Utility Theory

Ref: Chapter 16.1-3

Rational Decisions

- We have looked at a number of ways for agents to make rational decisions.
  - logic
  - probabilistic reasoning
  - statistical reasoning
- We can generalize *decision making* to the process of applying an Agent's preferences.
- We can generalize the notion of *preferences* to a utility function over all states
  - $U(s)$ assigns a number to each state, indicating the desirability of the state.

Expected Utility

- Each action *A* will have some set of possible outcomes.
  - the Agent has some estimate of the probability of each outcome.
- The agent can compute the *expected utility* of each action:

$$EU(A|E) = \sum_i P(Result_i(A)|Do(A), E)\, U(Result_i(A))$$

- E is everything the Agent knows (evidence).
- $Result_i$ is a state resulting from an action $Do(A)$

Maximum Expected Utility

- The general strategy of a rational agent is to pick the action that will maximize the expected utility of the resulting state.
  - our prior studies have shown the difficulty in making accurate predictions of *expected utility*.
- We have distilled decision making to the simple act of picking actions based on the utility (a simple number) assigned to the outcome.
  - and the probability of each outcome.

So What?

- In many cases we can't simplify things so that we have just a simple function that returns a number (the utility function).
  - so why does thinking about this help?
- It turns out that any agent that is acting rationally can be represented by a utility function.
  - our goal is not to find the function, but to study the implications of the existence of such a function.

Notation

$A \succ B$	A preferred to B
$A \sim B$	indifference between A and B
$A \succsim B$	B not preferred to A

- If actions are deterministic, A and B are specific outcome states.
- nondeterministic actions: A and B are *lotteries.*
  - probability distribution over set of possible outcomes states.

Lottery

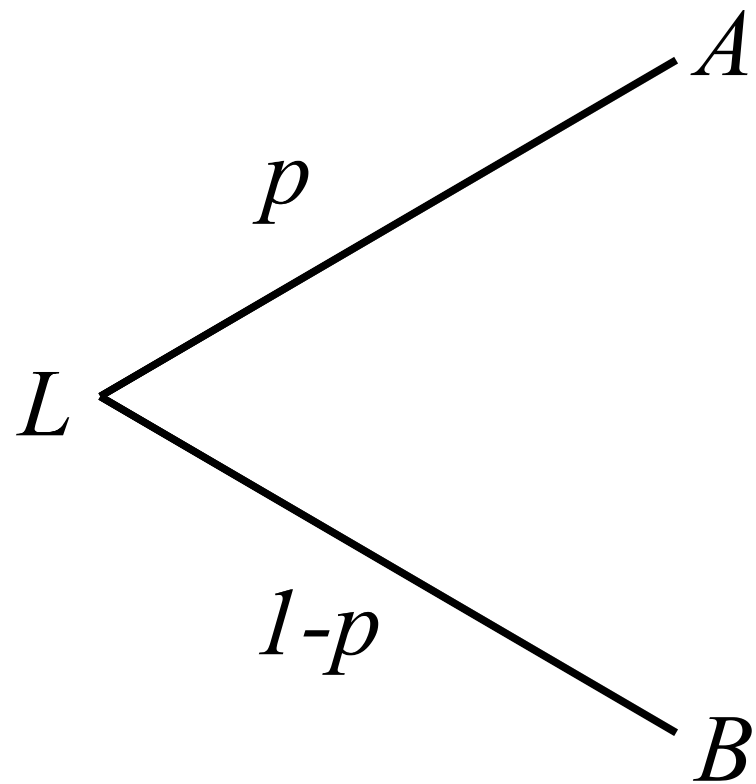
- A Lottery L with outcomes C_1, ... C_n that can occur with probabilites p_1, ... p_n:

$$L = [p_{1,} C_1 ; p_{2,} C_2 ; \ldots p_n , C_n]$$

- Each outcome C_i can be a state or another lottery.

Simple Lottery (2 outcome)

$$L = [\, p\,, A\,; (1 - p)\,, B\,]$$



Axioms of Utility Theory

- There are some commonsense constraints on the preferences and lotteries.
- These constraints represent (or attempt to represent) what we typically mean by "rational behavior".
- The idea here is to try to define what "rational behavior" is, and then immediately see how hard this can be.

