SYMPLECTIC GROUPOIDS FOR CLUSTER MANIFOLDS

Abstract. We prove some things. And we are happy! It seems to work!

Outline

(1) Intro to Poisson manifolds, symplectic groupoid and Poisson spray
(2) Intro to cluster algebra and compatible Poisson structures
(3) Cluster symplectic groupoid
(4) Totally positive cluster manifolds (definition of manifolds with corners [check D Joyce], associahedron of type A and generalized associahedron)
(5) Symplectic topology of the groupoid, and examples

1. Introduction

2. Poisson Geometry

In this section we recall the various definitions of Poisson manifolds and the construction of symplectic groupoids from the Poisson spray [5].

Definition 2.1. *A smooth Poisson manifold is a smooth manifold M equipped with one of the following three equivalent structures:*

(1) *A Lie bracket (called a Poisson bracket)*
$$\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$
satisfying the Leibniz rule
$$\{fg,h\} = f\{g,h\}+g\{f,h\}$$
is satisfied.
(2) *A Poisson bivector $\pi\in\mathfrak{X}^2(M)$ such that $[\pi,\pi]=0$ where*
$$[\cdot,\cdot] : \mathfrak{X}^p(M)\times\mathfrak{X}^q(M)\to\mathfrak{X}^{p+q-1}(M)$$
is the Schouten-Nijenhuis bracket.
(3) *A Poisson differental operator $d_\pi : \mathfrak{X}^p(M)\to\mathfrak{X}^{p+1}(M)$ with $d_\pi^2=0$.*

A holomorphic Poisson manifold is analogous where M is a holomorphic manifold.

Remark 2.2. *We shall denote a Poisson manifold by either (M,π) or $(M,\{,\})$.*

(1) *The Poisson bivector is related to the Poisson bracket by the following formula: $\{f,g\}=\pi(df\otimes dg)$, for $f,g\in C^\infty(M)$.*
(2) *The Poisson differential operator is related to the Poisson bivector by the following formula: $d_\pi X = [\pi,X]$ for $X\in\mathfrak{X}^p(M)$.*

The notion of the symplectic groupoid of a Poisson manifold was introduced independently by Weinstein [10], Karasëv [9] and Zakrzewski [11, 12]. It is closely related to Poisson sigma models [2] and quantization [7]. As a Lie groupoid, the symplectic groupoid integrates the Poisson Lie algebroid $\pi^\sharp : T^*M\to TM$ [10].

Definition 2.3. *A groupoid $G\rightrightarrows M$ consists of two sets G and M with the following maps:*

(1) *a surjective source map $\alpha : G\to M$ and a surjective target map $\beta : G\to M$;*
(2) *an injective identity map $\mathtt{1} : M\to G,\ \ x\mapsto \mathtt{1}_x$;*
(3) *an associative multiplication map $m : G_\beta\times_\alpha G\to G,\ \ (g,h)\mapsto gh$;*
(4) *and an involutive inversion map $i : G\to G,\ \ g\mapsto g^{-1}$*

that satisfies the following properties:

(1) $\alpha(\mathtt{1}_x)=\beta(\mathtt{1}_x)=x$*;*
(2) $\alpha(gh)=\alpha(g),\ \ \beta(gh)=\beta(h)$*;*

(3) $\alpha(g^{-1}) = \beta(g)$, $\beta(g^{-1}) = \alpha(g)$;
(4) $(1_x)^{-1} = 1_x$.

A Lie groupoid $G \rightrightarrows M$ has the following additional properties:

(1) *G and M are smooth manifolds;*
(2) *the source $\alpha : G \to M$ and the target $\beta : G \to M$ are surjective submersions;*
(3) *the multiplication map $m : G_\beta{\times}_\alpha G \to G$ is smooth;*
(4) *the inversion map $i : G \to G$ is smooth.*

Definition 2.4. *For a Poisson manifold (M, π), a symplectic groupoid is a symplectic manifold (G, ω) with a Lie groupoid structure $G \rightrightarrows M$ such that*

(1) *the source $\alpha : (G, \omega) \to (M, \pi)$ and the target $\beta : (G, \omega) \to (M, \pi)$ are Poisson maps;*
(2) *the graph of the multiplication map $\Gamma_m = \{(g, h, gh) \in G \times G \times G\}$ is a Lagrangian submanifold of $(G \times G \times G, \omega \oplus \omega \oplus -\omega)$.*

The problem of existence of symplectic groupoids given a Poisson manifold, or more generally the existence of Lie groupoids given a Lie algebroid, was solved by Crainic-Fernandes [3, 4]. There are a few notable examples of symplectic groupoids, e.g. the symplectic double of Poisson Lie groups [4] and the blow-up construction of log symplectic manifolds [6], but in general it is difficult to find explicit examples of symplectic groupoids. There is, however, a local construction of symplectic groupoids by Crainic-Marcut [5] and Cabrera-Marcut-Salazar [1], which utilizing the notion of Poisson sprays.

