Evolutionarily Stable Strategies and Replicator Dynamics in Asymmetric Two-Population Games

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2 authors:

Accinelli Elvio
Universidad Autónoma de San Luis Potosí

Edgar J. Sanchez Carrera
Università degli Studi di Urbino "Carlo Bo"

Chapter 1
Evolutionarily stable strategies and replicator dynamics in asymmetric two-population games

Elvio Accinelli and Edgar J. Sánchez Carrera

Abstract We analyze the main dynamical properties of the evolutionarily stable strategy ($\mathscr{ESS}$) for asymmetric two-population games of finite size and its corresponding replicator dynamics. We introduce a definition of $\mathscr{ESS}$ for two-population asymmetric games and a method of symmetrizing such an asymmetric game. We show that every strategy profile of the asymmetric game corresponds to a strategy in the symmetric game, and that every Nash equilibrium ($\mathscr{NE}$) of the asymmetric game corresponds to a (symmetric) $\mathscr{NE}$ of the symmetric version game. We study the (standard) replicator dynamics for the asymmetric game and we define the corresponding (non-standard) dynamics of the symmetric game. We claim that the relationship between $\mathscr{NE}$, $\mathscr{ESS}$ and the stationary states ($\mathscr{SS}$) of the dynamical system for the asymmetric game can be studied by analyzing the dynamics of the symmetric game.

1.1 Introduction

Evolutionary dynamics originally appeared in biology and then started to be used in economics. Evolutionary stability, introduced by Maynard Smith and Price (1973), is a criterion for the robustness of an incumbent strategy against the entry of individuals or mutants using a different strategy. The framework considered is a conflict within a homogenous population. This game is symmetric since all players have the same strategy set and the payoff for a given strategy depends only on the strategies being played and not on who is playing them.

Elvio Accinelli
Facultad de Economía, UASLP México
e-mail: elvio.accinell@eco.uaslp.mx

Edgar J. Sánchez Carrera
Department of Economics at the University of Siena
e-mail: sanchezcarre@unisi.it

Nevertheless, many economic applications come from multi-population rather than single-population dynamics on asymmetric environments. So, in most applications, the game is not symmetric and involves at least two players with different strategies and each player's role is represented by a different population. In the spirit of Nash's (1950) "mass action interpretation", each type of player is drawn from his or her "player-role population". For instance, the players may play the role of buyers or sellers, incumbents or entrants in oligopolistic markets, workers or firms, or the social relationships between migrants and residents; all of them with non-homogeneous behaviors on the state of the economy and different attitudes towards - and perceptions about - development efforts or environmental quality of the state of the economy and so forth.

Recall that, from the framework of symmetric games, there is a seminal refinement of the Nash equilibrium ($\mathscr{NE}$) concept that is the notion of Evolutionarily Stable Strategy ($\mathscr{ESS}$) (see Maynard Smith and Price (1973) and Maynard Smith (1974)). We know that every $\mathscr{ESS}$ is a stable strategy against mutants, i.e. the strategy is robust when the ***** population is invaded by a small population playing a different strategy. Furthermore, an $\mathscr{ESS}$ is an asymptotically stable steady state of the associated replicator dynamics. The relationship between $\mathscr{NE}$, $\mathscr{ESS}$ and the steady states ($\mathscr{SS}$) of the replicator dynamics are well known (see Weibull (1995)).

In this paper, we consider the evolution of two populations facing a conflictive situation modeled by an asymmetric normal form game. The main purpose of this work is to analyze the evolution and stability of the behaviors of the populations, involved in asymmetric games. Our approach is to symmetrize the asymmetric game because it give us the possibility to characterize the $\mathscr{ESS}$, using the well known properties of these strategies for the case of symmetric games. We introduce an approach to symmetrize a game that differs from the usual ones of symmetrizing a bimatrix game (see Hofbauer and Sigmund (1998) and Cressman (2003)).

We extend the concept of $\mathscr{ESS}$ for asymmetric two-population games, following the definition of Selten (1980) and Samuelson (1998), but in those papers it was not analyzed the evolutionary dynamics of such a population. We exhibit connections between $\mathscr{ESS}$, $\mathscr{NE}$ and $\mathscr{SS}$ for these two dynamics. Close to our argument is the one by Fishman (2008). In particular, we consider the necessity of requiring independence in the invader's frequencies that precludes "symmetrization". By symmetrizing the game, we get the advantage of generalizing the standard definition of $\mathscr{ESS}$ and its relationship with the stability of the dynamical equilibria of the replicator dynamics and with the strategic stability for asymmetric games. We note that, much of the topic of this paper can be generalized for cases of finite $(n > 2)$ asymmetric populations. However, to simplify the notation, we shall consider the case of two asymmetric populations.

