Recognising and evaluating the effectiveness of extortion in the Iterated Prisoner's Dilemma

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Abstract

The Iterated Prisoner's Dilemma is a model for rational and evolutionary interactive behaviour. It has applications both in the study of human social behaviour as well as in biology.

This game is used to understand when and how a rational individual might accept an immediate cost to their own utility for the direct benefit of another.

Much attention has been given to a class of strategies for this game, called Zero Determinant strategies. It has been theoretically shown that these strategies can "extort" any player.

In this work, an approach to identify if observed strategies are playing in a Zero Determinant way is described. Furthermore, experimental analysis of a large tournament with 204 strategies is considered. In this setting the most highly performing strategies do not play in a Zero Determinant way. This suggests that whilst the theory of Zero Determinant strategies indicates that memory is not of fundamental importance to the evolution of cooperative behaviour, this is incomplete.

1 Introduction

2 Recognising Extortion

In [2], given a match between 2 memory-one strategies, the concept of Zeto-Determinant (ZD) strategies is introduced. The main result of that paper showed that given two players $p, q \in \mathbb{R}^2$ a linear relationship between the players scores could be forced.

Assuming the utilities for player p are given by $S_x = (R, S, T, P)$ and for player q by $S_y = (R, T, S, P)$ and that the stationary scores of each player is given by S_X and S_Y respectively. The main result of [2] is that if

$$\tilde{p} = \alpha S_x + \beta S_y + \gamma \tag{1}$$

or

$$\tilde{q} = \alpha S_x + \beta S_y + \gamma \tag{2}$$

where $\tilde{p} = (1 - p_1, 1 - p_2, p_3, p_4)$ and $\tilde{q} = (1 - q_1, 1 - q_2, q_3, q_4)$ then:

$$\alpha S_X + \beta S_Y + \gamma = 0 \tag{3}$$

This work is interested with identifying a test for the reverse problem: given a $p \in \mathbb{R}^4$ how does one identify α, β, γ if they exist.

Note that equation 1 can be expressed linear algebraically as:

$$Mx = \tilde{p} \qquad x = (\alpha, \beta, \gamma) \tag{4}$$

with $M \in \mathbb{R}^{4\times 3}$ given by:

$$M = \begin{bmatrix} R & R & 1 \\ S & T & 1 \\ T & S & 1 \\ P & P & 1 \end{bmatrix} \tag{5}$$

Note that in general, equation (4) will not necessarily have a solution. From the Rouch-Capelli theorem []if there is a solution it is unique as $\text{rank}(M) = 3$ which the dimension of the variable x). Furthermore, removing a single row of M would ensure that the corresponding matrix is invertible. This corresponds to the fact that a ZD strategy is defined by only 3 of its values.

As an example, consider the known ZD strategy $p = (8/9, 1/2, 1/3, 0)$ from [3] which is referred to as `Extort-2`. In the standard case of $(R, S, T, P) = (3, 0, 5, 1)$ the inverse of $M_{(4)}$ (removing the last row of M) is given by:

$$M_{(4)}^{-1} = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{2}{5} \\ 1 & -\frac{2}{5} & -\frac{3}{5} \\ -5 & 3 & 3 \end{bmatrix} \tag{6}$$

This allows us to find the $x = (\alpha, \beta, \gamma)$ corresponding to $\tilde{p}$:

$$x = M_{(4)}^{-1}\tilde{p}_{(4)} = \begin{bmatrix} \frac{1}{18} & -\frac{1}{9} & \frac{1}{18} \end{bmatrix} \tag{7}$$

Using (4) gives that these values lead to the correct value for $p_4 = 0$ confirm that p_4 is a ZD strategy.

This approach could in fact be used to confirm that a given strategy p represents is ZD. However, in practice, if a closed form for p is not known, then due to measurement and/or numerical error this would not work.

Thus, an approach based on least squares [1] is proposed. This approach finds the best fitting $\bar{x} = (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ which minimises:

$$R^2 = \|Mx - \tilde{p}\|_2 = \sum_{i=1}^{4} \left((M\bar{x})_i - \tilde{p}_i\right)^2 \tag{8}$$

Note that R^2 which is the square of the Frobenius norm [1] becomes a measure of how close p is to being a ZD strategy.

