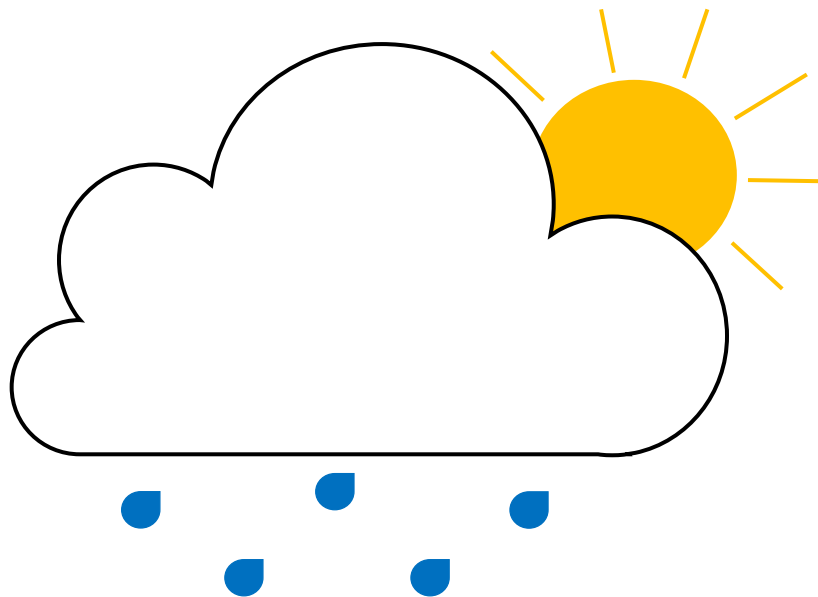# Predicting the Weather by Watching Aeroplanes:

Applying Refractive Techniques to Aeroplane ADSB Radio Wave Signals in the Lower Troposphere for Determining Relative Humidities.

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27th March 2019

Abstract

Relative humidities (RH) were measured during this investigation by treating them as homogenous values using plane position data observed at the University of Exeter on 26th August 2019 starting at 11:45am. The data was compared to model relative humidity curves for three different observed angle ranges at early and late times of observation. The least-squares difference was then obtained to find the best fitting model to the data, giving the corresponding relative humidity value for that data set. Generated RH values at early times (first 5000s) of observation were found to be $99 \pm 8\%$, $48 \pm 3\%$ and $38 \pm 3\%$, for observed angles 0.1° - 0.11°, 0.3° - 0.31° and 0.5° - 0.51° respectively: the actual RH value is 74 ± 4 [7]. Whereas, the generated RH value at late times (last 5000s) were found to be $11 \pm < 1\%$ for an observed angle range $0.3^{\circ} - 0.31^{\circ}$, RH values for the other two observed angle ranges could not be determined: the actual value RH for this time period is 77 ± 5 [7]. The generated RH values show differences with the known values due to the fact that the RH was treated as a constant value and should in fact be considered as a varying value that depends on other factors such as time, elevation angle and horizontal orientation. Understanding relative humidities allows to give better prediction for weather forecasting, potentially optimising food production for farmers and saving lives in extreme flooding events.

Contents

1 Introduction

The Earth's weather in the troposphere is a complicated dynamical system with many factors playing a crucial role in its development. Four of the most important are temperature, pressure, wind velocity and humidity. During this project, the relative humidity (RH) of the troposphere was investigated by treating it as constant (homogeneous) in order to obtain a relative humidity values. By obtaining values for the relative humidity many benefits can be obtained. One benefit might be that by knowing the relative humidity this could be used in our day-to-day lives to gain an idea of what the weather will be on a particular day, another is that this could give more accurate rainfall patterns allowing farmers to optimise weather conditions when planting crops. Furthermore, the prediction of heavy rainfall could be predicated — saving lives in the case of serious flooding. Thus, the relative humidity of the troposphere is of crucial interest to weather forecasters and this project was carried out in association with the Met Office to help develop their forecasting techniques.

At present, this is difficult to measure directly and only a small number of aircraft have such devices, for example Aircraft Meteorological Data Relay (AMDAR), on board [1] and these are often expensive which limits further opportunities using this method of measurement. An alternative approach in measuring humidity which has been proposed [1] is to use Automatic Dependent Surveillance Broadcast (ADSB) signals from air planes where many of these aircraft have this equipment on board. Therefore, these broadcasts containing aircraft information such as position, velocity, altitude and so on, can be continually obtained by ground monitoring facilities from many different aircraft. By studying how ADSB radio wave signals are refracted within the Earth's lower atmosphere, this could provide a way of obtaining the water vapour distribution thus give relative humidity values.

2 Theory

The refractive index, n is defined by

$$n = \frac{c}{v} \tag{1}$$

where c is the speed of an electromagnetic (EM) wave in vacuum and v is the phase velocity in a medium. An important relation that arises from the refractive is Snell's Law. This states that the refractive index multiplied by the sine of the angle between the incident EM wave and the normal to the interface surface between two different media of refractive indices n_1 and n_2 is conserved:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{2}$$

θ_1 is the angle between the incident ray and the normal (incident angle), θ_2 is the angle between the transmitted ray and the normal (refracted angle). A different form of Snell's Law (2) takes into account the radial length of the ray path and is derived as follows: if a radially stratified atmosphere is now considered, Figure 1 [2], Snell's Law (2) can be applied by considering these atmospheric layers as different media with different respective refractive indices.
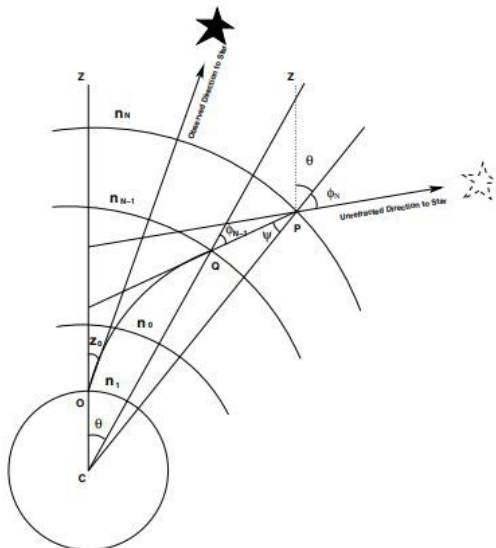
Figure 1: A visual illustration the path of a refracted EM ray as it propagates through a radially startified atmosphere. Image adapted from [2].

