Enriched Traced Monoidal Categories are Lax Functors out of Compact Categories

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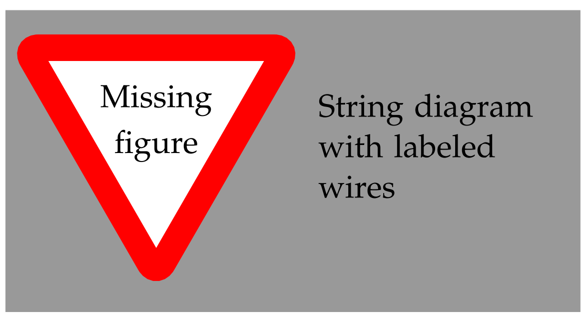
1 Introduction

Traced symmetric monoidal categories have been used to model processes with feedback or operators with fixed points (Ponto, Shulman). A graphical calculus for TSMCs was developed by Joyal, Street, and Verity, in which string diagrams of the form



represent compositions, i.e., new morphisms are constructed from old by specifying which outputs will be fed back into which inputs. In fact, these generalize Penrose diagrams in **Vect**, and the word *traced* originates in vector space terminology.

But notice that the above picture has another interpretation, in terms of 1-dimensional cobordisms between oriented 0-manifolds. A box in the picture includes only the data of two finite sets (X_-, X_+), drawn as input wires on the left and output wires on the right, which can be interpreted as an oriented 0-manifold. A string diagram then consists of boxes $X_1, \ldots, X_n$ wired together inside a larger box Y, and can be interpreted as a cobordism from $X_1 \sqcup \cdots \sqcup X_n$ to Y.

There is actually a bit more data in a string diagram for a TSMC $\mathscr{C}$; namely, each wire is labeled by an element of $\mathscr{O} := \mathrm{Ob}(\mathscr{C})$. The category of cobordisms is thus taken relative to a fixed set $\mathscr{O}$, i.e. 0-manifolds and cobordisms are coherently labeled by objects of $\mathscr{C}$.

We record these interpretations of string diagrams below.

Interpretations of string diagrams		
String diagram	Traced category $\mathscr{C}$	**Cob**/$\mathscr{O}$
Boxes	Morphisms	Labeled 0-manifolds
Diagram	Pasting diagram	Coherent cobordism
Nesting	Axioms of TSMCs	Composition law

The relationship between these interpretations is made precise in the following first main theorem.

Theorem 1.1. *The category* **TSMC** *is equivalent to the category of* **Cob**/$\mathscr{O}$*-algebras.*

This precise connection between objects in **Cob**/$\mathscr{O}$ and morphisms in the traced category $\mathscr{C}$ is made by a **Cob**/$\mathscr{O}$-algebra, i.e. a lax functor $P\colon \mathbf{Cob}/\mathscr{O} \to \mathbf{Set}$. For each object box (X_-, X_+) in a string diagram, there is a set of morphisms in $\mathscr{C}$, roughly $\mathrm{Hom}_{\mathscr{C}}(X_-, X_+)$, which can fill the box. The functor P assigns to each morphism in **Cob**/$\mathscr{O}$ a pasting diagram, which encodes a collection of compositions, monoidal products, and traces in $\mathscr{C}$. Finally the functor P provides a method for evaluating the pasting diagram on a set of morphisms of $\mathscr{C}$ filling the inner boxes to obtain a morphism filling the outer box, this encodes all of the axioms which specify how compositions, monoidal products, and traces must interact in $\mathscr{C}$. This process is intimately connected to the composition law in **Cob**/$\mathscr{O}$ which thus corresponds roughly to the set of axioms for the trace in $\mathscr{C}$.

1.1 Nesting properties and self-similarity in applications

1.2 Generalization

Theorem 1.2. *Let $\mathscr{C}$ be a compact closed category and $\mathscr{V}$ a symmetric monodical category. The category* $\mathrm{Lax}(\mathscr{C}, \mathscr{V})$ *of lax monodical functors is equivalent to the coslice category* $\mathscr{C}/\mathscr{V} - \mathbf{CompCat}$ *spanned by bijective on objects functors.*

Corollary 1.3. $\mathrm{Lax}(\mathrm{Int}(\mathscr{T}), \mathscr{V}) = \mathscr{V} - \mathbf{TSMC}_{\mathscr{T}/}$

2 Wiring Diagrams and 1 – Cob

2.1 Set-theoretic formulation of 1 – Cob, as free compact closed category on one object, as Int of the free TSMC on one object (Bij)