Definition 2.5. *For a Poisson manifold (M, π), a Poisson spray is a vector field $X \in \mathfrak{X}(T^*M)$ is a Poisson spray if*

(1) *for $(x, p) \in T^*M$,*
$$(\tau_M)_* \left(X_{(x,p)}\right) = \pi^\flat(p)$$
where $(\tau_M) : T^*M \to M$ is the projection;
(2) *X is homogeneous of degree 1, i.e.*
$$(m_\lambda)_*(X) = \lambda X$$
where $m_\lambda : T^*M \to T^*M$, $(x, p) \mapsto (x, \lambda p)$ is the fiberwise scaling map.

Theorem 2.6. [5, 1] *For a Poisson manifold (M, π) with a Poisson spray $X \in \mathfrak{X}(T^*M)$, a neighbourhood U of the zero section of T^*M is a local symplectic groupoid over (M, π) with the following structures:*

(1) *the source map $\alpha = \tau_M : U \to M$ is the bundle projection ;*
(2) *the target map is*
$$\beta : U \to M, \qquad \beta = \tau_M \circ \varphi_X^1$$
where $\varphi_X^1 : T^*M \to T^*M$ is the time-1-flow of the Poisson spray X;
(3) *the multiplication is the concatenation of the flow of X; and*
(4) *the symplectic form on U is*
$$\overline{\omega} = \int_0^1 (\varphi_X^s)^* \omega_0 ds.$$

Remark 2.7. *By local symplectic groupoid $G \rightrightarrows M$, we mean that the multiplication $m : G_\beta{\times}_\alpha G \to G$ may not be defined on the entirety of its domain, e.g. in this particular case the time-1-flow of the Poisson spray X may be outside the neighbourhood U.*

Remark 2.8. *Given a Poisson spray $X \in \mathfrak{X}(T^*M)$ on a Poisson manifold (M, π), there is actually a 1-parameter family of local symplectic groupoid structure on a neighourhood U of the zero section of T^*M with the following structures:*

(1) *the source map $\alpha = \tau_M : U \to M$ is the bundle projection ;*
(2) *the target map is*
$$\beta : U \to M, \qquad \beta = \tau_M \circ \varphi_X^t$$
where $\varphi_X^t : T^*M \to T^*M$ is the time-t-flow of the Poisson spray X;
(3) *the multiplication is the concatenation of the time-t-flow of X; and*

(4) *the symplectic form on U is*

$$\overline{\omega}_t = \frac{1}{t}\int_0^t (\varphi_X^s)^*\omega_0 ds.$$

The local symplectic groupoid $(U, \overline{\omega}_t)$ integrates the Poisson manifold $(M, t\pi)$ for $0 \le t \le 1$. In particular, the local symplectic groupoid in Theorem 2.6 is the case $t = 1$.

In local coordinates $T^*\mathbb{R}^n = \{(x_1, \dots, x_n, p_1, \dots, p_n)\}$, if the Poisson structure

$$\pi = \sum_{i>j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

then a Poisson spray is of the form

$$X = \sum_{i>j} \pi_{ij} p_i \frac{\partial}{\partial x_j} - \sum_{i>j} \pi_{ij} p_j \frac{\partial}{\partial x_i} + \sum_i f_i \frac{\partial}{\partial p_i}$$

where f_i's are fiberwise quadratic functions on $T^*\mathbb{R}^n$.