Following this approach it is straightforward to see that a strategic profile is an $\mathscr{ESS}$ if and only if it is a strict Nash equilibrium (see Balkenborg and Schlag (1995, 2007); Cressman (1992, 2003, 2006); Samuelson (1998); Selten (1980); Weibull (1995)) and that every $\mathscr{ESS}$ is an asymptotically stable steady state of the replicator dynamics (see Retchkiman (2007); Samuelson and Zhang (1992)).

The paper is organized as follows. Section 1.2 draws the notation and basic definitions to set up the baseline model, namely a two-player asymmetric normal-form game. Section 1.3 defines the $\mathscr{ESS}$ for our model. In section 1.4, we introduce the symmetric version of an asymmetric two population game. Section 1.5 studies the dynamics of our model. Section 1.6 states the relationships between $\mathscr{ESS}$, $\mathscr{NE}$ and $\mathscr{SS}$. Section 1.7 draws some concluding remarks.

1.2 The model

Consider a normal-form (strategic) game with a player set composed by individuals that comprise τ populations, namely residents R and migrants M i.e. $\tau = \{R, M\}$. Each population splits in different clubs denoted by n_i^τ with $i \in \{1, ..., k_\tau\}$, i.e. $(n_1^R, ..., n_{k_R}^R)$ and $(n_1^M, ..., n_{k_M}^M)$. The split depends on the strategy agents play or the behavior that agents follow. Strategies are in correspondence with the clubs. Individuals belonging to the n_i^τ club are called $i-$strategists. Thus, the set S^τ of pure strategies are $S^R = \left\{n_1^R, ..., n_{k_R}^R\right\}$ and $S^M = \left\{n_1^M, ..., n_{k_M}^M\right\}$. For each population $\tau \in \{M, R\}$ we represent the set of mixed strategies by

$$\Delta^\tau = \left\{ x \in R^{k_\tau} : \sum_{j=1}^{k_\tau} x_j = 1,\ x_j \geq 0,\ j = 1, ..., n_i \right\}$$

A profile distribution $x = (x_1, ..., x_{k_\tau}) \in \Delta^\tau$ can bee seen as the individual behavior of a player spending a part of his time x_j in the $n_j^\tau-$club. Hence, the population state represents the vector of individuals' share belonging to each club $i \in \{1, ..., k_\tau\}$, for all $\tau \in \{R, M\}$. The normal form representation of our described game is given by the next matrix payoff

$R \diagdown M$	y_1	$\cdots$	y_{k_M}
x_1	a_{11}, b_{11}	$\cdots$	a_{1k_M}, b_{1k_M}
$\vdots$	$\vdots$	$\ldots$	$\vdots$
x_{k_R}	$a_{k_R 1}, b_{k_R 1}$	$\cdots$	$a_{k_R k_M}, b_{k_R k_M}$

(1.1)

where a_{ij} denotes the payoff of an $i-$strategist from population R playing against a $j-$strategist from population M. Similarly, we define b_{ij} by replacing M by R and vice-versa.

The matching between individuals from different populations is random. The $i-$strategist's expected payoff, supposing that the $i-$strategist's belongs to the n_i^R-club from population R is given by

$$E^R(n_i^R \mid y) = \sum_{j=1}^{k_M} a_{ij} y_j,\ \forall\ n_i^R \in S^R$$

where x is the clubs' distribution for the other population M.

Similarly, the expected payoff of the $i-$strategist belonging to n_i^M-club from population M is given by

$$E^M(n_i^M/x) = \sum_{j=1}^{k_R} b_{ij}x_j,\ \forall n_i^M \in S^M$$

where x is the clubs' distribution for the other population R. Rational individuals follow the strategic profile that maximizes their expected payoffs.

1.3 The asymmetric game and the definition of $\mathscr{E}\mathscr{S}\mathscr{S}$

Consider the two-population normal form game

$$G = \left\{(\tau = \{R,M\}),\, S^\tau,\ (A = (a_{ij}),\, B = (b_{ij}))\right\} \tag{1.2}$$

where each population splits into clubs denoted by n_i^τ with $i \in \{1,...,k_\tau\}$ and $\tau = \{R,M\}$. Hence:

- The population of residents is the set: $R = \bigcup_{i=1}^{k_R} n_i^R$, and $\forall\ h \neq j\ n_h^R \bigcap n_j^R = \emptyset$.
- The population of migrants is the set: $M = \bigcup_{i=1}^{k_M} n_i^M$, and $\forall\ h \neq j\ n_h^M \bigcap n_j^M = \emptyset$.

