MATH 2270: Homework 8 Worksheet

due October 27, 2017

Instructions: Do the following problems on a separate sheet of paper. Show all of your work.

1. **Bases in subspaces of functions:**
   (a) In the vector space of all real valued functions, find a basis for the subspace H spanned by the set $\{\sin t, \sin 2t, (\cos t)(\sin t)\}$.
   (b) Consider the polynomials $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$. Is $\{p_1(t), p_2(t)\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

2. **Coordinate Systems:**
   (a) Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to the basis $\mathcal{B}$.

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

   (b) The set $\{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.
   (c) Use coordinate vectors to determine if the set of polynomials $\{1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3\}$ is linearly independent.
   (d) Is the function $T\colon \mathbb{P}_3 \to \mathbb{R}^4$ defined by $T(a_3t^3 + a_2t^2 + a_1t + a_0) = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 \end{bmatrix}^T$ a vector space isomorphism?

3. **Dimension**
   (a) Explain why the space $\mathbb{P}$ of all polynomials (of any degree) is an infinite dimensional vector space.
   (b) Let V be a finite dimensional vector space. Determine if each of the following statements is true or false and justify your answer.
      i. If there exists a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$, then $\dim V \leq p$.
      ii. If there exists a linearly independent set $\{\vec{v}_1, \ldots, \vec{v}_p\}$, then $\dim V \geq p$.
      iii. If every set of p elements in V fails to span V, then $\dim V > p$.
      iv. If every set of p elements in V is linearly dependent, then $\dim V > p$.

4. **Rank**
   (a) Let $T\colon V \to W$ be a linear transformation and assume that $\dim V = 10$ and $\dim W = 7$.
      i. If the dimension of $\ker T$ is 5, what is the dimension of the range of T? *(Recall that the range of a linear transformation $S\colon V \to W$ is the set of vectors*

$$\{\vec{w} \in W \mid \text{there is some } \vec{v} \in V \text{ such that } S(\vec{v}) = \vec{w}\}$$

In words, the range is the set of vectors in the codomain that are actually outputs of the function S.

ii. If the dimension of the range of T is 4, what is the dimension of $\ker T$?
iii. What is the minimum possible dimension of $\ker T$?
iv. What is the minimum possible dimension of the range of T?

(b) Let $T\colon \mathbb{P}_3 \to \mathbb{R}^2$ be the linear transformation defined by $T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. What is the dimension of the range of T? What is the dimension of $\mathbb{P}_3$? Without doing any further computations, determine the dimension of $\ker T$.

(c) Is it possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right-hand side constants? Is it possible for such a system to have a unique solution for every right-hand side? Explain.

5. **Change of bases and the matrix of a linear transformation**

(a) §4.7 problem 7

(b) §4.7 problem 9

(c) §4.7 problem 13

(d) **Extra Credit:** §4.7 problems 17 & 18 (*Use a calculator*)

(e) **Extra Credit:** Consider the linear transformation $D\colon \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p) = p' + 2p$ (here p' is the derivative of p). Let $\mathcal{B} = \{1, t, t^2, t^3\}$ and let $\mathcal{C} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$.

i. Let $p(t) = -2t^3 + 3t^2 - 10t + 1$. Find $[p(t)]_{\mathcal{C}}$.
ii. For the same $p(t)$ as in part (5(e)i), find $D(p(t))$.
iii. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. *Hint: It's probably easier to find them both directly than it is to find one and then compute its inverse.*
iv. What is the dimension of the image of D? What is the dimension of the kernel of D?
v. Find a basis for $\ker D$.
vi. Find the matrix for the linear transformation D with respect to the basis $\mathcal{B}$.
vii. Find the matrix for the linear transformation D with respect to the basis $\mathcal{C}$.