

UNIQUENESS OF SELF-SIMILAR SHRINKERS WITH ASYMPTOTICALLY CYLINDRICAL ENDS

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ABSTRACT. In this paper, we show the uniqueness of smooth embedded self-shrinkers asymptotic to infinite order to a generalized cylinder. Also, we construct non-rotationally symmetric self-shrinking ends asymptotic to a generalized cylinder with rate as fast as any given polynomial.

1. INTRODUCTION

A one-parameter family Σ_t of hypersurfaces in \mathbb{R}^{n+1} flows by mean curvature if, for $\mathbf{x} \in \Sigma_t$,

$$(1.1) \quad \partial_t \mathbf{x} = -H \mathbf{n},$$

where \mathbf{n} is the unit normal and H is the mean curvature given by $H = \operatorname{div} \mathbf{n}$. If, furthermore,

$$(1.2) \quad \Sigma_t = \sqrt{-t} \Sigma,$$

then the hypersurface Σ is called a self-shrinker. Thus, Σ satisfies the equation

$$(1.3) \quad H = \frac{1}{2} \langle \mathbf{x}, \mathbf{n} \rangle$$

for $\mathbf{x} \in \Sigma$. Here we use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product. Self-shrinkers play a crucial role in the study of mean curvature flow, since they describe all possible blow ups at a given singularity of mean curvature flow, cf. [14, 15], [17] and [23].

Throughout, B_r^{k+1} is the closed ball in the Euclidean space \mathbb{R}^{k+1} centered at the origin with radius r and $S_r^k \doteq \partial B_r^{k+1}$. Also, for simplicity, we will use S^k to denote the sphere $S_{\sqrt{2k}}^k$ and we call $S^k \times \mathbb{R}^{n-k}$ for $0 < k < n$ generalized cylinders. In this sequel to [22], we show the uniqueness of smooth embedded self-shrinkers asymptotic to infinite order to a generalized cylinder.

Theorem 1.1. *Given $R_0 > 0$ and $0 < k < n$, let Σ be a smooth self-shrinker defined by the normal graph of a C^2 uniformly bounded function u over $S^k \times (\mathbb{R}^{n-k} \setminus B_{R_0}^{n-k})$. Suppose that Σ is asymptotic to infinite order to $S^k \times \mathbb{R}^{n-k}$, that is, for all $l > 0$,*

$$(1.4) \quad \limsup_{r \rightarrow \infty} \|r^l u\|_{L^\infty(S^k \times S_r^{n-k})} = 0.$$

Then $u \equiv 0$ in $S^k \times (\mathbb{R}^{n-k} \setminus B_{R_0}^{n-k})$.

Theorem 1.1 gives a partial affirmative answer to the following well-known conjecture on the rigidity of cylinders:

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Conjecture 1.2 (see page 39 of [16]). *Let $\Sigma \subset \mathbb{R}^3$ be a smooth complete embedded self-shrinker with at most quadratic area growth. If one end of Σ is asymptotic to a cylinder, then Σ is isometric to $S^1 \times \mathbb{R}$.*

On the other hand, we show in section 4 that

Theorem 1.3. *Given $l > 0$, there exist $R > 0$ and a non-rotationally symmetric self-shrinking end which is given by the normal graph of a C^2 uniformly bounded function u over $S^{n-1} \times (R, \infty)$ with (1.4).*

This indicates that, without assuming the completeness of self-shrinkers, the infinite order asymptotic condition in Theorem 1.1 seems to be essential.

Colding-Minicozzi [4] proved that spheres and cylinders are the only generic singularities of mean curvature flow in \mathbb{R}^3 . It is conjectured that a similar classification in \mathbb{R}^{n+1} , $n \leq 6$, in which, modulo isometries, the sphere S^n and the generalized cylinders $S^k \times \mathbb{R}^{n-k}$, $0 < k < n$, are the only generic singularities. In the very recent paper [3], Colding-Ilmanen-Minicozzi showed that any complete smooth embedded self-shrinker in \mathbb{R}^{n+1} with finite entropy which is close to some generalized cylinder $S^k \times \mathbb{R}^{n-k}$ in a sufficiently large ball centered at the origin must itself be $S^k \times \mathbb{R}^{n-k}$. As an application, they proved the uniqueness of cylindrical tangent flows, provided that tangent flows are of multiplicity one. Moreover, under some mild conditions, the asymptotic behavior of mean curvature flow near a rotationally symmetric cylindrical singularity has been described in great details by Altschuler-Angenent-Giga [1], Gang-Knopf-Sigal [10] and Gang-Knopf [9]. Complementary to the previously mentioned results, Theorem 1.1, to the best of our knowledge, is the first statement on the rigidity at infinity of generalized cylinders.

We conclude the introduction by presenting a brief overview of the proof of Theorem 1.1. We follow essentially the same strategy as for the proof of the uniqueness of asymptotically conical self-shrinkers in [22]. First, we note that Theorem 1.1 can be restated as a strong unique continuation problem at infinity of a class of perturbed stability operators (see [4, 5]) for generalized cylinders. However, as explained in section 3 of [22], since the stability operator becomes highly singular and degenerate when approaching infinity, some well-known strong unique continuation theorems for elliptic operators (see [18], [19], and [11, 12]) do not seem to be applicable. Instead, in section 3.1, we introduce a new function defined on some space-time domain and relate Theorem 1.1 to a backwards uniqueness problem of parabolic equations. Then the key ingredients in the proof of the backwards uniqueness are the Carleman inequalities with suitable choices of weight functions inspired by those in [6, 7]; see section 2. On the other hand, in contrast with [22], the associated parabolic equations here turn out to be singular in the spherical direction as time goes to zero, which forces the weight functions to blow up at time zero. This is because that the cross sections of generalized cylinders shrink self-similarly to a point at time zero. To overcome this additional difficulty, we employ the self-similarity of the solutions in question to conclude the uniqueness via the Carleman inequalities.

2. CARLEMAN INEQUALITIES

In this section, we begin by deriving two integral identities with general weight functions. Then, adjusting the specific choices of weight functions in [7], we establish two Carleman inequalities which are the key ingredients in the proof of Theorem 1.1. In fact, one of them will be applied in the next section to conclude

the uniqueness, while the other is to show the desired exponential decay so that the use of the weight function in the previous inequality is legal.

2.1. Notations and Conventions. We will fix $0 < k < n$ and let ∇ and Δ denote the gradient and Laplacian, respectively, on \mathbb{R}^{n-k} . Similarly, we denote the Levi-Civita connection and Laplacian on S^k by $\bar{\nabla}$ and $\underline{\Delta}$ respectively. For convenience, we isometrically embed S^k into \mathbb{R}^{k+1} and will not distinguish vector fields on S^k with their push-forward by this embedding. Then we define the operators ∇_t and Δ_t acting on smooth functions on $S^k \times \mathbb{R}^{n-k} \times (0, T)$ by

$$\nabla_t \doteq \left(\frac{1}{\sqrt{t}} \bar{\nabla}, \nabla \right) \quad \text{and} \quad \Delta_t \doteq t^{-1} \underline{\Delta} + \Delta.$$

Also, let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product, $|\cdot|$ the Euclidean norm, and $d\mu$ the volume element on $S^k \times \mathbb{R}^{n-k}$. Finally, we set $\mathcal{E}_\tau^T \doteq S^k \times \mathbb{R}^{n-k} \times [\tau, T]$ for $0 < \tau < T$.

2.2. Two general L^2 Identities. We follow closely the computations in section 3 of [7] to derive two integral identities with general weight functions in Lemmas 2.1 and 2.2; see also section 6 of [8]. Namely, given a function $\phi \in C^\infty(\mathcal{E}_\tau^T)$, we consider the operator $\mathcal{L}_\phi \doteq e^\phi(\partial_t + \Delta_t)e^{-\phi}$. Then we decompose $t\mathcal{L}_\phi$ into the sum of a symmetric operator \mathcal{S}_ϕ and an antisymmetric operator \mathcal{A}_ϕ in $L^2(d\mu dt)$ which are given by

$$\mathcal{S}_\phi \doteq t\Delta_t + t \left(|\nabla_t \phi|^2 - \partial_t \phi \right) - \frac{1}{2} \quad \text{and} \quad \mathcal{A}_\phi \doteq t\partial_t - 2t \langle \nabla_t \phi, \nabla_t \cdot \rangle - t\Delta_t \phi + \frac{1}{2}.$$

We will compute the commutator $[\mathcal{S}_\phi, \mathcal{A}_\phi]$ and use integration by parts to get the first integral identity, which will be applied, choosing suitable weight functions ϕ , to control the weighted L^2 norm of functions by that of their images under $\mathcal{L}_0 = \partial_t + \Delta_t$.

Lemma 2.1. *Suppose that $\phi \in C^\infty(\mathcal{E}_\tau^T)$ is independent of the spherical variables. Then the following integral identity holds for all $h \in C_c^\infty(\mathcal{E}_\tau^T)$:*

$$\begin{aligned} (2.1) \quad & \iint_{\mathcal{E}_\tau^T} t^2 |\mathcal{L}_\phi h|^2 d\mu dt = \iint_{\mathcal{E}_\tau^T} \left(|\mathcal{S}_\phi h|^2 + |\mathcal{A}_\phi h|^2 + t |\nabla h|^2 \right) d\mu dt \\ & + \iint_{\mathcal{E}_\tau^T} 4t^2 \nabla \nabla \phi (\nabla h, \nabla h) d\mu dt + \iint_{\mathcal{E}_\tau^T} t \left(\partial_t \phi - |\nabla \phi|^2 \right) h^2 d\mu dt \\ & + \iint_{\mathcal{E}_\tau^T} t^2 \left(4 \nabla \nabla \phi (\nabla \phi, \nabla \phi) + \partial_t^2 \phi - \Delta^2 \phi - 2 \partial_t |\nabla \phi|^2 \right) h^2 d\mu dt \\ & + F(\tau, \phi, h) - F(T, \phi, h), \end{aligned}$$

where the function $F(t, \phi, h)$ is given by

$$F(t, \phi, h) = t^2 \int_{S^k \times \mathbb{R}^{n-k}} \left(|\nabla_t h|^2 + (\partial_t \phi - |\nabla \phi|^2) h^2 + \frac{h^2}{2t} \right) d\mu.$$

Proof. Since $t\mathcal{L}_\phi = \mathcal{S}_\phi + \mathcal{A}_\phi$, it follows from expanding squares that

$$(2.2) \quad \iint_{\mathcal{E}_\tau^T} t^2 |\mathcal{L}_\phi h|^2 d\mu dt = \iint_{\mathcal{E}_\tau^T} \left(|\mathcal{S}_\phi h|^2 + |\mathcal{A}_\phi h|^2 \right) d\mu dt + \iint_{\mathcal{E}_\tau^T} 2\mathcal{S}_\phi h \cdot \mathcal{A}_\phi h d\mu dt.$$

Furthermore, applying integration by parts to the last term on the right hand side of (2.2), we get

$$(2.3) \quad \iint_{\mathcal{E}_\tau^T} 2\mathcal{S}_\phi h \cdot \mathcal{A}_\phi h \, d\mu dt = \iint_{\mathcal{E}_\tau^T} h [\mathcal{S}_\phi, \mathcal{A}_\phi] h \, d\mu dt + F(\tau, \phi, h) - F(T, \phi, h).$$

Here we use the fact that \mathcal{S}_ϕ is symmetric and \mathcal{A}_ϕ is antisymmetric in $L^2(d\mu dt)$. Thus, substituting (2.3) back into the right hand side of (2.2), we arrive at that

$$(2.4) \quad \begin{aligned} \iint_{\mathcal{E}_\tau^T} t^2 |\mathcal{L}_\phi h|^2 \, d\mu dt &= \iint_{\mathcal{E}_\tau^T} (|\mathcal{S}_\phi h|^2 + |\mathcal{A}_\phi h|^2) \, d\mu dt \\ &+ \iint_{\mathcal{E}_\tau^T} h [\mathcal{S}_\phi, \mathcal{A}_\phi] h \, d\mu dt + F(\tau, \phi, h) - F(T, \phi, h). \end{aligned}$$