Applying equation (2) to the upper two layers in Figure 1 with refractive indices n_N and n_{N-1} the following is obtained

$$n_N \sin \phi_N = n_{N-1} \sin \psi \tag{3}$$

where ϕ_N and ψ are the incident and refracted angles respectively. By inspecting the triangle CQP in Figure 1 and using the law of sines, also noticing that $\sin \angle CQP \equiv \sin \phi_{N-1}$,

$$n_N r_N \sin \phi_N = n_{N-1} r_{N-1} \sin \phi_{N-1} \tag{4}$$

where phi_N and phi_{N-1} are the local zenith angles in Figure 1 with refractive indices n_N and n_{N-1} respectively, $r_N \equiv CP$ and $r_{N-1} \equiv CQ$ are the radial distance defined in Figure 1. Equation (4) can be rearranged to yield the radial distance r_N the subject

$$r_N = \frac{n_{N-1} r_{N-1} \sin \phi_{N-1}}{n_N \sin \phi_N} \tag{5}$$

where the radial distance r_N is now the subject of the equation.

The refractive index can vary in value from $n = 1$ for a vacuum to higher values for other media. Typically, more water vapour present in the air will cause the refractive index to increase because the phase velocity of an EM wave given in (1) will decrease. This leads to a useful alternative definition for the refractive index, n in terms of the refractivity, N given in parts per million (ppm)

$$n = 1 + (N \times 10^{-6}) \tag{6}$$

where n is expressed as the ppm excess above the value of unity. A formal definition of the relative humidity, RH [3] is given by

$$RH = \frac{P_w}{P_s} \times 100\% \tag{7}$$

where P_w is the actual/partial water vapour pressure and P_s is the "saturation water vapour pressure that is possible for a given temperature before condensation occurs" [3]. The refractivity of air is comprised of two parts: a dry component, N_{dry} and a wet component, N_{wet}.

$$N = N_{dry} + N_{wet} \tag{8}$$

By using further information from [2] equation (8) yields

$$N = \left(77.6 \frac{P}{T} - 5.6 \frac{P_w}{T} + 3.75 \times 10^5 \frac{P_w}{T^2} \right) \times 10^6 \tag{9}$$

where P is the total pressure (in hPa), T is the ambient air temperature (in K) and, as before in the definition of the refractivity (6), N is given in ppm. Expressions for P, T and P_w in equation (9) will now be outlined. Further information and derivations regarding these expressions can be found in [4] (for P and T) and [5] (for P_w). It can be shown that T varies as a function of altitude given by the following expression

$$T = T_b + \beta(H - H_b) \tag{10}$$

where T_b is the baseline temperature, H is the height above the baseline height H_b, both measure in km and β is the temperature gradient (Kkm^{-1}). In this case, the troposphere was considered thus, using the information from [4] $\beta = -6.5$Kkm^{-1}. An expression for P is given as,

$$P = P_b \left(1 + \frac{\beta}{T_b}(H - H_b) \right)^{-\frac{g M_0}{\beta R}} \tag{11}$$

where P_b is the baseline pressure, g is the acceleration due to gravity (in ms^2), M_0 is the molar mass of air (in kgmol^{-1}) and R is the gas constant (in JK^{-1}mol^{-1}). The partial pressure water vapour pressure P_w can be expressed as the Arden-Buck equation, which is adapted from [5]

$$P_w = \frac{RH}{100} 6.1121 \exp \left(\left(18.678 - \frac{T'}{234.5} \right) \left(\frac{T'}{257.14 + T'} \right) \right) \tag{12}$$

where T' is the temperature in *Celcius* (°C).

3 Method

3.1 Radio Interferometry

The refractive techniques discussed in Section 2 can be applied to ADSB signals transmitted from aircraft. These are 1090 MHz radio waves which are 112 bits long [6], containing aircraft information such as position, velocity, altitude, etc. Therefore, these will be refracted as an EM wave passing through the atmosphere. These broadcasts can be received by using radio interferometric techniques. There are two radio interferometers placed on the fourth and seventh floors of the Physics Building at the University of Exeter, this is illustrated in Figure 2.
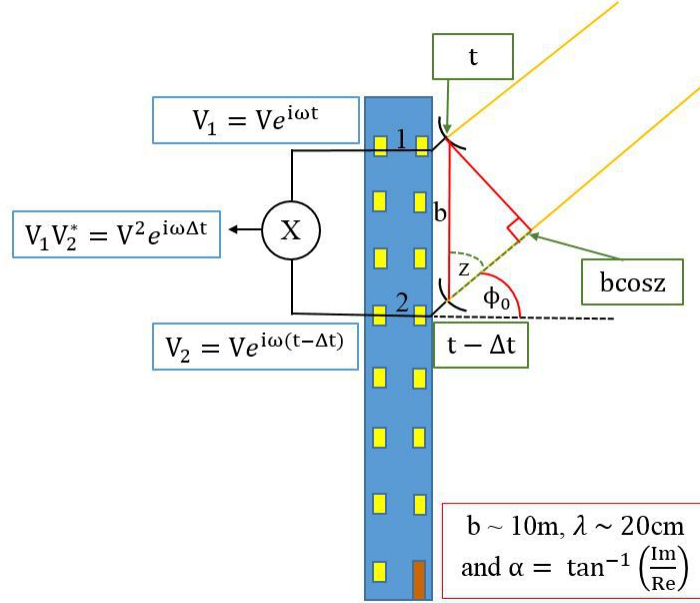
Figure 2: A diagram illustrating interferometry used to receive ADSB signals. t and $t - \Delta t$ are the times when the first and second radio interferometers, respectively received the ADSB signals, b is the baseline distance between the two receivers, V_1 and V_2 are the voltages of the first and second signals respectively, ϕ_0 is the observed angle, z is the zenith angle and $b \cos z$ is the path difference between the two signals. The value of b is known from internal project communications and λ is the wavelength of the ADSB signal.

