CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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Abstract. We prove that all quiver Grassmannians for preprojective and preinjective representations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. Introduction

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -something about torus actions and the universal cover -something about cell decompositions of quiver Grassmannians -acknowledgements?

2. Torus Action on Quiver Grassmannians

Let Q be an acylic quiver with vertices Q_0 and arrows Q_1 which we denote by $\alpha : i \to j$. Moreover, let W_Q be the free (non-abelian) group generated by Q_1. We denote by $\mathrm{Rep}(Q)$ the category of $\mathbb{C}$-representations of Q.

Definition 0.1. *The universal covering quiver $\tilde{Q}$ of Q is given by the vertices $\tilde{Q}_0 = Q_0 \times W_Q$ and the arrow set $\tilde{Q}_1 = Q_1 \times W_Q$ where $(\alpha, w) : (i, w) \to (j, w\alpha)$ for every $\alpha : i \to j$.*

We say that a representation $X \in \mathrm{Rep}(Q)$ can be lifted (to $\tilde{Q}$) if there exists a representation $\tilde{X} \in \mathrm{Rep}(\tilde{Q})$ such that $F_Q\tilde{X} = X$ where $F_Q : \mathrm{Rep}(\tilde{Q}) \to \mathrm{Rep}(Q)$ is the natural functor.

Lemma 0.1. *Every preprojective (resp. preinjective) representation of Q can be lifted to a representation of $\tilde{Q}$.*

Proof. This statement is clear for the simple representations S_q, $q \in Q_0$. Now every preprojective representation X can be obtained when applying a sequence of BGP-reflections [2] to a simple representation $S_{q'}$ of a quiver Q' whose underlying graph is the one of Q. Applying BGP-reflections to a source q of Q corresponds to applying BGP-reflections to all vertices (q, w) of $\tilde{Q}$. This leads the claim. The statement for preinjective representations follows in the same way. ☐

We choose a map $d : \tilde{Q}_0 \to \mathbb{Z}$ and fix a representation $X \in \mathrm{Rep}(Q)$. In any case, we can consider the decomposition $X_q = \bigoplus_{w \in W_Q} X_{(q,w)}$. We define a torus action on each $X_{(q,w)}$ via $t.x_{(q,w)} = t^{d(q,w)}x_{(q,w)}$ which can be extended linearly to each X_q. For a fixed a subspace U_q, we can define the subspace $t.U_q$. In general, this torus action induces no torus action on the Quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}(X)$ as $t.U = (t.U_q)_{q \in Q_0}$ is no subrepresentation of X for every $U \in \mathrm{Gr}_{\mathbf{e}}(X)$. Actually, for this the action has to satisfy $X_\alpha(t.U_i) \in t.U_j$ for every $\alpha : i \to j$ and every $x \in X_i$.

Lemma 0.2. *Fix an integer $c_\alpha \in \mathbb{Z}$ for every $\alpha \in Q_1$. If $X \in \mathrm{Rep}(Q)$ can be lifted, $d : \tilde{Q}_0 \to \mathbb{Z}$ induces a torus action on $\mathrm{Gr}_{\mathbf{e}}(X)$ if we have $d(j, w\alpha) - d(i, w) = c_\alpha$ for all $\alpha : i \to j$ and $w \in W_Q$.*

Proof. Since X can be lifted, we can write $X_\alpha : X_i \to X_j$ as block matrix consisting of linear maps $X_{(\alpha,w)} : X_{(i,w)} \to X_{(j,w\alpha)}$. Then the condition $X_\alpha(t.U_i) \in t.U_j$ translates into ...to be continued. ☐

Lemma 0.3. *There exists $d : \tilde{Q}_0 \to \mathbb{Z}$ such that $d(q, w) \neq d(q', w')$ for all q, q', w, w' with $\dim X_{q,w} \neq 0$ and $\dim X_{q',w'} \neq 0$.*

Theorem 1. $Gr_{\mathbf{e}}^Q(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} of type \mathbf{e}} Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$

Corollary 1.1. *affine bundles over $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$ if $Gr_{\mathbf{e}}^{Q}(X)$ is smooth*

$$\{U \in Gr_{\mathbf{e}}^{Q}(X) : \lim_{t\to 0} t \cdot U \in Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})\}$$

Question 1.1. *What are the ranks of these bundles? Poincaré polynomials?*

3. Representation Theory of $K(n)$

Denote by $K(n)$ the *n-Kronecker quiver* $1 \stackrel{n}{\longleftarrow} 2$ with vertices $K(n)_0 = \{1, 2\}$ and n arrows from vertex 2 to vertex 1.

