An Incomplete Overview of some Applications of Game Theory to Patient Flow

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ABSTRACT

Individual behaviour has an effect on the optimal control of queueing systems. This idea sits at the intersection of queueing theory and game theory. The role of this paper is to discuss some work in this area applied to healthcare systems.

One aspect considered is the effect of patient behaviour when choosing between service facilities. Various pieces of work will be discussed that have looked at this area.

A second aspect is the effect that managers may have when acting rationally in their control of queueing systems.

KEYWORDS. Patient Flow, Game Theory, Queueing Theory

Main Area: Healthcare

1. Introduction

Understanding the effect of selfish individuals in queueing systems can be traced back to a series of short communications between Leeman [11, 12] and Saaty [18].

In [1, 2, 4, 8, 13, 14, 21] a variety of models are discussed that obtain optimal and equilibrium behaviour in situations where individuals can choose between a best queue to join. Most of the work considers these systems in a theoretical context.

Most research where game theory is applied in healthcare has mainly concentrated on Emergency Departments (EDs) and how to deal with diversions of patients and ambulances. In [5] cooperative strategies for hospitals are considered, in order to reduce occurrences when ambulances are turned away due to the ED being full. In [3] a queueing network model of two EDs is proposed to study the network effect of ambulance diversion. Each ED aims to minimise the expected waiting time of its patients (walk-ins and ambulances) and chooses its diversion threshold based on the number of patients at its location. Decentralised decision making in the network is modelled as a non-cooperative game.

The degree of central control that should be exercised is a very important question to be considered by governments and/or policy makers. What is the effect of allowing individuals to choose service provider?

This is a question that is answered using the Game theoretical concept referred to as the *Price of Anarchy* (PoA). This was first defined in [10] but an excellent introduction is given in [17]. The PoA is defined as the ratio of the selfish utility to the optimal utility.

Section will discuss some PoA analysis applied to individuals choosing between queues. Section will discuss similar analysis in the case of selfish behaviour related to the management of queues. Section will conclude with a summary and thoughts for further work.

2. Choosing queues

This section describes how patient choices between various congestion affected services may be modelled. In particular the situation shown diagrammatically in Figure 1 is considered: patients have a choice amongst $M/M/c$ queues. Each $M/M/c$ queue corresponds to a (simple) model of a Hospital (or indeed any other public service facility).
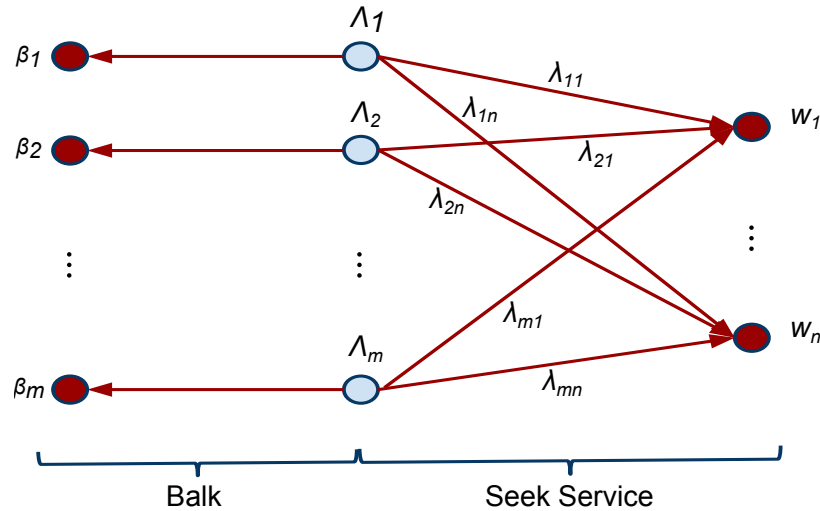

Figura 1: Routing patients from m hospitals to n services.

The parameters of this system are shown in Table 2.

Parameter	Interpretation
$m \in \mathbb{Z}$	Number of sources
$n \in \mathbb{Z}$	Number of service centers
$\beta \in \mathbb{R}_{\geq 0}^{m}$	Worth of service
$\Lambda \in \mathbb{R}_{\geq 0}^{\overline{m}}$	Demand rate
w_j for $j \in [n]$	A convex utility function
d_{ij} for $i \in [m]$, $j \in [n]$	Distance from source i to service center j
λ_{ij} for $i \in [m]$, $j \in [n]$	Traffic from source i to service center j

Figura 2: Parameters of model

There are two approaches to solving this problem: assuming that patients observe or not the system states before choosing a facility. A rigorous comparison of these two approaches for individuals choosing to join an $M/M/1$ queue is given in [20].

