**NUMERICAL NOTE**

When $A$ is symmetric and not too large, modern high-performance computer algorithms calculate eigenvalues and eigenvectors with great precision. They apply a sequence of similarity transformations to $A$ involving orthogonal matrices. The diagonal entries of the transformed matrices converge rapidly to the eigenvalues of $A$. (See the Numerical Notes in Section 5.2.) Using orthogonal matrices generally prevents numerical errors from accumulating during the process. When $A$ is symmetric, the sequence of orthogonal matrices combines to form an orthogonal matrix whose columns are eigenvectors of $A$.

A nonsymmetric matrix cannot have a full set of orthogonal eigenvectors, but the algorithm still produces fairly accurate eigenvalues. After that, nonorthogonal techniques are needed to calculate eigenvectors.

## PRACTICE PROBLEMS

1. Show that if $A$ is a symmetric matrix, then $A^2$ is symmetric.
2. Show that if $A$ is orthogonally diagonalizable, then so is $A^2$.

# 7.1 EXERCISES

Determine which of the matrices in Exercises 1–6 are symmetric.

**1.** $\begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix}$  **2.** $\begin{bmatrix} -3 & 5 \\ -5 & 3 \end{bmatrix}$

**3.** $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$  **4.** $\begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -2 \\ 3 & -2 & 0 \end{bmatrix}$

**5.** $\begin{bmatrix} -6 & 2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & -6 \end{bmatrix}$  **6.** $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Determine which of the matrices in Exercises 7–12 are orthogonal. If orthogonal, find the inverse.

**7.** $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$  **8.** $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

**9.** $\begin{bmatrix} -5 & 2 \\ 2 & 5 \end{bmatrix}$  **10.** $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

**11.** $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ \sqrt{5}/3 & -4/\sqrt{45} & -2/\sqrt{45} \end{bmatrix}$

**12.** $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ -.5 & .5 & -.5 & .5 \\ .5 & .5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$

Orthogonally diagonalize the matrices in Exercises 13–22, giving an orthogonal matrix $P$ and a diagonal matrix $D$. To save you time, the eigenvalues in Exercises 17–22 are: (17) 5, 2, −2; (18) 25, 3, −50; (19) 7, −2; (20) 13, 7, 1; (21) 9, 5, 1; (22) 2, 0.

**13.** $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$  **14.** $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

**15.** $\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$  **16.** $\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$

**17.** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$  **18.** $\begin{bmatrix} -2 & -36 & 0 \\ -36 & -23 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**19.** $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$  **20.** $\begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$

**21.** $\begin{bmatrix} 4 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 3 & 1 & 4 \end{bmatrix}$  **22.** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

**23.** Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Verify that 2 is an eigenvalue of $A$ and $\mathbf{v}$ is an eigenvector. Then orthogonally diagonalize $A$.

**24.** Let $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $A$. Then orthogonally diagonalize $A$.

In Exercises 25 and 26, mark each statement True or False. Justify each answer.

**25.** a. An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

b. If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{v} = 0$.

c. An $n \times n$ symmetric matrix has $n$ distinct real eigenvalues.

d. For a nonzero $\mathbf{v}$ in $\mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

**26.** a. Every symmetric matrix is orthogonally diagonalizable.

b. If $B = PDP^T$, where $P^T = P^{-1}$ and $D$ is a diagonal matrix, then $B$ is a symmetric matrix.

c. An orthogonal matrix is orthogonally diagonalizable.

d. The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

**27.** Suppose $A$ is a symmetric $n \times n$ matrix and $B$ is any $n \times m$ matrix. Show that $B^TAB$, $B^TB$, and $BB^T$ are symmetric matrices.

**28.** Show that if $A$ is an $n \times n$ symmetric matrix, then $(A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

**29.** Suppose $A$ is invertible and orthogonally diagonalizable. Explain why $A^{-1}$ is also orthogonally diagonalizable.

**30.** Suppose $A$ and $B$ are both orthogonally diagonalizable and $AB = BA$. Explain why $AB$ is also orthogonally diagonalizable.

**31.** Let $A = PDP^{-1}$, where $P$ is orthogonal and $D$ is diagonal, and let $\lambda$ be an eigenvalue of $A$ of multiplicity $k$. Then $\lambda$ appears $k$ times on the diagonal of $D$. Explain why the dimension of the eigenspace for $\lambda$ is $k$.

