CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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ABSTRACT. We prove that all quiver Grassmannians for preprojective and preinjective representations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. INTRODUCTION

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -something about torus actions and the universal cover -something about cell decompositions of quiver Grassmannians -acknowledgements?

2. TORUS ACTION ON QUIVER GRASSMANNIANS

Let Q be an acylic quiver with vertices Q_0 and arrows Q_1 which we denote by $\alpha : i \to j$. Moreover, let W_Q be the free (non-abelian) group generated by Q_1. We denote by $\mathrm{Rep}(Q)$ the category of $\mathbb{C}$-representations of Q.

Definition 2.1. *The universal covering quiver $\tilde{Q}$ of Q is given by the vertices $\tilde{Q}_0 = Q_0 \times W_Q$ and the arrow set $\tilde{Q}_1 = Q_1 \times W_Q$ where $(\alpha, w) : (i, w) \to (j, w\alpha)$ for every $\alpha : i \to j$.*

We say that a representation $X \in \mathrm{Rep}(Q)$ can be lifted (to $\tilde{Q}$) if there exists a representation $\tilde{X} \in \mathrm{Rep}(\tilde{Q})$ such that $F_Q\tilde{X} = X$ where $F_Q : \mathrm{Rep}(\tilde{Q}) \to \mathrm{Rep}(Q)$ is the natural functor.

Lemma 2.2. *Every preprojective (resp. preinjective) representation of Q can be lifted to a representation of $\tilde{Q}$.*

Proof. This statement is clear for the simple representations S_q, $q \in Q_0$. Now every preprojective representation X can be obtained when applying a sequence of BGP-reflections [2] to a simple representation $S_{q'}$ of a quiver Q' whose underlying graph is the one of Q. Applying BGP-reflections to a source q of Q corresponds to applying BGP-reflections to all vertices (q, w) of $\tilde{Q}$. This leads the claim. The statement for preinjective representations follows in the same way. $\square$

We choose a map $d : \tilde{Q}_0 \to \mathbb{Z}$ and fix a representation $X \in \mathrm{Rep}(Q)$. In any case, we can consider the decomposition $X_q = \bigoplus_{w \in W_Q} X_{(q,w)}$. We define a torus action on each $X_{(q,w)}$ via $t.x_{(q,w)} = t^{d(q,w)}x_{(q,w)}$ which can be extended linearly to each X_q. For a fixed a subspace U_q, we can define the subspace $t.U_q$. In general, this torus action induces no torus action on the Quiver Grassmannians $\mathrm{Gr}_\mathbf{e}(X)$ as $t.U = (t.U_q)_{q \in Q_0}$ is no subrepresentation of X for every $U \in \mathrm{Gr}_\mathbf{e}(X)$. Actually, for this the action has to satisfy $X_\alpha(t.U_i) \in t.U_j$ for every $\alpha : i \to j$ and every $x \in X_i$.

Lemma 2.3. *Fix an integer $c_\alpha \in \mathbb{Z}$ for every $\alpha \in Q_1$. If $X \in \mathrm{Rep}(Q)$ can be lifted, $d : \tilde{Q}_0 \to \mathbb{Z}$ induces a torus action on $\mathrm{Gr}_\mathbf{e}(X)$ if we have $d(j, w\alpha) - d(i, w) = c_\alpha$ for all $\alpha : i \to j$ and $w \in W_Q$.*

Proof. Since X can be lifted, we can write $X_\alpha : X_i \to X_j$ as block matrix consisting of linear maps $X_{(\alpha,w)} : X_{(i,w)} \to X_{(j,w\alpha)}$. Then the condition $X_\alpha(t.U_i) \in t.U_j$ translates into ...to be continued. $\square$

Lemma 2.4. *There exists $d : \tilde{Q}_0 \to \mathbb{Z}$ such that $d(q, w) \neq d(q', w')$ for all q, q', w, w' with $\dim X_{q,w} \neq 0$ and $\dim X_{q',w'} \neq 0$.*

Theorem 2.5. $Gr_\mathbf{e}^Q(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} \, of \, type \, \mathbf{e}} Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$

Corollary 2.6. *affine bundles over $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$ if $Gr_{\mathbf{e}}^{Q}(X)$ is smooth*

$$\{U \in Gr_{\mathbf{e}}^{Q}(X) : \lim_{t \to 0} t \cdot U \in Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})\}$$

Question 2.7. *What are the ranks of these bundles? Poincaré polynomials?*

3. Representation Theory of $K(n)$

Denote by $K(n)$ the *n-Kronecker quiver* $1 \xleftarrow{n} 2$ with vertices $K(n)_0 = \{1, 2\}$ and n arrows from vertex 2 to vertex 1.