By taking the conjugate product, $V_1 V_2^*$, the time average voltage, $V^2/2$ and the phase, $\alpha = \omega \Delta t = (2\pi b/\lambda) \cos z$, can be obtained to collect all the relevant aircraft information required for further analysis.

3.2 Data Analysis

In order to obtain constant, homogeneous relative humidity values, the first step was to use equation (5) in a computer program, provided by internal project communications, that calculates the radial path of the refracted ray (see Appendix A). This was achieved by using equations (10), (11) and (12) in equation (9) and modifying the program to obtain a more accurate refractivity value, N at each iteration of the propagating wave. This would then generate model 0% - 100% RH curves via inputting initial conditions: the observed angle above the horizon, ϕ_0 and increment angle step from the vertical, $\delta\theta$. These models can then be compared to the plane data positions, given in radial distance from the centre of the Earth, R (km) to the signal position and the increment angle, θ (rad), measured from the vertical through the centre of the Earth (where R and θ are illustrated in Section 2 Figure 1).

The chosen data set for analysis was 26nd August 2019 starting at 11:45am because this had a nice complete set of data that could be used for further data analysis and was recommended by internal project communications. This data set was then broken-up into early, first 5000s, and late, last 5000s, times of observation as it was anticipated that during the course of the day T_b and P_b would change. These baseline values were obtained from [7] (using nearby locations in Exeter) and an average for temperature, $\overline{T_b}$ and pressure, $\overline{P_b}$ was taken for the corresponding times to input into the computer program. Three observed angle ranges were considered: 0.1° - 0.11°, 0.3° - 0.31° and 0.5° - 0.51°. This gave a good idea of the behaviour of relative humidities at different altitudes. The average of these observed angle ranges, $\overline{\phi_0}$ was then taken from the sub-sectioned data sets in order to feed into the computer program generating the corresponding model curves.

Once these model curves had been generated, the data points (plane positions) would be interpolated onto each of the model relative humidity curves using another computer program (see Appendix B). Finally, a third computer program obtains the

least-squares difference to find the best fitting model curve that would be represent the plane position data outputting the corresponding relative humidity value in the process (see Appendix C).

The generated RH values would then be compared to the actual, average, RH value, $\overline{RH}$ on this day for the corresponding early and late times given by [7].

4   Results

Figure 3 shows two plots of R against θ for a given observed angle range of 0.1° - 0.11° for early (Figure 3a) and late (Figure 3b) times of observation. Furthermore, Figure 4 and Figure 5 show additional plots of these R and θ positions for different observed angle ranges of 0.3° - 0.31° and 0.5° - 0.51° respectively. Again, the early times of observation are shown for these different observed angles ranges in Figure 4a and Figure 5a at observed angle ranges of 0.3° - 0.31° and 0.5° - 0.51° respectively. The late times are shown in Figure 4b and Figure 5b for the observed angle ranges of 0.3° - 0.31° and 0.5° - 0.51° respectively. All these plot values are shown for the plane positions, in addition to the model relative humidity curves for 0%, 100% and the best fitting relative humidity curve for the plane positions.
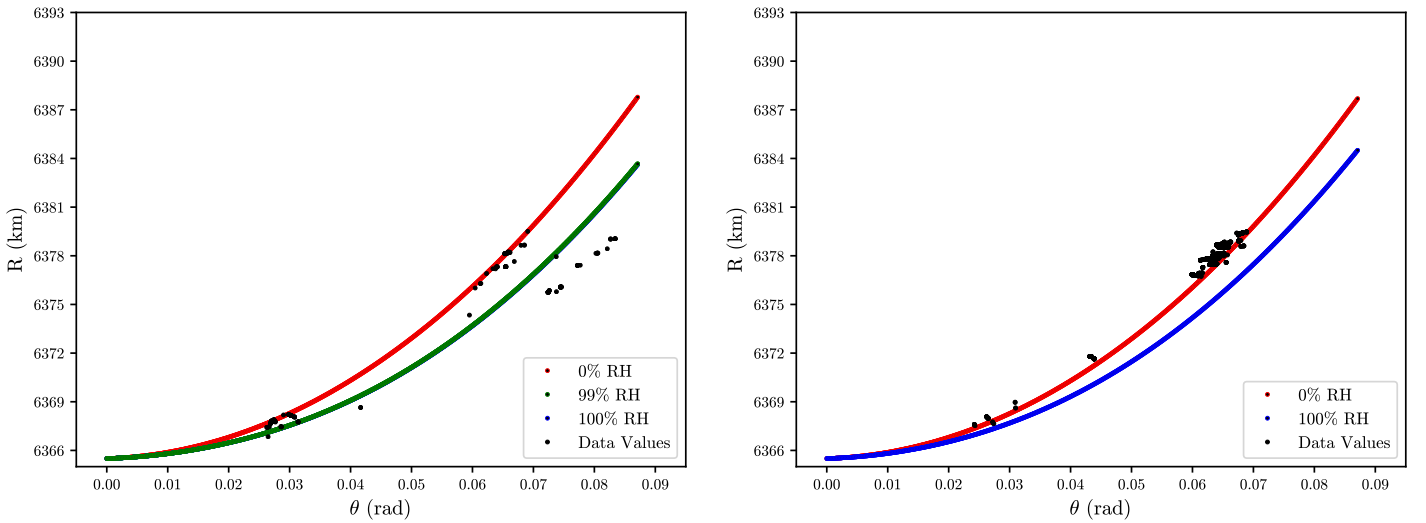
(a) R vs θ for early times (the first 5000s) of observation.   **(b)** R vs θ for late times (the last 5000s) of observation.