Let $p \in \Delta^R$ be the profile distribution of individuals' behavior from population R and let $q \in \Delta^M$ be the profile distribution of individuals' behavior in population M is at time t_0.

Let us postulate that an invasion occurs like a post-entry population at a post-period of time $t_1 > t_0$, by a small number of individuals of both types associated with an alternative strategy profile $(\bar{q},\bar{p})$. The profile distribution from population R after suffering a small mutation is

$$q_\varepsilon = (1-\varepsilon)q + \varepsilon\bar{q},$$

which is called the fitness of the post-entry population in M. Similarly, the profile distribution from population R after suffering a small mutation is

$$p_\varepsilon = (1-\varepsilon)p + \varepsilon\bar{p}.$$

Definition 1. Let $(p^*,q^*) \in \Delta^R \times \Delta^M$ be a profile of mixed strategies. We say that the profile (p^*,q^*) is an $\mathscr{E}\mathscr{S}\mathscr{S}$ for an asymmetric two-population normal form game G, if for each pair $(\bar{p},\bar{q}) \neq (p^*,q^*) \in \Delta^R \times \Delta^M$ there exists $\bar{\varepsilon}$ such that:

$$\begin{aligned} &1)\ E^R(p^*/q^*_\varepsilon) > E^R(\bar{p}/q^*_\varepsilon) \\ &2)\ E^M(q^*/p^*_\varepsilon) > E^M(\bar{q}/p^*_\varepsilon), \end{aligned} \tag{1.3}$$

for all ε, $0 < \varepsilon \leq \bar{\varepsilon}$, where $p^*_\varepsilon = (1-\varepsilon)p^* + \varepsilon\bar{p}$ and $q^*_\varepsilon = (1-\varepsilon)q^* + \varepsilon\bar{q}$ are the respective post-entry populations.

Hence, individuals' behavior who adopt an $\mathscr{ESS}$ brings more offspring (with higher fitness) than the mutant individuals' behavior from the post-entry population. It has already been noticed by Selten (1980) that an evolutionary stable strategy pair is not only stable when mutants appear in one of the populations but also if mutants appear in both populations. Definition 1 can be extended to the case of multipopulation models.

The following Theorem characterizes the $\mathscr{ESS}$ in terms of Nash equilibria (see, for instance, Swinkels J., 1992).

Proposition 1. *A profile x is $\mathscr{ESS}$ if and only if x is a strict Nash equilibrium.*

The evolutive properties of the $\mathscr{ESS}$ and its relationship with the set of Nash equilibria and the stationary states ($\mathscr{SS}$) of the replicator dynamics for the case of symmetric games are well known (see Hofbauer and Sigmund (1998); Weibull (1995)). Then, with the purpose of analyzing the dynamical properties of $\mathscr{ESS}$, we introduce the symmetric (one-population) version of the asymmetric two-population game G.

1.4 The symmetrized game

Consider the asymmetric two-population normal form game G (see (1.2)), where each population splits into clubs $n_1^R, ..., n_{k_R}^R$ and $n_1^M, ..., n_{k_M}^M$ and the payoff matrixes are A and B, respectively. Now, instead of pairwise matching, we consider the case that all players are interacting together, i.e. all players are "playing the field". Thus, the payoff of a player is determined by his own strategy and the strategies of all other players. So, the corresponding symmetrized one-population game is defined by as follows:

Let G be an asymmetric game defined by (1.2). Let $P = R \cup M$ be the big population. Let $N = \left\{ n_1^R, ..., n_{k_R}^R, n_1^M, ..., m_{k_M}^M \right\}$ be the set of pure strategy for P. The matrix payoff for the big population P is given by:

$$\Pi = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix} \tag{1.4}$$

where we assume that the elements of $A(\cdot)$ and $B(\cdot)$ are "well behaved" in the sense of being continuously differentiable.