Many object case/generalization Cob/$\mathscr{O}$

2.2 Drawings of morphisms in 1 – Cob as wiring diagrams, new way to visualize these

2.3 1 – Cob-algebras and applications

2.4 Definition of functors between TSMC and Cob/$\mathscr{O}$-algebras

Let $\mathscr{M}$ be a traced symmetric monoidal category with objects $\mathscr{O}$. We will define a $\mathbf{Cob}/\mathscr{O}$-algebra $\mathscr{P} = R(\mathscr{M})\colon \mathbf{Cob}/\mathscr{O} \to \mathbf{Set}$ as follows. For an object $X \in \mathrm{Ob}(\mathbf{Cob}/\mathscr{O})$, set

$$\mathscr{P}(X) := \mathrm{Hom}_{\mathscr{M}}(\overline{X_-}, \overline{X_+}).$$

We next consider morphisms.

Following Proposition **??** (borrow notation/setup from Abramsky instead) a morphism $\Phi\colon X \longrightarrow Y$ consists of a typed bijection

$$\varphi\colon X_- \sqcup Y_+ \xrightarrow{\cong} X_+ \sqcup Y_-,$$

together with a typed finite set S. Given an element $f \in \mathscr{P}(X)$ we must construct $\mathscr{P}(\Phi)(f) \in \mathscr{P}(Y)$. Let $\dim(\overline{S}) = \mathrm{Tr}^{\overline{S}}_{I,I}\big[\, \mathrm{id}_{\overline{S}}\, \big] \in \mathscr{S}_{\mathscr{M}}$. Then we use the formula

$$\mathscr{P}(\Phi)(f) := \mathrm{Tr}^{\overline{X_+}}_{\overline{Y_-},\overline{Y_+}}\Big[\big(f \otimes \mathrm{id}_{\overline{Y_+}}\big) \circ \overline{\varphi}\Big] \otimes \dim(\overline{S}).$$

Theorem 2.1. *The category* **TSMC** *is equivalent to the category of* **Cob**/$\mathscr{O}$*-algebras.*

Proof. □

Corollary 2.2. *Enriched setting?*

3 Preliminaries

Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories. Recall that a functor $F\colon\mathscr{C}\to\mathscr{D}$ is called *lax monoidal* if it is equipped with a morphism

$$I_D \xrightarrow{\epsilon} F(I_C)$$

and a natural transformation

$$F(X)\otimes_D F(Y) \xrightarrow{\mu_{X,Y}} F(X\otimes_C Y)$$

such that for all $X,Y,Z\in\mathscr{C}$, the diagram (suppressing associators)

$$\begin{tikzcd}
F(X)\otimes F(Y)\otimes F(Z) \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & F(X)\otimes F(Y\otimes Z) \arrow[d, "\mu"] \\
F(X\otimes Y)\otimes F(Z) \arrow[r, "\mu"'] & F(X\otimes Y\otimes Z)
\end{tikzcd}$$

commutes, and for all $X\in\mathscr{C}$ the two diagrams

$$\begin{tikzcd}
I_D\otimes F(X) \arrow[d, "\epsilon\otimes\mathrm{id}"'] & F(X) \arrow[l, "l_{F(X)}"'] \arrow[d, "F(l_X)"] \\
F(I_C)\otimes F(X) \arrow[r, "\mu"'] & F(I_C\otimes X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X)\otimes I_D \arrow[d, "\mathrm{id}\otimes\epsilon"'] & F(X) \arrow[l, "r_{F(X)}"'] \arrow[d, "F(r_X)"] \\
F(X)\otimes F(I_C) \arrow[r, "\mu"'] & F(X\otimes I_C)
\end{tikzcd}$$

commute. If ϵ and μ are isomorphisms, then F is *strong*. test

If $\mathscr{C}$ and $\mathscr{D}$ are symmetric monoidal, then F is a *lax symmetric monoidal functor* if it is lax monoidal, and commutes with the symmetries, in the sense that the diagram

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "\sigma"] \arrow[d, "\mu"'] & F(Y)\otimes F(X) \arrow[d, "\mu"] \\
F(X\otimes Y) \arrow[r, "F(\sigma)"'] & F(Y\otimes X)
\end{tikzcd}$$

commutes.

If F and G are lax monoidal functors (possibly symmetric), then a natural transformation $\alpha\colon F\to G$ is called a *monoidal transformation* if the diagrams

$$\begin{tikzcd}
F(X)\otimes F(Y) \arrow[r, "\alpha_X\otimes\alpha_Y"] \arrow[d, "\mu"'] & G(X)\otimes G(Y) \arrow[d, "\mu"] \\
F(X\otimes Y) \arrow[r, "\alpha_{X\otimes Y}"'] & G(X\otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_D \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] & \\
F(I_C) \arrow[rr, "\alpha_I"'] & & G(I_C)
\end{tikzcd}$$

commute.

