

The effect of homophily in hierarchical systems

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1 Mathematical formulation of model

- Given a hierarchical system with K levels.
- Level $0 \leq k < K$ has capacity C_k
- The first level ($k = 0$) has the most capacity and capacity is monotonically decreasing: $C_0 > C_1 > \dots > C_{K-1}$.
- There are 2 types of agents: $j \in \{0, 1\}$.

Consider a state space \mathcal{S} :

$$\mathcal{S} = \left\{ s \in \mathbb{Z}_{\geq 0}^{K \times 2} \left| \begin{array}{l} s_{i0} + s_{i1} \leq C_i \text{ for all } 0 \leq i \leq K-1 \\ s_{K-1,0} + s_{K-1,1} = C_{K-1} \\ \sum_{i=0}^{K-1} s_{i0} + s_{i1} \in \left\{ \sum_{i=0}^K C_i, \sum_{i=0}^K C_i - 1 \right\} \end{array} \right. \right\} \quad (1)$$

Where s_{ij} denotes the number of individuals of type j at level i .
For example,

- Let $K = 3$
- Let $C = (5, 3, 2)$

Then:

$$s = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$$

corresponds to a system with 3 agents of first type and 1 of second type at the first level, 2 of first type and 1 of second type at the second level and 1 of each type at the 3rd level.

The constraints on \mathcal{S} ensure that either all positions are filled or a single position is available. Thus at any stage either all spots are full and someone will retire or there will be a spot available and someone will be hired/promoted.

The size of the state space is then given by:

$$|S| = (C_{K-1} + 1) \cdot \prod_{i=0}^{K-2} (2C_i + 1) \quad (2)$$

Given two elements $s^{(1)}, s^{(2)} \in \mathcal{S}$ the transition rates are given by:

$$Q_{s_1, s_2} = \begin{cases} \mu_{ij}, & \text{if } s^{(2)} - s^{(1)} = -e_{ij} \text{ and } s_{i0}^{(1)} + s_{i1}^{(1)} = C_i \text{ for all } i \\ \max(rS_{ij} + S_{i,\bar{j}}, 1), & \text{if } s^{(2)} - s^{(1)} = e_{ij} - e_{i-1,j} \text{ and } s_{i0}^{(1)} + s_{i1}^{(1)} < C_i \text{ and } i > 0 \\ \lambda_j, & \text{if } s^{(2)} - s^{(1)} = e_{0j} \text{ and } s_{00}^{(1)} + s_{01}^{(1)} = C_0 - 1 \end{cases} \quad (3)$$

Where:

- μ_{ij} is the retirement rate of agents of type j at level i .
- $r > 1$ is a constant that reflects the homophily effect.
- λ_j is the hiring rate of individuals of type j .