Challenge Problem 3: Proof Techniques Applied

MATH2603: Discrete Mathematics

Overview: In this Challenge Problem, youll have a chance to apply your skills with the four main **proof techniques: direct proof, contrapositive, proof by cases, and proof by contradiction.** You'll combine all four of these techniques to prove the irrationality of $\sqrt{10}$

Instructions: Below are several exercises, each of which asks you to prove a specific statement. **Complete all of the exercises below.** Each solution you give should be a complete, clear, and correct proof that involves one (or two) of the four proof techniques discussed in class. Your proof must conform to all the standards we have set for proofs, including clear statements of the assumptions and good writing in the proof itself. Please clearly indicate which problem you are working by restating the propositions in your submission. Your submission can take the form Theorem. . . Proof. . . Theorem. . . Proof . . . etc.

Background/Review: In class, we proved that $\sqrt{2}$ is an irrational number. Our proof had several steps. The three big steps in our proof from class are formulated as theorems below. Each step consisted of proving the corresponding theorem.

Theorem 1. *If x is even, then x^2 is even.*

Theorem 2. *If x^2 is even, then x is even.*

Theorem 3. *$\sqrt{2}$ is irrational.*

Since we proved these statements in class, you may use Theorem 1 and Theorem 2 in your work below. You do not need to prove them.

The proof discussed in class can be adapted (with some extra work) to show that the square root of any integer is either irrational or an integer. The goal of this worksheet is to prove that $\sqrt{10}$ is an irrational number. The steps you take to prove this statement will provide a framework for proving that the square root of any integer (which is not a square) is irrational.

1. Prove the theorem stated below. Be sure to explicitly state what proof technique you are using; begin your proof by stating your assumptions; and end your proof by stating your conclusion.

   Theorem 4. *If $n \in \mathbb{Z}^+$ and $5 \mid n$, then $5 \mid n^2$.*

2. Prove the theorem below. Be sure to explicitly state what proof technique you are using; begin your proof by stating your assumptions; and end your proof by stating your conclusion.

   Theorem 5. *If $n \in \mathbb{Z}^+$ and $5 \mid n^2$, then $5 \mid n$.*

3. Prove the theorem below.

   Theorem 6. *$\sqrt{15}$ is irrational.*

Hint: Mimic our proof of the irrationality of $\sqrt{2}$, and make use of the theorems above (the ones you have proved, and the ones stated in the Background). Be careful about the conclusions you make!

Submitting your work: Your work must be neatly typed up using a system that supports mathematical notation. For example, you can use MS Word and its equation editor; or you can write your work in a Jupyter notebook using Markdown and LaTeX. Once it is written up, the work must be saved as a PDF file and then uploaded as a PDF to the area on Blackboard where the original assignment is located. Remember that the work is not actually submitted until you upload the file and click the "Submit" button. Grading and feedback will take place entirely on Blackboard. The following are not allowed: Submissions outside Blackboard (for example through email); files that are not in PDF form; and work that contains any handwriting, though you may *draw a diagram* neatly by hand, scan it, and include it in your submission.

Evaluation: Like all Challenge Problems, your work will be evaluated using the EMRN rubric. Please see the statement of this rubric in the syllabus for an explanation of how it is used. When applied to this Challenge Problem, the following criteria help to assign the grade:

- **E:** The submission consists clear, correct, and complete proofs using appropriate proof techniques that contains no major errors (computation, logic, syntax, or semantic). The submission is also exceptionally clear and the writeup is professional in its look and style. The solution would be at home in a professional lecture or publication.

- **M:** The submission consists of a clear, correct, and complete proofs using appropriate proof techniques that contains no major errors (computation, logic, syntax, or semantic) and which is neatly and professionally written up.

- **R:** The solution contains at least one, but not several, major errors (computation, logic, syntax, and/or semantic) that require revision. An "R" may also be given for writeups that do not expend sufficient effort to produce a good-looking writeup, or do not provide enough context for the submission to be self-contained.

- **N:** The solution has several significant errors; or the submission is missing large portions of the solution; or the solution is for a significantly altered version of the problem; or the submission is excessively cluttered, messy, difficult to read, or handwritten.

Possibly helpful LaTeXcommands: If you use LaTeXto write up your work, here are some helpful commands:

Symbol	Command to produce it (in math mode)
$\sqrt{x}$	`\sqrt{x}`
$3 \mid n$	`3 \mid n`
$\mathbb{Z}$	`\mathbb{Z}`

These LaTeXcommands will also work inside Markdown cells in Jupyter notebooks. As usual, you may create your submission inside a Jupyter notebook if you prefer. Please ask for help if you have any questions.