INTRODUCTION TO CLUSTER ALGEBRAS

Abstract. These are notes for a series of lectures presented at the ASIDE conference 2016.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [?] in 2002 as the culmination of their study of total positivity [?] and (dual) canonical bases. The topic of cluster algebras quickly grew into its own as a subject deserving independent study mainly fueled by its close relationship to many areas of mathematics. Here is a partial list of related topics: combinatorics [?], hyperbolic geometry [?, ?, ?], Lie theory [?], Poisson geometry [?], integrable systems [?, ?], representations of associative algebras [?, ?, ?, ?, ?, ?], mathematical physics [?, ?], and quantum groups [?, ?, ?, ?].

In these notes we will give an introduction to cluster algebras and a couple of the applications mentioned above. These notes are far from exhaustive and the references above only touch on the vast literature. Other overviews of cluster algebras can be found in the works [?] which will also provide additional references.

The paper is organized as follows. Section 2 gives several motivating examples from which we will abstract the definition of a cluster algebra. The reader will be guided towards providing ad-hoc proofs of some of the important theorems of cluster algebra theory in the exercises of that section. Section 3 contains several variations on the definition of cluster algebras with increasing general notions of coefficients culminating in cluster algebras defined over an arbitrary semifield. In Section 4 we describe the foundational results in the theory of cluster algebras and sketch or otherwise indicate the ideas behind the proofs of these results. Section 5 recalls the theory of Poisson structures compatible with a cluster algebra and describes how this naturally leads to a quantization of cluster algebras. Finally we conlclude with applications of the cluster algebra machinery to problems involving integrable systems in Section 6.

2. Motivating Examples

¡Write brief lead-in.¿

Example 2.1. *Markov numbers. Ad hoc proofs of: integrality and positivity.*

Example 2.2. $Gr_2(\mathbb{C}^n)$

Example 2.3. *Total positivity in SL_n. ¡Talk about double Bruhat cells?¿*

Example 2.4. *¡Include some example where y-variable mutations are natural.¿*

Exercise 2.5. *Anything illuminating to ask participants to do?*

3. A Unifying Concept: Cluster Algebras

We define (skew-symmetric) cluster algebras at three successive levels of generality, differing from one another in the allowable types of coefficients.

3.1. Trivial coefficients.

Definition 3.1. *A* seed *is a pair* $(\boldsymbol{x}, B)$ *where* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is an* n*-tuple of elements of a rational function field and* B *is a skew-symmetric integer* $n \times n$ *matrix. The vector* $\boldsymbol{x}$ *is called the* cluster *and the matrix* B *is called the* exchange matrix.

The following employs the notation $[a]_+ = \max(a, 0)$.

Definition 3.2. *Given a seed* $(\boldsymbol{x}, B)$ *and an integer* $k = 1, 2, \ldots, n$ *the* seed mutation μ_k *in direction* k *produces a new seed* $\mu_k(\boldsymbol{x}, B) = (\boldsymbol{x}', B')$ *where* $\boldsymbol{x}' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)$ *with*

$$x'_k = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k} \tag{3.1}$$

and B' *is defined by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{3.2}$$

In words, the mutation μ_k has the following effects:

(1) x_k changes to x'_k satisfying $x_k x'_k =$ (binomial in the other x_i),
(2) entries b_{ij} of B away from row and column k increase (resp. decrease) by $b_{ik}b_{kj}$ if b_{ik} and b_{kj} are both positive (resp. negative),
(3) row and column k of B are negated.

Definition 3.3. *Fix an ambient field* $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_n)$ *and an initial seed* $((x_1, \ldots, x_n), B)$. *The entries of the clusters of all seeds reachable from this one by a sequence of mutations are called the* cluster variable. *The* cluster algebra *associated with the initial seed is the subalgebra* $\mathcal{A}$ *of* F *generated by the set of all cluster variables.*

Example 3.4. *Type* A_2.

