Concepts Review

1. In terms of e and ln, $\pi^{\sqrt{3}} =$ _____. More generally $a^x =$ _____.

2. $\ln x = \log_a x$, where $a =$ _____.

3. $\log_a x$ can be expressed in terms of ln by $\log_a x =$ _____.

4. The derivative of the power function $f(x) = x^a$ is $f'(x) =$ _____; the derivative of the exponential function $g(x) = a^x$ is $g'(x) =$ _____.

Problem Set 6.4

In Problems 1–8, solve for x. Hint: $\log_a b = c \Leftrightarrow a^c = b$.

1. $\log_2 8 = x$

2. $\log_5 x = 2$

3. $\log_4 x = \frac{3}{2}$

4. $\log_x 64 = 4$

5. $2\log_9\left(\frac{x}{3}\right) = 1$

6. $\log_4\left(\frac{1}{2x}\right) = 3$

7. $\log_2(x+3) - \log_2 x = 2$

8. $\log_5(x+3) - \log_5 x = 1$

C *Use* $\log_a x = (\ln x)/(\ln a)$ *to calculate each of the logarithms in Problems 9–12.*

9. $\log_5 12$

10. $\log_7(0.11)$

11. $\log_{11}(8.12)^{1/5}$

12. $\log_{10}(8.57)^7$

C *In Problems 13–16, use natural logarithms to solve each of the exponential equations. Hint: To solve* $3^x = 11$, *take ln of both sides, obtaining* $x \ln 3 = \ln 11$; *then* $x = (\ln 11)/(\ln 3) \approx 2.1827$.

13. $2^x = 17$

14. $5^x = 13$

15. $5^{2s-3} = 4$

16. $12^{1/(\theta-1)} = 4$

In Problems 17–26, find the indicated derivative or integral.

17. $D_x(6^{2x})$

18. $D_x(3^{2x^2-3x})$

19. $D_x \log_3 e^x$

20. $D_x \log_{10}(x^3+9)$

21. $D_z[3^z \ln(z+5)]$

22. $D_\theta \sqrt{\log_{10}(3^{\theta^2-\theta})}$

23. $\int x2^{x^2}\,dx$

24. $\int 10^{5x-1}\,dx$

25. $\int_1^4 \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx$

26. $\int_0^1 (10^{3x} + 10^{-3x})\,dx$

In Problems 27–32, find dy/dx. *Note: You must distinguish among problems of the type* a^x, x^a, *and* x^x *as in Examples 5–7.*

27. $y = 10^{(x^2)} + (x^2)^{10}$

28. $y = \sin^2 x + 2^{\sin x}$

29. $y = x^{\pi+1} + (\pi+1)^x$

30. $y = 2^{(e^x)} + (2^e)^x$

31. $y = (x^2+1)^{\ln x}$

32. $y = (\ln x^2)^{2x+3}$

33. If $f(x) = x^{\sin x}$, find $f'(1)$.

C **34.** Let $f(x) = \pi^x$ and $g(x) = x^\pi$. Which is larger, $f(e)$ or $g(e)$? $f'(e)$ or $g'(e)$?

In Problems 35–40, first find the domain of the given function f and then find where it is increasing and decreasing, and also where it is concave upward and downward. Identify all extreme values and points of inflection. Then sketch the graph of $y = f(x)$.

35. $f(x) = 2^{-x}$

36. $f(x) = x2^{-x}$

37. $f(x) = \log_2(x^2+1)$

38. $f(x) = x\log_3(x^2+1)$

39. $f(x) = \int_1^x 2^{-t^2}\,dt$

40. $f(x) = \int_0^x \log_{10}(t^2+1)\,dt$

41. How are $\log_{1/2} x$ and $\log_2 x$ related?

42. Sketch the graphs of $\log_{1/3} x$ and $\log_3 x$ using the same coordinate axes.

C **43.** The magnitude M of an earthquake on the **Richter scale** is

$$M = 0.67\log_{10}(0.37E) + 1.46$$

where E is the energy of the earthquake in kilowatt-hours. Find the energy of an earthquake of magnitude 7. Of magnitude 8.