3. Cluster Algebras

Let $\tilde{B} = (B_{ij})$ be an $m \times n$ integer matrix with $m \ge n$. Write B for the upper $n \times n$ submatrix of $\tilde{B}$ and assume B is skew-symmetrizable, i.e. there exists an integer diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_n)$ with each $d_i > 0$ so that DB is skew-symmetric. We call such a matrix $\tilde{B}$ an $m \times n$ *exchange matrix.*

For $1 \le k \le n$, define the *mutation of $\tilde{B}$ in direction k* by $\mu_k \tilde{B} = (B'_{ij})$, where

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\ B_{ij} + [B_{ik}]_+ B_{kj} + B_{ik}[-B_{kj}]_+ & \text{otherwise.} \end{cases}$$

Above we used the notation $[a]_+ = \max\{a, 0\}$.

Let $\mathcal{F}$ be an extension field of $\mathbb{Q}$ of transcendence degree m. A *seed* in $\mathcal{F}$ is a pair $\Sigma = (\mathbf{x}, \tilde{B})$, where $\mathbf{x} = (x_1, \dots, x_m)$ is a transendence basis of $\mathcal{F}$ over $\mathbb{Q}$ called the *cluster* with entries called *cluster variables* and $\tilde{B}$ is an $m \times n$ exchange matrix. For $1 \le k \le n$, define the *mutation of Σ in direction k* by $\mu_k\Sigma = (\mathbf{x}', \mu_k \tilde{B})$, where $\mathbf{x}' = (x'_1, \dots, x'_m)$ given by the *exchange relations*

$$x'_i = \begin{cases} x_i & \text{if } i \ne k; \\ \frac{1}{x_k}\left(\prod_{i=1}^m x_i^{[B_{ik}]_+} + \prod_{i=1}^m x_i^{[-B_{ik}]_+}\right) & \text{if } i = k. \end{cases} \tag{1}$$

Observe that seed mutation is involutive, i.e. $\mu_k(\mu_k\Sigma) = \Sigma$. A seed Σ' is *mutation equivalent* to Σ if there exists a sequence of mutations which transforms Σ into Σ', in this case we write $\Sigma' \sim \Sigma$.

Definition 3.1. *Let Σ be a seed in $\mathcal{F}$. The* cluster algebra $\mathcal{A}(\Sigma)$ *is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds Σ' mutation equivalent to Σ.*

By iterating the exchange relations we appear to get elements of $\mathbb{Q}(x_1, \dots, x_m) \subset \mathcal{F}$, that is rational functions in $x_1, \dots, x_m$. The following result of Fomin and Zelevinsky known as "the Laurent phenomenon" shows that the cluster variables always take on a much simpler form.

Theorem 3.2. [?] *Let Σ be a seed in $\mathcal{F}$ and $\Sigma' \sim \Sigma$. Each cluster variable x'_i of Σ' is an element of the subring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}] \subset \mathcal{F}$.*

In fact, the situtation is even better: the initial cluster Laurent expansions of all cluster variables have positive integer coefficients.

Theorem 3.3. [?, ?] *Let Σ be a seed in $\mathcal{F}$ and $\Sigma' \sim \Sigma$. Each cluster variable x'_i of Σ' is an element of the subsemiring $\mathbb{Z}_{\ge 0}[x_1^{\pm 1}, \dots, x_m^{\pm 1}] \subset \mathcal{F}$.*

For x'_i a cluster variable from a seed $\Sigma' \sim \Sigma$, we write $x'_i(x_1, \dots, x_m)$ when we wish to emphasize that x'_i should be thought of as a function of $x_1, \dots, x_m$.

3.1. **The Cluster Manifold and Compatible Poisson Structures.** Fix the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. For an $m \times n$ exchange matrix $\tilde{B}$, define the *cluster chart* $\mathrm{Spec}(\mathbb{F}[x_1^{\pm 1}, \dots, x_m^{\pm 1}])$. Observe that $\Sigma = (\mathbf{x}, \tilde{B})$ is a seed in the field of rational functions on this cluster chart and thus we denote it by L_Σ. Then the exchange relation (1) provides a birational transformation between the cluster charts $\varphi_{\Sigma, \mu_k \Sigma} : L_\Sigma \to L_{\mu_k \Sigma}$ for $1 \le k \le n$. By composing these *elementary transition maps* for neighboring seeds we get a birational transformation between $\varphi_{\Sigma, \Sigma'} : L_\Sigma \to L_{\Sigma'}$ for any seeds $\Sigma \sim \Sigma'$.

