Abstract Algebra Summary

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1 The System of Natural Number

1.1 Prosuct Set

Definition 1.1.1. The product set $S \times T$ ofteo arbitrary sets S and T is a set of pairs $(s,t), s \in S, t \in T$.
In general $\prod S_i = S_1 \times S_2 \times \cdots \times S_r$ is the collection of r-tuples $(s_1, s_2, \cdots, s_r)$, where $s_i \in S_i$.
If $(s_1, s_2, \cdots, s_r)$ and $(s'_1, s'_2, \cdots, s'_r)$ are equal, we have $s_1 = s'_1, s_2 = s'_2, \cdots, s_r = s'_r$.

1.2 Mapping

Definition 1.2.1. A mapping α of set S onto set T if $\forall t \in T, \exists s \in S \Rightarrow \alpha(s) = t$, we also write the image of s in T as $s\alpha$ or s^α. The image set of S is denoted as $S\alpha$ or S^α.
If α is a one-to-one mapping, s is unique for every t, and we call a inverse mapping α^{-1} which is one-to-one of T onto S

Definition 1.2.2. Resultant or Product of mapping is denoted as $\alpha\beta$, where α maps set S onto set T and β maps set T onto set U. Mapping of S onto U can be written as $S(\alpha\beta) = (S\alpha)\beta$, and same for each element.

Remark. Mapping of a set into itself is called *transformations.*

Definition 1.2.3. Identity mapping, denoted $\mathbb{1}_S$, its product with any transformation α, $\mathbb{1}_S\alpha = \alpha = \alpha\mathbb{1}_S$

Theorem 1.2.1. *If α is a one-to-one mapping of S onto T and has inverse α^{-1}, then $\alpha\alpha^{-1} = \mathbb{1}_S$ and $\alpha^{-1}\alpha = \mathbb{1}_T$. Conversely, if $\alpha : S \mapsto T$ and $\beta : T \mapsto S$ such that $\alpha\beta = \mathbb{1}_S$ and $\beta\alpha = \mathbb{1}_T$, then α, β are one-to-one mapping and $\beta = \alpha^{-1}$*

1.3 Equivalence Relations

Definition 1.3.1. The equivalence relation $\sim$ of a pair of element satisfies:

1. a $\sim$ a (reflecive property).

2. a $\sim$ b $\Rightarrow$ b $\sim$ a (symmetric property)

3. a $\sim$ b and b $\sim$ c $\Rightarrow$ a $\sim$ c (transitive property)

Definition 1.3.2. An equivalence class

Theorem 1.3.1. *Two equivalence classes are either identical or mutually exclusive*

Proof. Suppose we have an equivalence class $[a]$ of element a, if $b \in [a]$, then $[b] \subseteq [a]$; hence by maximality of $[b]$, we conclude $[b] = [a]$. $\square$

Corollary 1.3.1.1. *The collection of distinct equivalence classes gives a decomposition of the set S into multually exculsive subsets. Conversely, suppose a set $S = \cup S_i$, where S_i are multually exclusive, we can define relation $\mathcal{R}$ as $a \sim b \iff$ subsets S_i, S_j containing a, b are identical.*

2 Groups

3 Rings

4 Fields