**32.** Suppose $A = PRP^{-1}$, where $P$ is orthogonal and $R$ is upper triangular. Show that if $A$ is symmetric, then $R$ is symmetric and hence is actually a diagonal matrix.

**33.** Construct a spectral decomposition of $A$ from Example 2.

**34.** Construct a spectral decomposition of $A$ from Example 3.

**35.** Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \mathbf{u}\mathbf{u}^T$.

a. Given any $\mathbf{x}$ in $\mathbb{R}^n$, compute $B\mathbf{x}$ and show that $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$, as described in Section 6.2.

b. Show that $B$ is a symmetric matrix and $B^2 = B$.

c. Show that $\mathbf{u}$ is an eigenvector of $B$. What is the corresponding eigenvalue?

**36.** Let $B$ be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a **projection matrix** (or an **orthogonal projection matrix**). Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}} = B\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

a. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

b. Let $W$ be the column space of $B$. Show that $\mathbf{y}$ is the sum of a vector in $W$ and a vector in $W^\perp$. Why does this prove that $B\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of $B$?

[**M**] Orthogonally diagonalize the matrices in Exercises 37–40. To practice the methods of this section, do not use an eigenvector routine from your matrix program. Instead, use the program to find the eigenvalues, and, for each eigenvalue $\lambda$, find an orthonormal basis for $\text{Nul}(A - \lambda I)$, as in Examples 2 and 3.

**37.** $\begin{bmatrix} 5 & 2 & 9 & -6 \\ 2 & 5 & -6 & 9 \\ 9 & -6 & 5 & 2 \\ -6 & 9 & 2 & 5 \end{bmatrix}$

**38.** $\begin{bmatrix} .38 & -.18 & -.06 & -.04 \\ -.18 & .59 & -.04 & .12 \\ -.06 & -.04 & .47 & -.12 \\ -.04 & .12 & -.12 & .41 \end{bmatrix}$

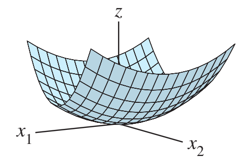
**39.** $\begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$

**40.** $\begin{bmatrix} 10 & 2 & 2 & -6 & 9 \\ 2 & 10 & 2 & -6 & 9 \\ 2 & 2 & 10 & -6 & 9 \\ -6 & -6 & -6 & 26 & 9 \\ 9 & 9 & 9 & 9 & -19 \end{bmatrix}$

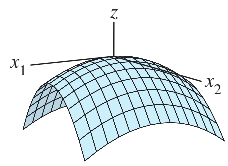
## SOLUTIONS TO PRACTICE PROBLEMS

**1.** $(A^2)^T = (AA)^T = A^TA^T$, by a property of transposes. By hypothesis, $A^T = A$. So $(A^2)^T = AA = A^2$, which shows that $A^2$ is symmetric.

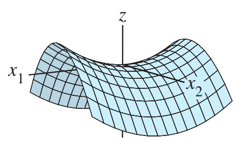
**2.** If $A$ is orthogonally diagonalizable, then $A$ is symmetric, by Theorem 2. By Practice Problem 1, $A^2$ is symmetric and hence is orthogonally diagonalizable (Theorem 2).



Positive definite



Negative definite



Indefinite

**PROOF** By the Principal Axes Theorem, there exists an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ such that

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{4}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. Since $P$ is invertible, there is a one-to-one correspondence between all nonzero $\mathbf{x}$ and all nonzero $\mathbf{y}$. Thus the values of $Q(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$ coincide with the values of the expression on the right side of (4), which is obviously controlled by the signs of the eigenvalues $\lambda_1, \ldots, \lambda_n$, in the three ways described in the theorem. ■

**EXAMPLE 6** Is $Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite?

**SOLUTION** Because of all the plus signs, this form "looks" positive definite. But the matrix of the form is

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

and the eigenvalues of $A$ turn out to be 5, 2, and $-1$. So $Q$ is an indefinite quadratic form, not positive definite. ■

The classification of a quadratic form is often carried over to the matrix of the form. Thus a **positive definite matrix** $A$ is a *symmetric* matrix for which the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite. Other terms, such as **positive semidefinite matrix**, are defined analogously.

