CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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Abstract. We prove that all quiver Grassmannians for preprojective and preinjective respesentations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. Introduction

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -acknowledgements?

2. Torus Action on Quiver Grassmannians

Definition 0.1. *universal cover* $\tilde{Q}$

Lemma 0.1. *how to lift exceptional representations of* Q *to* $\tilde{Q}$

Lemma 0.2. $d : \tilde{Q}_0 \to \mathbb{Z}$ *defines a torus action on* $Gr^Q_{\mathbf{e}}(X)$ *if* $d(q, w\rho) - d(q, w) = c_\rho$ *for all* $w \in W_{\tilde{Q}_1}$

Lemma 0.3. *There exists* $d : \tilde{Q}_0 \to \mathbb{Z}$ *such that* $d(q, w) \neq d(q', w')$ *for all* q, q', w, w' *with* $\dim X_{q,w} \neq 0$ *and* $\dim X_{q',w'} \neq 0$.

Theorem 1. $Gr^Q_{\mathbf{e}}(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} of type \mathbf{e}} Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\tilde{X})$

Corollary 1.1. *affine bundles over* $Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\tilde{X})$ *if* $Gr^Q_{\mathbf{e}}(X)$ *is smooth*

$$\{U \in Gr^Q_{\mathbf{e}}(X) : \lim_{t\to 0} t \cdot U \in Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\tilde{X})\}$$

Question 1.1. *What are the ranks of these bundles? Poincaré polynomials?*

3. Quiver Grassmannians of $\widetilde{K(n)}$

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$. For $m \geq 1$, let P_m be the preprojective representation of $K(n)$ with dimension vector (u_m, u_{m-1}).

Let $\tilde{P}_m$ be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 1.1. *There exist lifts* $\tilde{P}_{m-1,i}$ *for* $1 \leq i \leq n$ *of* P_{m-1} *to* $\widetilde{K(n)}$ *so that:*

(1) $\mathrm{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$;

(2) *For any proper subset* $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, *there exists a short exact sequence*

$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\ldots,i_k} \longrightarrow 0;$$

(3) *The lifts* $\tilde{P}_{m-1,i}$ *are pairwise orthogonal.*

(4) *All nontrivial proper subrepresentations of* $\tilde{P}_m^{(k)}$ *are preprojective.*

We will always choose the subset $\{1, \ldots, k\}$ when using Lemma 1.1.2 and thus we denote the cokernal simply by $\tilde{P}_m^{(k)}$.

Lemma 1.2. *If each* $Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\tilde{P}_{m-1,i})$ *has a cell decomposition, then* $Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\bigoplus \tilde{P}_{m-1,i_j})$ *has a cell decomposition.*

Lemma 1.3. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) *The subrepresentation* $\bigoplus_{i=1}^{k-1}\tilde{P}_{m-1,i}\oplus\bigoplus_{i=1}^{k}\tilde{P}_{m-2,i}\subset\bigoplus i=1^k\tilde{P}_{m-1,i}$ *is in* $(\tilde{P}_m^{(k)})^\perp$ *and*

$$\mathrm{Ext}(\bigoplus i=1^k\tilde{P}_{m-1,i},\tilde{P}_m^{(k)})\cong\mathrm{Ext}(\tilde{P}_{m-1}^{(k)},\tilde{P}_m^{(k)})$$

where $\tilde{P}_{m-1}^{(k)}$ *above denotes the cokernel of the inclusion.*

Corollary 1.2. *observe when fibers are empty*

Proposition 1.1. *Consider* $\psi: Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)\to\bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}}Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^{k}\tilde{P}_{m-1,i})\times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)}$. *Then the following hold:*

(1) *For* $V\subsetneq\tilde{P}_m^{(k)}$ *and* $U\subset\bigoplus_{i=1}^{k}\tilde{P}_{m-1,i}$, *we have* $\psi^{-1}(U,V)=\mathbb{A}^{\langle V,\bigoplus_{i=1}^{k}\tilde{P}_{m-1,i}/U\rangle}$.

(2) *If* $V=\tilde{P}_m^{(k)}$ *and the fiber is not empty, then* $\psi^{-1}(U,V)$ *is constant.*

Proof. (1) V is preprojective but $\bigoplus_{i=1}^{k}\tilde{P}_{m-1,i}/U$ is not unless $U=0$

(2)

$$0\longrightarrow[V,P/U]\longrightarrow[V,U]^1\longrightarrow[V,P]^1\longrightarrow[V,P/U]^1\longrightarrow 0$$

and the middle map is surjective. □

Theorem 2. *Every quiver Grassmannian of a preprojective or preinjective representation of* $K(n)$ *and* $\widetilde{K(n)}$ *has a cell decomposition.*

Question 2.1. *cells of* $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ *are in one-to-one correspondence with certain tuples of subgraphs for smaller* $\tilde{P}_\ell^{i_1,\dots,i_k}$

4. Compatible Pairs Label Cells in $Gr_{\mathbf{e}}^{Q}(P_m)$