An unobservable study is given in [6] where routing games [17] are used to study the system described. The routing game used is shown in 3.
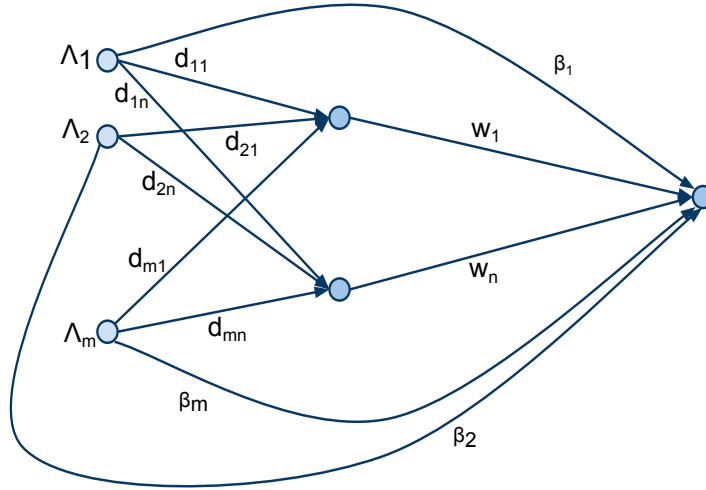

Figura 3: Routing game m hospitals to n services.

The corresponding cost function for a given traffic flow λ is given by:

$$C(\lambda) = \sum_{i=1}^{m} \alpha_i \sum_{j}^{n} d_{ij}\lambda_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{m} \lambda_{ij} w_j \left(\sum_{i=1}^{m} \lambda_{ij} \right) + \sum_{i=1}^{m} \beta_i \left(\Lambda_i - \sum_{j=1}^{n} \lambda_{ij} \right) \qquad (1)$$

The constant $\alpha_i \in \mathbb{R}_{\geq 0}$ is simply a weighting statistic for the relative importance of travel distances to the other factors (once again allowing for this coefficient to be dependent on population partitioning). The Nash flow corresponding to the flow at which all traffic from a given source travels on minimal paths can be shown to be given by the flow that minimises the following function:

$$\Phi(\lambda) = \sum_{i=1}^{m} \alpha_i \sum_{j}^{n} d_{ij}\lambda_{ij} + \sum_{j=1}^{n} \int_0^{\sum_{i=1}^{m} \lambda_{ij}} w_j(x)dx + \sum_{i=1}^{m} \beta_i \left(\Lambda_i - \sum_{j=1}^{n} \lambda_{ij} \right) \qquad (2)$$

To be able to obtain the PoA for a given instance the following mathematical program must be solved:

OPTMP: NASHMP:
minimise (1)   minimise (2)

such that:

$$\sum_{j=1}^{n} \lambda_{ij} \leq \Lambda_i \text{ for all } i \in [m] \tag{3}$$

$$\lambda_{ij} \in \mathbb{R}_{\geq 0}^{m \times n} \text{ for all } i \in [m], \ j \in [n] \tag{4}$$

$$\sum_{i=1}^{m} \lambda_{ij} < c_j \mu_j \text{ for all } j \in [n] \tag{5}$$

In [6] various theoretical results are proven with regards to the effect of worth of service on the PoA but also with regards to demand. The profile of Figure 4 is shown to hold in general.
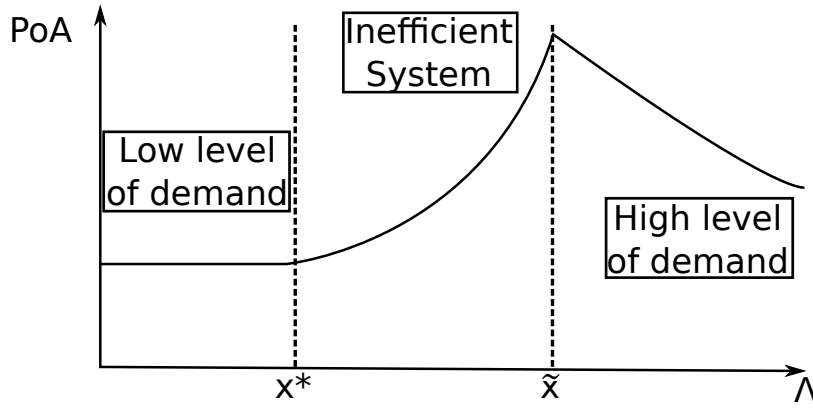

Figura 4: General profile of PoA.