The rest of the proof is devoted to the simplification of $h[\mathcal{S}_\phi, \mathcal{A}_\phi]h$. First, expanding the commutator $[\mathcal{S}_\phi, \mathcal{A}_\phi]$ implies that

$$(2.5) \quad \begin{aligned} h[\mathcal{S}_\phi, \mathcal{A}_\phi]h &= h[t\Delta_t, t\partial_t]h + h\left[t\left(|\nabla_t \phi|^2 - \partial_t \phi\right), t\partial_t\right]h - h[t\Delta_t, t\Delta_t \phi]h \\ &- 2h[t\Delta_t, t\langle \nabla_t \phi, \nabla_t \cdot \rangle]h - 2h\left[t\left(|\nabla_t \phi|^2 - \partial_t \phi\right), t\langle \nabla_t \phi, \nabla_t \cdot \rangle\right]h. \end{aligned}$$

Then we will simplify each term on the right hand side of (2.5). Namely, it is easy to see that the first two terms become

$$(2.6) \quad \begin{aligned} &h[t\Delta_t, t\partial_t]h + h\left[t\left(|\nabla_t \phi|^2 - \partial_t \phi\right), t\partial_t\right]h \\ &= -t\nabla \cdot (h\nabla h) + t|\nabla h|^2 - t\left(|\nabla_t \phi|^2 - \partial_t \phi\right)h^2 - t^2 h^2 \partial_t \left(|\nabla_t \phi|^2 - \partial_t \phi\right). \end{aligned}$$

Similarly, by the assumption that ϕ is independent of the spherical variables, we can simplify to find

$$\left\langle \nabla_t \phi, \nabla_t |\nabla_t \phi|^2 \right\rangle = 2\nabla \nabla \phi (\nabla \phi, \nabla \phi) \quad \text{and} \quad 2\langle \nabla_t \phi, \nabla_t (\partial_t \phi) \rangle = \partial_t |\nabla \phi|^2.$$

Thus, the last term gives that

$$(2.7) \quad 2h\left[t\left(|\nabla_t \phi|^2 - \partial_t \phi\right), t\langle \nabla_t \phi, \nabla_t \cdot \rangle\right]h = t^2 h^2 \partial_t |\nabla \phi|^2 - 4t^2 h^2 \nabla \nabla \phi (\nabla \phi, \nabla \phi).$$

Also, we can rewrite the third term to be

$$(2.8) \quad \begin{aligned} &-h[t\Delta_t, t\Delta_t \phi]h = -t^2 h^2 \Delta_t^2 \phi - 2t^2 h \langle \nabla_t (\Delta_t \phi), \nabla_t h \rangle \\ &= -t^2 h^2 \Delta_t^2 \phi - t^2 \nabla_t \cdot (\Delta_t \phi \nabla_t h^2) + 2t^2 h \Delta_t h \Delta_t \phi + 2t^2 \Delta_t \phi |\nabla_t h|^2. \end{aligned}$$

In the end, the most difficult term is the penultimate one, which we deal with as follows.

$$\begin{aligned} &-2h[t\Delta_t, t\langle \nabla_t \phi, \nabla_t \cdot \rangle]h = -2t^2 h \Delta_t \langle \nabla_t \phi, \nabla_t h \rangle + 2t^2 h \langle \nabla_t \phi, \nabla_t (\Delta_t h) \rangle \\ &= 2t^2 \{-h \Delta_t \langle \nabla_t \phi, \nabla_t h \rangle + \nabla_t \cdot (h \Delta_t h \nabla_t \phi) - h \Delta_t h \Delta_t \phi - \langle \nabla_t \phi, \nabla_t h \rangle \Delta_t h\} \\ &= 2t^2 \{\nabla_t \cdot (h \Delta_t h \nabla_t \phi) - \Delta_t (h \langle \nabla_t \phi, \nabla_t h \rangle) + 2\langle \nabla_t \langle \nabla_t \phi, \nabla_t h \rangle, \nabla_t h \rangle - h \Delta_t h \Delta_t \phi\}. \end{aligned}$$

Again, by the assumption that ϕ is independent of the spherical variables,

$$\begin{aligned} &2\langle \nabla_t \langle \nabla_t \phi, \nabla_t h \rangle, \nabla_t h \rangle = 2\nabla h (\langle \nabla \phi, \nabla h \rangle) + 2t^{-1} \bar{\nabla} h (\langle \nabla \phi, \nabla h \rangle) \\ &= \left\langle \nabla \phi, \nabla |\nabla h|^2 \right\rangle + 2\nabla \nabla \phi (\nabla h, \nabla h) + t^{-1} \left\langle \nabla \phi, \nabla |\bar{\nabla} h|^2 \right\rangle \\ &= \nabla \cdot (|\nabla_t h|^2 \nabla \phi) - |\nabla_t h|^2 \Delta \phi + 2\nabla \nabla \phi (\nabla h, \nabla h). \end{aligned}$$

Thus it follows that

$$(2.9) \quad \begin{aligned} -2h[t\Delta_t, t\langle \nabla_t \phi, \nabla_t \cdot \rangle]h &= 4t^2 \nabla \nabla \phi (\nabla h, \nabla h) - 2t^2 |\nabla_t h|^2 \Delta \phi - 2t^2 h \Delta_t h \Delta \phi \\ &\quad + 2t^2 \nabla_t \cdot \left\{ h \Delta_t h \nabla_t \phi - \nabla_t (h \langle \nabla_t \phi, \nabla_t h \rangle) + |\nabla_t h|^2 \nabla_t \phi \right\}. \end{aligned}$$

Therefore, combining (2.4)-(2.9), the integral identity (2.1) follows immediately from the divergence theorem. \square

Since the identity (2.1) in Lemma 2.1 only involves the radial part of the energy of h , we need to establish another integral identity to control the total energy of h and thus, in our applications, the weighted L^2 norm of the gradient of functions.

Lemma 2.2. *Suppose that ϕ is a smooth function defined on \mathcal{E}_τ^T . Then the following identity holds for $j \in \{1, 2\}$ and $h \in C_c^\infty(\mathcal{E}_\tau^T)$:*

$$(2.10) \quad \begin{aligned} &\iint_{\mathcal{E}_\tau^T} t^j \left(|\nabla_t h|^2 + |\nabla_t \phi|^2 h^2 \right) d\mu dt \\ &= \iint_{\mathcal{E}_\tau^T} t^j \left(2|\nabla_t \phi|^2 - \partial_t \phi \right) h^2 d\mu dt - \frac{1}{2} \iint_{\mathcal{E}_\tau^T} h \left(2t^j \mathcal{L}_\phi h + jt^{j-1} h \right) d\mu dt \\ &\quad + \frac{T^j}{2} \int_{S^k \times \mathbb{R}^{n-k}} h^2(T) d\mu - \frac{\tau^j}{2} \int_{S^k \times \mathbb{R}^{n-k}} h^2(\tau) d\mu. \end{aligned}$$

Proof. We start with the simple identity

$$(2.11) \quad \begin{aligned} (\partial_t + \Delta_t) h^2 &= 2h (\partial_t h + \Delta_t h) + 2|\nabla_t h|^2 \\ &= 2h \mathcal{L}_\phi h + 4h \langle \nabla_t \phi, \nabla_t h \rangle - 2 \left(|\nabla_t \phi|^2 - \partial_t \phi - \Delta_t \phi \right) h^2 + 2|\nabla_t h|^2. \end{aligned}$$

Then, multiplying t^j on both sides of (2.11) and integrating over \mathcal{E}_τ^T , it follows from the divergence theorem that

$$(2.12) \quad \begin{aligned} &\iint_{\mathcal{E}_\tau^T} t^j \left(h \mathcal{L}_\phi h - (|\nabla_t \phi|^2 - \partial_t \phi) h^2 \right) d\mu dt + \iint_{\mathcal{E}_\tau^T} t^j |\nabla_t h|^2 d\mu dt \\ &= -\frac{j}{2} \iint_{\mathcal{E}_\tau^T} t^{j-1} h^2 d\mu dt + \frac{T^j}{2} \int_{S^k \times \mathbb{R}^{n-k}} h^2(T) d\mu - \frac{\tau^j}{2} \int_{S^k \times \mathbb{R}^{n-k}} h^2(\tau) d\mu. \end{aligned}$$

Therefore, the lemma follows from adding the integration of $(t^j |\nabla_t \phi|^2 h^2)$ to both sides of (2.12). \square

2.3. Carleman inequalities to conclude the uniqueness. First, we choose $j = 1$ in Lemma 2.2 and set $h \doteq \xi e^\phi$. Note that

$$|\nabla_t \xi|^2 e^{2\phi} \leq 2|\nabla_t h|^2 + 2|\nabla_t \phi|^2 |\xi|^2 e^{2\phi}.$$

Thus, multiplying (2.10) by $1/2$ and combining with (2.1), it follows from the Cauchy-Schwarz inequality that

Lemma 2.3. *Given $R > 0$, let $\{\phi(t)\}_{t \in (0,1)}$ be a piecewise smooth family of smooth convex functions on $\mathbb{R}^{n-k} \setminus B_R^{n-k}$. Suppose that ϕ is in $C^1((\mathbb{R}^{n-k} \setminus B_R^{n-k}) \times (0, 1))$. Then, for all functions $\xi \in C_c^\infty(S^k \times (\mathbb{R}^{n-k} \setminus B_R^{n-k}) \times (0, 1))$, we have that*

$$(2.13) \quad \iint_{\mathcal{E}_0^1} \left(P_\phi |\xi|^2 + \frac{t}{4} |\nabla_t \xi|^2 \right) e^{2\phi} d\mu dt \leq 2 \iint_{\mathcal{E}_0^1} t^2 |\partial_t \xi + \Delta_t \xi|^2 e^{2\phi} d\mu dt,$$

where P_ϕ is the function on $(\mathbb{R}^{n-k} \setminus B_R^{n-k}) \times (0, 1)$ given by

$$P_\phi = t^2 \left(\partial_t^2 \phi - \Delta^2 \phi - 2\partial_t |\nabla \phi|^2 \right) + \frac{t}{2} \left(3\partial_t \phi - 4|\nabla \phi|^2 \right).$$

Next, we make a suitable choice of ϕ that satisfies the conditions in the above lemma and bound P_ϕ from below by a positive constant. Namely, given $\delta \in (3/4, 1)$ fixed, we define

$$\tau_0 \doteq \frac{2\delta(4\delta - 3)}{3\delta(4\delta - 3) + 32}$$

and a piecewise smooth function $\eta \in C^1([0, 1])$ by

$$\eta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \tau_0. \\ 1 - \frac{1}{32}\delta(4\delta - 3)(\tau_0^{-1}t - 1)^2 & \text{if } \tau_0 \leq t \leq 2\tau_0. \\ 1 + \frac{1}{32}\delta(4\delta - 3)(3 - 2\tau_0^{-1}t) & \text{if } 2\tau_0 \leq t \leq 1. \end{cases}$$

We remark that η is non-increasing and $\eta(1) = 0$. Then we choose the weight function ϕ in Lemma 2.3 to be

$$\phi_1(z, t) \doteq \phi_{1;\alpha}(z, t) \doteq \alpha\eta(t)t^{-\delta}|z|^{2\delta} \quad \text{for } \alpha > 0.$$

Note that ϕ_1 is a suitable variance of the weight function in the second Carleman inequality in section 4 of [7].

To verify that ϕ_1 satisfies our desired properties, we need to calculate the hessian of ϕ_1 and its derivatives involved in the expression of P_{ϕ_1} .