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$.

Theorem 3.1. *For each $m \geq 1$, there exist unique (up to isomorphism) indecomposable rigid representations P_m and I_m of $K(n)$ with dimension vectors (u_m, u_{m-1}) and (u_{m-1}, u_m) respectively. Moreover, any rigid representation of $K(n)$ is isomorphic to one of the form $P_m^{a_1} P_{m+1}^{a_2}$ or $I_m^{a_1} I_{m+1}^{a_2}$ for some $m \geq 1$ and some $a_1, a_2 \geq 0$.*

The representations P_m are called the *preprojective* representations of $K(n)$ and the representations I_m are called *preinjective*.

Lemma 3.2. *For any $m \geq 1$, we have*

$$\dim \operatorname{Hom}(P_m, P_m) = 1, \quad \dim \operatorname{Hom}(P_m, P_{m+1}) = n, \quad \dim \operatorname{Hom}(P_{m+1}, P_m) = 0.$$

Moreover, for $1 \leq k < n$ and linearly independent functions $f_1, \ldots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the map $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$ is injective.

Proof. Note that we have an Auslander-Reiten sequence

$$(1) \qquad\qquad 0 \longrightarrow P_{m-1} \longrightarrow P_m^n \xrightarrow{[f_1 \ \cdots \ f_n]} P_{m+1} \longrightarrow 0$$

with irreducible homomorphisms f_i which are also linearly independent. Thus they do not factor through a representation $Z \neq P_m$ which already means that they are injective, see for instance [1, Lemma 1.6]. If we pick $k < n$ linear independent homomorphisms $f_{i_1}, \ldots, f_{i_k}$, this also means that $[f_{i_1} \ \cdots \ f_{i_k}] : P_m^k \to P_{m+1}$ is injective. Indeed otherwise, at least one of the homomorphisms were forced to factor through a representation $Z \neq P_m$. $\qquad\square$

Write $P_{m+1}^{\{f_1, \ldots, f_k\}}$ for the cokernel of the map $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$, i.e. we have a short exact sequence

$$(2) \qquad\qquad 0 \longrightarrow P_m^k \xrightarrow{[f_1 \ \cdots \ f_k]} P_{m+1} \longrightarrow P_{m+1}^{\{f_1, \ldots, f_k\}} \longrightarrow 0.$$

Lemma 3.3. *For $m \geq 1$ and linearly independent functions $f_1, \ldots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the following hold:*

(1) $\operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \ldots, f_k\}})$ *is naturally isomorphic to* $\operatorname{Hom}(P_m, P_{m+1})/\mathbb{k}f_1 \oplus \cdots \oplus \mathbb{k}f_k$;

(2) $\operatorname{Ext}(P_{m+1}^{\{f_1, \ldots, f_k\}}, P_m)$ *is naturally isomorphic to* $\mathbb{k}f_1 \oplus \cdots \oplus \mathbb{k}f_k$;

(3) $\operatorname{Hom}(P_{m+1}^{\{f_1, \ldots, f_k\}}, P_m) = 0$.

Proof. Applying the functor $\operatorname{Hom}(P_m, -)$ to the sequence (2) gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(P_m, P_m^k) \xrightarrow{[f_1 \ \cdots \ f_k]^\circ} \operatorname{Hom}(P_m, P_{m+1}) \longrightarrow \operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \ldots, f_k\}}) \longrightarrow 0,$$

establishing (1).

To see (2), we apply the functor $\operatorname{Hom}(-, P_m)$ to the sequence (2) to get an isomorphism $\operatorname{Hom}(P_m^k, P_m) \cong \operatorname{Ext}(P_{m+1}^{\{f_1, \ldots, f_k\}}, P_m)$. Observe that $\operatorname{Hom}(P_m^k, P_m)$ is spanned by the coordinate projections $\pi_i : P_m^k \to P_m$

for $1 \leq i \leq k$ and each of these gives rise to a commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
P_m^{k-1} & =\!=\!=\!= & P_m^{k-1} \\
\downarrow & & \downarrow {\scriptstyle [f_1 \;\cdots\; \widehat{f_i} \;\cdots\; f_k]} \\
0 \to P_m^k \xrightarrow{[f_1 \;\cdots\; f_k]} & P_{m+1} \to P_{m+1}^{\{f_1,\ldots,f_k\}} \to 0 \\
\downarrow {\scriptstyle \pi_i} \quad\ulcorner & & \downarrow \qquad \| \\
0 \to P_m \xrightarrow{f_i} P_{m+1}^{\{f_1,\ldots,\widehat{f_i},\ldots,f_k\}} & \to & P_{m+1}^{\{f_1,\ldots,f_k\}} \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The bottom row of this diagram is the extension in $\operatorname{Ext}(P_{m+1}^{\{f_1,\ldots,f_k\}}, P_m)$ naturally identified with the generator f_i of $\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$.