Further to the theoretical consideration the PoA was calculated for a large case study using Hospital in Wales offering knee surgeries. A high PoA was calculated for the current demand.

To consider systems where individuals are able to observe the system there are two approaches: the first is to use a simulation based approach that allows individuals to choose their most desirable queue. One such approach that was considered specifically in the context of healthcare was considered in [7].

Given that individuals will consider a simple selfish decision rule this approach is relatively straightforward and can also be considered using straightforward analytical Markov models. The difficulty with this approach is appears when attempting to obtain the PoA.

In [19] various dynamic programming and approximate dynamic programming techniques are proposed that are able to not only give an optimal policy but also prove the following observation:

Selfish users make busier systems.

In the next section selfish congestion related decisions by managers will be considered.

3. Managing queues

The main proposition of the work requested by the ABUHB was to develop a mathematical model of bed occupancies at the CCUs at RG and NH. After an initial analysis of the data, behavioural aspects became apparent; for example, delaying patients discharge if there was no pressure on CCU beds, or admitting fewer patients if bed occupancy levels were high. As a result of this, a state-dependent queueing model has been developed, which includes the dependency of admission rate on actual occupancy [9]. This state dependent model was applied to both NH and RG separately. It is however obvious that the actions of one CCU impact on the other CCU, as diversion of patients from one CCU to the other sometimes occurs. A pictorial representation of the situation is given in Figure 5.
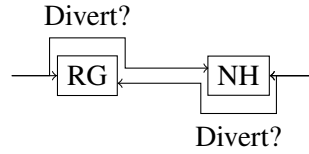

Figura 5: Diagrammatic representation of CCU interaction through patient diversion.

The capacity thresholds are denoted as $K_h \in \mathbb{Z}$ for $h \in \{\mathrm{NH}, \mathrm{RG}\}$. Note that $0 \le K_h \le c_h$.
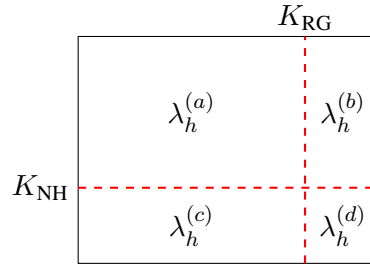

Figura 6: General arrival rates for each CCU at each region, where $h \in \{\mathrm{NH}, \mathrm{RG}\}$

Utilities will be of interest when this queueing theoretical model will be inserted in the game theoretical model. Throughput of patients is a natural choice of utility given that most hospitals are financially rewarded per served patient [16]. For each threshold pair $(K_{\mathrm{NH}}, K_{\mathrm{RG}})$, the utilisation rate U_h and throughput T_h can easily be obtained for each CCU: $h \in \{\mathrm{NH}, \mathrm{RG}\}$, using the following formulas:

$$U_h = \frac{\sum_{n=0}^{c_h} n P^{(h)}(n)}{c_h}$$

$$T_h = \mu_h \sum_{n=0}^{c_h} n P^{(h)}(n)$$

where $P^{(h)} = P^{(h)}(K_{\mathrm{NH}}, K_{\mathrm{RG}})$ is the steady state probability distribution function (obtained from the corresponding transition matrix $Q = Q(K_{\mathrm{NH}}, K_{\mathrm{RG}})$) for $h \in \{\mathrm{NH}, \mathrm{RG}\}$.

For $c_{\mathrm{NH}} = 8, c_{\mathrm{RG}} = 16, \lambda_{\mathrm{NH}} = (\lambda_{\mathrm{NH}}^{(a)}, \lambda_{\mathrm{NH}}^{(b)}, \lambda_{\mathrm{NH}}^{(c)}, \lambda_{\mathrm{NH}}^{(d)}) = (1.5, 3.74, 0, 0), \lambda_{\mathrm{RG}} = (\lambda_{\mathrm{RG}}^{(a)}, \lambda_{\mathrm{RG}}^{(b)}, \lambda_{\mathrm{RG}}^{(c)}, \lambda_{\mathrm{RG}}^{(d)}) = (2.24, 0, 3.74, 0)$ and $(K_{\mathrm{NH}}, K_{\mathrm{RG}}) = (6, 12)$, the steady state probabilities for each CCU are given in Figure **??**.

