Rensselaer

Decreasing Energy Aliasing in POD-ROMs

David Wells
Rensselaer Polytechnic Institute

In collaboration with:
T. Iliescu (VT), V. John (WIAS), S. Giere (WIAS)

June 28, 2017
Conference on Classical and Geophysical Fluid Dynamics

Outline

- Governing Equations
- A Little on Streamline-Upwind Petrov-Galerkin (SUPG)
- The SUPG-ROM
- Numerical Results
- Summary

Convection-Diffusion-Reaction Equation

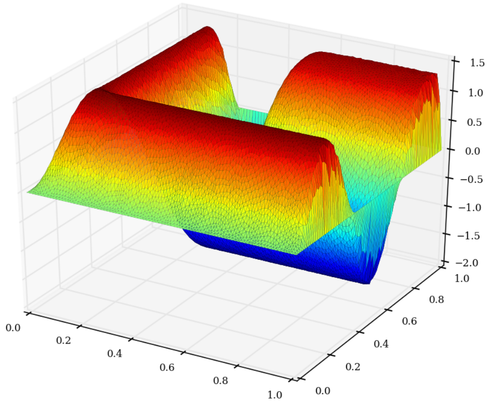
There are a few relevant model problems:

$$u_t - \varepsilon \Delta u + \vec{b} \cdot \nabla u + cu = f \tag{1}$$

$$-\varepsilon \Delta u + \vec{b} \cdot \nabla u + cu = f \tag{2}$$

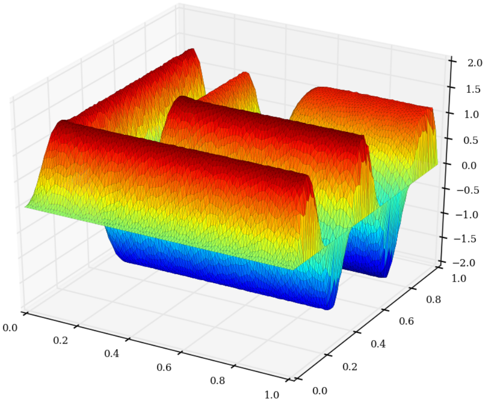
- $\varepsilon \ll 1$ and $\varepsilon \ll h$
- Small diffusion implies wavelike behavior (*convection-dominated*): the relevant length scale is $O(\sqrt{\varepsilon})$, which may be too small to resolve.

Convection-Diffusion-Reaction Equation



Third POD vector.

Convection-Diffusion-Reaction Equation



Fifth POD vector.

Stabilization by Streamline Upwinding

$$\varepsilon u_{xx} + u_x = 0, u(0) = 0, u(1) = 1 \tag{3}$$

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i+1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} = 1 \tag{4}$$

known to be an unstable discretization for small ε

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \frac{U_{i+1} - U_i}{h} = 1 \tag{5}$$

$$\varepsilon \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \frac{U_{i+1}}{2h} + \frac{U_{i+1}}{2h} - \frac{2U_i}{2h} + \frac{1}{2h}(U_{i-1} - U_{i-1}) = 1 \tag{6}$$

$$\left(\varepsilon + \frac{h}{2}\right) \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + \frac{U_{i+1} - U_{i-1}}{2h} = 1 \tag{7}$$

Using a forward difference (*upwinding*) results in an order 1 method with a much larger diffusion length scale ($O(\sqrt{h}) \gg O(\sqrt{\varepsilon})$).

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Stabilization by Streamline Upwinding (Petrov-Galerkin)

We can get a similar result for Finite Elements with the Petrov-Galerkin method by choosing test functions $\varphi + \delta\varphi_x$:

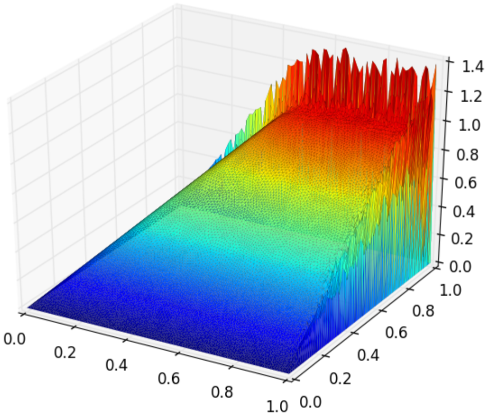
$$\varepsilon((\varphi + \delta\varphi_x)_x, u_x) + (\varphi + \delta\varphi_x, u_x) = (\varphi + \delta\varphi_x, f) \tag{8}$$

$$+(\varepsilon + \delta)(\varphi_x, u_x) + \delta(\varphi_{xx}, u_x) + (\varphi, u_x) = (\varphi + \delta\varphi_x, f) \tag{9}$$

$$\tag{10}$$

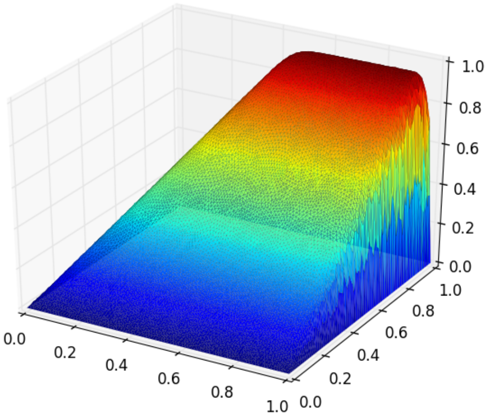
so the new viscous length scale is $O(\sqrt{\delta})$: if $\delta = O(h)$ then this can be resolved.

