

CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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ABSTRACT. We prove that all quiver Grassmannians for preprojective and preinjective representations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. INTRODUCTION

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -something about torus actions and the universal cover -something about cell decompositions of quiver Grassmannians -acknowledgements?

2. TORUS ACTION ON QUIVER GRASSMANNIANS

Let Q be an acyclic quiver with vertices Q_0 and arrows Q_1 which we denote by $\alpha : i \rightarrow j$. Moreover, let W_Q be the free (non-abelian) group generated by Q_1 . We denote by $\text{Rep}(Q)$ the category of \mathbb{C} -representations of Q .

Definition 0.1. *The universal covering quiver \tilde{Q} of Q is given by the vertices $\tilde{Q}_0 = Q_0 \times W_Q$ and the arrow set $\tilde{Q}_1 = Q_1 \times W_Q$ where $(\alpha, w) : (i, w) \rightarrow (j, w\alpha)$ for every $\alpha : i \rightarrow j$.*

We say that a representation $X \in \text{Rep}(Q)$ can be lifted (to \tilde{Q}) if there exists a representation $\tilde{X} \in \text{Rep}(\tilde{Q})$ such that $F_Q \tilde{X} = X$ where $F_Q : \text{Rep}(\tilde{Q}) \rightarrow \text{Rep}(Q)$ is the natural functor.

Lemma 0.1. *Every preprojective (resp. preinjective) representation of Q can be lifted to a representation of \tilde{Q} .*

Proof. This statement is clear for the simple representations S_q , $q \in Q_0$. Now every preprojective representation X can be obtained when applying a sequence of BGP-reflections [2] to a simple representation $S_{q'}$ of a quiver Q' whose underlying graph is the one of Q . Applying BGP-reflections to a source q of Q corresponds to applying BGP-reflections to all vertices (q, w) of \tilde{Q} . This leads the claim. The statement for preinjective representations follows in the same way. \square

We choose a map $d : \tilde{Q}_0 \rightarrow \mathbb{Z}$ and fix a representation $X \in \text{Rep}(Q)$. In any case, we can consider the decomposition $X_q = \bigoplus_{w \in W_Q} X_{(q,w)}$. We define a torus action on each $X_{(q,w)}$ via $t.x_{(q,w)} = t^{d(q,w)}x_{(q,w)}$ which can be extended linearly to each X_q . For a fixed a subspace U_q , we can define the subspace $t.U_q$. In general, this torus action induces no torus action on the Quiver Grassmannians $\text{Gr}_{\mathbf{e}}(X)$ as $t.U = (t.U_q)_{q \in Q_0}$ is no subrepresentation of X for every $U \in \text{Gr}_{\mathbf{e}}(X)$. Actually, for this the action has to satisfy $X_\alpha(t.U_i) \in t.U_j$ for every $\alpha : i \rightarrow j$ and every $x \in X_i$.

Lemma 0.2. *Fix an integer $c_\alpha \in \mathbb{Z}$ for every $\alpha \in Q_1$. If $X \in \text{Rep}(Q)$ can be lifted, $d : \tilde{Q}_0 \rightarrow \mathbb{Z}$ induces a torus action on $\text{Gr}_{\mathbf{e}}(X)$ if we have $d(j, w\alpha) - d(i, w) = c_\alpha$ for all $\alpha : i \rightarrow j$ and $w \in W_Q$.*

Proof. Since X can be lifted, we can write $X_\alpha : X_i \rightarrow X_j$ as block matrix consisting of linear maps $X_{(\alpha,w)} : X_{(i,w)} \rightarrow X_{(j,w\alpha)}$. Then the condition $X_\alpha(t.U_i) \in t.U_j$ translates into ...to be continued. \square

Lemma 0.3. *There exists $d : \tilde{Q}_0 \rightarrow \mathbb{Z}$ such that $d(q, w) \neq d(q', w')$ for all q, q', w, w' with $\dim X_{q,w} \neq 0$ and $\dim X_{q',w'} \neq 0$.*

Theorem 1. $Gr_{\mathbf{e}}^Q(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} \text{ of type } \mathbf{e}} Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$