Part (3) follows immediately from (2) and the observation that

$$\langle P_{m+1}^{\{f_1,\ldots,f_k\}}, P_m \rangle = \langle P_{m+1}, P_m \rangle - k \langle P_m, P_m \rangle = -k.$$

$\square$

Lemma 3.4. *Let $f_1, \ldots, f_n \in \operatorname{Hom}(P_m, P_{m+1})$ be linearly independent morphisms and write $g : P_{m-1} \to P_m$ for the composition of the inclusion from (1) with the projection $\pi_n : P_m^n \to P_m$ to the nth factor. Then $P_{m+1}^{\{f_1,\ldots,f_{n-1}\}} \cong P_m^{\{g\}}$.*

Proof. We have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & P_m^{n-1} & \to & P_m^n & \xrightarrow{\pi_n} & P_m & \to & 0 \\
& & \| & & \downarrow {\scriptstyle [f_1 \;\cdots\; f_n]} & & \downarrow {\scriptstyle f_n} & & \\
0 & \to & P_m^n \xrightarrow{[f_1 \;\cdots\; f_{n-1}]} & & P_{m+1} & \to & P_{m+1}^{\{f_1,\ldots,f_{n-1}\}} & \to & 0
\end{array}
$$

Since the vertical morphism on the left is an equality, the right hand square is a pullback. But this implies the kernels of the vertical morphisms coincide. The middle vertical morphism fits into the exact sequence (1) and so the kernel is P_{m-1}. It follows that there is a morphism $g : P_{m-1} \to P_m$ so that $P_{m+1}^{\{f_1,\ldots,f_{n-1}\}} \cong P_m^{\{g\}}$. $\square$

For any nonempty subset $I \subset [1, k]$, say with $|I| = \ell$, we get an exact sequence

$$0 \longrightarrow P_m^\ell \xrightarrow{[f_i]_{i \in I}} P_{m+1}^{\{f_j\}_{j \in [1,k]\setminus I}} \longrightarrow P_{m+1}^{\{f_1,\ldots,f_k\}} \longrightarrow 0.$$

Each such sequence has the following almost-split property for subrepresentations of $P_{m+1}^{\{f_1,\ldots,f_k\}}$.

Lemma 3.5. *For $m \geq 1$ and linearly independent functions $f_1, \ldots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, given any proper subrepresentation $V \subsetneq P_{m+1}^{\{f_1,\ldots,f_k\}}$, the upper pullback sequence*

$$
\begin{array}{ccccccccc}
0 & \to & P_m^\ell & \to & V^I & \to & V & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & P_m^\ell \xrightarrow{[f_i]_{i \in I}} & & P_{m+1}^{\{f_j\}_{j \in [1,k]\setminus I}} & \to & P_{m+1}^{\{f_1,\ldots,f_k\}} & \to & 0
\end{array}
$$

splits for every $I \subset [1, k]$.

Proof. When $I = \emptyset$, we have $\ell = 0$ and there is nothing to show, so assume I is a nontrivial subset of $[1, k]$. We proceed by simultaneous induction on k and $\ell = |I|$. For any nonzero $f \in \operatorname{Hom}(P_m, P_{m+1})$ the exact sequence

$$0 \longrightarrow P_m \xrightarrow{\ f\ } P_{m+1} \longrightarrow P_{m+1}^{\{f\}} \longrightarrow 0$$

