Probability

Defn: Outcomes are possible results of a hypothetical experiment.

ex. Dice roll. Outcomes: 1, 2, 3, 4, 5, 6 (integers)

ex. Items coming off production line.
Outcomes = "defective" and "non-defective"
(Categorical data)

ex. Change in ventricular volume.
Outcomes: all possible real numbers

Any recording of a specific atcome whether it is categorical or numerical is caled an observation.

Defn: S = Sample Space: The set of all possible outcomes of an experiment.

ex. Two ways to define the sample space of a 6-sided dice roll.

$$S_1 = \{1, 2, 3, 4, 5, 6\}$$

$$S_2 = \{\text{even}, \text{odd}\}$$

ex. Coin toss

$$S = \{H, T\}$$

ex. 2 consecutive coin tosses

$$S = \{HH, HT, TH, TT\}$$

ex. Number of students attending any given ECE 3530 lecture in Spring 2009.

$$S = \{x \mid 0 \le x \le 71\}$$

easier than enumerating all integers between 0 and 71.

ex. Life in years of an electronic Component

$$S = \{t \mid t \geq 0\}$$

<u>Defn.</u> To every point in the sample space, we assign a real number between 0 and 1 called its probability. The sum of the probabilities of all points in the sample space is 1.

ex. Fair dice $P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$

$$1 = 1/6 + 1/6 + \dots + 1/6$$

ex. A rigged dice is known to have probability 1/2 for the outcome 6 and all other outcome are known to be equally likely. What is the probability for outcome 2?

$$P(1) + P(2) + \dots + P(5) + P(6) = 1$$

$$5x + 1/2 = 1$$

$$P(2) = x = \frac{1 - 1/2}{5} = 1/10$$

<u>Defn</u> An <u>event</u> is a subset of the sample space. In other words, it is a set of outcomes but not necessarily all of them.

ex. Dice Roll $S = \{1,2,3,4,5,6\}$

event A : outcome is even $A = \{2,4,6\}$

event B : outcome is 5 $B = \{5\}$

event C : outcome is even or greater than 3

$$C = \{2,4,5,6\}$$

<u>Defn</u> : The probability of an event is the sum of the probabilities (weights) of all outcomes included in that event.

$P(A) = P(2) + P(4) + P(6) = 3/6$ Fair dice ex

$P(B) = P(5) = 1/6$

*How about the rigged dice example above?

Ex. A coin is twice. What is the probability that at least one head occurs.

$S = \{HH, HT, TH, TT\}$

$A = \{HH, HT, TH\}$

assuming a balanced coin all 4 atcomes are equaly likely; therefore $P(A) = 3 \times \frac{1}{4} = \frac{3}{4}$

<u>Defn</u>: Complement of an event A is all outcomes in the sample space that are not in A. Complement is denoted by A'

ex. $A = \{HH, HT, TH\}$ from above

$A' = \{TT\}$ $\quad P(A') = 1/4$

$$\boxed{P(A') = 1 - P(A)}$$

equivalently $P(A) + P(A') = P(S) = 1$

ex. Dice roll $\quad S = \{1, 2, 3, 4, 5, 6\}$

$B = \{2, 4, 6\}$ even atcome

$P(B) = 1/2$

$B' = \{1, 3, 5\}$

$P(B') = 1/2$

Defn: $A \cap B$ denotes the intersection of two events A and B. $A \cap B \equiv$ outcomes in both A and B

ex.: $A = \{1,2,5,6\}$ $B = \{2,4,6\}$

$A \cap B = \{2,6\}$

Defn: A and B are mutually exclusive if they have no elements in common. $A \cap B = \emptyset$ empty set

Note: $\emptyset$ empty set $P(\emptyset) = 0$

ex. $A = \{1,2,5,6\}$ and $B = \{3\}$ are mutually exclusive. $A \cap B = \emptyset$

Defn: $A \cup B$ denotes the union of two events A and B. $A \cup B \equiv$ outcomes in A or B or both.

ex. $A = \{2,3,5\}$ $B = \{4,5\}$ $C = \{1,6\}$

$P(A) = 3/6$ $P(B) = 2/6$ $P(C) = 2/6$

$A \cup C = \{1,2,3,5,6\}$ $P(A \cup C) = 5/6$

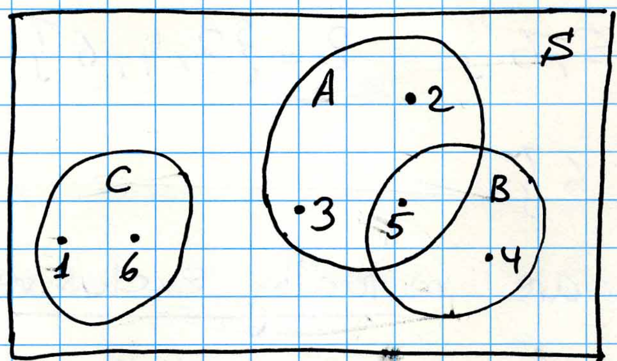
$P(A \cup C) = P(A) + P(C) = 3/6 + 2/6 = 5/6$

Note: A and C are mutually exclusive

$A \cup B = \{2,3,4,5\}$ We count 5 which occurs in both events only once

$$P(A \cup B) = \frac{4}{6} \neq \underbrace{P(A) + P(B) = \frac{3}{6} + \frac{2}{6}}_{\text{This counts 5 twice.}} \; !$$

Venn Diagrams: Helps us see relationships (unions, intersections) of events visually.



