Game Theory

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Overview

Normal Form Games

Game Theory: Introduction

Often decision analysis does not only depend on chance but on the decisions made by others: <u>interactive decision problems</u>.

Such decision problems are called <u>games</u>. The individuals making the decisions are called <u>players</u>.

2 Player Static Games

2 Player Static Games

We shall consider 2 player static games. Assume two players have two sets of available strategies: $S_1 = \{r_1, \ldots, r_m\}$ and $S_2 = \{s_1, \ldots, s_n\}$. Let $u_1(r, s)$, $u_2(r, s)$ be the utility gained by player 1 and 2 for a pair of strategies (s, r).

	s_1	s_2	$\ldots$	s_n
r_1	(u_1, u_2)	(u_1, u_2)	$\ldots$	(u_1, u_2)
r_2	(u_1, u_2)	(u_1, u_2)	$\ldots$	(u_1, u_2)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
r_m	(u_1, u_2)	(u_1, u_2)	$\ldots$	(u_1, u_2)

Both players aim to choose from their available strategies so as to maximise u_1 and u_2.

Example: Prisoner's Dilemma

Two criminal suspects have been caught. They have been isolated and are being questioned separately by the police. The following offer is made to both suspects:

- If one confesses that they both committed the crime then the confessor will be set free and the other will spend 5 years in jail.
- If both confess, then they will each get a 4 year sentence.
- If neither confess, then they will each spend 2 years in jail.

Example: Prisoner's Dilemma

Both players have 2 possible strategies:

- Keep quite (Q)
- Squeal (S)

	Q	S
Q	(-2,-2)	(-5,0)
S	(0,-5)	(-4,-4)

The "solution" of the game is (S, S). Both criminals squeal and go to prison for 4 years (Instead of 2).

Solving games using Dominance

We solved the prisoners' dilemma in an intuitively simple manner by observing the strategy S was always "better" then Q. We attempt to solve games by eliminating poor strategies for each player.

- A strategy for player 1, r_i is, strictly dominated by r_j if

$$u_1(r_i, s) < u_1(r_j, s) \text{ for all } s \in S_2$$

- A strategy for player 1, r_i is, weakly dominated by r_j if

$$u_1(r_i, s) \leq u_1(r_j, s) \text{ for all } s \in S_2$$

  and there exists a strategy $s_l \in S_2$ such that:

$$u_1(r_i, s_l) < u_1(r_j, s_l)$$

Example

Consider the following game:

	s_1	s_2
r_1	$(3, 3)$	$(2, 2)$
r_2	$(2, 1)$	$(2, 1)$

For player 2, s_1 weakly dominates s_2. For player 1, r_1 weakly dominates r_2. Thus (r_1, s_1) is the "solution" of this game.

Common Knowledge of Rationality

To solve a game by elimination of dominated strategies we have to assume that the players are rational. However, we can go further, if we also assume that:

- The players are rational.
- The players all know that the other players are rational.
- The players all know that the other players know that they are rational.
- ...

This chain of assumptions is called Common Knowledge of Rationality (CKR). By applying the CKR assumption, we can try to solve games by iterating the elimination of dominated strategies.

Example

	s_1	s_2	s_3
r_1	$(1, 0)$	$(1, 2)$	$(0, 1)$
r_2	$(0, 3)$	$(0, 1)$	$(2, 0)$

Initially player 1 has no dominated strategies. For player 2, s_3 is dominated by s_2. **Now**, r_2 is dominated by r_1. **Finally**, s_1 is dominated by s_2. Thus (r_1, s_2) is the "solution" of this game.

Pure Nash Equilibrium

(Pure) Nash Equilibrium

Importantly, certain games cannot be solved using the iterated elimination of dominated strategies:

	s_1	s_2	s_3
r_1	$(10, 0)$	$(5, 1)$	$(4, -2)$
r_2	$(10, 1)$	$(5, 0)$	$(1, -1)$

	s_1	s_2	s_3
r_1	$(1, 3)$	$(4, 2)$	$(2, 2)$
r_2	$(4, 0)$	$(0, 3)$	$(4, 1)$
r_3	$(2, 5)$	$(3, 4)$	$(5, 6)$

(exercise: why does iterated elimination fail here?)