Let **SymMonCat** denote the bicategory of symmetric monoidal categories and strong monoidal functors, and let $\mathrm{Lax}(\mathscr{C},\mathscr{D})$ denote the category of lax monoidal functors and monoidal transformations from $\mathscr{C}$ to $\mathscr{D}$. Let **CompCat** denote the full subcategory of **SymMonCat** spanned by the compact categories.

4 Profunctors

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Recall that a profunctor M from $\mathscr{C}$ to $\mathscr{D}$, written

$$\mathscr{C} \xrightarrow{M} \mathscr{D},$$

is defined to be a functor $M\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$. We can think of a profunctor as a sort of graded bimodule: for each object $c \in \mathscr{C}$ and $d \in \mathscr{D}$ there is a set $M(c,d)$ of elements in the bimodule, and given an element $m \in M(c,d)$ and morphisms $f\colon c' \to c$ in $\mathscr{C}$ and $g\colon d \to d'$ in $\mathscr{D}$, there are elements $g \cdot m \in M(c,d')$ and $m \cdot f \in F(c',d)$, such that $(g\cdot m)\cdot f = g\cdot(m\cdot f)$, and $g'\cdot(g\cdot m) = (g'\circ g)\cdot m$ and $(m\cdot f)\cdot f' = m\cdot(f\circ f')$ whenever they make sense.

If $F\colon \mathscr{C}' \to \mathscr{C}$ and $G\colon \mathscr{D}' \to \mathscr{D}$ are functors, and M is a profunctor as before, then there is a profunctor $M(F,G)$ from $\mathscr{C}'$ to $\mathscr{D}'$, defined to be the composite

$$\mathscr{C}'^{\mathrm{op}} \times \mathscr{D}' \xrightarrow{F^{\mathrm{op}}\times G} \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \xrightarrow{M} \mathbf{Set}.$$

In other words, for any objects $c \in \mathbf{C}'$ and $d \in \mathbf{D}'$, the profunctor $M(F,G)$ has elements $M(Fc,Gd)$, and if $m \in M(Fc,Gd)$ and $g\colon d \to d'$ is a morphism in $\mathscr{D}'$, then the element $m\cdot g$ in $M(F,G)$ is defined by the element $m\cdot G(g)$ in M, and similarly for the $\mathscr{C}'$ action.

Given two profunctors

$$\mathscr{C} \overset{M}{\underset{N}{\rightrightarrows}} \mathscr{D}$$

define a profunctor morphism $\phi\colon M \Rightarrow N$ to be a natural transformation. In other words, for each $c \in \mathscr{C}$ and $d \in \mathscr{D}$ there is a function $\phi_{c,d}\colon M(c,d) \to N(c,d)$ such that $\phi(f\cdot m\cdot g) = f\cdot\phi(m)\cdot g$ whenever it makes sense.

There is a tensor product of profunctors: given two profunctors

$$\mathscr{C} \xrightarrow{M} \mathscr{D} \xrightarrow{N} \mathscr{E}$$

define the profunctor $M\otimes N$ such that for objects $c \in \mathscr{C}$ and $e \in \mathscr{E}$, $(M\otimes N)(c,e)$ is the coequalizer of the diagram

$$\coprod_{d_1,d_2\in\mathscr{D}} M(c,d_1)\times\mathscr{D}(d_1,d_2)\times N(d_2,e) \rightrightarrows \coprod_{d\in\mathscr{D}} M(c,d)\times N(d,e)$$

where the two maps are given by the right action of $\mathscr{D}$ on M and by the left action of $\mathscr{D}$ on N. We can write elements of $(M\otimes N)(c,e)$ as tensors $m\otimes n$, where

$m \in M(c,d)$ and $n \in N(d,e)$ for some $d \in \mathscr{D}$. The coequalizer then implies that $(m \cdot f) \otimes n = m \otimes (f \cdot n)$ whenever the equation makes sense.

For any category $\mathscr{C}$, there is a profunctor $\mathrm{Hom}_{\mathscr{C}} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$, and these hom profunctors act as units for the tensor product. Precisely, if M is as above, there are canonical isomorphisms $\mathrm{Hom}_{\mathscr{C}} \otimes M \cong M \cong M \otimes \mathrm{Hom}_{\mathscr{D}}$.

