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Improving Boundary Derivative Recovery in Elliptic PDEs

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Outline

- Goals
- A Little on Superconvergence
- p-refinement results in 1D
- Extensions to 2D
- Summary

Goals

- Efficient numerical methods for elliptic PDEs
- Increased accuracy in (normal) boundary derivatives
- No postprocessing (achieve an energy estimate)
- Order $n+1$ data should lead to order $n+1$ derivatives

Goals

(mostly) bilinear elements and order 2 accurate (normal) second derivatives (at isolated points)

Assumptions

- Linear, constant coefficient elliptic PDEs, Dirichlet (or periodic) boundary conditions
- *Lots* of regularity
- Uniform grid (constant Δx, Δy) of tensor product elements (no triangles)
- Continuous finite element spaces
  - in 1D, degree n on interior elements, degree $n + p$ on nonperiodic edge elements
  - in 2D, bilinear on interior elements, degree $1 \otimes 1 + p$ on nonperiodic edge elements
- Optimal L^∞ estimates in 1D

$$\left\| \frac{d^m}{dx^m}(u - u^h) \right\|_{L^\infty} \leq C \left\| \frac{d^{m+1}}{dx^{m+1}} u \right\|_{L^\infty} \Delta x^{n+1-m}$$

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What's the plan?

- Static *p*-refinement: some cells have higher degree polynomial bases
- Global *h*-refinement: no mesh adaptivity (yet)
- Not really an *hp*-element method, but similar

What's the plan?

Picking the right type of p-refinement improves the boundary derivative convergence rates.

Superconvergence in 1D

Classic result: continuous FE approximation of $-u_{xx} = f$ with $V^h =$ piecewise polynomials of degree n has *zero error* at the knots (cell faces).

proof: (Arnold et al, 1972) Consider a knot $i\Delta x$:

$$\begin{aligned} |u^h(i\Delta x) - u(i\Delta x)| &= |a(G_{i\Delta x}, u^h - u)| \\ &= |a(G_{i\Delta x} - v^h, u^h - u)| \\ &\leq C\|G_{i\Delta x} - v^h\|_{H^1}\|u^h - u\|_{H^1} \end{aligned}$$

- $G_{i\Delta x}$ is the Green's function centered at $i\Delta x$
- $G_{i\Delta x}$ is piecewise linear $\Rightarrow G_{i\Delta x} \in V^h$
- Pick $G_{i\Delta x} = v^h$: no knot error!

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Superconvergence in 1D

Same idea for $-u_{xx} + bu_x + cu = f$ (Arnold et al, 1974):

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|u^h(i\Delta x) - u(i\Delta x)| &= |a(G_{i\Delta x}, u^h - u)| \\
&= |a(G_{i\Delta x} - v^h, u^h - u)| \\
&\leq C_1 \|G_{i\Delta x} - v^h\|_{H^1} \|u^h - u\|_{H^1} \\
&\leq C_2 \Delta x^{2k}.
\end{aligned}$$

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Superconvergence in 1D

Useful special case for CDR if the last cell consists of piecewise polynomials of degree $n + p$, $p \geq 1$, then

$$|u(1 - \Delta x) - u^h(1 - \Delta x)| \leq C\Delta x^{2n+1}$$

Proof:

- Same work as before: knot error is bounded by global error times Green's function (interpolation) error
- To the left of $1 - \Delta x$: derivatives of the Green's function scale like $O(\Delta x) \Rightarrow$ we gain an approximation order
- To the right of $1 - \Delta x$: $p \geq 1 \Rightarrow$ we gain (at least) one approximation order

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Improving boundary derivative convergence in $1D$