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$.

Theorem 2. *For each $m \geq 1$, there exist unique (up to isomorphism) indecomposable rigid representations P_m and I_m of $K(n)$ with dimension vectors (u_m, u_{m-1}) and (u_{m-1}, u_m) respectively. Moreover, any rigid representation of $K(n)$ is isomorphic to one of the form $P_m^{a_1} P_{m+1}^{a_2}$ or $I_m^{a_1} I_{m+1}^{a_2}$ for some $m \geq 1$ and some $a_1, a_2 \geq 0$.*

The representations P_m are called the *preprojective* representations of $K(n)$ and the representations I_m are called *preinjective*.

Lemma 2.1. *For any $m \geq 1$, we have*

$$\dim \operatorname{Hom}(P_m, P_m) = 1, \quad \dim \operatorname{Hom}(P_m, P_{m+1}) = n, \quad \dim \operatorname{Hom}(P_{m+1}, P_m) = 0.$$

Moreover, for $1 \leq k < n$ and linearly independent functions $f_1, \ldots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the map $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$ is injective.

Proof. Note that we have a Auslander-Reiten sequence

$$0 \longrightarrow P_{m-1} \longrightarrow P_m^n \stackrel{[f_1 \ \cdots \ f_n]}{\longrightarrow} P_{m+1} \longrightarrow 0$$

with irreducible homomorphisms f_i which are also linear independent. Thus they do not factor through a representation $Z \neq P_m$ which already means that they are injective, see for instance [1, Lemma 1.6]. If we pick $k < n$ linear independent homomorphisms $f_{i_1}, \ldots, f_{i_k}$, this also means that $[f_{i_1} \ \cdots \ f_{i_k}] : P_m^k \to P_{m+1}$ is injective. Indeed otherwise, at least one of the homomorphisms were forced to factor through a representation $Z \neq P_m$.

Write $P_{m+1}^{\{f_1,\ldots,f_k\}}$ for the cokernel of the map $[f_1 \ \cdots \ f_k] : P_m^k \to P_{m+1}$, i.e. we have a short exact sequence

$$0 \longrightarrow P_m^k \stackrel{[f_1 \ \cdots \ f_k]}{\longrightarrow} P_{m+1} \longrightarrow P_{m+1}^{\{f_1,\ldots,f_k\}} \longrightarrow 0. \tag{1}$$

□

Lemma 2.2. *For $m \geq 1$ and linearly independent functions $f_1, \ldots, f_k \in \operatorname{Hom}(P_m, P_{m+1})$, the following hold:*

1. $\operatorname{Hom}(P_m, P_{m+1}^{\{f_1,\ldots,f_k\}})$ *is naturally isomorphic to* $\operatorname{Hom}(P_m, P_{m+1})/\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$*;*
2. $\operatorname{Ext}(P_{m+1}^{\{f_1,\ldots,f_k\}}, P_m)$ *is naturally isomorphic to* $\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$*;*
3. $\operatorname{Hom}(P_{m+1}^{\{f_1,\ldots,f_k\}}, P_m) = 0$.

Proof. Applying the functor $\operatorname{Hom}(P_m, -)$ to the sequence (1) gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(P_m, P_m^k) \stackrel{[f_1 \ \cdots \ f_k]\circ}{\longrightarrow} \operatorname{Hom}(P_m, P_{m+1}) \longrightarrow \operatorname{Hom}(P_m, P_{m+1}^{\{f_1,\ldots,f_k\}}) \longrightarrow 0,$$

establishing (1).

To see (2), we apply the functor $\operatorname{Hom}(-, P_m)$ to the sequence (1) to get an isomorphism $\operatorname{Hom}(P_m^k, P_m) \cong \operatorname{Ext}(P_{m+1}^{\{f_1,\ldots,f_k\}}, P_m)$. Observe that $\operatorname{Hom}(P_m^k, P_m)$ is spanned by the coordinate projections $\pi_i : P_m^k \to P_m$

for $1 \le i \le k$ and each of these gives rise to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_m^{k-1} \arrow[r, equal] \arrow[d] & P_m^{k-1} \arrow[d, "{[f_1 \ \cdots \ \widehat{f_i} \ \cdots \ f_k]}"] & & \\
0 \arrow[r] & P_m^k \arrow[r, "{[f_1 \ \cdots \ f_k]}"] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\ulcorner"] & P_{m+1} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_m \arrow[r, "f_i"] \arrow[d] & P_{m+1}^{\{f_1,\dots,\widehat{f_i},\dots,f_k\}} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The bottom row of this diagram is the extension in $\mathrm{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}}, P_m)$ naturally identified with the generator f_i of $\Bbbk f_1 \oplus \cdots \oplus \Bbbk f_k$.