The symmetrized game version of the asymmetric game G is $G^s = \{P, N, \Pi\}$. For each asymmetric two-population game G, there exists a corresponding symmetric version G^s. It is worth to note that, these two versions are not equivalent in several aspects but every Nash equilibrium of the asymmetric game is a Nash equilibrium of its symmetric version. Our purpose is to characterize the main dynamics properties of the $\mathscr{ESS}$.

Let us consider the strategic profile $(p, q) \in \Delta^R \times \Delta^M$ and the profile distribution $x = (x_1, ..., x_{k_R+k_M})$ verifying the following identities:

$$x_i = \begin{cases} p_i \frac{|R|}{|R|+|M|} \text{ if } 1 \le i \le k_R \\ \\ q_i \frac{|M|}{|R|+|M|} \text{ if } k_R < i \le k_R + k_M \end{cases} \tag{1.5}$$

where $|\cdot|$ denotes the cardinality on the sets R and M defining the corresponding mixed strategy for the symmetric version G^s.

Proposition 2. *For each strategic profile $(p, q) \in \Delta^R \times \Delta^M$, there exists a mixed strategy $x \in \Delta^P$ of the corresponding one-population game, and vice-versa.*

Proof. Let $(p, q) \in \Delta^R \times \Delta^M$ be a strategic profile for the asymmetric game. Consider $x \in \Delta^P$ given by the expression (1.5), i.e. $x = \left(\frac{|R|}{|M|+|R|} p, \frac{|M|}{|M|+|R|} q\right)$. Thus, x is a mixed strategy for the symmetric game. To see the reciprocal, suppose that $x \in \Delta^P$. Since $x_i = \frac{|R_i|}{|R|+|M|}$ if $1 \le i \le k_R$ and $x_i = \frac{|M_i|}{|R|+|M|}$ if $k_R < i \le k_R + k_M$, we get $p_i = \frac{|R|+|M|}{|R|} x_i$, and $q_i = \frac{|R|+|M|}{|M|} x_i$.

Let us denote by $B_\tau(z)$ the set of best replies for the population $\tau = \{M, R\}$, where the profile distribution over the clubs in the opposite population $\tau' \neq \tau$ is given by z.

The following propositions offer an insight about the relationship between the set of $\mathscr{N}\mathscr{E}$ and the set of $\mathscr{E}\mathscr{S}\mathscr{S}$ for asymmetric games and their respective symmetric versions.

Proposition 3. *If the strategic profile (p^*, q^*) is a $\mathscr{N}\mathscr{E}$ of the original asymmetric two-population game, then the corresponding x^* defined by the expression (1.5) is the symmetric $\mathscr{N}\mathscr{E}$ in the corresponding symmetric version.*

Proof. Suppose that the profile (p^*, q^*) is a $\mathscr{N}\mathscr{E}$ of the asymmetric two-population game. Let $x^* = (x_1^*, ..., x_{k_M+k_R}^*)$ be the corresponding strategy in the corresponding symmetrized one-population game. Then, $p^* \in B_R(q^*)$ and $q^* \in B_M(p^*)$ implies that $x^* P x^* \ge y P x^*$, for all $y \in \Delta^P$ because

$$yPx^* = \frac{|M||R|}{(|M|+|R|)^2} \left(qB^T p^* + p^* A q\right) \le \frac{|M||R|}{(|M|+|R|)^2} \left(q^* B^T p^* + p^* A q^*\right) = x^* P x^*.$$

Proposition 4. *If the profile (p^*, q^*) is a strict Nash equilibrium for the asymmetric two population game, then the corresponding x^* is a strict Nash equilibrium for the symmetric version.*

Proof. Let (p^*, q^*) be a strict Nash equilibrium for the asymmetric two population game and let x^* be the corresponding profile for the symmetric version. Assume that there exist $y \neq x^* \in \Delta^p$, such that $y\Pi x^* = x^* \Pi x^*$. Using Proposition 2, there exist

$p \neq p^*$ such that $pAq^* \geq p^*Aq^*$ or, there exist $q \neq q^*$ such that $p^*Bq \geq p^*Bq$, which is in contradiction with our assumption.

Proposition 5. *If the profile (p^*, q^*) is an $\mathscr{ESS}$ for the asymmetric two-population game, then the corresponding x^* is an $\mathscr{ESS}$ for the symmetric version.*

Proof. Let (p^*, q^*) be an $\mathscr{ESS}$. By Proposition 1, (p^*, q^*) is a strict Nash equilibrium. From Proposition 4, the corresponding strategy x^* is a strict Nash equilibrium for the symmetric version and it is straightforward to see that the reciprocal of this Proposition does not hold.

