1.3 EXERCISES

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ **2.** $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy-graph: $\mathbf{u}$, $\mathbf{v}$, $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, and $\mathbf{u}-2\mathbf{v}$. Notice that $\mathbf{u}-\mathbf{v}$ is the vertex of a parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $-\mathbf{v}$.

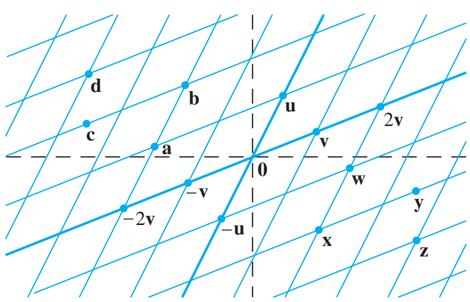
3. $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 1 **4.** $\mathbf{u}$ and $\mathbf{v}$ as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

6. $x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?



7. Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$

8. Vectors $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9.
$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

10.
$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 3 \\ -2x_1 - 7x_2 + 5x_3 &= 1 \\ 5x_1 + 4x_2 - 3x_3 &= 2 \end{aligned}$$

In Exercises 11 and 12, determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 13 and 14, determine if $\mathbf{b}$ is a linear combination of the vectors formed from the columns of the matrix A.

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

15. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

16. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}$. For what value(s) of h is $\mathbf{y}$ in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?

In Exercises 17 and 18, list five vectors in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on $\mathbf{v}_1$ and $\mathbf{v}_2$ used to generate the vector and list the three entries of the vector. Do not make a sketch.

17. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

18. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

19. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 18.

21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct a 3×3 matrix A, with nonzero entries, and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ is *not* in the set spanned by the columns of A.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[\,-4 \quad 3\,]$.

b. The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.

c. An example of a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the vector $\frac{1}{2}\mathbf{v}_1$.

d. The solution set of the linear system whose augmented matrix is $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}\,]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

e. The set Span $\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

24. a. When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ contains only the line through $\mathbf{u}$ and the origin, and the line through $\mathbf{v}$ and the origin.

b. Any list of five real numbers is a vector in $\mathbb{R}^5$.

c. Asking whether the linear system corresponding to an augmented matrix $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}\,]$ has a solution amounts to asking whether $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

d. The vector $\mathbf{v}$ results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector $\mathbf{v}$.

e. The weights $c_1, \dots, c_p$ in a linear combination $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ cannot all be zero.

25. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

a. Is $\mathbf{b}$ in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?

b. Is $\mathbf{b}$ in W? How many vectors are in W?

c. Show that $\mathbf{a}_1$ is in W. [*Hint:* Row operations are unnecessary.]

26. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is $\mathbf{b}$ in W?

b. Show that the second column of A is in W.

27. A mining company has two mines. One day's operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kilograms of silver, while one day's operation at mine #2 produces ore that contains 40 metric tons of copper and 380 kilograms of silver. Let $\mathbf{v}_1 = \begin{bmatrix} 30 \\ 600 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 40 \\ 380 \end{bmatrix}$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ represent the "output per day" of mine #1 and mine #2, respectively.

a. What physical interpretation can be given to the vector $5\mathbf{v}_1$?

b. Suppose the company operates mine #1 for x_1 days and mine #2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 240 tons of copper and 2824 kilograms of silver. Do not solve the equation.

c. [**M**] Solve the equation in (b).

28. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

a. How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?

b. Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.

c. [**M**] Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

29. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be points in $\mathbb{R}^3$ and suppose that for $j = 1, \dots, k$ an object with mass m_j is located at point $\mathbf{v}_j$. Physicists call such objects *point masses*. The total mass of the system of point masses is