Given any seed Σ, the transition maps above define the *cluster manifold* $M = M(\Sigma) = \bigcup_{\Sigma' \sim \Sigma} L_{\Sigma'}$. By construction we have $\mathcal{A}(\Sigma) \subset C^\infty(M)$ and any Poisson structure on $\mathcal{A}(\Sigma)$ naturally extends to give a Poisson structure on $C^\infty(M)$.

Definition 3.4. *A Poisson structure $\{\cdot,\cdot\} : \mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma)$ is* compatible *with the cluster algebra structure if, for each seed $\Sigma' \sim \Sigma$, the cluster variables in $\mathbf{x}'$ are* log-canonical *with respect to $\{\cdot,\cdot\}$, that is, there exists a skew-symmetric integer* coefficient matrix $\Omega' = (\Omega'_{ij})$ *so that*

$$\{x'_i, x'_j\} = \Omega'_{ij} x'_i x'_j \tag{2}$$

for $1 \le i, j \le m$.

Remark 3.5. *Suppose the cluster variables of a seed $\Sigma = (\mathbf{x}, \tilde{B})$ are log-canonical with respect to a Poisson bracket $\{\cdot,\cdot\} : \mathcal{A}(\Sigma) \times \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma)$ with coefficient matrix Ω. Then the compatibility of $\{\cdot,\cdot\}$, together with the exchange relations, imposes the condition $\tilde{B}^T \Omega = [D\ 0]$, where D is a skew-symmetrizing matrix for the upper $n \times n$ submatrix B of $\tilde{B}$ (see* [**?**, **?**] *for details).*

Theorem 3.6. [**?**] *Suppose the $m \times n$ exchange matrix $\tilde{B}$ of a seed Σ has full rank. Then there exists a Poisson structure Ω compatible with the cluster structure on $\mathcal{A}(\Sigma)$.*

4. Cluster Symplectic Groupoids

In this section we give an integration to a symplectic groupoid G of the Poisson structure on a cluster manifold $M(\Sigma)$. We build the cluster symplectic groupoid $G \rightrightarrows M$ by gluing together local groupoid charts $G_{\Sigma'} \rightrightarrows L_{\Sigma'}$, $\Sigma' \sim \Sigma$, along transition maps which lift the cluster mutations used to glue cluster charts of M.

Let $\Sigma = (\mathbf{x}, \tilde{B})$ be a seed and assume there exists a compatible Poisson structure on L_Σ with coefficient matrix $\Omega = (\Omega_{ij})$. Write π for the corresponding Poisson bivector on the cluster manifold M, i.e. in local coordinates on L_Σ we have $\pi = \sum_{i>j} \Omega_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$.

Consider the manifold $G_\Sigma = T^* L_\Sigma$, the cotangent bundle of L_Σ. Let $\mathbf{p} = (p_1, \dots, p_n)$ denote the cotangent coordinates of G_Σ. Define the map $\beta : G_\Sigma \to L_\Sigma$ by

$$\beta(\mathbf{x}, \mathbf{p}) = (s_1 x_1, \dots, s_n x_n), \quad \text{where} \quad s_i := e^{\sum_j \Omega_{ij} x_j p_j}. \tag{3}$$

Theorem 4.1. *The manifold G_Σ is a groupoid over L_Σ with source map the natural projection, target map β, multiplication given by*

$$(\mathbf{x}, \mathbf{p}) \cdot \big(\beta(\mathbf{x}, \mathbf{p}), \mathbf{p}'\big) = (\mathbf{x}, \mathbf{p}''), \quad p''_i = s_i p'_i + p_i,$$

inversion given by

$$(\mathbf{x}, \mathbf{p})^{-1} = \big(\beta(\mathbf{x}, \mathbf{p}), \mathbf{p}'\big), \quad p'_i = -s_i^{-1} p_i,$$

and identity map given by $1_\mathbf{x} = (\mathbf{x}, \mathbf{0})$.

Proof. It is clear that $1_\mathbf{x}$ gives the identity map for this multiplication and that the inversion map satisfies

$$(\mathbf{x}, \mathbf{p}) \cdot (\mathbf{x}, \mathbf{p})^{-1} = 1_\mathbf{x} = (\mathbf{x}, \mathbf{p})^{-1} \cdot (\mathbf{x}, \mathbf{p})$$

for all $(\mathbf{x}, \mathbf{p}) \in G_\Sigma$.

