FINAL PROJECT

Math 4030 is coming to a close. Rather than have a timed final exam, I'd like to give you all a chance for some reflection about this course. What follows is a "take home final." Hopefully these problems and open-ended questions are fun for you and that the document you produce can be of value in the future when you are teaching.

The requirements of the final are:

- Pick a topic from the following list: Due Friday the 18th (this week). Group work and group presentations are encouraged!
- Meet with me prior to your presentation to go over what you are learning and prepare for the presentation.
- Give your presentation during the final week of class.
- Write a short report: (in total 3 - 5 pages)

The written report is due via email at 5pm on Friday December 9th

  (1) Identify three topics or themes that you have learned about in this course. For each of them, describe some new insight that you gained from the course (whether it be about the material, or about ways you could teach/explain these topics to others.
  (2) One of the topics should be the subject of your project/presentation.
  (3) Also, write a biography of a mathematician who was known for their contributions to Algebra. I suggest you consider Emmy Noether, Sophie Germain, Carl Gauss, Evariste Galois, you may include others. Please provide references to three different websites or books (Wikipedia is ok). Mainly what I'm hoping you discover here is some interesting information about mathematicians and you can take this with you in your classes.

Remember that you will be presenting your topic to your peers - they'll know what we've learned in class, but they won't be familiar with the problem you chose, so be sure to give lots of examples and careful definitions.

We will do our presentations the final week of class. Sign ups for the day / time of your talk will be a first come / first served basis. Attendance on BOTH days is mandatory. It's important to support everyone in the class so I want full attendance and participation that week.

I would like everyone to schedule a time to come talk to me about their projects. You can drop by office hours or you can send me an email to schedule a time. If you are part of a group, I encourage you to come together, but this is not required. In this meeting we can talk about the project, what you have learned, what questions you have, and I can offer suggestions as to how you can give the best presentation. If you don't do this step, your grade on the final will be one letter grade lower. My schedule is very flexible so we will certainly be able to find a time to meet.

Email me your choice of topic by THIS FRIDAY, (preferable as soon as possible). Once I know what topics will be covered, I will prepare some preparatory materials to help you with your project. Since there are more topics than people, we will have multiple people on the same project. I hope that you can work together as a group, but if not, I can help guide the presentations so you can present different parts.

Rubric Your grade will be based off of how engaged you are with with the project.

"A" indicates serious involvement with the topic. You have more than a superficial understanding of the problem - you have solved the example problems and presented them clearly to the class. You have explored wikipedia, can comment on some of the history of the problem, and give an interesting and exciting presentation.

"B" indicates some mastery with the material. You have a rough understanding of what's going on, but some key parts of the definition / proofs are lacking.

"C" indicates very minimal effort.

Similar standards apply to the writing portion. The breakdown is presentation/writing at 60%/40%.

Remember, I really want you all to have fun with this, so feel free to pick a topic that you like - and feel free to ask me for as much help as you like. I'm happy to watch a practice presentation if you like.

1. The Topics

(1) **Irrational Numbers** We have seen lots of examples of numbers that are not rational. For instance, we proved that $\sqrt{2}$ is irrational by arguing by contradiction. However, we haven't seen much about say, why the number e is irrational. There are several proofs for the irrationality of e that are pretty approachable. I will post one on canvas, though you should feel free to search online to find others if you prefer. In your presentation I'd like you to give a proof that e is irrational and also talk about what a *transcendental number* is. Proving that π or e is transcendental is quite challenging actually, and wouldn't fit into the 15 minute presentation, but I'd like you to look at some of the proofs (posted on Canvas) and talk to the class about what you learned.

(2) **Algebraic Numbers form a field**: In class we saw lots of examples of algebraic numbers. Things like i, 3, $\sqrt{5}$ and $2+\sqrt[3]{5}$ are all examples of algebraic numbers. You could find the characteristic polynomial for these numbers without too much work. However what's true is that the algebraic numbers form a field! In other words - if α and β are two different algebraic numbers (say like $\sqrt[3]{5}$ and $\sqrt{2}$), then their sum $\alpha+\beta$ is also algebraic, and so is their product and also $1/\alpha$ is algebraic! This is really an impressive feat - but how could we ever prove something like this? As a first step you should work out what the characteristic polynomial is for $\alpha, \beta, \alpha+\beta, \alpha\beta$ and $1/\alpha$ in the case that $\alpha=\sqrt{2}$ and $\beta=\sqrt{3}$. In this case you can do everything pretty simply. But what if $\alpha=\sqrt[3]{5i}$ and $\beta=\sqrt{2}$? You should see that it's actually pretty challenging! If you choose this project you'll learn how to prove this amazing fact and how you can write down the characteristic polynomial for any combination of by using determinants of matrices. Let me know if you are interested in this project and we can meet to go over some of the basic ideas.

