NAME: MATH2270-004

December 11, 2017

Final Exam

(150 points)

Instructions: Do all the problems on **both sides** of each page. Show all your work and box your answers. If you get stuck on a problem, skip it and come back to it at the end.

1. **Definitions and Concepts:** Complete the following definitions in your own words, *using complete sentences.*

   (a) [**3 points**] A matrix is called *symmetric* if

   (b) [**3 points**] A number λ is called an *eigenvalue* of the matrix A if

   (c) [**3 points**] A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is called an *orthogonal set* if

   (d) [**3 points**] A function $T\colon V \to W$ is called a *linear transformation* if

   (e) [**3 points**] If A is a matrix, then the *nullspace* of A is

2. [**10 points**] Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. An orthogonal diagonalization of $A^T A$ is given by

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

Use this information to compute the singular value decomposition $A = U\Sigma V^T$. (*Hint: You can immediately write down* V *and* Σ *using information given above.*

3. **[5 points]** Let $T\colon \mathbb{R}^7 \to \mathbb{R}^4$ be a linear transformation. List all of the possible dimensions of the image (range) of T. In each case write down what the dimension of the kernel will be.

$\dim \operatorname{im} T$	$\dim \ker T$

4. **[10 points]** Compute a least squares solution $\hat{x}$ of $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

5. **Inventions:** In this problem, you will invent some things.

(a) **[3 points]** Give an example of a vector space in which every set of four vectors is linearly dependent.

(b) **[3 points]** Invent a matrix that is diagonalizable but not invertible.

(c) **[3 points]** Invent a basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ of $\mathbb{P}_2$ such that $[1 + t + t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(d) **[3 points]** Invent a vector space V whose objects are functions with $\dim V = 5$.

(e) **[3 points]** Invent a matrix A such that $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ is a solution of $A\vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) **[6 points]** Write down the corresponding quadratic form $\vec{x}^T A\vec{x}$.

(b) **[6 points]** Explain why the matrix from part (a) is diagonalizable. What does this mean that we could do to the quadratic form to make it simpler.

(c) **[6 points]** What is the determinant of the matrix A?

7. **[9 points]** Suppose A is a 4×4 matrix. You know the following:

- null A is 2 dimensional
- $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- $\det(A - 4I) = 0$

(a) What are the eigenvalues of A?

(b) Is A diagonalizable? Why or why not?

(c) If possible, write down the characteristic polynomial of A.

8. Let $A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & -1 & -2 & 3 \end{bmatrix}$ and consider the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$.

(a) **[6 points]** Compute the reduced echelon form of A.

(b) **[6 points]** Compute a basis for null A.

(c) **[3 points]** Write down a basis for col A.

(d) **[3 points]** Is T one-to-one? *(No explanation necessary)*

(e) **[3 points]** Is T onto? *(No explanation necessary)*

9. Consider the following symmetric matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

The eigenvalues of A are -2 and 7.

(a) **[8 points]** To save computation, you are given that $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are all eigenvectors of A. Use this information to write down bases for the eigenspaces coming from $\lambda = -2$ and $\lambda = 7$.

(b) **[3 points]** Using the information given above and what you found in (a), write down the characteristic polynomial of A in factored form.

(c) **[8 points]** Write down any matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Is it possible to choose P so that $P^{-1} = P^T$? If so, explain how you could find such a P (but don't do that computation).

10. **[8 points]** Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set in $\mathbb{R}^n$. Prove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a linearly dependent set in $\mathbb{R}^m$.

11. Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix} \right\}$ of $\mathbb{R}^3$.

(a) [**6 points**] Compute an orthogonal basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ of W.

(b) [**4 points**] Let $\vec{y} = \begin{bmatrix} -9 \\ 2 \\ 4 \end{bmatrix}$. Find the closest point in W to $\vec{y}$.

12. Consider the linear transformation $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ defined by $T(p(t)) = \begin{bmatrix} p(-1) \\ p(2) \end{bmatrix}$.

(a) **[6 points]** Find the matrix of T relative to the basis $\mathcal{B} = \{1+t+t^2, 2+t+t^2, -1+t^2\}$.

(b) **[4 points]** Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Write down the change of basis matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$.

Scratch Paper

"There is hardly any theory which is more elementary [than linear algebra], in spite of the fact that generations of professors and textbook writers have obscured its simplicity by preposterous calculations with matrices." – Dieudonne in "Foundations of Modern Analysis, Vol. 1"