Markov Chains Sage Sheet Solutions

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1. Discrete Markov Chains
   Navigate to the Sage code snippet at: `https://gist.github.com/drvinceknight/7b479fd1cd0868e4590c` and start a new Sage worksheet so that you can try a few of your own Sage commands.

   (a) Using the default inputs, what is the steady state distribution associated with this Markov chain (try and use the Sage "solve" command to verify this)?
   ANSWER:
   The steady state distribution is given by:

   $$\pi = \left(\frac{1}{3}, \frac{4}{9}, \frac{2}{9}\right)$$

   We can obtain this using the following Sage code:

```
var("pi1,pi2,pi3")
solve([.8*pi1+.1*pi2+.1*pi3= =pi1,.2*pi1+.7*pi2+.3*pi3= =   pi2,
            0*pi1+.2*pi2+.6*pi3= = pi3,pi1+pi2+pi3= = 1],[pi1,pi2,pi3])
```

   we can make things a bit more "generic" as follows:

```
A=matrix([[.8,.2,0],[.1,.7,.2],[.1,.3,.6]])
var("pi1,pi2,pi3")
solve([A[0,0]*pi1+A[1,0]*pi2+A[2,0]*pi3==pi1,
       A[0,1]*pi1+A[1,1]*pi2+A[2,1]*pi3==pi2,
       A[0,2]*pi1+A[1,2]*pi2+A[2,2]*pi3==pi3,
       pi1+pi2+pi3==1],[pi1,pi2,pi3])
```

   (b) How long does it seem to take to arrive at that state (try and use Sage to verify this)?
   ANSWER:
   It seems to take about 15 time steps. We can check this by trying the following code with various values of n:

```
n=11
A=matrix([[.8,.2,0],[.1,.7,.2],[.1,.3,.6]])
pi_0=vector([.2,.5,.3])
pi_0*A^n
```

2. Continuous Markov Chains
   The evolution of a Continuous Markov Chain obeys the following differential equation:

   $$\frac{d\pi(t)}{dt} = \pi(t)Q$$

   which has solution:

   $$\pi(t) = \pi(0)e^{Qt}$$

This requires the calculation of the exponential of a Matrix. This is not straightforward! One approach is to use the following formula:

$$e^M = \mathbb{I} + \sum_{k=1}^{\infty} \frac{M^k}{k!}$$

Navigate to the Sage code snippet at `http://interact.sagemath.org/node/71` and experiment with the code to see how good this approach is.

ANSWER:
We see that the default inputs require 21 terms to give a good approximation (to 4 decimal places). Playing around with different inputs should give different results.

(a) Consider the following Markov Chain:

$$Q = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 4 & -7 & 3 & 0 \\ 0 & 4 & -7 & 3 \\ 0 & 0 & 4 & -4 \end{pmatrix}$$

(b) Using the Sage "solve" command to obtain the steady state probabilities for this Markov Chain.

ANSWER:
The steady state probabilities are given by:

$$\pi = (.3657, .2743, .2057, .1543)$$

This can be obtained using the following code:

```
Q=matrix([[-3,3,0,0],[4,-7,3,0],[0,4,-7,3],[0,0,4,-4]])
var("pi1,pi2,pi3,pi4")
solve([Q[0,0]*pi1+Q[1,0]*pi2+Q[2,0]*pi3+Q[3,0]*pi4==0,
       Q[0,1]*pi1+Q[1,1]*pi2+Q[2,1]*pi3+Q[3,1]*pi4==0,
       Q[0,2]*pi1+Q[1,2]*pi2+Q[2,2]*pi3+Q[3,2]*pi4==0,
       Q[0,3]*pi1+Q[1,3]*pi2+Q[2,3]*pi3+Q[3,3]*pi4==0,
       pi1+pi2+pi3+pi4==1],[pi1,pi2,pi3,pi4])
```