VARIANCE

Expectation of a function of a random variable

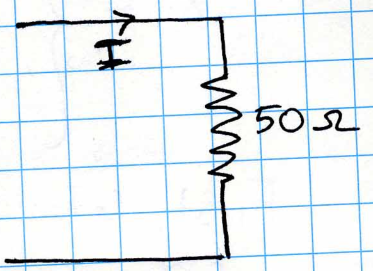
Let X be a random variable with probability distribution $f(x)$.
Let $g(X)$ be a function of the random variable X.
Then the expected value of $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x) \qquad \text{DISCRETE}$$

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad \text{CONTINUOUS}$$

Notice that $g(X)$ can be thought of as another random var.

Example:



A random current I flows through a resistor with $R = 50\,\Omega$. The probability density function for the current is given as

Power consumption (W)

$P = I^2 R$

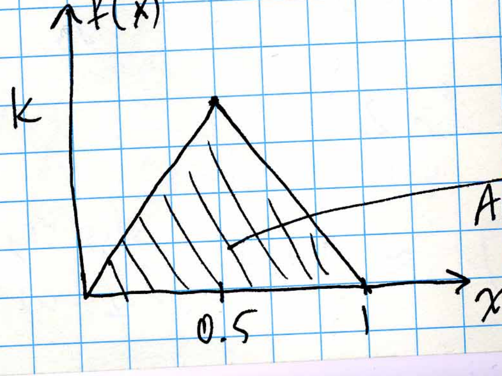
$$f(x) = \begin{cases} 2kx, & 0 \le x < 0.5 \\ 2k(1-x), & 0.5 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find the value of k which makes $f(x)$ a valid probability density function.

b) Find the expected value of the current I?

c) Find the expected value of the power dissipated over the resistor. Note: power $= I^2 R$

Soln.



For valid prob. density fn. we need $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Area under $f(x)$.

Triangle. $\frac{k}{2} = 1 \Rightarrow k = 2$

b) $$\mu_I = E[I] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{0.5} x \cdot 4x\, dx + \int_{0.5}^{1} x \cdot 4(1-x)\, dx$$

$$= \frac{4}{3} x^3 \Big|_0^{0.5} + \int_{0.5}^{1} 4x\, dx - \int_{0.5}^{1} 4x^2\, dx$$

$$= \frac{4}{3} x^3 \Big|_0^{0.5} + 2x^2 \Big|_{0.5}^{1} - \frac{4}{3} x^3 \Big|_{0.5}^{1}$$

$$= \frac{4}{3}(0.5)^3 + 2 - 2(0.5)^2 - \frac{4}{3}(1)^3 + \frac{4}{3}(0.5)^3$$

$$= \frac{\cancel{8}}{3} \times \frac{1}{\cancel{8}} + 2 - 2 \times \frac{1}{4} - \frac{4}{3} = \frac{1}{3} + 2 - \frac{1}{2} - \frac{4}{3}$$

$$= -1 + 2 - \frac{1}{2} = \frac{1}{2}.$$

This was what our intuition would expect since $f(x)$ was symmetric about 0.5

c) $$\mu_p = E[I^2 R] = \int_{-\infty}^{\infty} 50x^2 f(x)\, dx$$

$$= \int_0^{0.5} 50x^2 \cdot 4x\, dx + \int_{0.5}^{1} 50x^2\, 4(1-x)\, dx$$

Notice !!

$$\boxed{\mu_p \neq (\mu_I)^2 R}$$

$$= \frac{200}{4} x^4 \Big|_0^{0.5} + \int_{0.5}^{1} 200x^2\, dx - \int_{0.5}^{1} 200x^3\, dx$$

$$= 50x^4 \Big|_0^{0.5} + \frac{200}{3} x^3 \Big|_{0.5}^{1} - 50x^4 \Big|_{0.5}^{1}$$

$$= \frac{50}{16} - 0 + \frac{200}{3} - \frac{200}{3 \times 8} - 50 + \frac{50}{16} = 14.58W$$

A special case of expected value is when $g(X) = (X-\mu)^2$ where μ is the mean of the random variable X.

Defn: Let X be a r.v. with prob. dist. $f(x)$ and mean μ.

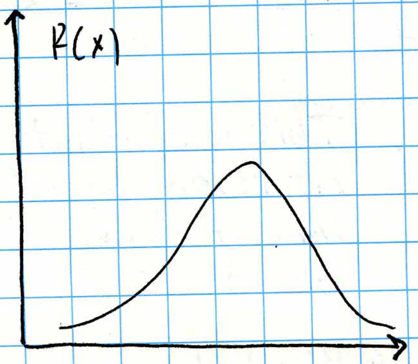
- The variance of X is

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x) \quad \text{DISCRETE}$$

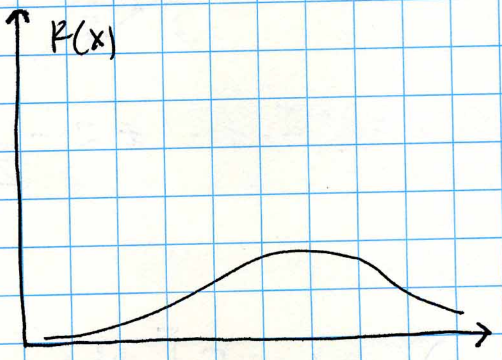
$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{CONTINUOUS}$$

- The positive square root of variance is called the standard deviation (σ).

Variance measures how spread out a distribution is.



