MATH2270: Final Exam Study Guide

The following is an overview of the material that will be covered on the final exam.

§1.1 Systems of Linear Equations

- Solving linear systems of equations by row reducing an augmented matrix.
- Determining the existence and uniqueness of solutions to a linear system.

§1.2 Row Reduction and Echelon Forms

- Row reduction of matrices to echelon or reduced echelon form.
- Identifying pivot positions in a matrix.
- Identifying basic and free variables in a linear system and giving a parametric description of the solution set.

§1.3 Vector Equations

- Be comfortable with performing basic vetor operations of addition and scalar multiplication.
- Writing a linear system of equations as a vector equation and vise versa.
- The definition of the span of a set of vectors in $\mathbb{R}^n$ and a geometric understanding of the span.
- The definition of a linear combination of vectors.

§1.4 The Matrix Equation $A\vec{x} = \vec{b}$

- Solving matrix equations.
- Translating between matrix equations, linear systems of equations, and vector equations.
- Theorem 4 says how all the concepts discussed thus far are connected.

§1.5 Solution Sets of Linear Systems

- Solutions to homogeneous and nonhomogeneous systems of equations.
- Describing the solution set of a linear system in parametric vector form.

§1.6 Applications of Linear Systems

- Balancing chemical equations.
- Network flow.

§1.7 Linear Independence

- Definition of linearly independent/dependent sets.
- Determining whether a given set of vectors is linearly independent or not.
- A geometric description of linearly independent sets.

§1.8 Introduction to Linear Transformations

- Definition of a linear transformation.
- Determining whether a given function is a linear transformation.

§1.9 The Matrix of a Linear Transformation

- Finding the matrix of a linear transformation.
- 1-1 and onto linear transformations (and characterizations thereof).

§2.1 Matrix Operations

- You should be comfortable with the basic algebra of matrices, including addition, scalar multiplication, and matrix multiplication.

§2.2 The Inverse of a Matrix

- You should be able to write down the inverse of a 2×2 matrix.
- Use our algorithm for computing the inverse of a matrix.
- Check whether two matrices are inverses of one another.

§2.3 Characterizations of Invertible Matrices

- Determine whether a matrix (or linear transformation) is invertible.

§2.4 Partitioned Matrices

- Matrix Computations with block matrices.

§2.8 Subspaces of $\mathbb{R}^n$ This section is subsumed by material in Ch. 4.

§2.9 Dimension and Rank This section is subsumed by material in Ch. 4.

§3.1 & §3.2 Determinants

- Be able to compute determinants using cofactor expansions. You should choose the row/column to expand in strategically.
- Be able to compute the determinant of a matrix, A, by row reducing A to an upper triangular matrix, and keeping track of how the row operations change the determinant.
- Know the defining properties of the determinant:
  1. The determinant is linear in each row
  2. The determinant is alternating (if you swap two rows, the determinant changes sign).
  3. $\det I_n = 1$.
- Know the other basic properties of the determinant:
  1. $\det(AB) = \det(A)\det(B)$
  2. $\det A^T = \det A$
  3. A is invertible if and only if $\det A \neq 0$.

§3.3 Cramer's Rule, Volume, and Linear Transformations

- Know Cramer's rule for computing the inverse of a matrix.
- Understand how the volume of a region in $\mathbb{R}^n$ changes under the application of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. Specifically, $\mathrm{vol}(T(R)) = \mathrm{vol}(R) \cdot |\det(T)|$.
- Understand how the determinant of a matrix relates to the volume of the parallelepiped determined by the columns of the matrix.

§4.1 Vector Spaces and Subspaces

- You should have some familiarity with abstract vector spaces, including but not limited to: $\mathbb{P}_n$, $\mathbb{P}$, $C(R)$, $C([a,b])$, $C^\infty(R)$, $C^\infty([a,b])$, and the solution set to certain differential equations (for an example, see §4.1 exercise 19).
- You should be comfortable with subspaces of these vector spaces.
- If $H, K \leq V$ (i.e., H and K are subspaces of V), you should know the definitions of the subspaces $H + K$ and $H \cap K$.

§4.2 Null Spaces, Column Spaces and Linear Transformations

- You should know the definition of a linear transformation and be able to check whether a function $f\colon V \to W$ is one.
- Understand linear transformations on arbitrary vector spaces.
- Know the definition of the kernel (or null space) of a linear transformation and be able to do computations involving kernels.
- You should be able to do all these things in more generality than just linear transformations on $\mathbb{R}^n$.

§4.3 Linearly Independent Sets: Bases

- Know what a basis is and be able to find a basis for a subspace (not just of $\mathbb{R}^n$, but of a general vector space).
- Be able to find a basis for the kernel of a linear transformation $T\colon V \to V$.
- This is one of the most important fundamental concepts about a vector space.

