CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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ABSTRACT. We prove that all quiver Grassmannians for preprojective and preinjective respesentations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. Introduction

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -acknowledgements?

2. Torus Action on Quiver Grassmannians

Definition 0.1. universal cover \tilde{Q}

Lemma 0.1. how to lift exceptional representations of Q to \tilde{Q}

Lemma 0.2. $d: \tilde{Q}_0 \to \mathbb{Z}$ defines a torus action on $Gr_{\mathbf{e}}^Q(X)$ if $d(q, w\rho) - d(q, w) = c_\rho$ for all $w \in W_{\tilde{Q}_1}$

Lemma 0.3. There exists $d: \tilde{Q}_0 \to \mathbb{Z}$ such that $d(q,w) \neq d(q',w')$ for all q,q',w,w' with dim $X_{q,w} \neq 0$ and dim $X_{q',w'} \neq 0$.

Theorem 1. $Gr_{\mathbf{e}}^Q(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}}oftype\mathbf{e}} Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$

Corollary 1.1. affine bundles over $Gr_{\tilde{e}}^{\tilde{Q}}(\tilde{X})$ if $Gr_{e}^{Q}(X)$ is smooth

$$\{U \in Gr_{\mathbf{e}}^Q(X): \lim_{t \to 0} t \cdot U \in Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})\}$$

Question 1.1. What are the ranks of these bundles? Poincaré polynomials?

3. Representation Theory of K(n)

Denote by K(n) the *n-Kronecker quiver* $1 \stackrel{n}{\longleftarrow} 2$ with vertices $K(n)_0 = \{1, 2\}$ and *n* arrows from vertex 2 to vertex 1.

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$.

Theorem 2. For each $m \ge 1$, there exist unique (up to isomorphism) indecomposable rigid representations P_m and I_m of K(n) with dimension vectors (u_m, u_{m-1}) and (u_{m-1}, u_m) respectively. Moreover, any rigid representation of K(n) is isomorphic to one of the form $P_m^{a_1}P_{m+1}^{a_2}$ or $I_m^{a_1}I_{m+1}^{a_2}$ for some $m \ge 1$ and some $a_1, a_2 \ge 0$.

The representations P_m are called the *preprojective* representations of K(n) and the representations I_m are called *preinjective*.

Lemma 2.1. For any $m \ge 1$, we have

$$\dim \text{Hom}(P_m, P_m) = 1$$
, $\dim \text{Hom}(P_m, P_{m+1}) = n$, $\dim \text{Hom}(P_{m+1}, P_m) = 0$.

Moreover, for $1 \le k < n$ and linearly independent functions $f_1, \ldots, f_k \in \text{Hom}(P_m, P_{m+1})$, the map $[f_1 \cdots f_k] : P_m^k \to P_{m+1}$ is injective.

Write $P_{m+1}^{\{f_1,\dots,f_k\}}$ for the cokernel of the map $[f_1\,\cdots\,f_k]:P_m^k\to P_{m+1}$, i.e. we have a short exact sequence

$$(1) 0 \longrightarrow P_m^k \stackrel{[f_1 \dots f_k]}{\longrightarrow} P_{m+1} \longrightarrow P_{m+1}^{\{f_1,\dots,f_k\}} \longrightarrow 0.$$

Lemma 2.2. For $m \geq 1$ and linearly independent functions $f_1, \ldots, f_k \in \text{Hom}(P_m, P_{m+1})$, the following

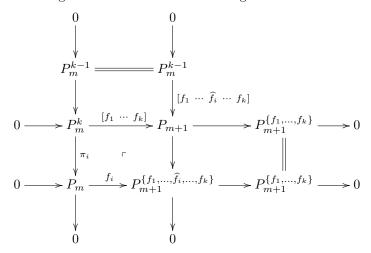
- (1) $\operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \dots, f_k\}})$ is naturally isomorphic to $\operatorname{Hom}(P_m, P_{m+1})/\mathbb{k}f_1 \oplus \dots \oplus \mathbb{k}f_k;$ (2) $\operatorname{Ext}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m)$ is naturally isomorphic to $\mathbb{k}f_1 \oplus \dots \oplus \mathbb{k}f_k;$ (3) $\operatorname{Hom}(P_{m+1}^{\{f_1, \dots, f_k\}}, P_m) = 0.$

