An Incomplete Overview of some Applications of Game Theory to Patient Flow

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ABSTRACT

Describe paper.

KEYWORDS. Patient Flow, Game Theory, Queueing Theory

Main Area: Healthcare

1. Introduction

- Review GT in HC in general (small);
- Review congestion type games;
- PoA;
- Structure.

2. Choosing queues

This section describes how patient choices between various congestion affected services may be modelled. In particular the situation shown diagrammatically in Figure **??** is considered: patients have a choice amongst $M/M/c$ queues.

Show diagram.

There are two approaches to solving this problem: assuming that patients observe or not the system states before choosing a facility. A rigorous comparison of these two approaches for individuals choosing to join an $M/M/1$ queue is given in [].

An unobservable study is given in [] where routing games [**?**] are used to study the system described. The routing game used is shown in **??**.

Show routing game image.

It can be shown that the cost for any given flow λ (denoting the amount of traffic from source i to facility j) corresponds to...

$$C(\lambda) = \sum_{i=1}^{m}\alpha_i\sum_{j}^{n}d_{ij}\lambda_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{m}\lambda_{ij}w_j\left(\sum_{i=1}^{m}\lambda_{ij}\right) + \sum_{i=1}^{m}\beta_i\left(\Lambda_i - \sum_{j=1}^{n}\lambda_{ij}\right) \tag{1}$$

Obtaining the Nash flow: ie the equilibrium behaviour.

$$\Phi(\lambda) = \sum_{i=1}^{m}\alpha_i\sum_{j}^{n}d_{ij}\lambda_{ij} + \sum_{j=1}^{n}\int_{0}^{\sum_{i=1}^{m}\lambda_{ij}} w_j(x)dx + \sum_{i=1}^{m}\beta_i\left(\Lambda_i - \sum_{j=1}^{n}\lambda_{ij}\right) \tag{2}$$

To be able to obtain the PoA for a given instance the following mathematical program must be solved:

OPTMP:	NASHMP:
minimise (1)	minimise (2)

such that:

$$\sum_{j=1}^{n}\lambda_{ij} \leq \Lambda_i \text{ for all } i \in [m] \tag{3}$$

$$\lambda_{ij} \in \mathbb{R}_{\geq 0}^{m\times n} \text{ for all } i \in [m],\ j \in [n] \tag{4}$$

$$\sum_{i=1}^{m}\lambda_{ij} < c_j\mu_j \text{ for all } j \in [n] \tag{5}$$

The constant $\alpha_i \in \mathbb{R}_{\geq 0}$ is simply a weighting statistic for the relative importance of travel distances to the other factors (once again allowing for this coefficient to be dependent on population partitioning).

In [] various theoretical results are proven. With regards to the effect of worth of service on the PoA but also with regards to demand. The profile of Figure **??** is shown to hold in general.

Pic

To consider systems where individuals are able to observe the system there are two approaches: the first is to use a simulation based approach that allows individuals to choose their most desirable queue. One such approach that was considered specifically in the context of healthcare was considered in [].

Given that individuals will consider a simple selfish decision rule this approach is relatively straightforward and can also be considered using straightforward analytical Markov models. The difficulty with this approach is appears when attempting to obtain the PoA. To carry this out an optimal policy must be obtained.

In [] various dynamic programming and approximate dynamic programming techniques are proposed that are able to not only give an optimal policy but also prove the following observation:

Selfish users make busier systems.

In the next section selfish congestion related decisions by managers will be considered.

3. CCU Work

The main proposition of the work requested by the ABUHB was to develop a mathematical model of bed occupancies at the CCUs at RG and NH. After an initial analysis of the data, behavioural aspects became apparent; for example, delaying patients discharge if there was no pressure on CCU beds, or admitting fewer patients if bed occupancy levels were high. As a result of this, a state-dependent queueing model has been developed, which includes the dependency of admission rate on actual occupancy [**?**]. This state dependent model was applied to both NH and RG separately. It is however obvious that the actions of one CCU impact on the other CCU, as diversion of patients from one CCU to the other sometimes occurs. A pictorial representation of the situation is given in Figure 1.