It remains to check associativity of the multiplication. Fix an element $(\mathbf{x}, \mathbf{p}) \in G_\Sigma$. Consider $(\mathbf{x}', \mathbf{p}'), (\mathbf{x}'', \mathbf{p}'') \in G_\Sigma$ with $\mathbf{x}' = \beta(\mathbf{x}, \mathbf{p})$ and $\mathbf{x}'' = \beta(\mathbf{x}', \mathbf{p}')$. Then we have

$$\big((\mathbf{x}, \mathbf{p}) \cdot (\mathbf{x}', \mathbf{p}')\big) \cdot (\mathbf{x}'', \mathbf{p}'') = (\mathbf{x}, \mathbf{p}'''), \quad p'''_i = e^{\sum_j \Omega_{ij} x_j (s_j p'_j + p_j)} p''_i + e^{\sum_j \Omega_{ij} x_j p_j} p'_i + p_i.$$

On the other hand we have

$$(\mathbf{x}, \mathbf{p}) \cdot \big((\mathbf{x}', \mathbf{p}') \cdot (\mathbf{x}'', \mathbf{p}'')\big) = (\mathbf{x}, \mathbf{p}'''), \quad p'''_i = e^{\sum_j \Omega_{ij} x_j p_j} (e^{\sum_j \Omega_{ij} x'_j p'_j} p''_i + p'_i) + p_i$$

and thus associativity holds. ∎

Write $\mu_k\Sigma = (\mathbf{x}', \tilde{B}')$. Define a map from G_Σ to $G_{\mu_k\Sigma}$, which by a slight abuse of notation we also denote $\varphi_{\Sigma,\mu_k\Sigma}$, as follows:

$$\varphi_{\Sigma,\mu_k\Sigma}(\mathbf{x}, \mathbf{p}) = (\mathbf{x}', \mathbf{p}'), \quad \mathbf{p}' = (p_1', \ldots, p_n'), \quad p_i' = \tag{4}$$

5. Totally Positive Cluster Manifolds

In this section we show that the totally nonnegative part $M_{\geq 0}(\Sigma)$ of a cluster manifold is a manifold with corners in the sense of [8]. Moreover, we show that the nonnegative cluster manifold is a union of symplectic leaves for any compatible Poisson structure on $\mathcal{A}(\Sigma)$. The symplectic leaves of $M_{\geq 0}(\Sigma)$ are naturally labelled by compatible subsets of cluster variables, where the number of cluster variables in the labeling set determines the corank of the symplectic leaf. Here there is a unique dense symplectic leaf and the boundary of $M_{\geq 0}(\Sigma)$ is again a union of symplectic leaves of lower dimension where the Poisson structure degenerates.

Theorem 5.1. *Let Σ be a seed. The 1-skeleton of $M_{\geq 0}(\Sigma)$ given by 0-dimensional and 1-dimensional symplectic leaves identifies with the exchange graph of $\mathcal{A}(\Sigma)$. Moreover, if Σ is a seed of finite-type, then $M_{\geq 0}(\Sigma)$ provides a realization of the generalized associahedron with the same Cartan type as Σ.*

Proof. The 0-dimensional symplectic leaves correspond to the vanishing of all cluster variables from a seed mutation equivalent to Σ. Then a 1-dimensional symplectic leaf whose boundaries correspond to seeds Σ' and Σ'' exactly corresponds to the non-vanishing of exchangable cluster variables x_k' and x_k''. But this is exactly the exchange graph of $\mathcal{A}(\Sigma)$.

When Σ is of finite-type, the realization of $M_{\geq 0}(\Sigma)$ as a simplicial complex, given by taking symplectic leaves as cells, is naturally dual to the cluster complex of $\mathcal{A}(\Sigma)$, i.e. $M_{\geq 0}(\Sigma)$ identifies with the associated generalized associahedron. □

6. Symplectic Topology of the Nonnegative Cluster Groupoid

Let $\mathcal{G}_{\geq 0}(\Sigma)$ denote the symplectic groupoid over $M_{\geq 0}(\Sigma)$. In this section we introduce a Poisson spray which may be used to construct $\mathcal{G}_{\geq 0}(\Sigma)$ and apply a Moser argument to show that up to symplectomorphism $\mathcal{G}_{\geq 0}(\Sigma)$ can be identified with $T^*M_{\geq 0}(\Sigma)$.

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