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ECE 3530

Homework 7

$$f(x,y)\begin{cases} e^{y}e^{-xe^{y}}, & 0<x<\infty \text{ and } 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

a)

$$z = e^{y}$$

$$\mu_X = E(X) = \int_0^\infty x\left[\int_1^e z e^{-xz}\frac{dz}{z}\right]dx$$

$$= \int_0^\infty x\left[\int_1^e e^{-xz}\,dz\right]dx$$

$$= \int_0^\infty x\left(\frac{e^{-xz}}{-x}\right)\Bigg|_{z=1}^{z=e} dx = \int_0^\infty -\left(e^{-xe}-e^{-x}\right)dx$$

$$e^{-\infty}=0$$

$$\mu_X = 1-\frac{1}{e}$$

b)

$$\sigma_Y^2 = E(Y^2)-\mu_Y^2$$

$$E(Y^2) = \int_0^1 y^2 f_Y(y)\,dy = \int_0^1 y^2\left[\int_0^\infty e^{y}e^{-xe^{y}}\,dx\right]dy$$

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(Y^2) = \int_0^1 y^2 \cdot 1 \cdot dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\mu_Y = \int_0^1 y\, f_Y(y)\, dy = \int_0^1 y \cdot 1 \cdot dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\sigma_Y^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$\sigma_Y = \frac{1}{12}$$

2.

$$f_{K,L}(k,l) = \begin{cases} ak^l, & k \in \{1,2,3\}, \ l \in \{1,2,3\} \\ 0, & \text{otherwise} \end{cases}$$

a)

L \ K	1	2	3	
1	a	2a	3a	$P_L(1)$
2	a	4a	9a	$P_L(2)$
3	a	8a	27a	$P_L(3)$
	$P_K(1)$	$P_K(2)$	$P_K(3)$	

b)

$$\sum_L \sum_K f(k,l) = a + a + a + 2a + 4a + 8a + 3a + 9a + 27a = 56a = 1$$

$$a = \frac{1}{56}$$

c)

$$f_K(k) = \sum_l P(l,k) = (a+a+a) + (2a+4a+8a) + (3a+9a+27a) = 1$$

$$f_l(l) = \sum_k P(l,k) = ($$

d)

$$f_{L|K}(l|k=1) = \sum_l \sum_{k=1} a+a+a = \frac{3}{56}$$

e)

$$P(1,1) = P_K(1)\,P_L(1)$$

$$\frac{1}{56} = \left(\frac{3}{56}\right)\left(\frac{6}{56}\right) = \frac{18}{56}$$

Not independent

5)

$$P_K(k) = \begin{cases} \frac{3}{56}, & k=1 \\ \frac{14}{56}, & k=2 \\ \frac{39}{56}, & k=3 \end{cases}$$

$$P_L(l) = \begin{cases} \frac{6}{56}, & 1 \\ \frac{14}{56}, & 2 \\ \frac{36}{56}, & 3 \end{cases}$$

$$\mu_K = E(K) = \sum_k k P(k)$$
$$= \left(1 \cdot \frac{3}{56}\right) + \left(2 \cdot \frac{14}{56}\right) + \left(3 \cdot \frac{39}{56}\right)$$
$$= \frac{3}{56} + \frac{28}{56} + \frac{117}{56}$$
$$= 2.64$$

$$\mu_L = E(L) = \sum_l l P(l)$$
$$= \left(1 \cdot \frac{6}{56}\right) + \left(2 \cdot \frac{14}{56}\right) + \left(\frac{36}{56} \cdot 3\right)$$
$$= 2.53$$

$$\sigma_{KL} = \sum_l \sum_k (k - \mu_K)(l - \mu_L) f(k,l)$$

$$= .0842$$

$$(1-2.64)(1-2.53)\frac{1}{56} + (1-2.64)(2-2.53)\cdot\frac{1}{56}$$
$$+ (1-2.64)(3-2.53)\frac{1}{56} + (2-2.64)(1-2.53)\frac{2}{56}$$
$$+ (2-2.64)(2-2.53)\frac{4}{56} + (2-2.64)(3-2.53)\frac{8}{56}$$
$$+ (3-2.64)(1-2.53)\frac{3}{56} + (3-2.64)(2-2.53)\frac{9}{56}$$
$$+ (3-2.64)(3-2.53)\, 27/56$$

2g)

$\sigma_k^2 = E(k^2) - \mu_k^2$

$\sigma_l^2 = E(L^2) - \mu_y^2$

$E(k^2) = \sum_k k^2 P(k)$

$E(L^2) = \sum_l l^2 P(l)$

$E(k^2) = (1^2)\left(\frac{3}{56}\right) + (2^2)\left(\frac{14}{56}\right) + (3^2)\left(\frac{39}{56}\right)$

$= 7.32$

$E(l^2) = (1^2)\left(\frac{6}{56}\right) + (2^2)\left(\frac{14}{56}\right) + (3^2)\left(\frac{36}{56}\right)$

$= 6.89$

$\sigma_K^2 = 7.32 - 6.97 = .35$

$\sigma_L^2 = 6.89 - 6.4 = .49$

$\sigma_K = \sqrt{.35} = .59$

$\sigma_L = \sqrt{.49} = .7$

$\rho_{KL} = \frac{\sigma_{KL}}{\sigma_K \sigma_L} = \frac{.0842}{(.59)(.7)}$

$\rho_{KL} = .204$

3.

component life in years
component life in years

Joint pdf of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$$

$P[0 < X < 1 \mid Y=2]$?