Part (3) follows immediately from (2) and the observation that

$$\langle P_{m+1}^{\{f_1,\dots,f_k\}}, P_m \rangle = \langle P_{m+1}, P_m \rangle - k \langle P_m, P_m \rangle = -k.$$

$\square$

For any nonempty subset $I \subset [1,k]$, say with $|I| = \ell$, we get an exact sequence

$$0 \longrightarrow P_m^\ell \xrightarrow{[f_i]_{i \in I}} P_{m+1}^{\{f_j\}_{j \in [1,k] \setminus I}} \longrightarrow P_{m+1}^{\{f_1,\dots,f_k\}} \longrightarrow 0.$$

Each such sequence has the following almost-split property for subrepresentations of $P_{m+1}^{\{f_1,\dots,f_k\}}$.

Lemma 2.3. *For $m \ge 1$ and linearly independent functions $f_1, \dots, f_k \in \mathrm{Hom}(P_m, P_{m+1})$, given any proper subrepresentation $V \subsetneq P_{m+1}^{\{f_1,\dots,f_k\}}$, the upper pullback sequence*

$$\begin{tikzcd}
0 \arrow[r] & P_m^\ell \arrow[r] \arrow[d, equal] & V^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^\ell \arrow[r, "{[f_i]_{i \in I}}"] & P_{m+1}^{\{f_j\}_{j \in [1,k] \setminus I}} \arrow[r] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] & 0
\end{tikzcd}$$

splits for every $I \subset [1,k]$.

Proof. When $I = \emptyset$ there is nothing to show, so assume I is a nontrivial subset of $[1,k]$.

We proceed by induction on k. For any nonzero $f \in \mathrm{Hom}(P_m, P_{m+1})$ the exact sequence

$$0 \longrightarrow P_m \xrightarrow{f} P_{m+1} \longrightarrow P_{m+1}^{\{f\}} \longrightarrow 0$$

is almost split, giving the claim in the case $k = 1$. For $k > 1$ and a nontrivial subset $I \subset [1, k]$, we have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & P_m^{k-\ell} \arrow[r, equal] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & P_m^{k-\ell} \arrow[r] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d, equal] & P_{m+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P_{m+1}^{\{f_i\}_{i\in I}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d] & P_{m+1}^{\{f_j\}_{j\in[1,k]\setminus I}} \arrow[r] \arrow[d] & P_{m+1}^{\{f_1,\dots,f_k\}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A proper subrepresentation $V \subset P_{m+1}^{\{f_1,\dots,f_k\}}$ gives rise, via pullbacks, to the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & P_m^{k-\ell} \arrow[r, equal] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & P_m^{k-\ell} \arrow[r] \arrow[d, "{[f_j]_{j\in[1,k]\setminus I}}"] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d, equal] & V^{[1,k]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V^{[1,k]\setminus I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_m^{\ell} \arrow[r, "{[f_i]_{i\in I}}"] \arrow[d] & V^{I} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the middle vertical sequence and the middle horizontal sequence split by induction. But then the lower horizontal sequence splits, establshing the claim for all $I \subset [1, k]$. $\square$

4. Quiver Grassmannians of $\widetilde{K(n)}$

Let $\tilde{P}_m$ be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 2.4. *There exist lifts $\tilde{P}_{m-1,i}$ for $1 \le i \le n$ of P_{m-1} to $\widetilde{K(n)}$ so that:*

1. $\mathrm{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$;
2. *For any proper subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, there exists a short exact sequence*
$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\dots,i_k} \longrightarrow 0;$$
3. *The lifts $\tilde{P}_{m-1,i}$ are pairwise orthogonal.*
4. *All nontrivial proper subrepresentations of $\tilde{P}_m^{(k)}$ are preprojective.*

We will always choose the subset $\{1, \dots, k\}$ when using Lemma 2.4.2 and thus we denote the cokernal simply by $\tilde{P}_m^{(k)}$.