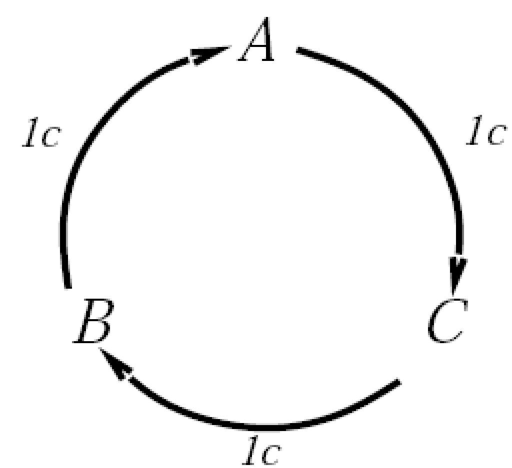
Transitivity

- Given any three states A,B and C, if the agent prefers A over B, and B over C, it must also prefer A over C.
- The following system is not transitive:

If $B \succ C$, then an agent who has C would pay (say) 1 cent to get B

If $A \succ B$, then an agent who has B would pay (say) 1 cent to get A

If $C \succ A$, then an agent who has A would pay (say) 1 cent to get C



Axioms

- **Orderability**: There is some ordering to outcomes.

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

- **Continuity:** In some state B between A and C in preference, there is some lottery over A and C that has the same preference as B