is almost split, giving the claim in the case $k = \ell = 1$. For $k > 1$ and a nontrivial proper subset $I \subsetneq [1, k]$, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & P_m^{k-\ell} & = & P_m^{k-\ell} & \longrightarrow & 0 & & \\
& & \downarrow{\scriptstyle [f_j]_{j \in [1,k] \setminus I}} & & \downarrow{\scriptstyle [f_j]_{j \in [1,k] \setminus I}} & & & & \\
0 & \longrightarrow & P_m^{\ell} & \xrightarrow{[f_i]_{i \in I}} & P_{m+1} & \longrightarrow & P_{m+1}^{\{f_i\}_{i \in I}} & \longrightarrow & 0 \\
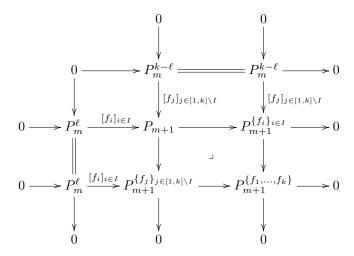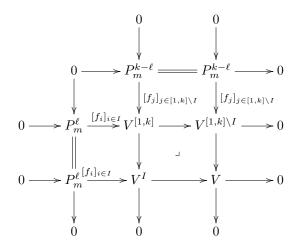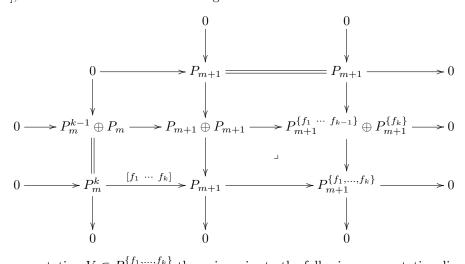& & \Big\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_m^{\ell} & \xrightarrow{[f_i]_{i \in I}} & P_{m+1}^{\{f_j\}_{j \in [1,k] \setminus I}} & \longrightarrow & P_{m+1}^{\{f_1, \ldots, f_k\}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

A proper subrepresentation $V \subset P_{m+1}^{\{f_1, \ldots, f_k\}}$ gives rise, via pullbacks, to the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & P_m^{k-\ell} & = & P_m^{k-\ell} & \longrightarrow & 0 & & \\
& & \downarrow{\scriptstyle [f_j]_{j \in [1,k] \setminus I}} & & \downarrow{\scriptstyle [f_j]_{j \in [1,k] \setminus I}} & & & & \\
0 & \longrightarrow & P_m^{\ell} & \xrightarrow{[f_i]_{i \in I}} & V^{[1,k]} & \longrightarrow & V^{[1,k] \setminus I} & \longrightarrow & 0 \\
& & \Big\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_m^{\ell} & \xrightarrow{[f_i]_{i \in I}} & V^I & \longrightarrow & V & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
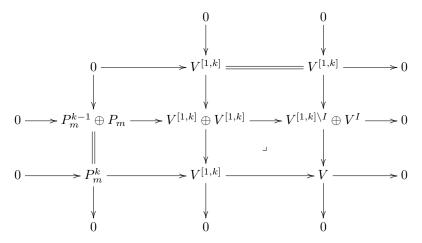& & 0 & & 0 & & 0 & &
\end{array}
$$

in which the middle vertical sequence and the middle horizontal sequence split by induction. But the images of $[f_i]_{i \in I}$ and $[f_j]_{j \in [1,k] \setminus I}$ intersect trivially inside $V^{[1,k]}$ and thus the lower horizontal sequence must split as well, establishing the claim for $I \subsetneq [1, k]$.

For $I = [1, k]$, we consider the commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & P_{m+1} & = & = & = & P_{m+1} & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & P_m^{k-1} \oplus P_m & \longrightarrow & P_{m+1} \oplus P_{m+1} & \longrightarrow & P_{m+1}^{\{f_1 \cdots f_{k-1}\}} \oplus P_{m+1}^{\{f_k\}} & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_m^k & \xrightarrow{[f_1 \cdots f_k]} & P_{m+1} & \longrightarrow & P_{m+1}^{\{f_1,\ldots,f_k\}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

A proper subrepresentation $V \subset P_{m+1}^{\{f_1,\ldots,f_k\}}$ then gives rise to the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & V^{[1,k]} & = & = & = & V^{[1,k]} & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & P_m^{k-1} \oplus P_m & \longrightarrow & V^{[1,k]} \oplus V^{[1,k]} & \longrightarrow & V^{[1,k]\setminus I} \oplus V^I & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_m^k & \longrightarrow & V^{[1,k]} & \longrightarrow & V & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

in which the middle vertical sequence splits and the middle horizontal sequence is split by induction. But then the lower horizontal sequence splits, establishing the claim for $I \subsetneq [1, k]$. $\qquad\square$

4. Quiver Grassmannians of $\widetilde{K(n)}$

Let $\tilde{P}_m$ be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 4.1. *There exist lifts $\tilde{P}_{m-1,i}$ for $1 \le i \le n$ of P_{m-1} to $\widetilde{K(n)}$ so that:*

(1) $\operatorname{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \operatorname{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$, *where each* $\operatorname{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$ *is one-dimensional;*

(2) *For any proper subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, there exists a short exact sequence*

$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\ldots,i_k} \longrightarrow 0;$$

(3) *The lifts $\tilde{P}_{m-1,i}$ are pairwise orthogonal.*

(4) *All nontrivial proper subrepresentations of $\tilde{P}_m^{i_1,\ldots,i_k}$ are preprojective.*

We will always choose the subset $\{1, \ldots, k\}$ when using Lemma 4.1.2 and thus we denote the cokernel simply by $\tilde{P}_m^{(k)}$.