(3) **Quadratic Reciprocity** This problem is all about whether x^2+1 is irreducible modulo p. For instance, mod 2 we see that it factors as $(x+1)^2$ whereas over $\mathbb{Z}_3$ it is irreducible. You should convince yourself that it factors if and only if $x^2=-1$ has a solution mod p. In other words we want to discover when this equation has a solution mod p. You should do some example, say with $p=2,3,5,7,11,13,17$ and see when you can solve this. You should see a pattern. If you choose this project I can help you discover why this is true and also show you the general ways in which we can figure out whether or not $x^2=a$ has a solution modulo p. For instance, how could you determine whether or not $x^2=18$ has a solution modulo 101? (One way would be to square all numbers $0,1,2,\ldots,100$, reduce and see if any of them were 18. That would be annoying and hard to do without a calculator). Instead you'll learn that there is a very quick way to do this using a Theorem called the Theorem of Quadratic Reciprocity. In this presentation you'll learn how to tell (with proof) whether $x^2=-1$ has a solution modulo p. This will involve some baby group theory.

(4) **Cyclotomic Polynomials** In this problem we'll talk about the polynomials of the form $x^n - 1$. Of course they all factor as $(x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ but which of them factor more? For instance, $x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$. It turns out that there are a certain number of building block that can be used to build any polynomial of the form $x^n - 1$. Consider

$$\phi_1 = x - 1$$
$$\phi_2 = x + 1$$
$$\phi_3 = x^2 + x + 1$$
$$\phi_4 = x^2 + 1$$
$$\phi_5 = x^4 + x^3 + x^2 + x + 1$$
$$\phi_6 = x^2 - x + 1$$

Then we have $x^6 - 1 = \phi_1\phi_2\phi_3\phi_6$ and $x^4 - 1 = \phi_1\phi_2\phi_4$. For instance look at the **Examples** section here `https://en.wikipedia.org/wiki/Cyclotomic_polynomial` to see that

$$\phi_{30} = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1.$$

What can we say about these polynomials? What do we know about the coefficients? How are they related to the n complex solutions to $x^n = 1$? In this project you'll learn how these polynomials are constructed and learn a bit about the roots of these polynomials.

(5) **Galois Theory** is the study of the symmetries of the roots of polynomial equations with rational coefficients. Suppose that $f(x)$ is such a polynomial. We say that a function $\sigma : \mathbb{C} \to \mathbb{C}$ is a *symmetry* of the roots of $f(x)$ if the following properties hold:
   (a) σ is 1-1 and onto
   (b) σ sends every root of $f(x)$ to a root (though it could be the same)
   (c) σ sends every rational number to itself
   (d) σ is additive and multiplicative.

We are most interested in the possible ways of shuffling the roots around.

Let's see what I mean here: For instance, consider the polynomial $f(x) = x^4 - 1$. Its roots are $\{1, -1, i, -i\}$. Now let's think about what symmetries it can have. Well we have to figure out what symmetries there are. Well by number (3) above, we know that $\sigma(1) = 1$ and $\sigma(-1) = -1$. So that just means that we are left with $\sigma(i)$ and $\sigma(-i)$. One option is that sigma sends $i \to i$ and $-i \to -i$, in which case we say that σ is the identity. However, there's another option - we could send $\sigma(i) = -i$ and $\sigma(-i) = i$. One can check that this satisfies all the various properties. So here we have that the symmetries of these four roots are just the identity and the one that flips i and $-i$.

As another example, consider the polynomial $(x^2 - 2)(x^2 + 1)$. This has roots $\pm\sqrt{2}$ and $\pm i$. Now none of these roots are rational, so we might have more freedom with our choice of sigma. Let's consider what $\sigma(\sqrt{2})$ can be. Well, I said that σ has to be additive and multiplicative. So that means that

$$\sigma(2) = \sigma(\sqrt{2}\sqrt{2}) = \sigma(\sqrt{2})\sigma(\sqrt{2}) = (\sigma(\sqrt{2}))^2$$

But remember that $\sigma(2) = 2$ since σ has to send every rational number to itself. Notice that this means that $\sigma(2)$ must satisfy

$$(\sigma(\sqrt{2}))^2 = 2.$$

But then the only possibilities for σ are that $\sigma(\sqrt{2}) = \pm\sqrt{2}$. Similarly, we see that $\sigma(i) = \pm i$. In other words it's not possible to send $i \to \sqrt{2}$ or vice versa.

In this project you'll look at the various roots of polynomials. For instance, what are the possible symmetries of the roots of $f(x) = x^4 - 2$ or $x^3 = 1$.

(6) **Another topic of your choosing** If you have another topic that you think would be good - please let me know, if I think it's suitable, I'll give you the go-ahead.