Nash Equilibrium

A (pure) Nash equilibrium is a pair of strategies $(\tilde{r}, \tilde{s})$ such that

$$u_1(\tilde{r}, \tilde{s}) \geq u_1(r, \tilde{s}) \text{ for all } r \in S_1$$

and

$$u_2(\tilde{r}, \tilde{s}) \geq u_2(\tilde{r}, s) \text{ for all } s \in S_2$$

Testing for Nash Equilibrium

One can find Nash equilibria by checking all strategy pairs and seeing if either player can improve their outcome.

	s_1	s_2	s_3
r_1	$(10, 0)$	$(5, 1)$	$(4, -2)$
r_2	$(10, 1)$	$(5, 0)$	$(1, -1)$

Nash Equilibria need not be unique!

Best response strategies

A strategy for player 1 r^* is a best response to some fixed strategy for player 2, s if:

$$u_1(r^*, s) \geq u_1(r, s) \text{ for all } r \in S_1$$

A strategy for player 2 s^* is a best response to some fixed strategy for player 1, r if:

$$u_2(r, s^*) \geq u_2(r, s) \text{ for all } s \in S_2$$

To use this definition to find Nash Equilibria we find for each player, the set of best responses to every possible strategy of the other player. We then look for pairs of strategies that are best responses to each other.

Example

	s_1	s_2	s_3
r_1	$(1, 3)$	$(4, 2)$	$(2, 2)$
r_2	$(4, 0)$	$(0, 3)$	$(4, 1)$
r_3	$(2, 5)$	$(3, 4)$	$(5, 6)$

Mixed Nash Equilibrium

Mixed Strategies

Importantly some games do not have **pure** Nash equilibria!
Consider the following game:

Two players each place a coin on a table, either “heads up” (strategy H) or “tails up” (strategy T). If the pennies match, player 1 wins, if the pennies differ, then player 2 wins.

	H	T
H	$(1, -1)$	$(-1, 1)$
T	$(-1, 1)$	$(1, -1)$

Mixed Strategies

In order to solve such games, we need to consider mixed strategies. I.e. we attach a distribution to the set of strategies of each player.

In the matching pennies example, let $\rho = (p, 1 - p)$ be the mixed strategy for player 1. I.e. player 1 plays H with probability p and plays T with probability $1 - p$.

Similarly let $\sigma = (q, 1 - q)$ be the mixed strategy for player 2. I.e. player 2 plays H with probability q and plays T with probability $1 - q$.

Mixed Strategies

Consider the payoff to player 1:

$$\begin{aligned} u_1(\rho, \sigma) &= pq - p(1-q) - (1-p)q + (1-p)(1-q) \\ &= 1 - 2q + 2p(2q-1) \\ &= (2q-1)(2p-1) \end{aligned}$$

- If $q < \frac{1}{2}$ then player 1s best response is to choose $p = 0$ (i.e. always play T).
- If $q > \frac{1}{2}$ then player 1s best response is to choose $p = 1$ (i.e. always play H).
- If $q = \frac{1}{2}$ then player 1s best response is to play any mixed strategy.

Mixed Strategies

Consider the payoff to player 2:

$$
\begin{aligned}
u_2(\rho, \sigma) &= -pq + p(1-q) + (1-p)q - (1-p)(1-q) \\
&= -1 + 2q - 2p(2q-1) \\
&= (2q-1)(1-2p)
\end{aligned}
$$

- If $p < \frac{1}{2}$ then player 2s best response is to choose $q = 1$ (i.e. always play H).
- If $p > \frac{1}{2}$ then player 2s best response is to choose $q = 0$ (i.e. always play T).
- If $p = \frac{1}{2}$ then player 2s best response is to play any mixed strategy.

Mixed Strategies

The only pair of strategies that are best responses to each other is $\rho = \sigma = \left(\frac{1}{2}, \frac{1}{2}\right)$.

This method of finding mixed Nash equilibria is called: the best response method. (Of course it also finds the pure Nash equilibria)

Exercise: Do the same exercise for the popular game "rock,paper scissors".