Our first step towards the goal: improving boundary derivative approximation orders for

$$-u_{xx} = f$$

$$\left| \frac{d^m}{dx^m}(u - u^h)(1) \right| \leq C\Delta x^{n+p+1-m}$$

- FE scheme on the boundary cell, y^h:
  - $y^h(1 - \Delta x) = u^h(1 - \Delta x) = u(1 - \Delta x)$
  - $y^h(1) = u^h(1) = u(1)$
- No knot errors
- Estimates on last cell *don't rely on the rest of the domain* (coupling has zero error)
- Equivalent, discretely, to just doing plain p-refinement on the last cell

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Improving boundary derivative convergence in 1D

For

$$-u_{xx} + bu_x + cu = f$$

$$\left| \frac{d^m}{dx^m}(u - u^h)(1) \right| \leq C_1 \Delta x^{n+p+1-m} + C_2 \Delta x^{2n-1}$$

- Same as the Laplace equation (C_1), but with a nonzero coupling error (saved by better knot estimate, $C_2 \Delta x^{2n-1}$)

$m = 2, n = 1, p = 2 \Rightarrow$ linears everywhere, cubics on boundary cells; a second order accurate second derivative on the boundary.

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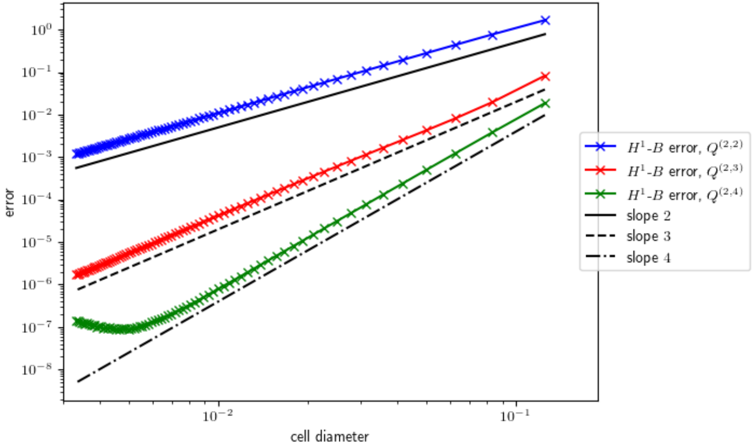
Numerical Results

Domain of [0, 1], with manufactured solution

$$u = \sin(10x)$$

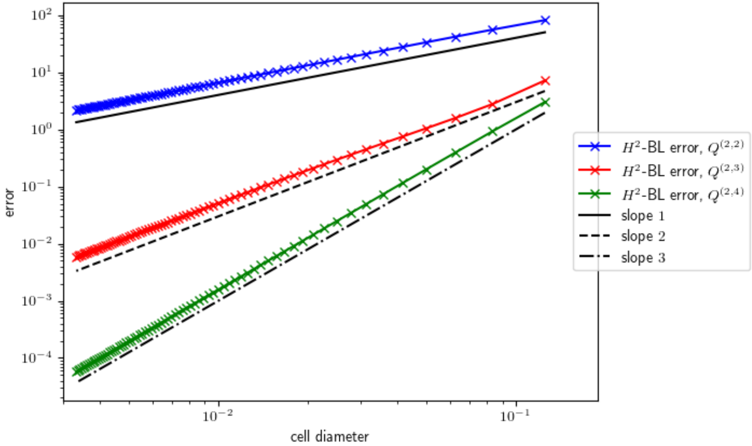
- Use `deal.II`'s *hp*-finite element support
- Bulk order 2, boundary orders 2, 3, and 4
- CDR equation
- Solve with GMRES

Numerical Results



Convergence rates for the p-refinement scheme of the first derivative on the boundary. We hit roundoff error near the end.

Numerical Results



Convergence rates for the p-refinement scheme of the second derivative on the boundary.

Extensions to 2D

With periodic boundary conditions in x and Dirichlet boundary conditions in y, $\sqrt{-1} = I$:

$$-\Delta u + \vec{b} \cdot \nabla u + cu = f \Rightarrow -\hat{u}_{yy} + b_2 \hat{u}_y + (k^2 + Ikb_1 + c)\hat{u} = \hat{f}_k$$

If we can handle the Fourier transform somehow then we are set.