Proof. Applying the functor $Hom(P_m, -)$ to the sequence (1) gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}(P_m, P_m^k) \stackrel{[f_1 \dots f_k] \circ}{\longrightarrow} \operatorname{Hom}(P_m, P_{m+1}) \longrightarrow \operatorname{Hom}(P_m, P_{m+1}^{\{f_1, \dots, f_k\}}) \longrightarrow 0,$$

establishing (1).

To see (2), we apply the functor $\operatorname{Hom}(-, P_m)$ to the sequence (1) to get an isomorphism $\operatorname{Hom}(P_m^k, P_m) \cong$ $\operatorname{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}},P_m)$. Observe that $\operatorname{Hom}(P_m^k,P_m)$ is spanned by the coordinate projections $\pi_i:P_m^k\to P_m$ for $1\leq i\leq k$ and each of these gives rise to a commutative diagram



The bottom row of this diagram is the extension in $\operatorname{Ext}(P_{m+1}^{\{f_1,\dots,f_k\}},P_m)$ naturally identified with the generator f_i of $\mathbb{k} f_1 \oplus \cdots \oplus \mathbb{k} f_k$.

Part (3) follows immediately from (2) and the observation that

$$\langle P_{m+1}^{\{f_1,\dots,f_k\}}, P_m \rangle = \langle P_{m+1}, P_m \rangle - k \langle P_m, P_m \rangle = -k.$$

For any nonempty subset $I \subset [1, k]$, say with $|I| = \ell$, we get an exact sequence

$$0 \longrightarrow P_m^{\ell} \overset{[f_i]_{i \in I}}{\longrightarrow} P_{m+1}^{\{f_j\}_{j \in [1,k] \backslash I}} \longrightarrow P_{m+1}^{\{f_1, \dots, f_k\}} \longrightarrow 0.$$

Each such sequence has the following almost-split property for subrepresentations of $P_{m+1}^{\{f_1,\ldots,f_k\}}$.

Lemma 2.3. For $m \ge 1$ and linearly independent functions $f_1, \ldots, f_k \in \text{Hom}(P_m, P_{m+1})$, given any proper subrepresentation $V \subseteq P_{m+1}^{\{f_1,\dots,f_k\}}$, the upper pullback sequence

$$0 \longrightarrow P_{m}^{\ell} \longrightarrow V^{I} \longrightarrow V \longrightarrow 0$$

$$\parallel \qquad \qquad \downarrow \qquad \qquad \downarrow$$

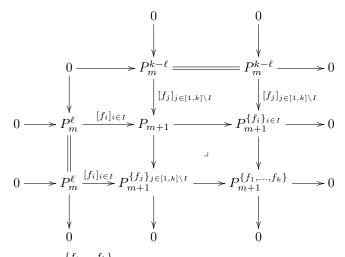
$$0 \longrightarrow P_{m}^{\ell} \stackrel{[f_{i}]_{i \in I}}{\longrightarrow} P_{m+1}^{\{f_{j}\}_{j \in [1,k] \setminus I}} \longrightarrow P_{m+1}^{\{f_{1}, \dots, f_{k}\}} \longrightarrow 0$$

splits for every $I \subset [1, k]$.