The rectangle denotes the sample space

Now we see visually why $P(A \cup B) \neq P(A) + P(B)$ while $P(A \cup C) = P(A) + P(C)$

Defn: For any two events A and B

$$P(A \cup B) = \underbrace{P(A) + P(B)}_{\text{This counts the elements in } A \cap B \text{ twice!}} - \underbrace{P(A \cap B)}_{\Rightarrow \text{ So subtract them once.}}$$

Another example: Let $D = \{2\}$ What is $P(A \cup D)$?

In our example: $P(A) = 3/6$ $P(B) = 2/6$ $P(A \cap B) = 1/6$

$P(A) + P(B) - P(A \cap B) = \dfrac{4}{6}$ which is what we expected.

If two events are mutually exclusive, then their intersection is ϕ so the formula simplifies:

In our example: $P(A) = 3/6$ $P(C) = 2/6$ $P(A \cap C) = 0$

$$P(A \cup C) = P(A) + P(C) - \cancel{P(A \cap C)}^{0}$$
$$= P(A) + P(C) = 3/6 + 2/6 = 5/6$$

Defn: If $A_1, A_2, A_3, \ldots$ are a sequence of mutually exclusive events then $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

Note: Remember the probability of an event A is the sum of the probabilities of the outcomes included in

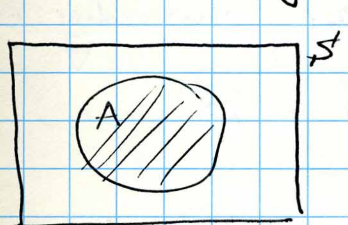
Now we can think of this as

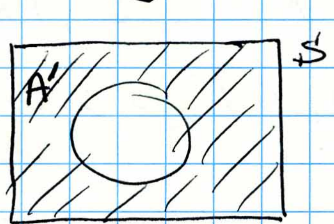
$$(*) \quad P(A) = P(O_1) + P(O_2) + P(O_3) + \ldots$$

where $O_1, O_2, O_3, \ldots$ are the outcomes included in A.

O_i are atomic events (can't break them down further) and by defn they are mutually exclusive $O_i \cap O_j = \emptyset$ if $i \neq j$. Hence, the result $(*)$ above.

Ex: Using venn diagrams to derive event relationships





$A \cap A' = \emptyset$

$A \cup A' = S$

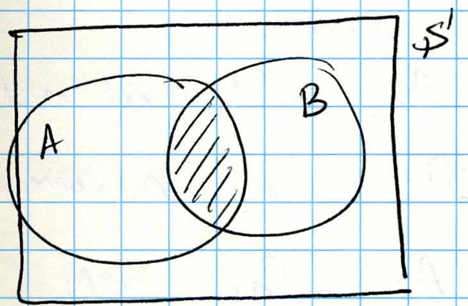
$(A')' = A$

$(S')' = \emptyset$

$(\emptyset)' = S$

$A \cap \emptyset = \emptyset$

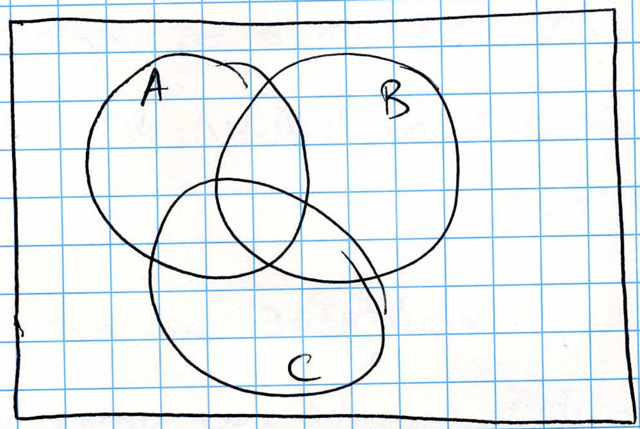
$A \cup \emptyset = A$



$A \cap B$

$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$



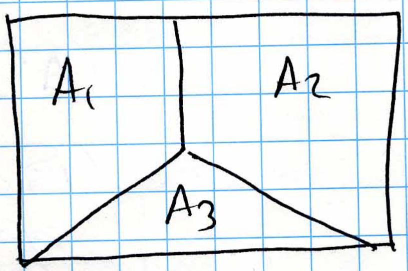
Think of how many times each area is counted.

$$P(A \cup B \cup C) = ?$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Defn. A collection of events $A_1, A_2, \ldots, A_n$ are called a partition of the sample space S if (1) $A_1, A_2, \ldots, A_n$ are mutually exclusive and (2) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

ex.



Playing cards

A_1 : odd numbered

A_2 : even numbered

A_3 : face cards

Notice if $A_1, A_2, \ldots A_n$ are a partition then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(S) = 1$$

Any event A and its complement A' form a partition of S.

Theorem: Law of total probability

$$P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$$

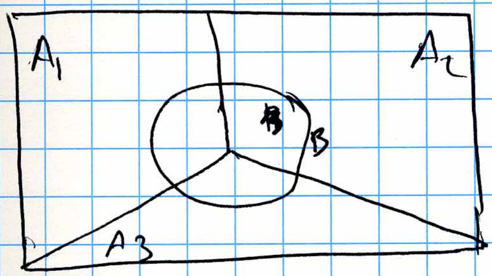
for any partition $A_1, A_2, \ldots, A_n$

Proof:
$$P(B) = P(B \cap S) = P(B \cap (A_1 \cup \ldots \cup A_n))$$
$$= P((B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n))$$

since $A_1, \ldots, A_n$ are mutually exclusive so are $A_1 \cap B, \ldots, A_n \cap B$

then $P(B) = P(B \cap A_1) + \cdots + P(B \cap A_n)$

ex.



Playing cards example

B = Card is hearts

$A_1 \cap B$ = card is hearts and odd numbered

$A_2 \cap B$ = " " " " even "

$A_3 \cap B$ = " " " " face card