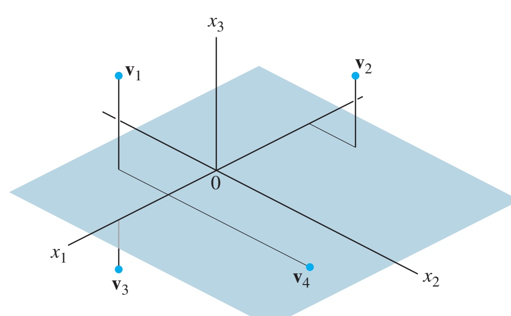
$$m = m_1 + \cdots + m_k$$

The *center of gravity* (or *center of mass*) of the system is

$$\bar{\mathbf{v}} = \frac{1}{m}[m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k]$$

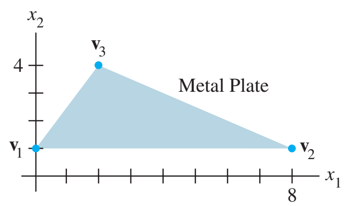
Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (2, -2, 4)$	4 g
$\mathbf{v}_2 = (-4, 2, 3)$	2 g
$\mathbf{v}_3 = (4, 0, -2)$	3 g
$\mathbf{v}_4 = (1, -6, 0)$	5 g



30. Let $\mathbf{v}$ be the center of mass of a system of point masses located at $\mathbf{v}_1, \dots, \mathbf{v}_k$ as in Exercise 29. Is $\mathbf{v}$ in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$? Explain.

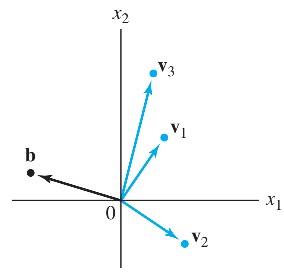
31. A thin triangular plate of uniform density and thickness has vertices at $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (8, 1)$, and $\mathbf{v}_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.



a. Find the (x, y)-coordinates of the center of mass of the plate. This "balance point" of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [*Hint:* Let w_1, w_2, and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

32. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{b}$ in $\mathbb{R}^2$, shown in the figure. Does the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ have a solution? Is the solution unique? Use the figure to explain your answers.



33. Use the vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\mathbf{w} = (w_1, \dots, w_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

a. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

b. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for each scalar c

34. Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

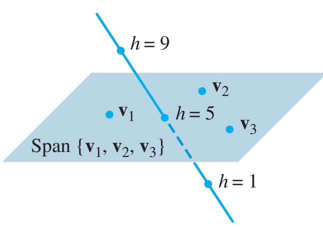
a. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

b. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

SOLUTIONS TO PRACTICE PROBLEMS

1. Take arbitrary vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$, and compute

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\
&= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\
&= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition}
\end{aligned}$$



The points $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ lie on a line that intersects the plane when $h = 5$.

2. The vector $\mathbf{y}$ belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h - 5$ must be 0. So $\mathbf{y}$ is in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

1.4 THE MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$, and (a) is true. If (d) is false, the last row of U is all zeros. Let $\mathbf{d}$ be any vector with a 1 in its last entry. Then $[\,U \quad \mathbf{d}\,]$ represents an *inconsistent* system. Since row operations are reversible, $[\,U \quad \mathbf{d}\,]$ can be transformed into the form $[\,A \quad \mathbf{b}\,]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

PRACTICE PROBLEMS

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit $\mathbf{b}$ as a specific linear combination of the columns of A.

2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.

1.4 EXERCISES

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$

7. $x_1\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

8. $z_1\begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2\begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3\begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

In Exercises 9 and 10, write the system first as a vector equation and then as a matrix equation.

9. $\begin{aligned} 5x_1 + x_2 - 3x_3 &= 8 \\ 2x_2 + 4x_3 &= 0 \end{aligned}$

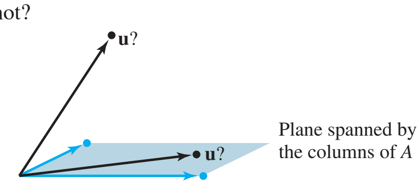
10. $\begin{aligned} 4x_1 - x_2 &= 8 \\ 5x_1 + 3x_2 &= 2 \\ 3x_1 - x_2 &= 1 \end{aligned}$

Given A and $\mathbf{b}$ in Exercises 11 and 12, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

11. $A = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & 2 \\ -3 & -7 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 12 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A? (See the figure.) Why or why not?