WEB

### NUMERICAL NOTE

A fast way to determine whether a symmetric matrix $A$ is positive definite is to attempt to factor $A$ in the form $A = R^T R$, where $R$ is upper triangular with positive diagonal entries. (A slightly modified algorithm for an LU factorization is one approach.) Such a *Cholesky factorization* is possible if and only if $A$ is positive definite. See Supplementary Exercise 7 at the end of Chapter 7.

### PRACTICE PROBLEM

Describe a positive semidefinite matrix $A$ in terms of its eigenvalues.

WEB

## 7.2 EXERCISES

1. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, when $A = \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix}$ and
   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

2. Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$, for $A = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and
   a. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ c. $\mathbf{x} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

3. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $10x_1^2 - 6x_1x_2 - 3x_2^2$ b. $5x_1^2 + 3x_1x_2$

4. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^2$.
   a. $20x_1^2 + 15x_1x_2 - 10x_2^2$ b. $x_1x_2$

5. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$
   b. $4x_1x_2 + 6x_1x_3 - 8x_2x_3$

6. Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.
   a. $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$
   b. $x_3^2 - 4x_1x_2 + 4x_2x_3$

7. Make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form $x_1^2 + 10x_1x_2 + x_2^2$ into a quadratic form with no cross-product term. Give $P$ and the new quadratic form.

8. Let $A$ be the matrix of the quadratic form

   $$9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$

   It can be shown that the eigenvalues of $A$ are 3, 9, and 15. Find an orthogonal matrix $P$ such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^TA\mathbf{x}$ into a quadratic form with no cross-product term. Give $P$ and the new quadratic form.

Classify the quadratic forms in Exercises 9–18. Then make a change of variable, $\mathbf{x} = P\mathbf{y}$, that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. Construct $P$ using the methods of Section 7.1.

9. $3x_1^2 - 4x_1x_2 + 6x_2^2$
10. $9x_1^2 - 8x_1x_2 + 3x_2^2$
11. $2x_1^2 + 10x_1x_2 + 2x_2^2$
12. $-5x_1^2 + 4x_1x_2 - 2x_2^2$
13. $x_1^2 - 6x_1x_2 + 9x_2^2$
14. $8x_1^2 + 6x_1x_2$
15. [M] $-2x_1^2 - 6x_2^2 - 9x_3^2 - 9x_4^2 + 4x_1x_2 + 4x_1x_3 + 4x_1x_4 + 6x_3x_4$
16. [M] $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 3x_1x_2 + 3x_3x_4 - 4x_1x_4 + 4x_2x_3$
17. [M] $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 9x_1x_2 - 12x_1x_4 + 12x_2x_3 + 9x_3x_4$
18. [M] $11x_1^2 - x_2^2 - 12x_1x_2 - 12x_1x_3 - 12x_1x_4 - 2x_3x_4$

19. What is the largest possible value of the quadratic form $5x_1^2 + 8x_2^2$ if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^T\mathbf{x} = 1$, that is, if $x_1^2 + x_2^2 = 1$? (Try some examples of $\mathbf{x}$.)

20. What is the largest value of the quadratic form $5x_1^2 - 3x_2^2$ if $\mathbf{x}^T\mathbf{x} = 1$?

In Exercises 21 and 22, matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

21. a. The matrix of a quadratic form is a symmetric matrix.
    b. A quadratic form has no cross-product terms if and only if the matrix of the quadratic form is a diagonal matrix.
    c. The principal axes of a quadratic form $\mathbf{x}^TA\mathbf{x}$ are eigenvectors of $A$.
    d. A positive definite quadratic form $Q$ satisfies $Q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
    e. If the eigenvalues of a symmetric matrix $A$ are all positive, then the quadratic form $\mathbf{x}^TA\mathbf{x}$ is positive definite.
    f. A Cholesky factorization of a symmetric matrix $A$ has the form $A = R^TR$, for an upper triangular matrix $R$ with positive diagonal entries.

22. a. The expression $\|\mathbf{x}\|^2$ is a quadratic form.
    b. If $A$ is symmetric and $P$ is an orthogonal matrix, then the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $\mathbf{x}^TA\mathbf{x}$ into a quadratic form with no cross-product term.
    c. If $A$ is a $2 \times 2$ symmetric matrix, then the set of $\mathbf{x}$ such that $\mathbf{x}^TA\mathbf{x} = c$ (for a constant $c$) corresponds to either a circle, an ellipse, or a hyperbola.
    d. An indefinite quadratic form is either positive semidefinite or negative semidefinite.
    e. If $A$ is symmetric and the quadratic form $\mathbf{x}^TA\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then the eigenvalues of $A$ are all negative.