1.5 The dynamics of the model

The symmetric version of the asymmetric game allows us to characterize the main dynamical properties of the asymmetric game, because these properties are well known in the symmetric case.

Consider the asymmetric two-population normal form game G (see 1.2).

Let $n_i^\tau(t)$ be the number of individuals at time t belonging to the $i-$club in the population τ. Let $p_i(t)$ be the share of individuals in the $i-$club from the population R and, similarly, let $q_i(t)$ the share of individuals in the $i-$club from the population M, at time t. Hence,

$$p_i(t) = \frac{n_i^R}{|R|}$$

and

$$q_i(t) = \frac{n_i^M}{|M|.}$$

The vector $(p(t), q(t))$ is the profile distribution (or population state) at time t. Furthermore, $p(t) \in \Delta^R$ and $q(t) \in \Delta^M$.

Recall that the members of the $i-$ club from population τ are called $i-$strategists from the population $\tau \in \{R, M\}$. Rational individuals choose strategies to maximize their expected payoffs. Let $z_0 = (p_0, q_0)$ be the strategic profile at time $t = 0$ for the asymmetric two-population game G. According to the rationality assumption, we define:

$$\begin{aligned} \dot{p}_i &= ((e_i^R - p)Aq)p_i, \; i = 1, ..., k_R \\ \dot{q}_i &= ((e_i^M - q)B^T p))q_i, \; i = 1, ..., k_M, \end{aligned} \tag{1.6}$$

where e_i^R is the $i-$canonical vector in $\mathbb{R}^R$ and e_i^M is the canonical $i-$th vector in the $\mathbb{R}^M$. The differential equation (1.6) represents the clubs' evolution for each population. For the system (1.6), a solution of the form $\xi(t, z_0) = (\xi_1(t, z_0), \xi_2(t, z_0))$ represents the evolution of the population states with initial state given by z_0.

From system (1.6), each time t the club of the $i-$strategists in each population increases if and only if the expected payoff of the $i-$strategy is greater than the average payoff, and reciprocally.

For each pair $(p(t), q(t))$ in G, there exists a corresponding mixed strategy $x(t)$ in the symmetric version G^s given by the expression (1.5).

The dynamical system (1.6) has a corresponding dynamical system, namely the replicator dynamics, (see Taylor and Jonker, 1978) of the symmetric one-population game given by

$$\dot{x}_i = ((e_i - x)Px)x_i \tag{1.7}$$

where $x = (x_1, ..., x_{k_R+k_M})$, x_i is given by the expression (1.5), and e_i is the $i-$ canonical vector in $\mathbb{R}^{k_R+k_M}$.

We analyse the relationship between $\mathcal{NE}$, $\mathcal{ESS}$ and $\mathcal{SS}$ of the system (1.6) of the symmetric version game G^s.

If a pair $(\bar{p}, \bar{q})$ is a stationary state of the system (1.6) then the corresponding $\bar{x}$ is a stationary state for the dynamical system (1.7). Furthermore, every strictly positive stationary state of the dynamical system (1.6) is a $\mathcal{NE}$ for the corresponding asymmetric two-population game. Every $\mathcal{NE}$ of an asymmetric two-population game is a stationary state for its corresponding dynamical system given by (1.6). Hence, we can conclude that the set of $\mathcal{NE}$ of an asymmetric two-population game is a subset of the set $\mathcal{SS}$ corresponding to the dynamical system (1.6). Every $\mathcal{NE}$ of a two-population game is a stationary state for the corresponding dynamical system (1.7).

1.6 Evolutionarily stable strategies and Liapunov's stability

Denote by $\mathcal{AS}$ the set of asymptotically stable steady states. From the well known relations between $\mathcal{ESS}$, $\mathcal{NE}$ and $\mathcal{SS}$ for the symmetric cases (see Weibull (1995)), the following relationship holds for every asymmetric two-population game

$$\mathcal{ESS} \subseteq \mathcal{AS}, \tag{1.8}$$

and

$$\mathcal{NE} \subseteq \mathcal{SS}. \tag{1.9}$$

Proposition 6. *For an asymmetric two-population game, if (p^*, q^*) is an asymptotically stable steady state of the dynamical system (1.6), then (p^*, q^*) is a $\mathcal{NE}$.*