§4.4 Coordinate Systems/ §4.7 Change of Coordinates

- Know how a basis for a vector space (or subspace) is related to a coordinate system on the vector space (or subspace).
- Be able to find the coordinates of $v \in V$ given a basis $\mathcal{B}$ for V.
- Given a linear transformation, $T\colon V \to V$, and a basis $\mathcal{B}$ for V, you should be able to compute the matrix for T with respect to the basis $\mathcal{B}$.
- Be able to compute the change of coordinates matrix between two bases, $\mathcal{B}$ and $\mathcal{C}$.
- Given a linear transformation $T\colon V \to V$, and two bases, $\mathcal{B}$ and $\mathcal{C}$ for V, you should be able to compute the matrix for T in the basis $\mathcal{B}$ and the basis $\mathcal{C}$. (This isn't covered in §4.7, but we talked about it in class. The book covers this in §5.4.)

§4.5 The Dimension of a Vector Space/ §4.6 Rank

- Know the definition of the dimension of a vector space (or subspace).
- Know the definition of rank of a matrix.
- Be able to compute a basis for $\operatorname{col} A$, $\ker A$, and $\operatorname{row} A$ for a matrix A.
- Be able to do basic calculations/deductions using dimension (i.e., relating the dimensions the kernel and image of a linear transformation to the dimension of the domain).

§5.1 Eigenvectors and Eigenvalues

- Know the definition of eigenvectors and eigenvalues.
- Be able to determine if a given vector is an eigenvector for a linear transformation.
- Be able to determine if a given number is an eigenvalue for a linear transformation.
- Be able to compute a basis for the eigenspace of a linear transformation associated to a specified eigenvalue.

§5.2 The Characteristic Equation

- Compute the characteristic polynomial of a matrix.
- Find the eigenvalues of a matrix (or linear transformation).

§5.3 Diagonalization

- Be able to diagonalize a matrix. That is, write $A = PDP^{-1}$. If the matrix is not diagonalizable, you should be able to determine that.
- Given a diagonalizable matrix, A, be able to find a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

§5.4 Eigenvectors and Linear Transformation

- Understand the connection between diagonalizing a matrix, and viewing a linear transformation in a "well chosen" basis.
- Given a linear transformation, $T\colon V \to V$, be able to find a basis of V (if one exists) for which the matrix of T with respect to $\mathcal{B}$ is diagonal.

§4.9 Application to Markov Chains/ §5.6 Discrete Dynamical Systems

- Know the following definitions: Markov chain, discrete dynamical system, stochastic matrix.
- Be able to compute the state of a dynamical system after a given amount of time for given initial conditions.
- Be able to find steady-state vectors for Markov chains and discrete dynamical systems.
- You should be able to find trajectories of a discrete dynamical system under iteration and classify the origin as an attractor/repeller/saddle.
- Be able to "diagonalize" a discrete dynamical system.

§5.7 Applications to Differential Equations

- Solve systems of first order linear differential equations by "diagonalization"

§6.1 Inner Product, Length, and Orthogonality

- Know the basic properties of the dot product and be able to do computations.
- Know the definitions of orthogonal vectors and the orthogonal complement of a subspace.
- Know theorem 3 on page 335 of the text book and be able to apply it.

§6.2 Orthogonal Sets/§6.3 Orthogonal Projections

- Know the definitions of orthogonal sets, orthonormal sets, and orthogonal projections onto subspaces. Be able to do computations.
- Is the set orthogonal? Orthonormal? You should be able to answer this.
- Find $\text{proj}_W v$ for a vector $v \in \mathbb{R}^n$ and a subspace W.
- Write $v \in \mathbb{R}^n$ as $v = w + z$ where $w \in W$ and $z \in W^\perp$.

§6.4 Gram-Schmidt

- Know what the Gram-Schmidt algorithm is and be able to apply it.

§6.5 Least Squares/§6.6 Applications to Linear Models

- You should be able to use least squares approximations to solve linear modeling problems.
- Be able to find a least squares fit to a line, quadratic, or other curve.

§6.7 Inner Product Spaces

- Know the definition of an inner product.
- Know basic examples of inner products including the ones we talked about in class.
- Know the "standard" inner product on $C[a, b]$ given by $\langle f, g \rangle = \int_a^b f \cdot g$ and be able to do computations with it.

§7.1 Diagonalization of Symmetric Matrices

- Know the spectral theorem for symmetric matrices.
- Be able to orthogonally diagonalize a symmetric matrix.

§7.2/§7.3 Quadratic Forms and Applications

- Just know what a quadratic form is and how they are connected to symmetric matrices.
- Be able to find the matrix for a quadratic form or find a formula for a quadratic form given its matrix.
- Be able to find an appropriate change of basis to "diagonalize" a quadratic form.
- Know how to find the maximum value a quadratic form attains on the unit ball and find where this happens.

§7.4/§7.5 Singular Value Decomposition

- Be able to compute a singular value decomposition for a given matrix.