FCDs for robust regression model

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Here are the full conditional distributions (FCD's) for this problem:

- For β_0:

$$p(\beta_0|\dots) \propto \left[\prod_i N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] N(\beta_0|0,10)$$

$$\propto \left[\prod_i \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i-\beta_1 x_i-\beta_0)^2\right)\right]\exp\left(-\frac{1}{2\times 10}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_i \tau_i(-2\beta_0[y_i-\beta_1 x_i]+\beta_0^2)\right)\exp\left(-\frac{1}{2\times 10}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left\{\beta_0^2\left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]-2\beta_0\left(\frac{1}{\sigma^2}\right)\sum_i \tau_i[y_i-\beta_1 x_i]\right\}\right)$$

$$p(\beta_0|\dots) = N\left(\left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]^{-1}\frac{1}{\sigma^2}\sum_i \tau_i[y_i-\beta_1 x_i], \left[\frac{1}{\sigma^2}\sum_i \tau_i+\frac{1}{10}\right]^{-1}\right)$$

- For β_1:

$$p(\beta_1|\dots) \propto \left[\prod_i N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] N(\beta_1|0,10)$$

$$\propto \left[\prod_i \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i-\beta_0-\beta_1 x_i)^2\right)\right]\exp\left(-\frac{1}{2\times 10}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\frac{1}{\sigma^2}\sum_i \tau_i\left(-2\beta_1 x_i[y_i-\beta_0]+\beta_1^2 x_i^2\right)\right)\exp\left(-\frac{1}{2\times 10}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_1^2\left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]-2\beta_1\frac{1}{\sigma^2}\sum_i \tau_i x_i[y_i-\beta_0]\right]\right)$$

$$p(\beta_1|\dots) = N\left(\left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]^{-1}\frac{1}{\sigma^2}\sum_i \tau_i x_i[y_i-\beta_0], \left[\frac{1}{\sigma^2}\sum_i \tau_i x_i^2+\frac{1}{10}\right]^{-1}\right)$$

- For τ_i:

$$p(\tau_i|\dots) \propto N\left(y_i|\beta_0+\beta_1 x_i, \frac{\sigma^2}{\tau_i}\right) Gamma\left(\tau_i|\frac{v}{2},\frac{v}{2}\right)$$

$$\propto \frac{1}{\sqrt{\frac{1}{\tau_i}}} \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right) \tau_i^{\frac{v}{2}-1} \exp\left(-\tau_i \left(\frac{v}{2}\right)\right)$$

$$\propto \tau_i^{\frac{1+v}{2}-1} \exp\left(-\tau_i \left[\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2 + \frac{v}{2}\right]\right)$$

$$p(\tau_i | \dots) = Gamma\left(\frac{1+v}{2}, \frac{\frac{1}{\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2 + v}{2}\right)$$

- For v:

$$p(v| \dots) \propto \left[\prod_i Gamma\left(\tau_i | \frac{v}{2}, \frac{v}{2}\right)\right] DiscUnif(v, 1{,}30)$$

This is the only tricky part of our code. To sample v, we rely on an independent Metropolis-Hastings algorithm. This entails choosing a new v (a number from 1,…,30) and accepting/rejecting it based on our threshold probability $p_t = min\left\{1, \frac{\prod_i Gamma\left(\tau_i | \frac{v_{new}}{2}, \frac{v_{new}}{2}\right)}{\prod_i Gamma\left(\tau_i | \frac{v_{old}}{2}, \frac{v_{old}}{2}\right)}\right\}$.

- For σ^2:

$$p\left(\frac{1}{\sigma^2}\middle| \dots\right) \propto \left[\prod_i N\left(y_i | \beta_0 + \beta_1 x_i, \frac{\sigma^2}{\tau_i}\right)\right] Gamma\left(\frac{1}{\sigma^2}\middle| a, b\right)$$

$$\propto \left[\prod_i \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\tau_i}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right)\right] \left(\frac{1}{\sigma^2}\right)^{a-1} \exp\left(-b\frac{1}{\sigma^2}\right)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{N+2a}{2}-1} \exp\left(-\frac{1}{\sigma^2}\left[\frac{1}{2}\sum_i \tau_i (y_i - \beta_0 - \beta_1 x_i)^2 + b\right]\right)$$

$$p\left(\frac{1}{\sigma^2}\middle| \dots\right) = Gamma\left(\frac{N+2a}{2}, \frac{1}{2}\sum_i \tau_i (y_i - \beta_0 - \beta_1 x_i)^2 + b\right)$$