As an example, Figure 1 shows R^2 for the vector $p = (8/9, p_2, 1/3, 0)$ for varying value of p_2. As expected, the lowest value is obtained for $p_2 = 1/2$.

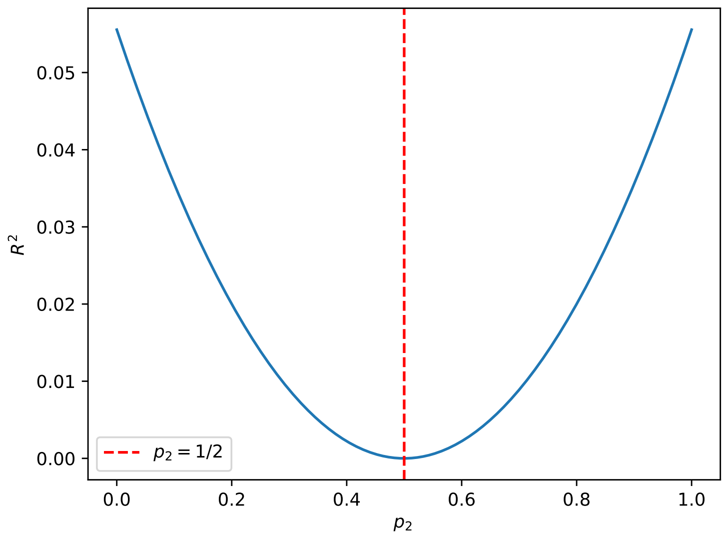

Figure 1: R^2 as defined in (8) for $p = (8/9, p_2, 1/3, 0)$.

In [2] a particular type of ZD is defined, if:

$$P(\alpha + \beta) + \gamma = 0 \tag{9}$$

then the player can ensure they get a score at χ times better than the opponent with. This extortion coefficient is given by:

$$\chi = \frac{-\beta}{\alpha} \tag{10}$$

Thus, if (9) holds and $\chi > 1$ a player is said to extort their opponent. Recalling (5), equation (9) corresponds to the last row of M, thus a player p extorts their opponent if and only if it is ZD and $\chi > 1$. Using the method of least squares, the strategy p is ZD for a given threshold of R^2 and χ can be approximated. Figure 2 illustrates this for four potential strategies: for all of these we see that for the strategy to be extortionate: as p_2 increases p_3 also increases (seemingly linearly).

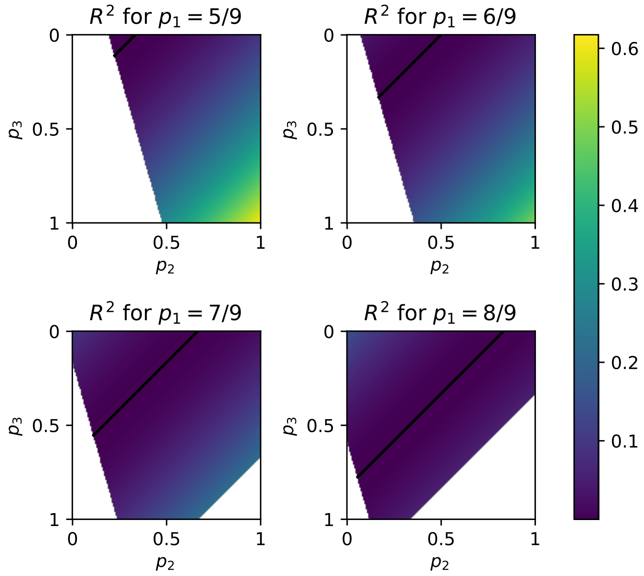

Figure 2: R^2 as a function of p_2, p_3 for strategies of the form: $p = (p_1, p_2, p_3, p_4)$. The solid line shows the values for which $R^2 < 10^{-6}$. Only values for which $\chi > 1$ are displayed, strategies outside of that region cannot be extortionate.

3 Numerical experiments

4 Conclusion

References

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[2] W. H. Press and F. J. Dyson. "Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent". In: *Proceedings of the National Academy of Sciences* 109.26 (May 2012), pp. 10409–10413. DOI: `10.1073/pnas.1206569109`.

[3] A. J. Stewart and J. B. Plotkin. "Extortion and cooperation in the Prisoner's Dilemma". In: *Proceedings of the National Academy of Sciences* 109.26 (June 2012), pp. 10134–10135. DOI: `10.1073/pnas.1208087109`.