Theorem 3.5. *Laurent phenomenon (minimal generality)*

3.2. Geometric type.

Geometric type cluster algebras generalize the previous setting by allowing clusters to contain so-called frozen variables in addition to cluster variables. The frozen variables do not mutate themselves, but they can take part in the exchange relations for cluster variables. An *extended cluster*, by convention, is typically written $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots x_m)$ where $x_1, \ldots, x_n$ are the cluster variable and $x_{n+1}, \ldots, x_m$ are the frozen variables. An *extended exchange matrix* is an $m \times n$ integers matrix $\tilde{B}$ with the property that its top $n \times n$ part is skew-symmetric.

Definition 3.6. *Fix a seed* $(\boldsymbol{x}, B)$ *with* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *an extended cluster and* $\tilde{B}$ *an extended exchange matrix. For an integer* $k = 1, 2, \ldots, n$*, the* seed mutation μ_k *in direction* k *produces a new seed* $\mu_k(\boldsymbol{x}, \tilde{B}) = (\boldsymbol{x}', \tilde{B}')$ *with* $\boldsymbol{x}' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_m)$. *The formulas for* x'_k *and the entries* $\tilde{b}'_{ij}$ *of* B' *are the same as in* (3.1) *and* (3.2)*, where in the former that the products now range from* 1 *to* m *instead of from* 1 *to* n.

In this setting, the cluster algebra is the subalgebra of the ambient field generated by all cluster variables reachable from a given seed, together with the frozen variables which are constant across all seeds.

There is an alternate formulation of cluster algebras of geometric type which focuses on the roles the frozen variables $x_{n+1}, \ldots, x_m$ play in the exchange relations, rather than on the variables themselves. Let $(\mathbf{x}, \tilde{B})$ be an extended seed. For $k = 1, \ldots, n$ let

$$y_k = \prod_{i=n+1}^{m} x_i^{b_{ik}}.$$

Define an operation (called auxiliary addition) $\oplus$ on Laurent monomials by

$$\prod_{i=n+1}^{m} x_i^{e_i} \oplus \prod_{i=n+1}^{m} x_i^{f_i} = \prod_{i=n+1}^{m} x_i^{\min(e_i, f_i)}.$$

Using this operation, we can extract positive and negative exponents as follows

$$\frac{y_k}{1 \oplus y_k} = \prod_{\substack{i=n+1,\ldots,m \\ b_{ik}>0}} x_i^{b_{ik}} \qquad \frac{1}{1 \oplus y_k} = \prod_{\substack{i=n+1,\ldots,m \\ b_{ik}<0}} x_i^{-b_{ik}}.$$

The change in going from the original exchange relation (3.1) to the one in geometric type, then, can be summarized by saying that the two terms of the binomial are enriched with coefficients $y_k/(1 \oplus y_k)$ and $1/(1 \oplus y_k)$. We now write

$$x'_k = \frac{y_k \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{(1 \oplus y_k) x_k}$$

where the products are for i from 1 to n.

3.3. **General coefficients.**

Exercise 3.7. *Compute all cluster variables and coefficient variables for cluster algebras associated to* $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ *with* $b, c \in \mathbb{Z}_{>0}$ *and* $bc \leq 3$. *Justify why attempting such a calculation is futile for* $bc \geq 4$.

Exercise 3.8. *Prove the Laurent phenomenon for rank 2 cluster algebras. ¡Include guiding hints and make the question more explicit.¿*

4. Basic Results

Theorem 4.1. *Laurent phenomenon.*

Theorem 4.2. *Finite-type classification.*

Theorem 4.3. *Positivity.*

5. Compatible Poisson Structures and Quantization

Definition 5.1. *Poisson algebra.*

Definition 5.2. *Log-canonical Poisson brackets.*

Definition 5.3. *Compatible Poisson structures on a cluster algebra.*

Theorem 5.4. *A compatible Poisson structure exists if and only if the exchange matrix has full rank. Moreover, the collection of all such Poisson structures is parametrized by an affine space of dimension ???.*

Definition 5.5. *Quantum cluster algebra.*

6. Applications to Integrable Systems

References

[FZ02] . Fomin and A. Zelevinsky, Cluster algebras I: Foundations