Queueing Theory Exercise Sheet Solutions

1. Fill in the gaps in the following table:

Statistic	Notation	$M/M/1$	$M/M/2$	$M/M/k$
Number of people in queue	L_q	$\frac{\rho^2}{1-\rho}$	$\frac{2\rho^3}{1-\rho^2}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2}$
Number of people in system	L_c	$\frac{\rho}{1-\rho}$	$\frac{2\rho}{1-\rho^2}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2}+\frac{\lambda}{\mu}$
Average waiting time in queue	W_q	$\frac{\rho}{\mu(1-\rho)}$	$\frac{\rho^2}{\mu(1-\rho^2)}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2\mu}$
Average time in system	W_c	$\frac{1}{\mu(1-\rho)}$	$\frac{1}{\mu(1-\rho^2)}$	$\frac{\left(\frac{\lambda}{\mu}\right)^{k}\pi_0}{kk!\left(1-\frac{\lambda}{k\mu}\right)^2\mu}+\frac{1}{\mu}$

2.
   - FIFO:

     $$\begin{aligned}\text{Total waiting time} &= 0+1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+(n-1))\\ &= \textstyle\sum_{k=1}^{n-1}\sum_{j=0}^{k} j = \sum_{k=1}^{n-1}\frac{k(k+1)}{2}\\ &= \tfrac{1}{2}\left(\textstyle\sum_{k=1}^{n-1}k^2+\sum_{k=1}^{n-1}k\right)\\ &= \tfrac{1}{2}\left(\frac{(n-1)n(2n-1)}{6}+\frac{n(n-1)}{2}\right)\\ &= \tfrac{1}{2}\left(\frac{(n-1)n(2n+2)}{6}\right) = \frac{(n-1)n(n+1)}{6}\end{aligned}$$

     However a total of n customers are served thus:

     $$W_q = \frac{(n-1)(n+1)}{6} = \frac{n-1^2}{6}$$

     as required.

   - LIFO

     $$\begin{aligned}\text{Total waiting time} &= 0+n+(n+(n-1))+\cdots+(n+\cdots+2)\\ &= \textstyle\sum_{k=0}^{n-2}\sum_{j=0}^{k}(n-j) = \sum_{k=0}^{n-2}\sum_{j=n-k}^{n} j = \sum_{k=0}^{n-2}\left(\sum_{j=0}^{n} j - \sum_{j=0}^{n-k-1} j\right)\\ &= \textstyle\sum_{k=0}^{n-2}\left(\frac{n(n+1)}{2}-\frac{(k-n)(1+k-n)}{2}\right) = \sum_{k=0}^{n-2}\frac{(k+1)(2n-k)}{2}\\ &= \tfrac{1}{2}\left(-\textstyle\sum_{k=0}^{n-2}k^2+(2n-1)\sum_{k=0}^{n-2}k+\sum_{k=0}^{n-2}2n\right)\\ &= \tfrac{1}{2}\left(-\frac{(n-2)(n-1)(2n-3)}{6}+\frac{(n-2)(n-1)(2n-1)}{2}+2n(n-1)\right) = \frac{(n-1)n(n+1)}{3}\end{aligned}$$

     However a total of n customers are served thus:

     $$W_q = \frac{(n-1)(n+1)}{3} = \frac{n-1^2}{3}$$

     as required.