Figure 3: Plots of radial distance, R against increment angle, θ from the vertical through the centre of the Earth for plane positions (shown in black) at a given observed angle range between 0.1° - 0.11° for early and late times of observation.
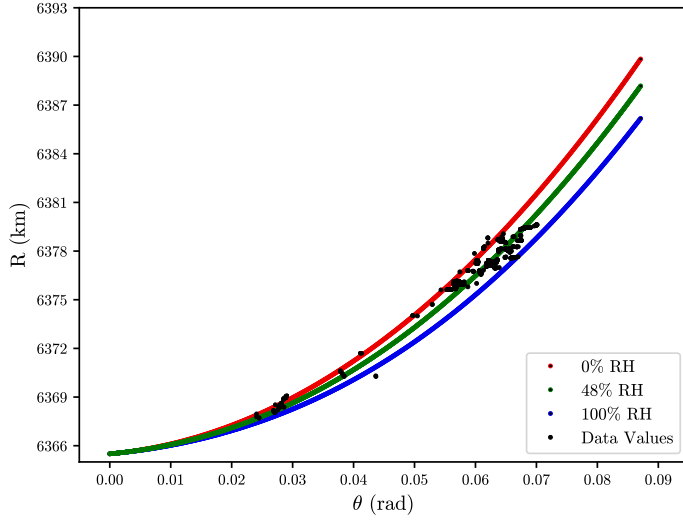
(a) R vs θ for early times (the first 5000s) of observation.

(b) R vs θ for late times (the last 5000s) of observation.

Figure 4: Plots of radial distance, R against increment angle, θ from the vertical through the centre of the Earth for plane positions (shown in black) at a given observed angle range between 0.3° - 0.31° for early and late times of observation.
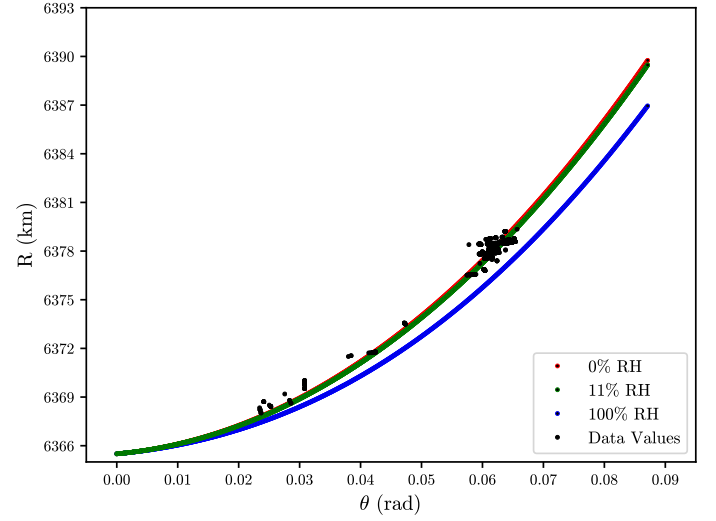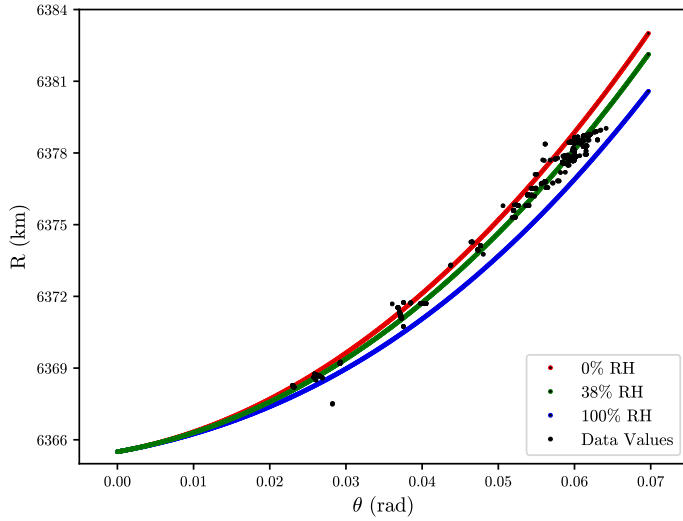

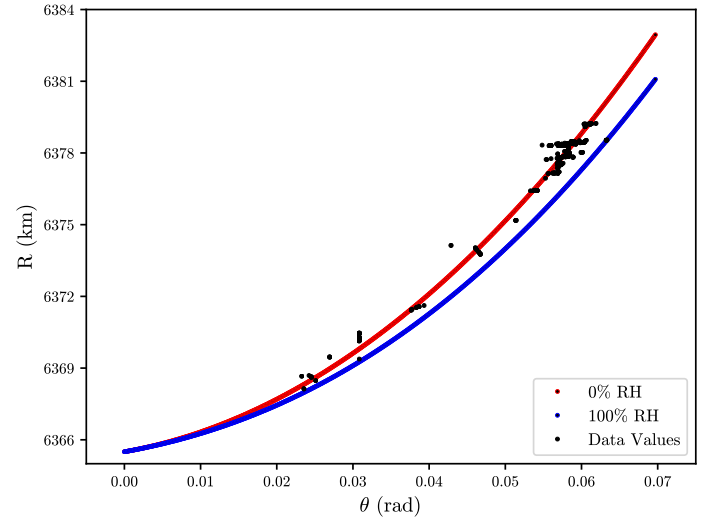**(a)** R vs θ for early times (the first 5000s) of observation.