Exercises 23 and 24 show how to classify a quadratic form $Q(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$, when $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\det A \neq 0$, without finding the eigenvalues of $A$.

23. If $\lambda_1$ and $\lambda_2$ are the eigenvalues of $A$, then the characteristic polynomial of $A$ can be written in two ways: $\det(A - \lambda I)$ and $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Use this fact to show that $\lambda_1 + \lambda_2 = a + d$ (the diagonal entries of $A$) and $\lambda_1\lambda_2 = \det A$.

24. Verify the following statements.
    a. $Q$ is positive definite if $\det A > 0$ and $a > 0$.
    b. $Q$ is negative definite if $\det A > 0$ and $a < 0$.
    c. $Q$ is indefinite if $\det A < 0$.

25. Show that if $B$ is $m \times n$, then $B^TB$ is positive semidefinite; and if $B$ is $n \times n$ and invertible, then $B^TB$ is positive definite.

26. Show that if an $n \times n$ matrix $A$ is positive definite, then there exists a positive definite matrix $B$ such that $A = B^TB$. [*Hint:* Write $A = PDP^T$, with $P^T = P^{-1}$. Produce a diagonal matrix $C$ such that $D = C^TC$, and let $B = PCP^T$. Show that $B$ works.]

27. Let $A$ and $B$ be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. [*Hint:* Consider quadratic forms.]

28. Let $A$ be an $n \times n$ invertible symmetric matrix. Show that if the quadratic form $\mathbf{x}^TA\mathbf{x}$ is positive definite, then so is the quadratic form $\mathbf{x}^TA^{-1}\mathbf{x}$. [*Hint:* Consider eigenvalues.]

**SG** Mastering: Diagonalization and Quadratic Forms 7–7

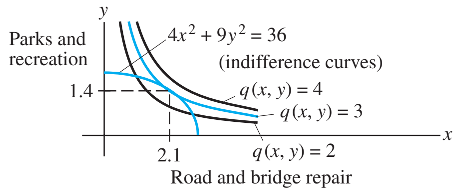



**FIGURE 4** The optimum public works schedule is (2.1, 1.4).

and define

$$x_1 = \frac{x}{3}, \quad x_2 = \frac{y}{2}, \qquad \text{that is,} \quad x = 3x_1 \quad \text{and} \quad y = 2x_2$$

Then the constraint equation becomes

$$x_1^2 + x_2^2 = 1$$

and the utility function becomes $q(3x_1, 2x_2) = (3x_1)(2x_2) = 6x_1x_2$. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then the problem is to maximize $Q(\mathbf{x}) = 6x_1x_2$ subject to $\mathbf{x}^T\mathbf{x} = 1$. Note that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

The eigenvalues of $A$ are $\pm 3$, with eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ for $\lambda = 3$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ for $\lambda = -3$. Thus the maximum value of $Q(\mathbf{x}) = q(x_1, x_2)$ is 3, attained when $x_1 = 1/\sqrt{2}$ and $x_2 = 1/\sqrt{2}$.

In terms of the original variables, the optimum public works schedule is $x = 3x_1 = 3/\sqrt{2} \approx 2.1$ hundred miles of roads and bridges and $y = 2x_2 = \sqrt{2} \approx 1.4$ hundred acres of parks and recreational areas. The optimum public works schedule is the point where the constraint curve and the indifference curve $q(x, y) = 3$ just meet. Points $(x, y)$ with a higher utility lie on indifference curves that do not touch the constraint curve. See Fig. 4. ■

## PRACTICE PROBLEMS

1. Let $Q(\mathbf{x}) = 3x_1^2 + 3x_2^2 + 2x_1x_2$. Find a change of variable that transforms $Q$ into a quadratic form with no cross-product term, and give the new quadratic form.
2. With $Q$ as in Problem 1, find the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, and find a unit vector at which the maximum is attained.

## 7.3 EXERCISES

In Exercises 1 and 2, find the change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into $\mathbf{y}^T D\mathbf{y}$ as shown.