Where is $\mathbf{u}$?

14. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$. Is $\mathbf{u}$ in the subset of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

15. Let $A = \begin{bmatrix} 3 & -1 \\ -9 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.

16. Repeat the requests from Exercise 15 with

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Exercises 17–20 refer to the matrices A and B below. Make appropriate calculations that justify your answers and mention an appropriate theorem.

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

17. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

18. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B above? Do the columns of B span $\mathbb{R}^3$?

19. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A above? Do the columns of A span $\mathbb{R}^4$?

20. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

21. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^4$? Why or why not?

22. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$? Why or why not?

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. The equation $A\mathbf{x} = \mathbf{b}$ is referred to as a *vector equation*.

b. A vector $\mathbf{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

c. The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix $[\,A \quad \mathbf{b}\,]$ has a pivot position in every row.

d. The first entry in the product $A\mathbf{x}$ is a sum of products.

e. If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^m$.

f. If A is an $m \times n$ matrix and if the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$, then A cannot have a pivot position in every row.

24. a. Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

b. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the set spanned by the columns of A.

c. Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

d. If the coefficient matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

e. The solution set of a linear system whose augmented matrix is $[\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}\,]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$.

f. If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

25. Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

26. Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$. It can be shown that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation $\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

27. Rewrite the (numerical) matrix equation below in symbolic form as a vector equation, using symbols $\mathbf{v}_1, \mathbf{v}_2, \ldots$ for the vectors and $c_1, c_2, \ldots$ for scalars. Define what each symbol represents, using the data given in the matrix equation.

$$\begin{bmatrix} -3 & 5 & -4 & 9 & 7 \\ 5 & 8 & 1 & -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

28. Let $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{v}$ represent vectors in $\mathbb{R}^5$, and let x_1, x_2, and x_3 denote scalars. Write the following vector equation as a matrix equation. Identify any symbols you choose to use.

$x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \mathbf{v}$

29. Construct a 3×3 matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

30. Construct a 3×3 matrix, not in echelon form, whose columns do *not* span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.

31. Let A be a 3×2 matrix. Explain why the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generalize your argument to the case of an arbitrary A with more rows than columns.

32. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? Explain. What about n vectors in $\mathbb{R}^m$ when n is less than m?

33. Suppose A is a 4×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the reduced echelon form of A? Justify your answer.

34. Let A be a 3×4 matrix, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in $\mathbb{R}^3$, and let $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. Suppose $\mathbf{v}_1 = A\mathbf{u}_1$ and $\mathbf{v}_2 = A\mathbf{u}_2$ for some vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^4$. What fact allows you to conclude that the system $A\mathbf{x} = \mathbf{w}$ is consistent? (*Note:* $\mathbf{u}_1$ and $\mathbf{u}_2$ denote vectors, not scalar entries in vectors.)

35. Let A be a 5×3 matrix, let $\mathbf{y}$ be a vector in $\mathbb{R}^3$, and let $\mathbf{z}$ be a vector in $\mathbb{R}^5$. Suppose $A\mathbf{y} = \mathbf{z}$. What fact allows you to conclude that the system $A\mathbf{x} = 5\mathbf{z}$ is consistent?

36. Suppose A is a 4×4 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ with the property that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Explain why the columns of A must span $\mathbb{R}^4$.

[**M**] In Exercises 37–40, determine if the columns of the matrix span $\mathbb{R}^4$.

37. $\begin{bmatrix} 7 & 2 & -5 & 8 \\ -5 & -3 & 4 & -9 \\ 6 & 10 & -2 & 7 \\ -7 & 9 & 2 & 15 \end{bmatrix}$ **38.** $\begin{bmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{bmatrix}$

39. $\begin{bmatrix} 10 & -7 & 1 & 4 & 6 \\ -8 & 4 & -6 & -10 & -3 \\ -7 & 11 & -5 & -1 & -8 \\ 3 & -1 & 10 & 12 & 12 \end{bmatrix}$

40. $\begin{bmatrix} 5 & 11 & -6 & -7 & 12 \\ -7 & -3 & -4 & 6 & -9 \\ 11 & 5 & 6 & -9 & -3 \\ -3 & 4 & -7 & 2 & 7 \end{bmatrix}$

41. [**M**] Find a column of the matrix in Exercise 39 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$.

