ECE 3530/CS 3130: Engineering Probability & Stats
Lecture Notes Spring 2015

Lecture #6

1 Independence

<u>Def'n:</u> *Independence of a Pair of Sets*
Sets A and B are independent if and only if $P[A \cap B] = P[A] P[B]$.

Example: Parking and Speeding.
The probability you get a ticket for an expired parking meter on any given day is $1/50$. The probability that you get a speeding violation on any given day is $1/200$. The probability that you get a parking meter ticket and speeding violation on the same day is $1/10,000$. Are the two events independent?
Solution: The question is the same as evaluating the truth of:

$$P[A \cap B] = P[A] P[B]$$

where A is the event you get a parking meter ticket, and B is the event you get a speeding violation. Note $A \cap B$ is the event both happen in the same day.

$$\frac{1}{10000} = \frac{1}{50}\frac{1}{200}$$

This statement is true. Therefore the two events are independent.

Example: Set independence
Consider $S = \{1, 2, 3, 4\}$ with equal probability, and events $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{4\}$.

1. Are A and B independent?
2. Are B and C independent?

Solution:

1. $P[A \cap B] = P[\{1\}] = 1/4 = (1/2)(1/2)$. Yes, A and B are independent.
2. $P[B \cap C] = P[\emptyset] = 0 \neq (1/2)(1/4)$. B and C are not independent.

Are B and C disjoint? Yes; but they are not independent. Independence is NOT about non-overlapping; orthogonal might be a better word. Actually, as we will show when discussing conditional independence, two sets are independent if knowing whether one has happened does not affect the probability of the other.

2 Conditional Probability

Def'n: *Conditional Probability,* $P[A|B]$
The conditional proabability of event A given that event B has occurred, denoted $P[A|B]$, is defined as:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

whenever $P[B] > 0$, and is undefined in the case when $P[B] = 0$.

You can read the vertical bar as the word "given". There is an implicit "has occurred" at the end of that phrase (after the given event B).

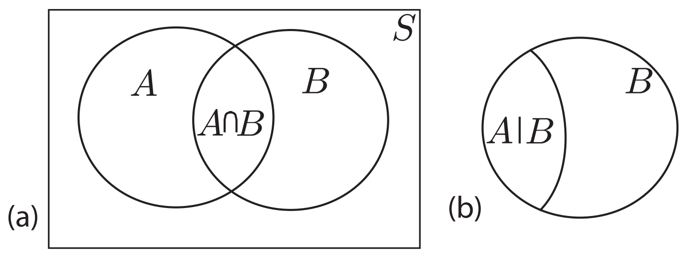

Figure 1: The original sample space shown in (a) is reduced to only the outcomes in event B when considering the probability of any event conditioned on B, as in (b). For example, part of the set A is no longer possible; the remaining outcomes in A can be called $A|B$.

Notes:

1. We're defining a new probability model, knowing more about the world. Instead of $P[\cdot]$, we call this model $P[\cdot|B]$. See Figure 1. All of our Axioms STILL APPLY, but with B as the sample space.

2. NOT TO BE SAID OUT LOUD because its not mathematically true in any sense. But you can remember which probability to put on the bottom, by thinking of the $|$ as $/$ – you know what to put in the denominator when you do division.

We use this definition to get the very important result, by multiplying both sides by $P[B]$, that

$$P[A \cap B] = P[A|B] P[B] \tag{1}$$

This is true for ANY two sets A and B. Recall $P[A \cap B] = P[A] P[B]$ if and only if A and B are independent. Thus, by the way, if A and B are independent, then $P[A|B] = P[A]$.

Relationship to Multiplication Rule The rule in Equation (1) is an extension of the multiplication rule, which was for counting events, to probabilities of events. Remember the definition of the multiplication rule says that "If there are n_1 ways of doing operation 1, and for each of those ways, there are n_2 ways of doing operation 2, then there are $n_1 n_2$ ways of doing the two together". In Equation (1), it essentially says,

- If there is a probability $P[B]$ of a first thing (B) happening, and when B occurs, there is a probability of $P[A|B]$ of a second thing (A) happening, then there is a probability $P[A \cap B] = P[A|B]\,P[B]$ of both events happening.

Tree Diagram for Conditional Probabilities The relationship in (1) makes explicit why we can multiply probabilities in a tree together to get the probability of events jointly occurring. See Figure 2.

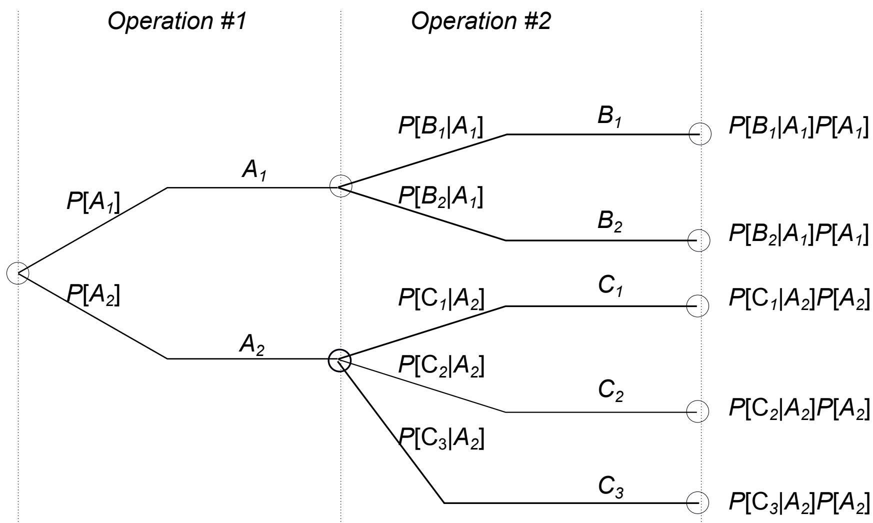

Figure 2: The probability written onto each branch is the probability conditioned on all the events listed to its left. Thus the probability of the intersection of all of the events from root to a "leaf" (the $\circ$ at the right end of the branch) is the product of the probabilities written from root to leaf.