1. $5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3 = 9y_1^2 + 6y_2^2 + 3y_3^2$
2. $3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3 = 5y_1^2 + 2y_2^2$
[*Hint:* $\mathbf{x}$ and $\mathbf{y}$ must have the same number of coordinates, so the quadratic form shown here must have a coefficient of zero for $y_3^2$.]

In Exercises 3–6, find (a) the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$, (b) a unit vector $\mathbf{u}$ where this maximum is attained, and (c) the maximum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$.

3. $Q(\mathbf{x}) = 5x_1^2 + 6x_2^2 + 7x_3^2 + 4x_1x_2 - 4x_2x_3$
(See Exercise 1.)

4. $Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$ (See Exercise 2.)

5. $Q(\mathbf{x}) = 5x_1^2 + 5x_2^2 - 4x_1x_2$

6. $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 + 3x_1x_2$

7. Let $Q(\mathbf{x}) = -2x_1^2 - x_2^2 + 4x_1x_2 + 4x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form $Q$ are 2, $-1$, and $-4$.]

8. Let $Q(\mathbf{x}) = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$. Find a unit vector $\mathbf{x}$ in $\mathbb{R}^3$ at which $Q(\mathbf{x})$ is maximized, subject to $\mathbf{x}^T\mathbf{x} = 1$. [*Hint:* The eigenvalues of the matrix of the quadratic form $Q$ are 9 and $-3$.]

9. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

10. Find the maximum value of $Q(\mathbf{x}) = -3x_1^2 + 5x_2^2 - 2x_1x_2$, subject to the constraint $x_1^2 + x_2^2 = 1$. (Do not go on to find a vector where the maximum is attained.)

11. Suppose $\mathbf{x}$ is a unit eigenvector of a matrix $A$ corresponding to an eigenvalue 3. What is the value of $\mathbf{x}^TA\mathbf{x}$?

12. Let $\lambda$ be any eigenvalue of a symmetric matrix $A$. Justify the statement made in this section that $m \le \lambda \le M$, where $m$ and $M$ are defined as in (2). [*Hint:* Find an $\mathbf{x}$ such that $\lambda = \mathbf{x}^TA\mathbf{x}$.]

13. Let $A$ be an $n \times n$ symmetric matrix, let $M$ and $m$ denote the maximum and minimum values of the quadratic form $\mathbf{x}^TA\mathbf{x}$, and denote corresponding unit eigenvectors by $\mathbf{u}_1$ and $\mathbf{u}_n$. The following calculations show that given any number $t$ between $M$ and $m$, there is a unit vector $\mathbf{x}$ such that $t = \mathbf{x}^TA\mathbf{x}$. Verify that $t = (1-\alpha)m + \alpha M$ for some number $\alpha$ between 0 and 1. Then let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$, and show that $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^TA\mathbf{x} = t$.

**[M]** In Exercises 14–17, follow the instructions given for Exercises 3–6.

14. $x_1x_2 + 3x_1x_3 + 30x_1x_4 + 30x_2x_3 + 3x_2x_4 + x_3x_4$

15. $3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4$

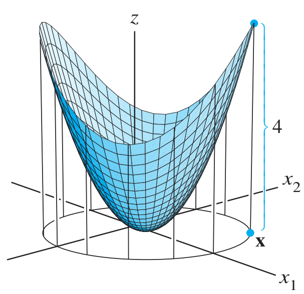
16. $4x_1^2 - 6x_1x_2 - 10x_1x_3 - 10x_1x_4 - 6x_2x_3 - 6x_2x_4 - 2x_3x_4$

17. $-6x_1^2 - 10x_2^2 - 13x_3^2 - 13x_4^2 - 4x_1x_2 - 4x_1x_3 - 4x_1x_4 + 6x_3x_4$

## SOLUTIONS TO PRACTICE PROBLEMS

1. The matrix of the quadratic form is $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. It is easy to find the eigenvalues, 4 and 2, and corresponding unit eigenvectors, $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So the desired change of variable is $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. (A common error here is to forget to normalize the eigenvectors.) The new quadratic form is $\mathbf{y}^TD\mathbf{y} = 4y_1^2 + 2y_2^2$.

2. The maximum of $Q(\mathbf{x})$ for $\mathbf{x}$ a unit vector is 4, and the maximum is attained at the unit eigenvector $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. [A common incorrect answer is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This vector maximizes the quadratic form $\mathbf{y}^TD\mathbf{y}$ instead of $Q(\mathbf{x})$.]



