11.2 The Quadratic Formula

In the last section, we added the method *completing the square* to our arsenal of tools to solve quadratic equations. Every tool that we have seen so far has had the downside of only working for a specific case of quadratics. This section introduces the method that trumps all others – *the quadratic formula*. While unwieldy and occasionally time consuming, the quadratic formula works in *all* cases and will directly give you the solutions of a quadratic equation. We will also discuss a specific part of the quadratic formula, namely the *discriminant* and discuss how it is used to categorize the solutions of an equation before we calculate them.

The Quadratic Formula Given a quadratic equation written in the form $ax^2+bx+c=0$ with $a \neq 0$, the solutions of the equation are given as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 11.2.1
Use the quadratic formula to solve the following:

$$2x^2 + 9x - 5 = 0$$

Example 11.2.2
Use the quadratic formula to solve the following:

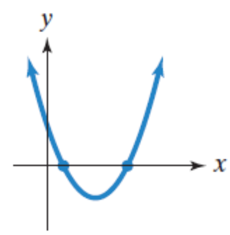
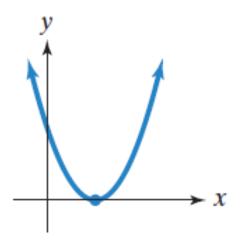
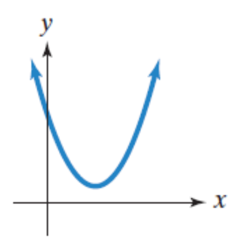
$$2x^2 = 6x - 1$$

Example 11.2.3
Use the quadratic formula to solve the following:

$$3x^2 + 5 = -6x$$

The Discriminant

The discriminant is the radicand in the quadratic formula. Typically we use the Greek letter Δ (delta) to represent the discriminant and define it as $\Delta = b^2 - 4ac$. We are able to use the determinant to determine what type of solutions an equation will have without actually solving for the solutions. In a class such as this, the discriminant isn't the most useful of tools, but it turns out that in higher level math classes the discriminant is invaluable.

Discriminant $b^2 - 4ac$	Kinds of Solutions to $ax^2 + bx + c = 0$	Graph of $y = ax^2 + bx + c$
$b^2 - 4ac > 0$	**Two unequal real solutions:** If a, b, and c are rational numbers and the discriminant is a perfect square, the solutions are *rational*. If the discriminant is not a perfect square, the solutions are *irrational* conjugates.	 Two x-intercepts
$b^2 - 4ac = 0$	**One solution (a repeated solution) that is a real number:** If a, b, and c are rational numbers, the repeated solution is also a rational number.	 One x-intercept
$b^2 - 4ac < 0$	**No real solution; two imaginary solutions:** The solutions are complex conjugates.	 No x-intercepts

Using the discriminant is a fairly straightforward process. Start by calculating the discriminant ($\Delta = b^2 - 4ac$) and then identify the category from the given table into which it falls.

Example 11.2.4
Use the discriminant to classify the types of solutions each equation has:

1. $x^2 + 6x + 9 = 0$

2. $2x^2 - 7x - 4 = 0$

3. $3x^2 - 2x + 4 = 0$

Working Backwards – Equations from Solutions

We know that in the process of factoring and solving a quadratic equation, we typically end up with a line that looks like $(x-a)(x-b)=0$ which we then break apart and solve yielding solutions of $x=a$ and $x=b$. However, what if we know the solutions and want to determine an equation that has those solutions? Notice than I say *an* equation. It is possible for there to be an infinite number of equations all with the same solutions.

Working Backwards If we have solutions given by $x=a$ and $x=b$, start by setting $(x-a)(x-b)=0$ regardless of what a and b are (they could be whole number, decimals, radicals, fractions, imaginary, complex, etc.). Then, FOIL the expression and rewrite in standard form. You can obtain other equations that have the same solutions by multiple (or dividing) the equation by some value.

Example 11.2.5
Find a quadratic equation with each of the following as roots:

1. $x=-3/5, x=1/4$

2. $x=-5\sqrt{2}, x=5\sqrt{2}$

Example 11.2.6
Find a quadratic equation with the root $-7i$.

Example 11.2.7
Find a *cubic* equation with the roots $0, 1$ and 3.

Example 11.2.8
Find a *cubic* equation with the roots $-1, 1$ and 2.