Lemma 2.5. *If each $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1,i})$ has a cell decomposition, then $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1,i_j})$ has a cell decomposition.*

Lemma 2.6. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) *The subrepresentation* $\bigoplus_{i=1}^{k-1} \tilde{P}_{m-1,i} \oplus \bigoplus_{i=1}^{k} \tilde{P}_{m-2,i} \subset \bigoplus i = 1^k \tilde{P}_{m-1,i}$ *is in* $(\tilde{P}_m^{(k)})^\perp$ *and*

$$\mathrm{Ext}(\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}, \tilde{P}_m^{(k)}) \cong \mathrm{Ext}(\tilde{P}_{m-1}^{(k)}, \tilde{P}_m^{(k)})$$

where $\tilde{P}_{m-1}^{(k)}$ *above denotes the cokernel of the inclusion.*

Lemma 2.7. *Consider* $0 \to \tilde{P}_m \to \tilde{P}_{m+1}^{(k)} \to \tilde{P}_{m+1}^{(k-1)} \to 0$ *There exists a short exact sequence*

$$0 \to \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k} \to \tilde{P}_m \to \tau \tilde{P}_{m+1}^{(k-1)} \to 0$$

where τ *denotes the Auslander-Reiten translate. Moreover, we have* $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m) \cong \mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = k$ *and* $\mathrm{Hom}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = 0$. [1]

Proof. Idea: Check this for $m = 1$ or $m = 2$ and apply BGP-reflections. □

Lemma 2.8. *Consider* $0 \to \tilde{P}_m \to \tilde{P}_{m+1}^{(k)} \to \tilde{P}_{m+1}^{(k-1)} \to 0$. *If* $U \subset \tilde{P}_m$ *such that* $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, *then we have* $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m/U) = 0$.

Proof. Idea: Every $U \subset \tilde{P}_m$ gives rise to a diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k} \arrow[r] & \tilde{P}_m \arrow[r] & \tau\tilde{P}_{m+1}^{(k-1)} \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u] & U \arrow[r] \arrow[u] & W \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

If $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \neq 0$, we have $W \neq 0$. Thus $\tau\tilde{P}_{m+1}^{(k-1)}/W$ is a proper factor. It follows that we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}/W) = 0$ by Auslander-Reiten-theory and because $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tau\tilde{P}_{m+1}^{(k-1)}) = k$.

Since we have $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k}) = 0$, it follows $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, (\tilde{P}_{m-1}^{n-1} \oplus \tilde{P}_{m-2}^{n-k})/V) = 0$. The first statement should be a consequence of Lemma 2.7. This yields the claim. □

Proposition 2.1. *Consider* $\psi : Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m) \to \bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}} Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}) \times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)}$. *Then the following hold:*

(1) *For* $V \subsetneq \tilde{P}_m^{(k)}$ *and* $U \subset \bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}$, *we have* $\psi^{-1}(U, V) = \mathbb{A}^{\langle V, \bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}/U\rangle}$.

(2) *If* $V = \tilde{P}_m^{(k)}$, *the fibre* $\psi^{-1}(U, V)$ *is non-empty and of constant dimension if and only if* $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$.

Proof. (1) V is preprojective but $\bigoplus_{i=1}^{k} \tilde{P}_{m-1,i}/U$ is not unless $U = 0$

(2) If $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) = 0$, the fibre is empty because the sequence is non-split. If $\mathrm{Ext}(\tilde{P}_m^{(k)}, U) \neq 0$, Lemma 2.8 yields $\mathrm{Ext}(\tilde{P}_m^{(k)}, \tilde{P}_{m-1}/U) = 0$. In particular, the dimension is constant if it is not empty and $\mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, U) \xrightarrow{\pi} \mathrm{Ext}(\tilde{P}_{m+1}^{(k-1)}, \tilde{P}_m)$ is surjective. But this already means that the fibre is not empty. □

Theorem 3. *Every quiver Grassmannian of a preprojective or preinjective representation of* $K(n)$ *and* $\widetilde{K(n)}$ *has a cell decomposition.*

Question 3.1. *cells of* $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ *are in one-to-one correspondence with certain tuples of subgraphs for smaller* $\tilde{P}_\ell^{i_1,\ldots,i_k}$

[1] introduce/consider/check indices, we have to mod out the corresponding reps in the covering

5. Compatible Pairs Label Cells in $Gr^Q_{\mathbf{e}}(P_m)$

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