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# Improving Boundary Derivative Recovery in Elliptic PDEs

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In collaboration with:
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September 30, 2017
SIAM Central States Meeting

# Goals

- Efficient numerical methods for elliptic PDEs
- Increased accuracy in (normal) boundary derivatives
- No postprocessing (achieve an energy estimate)
- Order $n+1$ data should lead to order $n+1$ derivatives

# Assumptions

- Linear, constant coefficient elliptic PDEs, Dirichlet (or periodic) boundary conditions
- *Lots* of regularity
- Uniform grid (constant $\Delta x$, $\Delta y$) of tensor product elements (no triangles)
- Continuous finite element spaces
  - in 1D, degree $n$ on interior elements, degree $n + p$ on nonperiodic edge elements
  - in 2D, bilinear on interior elements, degree $1 \otimes 1 + p$ on nonperiodic edge elements

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# What's the plan?

- Static *p*-refinement: some cells have higher degree polynomial bases
- Global *h*-refinement: no mesh adaptivity (yet)
- Not really an *hp*-element method, but similar

# What's the plan?

Picking the right type of $p$-refinement improves the boundary derivative convergence rates.

# Superconvergence in 1D

Classic idea for $-u_{xx} + bu_x + cu = f$ (Arnold et al, 1974):

$$\begin{aligned} |u^h(i\Delta x) - u(i\Delta x)| &= |a(G_{i\Delta x}, u^h - u)| \\ &= |a(G_{i\Delta x} - v^h, u^h - u)| \\ &\leq C_1 \|G_{i\Delta x} - v^h\|_{H^1} \|u^h - u\|_{H^1} \\ &\leq C_2 \Delta x^{2k}. \end{aligned}$$

*Special case*: $G_{L-\Delta x} = O(\Delta x)$ to the left of $L - \Delta x$. If we $p$-refine on the right we obtain

$$|u(L - \Delta x) - u^h(L - \Delta x)| \leq C \Delta x^{2n+1}.$$

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# Improving boundary derivative convergence in 1D

For

$$-u_{xx} + bu_x + cu = f$$

$$\left|\frac{d^m}{dx^m}(u - u^h)(L)\right| \leq C_1 \Delta x^{n+p+1-m} + C_2 \Delta x^{2n}$$

- First term: error on the last cell
- Second term: error from coupling to the rest of the domain (the knot estimate)

$m = 2, n = 1, p = 2 \Rightarrow$ linears everywhere, cubics on boundary cells; a second order accurate second derivative on the boundary.

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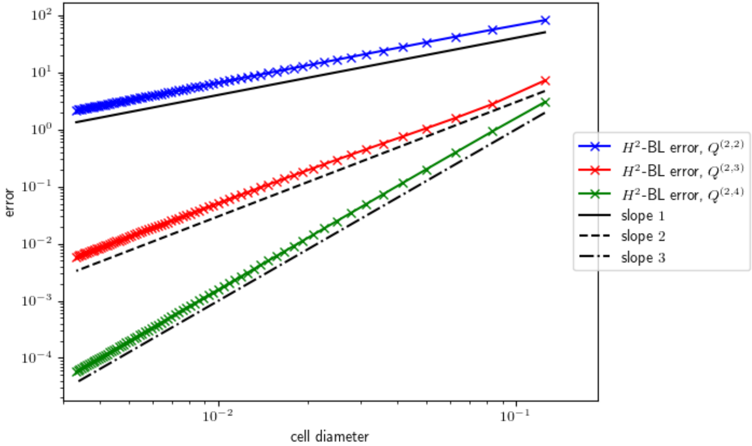
# Numerical Results

Domain of [0, 1], with manufactured solution

$$u = \sin(10x)$$

- Use `deal.II`'s *hp*-finite element support
- Bulk order 2, boundary orders 2, 3, and 4
- CDR equation
- Solve with GMRES

# Numerical Results




Convergence rates for the $p$-refinement scheme of the second derivative on the boundary.

# Extension to 2D at Mesh Knots

With periodic boundary conditions we can use Fourier mode interpolants as trial functions:

$$u^h(x,y) = \sum_{k=0}^{N-1} F_k(x)\hat{u}_k^h(y) : F_k(x) \in V^{\Delta x}([0,1]), \hat{u}^h(y) \in V^{\Delta y}([0,L]).$$

$$\int_0^L \hat{u}_{k,y}^h \bar{\phi}_y + b_1 \hat{u}_{k,y} \bar{\phi} + \frac{\lambda_{A,k}}{\lambda_{M,k}} \hat{u}_k^h \bar{\phi} dy = \int_0^L \left( \int_0^1 \frac{f(x,y) F_k(x)}{\lambda_{M,k}} dx \right) \bar{\phi} dy, \forall \phi(y) \in V^{\Delta y}.$$

Here $\lambda_{M,k}$ and $\lambda_{A,k}$ are eigenvalues of the (circulant) 1D mass and stiffness matrices.

The 1D result applies to each $\hat{u}_k^h(y)$, implying normal derivative superconvergence at mesh knots.

