The effect of homophily in hierarchical systems

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1 Mathematical formulation of base model

- Given a hierarchical system with K levels.
- Level $0 \leq k < K$ has capacity C_k
- The first level ($k = 0$) has the most capacity and capacity is monotonically decreasing: $C_0 \geq C_1 \geq \cdots \geq C_{k-2} \geq C_{k-1} = 1$.
- There are 2 types of agents: $j \in \{0, 1\}$.

Consider a state space $\mathcal{S}$:

$$\mathcal{S} = \left\{ s \in \mathbb{Z}_{\geq 0}^{K \times 2} \;\middle|\; \begin{array}{l} s_{i0} + s_{i1} \leq C_i \text{ for all } 0 \leq i \leq K-1 \\ s_{K-1} = (1, 0) \\ \sum_{i=0}^{K-1} s_{i0} + s_{i1} \in \left\{ \sum_{i=0}^{K-1} C_i, \sum_{i=0}^{K-1} C_i - 1 \right\} \end{array} \right\} \tag{1}$$

Where s_{ij} denotes the number of individuals of type j at level i.

For example,

- Let $K = 3$
- Let $C = (4, 3, 1)$

Then:

$$s = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

corresponds to a system with 3 agents of first type and 1 of second type at the first level, 2 of first type and 1 of second type at the second level and 1 of each type at the 3rd level.

The constraints on $\mathcal{S}$ ensure that either all positions are filled or a single position is available. Thus at any stage either all spots are full and someone will retire or there will be a spot available and someone will be hired/promoted.

The size of the state space is then given by:

$$|S| = \prod_{i=0}^{K-2} (2C_i + 1) \tag{2}$$

Given two elements $s^{(1)}, s^{(2)} \in \mathcal{S}$ the transition rates are given by:

$$Q_{s_1,s_2} = \begin{cases} \mu_{ij}, & \text{if } s^{(2)} - s^{(1)} = -e_{ij} \text{ and } s^{(1)}_{i0} + s^{(1)}_{i1} = C_i \text{ for all } i \\ \max(rs_{i+1,j} + s_{i+1,\bar{j}}, 1), & \text{if } s^{(2)} - s^{(1)} = e_{ij} - e_{i-1,j} \text{ and } s^{(1)}_{i0} + s^{(1)}_{i1} < C_i \text{ and } i > 0 \\ \lambda_j, & \text{if } s^{(2)} - s^{(1)} = e_{0j} \text{ and } s^{(1)}_{00} + s^{(1)}_{01} = C_0 - 1 \end{cases} \tag{3}$$

Where:

- μ_{ij} is the retirement rate of agents of type j at level i.
- $r > 1$ is a constant that reflects the homophily effect.
- λ_j is the hiring rate of individuals of type j.

2 Mathematical formulation of model with competence

The base model described in the previous section does not take into account the competence of the individuals. Let's consider the above example of:

- $K = 3$ and $C = (4, 3, 1)$

A possible state s for the above configuration when competence is taken into account is:

$$s = \begin{pmatrix} (1, 0.1) & (1, 0.2) & (1, 0.3) & (0, 0.5) \\ (1, 0.2) & (1, 0.3) & (0, 0.8) & \\ (1, 0.7) & & & \end{pmatrix}$$

Note that this is a possible state for the configurations of K and C but not the only one. The competence of each individual can be any real number between $[0, 1]$, thus there are an infinite number of possible states.

The set of possible states, denoted as S, is given by,

$$\mathcal{S} = \left\{ s \in (\{0,1\}, \mathbb{R}^{[0,1]})^{K \times C} \;\middle|\; \begin{array}{l} s_{i,j} = (l_{i,j}, c_{i,j}) \quad \forall i \in [0, K-1] \quad \forall j \in [C_i - 1, C_i] \\ \text{ where } l_{i,j} \in \{0,1\} \text{ and } c_{i,j} \in \mathbb{R}^{[0,1]} \\ |s_{k-1}| = 1 \\ l_{k-1,1} = 0 \end{array} \right\} \tag{4}$$

where $s_{i,j}$ is a tuple of $(l_{i,j}, c_{i,j})$ where $l_{i,j}$ is the type of individual j at level i and $c_{i,j}$ is the individual's competence.

Promotion in the competence model assumes that individuals of the same type give a random bonus to the competence of individuals like them. Thus for a state s where a promotion is happening at level i, the promotion probability of an individual j at level $i-1$ is given by:

$$P(i,j) = \frac{\sum_{t=1}^{C_{i-1}-1} \max((1 - |l_{i-i,t} - l_{i,j}|)\gamma_t, 1)\ c_{i,j}}{\sum_{\bar{j}=1}^{C_i} \sum_{t=1}^{C_{i-1}-1} \max((1 - |l_{i-i,t} - l_{i,j}|)\gamma_t, 1)\ c_{i,\bar{j}}}, \tag{5}$$

where $\gamma_t > 1$ is randomly sampled at each t from $[0, \Gamma]$.

Thus, the transition rates between two states are given by:

$$\begin{cases} \mu_{ij}, & |s_i| = C_i \text{ for all } i \\ P(i,j), & |s_i| < C_i \text{ and } i > 0 \\ \lambda_{(l,c)}, & |s_0| = C_0 - 1 \end{cases} \tag{6}$$

Where:

- μ_{ij} is the retirement rate of the agent j at level i.
- $\lambda_{(l,c)}$ is the hiring rate of an individual l that has a competence c.

The competence of the system is the total competence of the individuals in the system regarding their type.

$$\sum_{i=0}^{K-1} \sum_{j=1}^{C_i} c_{i,j} \tag{7}$$