$$A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1-p, C] \sim B$$

More Axioms

- **Substitutability**

$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

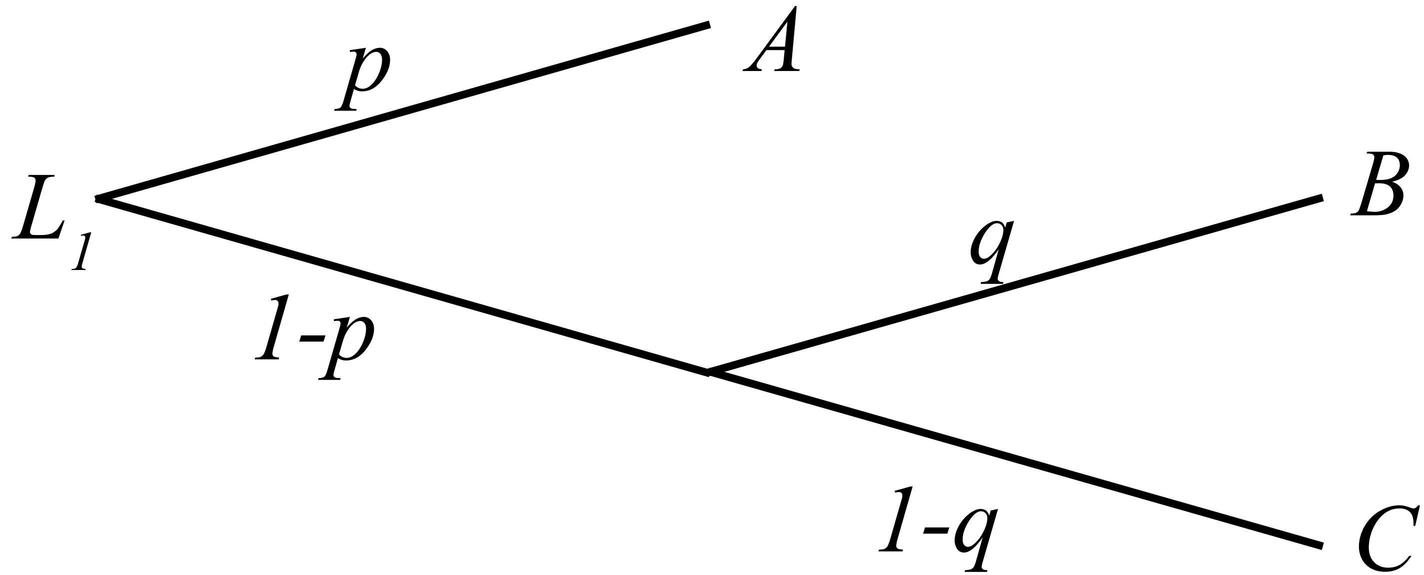
- **Monotonicity**

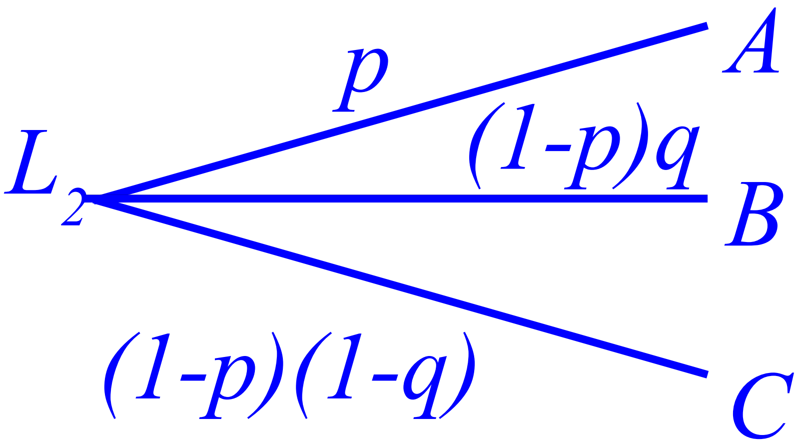
$$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succsim [q, A;\ 1-q, B])$$

- **Decomposability**
  - compound lotteries can always be combined into simpler ones.

$$L_1 = [p, A; (1-p), [q, B; 1-q, C]]$$

Decomposability





$$L_1 \sim L_2$$

$$L_2 = [p, A; (1-p)q, B; (1-p)(1-q), C]$$

Rational vs. Irrational

- An agent that violates any of the Axioms is not acting rationally.

- Important Note: The axioms do not mention the utility function!
  - They define rationality by placing constraints on preferences.
  - The assumption is that all agents have some mechanism for computing/acting on preferences.

Utility principle

- If an agent's preferences follow the Axioms, there exists a real valued function U that operates on states (a utility function):

$$U(A) \geq U(B) \quad \Leftrightarrow \quad A \succsim B$$

This means that regardless of how the agent operates (whether it explicitly uses a utility function or not), it's behavior can be represented by a utility function.

Maximum Expected Utility principle

- The utility of a lottery is the sum of the probability of each outcome times the utility of the outcome:

$$U([p_1, S_1; \; \ldots \; ; \; p_n, S_n]) = \sum_i p_i U(S_i)$$

So What? (again!)

- Knowing that any rational agent can be described by a utility function doesn't necessarily help design an agent.
- There are some situations where it does help.
- Consider an agent in a deterministic environment that does use an explicit utility function:
  - it doesn't matter what the scale of the function is.
  - we can apply any monotonic transformation and see the same behavior.

Money and utility

- Suppose you are faced with the following situation:

You spend 3 days/nights waiting in line to buy a PS3.

A: I will give you $3000 for the PS3 right now.

B: Or, you can sell your PS3 on ebay:

- with probability 80% you will get $4000 for it.
- with probability 20% you will get $0.

What Should You Do?

Another Scenario

I will give you one of two lottery tickets:

- **A:** 20% chance of winning $4000
- **B:** 25% chance of winning $3000

Expected Payoffs
(Expected Monetary Value)

- PS3:

  A: $3000

  B: (0.2 *$0) + (0.8 * $4000) = $3200

- Lottery ticket

  A: (0.8 * $0) + (0.2*$4000) = $800

  B: (0.75 * $0) + (0.25*$3000) = $750

If we assume $ = Utility, do humans act rationally?

A Clearer Example

- You can have \$1,000,000 or we can flip a coin: heads you get \$3,000,000, tails you get nothing.
- What would you choose?
- What would Donald Trump choose?

$$EU(gamble) = \tfrac{1}{2}U(S_k) + \tfrac{1}{2}U(S_k + \$3{,}000{,}000)$$

$$EU(takethemoney) = U(S_k + \$1{,}000{,}000)$$

Money is Funny

- Economists (who created utility theory) have come up with the following:
  - The utility of money is proportional to the log of the amount of money!
- The issue seems to be:

  risk-seeking vs. risk-adverse

- This all comes back to the Exploration/Exploitation issue...

Utility Scale

- The actual scale of a utility function is irrelevant.
- Any linear transformation will not change the agent's behavior:

$$U'(S) = k_1 + k_2 U(S)$$

- If actions are deterministic, any monotonic transformation will leave the behavior unchanged.
  - all that matters is the ordering of outcomes.

Utility Scales

- Normalized Utilities:
  - utility of 0 is "*worst possible catastrophe*"
  - utility of 1 is "*best possible outcome*"

- Medical and safety analysis
  - Micromort: one in a million chance of death.
  - QALY: Quality Adjusted Life Year