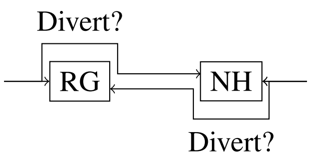

Figura 1: Diagrammatic representation of CCU interaction through patient diversion.

The capacity thresholds are denoted as $K_h \in \mathbb{Z}$ for $h \in \{\text{NH}, \text{RG}\}$. Note that $0 \leq K_h \leq c_h$.

Utilities will be of interest when this queueing theoretical model will be inserted in the game theoretical model. Throughput of patients is a natural choice of utility given that most hospitals are financially rewarded per served patient [**?**]. For each threshold pair $(K_{\text{NH}}, K_{\text{RG}})$, the utilisation rate U_h and throughput T_h can easily be obtained for each CCU: $h \in \{\text{NH}, \text{RG}\}$, using the following formulas:

$$U_h = \frac{\sum_{n=0}^{c_h} nP^{(h)}(n)}{c_h}$$

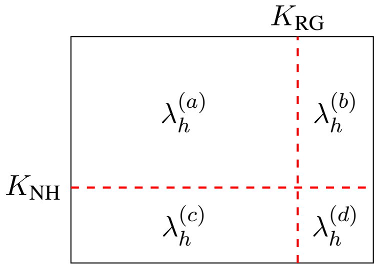

Figura 2: General arrival rates for each CCU at each region, where $h \in \{\mathrm{NH}, \mathrm{RG}\}$

