SYMPLECTIC GROUPOIDS FOR CLUSTER MANIFOLDS

Abstract. We prove some things. And we are happy! It seems to work!

Outline

(1) Intro to Poisson manifolds, symplectic groupoid and Poisson spray
(2) Intro to cluster algebra and compatible Poisson structures
(3) Cluster symplectic groupoid
(4) Totally positive cluster manifolds (definition of manifolds with corners [check D Joyce], associahedron of type A and generalized associahedron)
(5) Symplectic topology of the groupoid, and examples

1. Introduction

2. Poisson Geometry

In this section we recall the various incarnations of Poisson manifolds and the construction of symplectic groupoids from the Poisson spray [1].

Definition 2.1. *A smooth Poisson manifold is a smooth manifold M with Lie bracket*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that the Leibniz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

is satisfied. A holomorphic Poisson manifold is analogous in that M is a holomorphic manifold and we replace the space of smooth functions $C^\infty(M)$ with the space of holomorphic functions $\mathcal{O}(M)$.

3. Cluster Algebras

4. Cluster Symplectic Groupoids

In this section we give an integration to a symplectic groupoid $\mathcal{G}(\Sigma)$ of the Poisson structure on a cluster manifold $M(\Sigma)$.

5. Totally Positive Cluster Manifolds

In this section we show that the totally nonnegative part $M_{\geq 0}(\Sigma)$ of a cluster manifold is a manifold with corners in the sense of [2]. Moreover, we show that the nonnegative cluster manifold is a union of symplectic leaves for any compatible Poisson structure on $\mathcal{A}(\Sigma)$. The symplectic leaves of $M_{\geq 0}(\Sigma)$ are naturally labelled by compatible subsets of cluster variables, where the number of cluster variables in the labeling set determines the corank of the symplectic leaf. Here there is a unique dense symplectic leaf and the boundary of $M_{\geq 0}(\Sigma)$ is again a union of symplectic leaves of lower dimension where the Poisson structure degenerates.

Theorem 5.1. *Let Σ be a seed. The 1-skeleton of $M_{\geq 0}(\Sigma)$ given by 0-dimensional and 1-dimensional symplectic leaves identifies with the exchange graph of $\mathcal{A}(\Sigma)$. Moreover, if Σ is a seed of finite-type, then $M_{\geq 0}(\Sigma)$ provides a realization of the generalized associahedron with the same Cartan type as Σ.*

Proof. The 0-dimensional symplectic leaves correspond to the vanishing of all cluster variables from a seed mutation equivalent to Σ. Then a 1-dimensional symplectic leaf whose boundaries correspond to seeds Σ' and Σ'' exactly corresponds to the non-vanishing of exchangable cluster variables x'_k and x''_k. But this is exactly the exchange graph of $\mathcal{A}(\Sigma)$.

When Σ is of finite-type, the realization of $M_{\geq 0}(\Sigma)$ as a simplicial complex, given by taking symplectic leaves as cells, is naturally dual to the cluster complex of $\mathcal{A}(\Sigma)$, i.e. $M_{\geq 0}(\Sigma)$ identifies with the associated generalized associahedron. ◻

6. Symplectic Topology of the Nonnegative Cluster Groupoid

Let $\mathcal{G}_{\geq 0}(\Sigma)$ denote the symplectic groupoid over $M_{\geq 0}(\Sigma)$. In this section we introduce a Poisson spray which may be used to construct $\mathcal{G}_{\geq 0}(\Sigma)$ and apply a Moser argument to show that up to symplectomorphism $\mathcal{G}_{\geq 0}(\Sigma)$ can be identified with $T^*M_{\geq 0}(\Sigma)$.

References

[1] M. Crainic and I. Mărcuţ, *On the existence of symplectic realizations*, J. Symplectic Geom. **9** (2011), no. 4, 435–444.

[2] D. Joyce, *On manifolds with corners*, Advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 21, Int. Press, Somerville, MA, 2012, pp. 225–258.