CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

DYLAN RUPEL AND THORSTEN WEIST

Abstract. We prove that all quiver Grassmannians for preprojective and preinjective respesentations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. Introduction

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -acknowledgements?

2. Torus Action on Quiver Grassmannians

Definition 0.1. *universal cover* $\tilde{Q}$

Lemma 0.1. *how to lift exceptional representations of* Q *to* $\tilde{Q}$

Lemma 0.2. $d : \tilde{Q}_0 \to \mathbb{Z}$ *defines a torus action on* $Gr_{\mathbf{e}}^{Q}(X)$ *if* $d(q, w\rho) - d(q, w) = c_\rho$ *for all* $w \in W_{\tilde{Q}_1}$

Lemma 0.3. *There exists* $d : \tilde{Q}_0 \to \mathbb{Z}$ *such that* $d(q, w) \neq d(q', w')$ *for all* q, q', w, w' *with* $\dim X_{q,w} \neq 0$ *and* $\dim X_{q',w'} \neq 0$.

Theorem 1. $Gr_{\mathbf{e}}^{Q}(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} of type \mathbf{e}} Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$

Corollary 1.1. *affine bundles over* $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$ *if* $Gr_{\mathbf{e}}^{Q}(X)$ *is smooth*

$$\{U \in Gr_{\mathbf{e}}^{Q}(X) : \lim_{t \to 0} t \cdot U \in Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})\}$$

Question 1.1. *What are the ranks of these bundles? Poincaré polynomials?*

3. Representation Theory of $K(n)$

Denote by $K(n)$ the *n-Kronecker quiver* $1 \xleftarrow{n} 2$ with vertices $K(n)_0 = \{1, 2\}$ and n arrows from vertex 2 to vertex 1.

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$.

Theorem 2. *For each* $m \geq 1$, *there exist unique (up to isomorphism) indecomposable rigid representations* P_m *and* I_m *of* $K(n)$ *with dimension vectors* (u_m, u_{m-1}) *and* (u_{m-1}, u_m) *respectively. Moreover, any rigid representation of* $K(n)$ *is isomorphic to one of the form* $P_m^{a_1} P_{m+1}^{a_2}$ *or* $I_m^{a_1} I_{m+1}^{a_2}$ *for some* $m \geq 1$ *and some* $a_1, a_2 \geq 0$.

The representations P_m are called the *preprojective* representations of $K(n)$ and the representations I_m are called *preinjective*.

Lemma 2.1. *For any* $m \geq 1$, *we have*

$$\dim \mathrm{Hom}(P_m, P_m) = 1, \quad \dim \mathrm{Hom}(P_m, P_{m+1}) = n, \quad \dim \mathrm{Hom}(P_{m+1}, P_m) = 0.$$

Moreover, for $1 \leq k < n$ *and linearly independent functions* $f_1, \ldots, f_k \in \mathrm{Hom}(P_m, P_{m+1})$, *the map* $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$ *is injective.*

Write $P_{m+1}^{\{f_1,\ldots,f_k\}}$ for the cokernel of the map $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$, i.e. we have a short exact sequence

$$0 \longrightarrow P_m^k \xrightarrow{[f_1 \ \cdots \ f_k]} P_{m+1} \longrightarrow P_{m+1}^{\{f_1,\ldots,f_k\}} \longrightarrow 0. \tag{1}$$

Lemma 2.2. *For $m \geq 1$ and linearly independent functions $f_1, \dots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the following hold:*

(1) $\operatorname{Hom}(P_m, P_{m+1}^{\{f_1,\dots,f_k\}})$ *is naturally isomorphic to* $\operatorname{Hom}(P_m, P_{m+1})/\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$*;*
(2) $\operatorname{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}}, P_m)$ *is naturally isomorphic to* $\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$*;*
(3) $\operatorname{Hom}(P_{m+1}^{\{f_1,\dots,f_k\}}, P_m) = 0$.

Proof. Applying the functor $\operatorname{Hom}(P_m, -)$ to the sequence (1) gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(P_m, P_m^k) \overset{[f_1\ \cdots\ f_k]\circ}{\longrightarrow} \operatorname{Hom}(P_m, P_{m+1}) \longrightarrow \operatorname{Hom}(P_m, P_{m+1}^{\{f_1,\dots,f_k\}}) \longrightarrow 0,$$

establishing (1).

To see (2), we apply the functor $\operatorname{Hom}(-, P_m)$ to the sequence (1) to get an isomorphism $\operatorname{Hom}(P_m^k, P_m) \cong \operatorname{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}}, P_m)$. Observe that $\operatorname{Hom}(P_m^k, P_m)$ is spanned by the coordinate projections $\pi_i : P_m^k \to P_m$ for $1 \leq i \leq k$ and each of these gives rise to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_m^{k-1} \arrow[r, equal] \arrow[d] & P_m^{k-1} \arrow[d, "{[f_1\ \cdots\ \widehat{f_i}\ \cdots\ f_k]}"] & & \\
0 \arrow[r] & P_m^k \arrow[r, "{[f_1\ \cdots\ f_k]}"] \arrow[d, "\pi_i"] \arrow[dr, phantom, "\ulcorner"] & P_{m+1} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_m \arrow[r, "f_i"] \arrow[d] & P_{m+1}^{\{f_1,\dots,\widehat{f_i},\dots,f_k\}} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The bottom row of this diagram is the extension in $\operatorname{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}}, P_m)$ naturally identified with the generator f_i of $\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$.