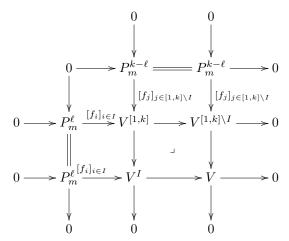
Proof. When $I = \emptyset$ there is nothing to show, so assume I is a nontrivial subset of [1, k]. We proceed by induction on k. For any nonzero $f \in \text{Hom}(P_m, P_{m+1})$ the exact sequence

$$0 \longrightarrow P_m \xrightarrow{f} P_{m+1} \longrightarrow P_{m+1}^{\{f\}} \longrightarrow 0$$

is almost split, giving the claim in the case k=1. For k>1 and a nontrivial subset $I\subset [1,k]$, we have the following commutative diagram



A proper subrepresentation $V \subset P_{m+1}^{\{f_1,\ldots,f_k\}}$ gives rise, via pullbacks, to the following commutative diagram



in which the middle vertical sequence and the middle horizontal sequence split by induction. But then the lower horizontal sequence splits, establishing the claim for all $I \subset [1, k]$.

4. Quiver Grassmannians of
$$\widetilde{K(n)}$$

Let \tilde{P}_m be a fixed lift of P_m to the universal cover K(n).

Lemma 2.4. There exist lifts $\tilde{P}_{m-1,i}$ for $1 \le i \le n$ of P_{m-1} to K(n) so that:

- (1) $\operatorname{Hom}_{Q}(P_{m-1}, P_{m}) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_{m});$ (2) For any proper subset $\{i_{1}, \ldots, i_{k}\} \subset \{1, \ldots, n\}$, there exists a short exact sequence

$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\dots,i_k} \longrightarrow 0;$$

- (3) The lifts $\tilde{P}_{m-1,i}$ are pairwise orthogonal.
- (4) All nontrivial proper subrepresentations of $\tilde{P}_m^{(k)}$ are preprojective.

We will always choose the subset $\{1, \ldots, k\}$ when using Lemma 2.4.2 and thus we denote the cokernal simply by $\tilde{P}_m^{(k)}$.

Lemma 2.5. If each $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1,i})$ has a cell decomposition, then $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1,i_j})$ has a cell decomposition.

Lemma 2.6. (1)
$$\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$$

(2) The subrepresentation
$$\bigoplus_{i=1}^{k-1} \tilde{P}_{m-1,i} \oplus \bigoplus_{i=1}^{k} \tilde{P}_{m-2,i} \subset \bigoplus i = 1^k \tilde{P}_{m-1,i}$$
 is in $(\tilde{P}_m^{(k)})^{\perp}$ and $\operatorname{Ext}(\bigoplus i = 1^k \tilde{P}_{m-1,i}, \tilde{P}_m^{(k)}) \cong \operatorname{Ext}(\tilde{P}_{m-1}^{(k)}, \tilde{P}_m^{(k)})$

where $\tilde{P}_{m-1}^{(k)}$ above denotes the cokernel of the inclusion.

Corollary 2.1. observe when fibers are empty

Proposition 2.1. Consider $\psi: Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m) \to \bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}} Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^k \tilde{P}_{m-1,i}) \times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)})$. Then the following

- (1) For $V \subsetneq \tilde{P}_m^{(k)}$ and $U \subset \bigoplus_{i=1}^k \tilde{P}_{m-1,i}$, we have $\psi^{-1}(U,V) = \mathbb{A}^{\langle V, \bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U \rangle}$. (2) If $V = \tilde{P}_m^{(k)}$ and the fiber is not empty, then $\psi^{-1}(U,V)$ is constant.

Proof. (1) V is preprojective but $\bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U$ is not unless U=0

$$0 \longrightarrow [V, P/U] \longrightarrow [V, U]^1 \longrightarrow [V, P]^1 \longrightarrow [V, P/U]^1 \longrightarrow 0$$

and the middle map is surjective.

Theorem 3. Every quiver Grassmannian of a preprojective or preinjective representation of K(n) and K(n)has a cell decomposition.

Question 3.1. cells of $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ are in one-to-one correspondence with certain tuples of subgraphs for $smaller \; \tilde{P}^{i_1,...,i_k}_{\boldsymbol{\ell}}$

5. Compatible Pairs Label Cells in $Gr_{\mathbf{e}}^Q(P_m)$