Example

	s_1	s_2
r_1	$(0,0)$	$(2,1)$
r_2	$(1,2)$	$(0,0)$

As before:

$$u_1(\rho,\sigma) = q + p(2-3q)$$
$$u_2(\rho,\sigma) = p + q(2-3p)$$

Best responses for player 1:

$$\rho^* = \begin{cases} (0,1) & \text{if } q > \frac{2}{3} \\ (1,0) & \text{if } q < \frac{2}{3} \\ (x, 1-x) \text{ with } 0 \leq x \leq 1 & \text{if } q = \frac{2}{3} \end{cases}$$

Example

	s_1	s_2
r_1	$(0,0)$	$(2,1)$
r_2	$(1,2)$	$(0,0)$

As before:

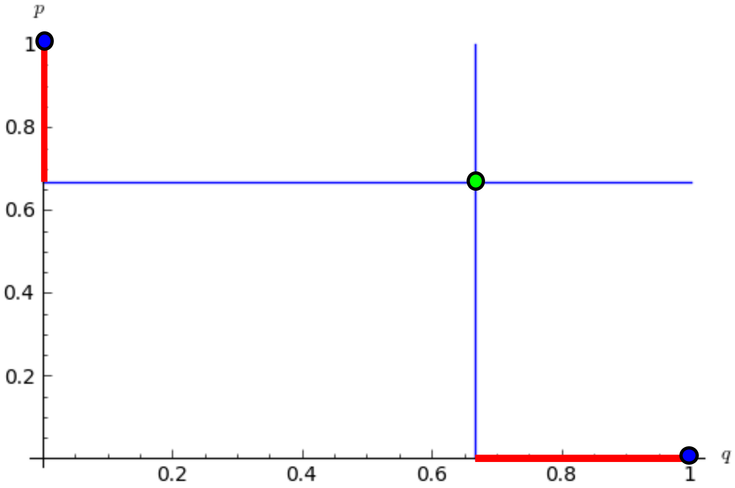
$$u_1(\rho,\sigma) = q + p(2-3q)$$
$$u_2(\rho,\sigma) = p + q(2-3p)$$

Best responses for player 2:

$$\sigma^* = \begin{cases} (0,1) & \text{if } p > \frac{2}{3} \\ (1,0) & \text{if } p < \frac{2}{3} \\ (y, 1-y) \text{ with } 0 \leq y \leq 1 & \text{if } p = \frac{2}{3} \end{cases}$$

Example

We plot both best responses:



Example

Thus for this example there are 3 Nash equilibria:

$$(r_1, s_2),\ (r_2, s_1) \text{ and } (\rho, \sigma) \text{ with } \rho = \sigma = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Equality of Payoffs

The support of a strategy ρ is the set $S(\rho)$ of all strategies for which ρ has non zero probability.

For example, if the strategy set is $\{A, B, C\}$ then the support of the mixed strategy $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$ is $\{A, B\}$. Similarly the support of the mixed strategy $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is $\{A, C\}$.

This leads to a very powerful result.

Equality of Payoffs Theorem

Let (ρ, σ) be a Nash equilibrium, and let S_1^* be the support of ρ. Then:

$$u_1(\rho, \sigma) = u_1(r, \sigma) \text{ for all } r \in S_1^*$$

Equality of Payoffs

Consider the matching pennies game. Let σ be the mixed strategy of player 2 with a chance of playing H of q and a chance of playing T with probability $(1-q)$. From the Equality of Payoffs theorem we have:

$$u_1(H,\sigma) = u_1(T,\sigma)$$
$$qu_1(H,H) + (1-q)u_1(H,T) = qu_1(T,H) + (1-q)u_1(T,T)$$
$$q - (1-q) = -q + (1-q)$$
$$q = \tfrac{1}{2}$$

Equality of Payoffs

Let ρ be the mixed strategy of player 1 with a chance of playing H of p and a chance of playing T with probability $1-p$.From the Equality of Payoffs theorem we also have:

$$u_2(\rho, H) = u_2(\rho, T)$$
$$pu_2(H, H) + (1-p)u_2(T, H) = pu_2(H, T) + (1-p)u_2(T, T)$$
$$-p + (1-p) = p - (1-p)$$
$$p = \frac{1}{2}$$

As expected.

Nash's Theorem

Every game that has a finite set of strategies has at least one Nash equilibrium (involving pure or mixed strategies).

(It can be shown that there is always an odd number of Nash equilibria.)