FCDs for regression parameters assuming that all $z_i's$ are known

Denis Valle

December 2017

As mentioned before, many of the models we will see here assume a Gaussian regression for the latent continuous variable z_i:

$$z_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$\beta_0, \beta_1 \sim N(0,100)$$

$$\frac{1}{\sigma^2} \sim Gamma(a, b)$$

As a result, the FCD's for the regression parameters are identical to those from a simple Gaussian regression:

- For β_0:

$$p(\beta_0 | \dots) \propto \left[\prod_i N(z_i | \beta_0 + \beta_1 x_i, \sigma^2)\right] N(\beta_0 | 0,100)$$

$$\propto \left[\prod_i \exp\left(-\frac{1}{2\sigma^2}(z_i - \beta_1 x_i - \beta_0)^2\right)\right] \exp\left(-\frac{1}{2 \times 100}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_i (z_i - \beta_1 x_i - \beta_0)^2 - \frac{1}{2 \times 100}\beta_0^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\frac{1}{\sigma^2}\sum_i (-2\beta_0(z_i - \beta_1 x_i) + \beta_0^2) + \frac{1}{100}\beta_0^2\right]\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_0^2\left[\frac{n}{\sigma^2} + \frac{1}{100}\right] - 2\beta_0 \frac{1}{\sigma^2}\sum_i (z_i - \beta_1 x_i)\right]\right)$$

$$p(\beta_0 | \dots) = N\left(\left[\frac{n}{\sigma^2} + \frac{1}{100}\right]^{-1} \frac{1}{\sigma^2}\sum_i (z_i - \beta_1 x_i), \left[\frac{n}{\sigma^2} + \frac{1}{100}\right]^{-1}\right)$$

- For β_1:

$$p(\beta_1|\dots) \propto \left[\prod_i N(z_i|\beta_0+\beta_1 x_i,\sigma^2)\right] N(\beta_1|0,100)$$

$$\propto \left[\prod_i \exp\left(-\frac{1}{2\sigma^2}(z_i-\beta_0-\beta_1 x_i)^2\right)\right]\exp\left(-\frac{1}{2\times 100}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_i (z_i-\beta_0-\beta_1 x_i)^2-\frac{1}{2\times 100}\beta_1^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\frac{1}{\sigma^2}\left(-2\beta_1\sum_i x_i(z_i-\beta_0)+\beta_1^2\sum_i x_i^2\right)+\frac{1}{100}\beta_1^2\right]\right)$$

$$\propto \exp\left(-\frac{1}{2}\left[\beta_1^2\left[\frac{1}{\sigma^2}\sum_i x_i^2+\frac{1}{100}\right]-2\beta_1\frac{1}{\sigma^2}\sum_i x_i(z_i-\beta_0)\right]\right)$$

$$p(\beta_1|\dots) = N\left(\left[\frac{1}{\sigma^2}\sum_i x_i^2+\frac{1}{100}\right]^{-1}\frac{1}{\sigma^2}\sum_i x_i(z_i-\beta_0), \left[\frac{1}{\sigma^2}\sum_i x_i^2+\frac{1}{100}\right]^{-1}\right)$$

- For $\frac{1}{\sigma^2}$:

$$p\left(\frac{1}{\sigma^2}\middle|\dots\right) \propto \left[\prod_i N(z_i|\beta_0+\beta_1 x_i,\sigma^2)\right] Gamma\left(\frac{1}{\sigma^2}\middle|a,b\right)$$

$$\propto \left[\prod_i \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{2\sigma^2}(z_i-\beta_0-\beta_1 x_i)^2\right)\right]\left(\frac{1}{\sigma^2}\right)^{a-1}\exp\left(-b\frac{1}{\sigma^2}\right)$$

$$\propto \left[\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left(-\frac{1}{\sigma^2}\frac{\sum_i (z_i-\beta_0-\beta_1 x_i)^2}{2}\right)\right]\left(\frac{1}{\sigma^2}\right)^{a-1}\exp\left(-b\frac{1}{\sigma^2}\right)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n+2a}{2}-1}\exp\left(-\frac{1}{\sigma^2}\left[\frac{\sum_i (z_i-\beta_0-\beta_1 x_i)^2}{2}+b\right]\right)$$

$$p\left(\frac{1}{\sigma^2}\middle|\dots\right) = Gamma\left(\frac{n+2a}{2},\frac{\sum_i (z_i-\beta_0-\beta_1 x_i)^2}{2}+b\right)$$