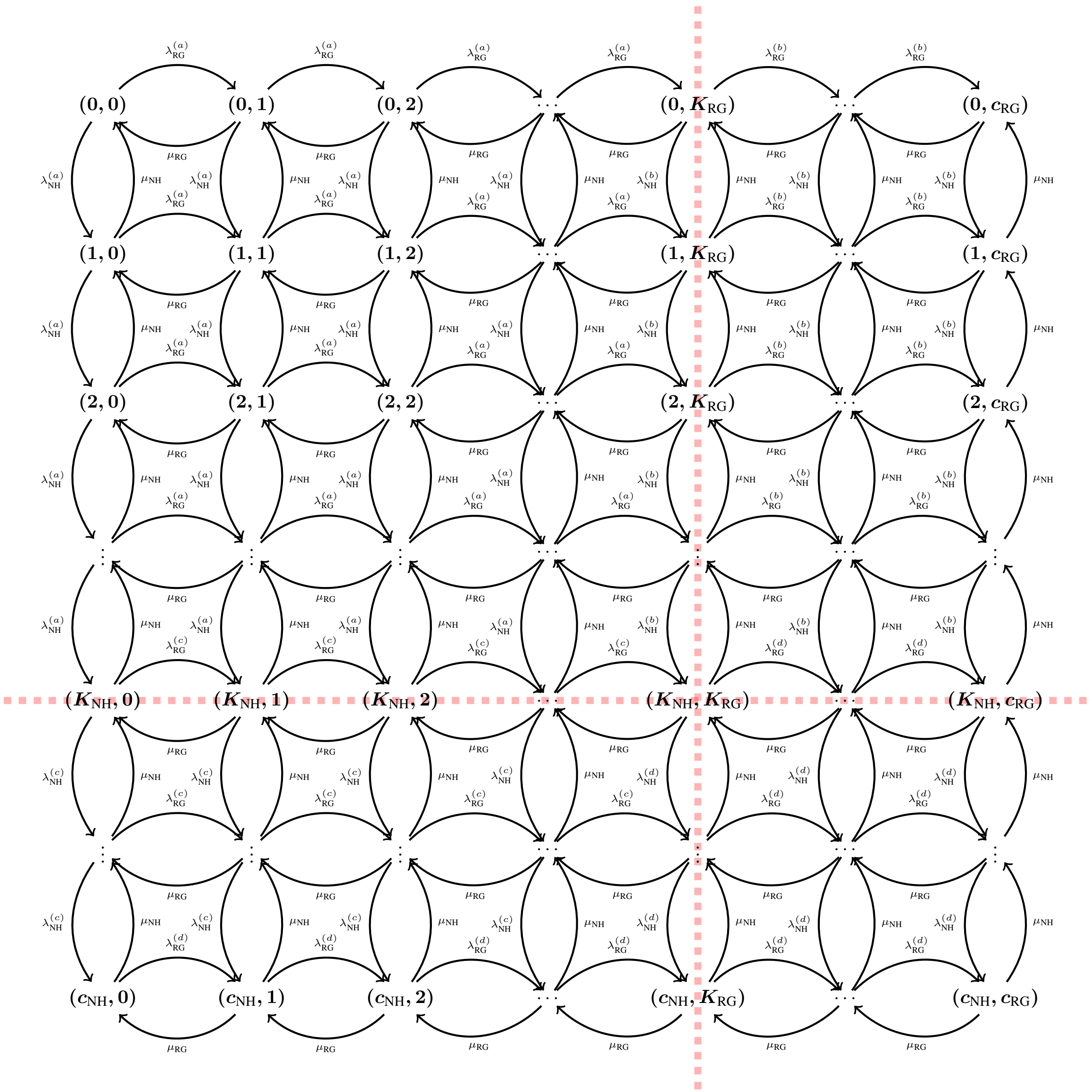

Figura 3: Generic Markov chain underpinning the queueing model of this paper

$$T_h = \mu_h \sum_{n=0}^{c_h} n P^{(h)}(n)$$

where $P^{(h)} = P^{(h)}(K_{\text{NH}}, K_{\text{RG}})$ is the steady state probability distribution function (obtained from the corresponding transition matrix $Q = Q(K_{\text{NH}}, K_{\text{RG}})$) for $h \in \{\text{NH}, \text{RG}\}$.

For $c_{\text{NH}} = 8, c_{\text{RG}} = 16, \lambda_{\text{NH}} = (\lambda_{\text{NH}}^{(a)}, \lambda_{\text{NH}}^{(b)}, \lambda_{\text{NH}}^{(c)}, \lambda_{\text{NH}}^{(d)}) = (1.5, 3.74, 0, 0), \lambda_{\text{RG}} = (\lambda_{\text{RG}}^{(a)}, \lambda_{\text{RG}}^{(b)}, \lambda_{\text{RG}}^{(c)}, \lambda_{\text{RG}}^{(d)}) = (2.24, 0, 3.74, 0)$ and $(K_{\text{NH}}, K_{\text{RG}}) = (6, 12)$, the steady state probabilities for each CCU are given in Figure **??**.

For all $h \in \{\text{NH}, \text{RG}\}$ minimise:

$$(U_h - t)^2$$

Subject to:

$$\begin{aligned} 0 \leq & K_h \leq c_h \\ & K_h \in \mathbb{Z} \end{aligned}$$

Figura 4: The optimisation problem underlying the game

This game is equivalent to a bi matrix game with restriction to pure strategies where both players aim to get their utilisation as close as possible to a certain target. As such a Nash Equilibrium is not guaranteed by traditional game theoretical results [**?**], but based on discussions with ABUHB, long term threshold policies are a realistic consideration.

The following result is a sufficient condition for the existence of an equilibrium:

Theorem.

Let $f_h(k) : [1, c_{\bar{h}}] \to [1, c_h]$ be the best response of player $h \in \{\text{NH}, \text{RG}\}$ to the diversion threshold of $\bar{h} \neq h$ ($\bar{h} \in \{\text{NH}, \text{RG}\}$).

If $f_h(k)$ is a non-decreasing function in k then the game of Figure 4 has at least one Nash Equilibrium.

proof

The function f_h is well defined as it maximises a continuous function over a finite discrete set (in case of multiple values that minimize U_h, it is assumed that f_h returns the lowest such value).

As such if f_h is non-decreasing then it is in fact a stepwise non-decreasing function. If we consider f_{NH} and f_{RG} plotted on the same axis (so that the domain of f_{NH} is the x-axis and the domain of f_{RG} is the y-axis) it is obvious to see that the functions must intersect at some point as shown in Figure 5.