Corollary 1.1. *affine bundles over $Gr_{\mathbf{e}}^{\tilde{Q}}(\tilde{X})$ if $Gr_{\mathbf{e}}^Q(X)$ is smooth*

$$\{U \in Gr_{\mathbf{e}}^Q(X) : \lim_{t \rightarrow 0} t \cdot U \in Gr_{\mathbf{e}}^{\tilde{Q}}(\tilde{X})\}$$

Question 1.1. *What are the ranks of these bundles? Poincaré polynomials?*

3. REPRESENTATION THEORY OF $K(n)$

Denote by $K(n)$ the n -Kronecker quiver $1 \xleftarrow{n} 2$ with vertices $K(n)_0 = \{1, 2\}$ and n arrows from vertex 2 to vertex 1.

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$.

Theorem 2. *For each $m \geq 1$, there exist unique (up to isomorphism) indecomposable rigid representations P_m and I_m of $K(n)$ with dimension vectors (u_m, u_{m-1}) and (u_{m-1}, u_m) respectively. Moreover, any rigid representation of $K(n)$ is isomorphic to one of the form $P_m^{a_1} P_{m+1}^{a_2}$ or $I_m^{a_1} I_{m+1}^{a_2}$ for some $m \geq 1$ and some $a_1, a_2 \geq 0$.*

The representations P_m are called the *preprojective* representations of $K(n)$ and the representations I_m are called *preinjective*.

Lemma 2.1. *For any $m \geq 1$, we have*

$$\dim \operatorname{Hom}(P_m, P_m) = 1, \quad \dim \operatorname{Hom}(P_m, P_{m+1}) = n, \quad \dim \operatorname{Hom}(P_{m+1}, P_m) = 0.$$

Moreover, for $1 \leq k < n$ and linearly independent functions $f_1, \dots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the map $[f_1 \cdots f_k] : P_m^k \rightarrow P_{m+1}$ is injective.

Proof. Note that we have a Auslander-Reiten sequence

$$0 \longrightarrow P_{m-1} \longrightarrow P_m^n \xrightarrow{[f_1 \cdots f_n]} P_{m+1} \longrightarrow 0$$

with irreducible homomorphisms f_i which are also linear independent. Thus they do not factor through a representation $Z \neq P_m$ which already means that they are injective, see for instance [1, Lemma 1.6]. If we pick $k < n$ linear independent homomorphisms f_{i_1}, \dots, f_{i_k} , this also means that $[f_{i_1} \cdots f_{i_k}] : P_m^k \rightarrow P_{m+1}$ is injective. Indeed otherwise, at least one of the homomorphisms were forced to factor through a representation $Z \neq P_m$.

Write $P_{m+1}^{\{f_1, \dots, f_k\}}$ for the cokernel of the map $[f_1 \cdots f_k] : P_m^k \rightarrow P_{m+1}$, i.e. we have a short exact sequence

$$(1) \quad 0 \longrightarrow P_m^k \xrightarrow{[f_1 \cdots f_k]} P_{m+1} \longrightarrow P_{m+1}^{\{f_1, \dots, f_k\}} \longrightarrow 0.$$

□

Lemma 2.2. *For $m \geq 1$ and linearly independent functions $f_1, \dots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the following hold:*

- (1) $\operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \dots, f_k\}})$ is naturally isomorphic to $\operatorname{Hom}(P_m, P_{m+1}) / \mathbb{k}f_1 \oplus \cdots \oplus \mathbb{k}f_k$;
- (2) $\operatorname{Ext}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m)$ is naturally isomorphic to $\mathbb{k}f_1 \oplus \cdots \oplus \mathbb{k}f_k$;
- (3) $\operatorname{Hom}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m) = 0$.

Proof. Applying the functor $\operatorname{Hom}(P_m, -)$ to the sequence (1) gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(P_m, P_m^k) \xrightarrow{[f_1 \cdots f_k]^\circ} \operatorname{Hom}(P_m, P_{m+1}) \longrightarrow \operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \dots, f_k\}}) \longrightarrow 0,$$

establishing (1).