Extensions to 2D

Linear elements in $x \Rightarrow$ centered differences in x:

$$D^0 U_i = \frac{U_{i+1} - U_{i-1}}{2\Delta x}$$
$$D^+ D^- U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2}$$

Extensions to 2D

- Discrete in x, continuous in y
- The solution in x is a superposition of grid eigenfunctions $\exp(Ikx_i) \Rightarrow$ calculate errors for each k and sum them up via inverse discrete Fourier transform

$$u(x_i, y) \approx \sum_{k=-N/2}^{N/2} \exp(Ikx_i)\, Y_k(y)$$

$$-Y_k''(y) + b_2 Y_k'(y) + \left(\frac{4\sin^2(k\Delta x/2)}{\Delta x^2} + \frac{b_1 I \sin(k\Delta x)}{\Delta x} + \frac{4 + 2\cos(k\Delta x)}{6} c \right) Y_k(y) = F_k(y).$$

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Extensions to 2D

$$\left| \frac{d^m}{dx^m} \left(u(x,y) - u^h(x,y) \right) \Big|_{(\delta_i, \delta_{N^*})} \right| \leq C_1 \Delta x^2 + C_2 \Delta y^{2+p-m}$$

where $N^* = 0$ or $N^* = N$.
proof outline:

- x-discretization is $O(\Delta x^2)$ from the Fourier solution; higher-frequency modes have a small contribution (regularity assumption)
- Derivatives of the Green's function scale like $O(k^{m-1})$: controlled by the decay of the Fourier coefficients (regularity assumption)
- Sums in the inverse Fourier transform converge with no loss in approximation order

Important: this only proves convergence at *knots*.

Numerical Results

Manufactured solution with $\vec{b} = (1,1)$ and $c = 2$:

$$u(x,y) = (y^3 + \exp(-y^2) + \sin(4.5y^2) + \sin(20y))(20\cos(4\pi x) + 0.1\sin(20\pi x) - 80\sin(6\pi x))$$

H^1-B error defined as

$$\max_{i,j} \left| (\nabla u^h \cdot \vec{n} - \nabla u \cdot \vec{n})(\delta_i, \delta_j) : (\delta_i, \delta_j) \in \partial\Omega^h \right|$$

and H^2-B error defined as

$$\max_{i,j} \left| (\vec{n}^T \Delta u^h \cdot \vec{n} - \vec{n}^T \Delta u \cdot \vec{n})(\delta_i, \delta_j) : (\delta_i, \delta_j) \in \partial\Omega^h \right|.$$

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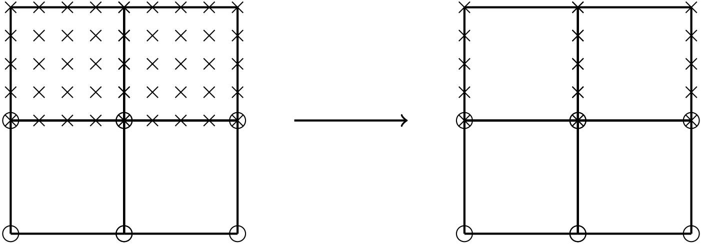
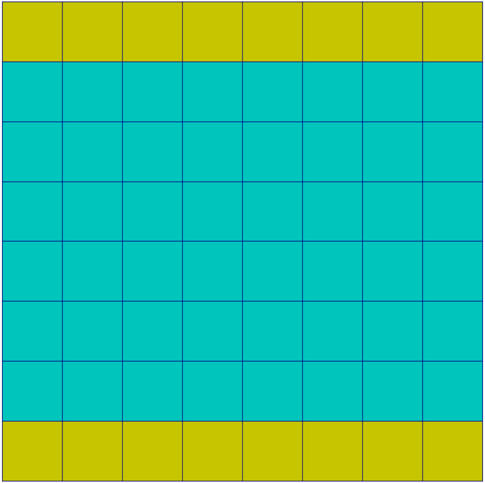
Numerical Results