Lemma 2.4. *For $\alpha > 0$ and $(z, t) \in (\mathbb{R}^{n-k} \setminus \{O\}) \times (0, 1)$,*

$$(2.14) \quad \nabla \phi_1 = 2\alpha\delta\eta t^{-\delta}|z|^{2\delta-2}z,$$

$$(2.15) \quad \nabla \nabla \phi_1 = 2\alpha\delta\eta t^{-\delta}|z|^{2\delta-4} \left(|z|^2 \text{Id}_{\mathbb{R}^{n-k}} + 2(\delta - 1)z \otimes z \right),$$

$$(2.16) \quad \partial_t \phi_1 = \alpha(t\eta' - \delta\eta)t^{-\delta-1}|z|^{2\delta},$$

$$(2.17) \quad \partial_t^2 \phi_1 = \alpha(t^2\eta'' - 2\delta t\eta' + \delta(\delta + 1)\eta)t^{-\delta-2}|z|^{2\delta},$$

$$(2.18) \quad \partial_t |\nabla \phi_1|^2 = 8\alpha^2\delta^2\eta(t\eta' - \delta\eta)t^{-2\delta-1}|z|^{4\delta-2},$$

$$(2.19) \quad \Delta^2 \phi_1 = 4\alpha\delta(\delta - 1)(n - k + 2\delta - 2)(n - k + 2\delta - 4)\eta t^{-\delta}|z|^{2\delta-4}.$$

Here η'' denotes the weak second derivative of η .

Since $\delta > 3/4$, it follows from (2.15) immediately that $\nabla \nabla \phi_1$ is always non-negative definite in $\mathbb{R}^{n-k} \setminus \{O\}$ for each $t > 0$. Moreover,

Lemma 2.5. *There exists $R_1 > R_0$, depending only on k, n and δ , such that $P_{\phi_1} \geq \alpha/4$ in $(\mathbb{R}^{n-k} \setminus B_{R_1}^{n-k}) \times (0, 1)$ for $\alpha > 0$ and $\phi_1 = \phi_{1;\alpha}$.*

Proof. For convenience, we group the terms in P_{ϕ_1} according to their powers of α . By Lemma 2.4, we can write $P_{\phi_1} = \alpha Q_1 + \alpha^2 Q_2$, where

$$\begin{aligned} Q_1 &\doteq \frac{1}{2} \left(2t^2\eta'' + (4\delta - 3)(\delta\eta - t\eta') + 2\delta(1 - \delta)\eta \right) t^{-\delta}|z|^{2\delta} \\ &\quad + 4\delta(1 - \delta)(n - k + 2\delta - 2)(n - k + 2\delta - 4)\eta t^{2-\delta}|z|^{2\delta-4}, \\ Q_2 &\doteq 8\delta^2(-2t\eta' + (2\delta - 1)\eta)\eta t^{1-2\delta}|z|^{4\delta-2}. \end{aligned}$$

Since $\delta > 3/4$ and η is non-negative and non-increasing on $[0, 1]$, it follows that $Q_2 \geq 0$ in $(\mathbb{R}^{n-k} \setminus \{O\}) \times (0, 1)$.

To bound Q_1 from below, we compute $(\delta\eta - t\eta')$ and $t^2\eta''$, that is,

$$\delta\eta - t\eta' = \begin{cases} \delta & \text{if } 0 \leq t \leq \tau_0. \\ \delta + \frac{\delta(4\delta-3)}{32\tau_0^2}(t-\tau_0)(\delta\tau_0 + (2-\delta)t) & \text{if } \tau_0 \leq t \leq 2\tau_0. \\ \delta + \frac{3}{32}\delta^2(4\delta-3) + \frac{\delta(1-\delta)(4\delta-3)t}{16\tau_0} & \text{if } 2\tau_0 \leq t \leq 1. \end{cases}$$

and

$$t^2\eta'' = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau_0. \\ -\frac{\delta(4\delta-3)t^2}{16\tau_0^2} & \text{if } \tau_0 \leq t \leq 2\tau_0. \\ 0 & \text{if } 2\tau_0 \leq t \leq 1. \end{cases}$$

Thus, recalling that $\delta \in (3/4, 1)$,

$$(2.20) \quad \delta\eta - t\eta' \geq \delta \quad \text{and} \quad t^2\eta'' \geq -\frac{1}{4}\delta(4\delta-3) \quad \text{for } 0 \leq t \leq 1.$$

On the other hand, since $0 \leq \eta \leq 1$ on $[0, 1]$, there exists $R > R_0$, depending only on δ , k and n , such that for $(z, t) \in (\mathbb{R}^{n-k} \setminus B_R^{n-k}) \times (0, 1)$,

$$(2.21) \quad \left| 4(1-\delta)(n-k+2\delta-2)(n-k+2\delta-4)\eta t^2 |z|^{-4} \right| \leq \frac{4\delta-3}{8}.$$

Hence, it follows from (2.20) and (2.21) that for $(z, t) \in (\mathbb{R}^{n-k} \setminus B_R^{n-k}) \times (0, 1)$,

$$(2.22) \quad Q_1 \geq \frac{1}{8}\delta(4\delta-3)t^{-\delta} |z|^{2\delta},$$

and, combining with the positivity of Q_2 , this implies Lemma 2.5 by enlarging R so that $\delta(4\delta-3)R^{2\delta} > 2$. \square

Therefore we have shown that

Proposition 2.6. *For $\alpha > 0$ and all $\xi \in C_c^\infty(S^k \times (\mathbb{R}^{n-k} \setminus B_{R_1}^{n-k}) \times (0, 1))$,*

$$(2.23) \quad \iint_{\mathcal{E}_0^1} \left(\alpha |\xi|^2 + t |\nabla_t \xi|^2 \right) e^{2\phi_1} d\mu dt \leq 8 \iint_{\mathcal{E}_0^1} t^2 |\partial_t \xi + \Delta_t \xi|^2 e^{2\phi_1} d\mu dt,$$

where $\phi_1 = \phi_{1;\alpha}$.

2.4. Carleman inequalities to imply the exponential decay. In what follows, let

$$\sigma(t) \doteq t e^{\frac{1-t}{3}} \quad \text{for } t > 0,$$

and we define, on $\mathbb{R}^{n-k} \times (0, \infty)$,

$$\phi_2(z, t) \doteq \phi_{2;\alpha}(z, t) \doteq -\frac{|z|^2}{8t} - \alpha \log \sigma(t) \quad \text{for } \alpha > 0.$$

The choice of ϕ_2 and the derivation of the Carleman inequality in Proposition 2.7 have some closely related precedents in section 3 of [7]; see also Proposition 6.1 of [8].

Proposition 2.7. *Given $\alpha > 2$, the following weighted inequality holds true for all functions $\xi \in C_c^\infty(S^k \times \mathbb{R}^{n-k} \times (0, 4))$:*

$$(2.24) \quad \iint_{\mathcal{E}_0^4} t \left(\alpha |\xi|^2 + t |\nabla_t \xi|^2 \right) e^{2\phi_2} d\mu dt \leq 10 \iint_{\mathcal{E}_0^4} t^2 |\partial_t \xi + \Delta_t \xi|^2 e^{2\phi_2} d\mu dt,$$

where $\phi_2 = \phi_{2;\alpha}$.

Proof. We choose the weight function ϕ in Lemmas 2.1 and 2.2 to be ϕ_2 and set $h = h_2 \doteq \xi e^{\phi_2}$. Then we calculate the hessian of ϕ_2 and its derivatives involved in (2.1) and (2.10). That is,

$$\begin{aligned} \nabla \phi_2 &= -\frac{z}{4t}, \quad \nabla \nabla \phi_2 = -\frac{1}{4t} \text{Id}_{\mathbb{R}^{n-k}}, \quad \partial_t |\nabla \phi_2|^2 = -\frac{|z|^2}{8t^3}, \\ \partial_t \phi_2 &= \frac{|z|^2}{8t^2} - \frac{\alpha}{t} + \frac{\alpha}{3}, \quad \partial_t^2 \phi_2 = -\frac{|z|^2}{4t^3} + \frac{\alpha}{t^2}. \end{aligned}$$

Thus, this implies that $4t^2 \nabla \nabla \phi_2 (\nabla h_2, \nabla h_2) = -t |\nabla h_2|^2$, $\Delta^2 \phi_2 = 0$, and

$$t^2 \left(4 \nabla \nabla \phi_2 (\nabla \phi_2, \nabla \phi_2) + \partial_t^2 \phi_2 - 2 \partial_t |\nabla \phi_2|^2 \right) + t \left(\partial_t \phi_2 - |\nabla \phi_2|^2 \right) = \frac{\alpha t}{3}.$$

Hence, it follows from Lemma 2.1 that

$$(2.25) \quad 3 \iint_{\mathcal{E}_0^4} t^2 |\partial_t \xi + \Delta_t \xi|^2 e^{2\phi_2} d\mu dt \geq \iint_{\mathcal{E}_0^4} \alpha t |\xi|^2 e^{2\phi_2} d\mu dt.$$

On the other hand, it follows from Lemma 2.2 (with $j = 2$) and the Cauchy-Schwarz inequality that

$$\begin{aligned} (2.26) \quad & \iint_{\mathcal{E}_0^4} t^2 |\nabla_t \xi|^2 e^{2\phi_2} d\mu dt \leq \iint_{\mathcal{E}_0^4} 2t^2 \left(|\nabla_t h_2|^2 + h_2^2 |\nabla \phi_2|^2 \right) d\mu dt \\ &= \iint_{\mathcal{E}_0^4} 2\alpha h_2^2 \left(t - \frac{t^2}{3} \right) d\mu dt - \iint_{\mathcal{E}_0^4} 2 \left(t h_2^2 + t^2 h_2 \mathcal{L}_{\phi_2} h_2 \right) d\mu dt \\ &\leq \iint_{\mathcal{E}_0^4} 2\alpha t |\xi|^2 e^{2\phi_2} d\mu dt + \iint_{\mathcal{E}_0^4} \frac{3t^2}{2\alpha} |\partial_t \xi + \Delta_t \xi|^2 e^{2\phi_2} d\mu dt. \end{aligned}$$

Therefore, the inequality (2.24) is implied directly from (2.25) and (2.26). \square

3. PROOF OF THEOREM 1.1

In this section, we will follow essentially the same strategy as in [22] to prove Theorem 1.1, however, here we need to deal with an additional difficulty which originates from the fact that the curvature of self-shrinking cylinders blows up along the whole \mathbb{R}^{n-k} at the singular time.

First, we reduce the uniqueness of asymptotically cylindrical shrinkers to a problem of backwards uniqueness for a class of parabolic inequalities in noncompact incomplete domains. Note that the parabolic inequalities we derive in section 3.1 (see (3.4)), which are linear perturbations of $(\partial_t + \Delta_t)$, become singular when time approaches zero, while the corresponding ones in [22] are regular and uniformly parabolic. Then, by the Carleman inequalities in Proposition 2.7, we show that any shrinker with the decay assumption (1.4) is indeed asymptotic to a generalized cylinder exponentially fast. Finally, we apply the Carleman inequalities in Proposition 2.6 to prove the backwards uniqueness of self-similar solutions to the associated parabolic inequalities. We make an important remark that, at present, we do not

address the backwards uniqueness of general solutions to the mean curvature flow and, in fact, we make essential use of the self-similarity in our argument to get around the involved asymptotic singularities.

