CELL DECOMPOSITION OF RANK 2 QUIVER GRASSMANNIANS

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Abstract. We prove that all quiver Grassmannians for preprojective and preinjective representations of a generalized Kronecker quiver admit a cell decomposition. We also provide a natural combinatorial labeling for these cells using compatible pairs in a maximal Dyck path.

1. Introduction

-something about cluster algebras -something about categorification and quiver Grassmannians -something about compatible pairs and combinatorial construction of cluster variables -statement of our results -something about torus actions and the universal cover -something about cell decompositions of quiver Grassmannians -acknowledgements?

2. Torus Action on Quiver Grassmannians

Let Q be an acylic quiver with vertices Q_0 and arrows Q_1 which we denote by $\alpha : i \to j$. Moreover, let W_Q be the free (non-abelian) group generated by Q_1. We denote by $\mathrm{Rep}(Q)$ the category of $\mathbb{C}$-representations of Q.

Definition 0.1. *The universal covering quiver $\tilde{Q}$ of Q is given by the vertices $\tilde{Q}_0 = Q_0 \times W_Q$ and the arrow set $\tilde{Q}_1 = Q_1 \times W_Q$ where $(\alpha, w) : (i, w) \to (j, w\alpha)$ for every $\alpha : i \to j$.*

We say that a representation $X \in \mathrm{Rep}(Q)$ can be lifted (to $\tilde{Q}$) if there exists a representation $\tilde{X} \in \mathrm{Rep}(\tilde{Q})$ such that $F_Q\tilde{X} = X$ where $F_Q : \mathrm{Rep}(\tilde{Q}) \to \mathrm{Rep}(Q)$ is the natural functor.

Lemma 0.1. *Every preprojective (resp. preinjective) representation of Q can be lifted to a representation of $\tilde{Q}$.*

Proof. This statement is clear for the simple representations S_q, $q \in Q_0$. Now every preprojective representation X can be obtained when applying a sequence of BGP-reflections [1] to a simple representation $S_{q'}$ of a quiver Q' whose underlying graph is the one of Q. Applying BGP-reflections to a source q of Q corresponds to applying BGP-reflections to all vertices (q, w) of $\tilde{Q}$. This leads the claim. The statement for preinjective representations follows in the same way. □

We choose a map $d : \tilde{Q}_0 \to \mathbb{Z}$ and fix a representation $X \in \mathrm{Rep}(Q)$. In any case, we can consider the decomposition $X_q = \bigoplus_{w \in W_Q} X_{(q,w)}$. We define a torus action on each $X_{(q,w)}$ via $t.x_{(q,w)} = t^{d(q,w)}x_{(q,w)}$ which can be extended linearly to each X_q. For a fixed a subspace U_q, we can define the subspace $t.U_q$. In general, this torus action induces no torus action on the Quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}(X)$ as $t.U = (t.U_q)_{q \in Q_0}$ is no subrepresentation of X for every $U \in \mathrm{Gr}_{\mathbf{e}}(X)$. Actually, for this the action has to satisfy $X_\alpha(t.U_i) \in t.U_j$ for every $\alpha : i \to j$ and every $x \in X_i$.

Lemma 0.2. *Fix an integer $c_\alpha \in \mathbb{Z}$ for every $\alpha \in Q_1$. If $X \in \mathrm{Rep}(Q)$ can be lifted, $d : \tilde{Q}_0 \to \mathbb{Z}$ induces a torus action on $\mathrm{Gr}_{\mathbf{e}}(X)$ if we have $d(j, w\alpha) - d(i, w) = c_\alpha$ for all $\alpha : i \to j$ and $w \in W_Q$.*

Proof. Since X can be lifted, we can write $X_\alpha : X_i \to X_j$ as block matrix consisting of linear maps $X_{(\alpha,w)} : X_{(i,w)} \to X_{(j,w\alpha)}$. Then the condition $X_\alpha(t.U_i) \in t.U_j$ translates into ...to be continued. □

Lemma 0.3. *There exists $d : \tilde{Q}_0 \to \mathbb{Z}$ such that $d(q, w) \neq d(q', w')$ for all q, q', w, w' with $\dim X_{q,w} \neq 0$ and $\dim X_{q',w'} \neq 0$.*

Theorem 1. $Gr^Q_{\mathbf{e}}(X)^T \cong \bigsqcup_{\tilde{\mathbf{e}} of type \mathbf{e}} Gr^{\tilde{Q}}_{\tilde{\mathbf{e}}}(\tilde{X})$

Corollary 1.1. *affine bundles over $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})$ if $Gr_{\mathbf{e}}^{Q}(X)$ is smooth*

$$\{U \in Gr_{\mathbf{e}}^{Q}(X) : \lim_{t\to 0} t\cdot U \in Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{X})\}$$

Question 1.1. *What are the ranks of these bundles? Poincaré polynomials?*

3. Quiver Grassmannians of $\widetilde{K(n)}$

Define Chebyshev polynomials u_k for $k \in \mathbb{Z}$ by the recursion $u_0 = 0$, $u_1 = 1$, $u_{k+1} = nu_k - u_{k-1}$. For $m \geq 1$, let P_m be the preprojective representation of $K(n)$ with dimension vector (u_m, u_{m-1}).

