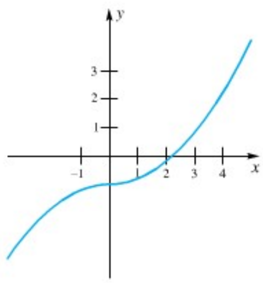
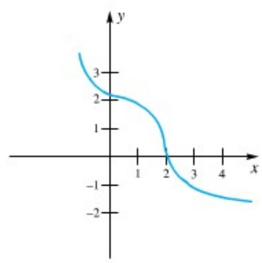
Problem Set 6.2

In Problems 1–6, the graph of $y = f(x)$ is shown. In each case, decide whether f has an inverse and, if so, estimate $f^{-1}(2)$.

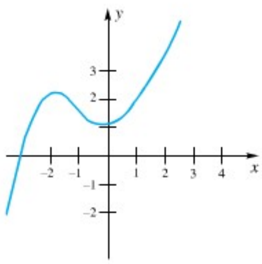
1.



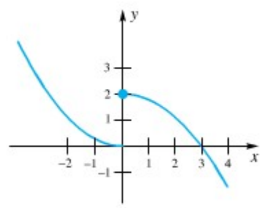
2.



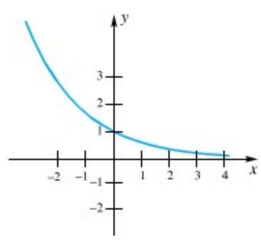
3.



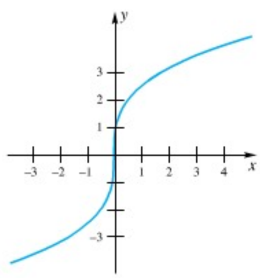
4.



5.



6.



In Problems 7–14, show that f has an inverse by showing that it is strictly monotonic (see Example 1).

7. $f(x) = -x^5 - x^3$ **8.** $f(x) = x^7 + x^5$

9. $f(\theta) = \cos\theta, 0 \le \theta \le \pi$

10. $f(x) = \cot x = \dfrac{\cos x}{\sin x}, 0 < x < \dfrac{\pi}{2}$

11. $f(z) = (z-1)^2, z \ge 1$

12. $f(x) = x^2 + x - 6, x \ge 2$

13. $f(x) = \displaystyle\int_0^x \sqrt{t^4 + t^2 + 10}\, dt$

14. $f(r) = \displaystyle\int_r^1 \cos^4 t\, dt$

In Problems 15–28, find a formula for $f^{-1}(x)$ and then verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (see Examples 2 and 3).

15. $f(x) = x + 1$ **16.** $f(x) = -\dfrac{x}{3} + 1$

17. $f(x) = \sqrt{x+1}$ **18.** $f(x) = -\sqrt{1-x}$

19. $f(x) = -\dfrac{1}{x-3}$ **20.** $f(x) = \sqrt{\dfrac{1}{x-2}}$

21. $f(x) = 4x^2, x \le 0$ **22.** $f(x) = (x-3)^2, x \ge 3$

23. $f(x) = (x-1)^3$ **24.** $f(x) = x^{5/2}, x \ge 0$

25. $f(x) = \dfrac{x-1}{x+1}$ **26.** $f(x) = \left(\dfrac{x-1}{x+1}\right)^3$

27. $f(x) = \dfrac{x^3+2}{x^3+1}$ **28.** $f(x) = \left(\dfrac{x^3+2}{x^3+1}\right)^5$

29. Find the volume V of water in the conical tank of Figure 8 as a function of the height h. Then find the height h as a function of volume V.

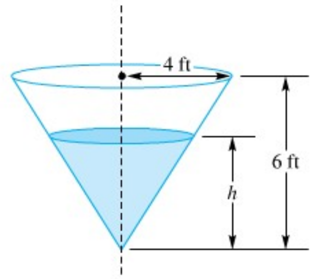


Figure 8

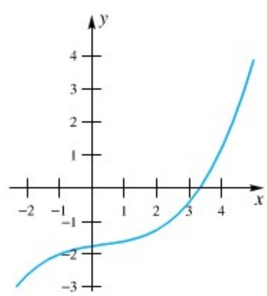
30. A ball is thrown vertically upward with velocity v_0. Find the maximum height H of the ball as a function of v_0. Then find the velocity v_0 required to achieve a height of H.

In Problems 31 and 32, restrict the domain of f so that f has an inverse, yet keeping its range as large as possible. Then find $f^{-1}(x)$. Suggestion: First graph f.

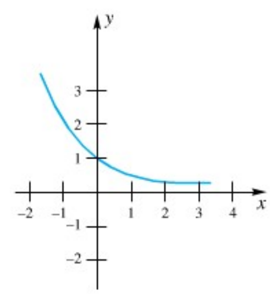
31. $f(x) = 2x^2 + x - 4$ **32.** $f(x) = x^2 - 3x + 1$

In each of Problems 33–36, the graph of $y = f(x)$ is shown. Sketch the graph of $y = f^{-1}(x)$ and estimate $(f^{-1})'(3)$.