42. [**M**] Find a column of the matrix in Exercise 40 that can be deleted and yet have the remaining matrix columns still span $\mathbb{R}^4$. Can you delete more than one column?

SG Mastering Linear Algebra Concepts: Span 1–18

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SOLUTIONS TO PRACTICE PROBLEMS

1. The matrix equation

$$\begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

is equivalent to the vector equation

$$3\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 1 \\ -8 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$$

which expresses $\mathbf{b}$ as a linear combination of the columns of A.

2.
$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$A(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 + 20 \\ 3 + 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$$

$$A\mathbf{u} + A\mathbf{v} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 11 \end{bmatrix} + \begin{bmatrix} 19 \\ -4 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \end{bmatrix}$$

1.5 EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. $$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

2. $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -2x_1 - 3x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 9x_3 &= 0 \end{aligned}$$

3. $$\begin{aligned} -3x_1 + 4x_2 - 8x_3 &= 0 \\ -2x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 0 \\ -3x_1 - 4x_2 + 2x_3 &= 0 \end{aligned}$$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. $$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_2 - 3x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -1x_1 + x_2 \qquad &= 0 \end{aligned}$$

In Exercises 7–12, describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the given matrix.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$

10. $\begin{bmatrix} -1 & -4 & 0 & -4 \\ 2 & -8 & 0 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

14. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5x_4$, $x_2 = 3 - 2x_4$, $x_3 = 2 + 5x_4$, with x_4 free. Use vectors to describe this set as a "line" in $\mathbb{R}^4$.

15. Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.

16. Describe and compare the solution sets of $x_1 - 2x_2 + 3x_3 = 0$ and $x_1 - 2x_2 + 3x_3 = 4$.

17. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

18. As in Exercise 17, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 \qquad &= -8 \end{aligned}$$

In Exercises 19 and 20, find the parametric equation of the line through **a** parallel to **b**.

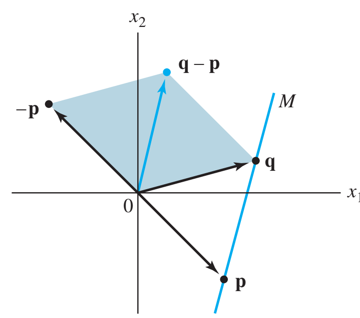
19. $\mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

20. $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

In Exercises 21 and 22, find a parametric equation of the line M through **p** and **q**. [*Hint*: M is parallel to the vector $\mathbf{q} - \mathbf{p}$. See the figure below.]

21. $\mathbf{p} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

22. $\mathbf{p} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$



In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A homogeneous equation is always consistent.

b. The equation $A\mathbf{x} = \mathbf{0}$ gives an explicit description of its solution set.

c. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through **v** parallel to **p**.

e. The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.

24. a. A homogeneous system of equations can be inconsistent.

b. If **x** is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in **x** is nonzero.

c. The effect of adding **p** to a vector is to move the vector in a direction parallel to **p**.

d. The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

e. If $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$.

25. Prove Theorem 6:

a. Suppose $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

b. Let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$. This shows that every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, with $\mathbf{p}$ a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ a solution of $A\mathbf{x} = \mathbf{0}$.

26. Suppose A is the 3×3 *zero* matrix (with all zero entries). Describe the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

27. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

In Exercises 28–31, (a) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and (b) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

28. A is a 3×3 matrix with three pivot positions.

29. A is a 4×4 matrix with three pivot positions.

30. A is a 2×5 matrix with two pivot positions.

31. A is a 3×2 matrix with two pivot positions.

32. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

33. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

34. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

35. Given $A = \begin{bmatrix} -1 & -3 \\ 7 & 21 \\ -2 & -6 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection. [*Hint*: Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

36. Given $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection.

37. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$. Why does this *not* contradict Theorem 6?