C **44.** The loudness of sound is measured in decibels in honor of Alexander Graham Bell (1847–1922), inventor of the telephone. If the variation in pressure is P pounds per square inch, then the loudness L in decibels is

$$L = 20\log_{10}(121.3P)$$

Find the variation in pressure caused by music at 115 decibels.

C **45.** In the equally tempered scale to which keyed instruments have been tuned since the days of J.S. Bach (1685–1750), the frequencies of successive notes C, C#, D, D#, E, F, F#, G, G#, A, A#, B, $\overline{\text{C}}$ form a geometric sequence (progression), with $\overline{\text{C}}$ having twice the frequency of C (C# is read C sharp). What is the ratio r between the frequencies of successive notes? If the frequency of A is 440, find the frequency of $\overline{\text{C}}$.

46. Prove that $\log_2 3$ is irrational. *Hint:* Use proof by contradiction.

GC **47.** You suspect that the xy-data that you collect lie on either an exponential curve $y = Ab^x$ or a power curve $y = Cx^d$. To check, you plot $\ln y$ against x on one graph and $\ln y$ against $\ln x$ on another graph. (Graphing calculators and CASs have options to make the vertical axis, or both the vertical and horizontal axes, a logarithmic scale.) Explain how these graphs will help you to come to a conclusion.

48. (An Amusement) Given the problem of finding y' if $y = x^x$, student A did the following:

Wrong 1

$$\begin{aligned} y &= x^x \\ y' &= x \cdot x^{x-1} \cdot 1 \qquad \begin{pmatrix}\text{misapplying the} \\ \text{Power Rule}\end{pmatrix} \\ &= x^x \end{aligned}$$

Student B did this:

Wrong 2

$$\begin{aligned} y &= x^x \\ y' &= x^x \cdot \ln x \cdot 1 \qquad \begin{pmatrix}\text{misapplying the} \\ \text{Exponential} \\ \text{Function Rule}\end{pmatrix} \\ &= x^x \ln x \end{aligned}$$

The sum $x^x + x^x \ln x$ is correct (Example 5), so

WRONG 1 + WRONG 2 = RIGHT

Show that the same procedure yields a correct answer for finding the derivative of $y = f(x)^{g(x)}$.

49. Convince yourself that $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ are not the same function. Then find $f'(x)$ and $g'(x)$. *Note:* When mathematicians write x^{x^x}, they mean $x^{(x^x)}$.

50. Consider $f(x) = \dfrac{a^x - 1}{a^x + 1}$ for fixed a, $a > 0$, $a \neq 1$. Show that f has an inverse and find a formula for $f^{-1}(x)$.

51. For a fixed $a > 1$, let $f(x) = x^a/a^x$ on $[0, \infty)$. Show:

(a) $\lim\limits_{x\to\infty} f(x) = 0$ *Hint:* Study $\ln f(x)$;

(b) $f(x)$ is maximized at $x_0 = a/\ln a$;

(c) $x^a = a^x$ has two positive solutions if $a \neq e$ and only one such solution if $a = e$;

(d) $\pi^e < e^\pi$.

52. Let $f_u(x) = x^u e^{-x}$ for $x \geq 0$. Show that for any fixed $u > 0$:

(a) $f_u(x)$ attains its maximum at $x_0 = u$;

(b) $f_u(u) > f_u(u + 1)$ and $f_{u+1}(u + 1) > f_{u+1}(u)$ imply

$$\left(\frac{u+1}{u}\right)^u < e < \left(\frac{u+1}{u}\right)^{u+1}$$

(c) $\dfrac{u}{u+1}e < \left(\dfrac{u+1}{u}\right)^u < e.$

Conclude from part (c) that $\lim\limits_{u\to\infty}\left(1 + \dfrac{1}{u}\right)^u = e$.

