HOMEWORK #5 solution

2. A power company is interested in analyzing how much power must be delivered to customers if an average voltage of exactly 110 V must be maintained over time and the variations in voltage are normally (or gaussian) distributed:

Voltage $V \sim n(v; \mu = 110, \sigma = 3)$ Volts

Power $W = (V - \mu)^2$

a) Find an expression for the pdf of W.

Hint: this is a scaled version of a random variable that has a χ^2 distribution.

b) Find the value of $f_W(w = 1\text{ V})$.

SOL'N: a) We define a new variable, Z, that we know has a chi-squared (χ^2) distribution:

$$Z \equiv \left(\frac{V - \mu}{\sigma}\right)^2$$

This variable differs from W by only a scaling factor, σ^2:

$$W = \sigma^2 Z$$

The following identity allows us to transform the probability density function (pdf) for Z into the pdf for W:

$$f_Y(y) = \frac{1}{|a|} f_X\left(x = \frac{y - b}{a}\right) \quad \text{when} \quad Y = aX + b$$

Here, z plays the role of x, and w plays the role of y, and we have $a = \sigma^2$ and $b = 0$.

Returning to the calculations, the pdf for Z is a χ^2 with $\nu = 1$ degree of freedom:

$$f_Z(z) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} z^{(\nu-2)/2} e^{-z/2} & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

The identity for the transformation from Z to W is as follows:

$$f_W(w) = \frac{1}{|\sigma^2|} f_Z\left(z = \frac{w}{\sigma^2}\right)$$

or

$$f_W(w) = \frac{1}{\sigma^2}\begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)}\left(\dfrac{w}{\sigma^2}\right)^{(\nu-2)/2} e^{-\left(\frac{w}{\sigma^2}\right)/2} & \dfrac{w}{\sigma^2} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Substituting $\nu = 1$ and using $\Gamma(1/2) = \sqrt{\pi}$, we have the following form:

$$f_W(w) = \frac{1}{\sigma^2}\begin{cases} \dfrac{\sqrt{\dfrac{\sigma^2}{w}}}{\sqrt{2\pi}} e^{-\left(\frac{w}{\sigma^2}\right)/2} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_W(w) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2 w}} e^{-w/2\sigma^2} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_W(w) = \begin{cases} \dfrac{1}{\sqrt{18\pi w}} e^{-w/18} & w > 0 \\ 0 & \text{otherwise} \end{cases}$$

b) We substitute $w = 1$ V into the answer for (a) and evaluate the result:

$$f_W(1 \text{ V}) = \frac{1}{\sqrt{18\pi}} e^{-1/18} = 0.1258$$