Proof. If $(p^*, q^*) \in AS$ for the dynamical system (1.6) then it is stationary state. If $p^* > 0$ and $q^* > 0$ then (p^*, q^*) is a $\mathcal{NE}$ for the asymmetric game. Now we consider the case where some strategy is absent in p^* or in q^*. Without loss of generality we assume that $p^*_j = 0$. Suppose that (p^*, q^*) is not a $\mathcal{NE}$. Then, there exists some pure strategy $j \notin supp(p^*)$ such that $E^R(e^R_j/q^*) = e^R_j A q^* > p^* A q^* = E^R(p^*/q^*)$. Assume that a perturbation affects the distribution p^* and that in the population R some $j-$strategist appears. The post-entry population at time t, is $p_\varepsilon(t) = (1 - \varepsilon(t))p^* + \varepsilon(t)e^R_j$. Substituting in the $j-$ differential equation of (1.6), we obtain

$$\dot{p}_{\varepsilon j} = \dot{\varepsilon} = [(e_j^R - p_\varepsilon)Aq^*]\varepsilon. \tag{1.10}$$

Define $F(\varepsilon) = (e_j^R - p_\varepsilon)Aq^*$. Note that $F(0) = (e_j^R - p^*)Aq^*$ and $F'(0) = (p^* - e_j^R)Aq^*$. The Taylor polynomial is $F(\varepsilon) = F(0) + F'(0)\varepsilon + 0(\varepsilon^2)$. Considering the first order approximation equation (1.10) gives

$$\dot{\varepsilon} = [(e_j^R - p^*)Aq^*]\varepsilon.$$

In the population R, the members in the n_j^R club increase, contradicting our claim that (p^*, q^*) is an asymptotically stable steady state with $n_j^R = 0$.

We now study the connection between $\mathscr{ESS}$ and the replicator dynamics in an asymmetric game. We will use the following Proposition (see Taylor and Jonker 1978).

Proposition 7. *For symmetric homogeneous population game every $\mathscr{ESS}$ is an asymptotically stable steady state of the replicator dynamics.*

Theorem 1. *For the asymmetric two-population game, we obtain the following chain of inclusions:*

$$\mathscr{ESS} \subseteq \mathscr{AS} \subseteq \mathscr{NE} \subseteq \mathscr{SS}.$$

Proof. Let (p^*, q^*) be an $\mathscr{ESS}$ for an asymmetric game and let x^* be the corresponding strategic profile in its symmetric version. So, from Proposition 1, it follows that (p^*, q^*) is a strict Nash equilibrium. By Proposition 4, it follows that the symmetric strategic profile of every strict Nash equilibrium of an asymmetric game is an strict $\mathscr{NE}$. Then x^* is a strict Nash equilibrium for the symmetric version, and then x^* is a $\mathscr{ESS}$. By Proposition 7, it follows that x^* is an asymptotically stable steady state of the replicator dynamics. Then, $(p^* q^*)$ is an asymptotically stable steady state for the asymmetric version and is a $\mathscr{NE}$.

Bomze, I. (1986) shows that every asymptotically stable steady state in the homogeneous population replicator dynamics corresponds to a Nash equilibrium that is trembling hand. However, using the symmetric version of a non-homogeneous asymmetric $n-$population the following Proposition holds:

Corollary 1. *Every $\mathscr{ESS}$ of a non-homogeneous asymmetric $n-$population game is trembling hand and isolate.*

Proof. Let (p^*, q^*) be an $\mathscr{ESS}$ for an asymmetric game and let x^* be the corresponding strategic profile in its symmetric version. By Theorem 1, it follows that every $\mathscr{ESS}$ is asymptotically stable for the symmetric version. Hence, x^* is asymptotically stable steady state for the symmetric version. By Bomze (1986), it follows that x^* is trembling hand and isolate equilibrium, and so (p^*, q^*) verifies this property in the original asymmetric game.

1.7 Concluding remarks

We extended the definition of evolutionarily stable strategies ($\mathscr{ESS}$) of symmetric games to asymmetric two-population games. We did it by taking as the strategy space for the symmetrized game the union of strategies from the two-population asymmetric game and assigning zero payoffs to all strategy combinations that belong to the same player position in the asymmetric game. Hence, evolutionary dynamics in a two-population asymmetric game can be analyzed using the well known properties of the replicator dynamics corresponding to the symmetric version of this game. This fact may have interest for economic theory and social analysis, where asymmetric games are useful to analyze the behavior of two populations engaged in non-cooperative games such as, buyers and suppliers, firms and workers or residents and migrant populations interacting in a given country or economy.

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