$\int_0^1 e^{-x-2}$

$e^{-2}\int_0^1 e^{-x}\,dx$

$e^{-2}\left[-e^{-x}\right]_0^1$

.0855

b)

$\int_0^\infty e^{-x-y}\,dx = e^{-y}\int_0^\infty e^{-x}\,dx = e^{-y}\left[-e^{-x}\right]_0^\infty = e^{-y}$

* same for x

Independent

4.

$X \sim u[0,1]$ or $f_X(X) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$Y \sim n(\mu_Y = 0, \sigma_Y = 1)$ or $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2}}$

a)

$Z = aX + bY$

find a and b in terms of μ_Z and σ_Z^2

$\sigma_Y^2 = 1 \quad \sigma_X^2 = \frac{1}{12} \quad \mu_X = 1/2 \quad \mu_Y = 0$

$\sigma_Z^2 = a\sigma_X^2 + b\sigma_Y^2$

$\sigma_Z^2 = a/12 + b$

$\mu_Z = \frac{a}{2} + b(0)$

$2\mu_Z = a$

$\sigma_Z^2 = \frac{2\mu_Z}{12} + b$

$\sigma_Z^2 = \frac{\mu_Z}{6} + b$

$\sigma_Z^2 - \frac{\mu_Z}{6} = b$

5. *Use info from #4

$Z = 6(X_1 + X_2 - 1)$

a) Find σ_Z^2, correct value for standard normal distribution?

$Z = 6X_1 + 6X_2 - 6$

The -6 shifts pdf left, has no effect on variance of Z and may be ignored.

$\sigma_{X_1}^2 = \frac{1}{12}$

$\sigma_{X_2}^2 = \frac{1}{12}$

$\sigma_Z^2 = 6^2 \sigma_{X_1}^2 + 6^2 \sigma_{X_1}^2 = \frac{36}{12} + \frac{36}{12} = 6$

variance for standard normal dist. is 1, so 6 is icorrect.

b)

Find $\rho_{X_1 Z} \equiv \frac{\sigma_{X_1 Z}}{\sigma_{X_1} \sigma_Z}$

*sub for z and work in terms of X_1 & X_2

$X_1 Z = X_1 6(X_1 + X_2 - 1) = 6(X_1^2 + X_1 X_2 - X_1)$

*calculate covariance

$\sigma_{X_1 Z} = E(X_1 Z) - \mu_{X_1} \mu_Z$

$E(X_1 Z) = 6[E(X_1^2) + E(X_1 X_2) - E(X_1)]$

$$E(X_1^2) = \sigma_1^2 + \mu_1^2 = \frac{1}{12} + \left(\frac{1}{2}\right)^2 = \frac{1}{3}$$

$$E(X_1 X_2) = E(X_1)E(X_2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$E(X_1) = \frac{1}{2}$$

*Put together

$$E(X_1 Z) = \mu_{X_1 Z} = 6\left(\frac{1}{3} + \frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}$$

* Now find mean of X_1 and Z

$\mu_{X_1} = \frac{1}{2}$ *since X_1 is uniform distribution on $[0,1]$

$$\mu_Z = E(6X_1 + 6X_2 - 6) = 6E(X_1) + 6E(X_2) - 6$$
$$= 6\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right) - 6 = 0$$

* product of means will be zero, therefore:

$$\sigma_{X_1 Z} = E(X_1 Z) = \frac{1}{2}$$

* Calculate variances of X_1 and Z

$\sigma_{X_1}^2 = \frac{1}{12}$ *since uniform dist. on $[0,1]$

* since mean of $Z = 0$:

$$\sigma_Z^2 = E(Z^2) = 36\left[E(X_1)^2 + E(X_2^2) + 1 + 2E(X_1 X_2) - 2E(X_1) - 2E(X_2)\right]$$

* Use earlier results and observe X_1 and X_2 is indepent, which implies $E(X_1X_2)$ is product of expected values.

$$\sigma_Z^2 = E(Z^2) = 36\left[\frac{1}{3} + \frac{1}{3} + 1 + \left(2 \cdot \frac{1}{2} \cdot \frac{1}{2}\right) - \left(2 \cdot \frac{1}{2}\right) - \left(2 \cdot \frac{1}{2}\right)\right]$$
$$= 36\left(\frac{1}{6}\right) = 6$$

* Take square root of above Z variance and calculate correlation.

$$\rho_{X_1 Z} = \frac{\sigma_{X_1 Z}}{\left(\sqrt{\frac{1}{12}}\right)\left(\sqrt{6}\right)} = \frac{\sqrt{2}}{2} = 0.7071$$