Given a category $\mathscr{C}$, there is a monoidal category $\mathrm{Prof}(\mathscr{C},\mathscr{C})$ of profunctors from $\mathscr{C}$ to itself and morphisms of profunctors, with the tensor product given above and $\mathrm{Hom}_{\mathscr{C}}$ as the monoidal unit. We would now like to investigate monoids in this monoidal category.

Suppose $M \in \mathrm{Prof}(\mathscr{C},\mathscr{C})$ has a monoid structure. The unit is a profunctor morphism $i \colon \mathrm{Hom}_{\mathscr{C}} \to M$. So for any $f \colon c \to d$ in $\mathscr{C}$ there is an element $i(f) \in M(c,d)$, such that $f \cdot i(g) \cdot h = i(f \circ g \circ h)$ whenever this makes sense. The multiplication $M \otimes M \to M$ is an operation assigning to any elements $m_1 \in M(c,d)$ and $m_2 \in M(d,e)$ an element $m_2 \bullet m_1 \in M(c,e)$, satisfying the following equations whenever they make sense:

$$(f \cdot m_2) \bullet (m_1 \cdot h) = f \cdot (m_2 \bullet m_1) \cdot h$$
$$(m_3 \cdot g) \bullet m_1 = m_3 \bullet (g \cdot m_1)$$
$$m \bullet i(f) = m \cdot f \quad \text{and} \quad i(g) \bullet m = g \cdot m$$

Lemma 4.1. *There is an equivalence of categories* $\mathbf{Mon}(\mathrm{Prof}(\mathscr{C},\mathscr{C})) \simeq (\mathscr{C}/\mathbf{Cat})_{b.o.o.}$ *between the category of monoids in* $\mathrm{Prof}(\mathscr{C},\mathscr{C})$ *and the full subcategory of the coslice category* $\mathscr{C}/\mathbf{Cat}$ *spanned by the bijective-on-objects functors.*

Proof. Simple to check. The unit provides the identities and the functor from $\mathscr{C}$, while the multiplication provides the composition. □

Now suppose $\mathscr{C}$ and $\mathscr{D}$ are monoidal categories. We will write

$$a_{c,d,e} \colon (c \otimes d) \otimes e \to c \otimes (d \otimes e), \quad \lambda_c \colon I \otimes c \to c, \quad \rho_c \colon c \otimes I \to c$$

for the associator and left and right unitor isomorphisms, respectively, leaving it to context to make clear whether we are in $\mathscr{C}$ or $\mathscr{D}$.

A *monoidal profunctor* M from $\mathscr{C}$ to $\mathscr{D}$ is an ordinary profunctor such that the functor $M \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$ is equipped with a lax-monoidal structure, with the cartesian monoidal structure on $\mathbf{Set}$. In the bimodule notation, this means that there is an associative operation assigning to any elements $m_1 \in M(c_1,c_1')$ and $m_2 \in M(c_2,c_2')$ an element $m_1 \boxtimes m_2 \in M(c_1 \otimes c_2, c_1' \otimes c_2')$ such that

$$(f_1 \cdot m_1 \cdot g_1) \boxtimes (f_2 \cdot m_2 \cdot g_2) = (f_1 \otimes f_2) \cdot (m_1 \boxtimes m_2) \cdot (g_1 \otimes g_2),$$

as well as a distinguished element $I_M \in M(I,I)$ such that $\lambda_d \cdot (I_M \boxtimes m) \cdot \lambda_c^{-1} = m = \rho_d \cdot (m \boxtimes I_M) \cdot \rho_c^{-1}$ for any $m \in M(c,d)$.

A monoidal profunctor morphism $\phi \colon M \to N$ is simply a monoidal transformation. Spelling this out in bimodule notation, ϕ is an ordinary morphism of profunctors such that $\phi(m_1 \boxtimes m_2) = \phi(m_1) \boxtimes \phi(m_2)$ and $\phi(I_M) = I_N$. We will denote the category of monoidal profunctors from $\mathscr{C}$ to $\mathscr{D}$ and monoidal profunctor morphisms as $\mathrm{MProf}(\mathscr{C}, \mathscr{D})$.

A unit for a monoidal profunctor $M \in \mathrm{MProf}(\mathscr{C}, \mathscr{C})$ is a unit $i\colon \mathrm{Hom}_{\mathscr{C}} \to M$ in $\mathrm{Prof}(\mathscr{C}, \mathscr{C})$ such that, additionally, $i(\mathrm{id}_{I_{\mathscr{C}}}) = I_M$ and $i(f \otimes g) = i(f) \boxtimes i(g)$ for any morphisms f and g in $\mathscr{C}$. Similarly, a multiplication on M is as above, with the additional conditions

$$I_M \bullet I_M = I_M$$
$$(m_1 \boxtimes m_1') \bullet (m_2 \boxtimes m_2') = (m_1 \bullet m_2) \boxtimes (m_1' \bullet m_2')$$

for any $m_1 \in M(c,d)$, $m_1' \in M(c',d')$, $m_2 \in M(d,e)$, and $m_2' \in M(d'e')$.