The maximum value of $Q(\mathbf{x})$ subject to $\mathbf{x}^T\mathbf{x} = 1$ is 4.

# 7.4 THE SINGULAR VALUE DECOMPOSITION

The diagonalization theorems in Sections 5.3 and 7.1 play a part in many interesting applications. Unfortunately, as we know, not all matrices can be factored as $A = PDP^{-1}$ with $D$ diagonal. However, a factorization $A = QDP^{-1}$ *is* possible for any $m \times n$ matrix $A$! A special factorization of this type, called the *singular value decomposition*, is one of the most useful matrix factorizations in applied linear algebra.

The singular value decomposition is based on the following property of the ordinary diagonalization that can be imitated for rectangular matrices: The absolute values of the eigenvalues of a symmetric matrix $A$ measure the amounts that $A$ stretches or shrinks

Moler, C. B., and D. Morrison, "Singular Value Analysis of Cryptograms." *Amer. Math. Monthly* **90** (1983), pp. 78–87.

Strang, Gilbert, *Linear Algebra and Its Applications,* 4th ed. (Belmont, CA: Brooks/Cole, 2005).

Watkins, David S., *Fundamentals of Matrix Computations* (New York: Wiley, 1991), pp. 390–398, 409–421.

### PRACTICE PROBLEM

WEB

Given a singular value decomposition, $A = U\Sigma V^T$, find an SVD of $A^T$. How are the singular values of $A$ and $A^T$ related?

## 7.4 EXERCISES

Find the singular values of the matrices in Exercises 1–4.

**1.** $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **2.** $\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$

**3.** $\begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix}$ **4.** $\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$

Find an SVD of each matrix in Exercises 5–12. [*Hint:* In Exercise 11, one choice for $U$ is $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$. In Exercise 12, one column of $U$ can be $\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.]

**5.** $\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

**7.** $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

**9.** $\begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$ **10.** $\begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$

**11.** $\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

**13.** Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ [*Hint:* Work with $A^T$.]

**14.** In Exercise 7, find a unit vector $\mathbf{x}$ at which $A\mathbf{x}$ has maximum length.

**15.** Suppose the factorization below is an SVD of a matrix $A$, with the entries in $U$ and $V$ rounded to two decimal places.

$$A = \begin{bmatrix} .40 & -.78 & .47 \\ .37 & -.33 & -.87 \\ -.84 & -.52 & -.16 \end{bmatrix} \begin{bmatrix} 7.10 & 0 & 0 \\ 0 & 3.10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} .30 & -.51 & -.81 \\ .76 & .64 & -.12 \\ .58 & -.58 & .58 \end{bmatrix}$$

a. What is the rank of $A$?

b. Use this decomposition of $A$, with no calculations, to write a basis for Col $A$ and a basis for Nul $A$. [*Hint:* First write the columns of $V$.]

**16.** Repeat Exercise 15 for the following SVD of a $3 \times 4$ matrix $A$:

$$A = \begin{bmatrix} -.86 & -.11 & -.50 \\ .31 & .68 & -.67 \\ .41 & -.73 & -.55 \end{bmatrix} \begin{bmatrix} 12.48 & 0 & 0 & 0 \\ 0 & 6.34 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} .66 & -.03 & -.35 & .66 \\ -.13 & -.90 & -.39 & -.13 \\ .65 & .08 & -.16 & -.73 \\ -.34 & .42 & -.84 & -.08 \end{bmatrix}$$

In Exercises 17–24, $A$ is an $m \times n$ matrix with a singular value decomposition $A = U\Sigma V^T$, where $U$ is an $m \times m$ orthogonal matrix, $\Sigma$ is an $m \times n$ "diagonal" matrix with $r$ positive entries and no negative entries, and $V$ is an $n \times n$ orthogonal matrix. Justify each answer.

**17.** Suppose $A$ is square and invertible. Find a singular value decomposition of $A^{-1}$.

**18.** Show that if $A$ is square, then $|\det A|$ is the product of the singular values of $A$.

**19.** Show that the columns of $V$ are eigenvectors of $A^TA$, the columns of $U$ are eigenvectors of $AA^T$, and the diagonal entries of $\Sigma$ are the singular values of $A$. [*Hint:* Use the SVD to compute $A^TA$ and $AA^T$.]