3.1. Reduction to a problem of backwards uniqueness. First, since u is assumed to be C^2 uniformly bounded on its domain, it follows from Lemma 2.4 in [22] that

$$(3.1) \quad \Delta_1 u - \frac{1}{2} \langle \mathbf{x}, \nabla_1 u \rangle + u = Q(\mathbf{x}, u, \nabla_1 u, \nabla_1 \nabla_1 u),$$

where $\mathbf{x} \in S^k \times (\mathbb{R}^{n-k} \setminus B_{R_0}^{n-k})$ and the nonlinear term Q satisfies that

$$|Q(\mathbf{x}, u, \nabla_1 u, \nabla_1 \nabla_1 u)| \leq C_0 |\mathbf{x}| (|u| + |\nabla_1 u|)$$

for some $C_0 > 0$ depending only on k, n and $\|u\|_{C^2}$. From now on, we call (3.1) to be the self-shrinker equation. Thus, by the Schauder interior estimates, cf. Theorem 6.2 in [13], the L^∞ decay assumption (1.4) implies that for each $l > 0$,

$$(3.2) \quad \limsup_{r \rightarrow \infty} \|r^l u\|_{C^2(S^k \times (B_{2r}^{n-k} \setminus B_r^{n-k}))} = 0.$$

Hence, invoking Lemma 2.4 in [22] again, given $\epsilon > 0$, there exists $R > R_0$, depending only on k, n, u and ϵ , such that

$$(3.3) \quad |Q(\mathbf{x}, u, \nabla_1 u, \nabla_1 \nabla_1 u)| \leq \epsilon |\mathbf{x}|^{-2} (|u| + |\nabla_1 u|)$$

on $S^k \times (\mathbb{R}^{n-k} \setminus B_R^{n-k})$.

Theorem 1.1 can be restated as a strong unique continuation theorem for the elliptic equation (3.1) in some neighborhood of infinity. However, as explained in section 3 of [22], since the elliptic operator involved in (3.1) becomes highly singular and degenerate as $|\mathbf{x}| \rightarrow \infty$, it seems that some well-known unique continuation theorems for elliptic equations (for instance, see [11, 12], [18], [19] and [21]) cannot be applied to our problem directly. In fact, the deeper reason behind this is the mixed homogeneity feature of the operator on the left hand side of (3.1) and thus, to overcome this difficulty, we will introduce a new function v on some space-time domain, so that the equation for v possesses scaling invariances. More precisely, we define the function $v : S^k \times (\mathbb{R}^{n-k} \setminus B_{R_0}^{n-k}) \times (0, 1] \rightarrow \mathbb{R}$ by $v(\theta, z, t) \doteq tu(\theta, z/\sqrt{t})$. Thus, it follows from the equation (3.1) for u and the estimate (3.3) that

Lemma 3.1. *Given $\epsilon_1 > 0$, there exists $R_2 > 0$, depending only on k, n, u and ϵ_1 , such that*

$$(3.4) \quad |\partial_t v + \Delta_t v| \leq \epsilon_1 (|v| + |\nabla_t v|) \quad \text{on} \quad S^k \times (\mathbb{R}^{n-k} \setminus B_{R_2}^{n-k}) \times (0, 1].$$

Remark 3.2. Comparing with the uniformly parabolic inequality (3.5) in [22] for asymptotically conical shrinkers, observe that here we do not resolve the singularity along \mathbb{R}^{n-k} in (3.1) completely. Instead, it is transferred to the spherical direction as time approaches to zero.

If necessary, appealing to the classical strong unique continuation theorems in [11, 12], it suffices to prove the following backwards uniqueness theorem so to conclude Theorem 1.1.

Theorem 3.3. *There exist $R_3 > R_0$ and $0 < T_0 < 1$, depending only on k, n and u , such that v vanishes in $S^k \times (\mathbb{R}^{n-k} \setminus B_{R_3}^{n-k}) \times (0, T_0)$.*

3.2. Exponential decay. Since the weight function ϕ_1 in the Carleman inequalities in Proposition 2.6 grows fast when $t \rightarrow 0$ and $z \in \mathbb{R}^{n-k}$ tends to infinity, we need to prove that v and $\nabla_t v$ decay at least as rapidly as ϕ_1 . To achieve this, we employ a scaling argument and the Carleman inequalities in Proposition 2.7.

Lemma 3.4. *There exist $\beta > 0$ and $R_4 > R_0$, depending only on k, n and u , such that*

$$(3.5) \quad |v(\theta, z, t)| + |\nabla_t v(\theta, z, t)| \leq \beta^{-1} e^{-\frac{\beta|z|^2}{t}}$$

for $(\theta, z, t) \in S^k \times (\mathbb{R}^{n-k} \setminus B_{R_4}^{n-k}) \times (0, 1]$.

Proof. Below we broadly follow the arguments in [7]. First, choosing $\varepsilon_1 = 1$ in Lemma 3.1, there exists $r_1 > 1$, depending only on k, n and u , such that

$$(3.6) \quad |\partial_t v + \Delta_t v| \leq |v| + |\nabla_t v| \quad \text{on } S^k \times (\mathbb{R}^{n-k} \setminus B_{r_1}^{n-k}) \times (0, 1].$$

Given $z_0 \in \mathbb{R}^{n-k} \setminus B_{2r_1}^{n-k}$ and $0 < t_0 < 1/4$, we define the function v_{z_0, t_0} on $S^k \times B_{2r}^{n-k} \times (0, 4)$ by

$$v_{z_0, t_0}(\theta, z, t) = v(\theta, z_0 + \sqrt{t_0}z, t_0 t),$$

where $r = |z_0|/(4\sqrt{t_0}) > 1$. Thus it follows from (3.6) that

$$(3.7) \quad |\partial_t v_{z_0, t_0} + \Delta_t v_{z_0, t_0}| \leq \sqrt{t_0} (|v_{z_0, t_0}| + |\nabla_t v_{z_0, t_0}|).$$

Next, we introduce two cut-off functions along the time direction and \mathbb{R}^{n-k} respectively. Namely, for $0 < \varepsilon < 1/4$, we define $\chi_\varepsilon(t) : [0, 4] \rightarrow [0, 1]$ which satisfies that $\chi_\varepsilon \equiv 1$ in $[2\varepsilon, 2]$ while $\chi_\varepsilon \equiv 0$ in $[0, \varepsilon] \cup [3, 4]$, and $|\chi'_\varepsilon| \leq 2$ in $(2, 3)$ and $|\chi'_\varepsilon| \leq 2/\varepsilon$ in $(\varepsilon, 2\varepsilon)$. Similarly, we define the cut-off function $\psi_r(z) : \mathbb{R}^{n-k} \rightarrow [0, 1]$ such that $\psi_r \equiv 1$ in B_r^{n-k} and $\psi_r \equiv 0$ in $\mathbb{R}^{n-k} \setminus B_{2r}^{n-k}$ with bounds $|\nabla \psi_r| \leq 3/r$ and $|\Delta \psi_r| \leq 6/r^2$. Then we choose the test function ξ in Proposition 2.7 to be $\xi_{\varepsilon, r} \doteq \chi_\varepsilon \psi_r v_{z_0, t_0}$. Observe that, by (3.7),

$$\begin{aligned} |(\partial_t + \Delta_t) \xi_{\varepsilon, r}| &\leq \sqrt{t_0} (|\xi_{\varepsilon, r}| + |\nabla_t \xi_{\varepsilon, r}|) + |\chi'_\varepsilon| |\psi_r v_{z_0, t_0}| \\ &\quad + 2 |\chi_\varepsilon| (|v_{z_0, t_0}| + |\nabla v_{z_0, t_0}|) (|\nabla \psi_r| + |\Delta \psi_r|). \end{aligned}$$

Assuming that $t_0 < 10^{-3}$, it follows from Proposition 2.7 that for each $\alpha > 1$,

$$\begin{aligned} &\iint_{\mathcal{E}_0^4} t \left(\alpha |\xi_{\varepsilon, r}|^2 + t |\nabla_t \xi_{\varepsilon, r}|^2 \right) e^{2\phi_2} d\mu dt \\ (3.8) \quad &\leq C \iint_{\mathcal{E}_\varepsilon^{2\varepsilon}} |\psi_r v_{z_0, t_0}|^2 e^{2\phi_2} d\mu dt + C \iint_{\mathcal{E}_2^3} t^2 |\psi_r v_{z_0, t_0}|^2 e^{2\phi_2} d\mu dt \\ &\quad + C \int_\varepsilon^3 \int_{A(r, 2r)} t^2 \left(|v_{z_0, t_0}|^2 + |\nabla v_{z_0, t_0}|^2 \right) e^{2\phi_2} d\mu dt, \end{aligned}$$

where $C > 1$ is some universal constant and $A(R, \tilde{R}) \doteq S^k \times (\mathring{B}_{\tilde{R}}^{n-k} \setminus B_R^{n-k})$ for $\tilde{R} > R$. Since u is assumed to decay of infinite order, it follows that v decays uniformly of infinite order in time as $t \rightarrow 0$ and thus, so does v_{z_0, t_0} . On the other hand, $e^{2\phi_2}$ is asymptotic to $t^{-2\alpha}$, as $t \rightarrow 0$. Hence, the first integral on the right hand side of (3.8) converges to zero, as $\varepsilon \rightarrow 0$, and it follows from the Monotone

Convergence Theorem and the monotonicity of σ that

$$(3.9) \quad \begin{aligned} & \iint_{\mathcal{E}_0^4} t \left(\alpha |\psi_r v_{z_0, t_0}|^2 + t |\nabla_t (\psi_r v_{z_0, t_0})|^2 \right) \sigma^{-2\alpha} e^{-\frac{|z|^2}{4t}} d\mu dt \\ & \leq C' r^{n-k} \left(\sigma^{-2\alpha}(2) + \int_0^3 t^2 \sigma^{-2\alpha} e^{-\frac{r^2}{4t}} dt \right), \end{aligned}$$

where $C' > 1$ depends only on k, n, C and $\|u\|_{C^1}$.

Finally, let $\beta_1 \doteq e \log \sigma(2)/16$ and we choose $\alpha = 1 + \beta_1 r^2 / \log \sigma(2)$. Since σ is increasing in $(0, 3)$ and $\sigma(1) = 1$,

$$\begin{aligned} & \sigma^{-2\alpha}(2) + \int_0^3 t^2 \sigma^{-2\alpha} e^{-\frac{r^2}{4t}} dt \\ & \leq e^{-2\beta_1 r^2} + \int_0^1 e^{-\frac{2\beta_1 r^2}{\log \sigma(2)} \log t} e^{-\frac{r^2}{4t}} dt + e^{\frac{4}{3}} \int_1^3 e^{-\frac{r^2}{4t}} dt \\ & \leq e^{-2\beta_1 r^2} + e^{-\frac{r^2}{8}} + 2e^{\frac{4}{3}} e^{-\frac{r^2}{12}}, \end{aligned}$$

where we use the fact that $0 < -\log t \leq 1/(et)$ for $0 < t \leq 1$. Hence, the inequality (3.9) gives

$$\int_{1/2}^1 \int_{S^k \times B_1^{n-k}} \left(|v_{z_0, t_0}|^2 + |\nabla_t v_{z_0, t_0}|^2 \right) d\mu dt \leq C'' e^{-\beta_2 r^2}$$

for $\beta_2 \doteq \min\{\beta_1, 1/24\}$, and $C'' > 1$, depending on k, n, β_1 and C' . Therefore, it follows from Theorems 7.36 and 4.8 in [20] that for $z_0 \in \mathbb{R}^{n-k} \setminus B_{2r_1}^{n-k}$ and $t_0 \in (0, 10^{-3})$,

$$|v(\theta, z_0, t_0/2)| + |\nabla_1 v(\theta, z_0, t_0/2)| \leq C''' e^{-\frac{\beta_2 |z_0|^2}{32t_0}},$$

where $C''' > 1$ depends on k, n and C'' , and this implies the estimate (3.5) from the self-similarity of v . \square

3.3. Backwards uniqueness. We will make use of the Carleman inequalities in Proposition 2.6 and the self-similarity of v to prove Theorem 3.3.