(b) R vs θ for late times (the last 5000s) of observation.

Figure 5: Plots of radial distance, R against increment angle, θ from the vertical through the centre of the Earth for plane positions (shown in black) at a given observed angle range between 0.5° - 0.51° for early and late times of observation.

Table 1 compares the obtained values for the relative humidity from the best fitting (green) curves in Figure 3a, Figure 4a and Figure 5a to the mean of the actual relative humidities obtained from [7] along with their estimated errors for each of the observed angle ranges at early times of observation.

Table 2 compares the obtained values for the relative humidity from the best fitting (green) curves in Figure 3b, Figure 4b and Figure 5b to the mean of the actual relative humidities obtained from [7] along with their estimated errors for each of the observed angle ranges at late times of observation.

Table 1: A table comparing generated relative humidities (RH) from an interpolative and least squares method for the various different observed angle, ϕ_0 ranges investigated to the mean of the actual RH values given by [7] for the early times of observation (first 5000s).

Observed Angle Range ϕ_0 (°)	Generated RH (%)	Actual RH (%) [7]
0.10 - 0.11	99 ± 8	74 ± 4
0.30 - 0.31	48 ± 3	74 ± 4
0.50 - 0.51	38 ± 3	74 ± 4

Table 2: A table comparing generated relative humidities (RH) from an interpolative and least squares method for the various different observed angle, ϕ_0 ranges investigated to the mean of the actual RH values given by [7] for the late times of observation (last 5000s).

Observed Angle Range ϕ_0 (°)	Generated RH (%)	Actual RH (%) [7]
0.10 - 0.11	-	77 ± 5
0.30 - 0.31	$11 \pm\; <1$	77 ± 5
0.50 - 0.51	-	77 ± 5

The errors for the actual relative humidities for the early and late times of observation, given by [7], were estimated by using the error on the mean value of the relative humidity $\delta\overline{RH}$,

$$\delta\overline{RH} = \frac{\sigma(RH)}{\sqrt{N_{RH}}} \tag{13}$$

where $\sigma(RH)$ is the standard deviation of the average actual relative humidity $(\overline{RH})$ and N_{RH} is the number of RH values used to determine these averages.

However, the errors for the generated relative humidities were estimated in Table 1 and Table 2 by using the error on the mean for the baseline temperature $\overline{T_b}$ [7], baseline pressure $\overline{P_b}$ [7] and observed angle $\overline{\phi_0}$. The following were used to obtain these errors

$$\delta\overline{T_b} = \frac{\sigma(T_b)}{\sqrt{N_{T_b}}} \tag{14}$$

$$\delta\overline{P_b} = \frac{\sigma(P_b)}{\sqrt{N_{P_b}}} \tag{15}$$

$$\delta\overline{\phi_0} = \frac{0.03}{\sqrt{N_{\phi_0}}} \tag{16}$$

where $\sigma(T_b)$, $\sigma(P_B)$ are the standard deviations of the baseline temperature and pressure respectively, 0.03 is the known error in the observed angle as given by internal project communications, N_{T_b}, N_{P_b} and N_{ϕ_0} are the number of baseline temperature, pressure and observed angle readings considered.

These estimated errors given in (14), (15) and (16) were added to the mean values for the baseline temperature, baseline pressure and observed angle when calculating the refracted ray path using the appropriate computer program. This gave an upper bound on the relative humidity value. Thus, the difference between the upper bound of the relative humidity and the relative humidity generated by using the average of these values gave an estimated error for the generated relative humidity value.

5 Discussion

The results for the determined relative humidities given for the early and late part of the observations in Table 1 and Table 2 respectively show that the relative humidity does vary as a function of time. This is due to the fact that later in the day, the temperature and pressure decreases causing a change in the relative humidity as anticipated in Section 2.

By considering the early part of the day, the relative humidities given in Table 1 show discrepencies when compared with the actual relative humidities. The corresponding graphs displaying these best fitting relative humidity curves in Figure 3a, Figure 4a and Figure 5a visually show these discrepancies. For example, in Figure 4a and Figure 5a show that indeed the plane positions are grouped towards a lower relative humidity. However, when compared to the actual values displayed in Table 1 these generated values are much lower. This could be due to the fact that the relative humidity cannot be treated

as a constant, homogeneous value. Instead, the relative humidity could be varying as a function of many different factors; such as temperature, pressure, height, etc.