Let $\tilde{P}_m$ be a fixed lift of P_m to the universal cover $\widetilde{K(n)}$.

Lemma 1.1. *There exist lifts $\tilde{P}_{m-1,i}$ for $1 \leq i \leq n$ of P_{m-1} to $\widetilde{K(n)}$ so that:*

1. $\mathrm{Hom}_Q(P_{m-1}, P_m) \cong \bigoplus_{i=1}^n \mathrm{Hom}_{\tilde{Q}}(\tilde{P}_{m-1,i}, \tilde{P}_m)$;
2. *For any proper subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, there exists a short exact sequence*
$$0 \longrightarrow \tilde{P}_{m-1,i_1} \oplus \cdots \oplus \tilde{P}_{m-1,i_k} \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_m^{i_1,\ldots,i_k} \longrightarrow 0;$$
3. *The lifts $\tilde{P}_{m-1,i}$ are pairwise orthogonal.*
4. *All nontrivial proper subrepresentations of $\tilde{P}_m^{(k)}$ are preprojective.*

We will always choose the subset $\{1, \ldots, k\}$ when using Lemma 1.1.2 and thus we denote the cokernal simply by $\tilde{P}_m^{(k)}$.

Lemma 1.2. *If each $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_{m-1,i})$ has a cell decomposition, then $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\bigoplus \tilde{P}_{m-1,i_j})$ has a cell decomposition.*

Lemma 1.3. (1) $\tilde{P}_m^{(n-1)} \cong \tilde{P}_{m-1}^{(1)}$

(2) *The subrepresentation $\bigoplus_{i=1}^{k-1} \tilde{P}_{m-1,i} \oplus \bigoplus_{i=1}^{k} \tilde{P}_{m-2,i} \subset \bigoplus i = 1^k \tilde{P}_{m-1,i}$ is in $(\tilde{P}_m^{(k)})^\perp$ and*
$$\mathrm{Ext}(\bigoplus i = 1^k \tilde{P}_{m-1,i}, \tilde{P}_m^{(k)}) \cong \mathrm{Ext}(\tilde{P}_{m-1}^{(k)}, \tilde{P}_m^{(k)})$$
where $\tilde{P}_{m-1}^{(k)}$ above denotes the cokernel of the inclusion.

Corollary 1.2. *observe when fibers are empty*

Proposition 1.1. *Consider $\psi : Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m) \to \bigsqcup_{\tilde{\mathbf{f}}+\tilde{\mathbf{g}}=\tilde{\mathbf{e}}} Gr_{\tilde{\mathbf{f}}}^{\tilde{Q}}(\bigoplus_{i=1}^k \tilde{P}_{m-1,i}) \times Gr_{\tilde{\mathbf{g}}}^{\tilde{Q}}(\tilde{P}_m^{(k)}$. Then the following hold:*

1. *For $V \subsetneq \tilde{P}_m^{(k)}$ and $U \subset \bigoplus_{i=1}^k \tilde{P}_{m-1,i}$, we have $\psi^{-1}(U, V) = \mathbb{A}^{\langle V, \bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U\rangle}$.*
2. *If $V = \tilde{P}_m^{(k)}$ and the fiber is not empty, then $\psi^{-1}(U, V)$ is constant.*

Proof. (1) V is preprojective but $\bigoplus_{i=1}^k \tilde{P}_{m-1,i}/U$ is not unless $U = 0$

(2)
$$0 \longrightarrow [V, P/U] \longrightarrow [V, U]^1 \longrightarrow [V, P]^1 \longrightarrow [V, P/U]^1 \longrightarrow 0$$
and the middle map is surjective. ☐

Theorem 2. *Every quiver Grassmannian of a preprojective or preinjective representation of $K(n)$ and $\widetilde{K(n)}$ has a cell decomposition.*

Question 2.1. *cells of $Gr_{\tilde{\mathbf{e}}}^{\tilde{Q}}(\tilde{P}_m)$ are in one-to-one correspondence with certain tuples of subgraphs for smaller $\tilde{P}_\ell^{i_1,\ldots,i_k}$*

4. Compatible Pairs Label Cells in $Gr_{\mathbf{e}}^{Q}(P_m)$

References

[1] Bernstein, I. N., Gelfand, I. M., Ponomarev, V. A.: Coxeter functors, and Gabriel's theorem. Russian Mathematical Surveys **28**(2), 17-32 (1973).