Proof of Theorem 3.3. First, choosing $\varepsilon_1 = 1/8$ in Lemma 3.1, there exists $r_1 > \max\{1, R_4\}$ such that

$$(3.10) \quad |\partial_t v + \Delta_t v| \leq \frac{1}{8} (|v| + |\nabla_t v|) \quad \text{on} \quad S^k \times (\mathbb{R}^{n-k} \setminus B_{r_1}^{n-k}) \times (0, 1].$$

Then we introduce two cut-off functions along the time direction and \mathbb{R}^{n-k} in order to construct suitable test functions in Proposition 2.6. For $0 < \varepsilon < 1/8$ and $\alpha > 8$, let $\chi_{\varepsilon, \alpha}(t) : [0, 1] \rightarrow [0, 1]$ satisfy that $\chi_{\varepsilon, \alpha} \equiv 1$ in $[2\varepsilon, 1 - 2/\alpha]$ while $\chi_{\varepsilon, \alpha} \equiv 0$ in $[0, \varepsilon] \cup [1 - 1/\alpha, 1]$, and $|\chi'_{\varepsilon, \alpha}| \leq 2/\varepsilon$ in $(\varepsilon, 2\varepsilon)$ and $|\chi'_{\varepsilon, \alpha}| \leq 2\alpha$ in $(1 - 2/\alpha, 1 - 1/\alpha)$. Also, for $r > r_1 + 1$, we define the function $\psi_r(z) : \mathbb{R}^{n-k} \rightarrow [0, 1]$ such that $\psi_r \equiv 1$ in $B_r^{n-k} \setminus B_{r_1+1}^{n-k}$ and $\psi_r \equiv 0$ in $B_{r_1}^{n-k} \cup (\mathbb{R}^{n-k} \setminus B_{2r}^{n-k})$ with $|\nabla \psi_r| + |\Delta \psi_r| \leq L$ for some $L = L(k, n)$. Now we choose ξ in Proposition 2.6 to be $\xi_{\varepsilon, \alpha, r} \doteq \chi_{\varepsilon, \alpha} \psi_r v$. Note that, by the inequality (3.10) for v ,

$$\begin{aligned} |(\partial_t + \Delta_t) \xi_{\varepsilon, \alpha, r}| & \leq \frac{1}{8} (|\xi_{\varepsilon, \alpha, r}| + |\nabla_t \xi_{\varepsilon, \alpha, r}|) + |\chi'_{\varepsilon, \alpha}| |\psi_r v| \\ & \quad + 2 |\chi_{\varepsilon, \alpha}| (|v| + |\nabla v|) (|\nabla \psi_r| + |\Delta \psi_r|). \end{aligned}$$

Thus, it follows from Proposition 2.6 that for $\alpha > 8$,

$$\begin{aligned}
& \iint_{\mathcal{E}_0^1} \left(\alpha |\xi_{\varepsilon, \alpha, r}|^2 + t |\nabla_t \xi_{\varepsilon, \alpha, r}|^2 \right) e^{2\phi_1} d\mu dt \\
& \leq C \iint_{\mathcal{E}_{\varepsilon}^{2\varepsilon}} |\psi_r v|^2 e^{2\phi_1} d\mu dt + C\alpha^2 \iint_{\mathcal{E}_{1-2/\alpha}^{1-1/\alpha}} |\psi_r v|^2 e^{2\phi_1} d\mu dt \\
(3.11) \quad & + C \int_0^1 \int_{A(r_1, r_1+1)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\phi_1} d\mu dt \\
& + C \int_0^1 \int_{A(r, 2r)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\phi_1} d\mu dt,
\end{aligned}$$

where $C > 1$ depends only on L . Observe that, by Lemma 3.4, given $\alpha > 8$, there exists $\epsilon_0 \in (0, \tau_0/2)$, depending only on δ, r_1, α and β , such that if $0 < \epsilon < \epsilon_0$,

$$|v|^2 e^{2\phi_1} \leq \beta^{-2} \exp \left\{ -\frac{|z|^2}{\epsilon} (\beta - 2\alpha\epsilon^{1-\delta} r_1^{2\delta-2}) \right\} \leq \beta^{-2} \exp \left(-\frac{\beta r_1^2}{2\epsilon} \right)$$

on $A(r_1, 2r) \times [\epsilon, 2\epsilon]$. Thus, invoking the Dominant Convergence Theorem, the first term on the right hand side of (3.11) converges to zero, as $\epsilon \rightarrow 0$. Similarly, given $\alpha > 8$, there exists $r_0 > r_1$, depending only on α, β and δ , such that if $r > r_0$,

$$\left(|v|^2 + |\nabla v|^2 \right) e^{2\phi_1} \leq \beta^{-2} \exp \left\{ -\frac{2r^2}{t} (\beta - 4\alpha r^{2\delta-2}) \right\} \leq \beta^{-2} e^{-\beta r^2}$$

on $A(r, 2r) \times (0, 1]$. This implies that the last term in the right side of (3.11) converges to zero, as $r \rightarrow \infty$. On the other hand, we can bound the second integration on the right hand side of (3.11) uniformly in α and r . Namely, note that there exists $\gamma_0 = \gamma_0(\delta) > 0$ such that for all $\alpha > 8$ and $r > r_1 + 1$,

$$|v|^2 e^{2\phi_1} \leq \beta^{-2} e^{-2\beta|z|^2 + \gamma_0|z|^{2\delta}} \quad \text{on } A(r_1, 2r) \times [1 - 2\alpha^{-1}, 1 - \alpha^{-1}].$$

Then it follows that

$$\iint_{\mathcal{E}_{1-2/\alpha}^{1-1/\alpha}} |\psi_r v|^2 e^{2\phi_1} d\mu dt \leq L' \alpha^{-1}$$

for some L' depending only on k, n, δ, r_1, β and γ_0 . Hence, sending $\varepsilon \rightarrow 0$ and $r \rightarrow \infty$, it follows from the Monotone Convergence Theorem that for all $\alpha > 8$,

$$\begin{aligned}
& \int_0^{\frac{3}{4}} \int_{A(r_1+1, \infty)} \left(\alpha |v|^2 + t |\nabla_t v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
(3.12) \quad & \leq C' \int_0^1 \int_{A(r_1, r_1+1)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt + C' \alpha,
\end{aligned}$$

where the constant C' depends only on C and L' .

Since the function $t^{-\delta} |z|^{2\delta}$ on $A(r_1, r_1 + 1) \times (0, 1)$ blows up as $t \rightarrow 0$, in order to conclude $v \equiv 0$ by sending $\alpha \rightarrow \infty$, we will need to bound the integration on the right hand side of (3.12) in term of that on the left hand side. On one hand, it is

implied from the relation between v and u and changing variables that

$$\begin{aligned}
& \int_0^{\tau_0} \int_{A(r_1+1, \infty)} \left(\alpha |v|^2 + t |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
&= \int_0^{\tau_0} \int_{A((r_1+1)/\sqrt{t}, \infty)} \left(\alpha |u|^2 + |\nabla u|^2 \right) t^{\frac{n-k+4}{2}} e^{2\alpha |z|^{2\delta}} d\mu dt \\
&= \lambda^{k-n-6} \int_0^{\lambda^2 \tau_0} \int_{A(\lambda(r_1+1)/\sqrt{t}, \infty)} \left(\alpha |u|^2 + |\nabla u|^2 \right) t^{\frac{n-k+4}{2}} e^{2\alpha |z|^{2\delta}} d\mu dt
\end{aligned}$$

for any $\lambda \in (0, 1)$. On the other, by similar reasoning,

$$\begin{aligned}
& \int_0^{\lambda^2 \tau_0} \int_{A(r_1, r_1+1)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
&\leq \lambda^2 \tau_0 \int_0^{\lambda^2 \tau_0} \int_{A(r_1/\sqrt{t}, (r_1+1)/\sqrt{t})} \left(|u|^2 + |\nabla u|^2 \right) t^{\frac{n-k+4}{2}} e^{2\alpha |z|^{2\delta}} d\mu dt.
\end{aligned}$$

Thus, by choosing $\lambda = \min\{1, r_1/(2r_1+2), 1/(2C')\}$ and defining $\tau_1 \doteq \lambda^2 \tau_0$,

$$\begin{aligned}
(3.13) \quad & \int_0^{\tau_1} \int_{A(r_1, r_1+1)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
& \leq \frac{1}{2C'} \int_0^{\tau_0} \int_{A(r_1+1, \infty)} \left(\alpha |v|^2 + t |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt.
\end{aligned}$$

Note that, from Lemma 3.4, the integration on the right hand side of (3.13) is finite and thus can be absorbed into the left side of (3.12) which gives that

$$\begin{aligned}
(3.14) \quad & \frac{1}{2} \int_0^1 \int_{A(r_1+1, \infty)} \left(\alpha |v|^2 + t |\nabla_t v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
& \leq C' \int_{\tau_1}^1 \int_{A(r_1, r_1+1)} t^2 \left(|v|^2 + |\nabla v|^2 \right) e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt + C' \alpha \\
& \leq C'' e^{2\alpha\tau_1^{-\delta} (r_1+1)^{2\delta}} + C' \alpha
\end{aligned}$$

for some C'' depending only on k, n, r_1, C' and $\|u\|_{C^1}$. Hence it follows that

$$\begin{aligned}
(3.15) \quad & \int_0^{\frac{\tau_1}{4}} \int_{A(r_1+1, \infty)} |v|^2 d\mu dt \\
& \leq e^{-4\alpha\tau_1^{-\delta} (r_1+1)^{2\delta}} \int_0^{\frac{\tau_1}{4}} \int_{A(r_1+1, \infty)} |v|^2 e^{2\alpha\eta t^{-\delta} |z|^{2\delta}} d\mu dt \\
& \leq 2C'' \alpha^{-1} e^{-2\alpha\tau_1^{-\delta} (r_1+1)^{2\delta}} + 2C' e^{-4\alpha\tau_1^{-\delta} (r_1+1)^{2\delta}}.
\end{aligned}$$

Therefore, letting $\alpha \rightarrow \infty$, we conclude that $v \equiv 0$ in $A(r_1+1, \infty) \times [0, \tau_1/4]$. \square

4. EXAMPLES OF ASYMPTOTICALLY CYLINDRICAL SELF-SHRINKING ENDS

For each $l > 0$, we construct non-rotationally symmetric self-shrinkers asymptotic to the generalized cylinder $S^{n-1} \times \mathbb{R}$ with the rate (1.4). We will establish weighted Schauder estimates for the linearized operator of self-shrinkers which allow us to solve the self-shrinker equation by the contraction principle. The method here is analogous to that of constructions of non-conical minimal hypersurfaces

with isolated singularities in [2]. However, we cannot apply their arguments and estimates directly, since the linearized operator for $S^{n-1} \times \mathbb{R}$ (see [4, 5]) given by

$$\mathcal{L} \doteq \frac{\partial^2}{\partial z^2} + \underline{\Delta} - \frac{z}{2} \frac{\partial}{\partial z} + 1,$$

behaves very differently from that of minimal hypersurfaces as $z \rightarrow \infty$.

4.1. The linearized problem for self-shrinkers. We seek decaying solutions to the inhomogeneous problem $\mathcal{L}w = f$, provided that f satisfies certain decay conditions. To achieve this, we first use the method of separation of variables to find weak solutions explicitly, then employ the regularity theory for elliptic equations to derive weighted $C^{2,\gamma}$ estimates for some $\gamma \in (0, 1)$.