38. Let A be an $m \times n$ matrix and let $\mathbf{w}$ be a vector in $\mathbb{R}^n$ that satisfies the equation $A\mathbf{x} = \mathbf{0}$. Show that for any scalar c, the vector $c\mathbf{w}$ also satisfies $A\mathbf{x} = \mathbf{0}$. [That is, show that $A(c\mathbf{w}) = \mathbf{0}$.]

39. Let A be an $m \times n$ matrix, and let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ with the property that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. Explain why $A(\mathbf{v} + \mathbf{w})$ must be the zero vector. Then explain why $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ for each pair of scalars c and d.

40. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution. Does there exist a vector $\mathbf{y}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ has a unique solution? Discuss.

SOLUTIONS TO PRACTICE PROBLEMS

1. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad\; + 3x_3 &= 4 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

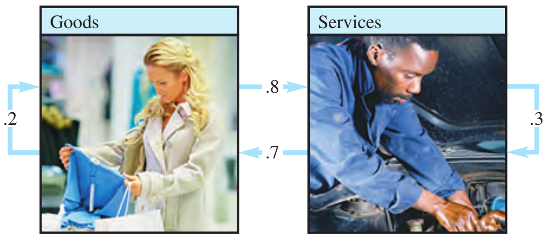
Thus $x_1 = 4 - 3x_3$, $x_2 = -1 + 2x_3$, with x_3 free. The general solution in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \underset{\uparrow \atop \mathbf{p}}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}} + x_3 \underset{\uparrow \atop \mathbf{v}}{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

The intersection of the two planes is the line through $\mathbf{p}$ in the direction of $\mathbf{v}$.

1.6 EXERCISES

1. Suppose an economy has only two sectors: Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.



2. Find another set of equilibrium prices for the economy in Example 1. Suppose the same economy used Japanese yen instead of dollars to measure the values of the various sectors' outputs. Would this change the problem in any way? Discuss.

3. Consider an economy with three sectors: Fuels and Power, Manufacturing, and Services. Fuels and Power sells 80% of its output to Manufacturing, 10% to Services, and retains the rest. Manufacturing sells 10% of its output to Fuels and Power, 80% to Services, and retains the rest. Services sells 20% to Fuels and Power, 40% to Manufacturing, and retains the rest.
   a. Construct the exchange table for this economy.
   b. Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.
   c. [**M**] Find a set of equilibrium prices when the price for the Services output is 100 units.

4. Suppose an economy has four sectors: Mining, Lumber, Energy, and Transportation. Mining sells 10% of its output to Lumber, 60% to Energy, and retains the rest. Lumber sells 15% of its output to Mining, 50% to Energy, 20% to Transportation, and retains the rest. Energy sells 20% of its output to Mining, 15% to Lumber, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Mining, 10% to Lumber, 50% to Energy, and retains the rest.
   a. Construct the exchange table for this economy.
   b. [**M**] Find a set of equilibrium prices for the economy.

5. An economy has four sectors: Agriculture, Manufacturing, Services, and Transportation. Agriculture sells 20% of its output to Manufacturing, 30% to Services, 30% to Transportation, and retains the rest. Manufacturing sells 35% of its output to Agriculture, 35% to Services, 20% to Transportation, and retains the rest. Services sells 10% of its output to Agriculture, 20% to Manufacturing, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to Agriculture, 30% to Manufacturing, 20% to Services, and retains the rest.
   a. Construct the exchange table for this economy.
   b. [**M**] Find a set of equilibrium prices for the economy if the value of Transportation is \$10.00 per unit.
   c. The Services sector launches a successful "eat farm fresh" campaign, and increases its share of the output from the Agricultural sector to 40%, whereas the share of Agricultural production going to Manufacturing falls to 10%. Construct the exchange table for this new economy.
   d. [**M**] Find a set of equilibrium prices for this new economy if the value of Transportation is still \$10.00 per unit. What effect has the "eat farm fresh" campaign had on the equilibrium prices for the sectors in this economy?