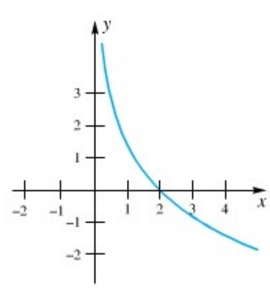
33.



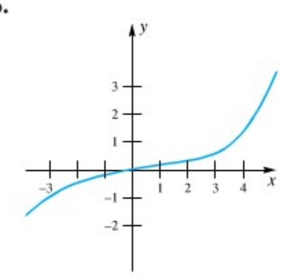
34.



35.



36.



In Problems 37–40, find $(f^{-1})'(2)$ by using Theorem B (see Example 4). Note that you can find the x corresponding to $y = 2$ by inspection.

37. $f(x) = 3x^5 + x - 2$ **38.** $f(x) = x^5 + 5x - 4$

39. $f(x) = 2 \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

40. $f(x) = \sqrt{x + 1}$

41. Suppose that both f and g have inverses and that $h(x) = (f \circ g)(x) = f(g(x))$. Show that h has an inverse given by $h^{-1} = g^{-1} \circ f^{-1}$.

42. Verify the result of Problem 41 for $f(x) = 1/x$, $g(x) = 3x + 2$.

43. If $f(x) = \int_0^x \sqrt{1 + \cos^2 t}\, dt$, then f has an inverse. (Why?) Let $A = f(\pi/2)$ and $B = f(5\pi/6)$. Find

(a) $(f^{-1})'(A)$, (b) $(f^{-1})'(B)$,

(c) $(f^{-1})'(0)$.

44. Let $f(x) = \dfrac{ax + b}{cx + d}$ and assume $bc - ad \neq 0$.

(a) Find the formula for $f^{-1}(x)$.

(b) Why is the condition $bc - ad \neq 0$ needed?

(c) What condition on a, b, c, and d will make $f = f^{-1}$?

45. Suppose that f is continuous and strictly increasing on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. If $\int_0^1 f(x)\, dx = \frac{2}{5}$, calculate $\int_0^1 f^{-1}(y)\, dy$. *Hint:* Draw a picture.

EXPL **46.** Let f be continuous and strictly increasing on $[0, \infty)$ with $f(0) = 0$ and $f(x) \to \infty$ as $x \to \infty$. Use geometric reasoning to establish **Young's Inequality.** For $a > 0$, $b > 0$,

$$ab \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(y)\, dy$$

What is the condition for equality?

EXPL **47.** Let $p > 1$, $q > 1$, and $1/p + 1/q = 1$. Show that the inverse of $f(x) = x^{p-1}$ is $f^{-1}(y) = y^{q-1}$ and use this together with Problem 46 to prove **Minkowski's Inequality:**

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad a > 0, b > 0$$

Answers to Concepts Review: **1.** $f(x_1) \neq f(x_2)$ **2.** x; $f^{-1}(y)$ **3.** monotonic; increasing; decreasing **4.** $(f^{-1})'(y) = 1/f'(x)$

6.3 The Natural Exponential Function

The graph of $y = f(x) = \ln x$ was obtained at the end of Section 6.1 and is reproduced in Figure 1. The natural logarithm function is differentiable (hence continuous) and increasing on its domain $D = (0, \infty)$; its range is $R = (-\infty, \infty)$. It is, in fact, precisely the kind of function studied in Section 6.2, and therefore has an inverse $\ln^{-1}$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. This function is so important that it is given a special name and a special symbol.

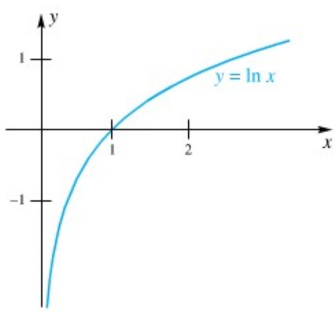


Figure 1

Definition

The inverse of ln is called the **natural exponential function** and is denoted by exp. Thus,

$$x = \exp y \iff y = \ln x$$

It follows immediately from this definition that

1. $\exp(\ln x) = x, \quad x > 0$
2. $\ln(\exp y) = y, \quad$ for all y

Since exp and ln are inverse functions, the graph of $y = \exp x$ is just the graph of $y = \ln x$ reflected across the line $y = x$ (Figure 2).

But why the name *exponential function*? You will see.

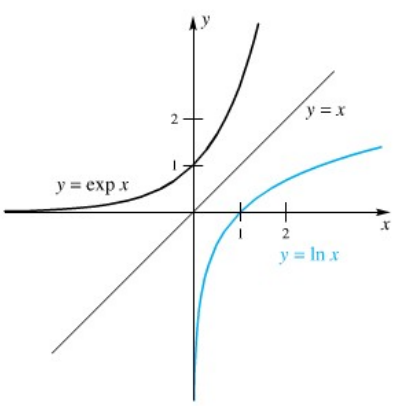


Figure 2

Properties of the Exponential Function

We begin by introducing a new number, which, like π, is so important in mathematics that it gets a special symbol, e. The letter e is appropriate since Leonhard Euler first recognized the significance of this number.

Definition

The letter e denotes the unique positive real number such that $\ln e = 1$.