4.1.1. Existence of weak solutions. Suppose the eigenvalues of the Laplacian on S^{n-1} are

$$0 = \mu_0 < \mu_1 \leq \cdots \leq \mu_{j-1} \leq \mu_j \leq \cdots \rightarrow \infty,$$

and the corresponding eigenfunctions $\{\varphi_j\}_{j \geq 0}$ are chosen to form an orthonormal basis for $L^2(S^{n-1})$. We denote the inner product on $L^2(S^{n-1})$ by $\langle \cdot, \cdot \rangle_{L^2}$. Thus the Fourier expansion of w is given by

$$w = \sum_{j=0}^{\infty} a_j(z) \varphi_j(\theta), \quad \text{where} \quad a_j = \langle w, \varphi_j \rangle_{L^2}.$$

Moreover, w is a H^1 weak solution of $\mathcal{L}w = f$ on $A_+(R, \tilde{R}) \doteq S^{n-1} \times (R, \tilde{R})$, i.e., for all $\xi \in C_c^\infty(A_+(R, \tilde{R}))$,

$$- \int_{A_+(R, \tilde{R})} (\partial_z w \partial_z \xi + \langle \nabla w, \nabla \xi \rangle - w \xi) e^{-\frac{z^2}{4}} d\mu = \int_{A_+(R, \tilde{R})} f \xi e^{-\frac{z^2}{4}} d\mu,$$

if and only if, for each $j \geq 0$,

$$(4.1) \quad \mathcal{M}_j a_j \doteq a_j'' - \frac{1}{2} z a_j' + (1 - \mu_j) a_j = f_j, \quad \text{where} \quad f_j = \langle f, \varphi_j \rangle_{L^2},$$

and

$$\sum_{j=0}^{\infty} (1 + \mu_j) \|a_j\|_{L^2(R, \tilde{R})}^2 + \sum_{j=0}^{\infty} \|a_j'\|_{L^2(R, \tilde{R})}^2 < \infty.$$

To solve (4.1) by reduction of orders, we need to study the kernel of \mathcal{M}_j .

Lemma 4.1. *There exists $R_5 > 1$ such that for each $j \geq 0$, the equation $\mathcal{M}_j b = 0$ possesses a unique positive solution $b = b_j$ on (R_5, ∞) which satisfies $b_j(R_5) = 1$ and*

$$(4.2) \quad C_1^{-1} z^{2(1-\mu_j)} \leq b_j(z) \leq C_1 z^{2(1-\mu_j)}$$

for some $C_1 = C_1(j) > 0$.

Proof. Through the proof, we fix $j \geq 0$ and define $\tilde{b}(z) \doteq z^{2(\mu_j-1)} b(z)$. Then, finding solutions of $\mathcal{M}_j b = 0$ with the desired asymptotics (4.2) is equivalent to finding solutions of the equation

$$(4.3) \quad \tilde{\mathcal{M}}_j \tilde{b} \doteq \tilde{b}'' - \left(\frac{1}{2} z + 4(\mu_j - 1) \frac{1}{z} \right) \tilde{b}' + 2(\mu_j - 1)(2\mu_j - 1) \frac{\tilde{b}}{z^2} = 0,$$

which satisfy that \tilde{b} and its reciprocal stay positive and bounded as $z \rightarrow \infty$. Such solutions will be obtained as limits of sequence of solutions to (4.3) on finite intervals with suitable initial conditions.

Given $R > 0$, by the local theory of linear ODEs, there exist $\bar{R} \in [0, R)$ and a unique solution \tilde{b}_R of (4.3) on $(\bar{R}, R]$ with initial conditions $\tilde{b}_R(R) = 1$ and $\tilde{b}'_R(R) = 0$, such that $(\bar{R}, R]$ is the maximal interval on which \tilde{b}_R exists and its image is contained in $(1/2, 2)$. Thus, the equation (4.3) gives that

$$(4.4) \quad \frac{d}{dz} \left\{ \tilde{b}'_R z^{4(1-\mu_j)} e^{-\frac{z^2}{4}} \right\} = -2(\mu_j - 1)(2\mu_j - 1) \tilde{b}_R z^{2(1-2\mu_j)} e^{-\frac{z^2}{4}}.$$

Integrating (4.4) with respect to z and using $\tilde{b}'_R(R) = 0$, we get

$$\begin{aligned} \left| \tilde{b}'_R(z) \right| &\leq 2|\mu_j - 1| |2\mu_j - 1| e^{\frac{z^2}{4}} \int_z^R \tilde{b}_R \rho^{-2} (\rho z^{-1})^{4(1-\mu_j)} e^{-\frac{\rho^2}{4}} d\rho \\ &\leq 4|\mu_j - 1| |2\mu_j - 1| z^{-5} e^{\frac{z^2}{4}} \int_z^R \rho^3 e^{-\frac{\rho^2}{4}} d\rho \end{aligned}$$

for $z \in (\bar{R}, R]$. Moreover, since

$$\int_z^R \rho^3 e^{-\frac{\rho^2}{4}} d\rho = 2z^2 e^{-\frac{z^2}{4}} + 8e^{-\frac{z^2}{4}} - 2R^2 e^{-\frac{R^2}{4}} - 8e^{-\frac{R^2}{4}},$$

we conclude that

$$(4.5) \quad \left| \tilde{b}'_R(z) \right| \leq C z^{-3} \quad \text{for} \quad \max\{1, \bar{R}\} < z \leq R,$$

where C depends only on j . Furthermore, by (4.3), $z^2 |\tilde{b}''_R(z)|$ is uniformly bounded (depending only on j) for $\max\{1, \bar{R}\} < z \leq R$. In order to pass \tilde{b}_R to limits as $R \rightarrow \infty$, we need to bound \bar{R} from above independent of R . If $\bar{R} \geq 1$, integrating (4.5) gives that

$$(4.6) \quad \left| \tilde{b}_R(R) - \tilde{b}_R(\bar{R}) \right| \leq \frac{C}{2} \left(\frac{1}{\bar{R}^2} - \frac{1}{R^2} \right).$$

By the maximality of the interval $(\bar{R}, R]$, $\tilde{b}_R(\bar{R}) = 1/2$ or 2 . Invoking the initial data $\tilde{b}_R(R) = 1$, it is implied from (4.6) that $\bar{R} \leq \sqrt{C}$ and thus $\bar{R} \leq \max\{1, \sqrt{C}\}$. Hence, sending $R \rightarrow \infty$, there is a subsequence of \tilde{b}_R converging to a solution \tilde{b} of (4.3) on (\bar{R}, ∞) with image in $(1/2, 2)$. Here \bar{R} is defined to be $\max\{1, \sqrt{C}\}$. On the other hand, since (4.3) is linear and regular whenever $z \neq 0$, it follows that \tilde{b} can be extended to a smooth solution on $(0, \infty)$.

For the positivity, we will show that if $\mathcal{M}_j b = 0$ and $|b(z)| \leq L e^{Lz}$ on $(0, \infty)$ for some $L > 0$, then b is never zero on $[2\sqrt{3}, \infty)$ unless b is identically zero. To see this, we first derive the bound on b' . It is easy to see that

$$(4.7) \quad b'(z) = C' + \frac{1}{2} z b(z) + \frac{1}{2} \int_1^z (2\mu_j - 3) b(\rho) d\rho,$$

where C' is an arbitrary constant. Thus, $|b'(z)| \leq L' e^{L'z}$ for some L' depending on j and L . Next, letting $\bar{b}(z) \doteq z^{-4} b(z)$,

$$(4.8) \quad \mathcal{M}_j \bar{b} \doteq \bar{b}'' - \left(\frac{z}{2} - \frac{8}{z} \right) \bar{b}' + \left(-1 - \mu_j + \frac{12}{z^2} \right) \bar{b} = 0.$$

Assume that $b(R) = 0$ for some $R \geq 2\sqrt{3}$. Then $\bar{b}(R) = 0$ and multiplying $\bar{b}z^8e^{-z^2/4}$ on both sides of (4.8) and integrating over (R, ∞) give that

$$(4.9) \quad \int_R^\infty |\bar{b}'| z^8 e^{-\frac{z^2}{4}} dz + \int_R^\infty \left(1 + \mu_j - \frac{12}{z^2}\right) \bar{b}^2 z^8 e^{-\frac{z^2}{4}} dz = 0.$$

Here we use the bounds on b and b' to apply integration by parts. Since $(1 + \mu_j - 12/z^2)$ is non-negative for $z \geq 2\sqrt{3}$, it follows from (4.9) that $\bar{b} \equiv 0$ in $[R, \infty)$. Thus, b either has no zero points on $[2\sqrt{3}, \infty)$ or must be identically zero. Hence we can choose R_5 to be $2\sqrt{3}$ and b_j to be $b/b(2\sqrt{3})$.

Finally, if b_j and \hat{b}_j are two solutions of the initial value problem $\mathcal{M}_j b = 0$ and $b(R_5) = 1$ with the asymptotics (4.2), the difference $b_j - \hat{b}_j$ satisfies $\mathcal{M}_j(b_j - \hat{b}_j) = 0$ vanishing at $z = R_5$ with the same asymptotics. It follows from the previous argument that $b_j \equiv \hat{b}_j$ in $[R_5, \infty)$ and this shows the uniqueness of solutions in the lemma. \square

Given $m > 0$, we define

$$\|f\|_m^2 \doteq \int_{A_+(R_5, \infty)} |f|^2 z^{2m+5} d\mu = \sum_{j=0}^\infty \int_{R_5}^\infty |f_j|^2 z^{2m+5} dz.$$

Suppose that the function f satisfies $\|f\|_m < \infty$. Recall that we are only interested in decaying solutions of $\mathcal{L}w = f$. Thus, using the method of reduction of orders, we solve (4.1) for each $j \geq 0$, that is,

$$(4.10) \quad a_j(z) = \beta_j b_j(z) - b_j(z) \int_{z_j}^z b_j^{-2}(r) e^{\frac{r^2}{4}} \int_r^\infty b_j(\rho) f_j(\rho) e^{-\frac{\rho^2}{4}} d\rho dr,$$

where β_j and z_j are arbitrary constants. Next, we prove decay estimates for a_j by choosing suitable lower limits z_j .

Lemma 4.2. *Given $m > 0$ and $j \geq 0$, there exists $C_2 > 0$, depending only on μ_j , C_1 , R_5 and m , such that*

(1) *if $2\mu_j < m + 3$, choosing $z_j = \infty$ in (4.10),*

$$(4.11) \quad |a_j(z) - \beta_j b_j(z)| \leq C_2 \|f\|_m z^{-m-1} \quad \text{on } [R_5, \infty).$$

(2) *if $2\mu_j > m + 3$, choosing $z_j = R_5$ in (4.10),*

$$(4.12) \quad |a_j(z) - \beta_j b_j(z)| \leq C_2 \|f\|_m z^{-m-1} \quad \text{on } [R_5, \infty).$$

Proof. First, we assume that $2\mu_j < m + 3$. Then, for $z \geq R_5$,

$$\begin{aligned} & b_j(z) \int_z^\infty b_j^{-2}(r) e^{\frac{r^2}{4}} \int_r^\infty b_j(\rho) |f_j(\rho)| e^{-\frac{\rho^2}{4}} d\rho dr \\ & \leq C z^{2(1-\mu_j)} \int_z^\infty r^{4(\mu_j-1)} \int_r^\infty \rho^{2(1-\mu_j)} |f_j(\rho)| d\rho dr \\ & \leq C \|f\|_m z^{2(1-\mu_j)} \int_z^\infty r^{4(\mu_j-1)} \left(\int_r^\infty \rho^{-4\mu_j-2m-1} d\rho \right)^{\frac{1}{2}} dr \\ & \leq C \|f\|_m z^{2(1-\mu_j)} \int_z^\infty r^{2\mu_j-m-4} dr \leq C \|f\|_m z^{-m-1}. \end{aligned}$$

Here C depends only on μ_j , C_1 and m , and we use (4.2) in the first inequality and the Cauchy-Schwarz inequality in the second inequality. This proves the first estimate (4.11) in the lemma.