**20.** Show that if $A$ is an $n \times n$ positive definite matrix, then an orthogonal diagonalization $A = PDP^T$ is a singular value decomposition of $A$.

**21.** Show that if $P$ is an orthogonal $m \times m$ matrix, then $PA$ has the same singular values as $A$.

**22.** Justify the statement in Example 2 that the second singular value of a matrix $A$ is the maximum of $\|A\mathbf{x}\|$ as $\mathbf{x}$ varies over all unit vectors orthogonal to $\mathbf{v}_1$, with $\mathbf{v}_1$ a right singular vector corresponding to the first singular value of $A$. [*Hint:* Use Theorem 7 in Section 7.3.]

**23.** Let $U = [\,\mathbf{u}_1 \ \cdots \ \mathbf{u}_m\,]$ and $V = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]$, where the $\mathbf{u}_i$ and $\mathbf{v}_i$ are as in Theorem 10. Show that

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T.$$

**24.** Using the notation of Exercise 23, show that $A^T \mathbf{u}_j = \sigma_j \mathbf{v}_j$ for $1 \le j \le r = \text{rank } A$.

**25.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Describe how to find a basis $\mathcal{B}$ for $\mathbb{R}^n$ and a basis $\mathcal{C}$ for $\mathbb{R}^m$ such that the matrix for $T$ relative to $\mathcal{B}$ and $\mathcal{C}$ is an $m \times n$ "diagonal" matrix.

[**M**] Compute an SVD of each matrix in Exercises 26 and 27. Report the final matrix entries accurate to two decimal places. Use the method of Examples 3 and 4.

**26.** $A = \begin{bmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{bmatrix}$

**27.** $A = \begin{bmatrix} 6 & -8 & -4 & 5 & -4 \\ 2 & 7 & -5 & -6 & 4 \\ 0 & -1 & -8 & 2 & 2 \\ -1 & -2 & 4 & 4 & -8 \end{bmatrix}$

**28.** [**M**] Compute the singular values of the $4 \times 4$ matrix in Exercise 9 in Section 2.3, and compute the condition number $\sigma_1/\sigma_4$.

**29.** [**M**] Compute the singular values of the $5 \times 5$ matrix in Exercise 10 in Section 2.3, and compute the condition number $\sigma_1/\sigma_5$.

### SOLUTION TO PRACTICE PROBLEM

If $A = U\Sigma V^T$, where $\Sigma$ is $m \times n$, then $A^T = (V^T)^T \Sigma^T U^T = V \Sigma^T U^T$. This is an SVD of $A^T$ because $V$ and $U$ are orthogonal matrices and $\Sigma^T$ is an $n \times m$ "diagonal" matrix. Since $\Sigma$ and $\Sigma^T$ have the same nonzero diagonal entries, $A$ and $A^T$ have the same nonzero singular values. [*Note:* If $A$ is $2 \times n$, then $AA^T$ is only $2 \times 2$ and its eigenvalues may be easier to compute (by hand) than the eigenvalues of $A^TA$.]

## 7.5 APPLICATIONS TO IMAGE PROCESSING AND STATISTICS

The satellite photographs in this chapter's introduction provide an example of multidimensional, or *multivariate*, data—information organized so that each datum in the data set is identified with a point (vector) in $\mathbb{R}^n$. The main goal of this section is to explain a technique, called *principal component analysis*, used to analyze such multivariate data. The calculations will illustrate the use of orthogonal diagonalization and the singular value decomposition.

Principal component analysis can be applied to any data that consist of lists of measurements made on a collection of objects or individuals. For instance, consider a chemical process that produces a plastic material. To monitor the process, 300 samples are taken of the material produced, and each sample is subjected to a battery of eight tests, such as melting point, density, ductility, tensile strength, and so on. The laboratory report for each sample is a vector in $\mathbb{R}^8$, and the set of such vectors forms an $8 \times 300$ matrix, called the **matrix of observations**.

Loosely speaking, we can say that the process control data are eight-dimensional. The next two examples describe data that can be visualized graphically.

**EXAMPLE 1** An example of two-dimensional data is given by a set of weights and heights of $N$ college students. Let $\mathbf{X}_j$ denote the **observation vector** in $\mathbb{R}^2$ that lists the weight and height of the $j$th student. If $w$ denotes weight and $h$ height, then the matrix