Game Theory Exercise Sheet SOLUTIONS

This sheet was last updated on October 15, 2012.

1.

	s_1	s_2	s_3
r_1	$(\underline{6}, \underline{3})$	$(2, 2)$	$(2, 2)$
r_2	$(4, 0)$	$(0, 3)$	$(\underline{4}, \underline{5})$
r_3	$(2, 3)$	$(\underline{3}, \underline{4})$	$(3, 2)$

(a)

	s_1	s_2
r_1	$(\underline{7}, -2)$	$(\underline{4}, \underline{0})$
r_2	$(1, -5)$	$(0, \underline{-4})$
r_3	$(4, \underline{-1})$	$(3, -5)$
r_4	$(6, -7)$	$(\underline{4}, \underline{-5})$

(b)

	s_1	s_2	s_3
r_1	$(160, \underline{2})$	$(\underline{205}, \underline{2})$	$(44, \underline{2})$
r_2	$(175, 1)$	$(180, .5)$	$(45, \underline{5})$
r_3	$(\underline{201}, 3)$	$(204, 4)$	$(\underline{50}, \underline{10})$
r_4	$(120, 4)$	$(107, \underline{6})$	$(49, 2)$

(c)

	s_1	s_2	s_3
r_1	$(0, 0)$	$(-1, 1)$	$(1, -1)$
r_2	$(1, -1)$	$(0, 0)$	$(-1, 1)$
r_3	$(-1, 1)$	$(1, -1)$	$(0, 0)$

(d)

Since the number of Nash Equilibria for any given game is odd, we expect to not have identified all equilibria for $(b), (c)$ and (d).

2. The bi-matrix representation is given by:

	100	99	98	...	3	2
100	$(100, 100)$	$(97, 101)$	$(96, 100)$	...	$(1, 5)$	$(0, 4)$
99	$(101, 97)$	$(99, 99)$	$(96, 100)$	...	$(1, 5)$	$(0, 4)$
98	$(100, 96)$	$(100, 96)$	$(98, 98)$	...	$(1, 5)$	$(0, 4)$
$\vdots$	...	...	...	$\ddots$	$\vdots$	$\vdots$
3	$(5, 1)$	$(5, 1)$	$(5, 1)$	...	$(3, 3)$	$(0, 4)$
2	$(4, 0)$	$(4, 0)$	$(4, 0)$	...	$(4, 0)$	$(2, 2)$

This game is immediate to solve with dominance and so the Nash equilibrium is $(2, 2)$.

3. We have the bi-matrix game representation:

	R	P	S
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

There is no pure Nash equilibrium and it is immediate to see that no mixed strategy will have support of size 2. Indeed, assume that a mixed strategy for player 1 does not play “scissors”. Player 2 would have an immediate benefit of playing the pure strategy “paper” (as he’ll never lose). This can be shown mathematically.

Thus the mixed strategy for player 1, ρ, will be of the form:

$$\rho = (p, q, 1 - p - q)$$

The mixed strategy for player 2, σ, will be of the form:

$$\sigma = (u, v, 1 - u - v)$$

Using the equality of payoffs theorem, we have:

$$u_1(R, \sigma) = u_1(S, \sigma) = u_1(T, \sigma) \tag{1}$$

and

$$u_2(\rho, R) = u_2(\rho, S) = u_2(\rho, T) \tag{2}$$

We have:

$$\begin{aligned} u_1(R, \sigma) &= -v + 1 - u - v \ (a) \\ u_1(P, \sigma) &= u - 1 + u + v \quad (b) \\ u_1(S, \sigma) &= -u + v \qquad\quad (c) \end{aligned} \tag{3}$$

Combining (1) and (3) gives:

$$\begin{aligned} (a) = (b) &\Rightarrow 3u + 3v = 2 \\ (a) = (c) &\Rightarrow 3v = 1 \\ (b) = (c) &\Rightarrow 3u = 1 \end{aligned}$$

Thus $\sigma = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as expected. A similar approach using (3) gives the expected result for ρ.