A similar argument gives that, assuming that $2\mu_j > m + 3$,

$$\begin{aligned} b_j(z) & \int_{R_5}^z b_j^{-2}(r) e^{\frac{r^2}{4}} \int_r^\infty b_j(\rho) |f_j(\rho)| e^{-\frac{\rho^2}{4}} d\rho dr \\ & \leq C' \|f\|_m z^{2(1-\mu_j)} \int_{R_5}^z r^{2\mu_j-m-4} dr \leq C' \|f\|_m z^{-m-1} \end{aligned}$$

for $z \geq R_5$, where C' depends on μ_j , C_1 , R_5 and m . Thus the second estimate (4.12) follows immediately. \square

However, without estimates on C_2 , Lemma 4.2 is not enough to conclude the sum of $\|a_j\|_{L^2}^2$ over j converges for appropriate choices of β_j . Thus we need to obtain improved estimates on $a_j - \beta_j b_j$ for sufficiently large j .

Lemma 4.3. *Given $m > 0$, there exists $J_1 = J_1(n, m) \geq 0$ such that if $j > J_1$,*

$$\sqrt{\mu_j} \|(a_j - \beta_j b_j) z^m\|_{L^2(R_5, \infty)} + \|(a'_j - \beta_j b'_j) z^m\|_{L^2(R_5, \infty)} \leq C_3 \|f_j z^{m+\frac{5}{2}}\|_{L^2(R_5, \infty)}$$

for some universal constant C_3 .

Proof. Let $\tilde{a}_j \doteq z^m(a_j - \beta_j b_j)$. Since $\mathcal{M}_j(a_j - \beta_j b_j) = f_j$, the equation for \tilde{a}_j is given by

$$(4.13) \quad \tilde{a}_j'' - \left(\frac{z}{2} + \frac{2m}{z}\right) \tilde{a}_j' + \left(1 - \mu_j + \frac{m}{2} + m(m+1) \frac{1}{z^2}\right) \tilde{a}_j = z^m f_j.$$

The estimate (4.12) implies that $|\tilde{a}_j(z)|$ decays like $1/z$, and thus it follows from (4.13) that $|\tilde{a}_j'(z)|$ grows at most like $\log z$.

Then we define $R(t) = R_5 \sqrt{1-t}$ and

$$c_j(z, t) \doteq (1-t)^{\frac{m}{2}+1-\mu_j} \tilde{a}_j\left(\frac{z}{\sqrt{1-t}}\right)$$

for $t \in [0, 1/2]$ and $z \in (R(t), \infty)$. Thus, $|c_j(z, t)|$ decays like $1/z$ and $|\partial_z c_j(z, t)|$ grows at most like $\log z$. Moreover, c_j satisfies that

$$(4.14) \quad \frac{\partial c_j}{\partial t} - \frac{\partial^2 c_j}{\partial z^2} + \frac{2m}{z} \frac{\partial c_j}{\partial z} - m(m+1) \frac{c_j}{z^2} = -\frac{z^m}{(1-t)^{\mu_j}} f_j\left(\frac{z}{\sqrt{1-t}}\right).$$

Assuming $2\mu_j > m+3$, then $c_j(R(t), t) = 0$ for all $t \in [0, 1/2]$. Thus, given $\tilde{R} > 2R_5$, multiplying c_j on both sides of (4.14) and integrating over $(R(t), \tilde{R})$ followed by that over $[0, 1/2]$, it follows from integration by parts that

$$\begin{aligned} & - \int_0^{\frac{1}{2}} \int_{R(t)}^{\tilde{R}} c_j f_j \left(\frac{z}{\sqrt{1-t}}\right) \frac{z^m}{(1-t)^{\mu_j}} dz dt = \int_0^{\frac{1}{2}} \left(\frac{m}{\tilde{R}} c_j^2(\tilde{R}, t) - \frac{1}{2} \frac{\partial c_j^2}{\partial z}(\tilde{R}, t) \right) dt \\ & + \frac{1}{2} \int_{R(\frac{1}{2})}^{\tilde{R}} c_j^2\left(z, \frac{1}{2}\right) dz - \frac{1}{2} \int_{R(0)}^{\tilde{R}} c_j^2(z, 0) dz + \int_0^{\frac{1}{2}} \int_{R(t)}^{\tilde{R}} \left(\left| \frac{\partial c_j}{\partial z} \right|^2 - \frac{m^2 c_j^2}{z^2} \right) dz dt. \end{aligned}$$

We pass $\tilde{R} \rightarrow \infty$ and, invoking the bounds on c_j and $\partial_z c_j$ again, the Dominant Convergence Theorem and Monotone Convergence Theorem imply that

$$(4.15) \quad \begin{aligned} & - \int_0^{\frac{1}{2}} \int_{R(t)}^\infty c_j f_j \left(\frac{z}{\sqrt{1-t}}\right) \frac{z^m}{(1-t)^{\mu_j}} dz dt = \frac{1}{2} \int_{R(\frac{1}{2})}^\infty c_j^2\left(z, \frac{1}{2}\right) dz \\ & - \frac{1}{2} \int_{R(0)}^\infty c_j^2(z, 0) dz + \int_0^{\frac{1}{2}} \int_{R(t)}^\infty \left| \frac{\partial c_j}{\partial z} \right|^2 dz dt - \int_0^{\frac{1}{2}} \int_{R(t)}^\infty \frac{m^2 c_j^2}{z^2} dz dt. \end{aligned}$$

Next, we rewrite and estimate each term in (4.15) as integrations of \tilde{a}_j and its derivatives. Recalling the definition of c_j , a straightforward verification gives that

$$\frac{1}{2} \int_{R(\frac{1}{2})}^{\infty} c_j^2 \left(z, \frac{1}{2} \right) dz - \frac{1}{2} \int_{R(0)}^{\infty} c_j^2(z, 0) dz = \frac{1}{2} \left(2^{2\mu_j - m - \frac{5}{2}} - 1 \right) \int_{R_5}^{\infty} \tilde{a}_j^2 dz.$$

If $2\mu_j > m + 3$, then the last term on the right hand side of (4.15) becomes

$$\begin{aligned} \int_0^{\frac{1}{2}} \int_{R(t)}^{\infty} \frac{m^2 c_j^2}{z^2} dz dt &= \int_0^{\frac{1}{2}} (1-t)^{m+\frac{3}{2}-2\mu_j} dt \int_{R_5}^{\infty} \frac{m^2}{z^2} \tilde{a}_j^2 dz \\ &\leq \frac{2m^2}{4\mu_j - 2m - 5} \left(2^{2\mu_j - m - \frac{5}{2}} - 1 \right) \int_{R_5}^{\infty} \tilde{a}_j^2 dz. \end{aligned}$$

For the penultimate term on the right hand side of (4.15), it is easy to see

$$\begin{aligned} \int_0^{\frac{1}{2}} \int_{R(t)}^{\infty} \left| \frac{\partial c_j}{\partial z} \right|^2 dz dt &= \int_0^{\frac{1}{2}} (1-t)^{m+\frac{3}{2}-2\mu_j} dt \int_{R_5}^{\infty} |\tilde{a}_j'|^2 dz \\ &= \frac{2}{4\mu_j - 2m - 5} \left(2^{2\mu_j - m - \frac{5}{2}} - 1 \right) \int_{R_5}^{\infty} |\tilde{a}_j'|^2 dz. \end{aligned}$$

Finally, the left hand side of (4.15) can be bounded by

$$\begin{aligned} & - \int_0^{\frac{1}{2}} \int_{R(t)}^{\infty} c_j f_j \left(\frac{z}{\sqrt{1-t}} \right) \frac{z^m}{(1-t)^{\mu_j}} dz dt \\ &= - \int_0^{\frac{1}{2}} (1-t)^{m+\frac{3}{2}-2\mu_j} dt \int_{R_5}^{\infty} \tilde{a}_j f_j z^m dz \\ &\leq \frac{1}{4\mu_j - 2m - 5} \left(2^{2\mu_j - m - \frac{5}{2}} - 1 \right) \int_{R_5}^{\infty} (\tilde{a}_j^2 + f_j^2 z^{2m}) dz. \end{aligned}$$

Here we use the Cauchy-Schwarz inequality in the last inequality. Note that the L^2 norm of \tilde{a}_j over (R_5, ∞) is finite. Thus it follows from (4.15) that

$$\begin{aligned} (4.16) \quad & \left(\frac{1}{2} - \frac{2m^2 + 1}{4\mu_j - 2m - 5} \right) \int_{R_5}^{\infty} \tilde{a}_j^2 dz + \frac{2}{4\mu_j - 2m - 5} \int_{R_5}^{\infty} |\tilde{a}_j'|^2 dz \\ & \leq \frac{1}{4\mu_j - 2m - 5} \int_{R_5}^{\infty} f_j^2 z^{2m+5} dz, \end{aligned}$$

provided that $2\mu_j > m + 3$. It is easy to deduce from (4.16) that there exists $J \geq 0$, depending only on n and m , and a universal constant $C > 0$ such that for $j > J$,

$$(4.17) \quad \int_{R_5}^{\infty} (\mu_j \tilde{a}_j^2 + |\tilde{a}_j'|^2) dz \leq C \int_{R_5}^{\infty} f_j^2 z^{2m+5} dz.$$

□

Given $J \geq 0$, we define the projection operator $\mathcal{P}_J : L^2(S^{n-1}) \rightarrow L^2(S^{n-1})$ by

$$\mathcal{P}_J(g) = \sum_{j=J}^{\infty} \langle g, \varphi_j \rangle_{L^2} \varphi_j,$$

and the extension operator $\mathcal{H}_J : L^2(S^{n-1}) \rightarrow C^\infty(S^{n-1} \times (R_5, \infty))$ by

$$\mathcal{H}_J(g) = \sum_{j=J}^{\infty} \langle g, \varphi_j \rangle_{L^2} b_j \varphi_j.$$

Therefore, we summarize the results of Lemmas 4.1, 4.2 and 4.3 into the following proposition.

Proposition 4.4. *Suppose $m > 0$ with $2\mu_{J-1} - 3 < m < 2\mu_J - 3$ for some J , and functions $g \in L^2(S^{n-1})$ with $\mathcal{H}_J(g) \in \cap_{R>R_5} H^1(A_+(R_5, R))$ and $f \in L^2(A_+(R_5, \infty))$ with finite $\|f\|_m$. Then there is a weak solution w of $\mathcal{L}w = f$ in $\cap_{R>R_5} H^1(A_+(R_5, R))$ with $\mathcal{P}_J(w(\cdot, R_5)) = \mathcal{P}_J(g)$. Indeed, w is of the explicit form $w = \mathcal{H}_J(g) + \sum_{j=0}^{\infty} I_j \varphi_j$, where, if $j \leq J-1$,*

$$I_j(z) = b_j(z) \int_z^\infty b_j^{-2}(r) e^{\frac{r^2}{4}} \int_r^\infty b_j(\rho) f_j(\rho) e^{-\frac{\rho^2}{4}} d\rho dr,$$

and if $j \geq J$,

$$I_j(z) = -b_j(z) \int_{R_5}^z b_j^{-2}(r) e^{\frac{r^2}{4}} \int_r^\infty b_j(\rho) f_j(\rho) e^{-\frac{\rho^2}{4}} d\rho dr.$$

Moreover, there exists $C_4 > 0$, depending only on n, m, C_2 and C_3 , such that

$$(4.18) \quad \sup_{z \geq R_5} z^m \|w(\cdot, z) - \mathcal{H}_J(g)(\cdot, z)\|_{L^2} \leq C_4 \|f\|_m.$$

