# MATH 2270: Midterm 3 Practice Answers

The following are practice problems for the third exam.

1. Inventions:

   (a) Invent a 2-dimensional subspace of $\mathbb{P}_3$. Possible Answer:Span$\{t, t^2\}$

   (b) Invent a linearly independent set consisting of infinitely many vectors in $\mathbb{P}$. Possible Answer: $\{1, t, t^2, t^3, \ldots\}$

   (c) Invent a linear transformation $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ such that $\ker T$ is one-dimensional. Possible Answer: $T(p(t)) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}$

   (d) Invent a basis for the subspace $H$ of $\mathbb{P}_3$ defined by $H = \{p(t) \in \mathbb{P}_3 \mid p(1) = 0\}$. Possible Answer: $\{(t-1), (t-1)^2, (t-1)^3\}$

   (e) Let $\mathcal{S} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$. Invent a basis $\mathcal{B}$ of $\mathbb{P}_2$ so that the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{S}}{\mathcal{P}}$ is upper triangular. Possible Answer: $\mathcal{B} = \{2, 3+4t, 1+t+t^2\}$

   (f) Find a vector space $V$ so that for any linear transformation $T\colon V \to \mathbb{R}^3$, the dimension of the kernel of $T$ is at least 2. Possible Answer: $V = \mathbb{R}^5$

   (g) Invent a $3 \times 3$ matrix $A$ so that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvectors of $A$ with eigenvalue $\lambda = 1$. Possible Answer:

   $$\begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   (h) Invent a $2 \times 2$ matrix $A$ whose only eigenvalue is 3. Possible Answer:

   $$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

2. Answer each of the following true/false questions, and then give an explanation of your reasoning.

   (a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$. False

   (b) $\mathbb{P}^2$ is a subspace of $\mathbb{P}^3$. True

   (c) $C([0, 1])$ is a subspace of $C([0, 2])$. False

   (d) A change of basis matrix is never invertible. False

   (e) Eigenspaces for a linear transformation $T\colon V \to V$ are subspaces that are $T$-invariant. True

   (f) A linear transformation of an $n$-dimensional vector space has at least $n$ different eigenvalues. False

   (g) If $T\colon V \to V$ is a linear transformation, then $T$ is represented by a unique matrix. False

3. Determine whether or not each of the following linear transformations has a non-trivial eigenspace. If it does, describe the eigenspace and associated eigenvalue. Explain your reasoning.

   (a) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that rotates counter-clockwise about the origin by $\pi/2$. No. There are no eigenspaces

   (b) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that reflects across the line $y = x$. Yes. The subspace $\mathrm{Span}\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$ is an eigenspace

   (c) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that takes a vector $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}x\\2y\end{smallmatrix}\right]$. Yes. $H = \mathrm{Span}\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\}$ is an eigenspace, as is $H' = \mathrm{Span}\{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$

   (d) The linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ that reflects in the $xy$-plane. Yes. The $xy$-plane is fixed by the linear transformation, so it's an eigenspace with eigenvalue 1.

4. Show that $H = \{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is a subspace of $C(\mathbb{R})$. This question was on a FFT.

5. Consider the linear transformation $D\colon \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p) = p'$. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ and let $\mathcal{C} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$.

   (a) Find the matrix for the linear transformation $D$ with respect to the basis $\mathcal{B}$. Answer:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) Let $p(t) = -2t^3 + 3t^2 - 10t + 1$. Find $[p(t)]_{\mathcal{C}}$. Answer: $\left[\begin{smallmatrix}11\\-13\\5\\-2\end{smallmatrix}\right]$

   (c) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. Answer:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Compute the characteristic polynomial of $A$. Then factor it to find the eigenvalues of $A$. Answer: $(\lambda - 3)^2(\lambda - 2)$

   (b) For each eigenvalue, $\lambda$, find a basis for the corresponding eigenspace, $V^{(\lambda)}$. Possible Answer: $V^{(2)} = \mathrm{Span}\{\left[\begin{smallmatrix}-1\\1\\0\end{smallmatrix}\right]\}, V^{(3)} = \mathrm{Span}\{\left[\begin{smallmatrix}-2\\0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]\}$

(c) Use the computations from parts (a) and (b) to write $A = PDP^{-1}$, where $D$ is a diagonal matrix, and $P$ is an invertible matrix. You do not need to compute $P^{-1}$. Possible Answer:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

7. Let $H$ be the subspace of $C(\mathbb{R})$ (continuous functions $\mathbb{R} \to \mathbb{R}$) spanned by $\{\sinh x, \cosh x\}$. Recall (if you don't already know) that $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$ (note the lack of a minus sign). Consider the linear transformation $D \colon H \to H$ defined by $D(f) = f'$, where $f'$ denotes the derivative of $f$.

   (a) Compute the matrix of $D$ with respect to the basis, $\mathcal{B} = \{\sinh x, \cosh x\}$. *You do not need to show that $\mathcal{B}$ is a basis for $H$.* Answer:

   $$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   (b) Use the techniques of Chapter 5 to find a basis for $H$ in which the matrix for $D$ is diagonal. Possible Answer: $\mathcal{C} = \{\sinh x + \cosh x, \sinh x - \cosh x\}$

8. Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x) = Ax$ where $A$ is a $3 \times 3$ matrix whose eigenvalues are $1, 3$, and $-2$. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the matrix for $T$ with respect to $\mathcal{B}$ is diagonal? Why or why not? Answer: Yes

9. Give an example of a $2 \times 2$ matrix that is not diagonalizable. Possible Answer: $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]$

10. Define $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where $A = \left[\begin{smallmatrix} -6 & -2 \\ 4 & 0 \end{smallmatrix}\right]$. Find the matrix for $T$ with respect to the basis $\mathcal{B} = \{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]\}$. Answer: $\left[\begin{smallmatrix} -4 & 0 \\ 2 & -2 \end{smallmatrix}\right]$

11. §4.9 Exercises 4 & 14.

12. §5.6 Exercises 1 & 10.