To see (2), we apply the functor $\operatorname{Hom}(-, P_m)$ to the sequence (1) to get an isomorphism $\operatorname{Hom}(P_m^k, P_m) \cong \operatorname{Ext}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m)$. Observe that $\operatorname{Hom}(P_m^k, P_m)$ is spanned by the coordinate projections $\pi_i : P_m^k \rightarrow P_m$

for $1 \leq i \leq k$ and each of these gives rise to a commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & P_m^{k-1} & \xlongequal{\quad} & P_m^{k-1} & & & \\
 & \downarrow & & \downarrow & & & \\
 & & & & [f_1 \ \cdots \ \widehat{f_i} \ \cdots f_k] & & \\
 0 \longrightarrow & P_m^k & \xrightarrow{[f_1 \ \cdots f_k]} & P_{m+1} & \longrightarrow & P_{m+1}^{\{f_1, \dots, f_k\}} & \longrightarrow 0 \\
 & \downarrow \pi_i & \lrcorner & \downarrow & & \parallel & \\
 0 \longrightarrow & P_m & \xrightarrow{f_i} & P_{m+1}^{\{f_1, \dots, \widehat{f_i}, \dots, f_k\}} & \longrightarrow & P_{m+1}^{\{f_1, \dots, f_k\}} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
 \end{array}$$

The bottom row of this diagram is the extension in $\text{Ext}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m)$ naturally identified with the generator f_i of $\mathbb{k}f_1 \oplus \cdots \oplus \mathbb{k}f_k$.

Part (3) follows immediately from (2) and the observation that

$$\langle P_{m+1}^{\{f_1, \dots, f_k\}}, P_m \rangle = \langle P_{m+1}, P_m \rangle - k \langle P_m, P_m \rangle = -k.$$

□

For any nonempty subset $I \subset [1, k]$, say with $|I| = \ell$, we get an exact sequence

$$0 \longrightarrow P_m^\ell \xrightarrow{[f_i]_{i \in I}} P_{m+1}^{\{f_j\}_{j \in [1, k] \setminus I}} \longrightarrow P_{m+1}^{\{f_1, \dots, f_k\}} \longrightarrow 0.$$

Each such sequence has the following almost-split property for subrepresentations of $P_{m+1}^{\{f_1, \dots, f_k\}}$.

Lemma 2.3. *For $m \geq 1$ and linearly independent functions $f_1, \dots, f_k \in \text{Hom}(P_m, P_{m+1})$, given any proper subrepresentation $V \subsetneq P_{m+1}^{\{f_1, \dots, f_k\}}$, the upper pullback sequence*

$$\begin{array}{ccccccc}
 0 \longrightarrow & P_m^\ell & \longrightarrow & V^I & \longrightarrow & V & \longrightarrow 0 \\
 & \parallel & & \downarrow & \lrcorner & \downarrow & \\
 0 \longrightarrow & P_m^\ell & \xrightarrow{[f_i]_{i \in I}} & P_{m+1}^{\{f_j\}_{j \in [1, k] \setminus I}} & \longrightarrow & P_{m+1}^{\{f_1, \dots, f_k\}} & \longrightarrow 0
 \end{array}$$

splits for every $I \subset [1, k]$.

Proof. When $I = \emptyset$ there is nothing to show, so assume I is a nontrivial subset of $[1, k]$.

We proceed by induction on k . For any nonzero $f \in \text{Hom}(P_m, P_{m+1})$ the exact sequence

$$0 \longrightarrow P_m \xrightarrow{f} P_{m+1} \longrightarrow P_{m+1}^{\{f\}} \longrightarrow 0$$

is almost split, giving the claim in the case $k = 1$. For $k > 1$ and a nontrivial subset $I \subset [1, k]$, we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P_m^{k-\ell} & \xlongequal{\quad} & P_m^{k-\ell} & \longrightarrow & 0 \\
 & \downarrow & \downarrow [f_j]_{j \in [1, k] \setminus I} & & \downarrow [f_j]_{j \in [1, k] \setminus I} & & \\
 0 & \longrightarrow & P_m^\ell & \xrightarrow{[f_i]_{i \in I}} & P_{m+1} & \longrightarrow & P_{m+1}^{\{f_i\}_{i \in I}} \longrightarrow 0 \\
 & \parallel & \downarrow & \lrcorner & \downarrow & & \\
 0 & \longrightarrow & P_m^\ell & \xrightarrow{[f_i]_{i \in I}} & P_{m+1}^{\{f_j\}_{j \in [1, k] \setminus I}} & \longrightarrow & P_{m+1}^{\{f_1, \dots, f_k\}} \longrightarrow 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