Furthermore, by considering Figure 3a the plane positions seem to be forming two distinct relative humidity curves. This shows that the relative humidity does indeed seem to be different for a specific observed angle range. In this case, there is a shallow observed angle range 0.1° - 0.11° above the horizon. From the discussion presented in Section 2 (9) this is where most of the refraction occurs due to the denser atmosphere nearer the Earth's surface. Therefore, it would be expected that more refraction would occur because the ADSB signal spends more time in the lower part of the atmosphere for these small observed angles above the horizon according to Section 2 Figure 1. This is shown by the bottom set of plane position data values in Figure 3a. However, there is also a set of points that indicate a higher relative humidity. This was investigated further and additional measurements, using the latitude and longitude plane positions, show that the relative humidity, as well as depending on time, also depends on upon the horizontal direction (azimuthal angle). Figure 6 shows a plot of latitude against longitude for the observed angle range 0.1° - 0.11° for early observation times.
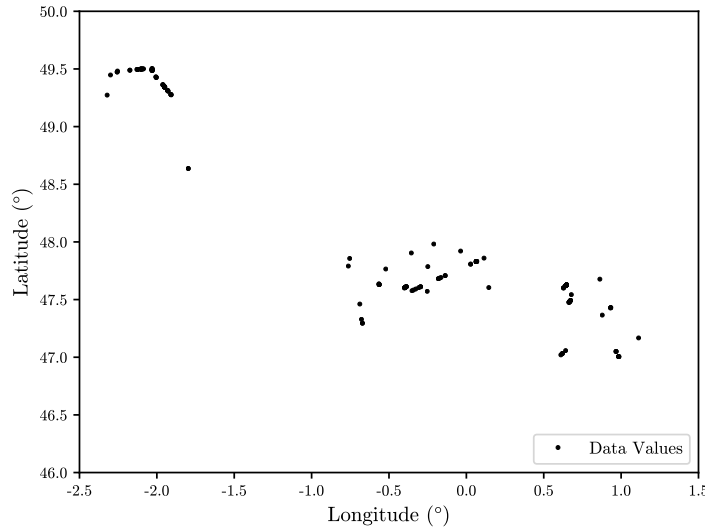
Figure 6: A plot showing the latitude and longitude plane positions during the early part of observations (first 5000s) for the observed angle range 0.1° - 0.11°.

Figure 6 shows that for this given observed angle range there are three distinct horizontal regions for these plane positions. This shows that the relative humidity also depends upon the horizontal orientation as well as time.

Analysing the results for the determined relative humidities given for the late part of the observations in Table 2 show further discrepancies when compared to the actual relative humidities. Two of these relative humidities could not be determined due to the nature of the graphs displayed in Figure 3b, and Figure 5b. These plots show that the plane positions would yield a best fit relative humidity curve that would be less than 0%. This further indicates that the relative humidity cannot be treated as an absolute constant value and that the gradient of the relative humidity curves in addition to how they curves vary with height should be considered.

By considering the refractivity as a function of altitude, this would mean that a higher ground level relative humidity would give a steeper fall-off gradient with height due to the refractivity relations given in Section 2. Indeed by analysing the corresponding refractivity curve against height for Figure 3b the following plot is generated in Figure 7.
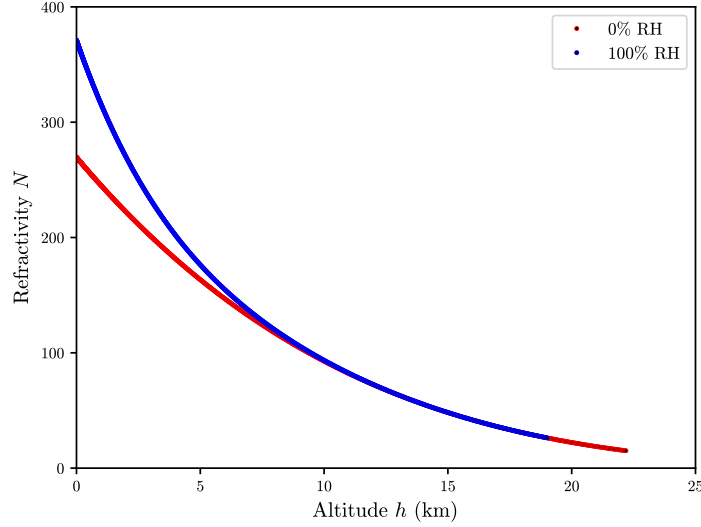
Figure 7: A plot showing the refractivity N as a function of altitude h (km) for 0% and 100% relative humidities at the observed angle range $0.1°$ - $0.11°$ for late times of observation (last 5000s).

So, the plots Figure 3b, Figure 4b and Figure 5b show that the generated relative humidities are less than 0%. According to Figure 7 this would mean a negative absolute value of relative humidity with a shallower gradient refractivity curve. Therefore, a systematic shift would be required for the plane positions, so that a model refractivity curve with a low value of relative humidity could be used to represent the data. This would display the same shallow gradient characteristics of the plane position data for low relative humidities, but translated to a higher refractivity/relative humidity. This change in relative humidity with height above ground level for the translated curve, would result in an initially rising relative humidity with altitude. However, this would have to fall off after some height as this translated refractivity curve, with a shallow gradient, would intersect the 100% refractivity curve after some altitude (which has a steeper fall off gradient) according to Figure 7 - the translated curve would not be able to go above 100% due to the definition of relative humidity given in Section 2 equation (7). Therefore, this means that the relative humidity would initially rise and then fall off again with altitude when this intersection between the two refractivity curves occurs. Furthermore, this analysis indicates that the absolute value of the relative humidity at ground level may not be the ultimate factor, but the gradient (i.e. how the humidity behaves as a function of height) could play a more vital role when determining relative humidities.

6   Conclusion

Overall, the results presented in Section 4 show some interesting properties of relative humidity that were previously not well known. For example, as well as depending on time and vertical elevation angle, the relative humidity also depends on the horizontal orientation. Thus, by looking in a different direction, different weather systems could be observed at the same as displayed in Section 4 Figure 3a. Although the generated relative humidities presented in Section 4 Table 1 and Table 2 were not in conjunction with the actual values from [7], it was shown that the absolute value may not be the only way of describing these values. By analysing the corresponding refractivity curves, such as Figure 7 in Section 5 for Figure 3b, it was indicated that the gradient of the R vs θ curves should be taken into account. Therefore, instead of the relative humidity been treated as a constant, homogeneous value, this should be treated as a variable factor shares a relation with time, horizontal orientation and altitude height. Ultimately, this would give more accurate and reliable relative humidity values; thus enabling better precision for the prediction of the weather by forecasting companies such as the Met Office.