With Law of Total Probability Given that B occurs, now we know that either $A \cap B$ occurs, or $A^c \cap B$ occurs. For events A and B, using the law of total probability and the partition A, A^C,

$$P[A|B] \triangleq \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B]}{P[A \cap B] + P[A^c \cap B]}$$

Note: Conditional probability almost always has the form, $\frac{x}{x+y}$. If $P[A|B] = \frac{x}{x+y}$ then $P[A^c|B] = \frac{y}{x+y}$. Note the two terms add to one.

Example: Coin then Die

Your assistant rolls a fair coin. If it is heads, he rolls a fair 4-sided die.

If it is tails, he rolls a fair 6-sided die. In either case, he emails you the number on the die, but not what die he rolled or the coin outcome. You get the number 3 by email. What is the probability that his coin toss was heads?
Solution: Draw a tree diagram to help with this one. $A =$ heads on the coin, $B =$ getting the number 3. The probability is

$$P[A|B] = \frac{P[A \cap B]}{P[A \cap B] + P[A^c \cap B]}$$

Example: Three Card Monte

(Credited to Prof. Andrew Yagle, U. of Michigan.)

There are three two-sided cards: red/red, red/yellow, yellow/yellow. The cards are mixed up and shuffled, one is selected at random, and you look at one side of that card at random. You see red. **What is the probability that the other side is red?**

Three possible lines of reasoning on this:

1. Bottom card is red only if you chose the red/red card: $P = 1/3$.
2. You didn't pick the yellow/yellow card, so either the red/red card or the red/yellow card: $P = 1/2$.
3. There are five sides which we can't see, two red and three yellow: $P = 2/5$.

Which is correct?
Solution: BR = Bottom red; TR = Top red; BY = Bottom yellow; TY = Top yellow.

$$\begin{aligned} P[BR|TR] &= \frac{P[BR \text{ and } TR]}{P[TR]} \\ &= \frac{P[BR \text{ and } TR]}{P[BR \text{ and } TR] + P[BY \text{ and } TR]} \\ &= \frac{2/6}{2/6 + 1/6} = \frac{1/3}{1/2} = 2/3. \end{aligned}$$

2.1 Independence and Conditional Probability

We know that for any two sets A and B, that $P[A \cap B] = P[A|B]\,P[B]$. Recall that independent sets have the property $P[A \cap B] = P[A]\,P[B]$. So, independent sets also have the property that

$$\begin{aligned} P[A|B]\,P[B] &= P[A]\,P[B] \\ P[A|B] &= P[A] \end{aligned}$$

as long as $P[B] > 0$.

Thus the following are equivalent:

1. $P[A \cap B] = P[A]\,P[B]$,
2. $P[A|B] = P[A]$, and
3. $P[B|A] = P[B]$,

If one is true, all of them are true. If one is false, all are false.

Example: Three different dice
There are three dice in a box, one a four-sided die, one a six-sided die, and one a 10-sided die. Each is fair, that is, each number on a die is equally likely to come up when rolled. One die is picked at random from the three, and then rolled.

1. What is the sample space? How many elements does it have?
2. What is the probability of getting the 4-sided die and rolling a 1?
3. What is the probability of rolling a 1?
4. What is the probability of rolling a 10?

Solution:

1. The sample space could be just $\{1, \ldots, 10\}$. Or, it could be all of the dice-number pairs, eg, $\{(4sided, 1), \ldots (4sided, 4), (6sided, 1), \ldots (6sided, 6), (10sided, 1), \ldots (10sided, 10)$ The former has 10, the latter 20.

2. The probability of getting the 4-sided die and rolling a 1 is

   $P[roll = 1 \cap 4sided] = P[4 - sided]\,P[roll = 1 \,|\, 4 - sided] = (1/3)(1/4) = 1/12.$

3. There are three ways of rolling a 1. Using the law of total probability:

   $P[roll = 1] = P[roll = 1 \cap 4sided] + P[roll = 1 \cap 6sided] + P[roll = 1 \cap 10sided]$

   We have the $P[roll = 1 \cap 4sided]$ above. Similarly, $P[roll = 1 \cap 6sided] = (1/3)(1/6)$ and $P[roll = 1 \cap 10sided] = (1/3)(1/10)$. So

   $$P[roll = 1] = \frac{1}{12} + \frac{1}{18} + \frac{1}{30} \approx 0.1722$$

4. There's only one way of rolling a 10.

   $P[roll = 10] = P[roll = 10 \cap 10sided] = P[10sided]\,P[roll = 10 \,|\, 10sided] = (1/3)(1/10) = 1/30.$

Note a tree diagram is really useful for this problem.