A proper subrepresentation $V \subset P_{m+1}^{\{f_1, \dots, f_k\}}$ gives rise, via pullbacks, to the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P_m^{k-\ell} & \xlongequal{\quad} & P_m^{k-\ell} & \longrightarrow & 0 \\
 & \downarrow & \downarrow [f_j]_{j \in [1, k] \setminus I} & & \downarrow [f_j]_{j \in [1, k] \setminus I} & & \\
 0 & \longrightarrow & P_m^\ell & \xrightarrow{[f_i]_{i \in I}} & V^{[1, k]} & \longrightarrow & V^{[1, k] \setminus I} \longrightarrow 0 \\
 & \parallel & \downarrow & \lrcorner & \downarrow & & \\
 0 & \longrightarrow & P_m^\ell & \xrightarrow{[f_i]_{i \in I}} & V^I & \longrightarrow & V \longrightarrow 0 \\
 & \downarrow & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

in which the middle vertical sequence and the middle horizontal sequence split by induction. But then the lower horizontal sequence splits, establishing the claim for all $I \subset [1, k]$. \square

4. QUIVER GRASSMANNIANS OF $\widetilde{K(n)}$

Let \tilde{P}_m be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 2.4. *There exist lifts $\tilde{P}_{m-1, i}$ for $1 \leq i \leq n$ of P_{m-1} to $\widetilde{K(n)}$ so that:*

- (1) $\text{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \text{Hom}_{\tilde{Q}}(\tilde{P}_{m-1, i}, \tilde{P}_m)$;
- (2) *For any proper subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, there exists a short exact sequence*

$$0 \longrightarrow \tilde{P}_{m-1, i_1} \oplus \dots \oplus \tilde{P}_{m-1, i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1, \dots, i_k} \longrightarrow 0;$$

- (3) *The lifts $\tilde{P}_{m-1, i}$ are pairwise orthogonal.*
- (4) *All nontrivial proper subrepresentations of $\tilde{P}_m^{(k)}$ are preprojective.*

We will always choose the subset $\{1, \dots, k\}$ when using Lemma 2.4.2 and thus we denote the cokernel simply by $\tilde{P}_m^{(k)}$.

Lemma 2.5. *If each $\text{Gr}_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1, i})$ has a cell decomposition, then $\text{Gr}_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1, i_j})$ has a cell decomposition.*

Lemma 2.6. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) The subrepresentation $\bigoplus_{i=1}^{k-1} \tilde{P}_{m-1,i} \oplus \bigoplus_{i=1}^k \tilde{P}_{m-2,i} \subset \bigoplus_{i=1}^k \tilde{P}_{m-1,i}$ is in $(\tilde{P}_m^{(k)})^\perp$ and

$$\text{Ext}(\bigoplus_{i=1}^k \tilde{P}_{m-1,i}, \tilde{P}_m^{(k)}) \cong \text{Ext}(\tilde{P}_{m-1}^{(k)}, \tilde{P}_m^{(k)})$$

where $\tilde{P}_{m-1}^{(k)}$ above denotes the cokernel of the inclusion.

Lemma 2.7. Consider $0 \rightarrow \tilde{P}_m \rightarrow \tilde{P}_{m+1}^{(k)} \rightarrow \tilde{P}_{m+1}^{(k-1)} \rightarrow 0$. If $U \subset P_m$ such that $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, then we have $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$.

