MATH1220: Midterm 3 Practice Problems

The following are practice problems for the second exam.

1. Compute the following limits:

   (a) $\lim_{x\to 1} \frac{x^2-2x+1}{\sin(\pi x)} = 0$

   (b) $\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} = 1$

   (c) $\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = 0$

   (d) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

2. Evaluate the following integrals:

   (a) $\int_{-\infty}^{1} e^{4x}\,dx = \frac{1}{4}e^4$

   (b) $\int_{5}^{\infty} \frac{x}{1+x^2}\,dx = \infty$

   (c) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2} = \frac{2\pi}{81}$ Obtain this answer usinga trig substitution.

   (d) $\int_{0}^{3} \frac{dx}{x^2-2x-3} dx$ Answer: Diverges

   (e) $\int_{-4}^{0} \frac{dx}{(x+3)^2}$ Answer: Diverges

3. Write an explicit formula for the n-th term of the sequence. Then determine whether the sequence converges or diverges. If it converges, find what number it converges to:

   (a) $a_1 = 7$, $a_{n+1} = a_n\left(\frac{2}{3}\right)$ Answer: $a_n = 7(2/3)^{n-1}$

   (b) $-1, 2, 5, 8, 11, \ldots$ Answer: $a_n = -4+3n$

   (c) $0, \frac{1}{4}, \frac{2}{6}, \frac{3}{8}, \frac{4}{10}, \frac{5}{12}, \frac{6}{14}, \ldots$ Answer: $a_n = \frac{n-1}{2n}$

4. Find the limit of the sequence $a_n = \frac{2n^3}{5n^3-2n+2} = \frac{2}{5}$.

5. Show that the sequence $a_n = \frac{n}{n+1}\left(2-\frac{1}{n^2}\right)$ converges using the monotone sequence theorem. Answer: a_n is an increasing sequence because $\frac{n}{n+1}$ is increasing as is $2-\frac{1}{n^2}$. Also, a_n is bounded above by 2 since $a_n = \frac{n}{n+1}\left(2-\frac{1}{n^2}\right) \leq 1\cdot 2 = 2$. Hence it converges.

6. Determine the convergence/divergence of the following series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ Answer: Diverges by the divergence test (the terms don't go to zero)

   (b) $\displaystyle\sum_{k=1}^{\infty} \left[5\left(\frac{1}{2}\right)^k - 3\left(\frac{1}{7}\right)^k\right]$ Answer: Converges because it's a difference of convergent geometric series. It converges to $\displaystyle\frac{5}{1-1/2} - \frac{3}{1-1/7}$

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{(k+2)k}$ Answer: Converges by limit comparison test.

   (d) $\displaystyle\sum_{k=1}^{\infty} \ln(k/(k+1))$ Answer: Diverges because it's a collapsing series and the partial sums don't converge.

   (e) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2k^2+1}$ Answer: Converges by limit comparison test

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3}$ Answer: Diverges by limit comparison test.

   (g) $\displaystyle\sum_{k=1}^{\infty} k\sin(1/k)$ Answer: Diverges by the divergence test. $\displaystyle\lim_{k\to\infty} k\sin(1/k) = \lim_{k\to\infty} \frac{\sin(1/k)}{1/k} = \lim_{x\to 0} \frac{\sin x}{x}$.

   (h) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[5]{3n^4+3}}{n^2}$ Answer: Converges by limit comparison test.

   (i) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k+k}{k!}$ Answer: Converges by ratio test.

   (j) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{2^k}$ Answer: Converges by ratio test.

   (k) $\displaystyle\sum_{k=1}^{\infty} \frac{4^{2n}}{n!}$ Answer: Converges by ratio test.

   (l) $\displaystyle\frac{\ln 2}{2^2} + \frac{\ln 3}{3^3} + \frac{\ln 4}{4^4} + \frac{\ln 5}{5^5} + \ldots$ Answer: Converges by ratio test. Computing the limit is difficult because it involves several tricks.

7. Determine whether each of the following is absolutely convergent, conditionally convergent, or divergent:

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ Answer: Conditionally convergent by alternating series test and absolute ratio test.

(b) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{e^k}$ Answer: Absolutely convergent by absolute ratio test.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n-1}{n}$ Answer: Divergent by alternating series test (terms don't go to zero).