

INTRODUCTION TO CLUSTER ALGEBRAS

ABSTRACT. These are notes for a series of lectures presented at the ASIDE conference 2016.

1. INTRODUCTION

Write overview of cluster algebras and broad history.

2. MOTIVATING EXAMPLES

Write brief lead-in.

Example 2.1. Markov numbers. Ad hoc proofs of: integrality and positivity.

Example 2.2. $Gr_2(\mathbb{C}^n)$

Example 2.3. Total positivity in SL_n . Talk about double Bruhat cells?

Example 2.4. Include some example where y -variable mutations are natural.

Exercise 2.5. Anything illuminating to ask participants to do?

3. A UNIFYING CONCEPT: CLUSTER ALGEBRAS

We define (skew-symmetric) cluster algebras at three successive levels of generality, differing from one another in the allowable types of coefficients.

3.1. Trivial coefficients.

Definition 3.1. A seed is a pair (\mathbf{x}, B) where $\mathbf{x} = (x_1, \dots, x_n)$ is an n -tuple of elements of a rational function field and B is a skew-symmetric integer $n \times n$ matrix. The vector \mathbf{x} is called the cluster and the matrix B is called the exchange matrix.

The following employs the notation $[a]_+ = \max(a, 0)$.

Definition 3.2. Given a seed (\mathbf{x}, B) and an integer $k = 1, 2, \dots, n$ the seed mutation μ_k in direction k produces a new seed $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$ where $\mathbf{x}' = (x_1, \dots, x_{k-1}, x'_k, x_{k+1}, \dots, x_n)$ with

$$(3.1) \quad x'_k = \frac{\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{x_k}$$

and B' is defined by

$$(3.2) \quad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik} [-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

In words, the mutation μ_k has the following effects:

- (1) x_k changes to x'_k satisfying $x_k x'_k = (\text{binomial in the other } x_i)$,
- (2) entries b_{ij} of B away from row and column k increase (resp. decrease) by $b_{ik} b_{kj}$ if b_{ik} and b_{kj} are both positive (resp. negative),

(3) row and column k of B are negated.

Definition 3.3. Fix an ambient field $\mathcal{F} = \mathbb{C}(x_1, \dots, x_n)$ and an initial seed $((x_1, \dots, x_n), B)$. The entries of the clusters of all seeds reachable from this one by a sequence of mutations are called the cluster variable. The cluster algebra associated with the initial seed is the subalgebra \mathcal{A} of \mathcal{F} generated by the set of all cluster variables.

Example 3.4. Type A_2 .

Theorem 3.5. Laurent phenomenon (minimal generality)

3.2. Geometric type. Geometric type cluster algebras generalize the previous setting by allowing clusters to contain so-called frozen variables in addition to cluster variables. The frozen variables do not mutate themselves, but they can take part in the exchange relations for cluster variables. An *extended cluster*, by convention, is typically written $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ where x_1, \dots, x_n are the cluster variable and x_{n+1}, \dots, x_m are the frozen variables. An *extended exchange matrix* is an $m \times n$ integers matrix \tilde{B} with the property that its top $n \times n$ part is skew-symmetric.

Definition 3.6. Fix a seed (\mathbf{x}, B) with $\mathbf{x} = (x_1, \dots, x_m)$ an extended cluster and \tilde{B} an extended exchange matrix. For an integer $k = 1, 2, \dots, n$, the seed mutation μ_k in direction k produces a new seed $\mu_k(\mathbf{x}, \tilde{B}) = (\mathbf{x}', \tilde{B}')$ with $\mathbf{x}' = (x_1, \dots, x_{k-1}, x'_k, x_{k+1}, \dots, x_m)$. The formulas for x'_k and the entries \tilde{b}'_{ij} of B' are the same as in (3.1) and (3.2), where in the former that the products now range from 1 to m instead of from 1 to n .

In this setting, the cluster algebra is the subalgebra of the ambient field generated by all cluster variables reachable from a given seed, together with the frozen variables which are constant across all seeds.

There is an alternate formulation of cluster algebras of geometric type which focuses on the roles the frozen variables x_{n+1}, \dots, x_m play in the exchange relations, rather than on the variables themselves. Let (\mathbf{x}, \tilde{B}) be an extended seed. For $k = 1, \dots, n$ let

$$y_k = \prod_{i=n+1}^m x_i^{b_{ik}}.$$

Define an operation (called auxiliary addition) \oplus on Laurent monomials by

$$\prod_{i=n+1}^m x_i^{e_i} \oplus \prod_{i=n+1}^m x_i^{f_i} = \prod_{i=n+1}^m x_i^{\min(e_i, f_i)}.$$

Using this operation, we can extract positive and negative exponents as follows

$$\frac{y_k}{1 \oplus y_k} = \prod_{\substack{i=n+1, \dots, m \\ b_{ik} > 0}} x_i^{b_{ik}} \quad \frac{1}{1 \oplus y_k} = \prod_{\substack{i=n+1, \dots, m \\ b_{ik} < 0}} x_i^{-b_{ik}}.$$

The change in going from the original exchange relation (3.1) to the one in geometric type, then, can be summarized by saying that the two terms of the binomial are enriched with coefficients $y_k/(1 \oplus y_k)$ and $1/(1 \oplus y_k)$. We now write

$$x'_k = \frac{y_k \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}}{(1 \oplus y_k)x_k}$$

where the products are for i from 1 to n .

3.3. General coefficients.

Exercise 3.7. *Compute all cluster variables and coefficient variables for cluster algebras associated to $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ with $b, c \in \mathbb{Z}_{>0}$ and $bc \leq 3$. Justify why attempting such a calculation is futile for $bc \geq 4$.*

Exercise 3.8. *Prove the Laurent phenomenon for rank 2 cluster algebras. (Include guiding hints and make the question more explicit.)*

4. BASIC RESULTS

Theorem 4.1. *Laurent phenomenon.*

Theorem 4.2. *Finite-type classification.*

Theorem 4.3. *Positivity.*

5. COMPATIBLE POISSON STRUCTURES AND QUANTIZATION

Definition 5.1. *Poisson algebra.*

Definition 5.2. *Log-canonical Poisson brackets.*

Definition 5.3. *Compatible Poisson structures on a cluster algebra.*

Theorem 5.4. *A compatible Poisson structure exists if and only if the exchange matrix has full rank. Moreover, the collection of all such Poisson structures is parametrized by an affine space of dimension ???.*

Definition 5.5. *Quantum cluster algebra.*

6. APPLICATIONS TO INTEGRABLE SYSTEMS

REFERENCES

[FZ02] . Fomin and A. Zelevinsky, Cluster algebras I: Foundations