Lower variance
Tighter $f(x)$



Higher variance
More spread out $f(x)$

Example: Compute the variance of the current random variable from the previous example.

Remember $f(x) = \begin{cases} 4x & , 0 \le x \le 0.5 \\ 4-4x & , 0.5 \le x \le 1 \\ 0 & , \text{elsewhere.} \end{cases}$

$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-0.5)^2 f(x)\,dx$$

μ computed as 0.5 previously

$$\sigma^2 = \int_0^{0.5} (x-0.5)^2 4x\,dx + \int_{0.5}^{1} (x-0.5)^2 (4-4x)\,dx$$

$$= \int_0^{0.5} 4x^3 - 4x^2 + x\,dx + \int_{0.5}^{1} -4x^3 + 8x^2 - 5x + 1\,dx$$

$$= x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 \Big|_0^{0.5} + -x^4 + \frac{8}{3}x^3 - \frac{5}{2}x^2 + x \Big|_{0.5}^{1}$$

$$= \frac{1}{16} - \frac{4}{3}\times\frac{1}{8} + \frac{1}{2}\times\frac{1}{4} - 1 + \frac{1}{16} + \frac{8}{3} - \frac{1}{3}$$

$$- \frac{5}{2} + \frac{5}{2}\times\frac{1}{4} + 1 - 0.5 = 0.0417$$

A faster (sometimes) way to compute σ^2

$$\sigma^2 = E[X^2] - \mu^2.$$

<u>Proof</u>: $\sigma^2 = \int_{-\infty}^{\infty} (X-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)\,dx$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \int_{-\infty}^{\infty} 2\mu x\, f(x)\,dx + \int_{-\infty}^{\infty} \mu^2 f(x)\,dx$$

1st term: $E[X^2]$

2nd term $-2\mu \int_{-\infty}^{\infty} x f(x)\,dx = -2\mu^2$

3rd term $\mu^2 \int_{-\infty}^{\infty} f(x)\,dx = \mu^2 \cdot 1.$

μ is a constant and can be moved out of the integral

So $\sigma^2 = E[X^2] - \mu^2.$

<u>Back to our example</u>:

$$E[X^2] = \int_0^{0.5} x^2 4x\,dx + \int_{0.5}^{1} x^2 (4-4x)\,dx$$

$$= x^4 \Big|_0^{0.5} + \frac{4}{3}x^3 \Big|_{0.5}^{1} - x^4 \Big|_{0.5}^{1}$$

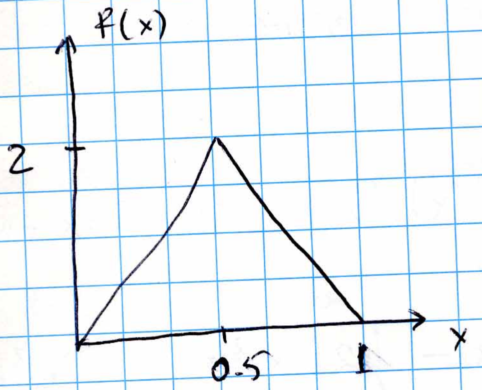
$$= \frac{1}{16} + \frac{4}{3} - \frac{4}{3} \times \frac{1}{8} - 1 + \frac{1}{16} = 0.2917$$

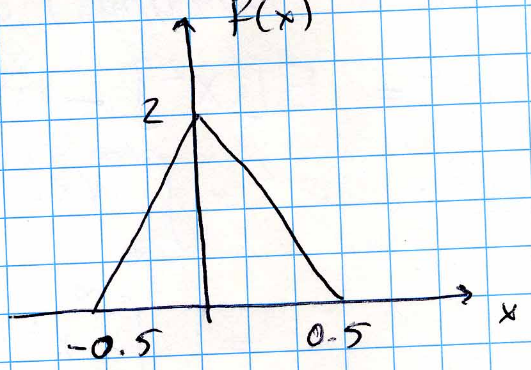
Therefore, $\sigma^2 = E[x^2] - \mu^2 = 0.2917 - (0.5)^2$

$$= 0.0417. \text{ (same as before)}$$

Some observations:

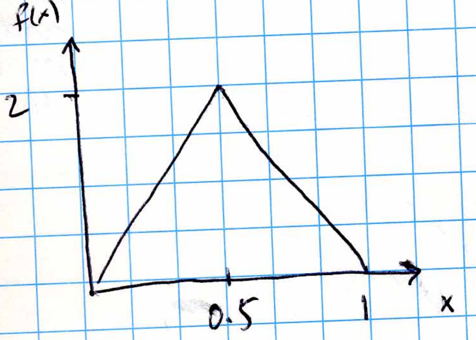
* Only the shape of $f(x)$ determines its variance, shifting $f(x)$ left or right doesn't affect σ^2

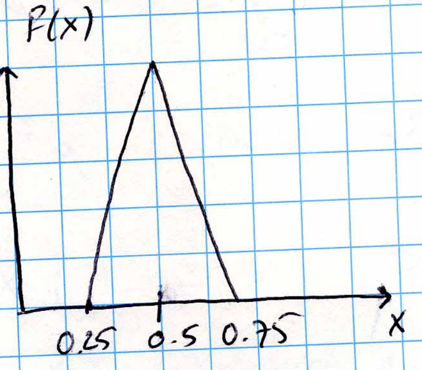




Both of these have the same σ^2.

* Tighter distributions have smaller variance.





$\sigma^2 = 0.0417$ $\qquad$ $\sigma^2 = 0.0104$