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[7] Met Office Weather Observations Website, 22nd November 2019, `https://wow.metoffice.gov.uk/`.

Appendix A: Code for Calculating the Path of a Refracted Ray

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <string.h>
#include "nrutil.h"

main(argc, argv)
int argc;
char **argv;
{

    int ndata;
    char curr;
    int i, j;
    double pi;
    double *r,*rray,*theta,*phi,*phib,*n,*ndry,*nwet,*ninf;
    double *T, *P, *init_P; // creating arrays for temp, and pressure
    double *val_1, *val_2, *P_w; // creating arrays for the exponent values for the wet pressure contribution
    double input_phi0, phi0,r0;
    double input_dtheta, dtheta;
    double RH;
    double beta, Pb, Tb; // declaring the temperature gradient, sea-level pressure and temp respectively
    double grav, R, M_0; // declaring the acceleration due to gravity, the gas constant and molar mass of air
    double logscale;
    int ntheta;
    FILE *fp,*gp;
    char infile[80],outfile[80];

    FILE *files[101]; // a file pointer, pointing to 101 RH data files for a given observed angle

    if (argc!=3)
        { fprintf(stderr,"\n Usage: refractray phi0(deg) dtheta(rad)\n\n");
        exit(1);
    }

    /* sscanf(*++argv,"%s",infile);*/  /* input file name */
    // sscanf(*++argv,"%s",outfile);  /* output file name */
    sscanf(*++argv,"%lf",&input_phi0);     /* initial co-zenith angle */
    sscanf(*++argv,"%lf",&input_dtheta);     /* increment in theta */
    // sscanf(*++argv,"%lf",&RH); /* relative humidity in percent */

    for(int i = 0; i < 101; i++) {
        RH = i;
        phi0 = input_phi0;
        dtheta = input_dtheta;

        sprintf(outfile, "%.3fphi0_%03dRH.dat", phi0, i); /* output file name */
        files[i] = fopen(outfile, "w"); /* list of output files with output file name given above */


        pi=acos(-1.0);
        phi0=pi/2.0-phi0*pi/180.0;
        beta = -6.5; // temp gradient in K(km^-1)
        Pb = 1018; // sea-level pressure in hPa (1013.25)
        Tb = 293.15; // sea-level temp in K (288.15)
        grav = 9.80665;
        R = 8.31432;
        M_0 = 0.028964420; // in kg(mol)^-1
        r0=6365.5;

        ntheta=(int)5.0*pi/180.0/dtheta;
```

```c
        phi=dvector(0,ntheta-1);
        phib=dvector(0,ntheta-1);
        rray=dvector(0,ntheta-1);
        n=dvector(0,ntheta-1);
        theta=dvector(0,ntheta-1);

        T = dvector(0,ntheta-1); // initialising an array for temperatures
        P = dvector(0,ntheta-1); // intitalising an array for pressures
        init_P = dvector(0,ntheta-1);
        val_1 = dvector(0,ntheta-1);
        val_2 = dvector(0,ntheta-1);
        P_w = dvector(0,ntheta-1);

        val_1[0] = 18.678 - ((Tb - 273.15) / 234.5);
        val_2[0] = (Tb - 273.15) / (Tb - 16.01);
        P_w[0] = (RH / 100) * 6.1121 * exp(val_1[0] * val_2[0]);

        theta[0]=0.0;
        rray[0]=r0;
        phi[0]=phi0;
        // printf("First phi value is: %f\n", phi[0]);
        T[0] = Tb;
        n[0]=1.0 + (77.6*Pb)/Tb/1e6 - (5.6*P_w[0])/Tb/1e6 + (3.75e5*P_w[0])/pow(Tb, 2)/1e6;
        phib[0]=pi/2.0-phi0;

        // printf("phi angle is phi[%f]\n", phi[i]);
        for (j=1;j<=ntheta;j++)
            {

                theta[j]=dtheta*j;
                // printf("theta angle is theta[%f]\n", theta[j]);
                rray[j]=rray[j-1]*sin(phi[j-1]-theta[j-1])/sin(phi[j-1]-theta[j]);

                T[j] = (beta * (rray[j] - rray[0])) + Tb;
                init_P[j] = (((beta * (rray[j] - rray[0])) + Tb)) / Tb; // (in hPa)
                double exp_P = (-grav * M_0 * 1000) / (beta * R);
                P[j] = Pb * pow(init_P[j], exp_P);

                val_1[j] = 18.678 - ((T[j] - 273.15) / 234.5);
                val_2[j] = (T[j] - 273.15) / (T[j] - 16.01);
                P_w[j] = (RH / 100) * 6.1121 * exp(val_1[j] * val_2[j]);

                n[j] = 1 + (77.6 * P[j]/T[j])/1e6 - (5.6 * P_w[j])/T[j]/1e6 + (3.75e5 * P_w[j])/pow(T[j], 2)/1e6;

                phi[j]=asin(n[0]*rray[0]*sin(phi[0])/n[j]/rray[j])+theta[j];
                phib[j]=atan((rray[j]*cos(theta[j])-rray[0])/(rray[j]*sin(theta[j]))); /* bhatdotrhatv */
                }
        gp=fopen(outfile,"w");
        // fprintf(gp,"#theta R n z T P_w\n");
        for (j=0;j<ntheta;j++)
            {
            fprintf(gp,"%.16g %.16g %.16g %.16g %.16g %.16g\n",theta[j],rray[j],n[j],phi[j]-theta[j],T[j],P_w[j]);
            }

        free_dvector(theta,0,ntheta-1);
        free_dvector(rray,0,ntheta-1);
        free_dvector(phi,0,ntheta-1);
        free_dvector(phib,0,ntheta-1);
        free_dvector(n,0,ntheta-1);

        free_dvector(T,0,ntheta-1); // free the memory allocation of the T vector
        free_dvector(P,0,ntheta-1);
        free_dvector(init_P,0,ntheta-1);
        free_dvector(val_1,0,ntheta-1);
        free_dvector(val_2,0,ntheta-1);
        free_dvector(P_w,0,ntheta-1);
    }
}
```

Appendix B: Code for Calculating Interpolated Radial Distance Values

