Proof of optimal values of SSError

We start by obtaining an expression for SSError in terms of α, β and a measured $\|Cx - p^*\|_2^2$ where $x = (\alpha, \beta)$:

$$\|Cx - p^*\|_2^2 = (-P(\alpha + \beta) + R\alpha + R\beta - p_1^*)^2 + (-P(\alpha + \beta) + S\alpha + T\beta - p_2^*)^2 + (-P(\alpha + \beta) + S\beta + T\alpha - p_3^*)^2 \quad (1)$$

In order to find SSError we need to optimise the above in terms of α, β.
Differentiating (1), with respect to α and equating to 0 we obtain following value for α:

$$\alpha = \frac{-3P^2\beta + 2PR\beta + 2PS\beta + 2PT\beta - Pp_1^* - Pp_2^* - Pp_3^* - R^2\beta + Rp_1^* - 2ST\beta + Sp_2^* + Tp_3^*}{3P^2 - 2PR - 2PS - 2PT + R^2 + S^2 + T^2} \quad (2)$$

Differentiating (1), with respect to β and equating to 0, and substituting (2) we obtain the following value for β:

$$\beta = \frac{-3P^2 p_2^* + 3P^2 p_3^* + 2PR p_2^* - 2PR p_3^* - PS p_1^* + PS p_2^* - 3PS p_3^* + PT p_1^* + 3PT p_2^* - PT p_3^* - R^2 p_2^* + R^2 p_3^* + RS p_1^* - RT p_1^* + S^2 p_3^* - ST p_2^* + ST p_3^* - T^2 p_2^*}{(S - T)(6P^2 - 4PR - 4PS - 4PT + 2R^2 + S^2 + 2ST + T^2)} \quad (3)$$

Substituting this back in to (2) we obtain:

$$\alpha = \frac{3P^2 p_2^* - 3P^2 p_3^* - 2PR p_2^* + 2PR p_3^* - PS p_1^* - 3PS p_2^* + PS p_3^* + PT p_1^* - PT p_2^* + 3PT p_3^* + R^2 p_2^* - R^2 p_3^* + RS p_1^* - RT p_1^* + S^2 p_2^* + ST p_2^* - ST p_3^* - T2 p_3^*}{6P^2 S - 6P^2 T - 4PRS + 4PRT - 4PS^2 + 4PT^2 + 2R^2 S - 2R^2 T + S^3 + S^2 T - ST^2 - T^3} \quad (4)$$

Substituting (3-4) in to (1) gives the required expression for SSError.
The condition $\chi > 1$ corresponds to $-\beta > \alpha$:

$$-\beta - \alpha = \frac{2p_1^*(P - R) + (p_2^* + p_3^*)(2P - S - T)}{6P^2 - 4PR - 4PS - 4PT + 2R^2 + S^2 + 2ST + T^2} \quad (5)$$

However, the denominator of (5) is positive as:

$$(6P^2 - 4PR - 4PS - 4PT + 2R^2 + S^2 + 2ST + T^2) = (T - 2P - S - \sqrt{2}i(P - R))(T - 2P - S + \sqrt{2}i(P - R)) \quad (6)$$

so it is a quadratic in T with no real roots and leading coefficient 1. So $\chi > 1$ corresponds to:

$$(2p_1^*(P - R) + (p_2^* + p_3^*)(2P - S - T)) > 0 \quad (7)$$

which gives to the required expression because $2(P - R) < 0$ by the definition of the Prisoner's Dilemma.