Balance the chemical equations in Exercises 6–11 using the vector equation approach discussed in this section.

6. Aluminum oxide and carbon react to create elemental aluminum and carbon dioxide:

   $Al_2O_3 + C \rightarrow Al + CO_2$

   [For each compound, construct a vector that lists the numbers of atoms of aluminum, oxygen, and carbon.]

7. Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):

   $NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2$

8. Limestone, $CaCO_3$, neutralizes the acid, H_3O, in acid rain by the following unbalanced equation:

   $H_3O + CaCO_3 \rightarrow H_2O + Ca + CO_2$

9. Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs). The unbalanced equation is

   $B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S$

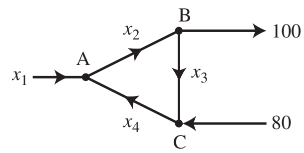
10. [**M**] If possible, use exact arithmetic or a rational format for calculations in balancing the following chemical reaction:

    $PbN_6 + CrMn_2O_8 \rightarrow Pb_3O_4 + Cr_2O_3 + MnO_2 + NO$

11. [**M**] The chemical reaction below can be used in some industrial processes, such as the production of arsene (AsH_3). Use exact arithmetic or a rational format for calculations to balance this equation.

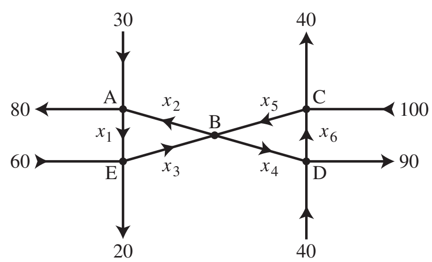
    $$MnS + As_2Cr_{10}O_{35} + H_2SO_4 \rightarrow HMnO_4 + AsH_3 + CrS_3O_{12} + H_2O$$

12. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for x_4?



13. a. Find the general flow pattern of the network shown in the figure.

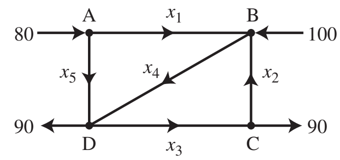
b. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4, and x_5.



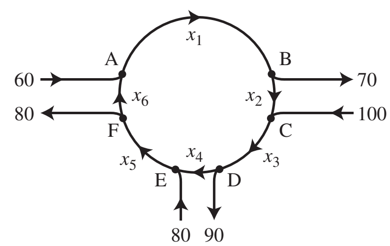
14. a. Find the general traffic pattern of the freeway network shown in the figure. (Flow rates are in cars/minute.)

b. Describe the general traffic pattern when the road whose flow is x_5 is closed.

c. When $x_5 = 0$, what is the minimum value of x_4?



15. Intersections in England are often constructed as one-way "roundabouts," such as the one shown in the figure. Assume that traffic must travel in the directions shown. Find the general solution of the network flow. Find the smallest possible value for x_6.



SOLUTIONS TO PRACTICE PROBLEMS

1. Write the percentages as decimals. Since all output must be taken into account, each column must sum to 1. This fact helps to fill in any missing entries.

Distribution of Output from:			
Agriculture	**Mining**	**Manufacturing**	**Purchased by:**
.65	.20	.20	Agriculture
.05	.10	.30	Mining
.30	.70	.50	Manufacturing

2. Since $x_5 \le 500$, the equations D and A for x_1 and x_2 imply that $x_1 \ge 100$ and $x_2 \le 700$. The fact that $x_5 \ge 0$ implies that $x_1 \le 600$ and $x_2 \ge 200$. So, $100 \le x_1 \le 600$, and $200 \le x_2 \le 700$.

1.7 LINEAR INDEPENDENCE

The homogeneous equations in Section 1.5 can be studied from a different perspective by writing them as vector equations. In this way, the focus shifts from the unknown solutions of $A\mathbf{x} = \mathbf{0}$ to the vectors that appear in the vector equations.