Part (3) follows immediately from (2) and the observation that

$$\langle P_{m+1}^{\{f_1,\dots,f_k\}}, P_m \rangle = \langle P_{m+1}, P_m \rangle - k \langle P_m, P_m \rangle = -k.$$

$\square$

For any nonempty subset $I \subset [1, k]$, say with $|I| = \ell$, we get an exact sequence

$$0 \longrightarrow P_m^\ell \overset{[f_i]_{i\in I}}{\longrightarrow} P_{m+1}^{\{f_j\}_{j\in[1,k]\setminus I}} \longrightarrow P_{m+1}^{\{f_1,\dots,f_k\}} \longrightarrow 0.$$

Each such sequence has the following almost-split property for subrepresentations of $P_{m+1}^{\{f_1,\dots,f_k\}}$.

Lemma 2.3. *For $m \geq 1$ and linearly independent functions $f_1, \dots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, given any proper subrepresentation $V \subsetneq P_{m+1}^{\{f_1,\dots,f_k\}}$, the upper pullback sequence*

$$\begin{tikzcd}
0 \arrow[r] & P_m^\ell \arrow[r] \arrow[d, equal] & V^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^\ell \arrow[r, "{[f_i]_{i\in I}}"] & P_{m+1}^{\{f_j\}_{j\in[1,k]\setminus I}} \arrow[r] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] & 0
\end{tikzcd}$$

splits for every $I \subset [1, k]$.

Proof. When $I = \emptyset$ there is nothing to show, so assume I is a nontrivial subset of $[1, k]$.

We proceed by induction on k. For any nonzero $f \in \operatorname{Hom}(P_m, P_{m+1})$ the exact sequence

$$0 \longrightarrow P_m \overset{f}{\longrightarrow} P_{m+1} \longrightarrow P_{m+1}^{\{f\}} \longrightarrow 0$$

is almost split, giving the claim in the case $k = 1$. For $k > 1$ and a nontrivial subset $I \subset [1, k]$, we have the following commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & P_m^{k-\ell} \arrow[r, equal] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & P_m^{k-\ell} \arrow[r] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d, equal] & P_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P_{m+1}^{\{f_i\}_{i\in I}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d] & P_{m+1}^{\{f_j\}_{j\in[1,k]\setminus I}} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

A proper subrepresentation $V \subset P_{m+1}^{\{f_1,\dots,f_k\}}$ gives rise, via pullbacks, to the following commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & P_m^{k-\ell} \arrow[r, equal] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & P_m^{k-\ell} \arrow[r] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d, equal] & V^{[1,k]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V^{[1,k]\setminus I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d] & V^{I} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

in which the middle vertical sequence and the middle horizontal sequence split by induction. But then the lower horizontal sequence splits, establshing the claim for all $I \subset [1, k]$. $\square$

4. Quiver Grassmannians of $\widetilde{K(n)}$

Let $\tilde{P}_m$ be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 2.4. *There exist lifts $\tilde{P}_{m-1,i}$ for $1 \le i \le n$ of P_{m-1} to $\widetilde{K(n)}$ so that:*

(1) $\mathrm{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$;
(2) *For any proper subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, there exists a short exact sequence*
$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\dots,i_k} \longrightarrow 0;$$
(3) *The lifts $\tilde{P}_{m-1,i}$ are pairwise orthogonal.*
(4) *All nontrivial proper subrepresentations of $\tilde{P}_m^{(k)}$ are preprojective.*

We will always choose the subset $\{1, \dots, k\}$ when using Lemma 2.4.2 and thus we denote the cokernal simply by $\tilde{P}_m^{(k)}$.

Lemma 2.5. *If each $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1,i})$ has a cell decomposition, then $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1,i_j})$ has a cell decomposition.*

Lemma 2.6. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) *The subrepresentation* $\bigoplus_{i=1}^{k-1}\tilde{P}_{m-1,i}\oplus\bigoplus_{i=1}^{k}\tilde{P}_{m-2,i}\subset\bigoplus i=1^k\tilde{P}_{m-1,i}$ *is in* $(\tilde{P}_m^{(k)})^\perp$ *and*

$$\mathrm{Ext}(\bigoplus i=1^k\tilde{P}_{m-1,i},\tilde{P}_m^{(k)})\cong\mathrm{Ext}(\tilde{P}_{m-1}^{(k)},\tilde{P}_m^{(k)})$$

where $\tilde{P}_{m-1}^{(k)}$ *above denotes the cokernel of the inclusion.*

Corollary 2.1. *observe when fibers are empty*

Proposition 2.1. *Consider* $\psi:Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)\to\bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}}Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^k\tilde{P}_{m-1,i})\times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)}$. *Then the following hold:*

(1) *For* $V\subsetneq\tilde{P}_m^{(k)}$ *and* $U\subset\bigoplus_{i=1}^k\tilde{P}_{m-1,i}$, *we have* $\psi^{-1}(U,V)=\mathbb{A}^{\langle V,\bigoplus_{i=1}^k\tilde{P}_{m-1,i}/U\rangle}$.

(2) *If* $V=\tilde{P}_m^{(k)}$ *and the fiber is not empty, then* $\psi^{-1}(U,V)$ *is constant.*

Proof. (1) V is preprojective but $\bigoplus_{i=1}^k\tilde{P}_{m-1,i}/U$ is not unless $U=0$

(2)

$$0\longrightarrow[V,P/U]\longrightarrow[V,U]^1\longrightarrow[V,P]^1\longrightarrow[V,P/U]^1\longrightarrow 0$$

and the middle map is surjective. $\square$

Theorem 3. *Every quiver Grassmannian of a preprojective or preinjective representation of* $K(n)$ *and* $\widetilde{K(n)}$ *has a cell decomposition.*

Question 3.1. *cells of* $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ *are in one-to-one correspondence with certain tuples of subgraphs for smaller* $\tilde{P}_\ell^{i_1,\ldots,i_k}$

5. Compatible Pairs Label Cells in $Gr_{\mathbf{e}}^{Q}(P_m)$