Elimination of nonnormal p-refinement:



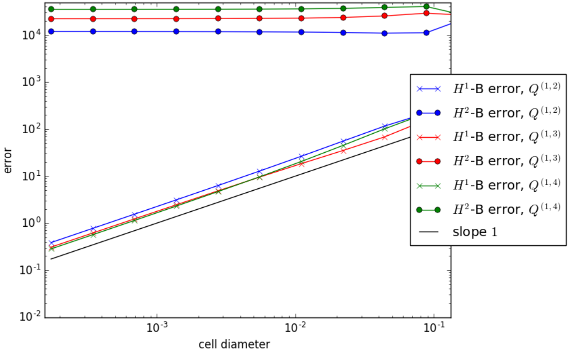
Depiction of normal p-refinement scheme, with elimination of extra DoFs. Linear DoFs are circles, quartic are crosses.

Numerical Results



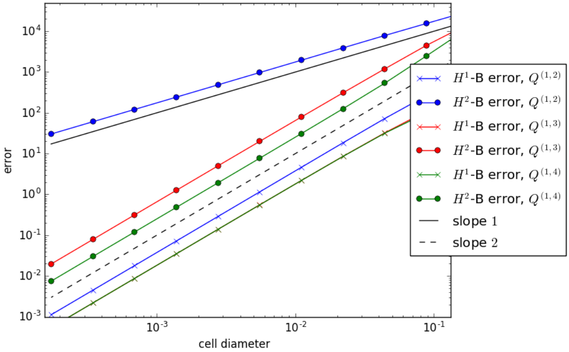
Depiction of the square grid with bulk cells in cyan and p-refined cells in yellow.

Numerical Results



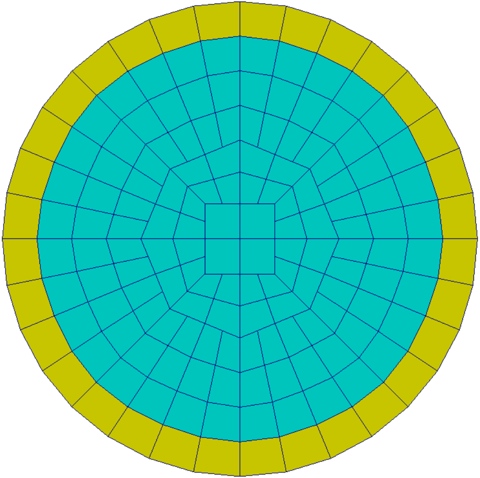
Convergence rates with anisotropic p-refinement. We do not recover any higher order accurate derivatives.

Numerical Results



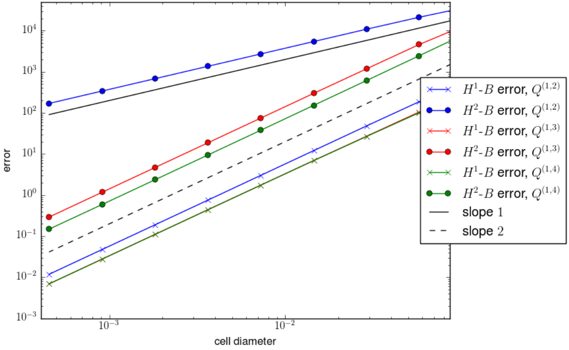
Convergence rates with isotropic p-refinement. The 1D theory applies at the knots.

Numerical Results



A circular grid with a geometry described near the boundary in polar coordinates, in the middle with Cartesian coordinates, and a transfinite interpolation in-between.

Numerical Results



Rates of convergence for the circular grid.

Summary and Outlook

- p-refinement error estimates involve a coupling term and a local term
- We can achieve higher-order derivative approximation at isolated points
- Future work: escape the Fourier framework, higher-order bulk elements, better tools for describing arbitrary geometries

Thank You!