This point of intersection corresponds to a Nash Equilibrium of Figure 4. $\square$

4. EU and EMV Interface

Describe simple Markov model for hospital. Describe routing game model for thing.

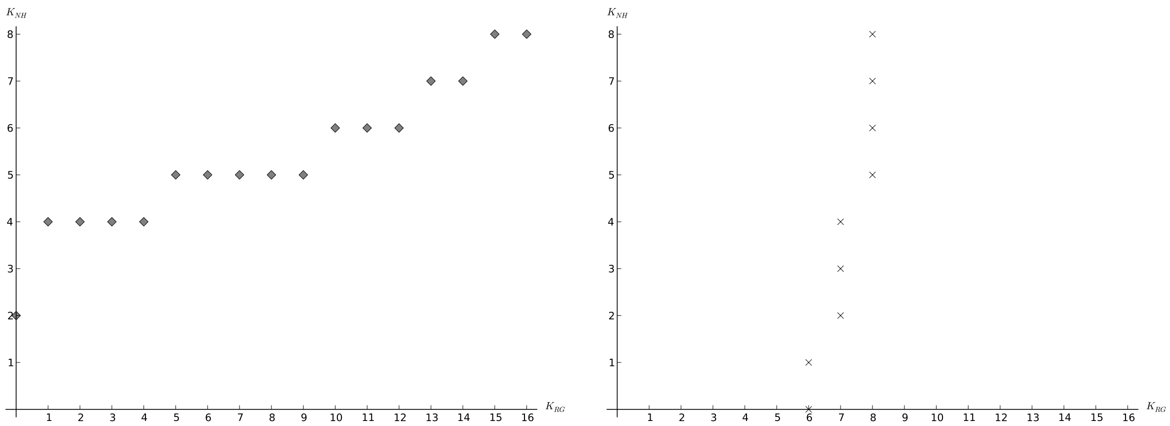

Figura 5: Plots of $f_{\mathrm{NH}}(K_{\mathrm{RG}})$ and $f_{\mathrm{RG}}(K_{\mathrm{NH}})$

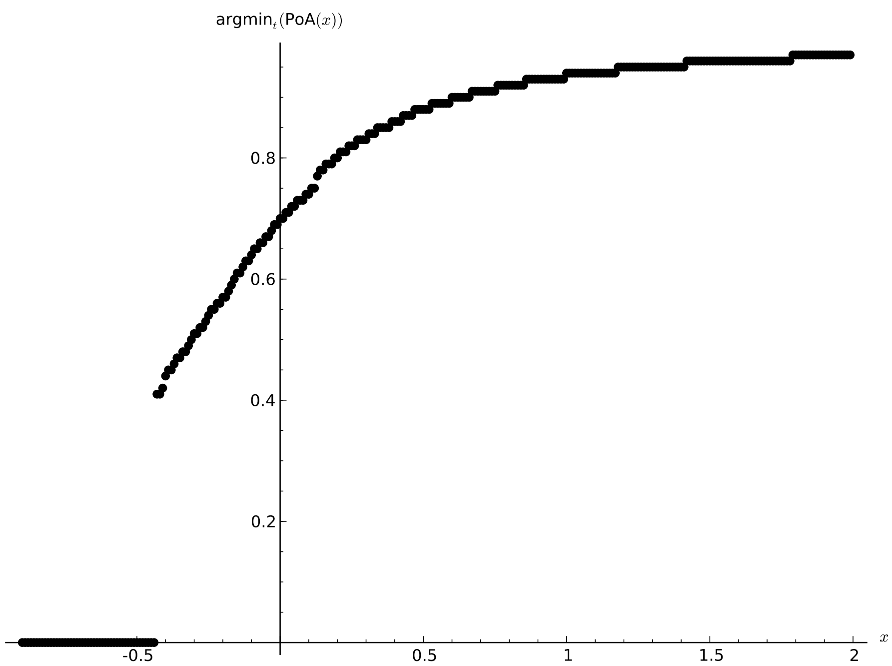

Figura 6: Lowest value of t ensuring PoA$= 1$

5. Conclusions

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