GC **53.** Find $\lim\limits_{x\to 0^+} x^x$. Also find the coordinates of the minimum point for $f(x) = x^x$ on $[0, 4]$.

GC **54.** Draw the graphs of $y = x^3$ and $y = 3^x$ using the same axes and find all their intersection points.

CAS **55.** Evaluate $\displaystyle\int_0^{4\pi} x^{\sin x}\,dx$.

CAS **56.** Refer to Problem 49. Draw the graphs of f and g using the same axes. Then draw the graphs of f' and g' using the same axes.

Our graphing experience so far has been restricted to using standard (linear) coordinate spacings. When working with exponential and logarithmic functions it may be more instructive to use logarithmic and log–log scales. We explore these techniques in Problems 57 and 58.

GC **57.** On a single set of axes, use your calculator to draw the graphs of $y = 2^x$, $y = 3^x$, and $y = 4^x$ over the interval $0 < x < 4$. Do the same for the inverse functions $y = \log_2 x$, $y = \log_3 x$, and $y = \log_4 x$. If we use a computer graphing program that permits the use of semilog axes (a logarithmic scale on the y-axis and a normal scale on the x-axis) to graph the functions $y = 2^x$, $y = 3^x$, and $y = 4^x$ over the region $-5 < x < 5$ (Figure 3), we get three lines.

(a) Identify each of the lines in Figure 3.

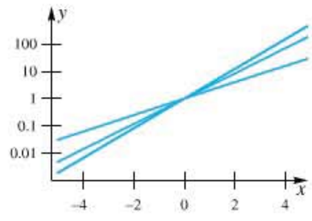

Figure 3

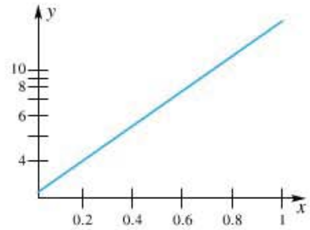

Figure 4

(b) Noting that, if $y = Cb^x$ then $\ln y = \ln C + x \ln b$, explain why all the curves in Figure 3 are lines through the point $(0, 1)$.

(c) Based on the semilog plot given by Figure 4, determine the C and b in the equation $y = Cb^x$.

58. If we use log scaling for the x-axis as well as the y-axis (called a log–log plot) and graph several power functions, we will also get lines. Using the result that, upon taking logs, $y = Cx^r$ becomes $\log y = \log C + r \log x$, identify the equations that are graphed in Figure 5.

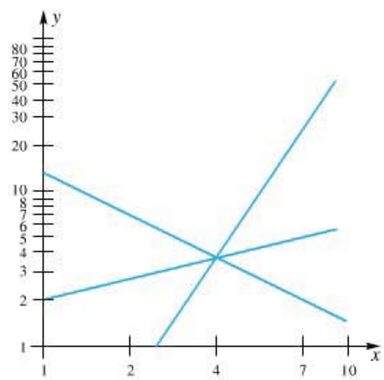

Figure 5

Answers to Concepts Review: **1.** $e^{\sqrt{3}\ln\pi}$; $e^{x\ln a}$ **2.** e **3.** $(\ln x)/(\ln a)$ **4.** ax^{a-1}; $a^x \ln a$

6.5 Exponential Growth and Decay

At the beginning of 2004, the world's population was about 6.4 billion. It is said that by the year 2020 it will reach 7.9 billion. How are such predictions made?

To treat the problem mathematically, let $y = f(t)$ denote the size of the population at time t, where t is the number of years after 2004. Actually, $f(t)$ is an integer, and its graph "jumps" when someone is born or someone dies. However, for a large population, these jumps are so small relative to the total population that we will not go far wrong if we pretend that f is a nice differentiable function.