4. Recall:

	Attack Bomber 1	Attack Bomber 2
Transport with Bomber 1	$(80, -80)$	$(100, -100)$
Transport with Bomber 2	$(100, -100)$	$(60, -60)$

There is clearly no pure Nash equilibria. Let the bombers use bomber 1 with probability p (thus they use bomber 2 with probability $1-p$). We denote the mixed strategy of the bombers by $\rho = \{p, 1 - p\}$. Let the fighter attack bomber 1 with probability q (thus the fighter attacks bomber 2 with probability $1 - q$). We denote the mixed strategy of the fighter by $\sigma = \{q, 1 - q\}$. We could use the equality of payoffs theorem to solve this problem. Let us however, consider a direct approach by looking at best responses:

$$\begin{aligned} u_1(\rho, \sigma) &= 80pq + 100(p(1 - q) + q(1 - p)) + 60(1 - q)(1 - p) \\ &= 20(3 + 2p + 2q - 3pq) \\ &= 20(p(2 - 3q) + 3 + 2q) \end{aligned}$$

We immediately see that:

- If $q < \frac{2}{3}$ then player 1s best response is to choose $p = 1$.
- If $q > \frac{2}{3}$ then player 1s best response is to choose $p = 0$.
- If $q = \frac{2}{3}$ then player 1s best response is to play any mixed strategy.

Similarly we have:

$$\begin{aligned}
u_2(\rho, \sigma) &= -(80pq + 100(p(1-q) + q(1-p)) + 60(1-q)(1-p)) \\
&= -(20(3 + 2p + 2q - 3pq)) \\
&= 20(q(3p-2) - 3 - 2p)
\end{aligned}$$

and we have:

- If $p < \frac{2}{3}$ then player 1s best response is to choose $q = 0$.
- If $p > \frac{2}{3}$ then player 1s best response is to choose $q = 1$.
- If $p = \frac{2}{3}$ then player 1s best response is to play any mixed strategy.

The only strategies that are best responses to each other is $\rho = \sigma = \left(\frac{2}{3}, \frac{1}{3}\right)$.

5. Using the equality of payoffs theorem identify all the Nash equilibria for the following games: (a)

	s_1	s_2
r_1	$(0,0)$	$(2,1)$
r_2	$(1,2)$	$(0,0)$

The pure Nash equilibria are given by (r_2, s_1) and (r_1, s_2). Consider the mixed strategies $\rho = (p, 1-p)$ and $\sigma = (q, 1-q)$. By the equality of payoff theorem we have:

$$u_1(r_1, \sigma) = u_1(r_2, \sigma)$$

and

$$u_2(\rho, s_1) = u_2(\rho, s_2)$$

The first equation is equivalent to:

$$2(1-q) = q$$

which gives $q = \frac{2}{3}$. Similarly we get $p = \frac{2}{3}$. Thus $\rho = \sigma = \left(\frac{2}{3}, \frac{1}{3}\right)$.

(b)

	s_1	s_2
r_1	$(3,3)$	$(3,2)$
r_2	$(2,2)$	$(5,6)$
r_3	$(0,3)$	$(6,1)$

The pure Nash equilibria is (r_1, s_1). Consider the mixed strategies $\rho = (p, q, 1-p-q)$ and $\sigma = (u, 1-u)$. The difficult part of this problem is to identify the various different supports that ρ may have (it is obvious that the size of the support of σ is 2). Let us first consider supports of size 2:

- Assume that the support of ρ is $\{r_1, r_2\}$:
  Using the equality of payoffs theorem we have:

  $$u_1(r_1, \sigma) = u_1(r_2, \sigma)$$

  and

  $$u_2(\rho, s_1) = u_2(\rho, s_2)$$

  this gives:

  $$u_1(r_1, \sigma) = u_1(r_2, \sigma) \Rightarrow 3(u + 1 - u) = 2u + 5(1 - u) \Rightarrow u = \frac{2}{3}$$

  and (recalling that in this case we have $\rho = (p, 1 - p, 0)$)

  $$u_2(\rho, s_1) = u_2(\rho, s_2) \Rightarrow 3p + 2(1 - p) = 2p + 6(1 - p) \Rightarrow p = \frac{4}{5}$$