Lemma 4.2. *If each $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1,i})$ has a cell decomposition, then $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1,i_j})$ has a cell decomposition.*

Lemma 4.3. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) *The subrepresentation $\bigoplus_{i=1}^{k-1} \tilde{P}_{m-1,i} \oplus \bigoplus_{i=i}^{k} \tilde{P}_{m-2,i} \subset \bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}$ is in $(\tilde{P}_m^{(k)})^\perp$ and*

$$\mathrm{Ext}(\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}, \tilde{P}_m^{(k)}) \cong \mathrm{Ext}(\tilde{P}_{m-1}^{(k)}, \tilde{P}_m^{(k)})$$

where $\tilde{P}_{m-1}^{(k)}$ above denotes the cokernel of the inclusion.

Lemma 4.4. *Consider $0 \to \tilde{P}_m \to \tilde{P}_{m+1}^{(k)} \to \tilde{P}_{m+1}^{(k-1)} \to 0$ There exists a short exact sequence*

$$0 \to \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k} \to \tilde{P}_m \to \tau \tilde{P}_{m+1}^{(k-1)} \to 0$$

where τ denotes the Auslander-Reiten translate. Moreover, we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m) \cong \mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = k$ and $\mathrm{Hom}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = 0$. [1]

Proof. Idea: Check this for $m = 1$ or $m = 2$ and apply BGP-reflections. □

Lemma 4.5. *Consider $0 \to \tilde{P}_m \to \tilde{P}_{m+1}^{(k)} \to \tilde{P}_{m+1}^{(k-1)} \to 0$. If $U \subset \tilde{P}_m$ such that $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, then we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$.*

Proof. Idea: Every $U \subset \tilde{P}_m$ gives rise to a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k} & \longrightarrow & \tilde{P}_m & \longrightarrow & \tau\tilde{P}_{m+1}^{(k-1)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & V & \longrightarrow & U & \longrightarrow & W & \longrightarrow & 0
\end{array}
$$

If $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, we have $W \neq 0$. Thus $\tau\tilde{P}_{m+1}^{(k-1)}/W$ is a proper factor. It follows that we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}/W) = 0$ by Auslander-Reiten-theory and because $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = k$.

Since we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k}) = 0$, it follows $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, (\tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k})/V) = 0$. The first statement should be a consequence of Lemma 4.4. This yields the claim.

□

Proposition 4.6. *Consider $\psi : Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m) \to \bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}} Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}) \times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)})$. Then the following hold:*

(1) *For $V \subsetneq \tilde{P}_m^{(k)}$ and $U \subset \bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}$, we have $\psi^{-1}(U, V) = \mathbb{A}^{\langle V, \bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}/U\rangle}$.*
(2) *If $V = \tilde{P}_m^{(k)}$, the fibre $\psi^{-1}(U, V)$ is non-empty and of constant dimension if and only if $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$.*

Proof. (1) V is preprojective but $\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}/U$ is not unless $U = 0$

(2) If $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) = 0$, the fibre is empty because the sequence is non-split. If $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$, Lemma 4.5 yields $\mathrm{Ext}(\tilde{P}_m^{(k)}, \tilde{P}_{m-1}/U) = 0$. In particular, the dimension is constant if it is not empty and $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \xrightarrow{\pi} \mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m)$ is surjective. But this already means that the fibre is not empty. □

Theorem 4.7. *Every quiver Grassmannian of a preprojective or preinjective representation of $K(n)$ and $\widetilde{K(n)}$ has a cell decomposition.*

Question 4.8. *cells of $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ are in one-to-one correspondence with certain tuples of subgraphs for smaller $\tilde{P}_\ell^{i_1,\ldots,i_k}$*

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[1] introduce/consider/check indices, we have to mod out the corresponding reps in the covering

5. Compatible Pairs Label Cells in $Gr_{\mathbf{e}}^{Q}(P_m)$

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