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# Numerical Results

Manufactured solution with $\vec{b} = (1, 1)$ and $c = 2$:

$$u(x, y) = (y^3 + \exp(-y^2) + \sin(4.5y^2) + \sin(20y))(20\cos(4\pi x) + 0.1\sin(20\pi x) - 80\sin(6\pi x))$$

$H^1$-$B$ error defined as

$$\max_{i,j} \left| (\nabla u^h \cdot \vec{n} - \nabla u \cdot \vec{n})(\delta_i, \delta_j) : (\delta_i, \delta_j) \in \partial\Omega^h \right|$$

and $H^2$-$B$ error defined as

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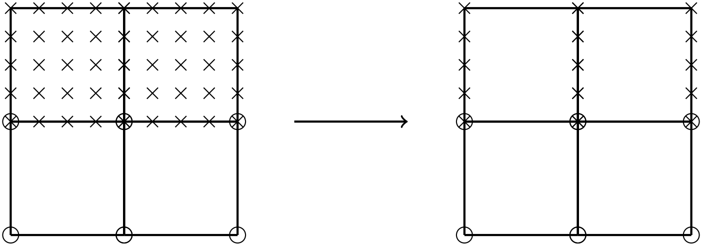
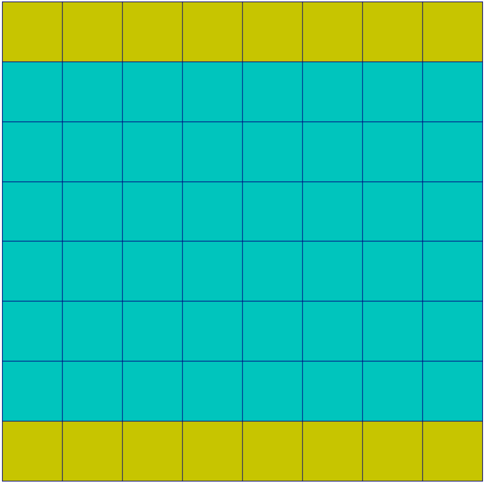
# Numerical Results

Elimination of nonnormal $p$-refinement:



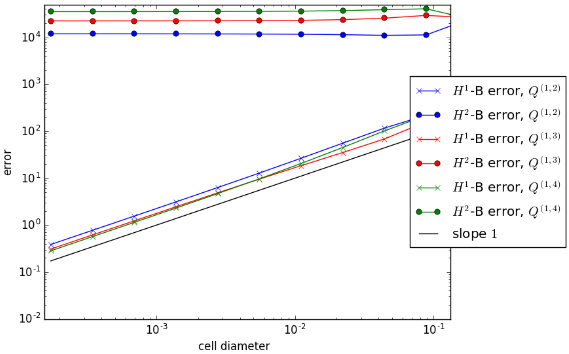
Depiction of normal $p$-refinement scheme, with elimination of extra DoFs. Linear DoFs are circles, quartic are crosses.

# Numerical Results



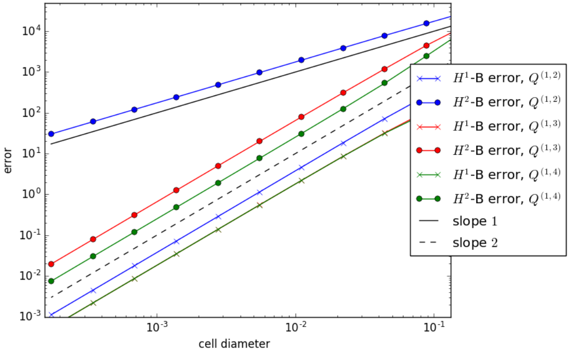
Depiction of the square grid with bulk cells in cyan and $p$-refined cells in yellow.

# Numerical Results




Convergence rates with anisotropic $p$-refinement. We do not recover any higher order accurate derivatives.

# Numerical Results




Convergence rates with isotropic $p$-refinement. The 1D theory applies at the knots.

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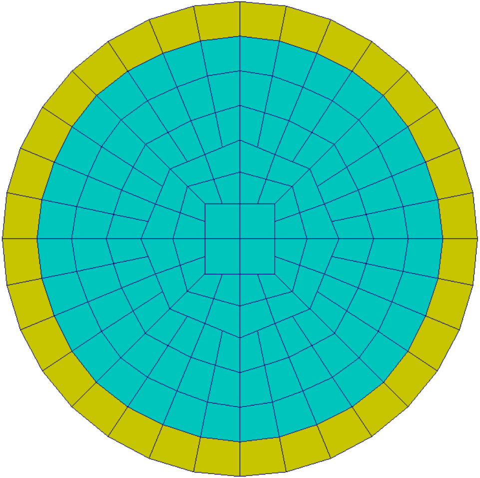

A circular grid with a geometry described near the boundary in polar coordinates, in the middle with Cartesian coordinates, and a transfinite interpolation in-between.

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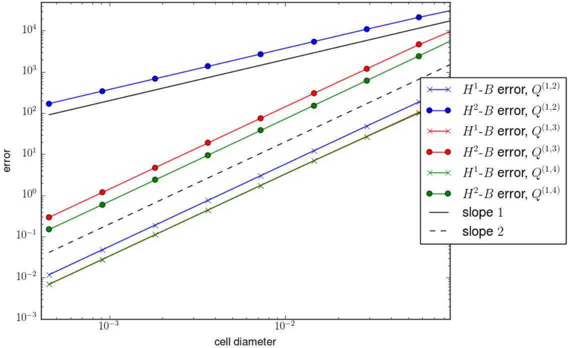


Rates of convergence for the circular grid.

# Summary and Outlook

- $p$-refinement error estimates involve a coupling term and a local term
- We can achieve higher-order derivative approximation at isolated points
- Future work: escape the Fourier framework, higher-order bulk elements, better tools for describing arbitrary geometries

Thank You!