4.1.2. Weighted Schauder estimates. Since the self-shrinker equation is quasilinear, it is more convenient to develop a weighted Schauder theory, i.e., weighted Hölder estimates, for the linearized operator \mathcal{L} . First, we introduce suitable Hölder spaces for functions and tensors on Riemannian manifolds. Let $P_{\mathbf{x}, \mathbf{y}}$ denote the parallel transport along the minimizing geodesic from \mathbf{x} to \mathbf{y} in $S^{n-1} \times \mathbb{R}$. Also, the geodesic distance in $S^{n-1} \times \mathbb{R}$ is denoted by d . Then, given $0 < \gamma < 1$ and $j \in \{0, 1, 2\}$, we define

$$\|\psi\|_{j, \gamma, \Omega} \doteq \sup_{\mathbf{x} \in \Omega} \sum_{i=0}^j |\nabla_1^i \psi(\mathbf{x})|_c + \sup_{\mathbf{x}, \mathbf{y} \in \Omega, \mathbf{x} \neq \mathbf{y}} \frac{|\nabla_1^j \psi(\mathbf{x}) - P_{\mathbf{x}, \mathbf{y}}^* \nabla_1^j \psi(\mathbf{y})|_c}{d(\mathbf{x}, \mathbf{y})^\gamma}$$

for all functions ψ on a subset $\Omega \subset S^{n-1} \times \mathbb{R}$. Here, $|\cdot|_c$ denotes the length of tensors induced by the cylindrical metric on $S^{n-1} \times \mathbb{R}$. And the space $C^{j, \gamma}(\Omega)$ is defined to be the set of all functions ψ on Ω with finite $\|\psi\|_{j, \gamma, \Omega}$. For simplicity, we use $\|\psi\|_{j, \gamma, r}$ to denote $\|\psi\|_{j, \gamma, \overline{A_+(r, 2r)}}$. We remark that it is equivalent to define the norm to take the second sup over points \mathbf{x} and \mathbf{y} with $d(\mathbf{x}, \mathbf{y})$ small. Also, by the rotational and translational symmetries of $S^{n-1} \times \mathbb{R}$, there exist ε_2 and C_5 , depending only on n and $\gamma \in (0, 1)$, such that for $j \in \{0, 1, 2\}$, $\mathbf{x} \in S^{n-1} \times \mathbb{R}$ and any domain $\Omega_{\mathbf{x}} \subset \exp_{\mathbf{x}}(B_{2\varepsilon_2}^n)$,

$$C_5^{-1} \|\psi\|_{j, \gamma, \Omega_{\mathbf{x}}} \leq \|\exp_{\mathbf{x}}^* \psi\|_{C^{j, \gamma}(\exp_{\mathbf{x}}^{-1}(\Omega_{\mathbf{x}}))} \leq C_5 \|\psi\|_{j, \gamma, \Omega_{\mathbf{x}}},$$

where \exp is the exponential map of $S^{n-1} \times \mathbb{R}$.

In addition to the hypotheses of Proposition 4.4, we make further $C^{0, \gamma}$ and $C^{2, \gamma}$ regularity assumptions on f and g respectively. Thus, if $\exp_{\mathbf{x}}(B_{\varepsilon_2}^n)$ is contained in $A_+(R_5, \infty)$, we can apply the local boundedness (Theorem 8.17 in [13]) and the Schauder interior estimate (Theorem 6.2 in [13]) to $\exp_{\mathbf{x}}^* w$ in $B_{\varepsilon_2}^n$. Otherwise, there exists $\tilde{\mathbf{x}} \in \partial A_+(R_5, \infty)$ such that $\exp_{\mathbf{x}}(B_{\varepsilon_2}^n) \cap \overline{A_+(R_5, \infty)}$ is contained in $\exp_{\tilde{\mathbf{x}}}(B_{2\varepsilon_2}^n) \cap \overline{A_+(R_5, \infty)}$, then we can use the boundary boundedness (Theorem 8.25 in [13]) and the Schauder boundary estimate (Lemma 6.4 and Theorem 6.6 in [13]) to $\exp_{\tilde{\mathbf{x}}}^* w$ in a half of $B_{2\varepsilon_2}^n$. Hence, combining with (4.11) and (4.18), we conclude that

Proposition 4.5. *Given $m > 0$ with $2\mu_{J-1} - 3 < m < 2\mu_J - 3$ for some $J > 0$, $\gamma \in (0, 1)$, and functions $g \in C^{2,\gamma}(S^{n-1})$ with $\mathcal{H}_J(g) \in C^{2,\gamma}(A_+(R_5, \infty))$ and $f \in C^{0,\gamma}(A_+(R_5, \infty))$ with*

$$\sup_{r>R_5} r^{m+4} \|f\|_{0,\gamma,r} < \infty,$$

there exist $m_0 > 0$, depending only on n and γ , and $C_6 > 0$, depending only on $n, m, \gamma, \varepsilon_2, C_4$ and C_5 , such that the solution w in Proposition 4.4 is in $\cap_{R>R_5} C^{2,\gamma}(A_+(R_5, R))$ and satisfies that

$$(4.19) \quad \sup_{r>R_5} r^{m-m_0} \|w - \mathcal{H}_J(g)\|_{2,\gamma,r} \leq C_6 \sup_{r>R_5} r^{m+4} \|f\|_{0,\gamma,r}.$$

4.2. A fixed point theorem. To solve the self-shrinker equation, it is equivalent to find fixed points of the operator \mathcal{T} , which is roughly defined in the following way: given a function ψ on some domain $\Omega \subset S^{n-1} \times \mathbb{R}$, $\mathcal{T}\psi$ is a solution of $\mathcal{L}(\mathcal{T}\psi) = Q(\mathbf{x}, \psi, \nabla_1 \psi, \nabla_1 \nabla_1 \psi)$ in Ω with suitable boundary conditions. Here we recall that Q is the nonlinear term in (3.1). The key is to prove that \mathcal{T} is a contraction mapping on some closed subset of an appropriate Banach space.

We start with deriving the structural inequalities for the nonlinear term Q . It follows from the proof of Lemma 2.4 in [22] that there exist K and N such that

$$Q(\mathbf{x}, \psi, \nabla_1 \psi, \nabla_1 \nabla_1 \psi) = K(\mathbf{x}, \psi, \nabla_1 \psi) * \nabla_1 \nabla_1 \psi + N(\mathbf{x}, \psi, \nabla_1 \psi),$$

where $*$ denotes contractions by the induced metric on $S^{n-1} \times \mathbb{R}$. Moreover, there exists a function Λ on $S^{n-1} \times \mathbb{R}$ of at most linear growth such that, at $\mathbf{x} \in S^{n-1} \times \mathbb{R}$, $-1 \leq z \leq 1$ and $\mathbf{p} \in B_1^{n+1}$, the following inequalities, which are similar to those in [2], hold for $K(\mathbf{x}, z, \mathbf{p})$ and $N(\mathbf{x}, z, \mathbf{p})$:

$$(4.20) \quad \begin{aligned} |N| + |N_{\mathbf{x}}| &\leq \Lambda(|z| + |\mathbf{p}|)^2, \\ |N_z| + |N_{\mathbf{p}}| + |N_{\mathbf{x}z}| + |N_{\mathbf{x}\mathbf{p}}| + |K| + |K_{\mathbf{x}}| &\leq \Lambda(|z| + |\mathbf{p}|), \\ |N_{zz}| + |N_{z\mathbf{p}}| + |N_{\mathbf{p}\mathbf{p}}| + |K_z| + |K_{\mathbf{p}}| \\ &+ |K_{\mathbf{x}z}| + |K_{\mathbf{x}\mathbf{p}}| + |K_{zz}| + |K_{z\mathbf{p}}| + |K_{\mathbf{p}\mathbf{p}}| \leq \Lambda, \end{aligned}$$

where the subscripts denote partial differentiation.

Therefore, Theorem 1.3 follows immediately from the following theorem.

Theorem 4.6. *Given $0 < \gamma < 1$ and $m > 2m_0 + 5$ with $2\mu_{J-1} - 3 < m < 2\mu_J - 3$ for some $J > 0$, there exists $\epsilon_3 > 0$, depending only on n, γ, m, C_5, C_6 and $\Lambda/|\mathbf{x}|$, such that if $g \in C^{2,\gamma}(S^{n-1})$ with $\mathcal{P}_J(g) \neq 0$ and*

$$(4.21) \quad \sup_{r>R_5} r^{m-m_0} \|\mathcal{H}_J(g)\|_{2,\gamma,r} \leq \epsilon_3,$$

then there exists a solution u of (3.1) with $\mathcal{P}_J(u(\cdot, R_5)) = \mathcal{P}_J(g)$ which satisfies that

$$(4.22) \quad \sup_{r>R_5} r^{m-m_0} \|u\|_{2,\gamma,r} \leq 2\epsilon_3.$$

Proof. Throughout, we fix $\gamma \in (0, 1)$ and $m > 2m_0 + 5$ with $2\mu_{J-1} < m + 3 < 2\mu_J$ for some $J > 0$. Let X denote the subspace of $C^{2,\gamma}(A_+(R_5, \infty))$ for which

$$\|\psi\|_X \doteq \sup_{r>R_5} r^{m-m_0} \|\psi\|_{2,\gamma,r}$$

is finite. Then X is a Banach space with the norm $\|\cdot\|_X$. Moreover, for each $\epsilon \in (0, 1/2)$ and $g \in C^{2,\gamma}(S^{n-1})$, the set

$$X_{\epsilon,g} \doteq \{\psi \in X, \|\psi\|_X \leq 2\epsilon, \mathcal{P}_J(\psi(\cdot, R_5)) = \mathcal{P}_J(g)\}^1$$

is closed in X .

For simplicity, we set up the notation that

$$E_\psi \doteq K(\mathbf{x}, \psi, \nabla_1 \psi) * \nabla_1 \nabla_1 \psi + N(\mathbf{x}, \psi, \nabla_1 \psi) \quad \text{for } \psi \in X.$$

Thus, taking ψ and $\varphi \in X_{\epsilon,g}$, we deduce from the structural inequalities (4.20) that

$$(4.23) \quad \|E_\psi\|_{0,\gamma,r} \leq C\Lambda r^{2m_0-2m} \|\psi\|_X^2,$$

$$(4.24) \quad \|E_\psi - E_\varphi\|_{0,\gamma,r} \leq C\Lambda r^{2m_0-2m} (\|\psi\|_X + \|\varphi\|_X) \|\psi - \varphi\|_X,$$

for some $C = C(n, \gamma) > 0$. Since $m > 2m_0 + 5$ and Λ grows at most linearly, invoking (4.23) and Proposition 4.5, we can define the operator $\mathcal{T}_g : X_{\epsilon,g} \rightarrow X$ such that $\mathcal{T}_g \psi$ is the solution of $\mathcal{L}(\mathcal{T}_g \psi) = E_\psi$ constructed in Proposition 4.4 with boundary condition $\mathcal{P}_J(\mathcal{T}_g \psi(\cdot, R_5)) = \mathcal{P}_J(g)$. Again, Proposition 4.5, together with the estimates (4.23) and (4.24), implies that for all ψ and φ in $X_{\epsilon,g}$,

$$(4.25) \quad \|\mathcal{T}_g \psi - \mathcal{H}_J(g)\|_X \leq C'\epsilon^2,$$

$$(4.26) \quad \|\mathcal{T}_g \psi - \mathcal{T}_g \varphi\|_X \leq C'\epsilon \|\psi - \varphi\|_X,$$

where $C' > 0$ depends only on C_6 , C and $\Lambda/|\mathbf{x}|$. Therefore, if $\epsilon < 1/C'$ and $\|\mathcal{H}_J(g)\|_X \leq \epsilon$, then, by (4.25) and (4.26), \mathcal{T}_g is a contraction mapping from $X_{\epsilon,g}$ into itself and thus has a unique fixed point, that is a solution of the self-shrinker equation (3.1) in $X_{\epsilon,g}$. \square

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¹ $\psi(\cdot, R_5)$ is the restriction of ψ onto $\partial A_+(R_5, \infty)$ by the trace operator.

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