This game is equivalent to a bi matrix game with restriction to pure strategies where both players aim to get their utilisation as close as possible to a certain target. As such a Nash Equilibrium is
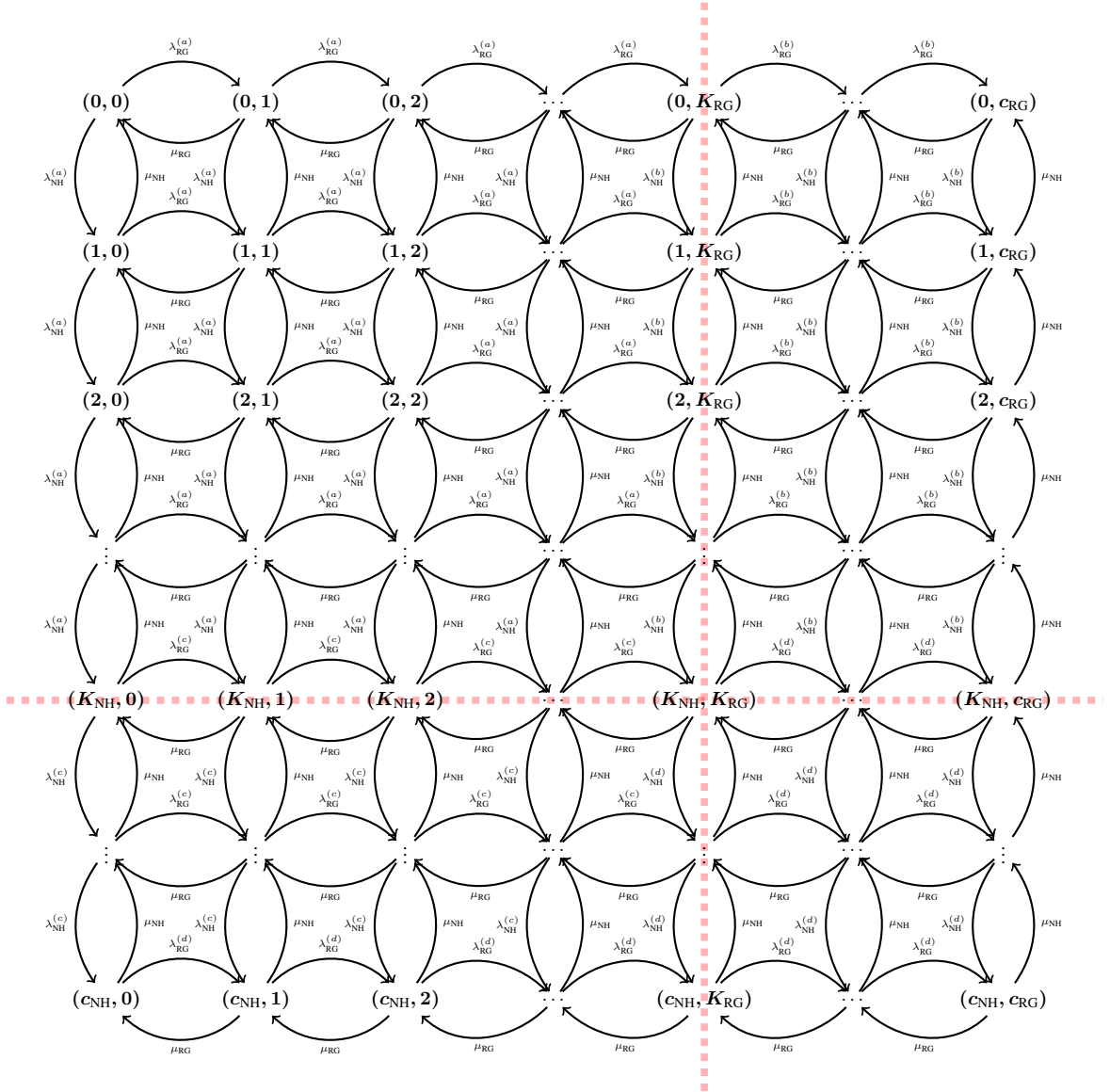
Figura 7: Generic Markov chain underpinning the queueing model of this paper

For all $h \in \{\text{NH}, \text{RG}\}$ minimise:

$$(U_h - t)^2$$

Subject to:

$$0 \leq K_h \leq c_h$$
$$K_h \in \mathbb{Z}$$

Figura 8: The optimisation problem underlying the game

not guaranteed by traditional game theoretical results [15], but based on discussions with ABUHB, long term threshold policies are a realistic consideration.

The following result is a sufficient condition for the existence of an equilibrium:

Theorem.

Let $f_h(k) : [1, c_{\bar{h}}] \rightarrow [1, c_h]$ be the best response of player $h \in \{\text{NH}, \text{RG}\}$ to the diversion threshold of $\bar{h} \neq h$ ($\bar{h} \in \{\text{NH}, \text{RG}\}$).