Lemma 4.2. *Let $\mathscr{C}$ be a monoidal category. There is an equivalence of categories $\mathbf{Mon}(\mathrm{MProf}(\mathscr{C}, \mathscr{C})) \cong (\mathscr{C}/\mathbf{MonCat})_{b.o.o.}$ between the category of monoids in $\mathrm{MProf}(\mathscr{C}, \mathscr{C})$ and the full subcategory of the coslice category $\mathscr{C}/\mathbf{MonCat}$ spanned by the bijective-on-objects functors.*

5 Compact closed categories

Let $\mathscr{C}$ be a compact closed category.

Proposition 5.1. *There are functors*

$$\mathrm{MProf}(1, \mathscr{C}) \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{MProf}(\mathscr{C}, \mathscr{C})$$

Proof. For any $M\colon \mathscr{C} \to \mathbf{Set}$, define $FM\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ by $FM(A,B) = M(A^* \otimes B)$. In the other direction, for $N\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$, define $UN(A) = N(1,A)$. □

Proposition 5.2. *Let $N \in \mathrm{MProf}(\mathscr{C}, \mathscr{C})$ be a monoidal profunctor equipped with a unit $\eta\colon \mathrm{Hom}_{\mathscr{C}} \to N$. Then N has a canonical multiplication $\mu\colon N \otimes N \to N$ making N a monoid in $\mathrm{MProf}(\mathscr{C}, \mathscr{C})$.*

Proof. We can define a multiplication on N by the following formula: given any $n_1 \in N(c,d)$ and $n_2 \in N(d,e)$,

$$n_2 \bullet n_1 = (\mathrm{id}_c \otimes \eta_d) \cdot (n_1 \boxtimes i(\mathrm{id}_{d^*}) \boxtimes n_2) \cdot (\epsilon_d \otimes \mathrm{id}_e)$$

□

Proposition 5.3. *For any $M \in \mathrm{MProf}(1, \mathscr{C})$, $FM \in \mathrm{Prof}(\mathscr{C}, \mathscr{C})$ has a canonical unit.*

Corollary 5.4. *The functor F factors canonically through the category $\mathbf{Mon}(\mathrm{MProf}(\mathscr{C}, \mathscr{C}))$ of monoid objects.*

Proposition 5.5. *The functors F and U induce an equivalence of categories* $\mathrm{MProf}(1,\mathscr{C}) \simeq \mathbf{Mon}(\mathrm{MProf}(\mathscr{C},\mathscr{C}))$.

Corollary 5.6. *There is an equivalence of categories* $\mathrm{Lax}(\mathscr{C},\mathbf{Set}) \simeq (\mathscr{C}/\mathbf{CompCat})_{b.o.o.}$.

6 Traced Monoidal Categories

Recall from [**?**]

- Let $\mathscr{D}$ be a traced symmetric monoidal category, and $F\colon\mathscr{C} \to \mathscr{D}$ a fully faithful symmetric monoidal functor. Then $\mathscr{C}$ has a unique trace for which F is a traced functor.
- Any compact category has a canonical trace, defining a functor $U\colon\mathbf{CompCat} \to \mathbf{TrCat}$.
- The Int construction $\mathrm{Int}\colon\mathbf{TrCat} \to \mathbf{CompCat}$ is left 2-adjoint to U. For any traced symmetric monoidal category $\mathscr{C}$, the unit $\mathscr{C} \to \mathrm{Int}(\mathscr{C})$ is fully faithful.

Lemma 6.1. *Let $\mathscr{D}$ be a traced symmetric monoidal category, and $F\colon\mathscr{C} \to \mathscr{D}$ a fully faithful symmetric monoidal functor. Then, using the unique trace on $\mathscr{C}$ making F a traced functor, the functor* $\mathrm{Int}(\mathscr{C}) \to \mathscr{D}$ *which is adjunct to F is also fully faithful.*

Proposition 6.2. *Let $\mathscr{C}$ be a traced symmetric monoidal category. Then the Int construction provides an equivalence of categories*

$$(\mathscr{C}/\mathbf{TrCat})_{b.o.o.} \simeq (\mathrm{Int}(\mathscr{C})/\mathbf{CompCat})_{b.o.o.}$$