  Thus this support gives the mixed Nash equilibium: $\left(\left\{\frac{4}{5}, \frac{1}{5}, 0\right\}, \left\{\frac{2}{3}, \frac{1}{3}\right\}\right)$

- Assume that the support of ρ is $\{r_2, r_3\}$:
  Using the equality of payoffs theorem we have:

  $$u_1(r_2, \sigma) = u_1(r_3, \sigma)$$

  and

  $$u_2(\rho, s_1) = u_2(\rho, s_2)$$

  this gives:

  $$u_1(r_2, \sigma) = u_1(r_3, \sigma) \Rightarrow 2u + 5(1 - u) = 0u + 6(1 - u) \Rightarrow u = \frac{1}{3}$$

  and (recalling that in this case we have $\rho = (0, q, 1 - q)$)

  $$u_2(\rho, s_1) = u_2(\rho, s_2) \Rightarrow 3q + 3(1 - q) = 6q + (1 - q) \Rightarrow q = \frac{1}{3}$$

  Thus this support gives the mixed Nash equilibium: $\left(\left\{0, \frac{1}{3}, \frac{2}{3}\right\}, \left\{\frac{1}{3}, \frac{2}{3}\right\}\right)$

- Assume that the support of ρ is $\{r_1, r_3\}$:
  Using the equality of payoffs theorem we have:

  $$u_1(r_1, \sigma) = u_1(r_3, \sigma)$$

  and

  $$u_2(\rho, s_1) = u_2(\rho, s_2)$$

  this gives:

  $$u_1(r_1, \sigma) = u_1(r_3, \sigma) \Rightarrow 3u + 3(1 - u) = 0u + 6(1 - u) \Rightarrow u = \frac{1}{2}$$

  and (recalling that in this case we have $\rho = (p, 0, 1 - p)$)

  $$u_2(\rho, s_1) = u_2(\rho, s_2) \Rightarrow 3p + 3(1 - p) = 2p + (1 - p) \Rightarrow p = 2$$

  However, this last value is not consistent with probabilities! Thus, this support does not have a Nash equilibrium.

We are left with having to consider one last support: $\{r_1, r_2, r_3\}$. It should be apparent that this case will simplify to one of the previous cases. Thus, we have found all the Nash equilibria:

$$(r_1, s_1), \quad \left(\left\{\frac{4}{5}, \frac{1}{5}, 0\right\}, \left\{\frac{2}{3}, \frac{1}{3}\right\}\right) \text{ and } \left(\left\{0, \frac{1}{3}, \frac{2}{3}\right\}, \left\{\frac{1}{3}, \frac{2}{3}\right\}\right)$$

6. (a) Assuming "walking in to each other" gives both players a utility of -1 and "avoiding each other" a utility of 1, the bi matrix representation of this game is:

	L	R
L	$(1,1)$	$(-1,-1)$
R	$(-1,-1)$	$(1,1)$

where $L, ; R$ represent the step left and right strategies respectively.

(b) Using best responses we have:

	L	R
L	$(\underline{1},\underline{1})$	$(-1,-1)$
R	$(-1,-1)$	$(\underline{1},\underline{1})$

thus the two pure Nash equilibria are $\{L, L\}$ and $\{R, R\}$.

(c) Assume player 1, plays the mixed strategy $\rho = (p, 1-p)$ and player 2 plays the mixed strategy $\sigma = (q, 1-q)$. By the equality of payoffs theorem we have:

$$\begin{aligned} u_1(L,\sigma) &= u_1(R,\sigma) && \text{and} && u_2(\rho, L) = u_2(\rho, R) \\ q + (1-q)(-1) &= q(-1) + (1-q) && \text{and} && p + (1-p)(-1) = p(-1) + 1 - p \\ q &= \tfrac{1}{2} && \text{and} && p = \tfrac{1}{2} \end{aligned}$$

thus $p = q = \frac{1}{2}$ The mixed Nash equilibria is $\left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$