```python
import numpy as np
import os

theta_vals_data = []
R_vals_data = []
theta_errs_data = []
R_errs_data = []

data_values = input("Input the data file you wish to read in: ")
data_fptr = open(data_values, "r", newline=None) # opening the file with the observed data

data_result_list = data_fptr.readlines() # creating a list containing observed data results

for result in data_result_list:
    datum = result.split()
    for i in range(0, len(datum)):
        if i == 0:
            theta_vals_data.append(float(datum[i]))
        elif i == 1:
            R_vals_data.append(float(datum[i]))

data_fptr.close()

filenames = os.listdir(os.curdir)
filenames = os.listdir(r"/Users/matthewevans/Documents/cprogramming/Graphs/PHY3138/PHY3147/phi0_0.505_late")
file_list = []

[file_list.append(filename) for filename in filenames]


R_model_vals = [] # a list of interpolated R values

for filename in file_list:
    fpath = os.path.join(r"/Users/matthewevans/Documents/cprogramming/Graphs/PHY3138/PHY3147/phi0_0.505_late", \
    filename) # reads files from another directory
    fptr = open(fpath, "r", newline = None)
    list_of_results = fptr.readlines()

    data = []
    theta_vals = []
    R_vals = []
    theta_errs = []
    R_errs = []

    for result in list_of_results:
        data = result.split()
        for i in range(0, len(data)):
            if i == 0:
                theta_vals.append(float(data[i]))
            elif i == 1:
                R_vals.append(float(data[i]))

    fptr.close()

    for i in range(0, len(theta_vals_data)):
        found_flag = 0
        R_model = 0
        for j in range(0, len(theta_vals)):
            if theta_vals[j] >= theta_vals_data[i] and found_flag == 0:


                R_model = R_vals[j-1] + ((R_vals[j] - R_vals[j-1]) / (theta_vals[j] - theta_vals[j-1])) * \
                (theta_vals_data[i] - theta_vals[j-1])
                R_model_vals.append(R_model)

                found_flag = 1

N = len(data_result_list)

model_data = input("Enter in the file where the RH interpolated values should be stored: ")
my_file = open(model_data, "w") # opening the interpolated R values data file e.g. 'test_model.dat'
```

```python
counter = 0
for number in R_model_vals:
    if counter % N == 0 and counter != 0:
        my_file.write("\n")
    my_file.write(str(number))
    my_file.write(" ")

    counter += 1

my_file.close()
```

Appendix C: Code for Calculating Least-Squares Differences

```python
import numpy as np
import os
import pandas as pd

theta_vals_data = []
R_vals_data = []
theta_errs_data = []
R_errs_data = []

data_values = input("Input the data file you wish to read in: ")
fptr = open(data_values, "r", newline=None) # opening the R data values

list_of_results = fptr.readlines()

data = []

for result in list_of_results:
    datum = result.split()
    for i in range(0, len(datum)):
        if i == 0:
            theta_vals_data.append(float(datum[i]))
        elif i == 1:
            R_vals_data.append(float(datum[i]))
        elif i == 2:
            theta_errs_data.append(float(datum[i]))
        elif i == 3:
            R_errs_data.append(float(datum[i]))


fptr.close()

R_vals_model = []

model_data = input("Enter in the file with the RH model data: ")
fptr2 = open(model_data, "r", newline=None) # opening the interpolated R values 'test_model.dat'

list_of_results2 = fptr2.readlines()

data = []

for result in list_of_results2:
    datum = result.split()
    for i in range(0, len(datum)):
        R_vals_model.append(float(datum[i]))


fptr2.close()


data_values = np.array(R_vals_data) # creating an array for the R data values
N = len(data_values)
print(N)

data_errors = np.array(R_errs_data) # creating an array for the R errors

model = np.array(R_vals_model) # creating a 2D array for the model (interpolated) values
shape = (101, N)
model_values = model.reshape(shape)


residual_values = []
```

```python
ls_values = []
for i in range(0, len(model_values)):

    residual = 0
    ls_value = 0
    for j in range(0, N):
        residual = data_values[j] - model_values[i][j]
        ls_value += (residual ** 2) # remember to divide by errors when we get data!
        if j == (N - 1):
            residual_values.append(residual)
            ls_values.append(ls_value)


min_value = min(ls_values)

lowest_res_vals = []
for i in range(0, len(ls_values)): # prints out the minimum values and index number
    if ls_values[i] == min_value:
        print("The minimum value is: {:f} ".format(ls_values[i]))
        print("The corresponding (index number) RH is: {:d} ".format(i))

        lowest_res = 0
        for j in range(0, N):
            lowest_res = data_values[j] - model_values[i][j]
            lowest_res_vals.append(lowest_res)

output_file = input("Enter in the file name you wish to write ls values to: ") # e.g. 'ls_test.dat'
my_file = open(output_file, "w")

for number in ls_values:
    my_file.write(str(number))
    my_file.write("\n")

my_file.close()
```