exist (since order doesn't matter in a set). As another example, in many card games you are dealt a hand which you

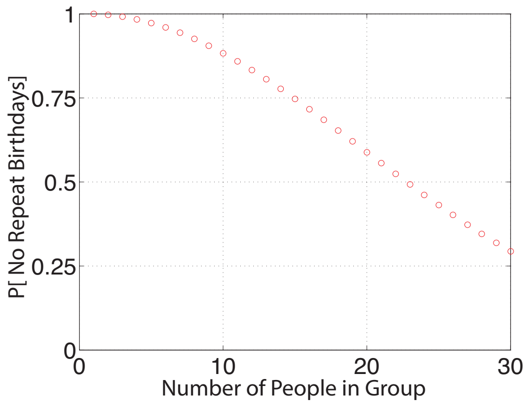

Figure 1: The probability of having no people with matching birth dates in a group of people, vs. the number in the group. Note the probability is less than half when the class is bigger than 23.

can re-order as you wish, thus the order you were dealt the hand doesn't matter.

Def'n: *Combinations*
The number of (unordered) combinations of size r taken from n distinct objects is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

We can think of permutations as overcounting in the case when order doesn't matter. If order doesn't matter and we order r distinct objects from a set of n, we'd count them as $n!/(n-r)!$. But for each ordering of length r, somewhere else in the permutation set is a different ordering of the same r objects. Since there are $r!$ ways to order these r objects, the formula ${}_nP_r$ overcounts the number of combinations by a factor of $r!$, or:

$$\binom{n}{r} ={}_n P_r \frac{1}{r!} = \frac{n!}{(n-r)!}\frac{1}{r!}.$$

Example: 5-card Poker
How many ways are there to be dealt a 5-card hand from a standard 52-card deck, where order doesn't matter? Note that in poker, all 52 cards are distinct.
Solution: We are taking $r = 5$ cards from a $n = 52$ distinct objects. Thus

$$\binom{52}{5} = \frac{52!}{(52-5)!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960$$

Example: Poker
Evaluate the probabilities of being dealt a "flush" in a five card poker

hand. Note the The standard deck has 52 cards, 13 cards of each suit; and there are four suits (hearts, diamonds, clubs, and spades). The thirteen cards are, in order, A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. The ace (A) can also be higher than the king (K). A flush is any 5 cards of the same suit. Note that this definition of flush includes the royal flush.
Solution:

1. How many different hands are there? Order doesn't matter in a poker hand, so this is a combination problem. A: 52 choose 5, or 2,598,960.

2. $P[Flush]$? (A flush is any 5 cards of the same suit, not including any straight flushes.) There are really two "operations" you do to build a flush. First, you select the suit. There are four suits, and you choose one. Then there are 13 of each suit, of which you choose five cards. so
$$\binom{13}{5}\binom{4}{1} = 1287(4) = 5148$$
ways to have a flush.

Example: Pass / Try Again / Fail
There are 9 PhD students who take a qualifying exam. Their grade will be, "pass", "retake", or "fail". The program has determined that two students will pass, three will retake, and four will fail. How many ways are there for the eight students to receive grades?
Solution: There are two operations to be performed. First, we must select the two students from 9 who will pass, which can occur in $\binom{9}{2}$ ways. Second, we must select the three students from the remaining 7 who will get a retake grade, which can occur in $\binom{7}{3}$ ways. The remaining four will fail. Thus there are
$$\binom{9}{2}\binom{7}{3} = \frac{9!}{(9-2)!2!}\frac{7!}{(7-3)!3!} = \frac{9!}{2!3!4!} = 1260$$

Note there is a reason the 7! cancels in the two fractions. This is my preferred way of doing problems which the Walpole book describes as "partitioning n objects into r cells", given in Theorem 2.5.