10.7 Complex Numbers

The Imaginary Unit i

Before we define complex numbers, we need to define *imaginary numbers*. When solving some quadratic (or higher degree) equations, we come across solutions that don't necessarily seem to work. Consider the quadratic $y = x^2 + 1$. If we were to factor that quadratic to find the zeroes (roots/solutions), we see that it, well, it doesn't factor. We could use the squareroot property to solve it though.

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \end{aligned}$$

We define the *imaginary unit* i as the principal root of the above equation: $i = \sqrt{-1}$. Using this imaginary unit, we can further reduce radicals that we previously could not.

Taking the Square Root of a Negative Real Number
If b is a positive real number, then

$$\sqrt{-b} = \sqrt{b \cdot -1} = \sqrt{b} \cdot \sqrt{-1} = \sqrt{b}i$$

Example 10.7.1. Find each of the following:

1. $\sqrt{-9}$

$= \sqrt{9} \cdot \sqrt{-1}$

$= 3i$

2. $\sqrt{-43}$

$= \sqrt{43}\sqrt{-1}$ 43 is a prime number

$= \sqrt{43}\, i$

Complex Numbers

Complex numbers are named such because they are composed of two parts – a *real* part (a) and an *imaginary* part ($b\imath$) where both a and b are real numbers. The symbol $\mathbb{R}$ is used to represent the set of real numbers and the symbol $\mathbb{C}$ is used to represent the set of complex numbers. While there isn't a standard symbol to be used for imaginary numbers, the convention is to use $i\mathbb{R}$.

The set $\mathbb{C}$ of complex numbers is given as

$$\mathbb{C} = \{a + bi \mid a \in \mathbb{R}, b \in \mathbb{R}\}$$

Good to know, but no need to memorize!

Operations on Complex Numbers

Operations on complex numbers can be fairly straightforward. As far as addition and subtraction go, treat the real and imaginary parts as like terms and combine as appropriate.

General Method for Adding, Subtracting and Multiplying

1. $(a + bi) + (c + di) = (a + c) + (b + d)i$
2. $(a + bi) - (c + di) = (a - c) + (b - d)i$
3. $(a + bi)(c + di) = (ac - db) + (ad + db)i$

Don't memorize! Pretend that the i is an x and combine like terms like normal!

While the definition for multiplying complex numbers looks unpleasant, it follows the FOIL method that we have employed throughout the semester. The only difference is that we will encounter an i^2 and will need to deal with that.

We know that $i = \sqrt{-1}$ by definition. If we want to find i^2, use the definition as follows:

$$i^2 = \sqrt{-1}^2 = -1$$

We will use a similar method to find other powers of i later in this section.

Example 10.7.2. Let $a = 3 - 4i$ and $b = 6 + 2i$. Find each of the following:

1. $a + b$

$= (3-4i) + (6+2i)$

$= 3 + 6 - 4i + 2i$

$= 9 - 2i$

2. $a - b$

$= (3-4i) - (6+2i)$

$= 3 - 4i - 6 - 2i$

$= 3 - 6 - 4i - 2i \quad = -3 - 6i$

3. $a \cdot b$

$= (3-4i)(6+2i)$

$= 3 \cdot 6 + 3 \cdot 2i + (-4i)(6) + (-4i)(2i)$

$= 18 + 6i - 24i - 8i^2$ $\quad i^2 = -1$

$= 18 + 6i - 24i + 8 \quad = 26 - 18i$

Example 10.7.3. Let $a = -2 + 3i$ and $b = -5 - 7i$. Find each of the following:

1. $a + b$

$= (-2+3i) + (-5-7i)$

$= -2 - 5 + 3i - 7i$

$= -7 - 4i$

2. $a - b$

$= (-2+3i) - (-5-7i)$

$= (-2+3i) + (5+7i)$

$= -2 + 5 + 3i + 7i \quad = 3 + 10i$

3. $a \cdot b$

$(-2+3i)(-5-7i)$

$= (-2)(-5) + (-2)(-7i) + (3i)(-5) + (3i)(-7i)$

$= 10 + 14i - 15i - 21i^2$ $\quad i^2 = -1$

$= 10 + 14i - 15i + 21 \quad = 31 - i$

Dividing Complex Numbers

In Chapter 10, we simplified expressions involving radicals – this included dividing radical expressions by other radical expressions. In order to do so, we used the *conjugate* to remove (*rationalize*) the radical from the denominator. Dividing complex numbers uses a similar technique.

Finding the Conjugate of a Complex Number
The conjugate of a complex number is found much like the conjugate of a radical expression. Identify the sign (or operation) and change to its inverse. In general, the conjugate of a complex number $z = a + bi$ is given as $\bar{z} = a - bi$.

Example 10.7.4. Find the complex conjugate of each.

1. $12 - 4i$ Change − to +

$= 12 + 4i$

2. $-3 + 2i$ Change + to −

$= -3 - 2i$

don't memorize! just multiply!

Dividing Complex Numbers
Dividing complex numbers involves multiplying and dividing by the conjugate of the denominator. In general, we say

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

Example 10.7.5. Divide and write as a complex number in the form $a + bi$.

$$\frac{6+2i}{4-3i}$$

$$= \frac{(6+2i)}{(4-3i)} \cdot \frac{(4+3i)}{(4+3i)}$$

diff. of squares

$$= \frac{6 \cdot 4 + 6 \cdot 3i + 2i \cdot 4 + 2i \cdot 3i}{4^2 - (3i)^2}$$

$i^2 = -1$

$$= \frac{24 + 18i + 8i + 6i^2}{16 + 9}$$

$$= \frac{24 + 26i - 6}{25} = \frac{18 + 26i}{25} = \frac{18}{25} + \frac{26}{25}i$$

Not the correct form

correct form $(a+bi)$

Example 10.7.6. Divide and write as a complex number in the form $a + bi$.

$$\frac{3-2i}{4i}$$

Conjugate of $4i$? $-4i$

$\frac{(3-2i)}{4i} \cdot \frac{-4i}{-4i}$

$= \frac{-4i \cdot 3 - 2i(-4i)}{-16i^2}$

$i^2 = -1$

$= \frac{-12i + 8i^2}{16} = \frac{-8-12i}{16} = \frac{-8}{16} - \frac{12}{16}i = \frac{-1}{2} - \frac{3}{4}i$

Reduce!

Powers of i

We used the fact that $i^2 = -1$ to multiply complex numbers, but what about other powers of i? The first few are easy enough to compute:

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= \sqrt{-1}^2 = -1 \\ i^3 &= i^2 \cdot i = -1 \cdot i = -i \\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1 \end{aligned}$$

After the fourth power, the values become cyclical – that is, they begin to repeat: $i^5 = i, i^6 = -1, i^7 = -i, \ldots$.

Use your exponent rules!

Simplifying Powers of i
Use the fact that $i^4 = 1$ and that $1^m = 1$ for any value of m in order to simplify powers of i. Start by finding the largest multiple of 4 less than the exponent. Break the original exponent into two problems using exponent rules – one that is a multiple of 4 and one that is the remainder. The factor that has a multiple of 4 becomes 1, leaving you with the second factor.

Example 10.7.7. Simplify each of the following:

1. i^{16}

$i^4 = 1$

$= (i^4)^4$

$= (1)^4 = 1$

2. i^{25}

$= i^{24} \cdot i^1$

$= (i^4)^6 \cdot i$

$= 1^6 \cdot i = i$

3. i^{35}

$= i^{32} \cdot i^3$

$= (i^4)^8 \cdot i^3$

$= 1^8 \cdot i^3$

$= i^3 = -i$