If $f_h(k)$ is a non-decreasing function in k then the game of Figure 8 has at least one Nash Equilibrium.

proof

The function f_h is well defined as it maximises a continuous function over a finite discrete set (in case of multiple values that minimize U_h, it is assumed that f_h returns the lowest such value).

As such if f_h is non-decreasing then it is in fact a stepwise non-decreasing function. If we consider f_{NH} and f_{RG} plotted on the same axis (so that the domain of f_{NH} is the x-axis and the domain of f_{RG} is the y-axis) it is obvious to see that the functions must intersect at some point as shown in Figure 9.
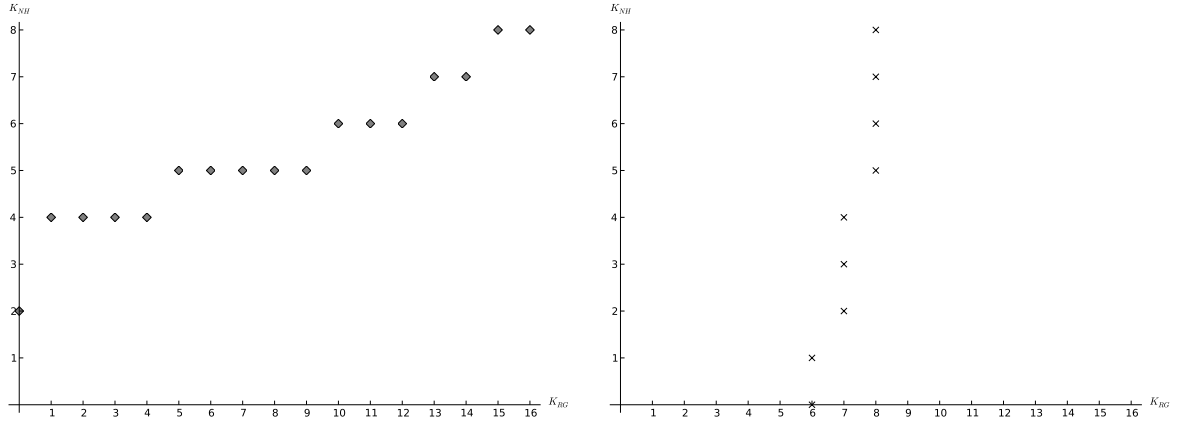

Figura 9: Plots of $f_{\text{NH}}(K_{\text{RG}})$ and $f_{\text{RG}}(K_{\text{NH}})$

This point of intersection corresponds to a Nash Equilibrium of Figure 8. □

4. Conclusions

- Describe potential
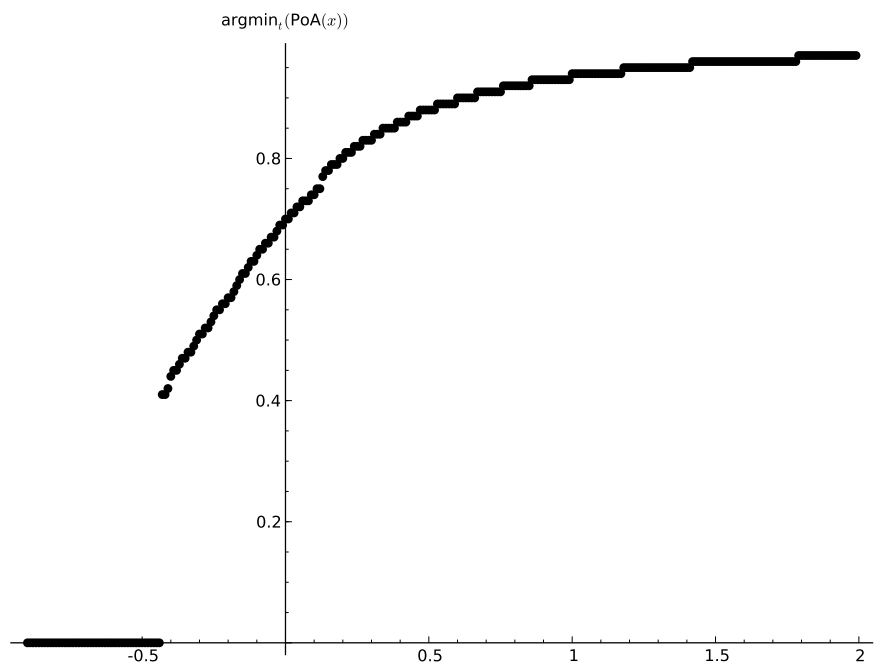- Describe further work
- Discuss potential for heuristic approaches

Figura 10: Lowest value of t ensuring PoA= 1

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