Stabilization by SUPG



$f = 1$, $\varepsilon = 10^{-3}$, $\vec{b} = (\cos(\pi/3), \sin(\pi/3))$, and $c = 10^{-3}$: no stabilization.

Stabilization by SUPG



$f = 1$, $\varepsilon = 10^{-3}$, $\vec{b} = (\cos(\pi/3), \sin(\pi/3))$, and $c = 10^{-3}$, with SUPG, $\delta = h$.

The SUPG-ROM

- SUPG helps us avoid instabilities arrising from unresolved modes
- Unresolved modes $\rightarrow$ aliasing onto the resolved parts (wiggles)
- ROMs, by definition, have lots of unresolved modes!

Plan: add SUPG to a POD-ROM for the convection-diffusion-reaction equation, with analysis done with both FEM and POD estimates.

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Error estimates

We assume a centered trajectory u_s, ROM approximating the fluctuations u_r, and projection onto the POD space P_r:

$$\begin{aligned} u - (u_s + u_r) &= (u - P_r(u_h)) + (P_r(u_h) - (u_s + u_r)) \\ &:= \eta - \phi_r \end{aligned} \tag{11}$$

Error estimate of the form (Giere, 2015 [1])

$$\begin{aligned} |||\phi_r|||_{\mathrm{SUPG},r} \leq C\Bigg[&\left(1 + \frac{1}{\sqrt{\delta}} + \sqrt{\delta}\right) \|\eta\| + \sqrt{\varepsilon}\|\nabla\eta\| \\ &+ \sqrt{\delta}\|\vec{b} \cdot \nabla\eta\| + \sqrt{\delta}\sqrt{\sum_T \|\Delta\eta\|^2_{0,T}}\Bigg]. \end{aligned} \tag{12}$$

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The POD Option

Seminorm bound on derivatives: [2, 4]

$$\sum_{n=0}^{R-1} \left| u_n - \sum_{j=0}^{r-1} \langle u_n, \phi_j \rangle \phi_j \right|_s^2 = \sum_{j=r}^{R-1} |\phi_j|_s^2 \sigma_j^2. \tag{13}$$

$$\|\eta\| \leq C_1 \sqrt{\sum_{j=r}^{R-1} \sigma_j^2} := C_1 \Lambda_0 \tag{14}$$

$$\|\nabla\eta\| \leq C_1 \sqrt{\sum_{j=r}^{R-1} \sigma_j^2 |\varphi_j|_1} := C_1 \Lambda_1 \tag{15}$$

$$\sqrt{\sum_{T \in \mathcal{T}_h} \|\delta\eta\|_T^2} \leq C_1 \sqrt{\sum_{j=r}^{R-1} \sigma_j^2 |\varphi_j|_2} := C_1 \Lambda_2. \tag{16}$$

$$\delta = \frac{\Lambda_0}{\Lambda_2 \varepsilon + \Lambda_0 + \Lambda_1}. \tag{17}$$

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The Finite Element Option

Another option is to use finite element estimates [3] with order m polynomials and s derivatives:

$$|\eta|_s \leq Ch^{m+1/2-s} \tag{18}$$

Solve the optimization problem for δ:

$$\delta = \frac{\Lambda_0 h^2 + h^{m+\frac{5}{2}}}{\left(h^2 + \varepsilon + h\right)\Lambda_0 + \varepsilon h^{m+\frac{1}{2}} + h^{m+\frac{5}{2}} + h^{m+\frac{3}{2}}} \tag{19}$$

If $m \geq 2$:

$$\delta \approx \frac{\Lambda_0 h^2}{h\Lambda_0} = h \tag{20}$$

We are back where we started! In practice the two stabilization parameters are within about 10%.

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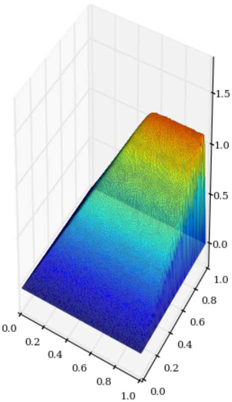
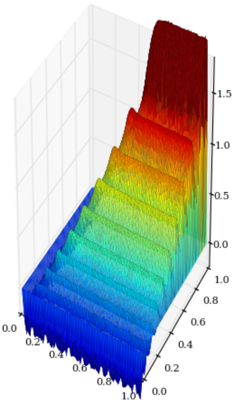
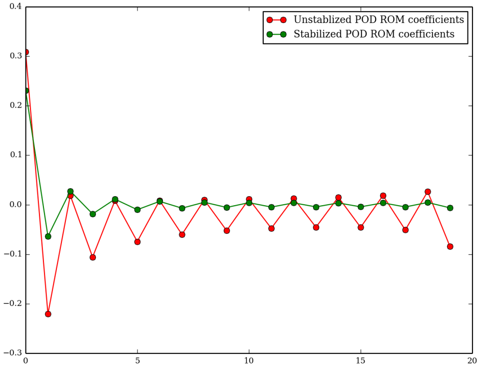
POD-ROM and SUPG-ROM





Stabilized (left) and unstabilized (right) POD-ROMs at $t = 1.0$.

Comparison of POD coefficients



Coefficients at $t = 1.0$ for the stabilized and unstabilized models.

Summary

- ROMs for convection-dominated problems have aliasing issues when scales are unresolved.
- We can overcome this problem with SUPG; tentatively, this implies that the usual FEM stabilization methods may be useful.

Thank You!

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