Proof. First observe that $\tilde{P}_m \in {}^\perp \tilde{P}_{m+1}^{(k-1)}$ and $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m) = k$. Assume that U is preprojective with $\text{Hom}(U, \tilde{P}_{m+1}^{(k-1)}) = 0$ (check again). This yields $\text{Ext}(\tilde{P}_m/U, \tilde{P}_{m+1}^{(k-1)}) = 0$. Consider

$$0 \longrightarrow [\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U] \longrightarrow [\tilde{P}_{m+1}^{(k-1)}, U]^1 \xrightarrow{\pi} [\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m]^1 \longrightarrow [\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U]^1 \longrightarrow 0.$$

Since $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m) = k$, we have that π is surjective if $\pi \neq 0$. If $U = U_1 \oplus \dots \oplus U_r$ is a decomposition into indecomposable representations, we can write $\pi = (\pi_1, \dots, \pi_r)$ and it suffices to show that $\pi_j \neq 0$ for at least one $j = 1, \dots, r$. Thus we can without loss of generality assume that U is indecomposable (check again). This means that $U \cong \tilde{P}_i$ for some $i = 1, \dots, m$. Moreover, \tilde{P}_{m+1}/U is indecomposable (details).

If \tilde{P}_{m+1}/U is exceptional, $\text{Ext}(\tilde{P}_m/U, \tilde{P}_{m+1}^{(k-1)}) = 0$ together with [4, Theorem 4.1] yields that either $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$ or $\text{Hom}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$. Thus $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) \neq 0$ would imply that $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) = 0$. But this contradicts the assumption and the claim follows.

Thus assume that \tilde{P}_{m+1}/U has self-extensions. Since $\text{Ext}(\tilde{P}_m/U, \tilde{P}_{m+1}^{(k-1)}) = 0$, from [3, Lemma 4.1] we obtain that every homomorphism $\tilde{P}_{m+1}^{(k-1)} \rightarrow \tilde{P}_m/U$ is injective or surjective. By a dimension count it needs to be surjective (case $k = 1$?). Such a homomorphism would yield $\text{Ext}(\tilde{P}_m/U, \tilde{P}_{m+1}^{(k-1)}) \twoheadrightarrow \text{Ext}(\tilde{P}_m/U, \tilde{P}_m/U) \neq 0$ which contradicts the further observations. Thus $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, already yields $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$ in this case.

Case $\text{Hom}(U, P) \neq 0$

□

Proposition 2.1. Consider $\psi : Gr_{\mathbf{e}}^{\tilde{Q}}(\tilde{P}_m) \rightarrow \bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\mathbf{e}} Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^k \tilde{P}_{m-1,i}) \times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)})$. Then the following hold:

- (1) For $V \subsetneq \tilde{P}_m^{(k)}$ and $U \subset \bigoplus_{i=1}^k \tilde{P}_{m-1,i}$, we have $\psi^{-1}(U, V) = \mathbb{A}^{\langle V, \bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U \rangle}$.
- (2) If $V = \tilde{P}_m^{(k)}$, the fibre $\psi^{-1}(U, V)$ is non-empty and of constant dimension if and only if $\text{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$.

Proof. (1) V is preprojective but $\bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U$ is not unless $U = 0$

(2) If $\text{Ext}(\tilde{P}_m^{(k)}, U) = 0$, the fibre is empty because the sequence is non-split. If $\text{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$, Lemma 2.7 yields $\text{Ext}(\tilde{P}_m^{(k)}, \tilde{P}_{m-1}/U) = 0$. In particular, the dimension is constant if it is not empty and $\text{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \xrightarrow{\pi} \text{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m)$ is surjective. But this already means that the fibre is not empty. □

Theorem 3. Every quiver Grassmannian of a preprojective or preinjective representation of $K(n)$ and $\widetilde{K(n)}$ has a cell decomposition.

Question 3.1. cells of $Gr_{\mathbf{e}}^{\tilde{Q}}(\tilde{P}_m)$ are in one-to-one correspondence with certain tuples of subgraphs for smaller $\tilde{P}_\ell^{i_1, \dots, i_k}$

5. COMPATIBLE PAIRS LABEL CELLS IN $Gr_{\mathbf{e}}^Q(P_m)$

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