

2.3 EXERCISES

Unless otherwise specified, assume that all matrices in these exercises are $n \times n$. Determine which of the matrices in Exercises 1–10 are invertible. Use as few calculations as possible. Justify your answers.

1. $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$
2. $\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 8 & 5 & -3 \end{bmatrix}$
4. $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$
5. $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 0 & 4 \\ -4 & 0 & 7 \end{bmatrix}$
6. $\begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ -3 & 6 & 0 \end{bmatrix}$
7. $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix}$
8. $\begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
9. [M] $\begin{bmatrix} 4 & 0 & -3 & -7 \\ -6 & 9 & 9 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 4 & -1 \end{bmatrix}$
10. [M] $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$

In Exercises 11 and 12, the matrices are all $n \times n$. Each part of the exercises is an *implication* of the form “If (statement 1), then (statement 2).” Mark an implication as True if the truth of (statement 2) *always* follows whenever (statement 1) happens to be true. An implication is False if there is an instance in which (statement 2) is false but (statement 1) is true. Justify each answer.

11. a. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.
 b. If the columns of A span \mathbb{R}^n , then the columns are linearly independent.
 c. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
 d. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then A has fewer than n pivot positions.
 e. If A^T is not invertible, then A is not invertible.
12. a. If there is an $n \times n$ matrix D such that $AD = I$, then $DA = I$.
 b. If the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n into \mathbb{R}^n , then the row reduced echelon form of A is I .
 c. If the columns of A are linearly independent, then the columns of A span \mathbb{R}^n .
 d. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n , then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.
 e. If there is a \mathbf{b} in \mathbb{R}^n such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution is unique.
13. An $m \times n$ **upper triangular matrix** is one whose entries *below* the main diagonal are 0's (as in Exercise 8). When is a square upper triangular matrix invertible? Justify your answer.
14. An $m \times n$ **lower triangular matrix** is one whose entries *above* the main diagonal are 0's (as in Exercise 3). When is a square lower triangular matrix invertible? Justify your answer.
15. Is it possible for a 4×4 matrix to be invertible when its columns do not span \mathbb{R}^4 ? Why or why not?
16. If an $n \times n$ matrix A is invertible, then the columns of A^T are linearly independent. Explain why.
17. Can a square matrix with two identical columns be invertible? Why or why not?
18. Can a square matrix with two identical rows be invertible? Why or why not?
19. If the columns of a 7×7 matrix D are linearly independent, what can be said about the solutions of $D\mathbf{x} = \mathbf{b}$? Why?
20. If A is a 5×5 matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every \mathbf{b} in \mathbb{R}^5 , is it possible that for some \mathbf{b} , the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution? Why or why not?
21. If the equation $C\mathbf{u} = \mathbf{v}$ has more than one solution for some \mathbf{v} in \mathbb{R}^n , can the columns of the $n \times n$ matrix C span \mathbb{R}^n ? Why or why not?
22. If $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute. Explain why.
23. Assume that F is an $n \times n$ matrix. If the equation $F\mathbf{x} = \mathbf{y}$ is inconsistent for some \mathbf{y} in \mathbb{R}^n , what can you say about the equation $F\mathbf{x} = \mathbf{0}$? Why?
24. If an $n \times n$ matrix G cannot be row reduced to I_n , what can you say about the columns of G ? Why?
25. Verify the boxed statement preceding Example 1.
26. Explain why the columns of A^2 span \mathbb{R}^n whenever the columns of an $n \times n$ matrix A are linearly independent.
27. Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is A . You cannot use Theorem 6(b), because you cannot assume that A and B are invertible. [Hint: There is a matrix W such that $ABW = I$. Why?]
28. Let A and B be $n \times n$ matrices. Show that if AB is invertible, so is B .
29. If A is an $n \times n$ matrix and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, what else can you say about this transformation? Justify your answer.

30. If A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some \mathbf{b} , then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one. What else can you say about this transformation? Justify your answer.
31. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n . Without using Theorems 5 or 8, explain why each equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.
32. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Without using the Invertible Matrix Theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each \mathbf{b} in \mathbb{R}^n .

In Exercises 33 and 34, T is a linear transformation from \mathbb{R}^2 into \mathbb{R}^2 . Show that T is invertible and find a formula for T^{-1} .

33. $T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$
34. $T(x_1, x_2) = (2x_1 - 8x_2, -2x_1 + 7x_2)$
35. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transformation. Explain why T is both one-to-one and onto \mathbb{R}^n . Use equations (1) and (2). Then give a second explanation using one or more theorems.
36. Suppose a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n . Can T map \mathbb{R}^n onto \mathbb{R}^n ? Why or why not?
37. Suppose T and U are linear transformations from \mathbb{R}^n to \mathbb{R}^n such that $T(U(\mathbf{x})) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Is it true that $U(T(\mathbf{x})) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n ? Why or why not?
38. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transformation, and let S and U be functions from \mathbb{R}^n into \mathbb{R}^n such that $S(T(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for all \mathbf{x} in \mathbb{R}^n . Show that $U(\mathbf{v}) = S(\mathbf{v})$ for all \mathbf{v} in \mathbb{R}^n . This will show that T has a unique inverse, as asserted in Theorem 9. [Hint: Given any \mathbf{v} in \mathbb{R}^n , we can write $\mathbf{v} = T(\mathbf{x})$ for some \mathbf{x} . Why? Compute $S(\mathbf{v})$ and $U(\mathbf{v})$.]
39. Let T be a linear transformation that maps \mathbb{R}^n onto \mathbb{R}^n . Show that T^{-1} exists and maps \mathbb{R}^n onto \mathbb{R}^n . Is T^{-1} also one-to-one?
40. Suppose T and S satisfy the invertibility equations (1) and (2), where T is a linear transformation. Show directly that S is a linear transformation. [Hint: Given \mathbf{u}, \mathbf{v} in \mathbb{R}^n , let $\mathbf{x} = S(\mathbf{u})$, $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x}) = \mathbf{u}$, $T(\mathbf{y}) = \mathbf{v}$. Why? Apply S to both sides of the equation $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. Also, consider $T(c\mathbf{x}) = cT(\mathbf{x})$.]

41. [M] Suppose an experiment leads to the following system of equations:

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.249 \\ 1.6x_1 + 1.1x_2 &= 6.843 \end{aligned} \quad (3)$$

- a. Solve system (3), and then solve system (4), below, in which the data on the right have been rounded to two decimal places. In each case, find the *exact* solution.

$$\begin{aligned} 4.5x_1 + 3.1x_2 &= 19.25 \\ 1.6x_1 + 1.1x_2 &= 6.84 \end{aligned} \quad (4)$$

- b. The entries in system (4) differ from those in system (3) by less than .05%. Find the percentage error when using the solution of (4) as an approximation for the solution of (3).
- c. Use a matrix program to produce the condition number of the coefficient matrix in (3).

Exercises 42–44 show how to use the condition number of a matrix A to estimate the accuracy of a computed solution of $A\mathbf{x} = \mathbf{b}$. If the entries of A and \mathbf{b} are accurate to about r significant digits and if the condition number of A is approximately 10^k (with k a positive integer), then the computed solution of $A\mathbf{x} = \mathbf{b}$ should usually be accurate to at least $r - k$ significant digits.

42. [M] Let A be the matrix in Exercise 9. Find the condition number of A . Construct a random vector \mathbf{x} in \mathbb{R}^4 and compute $\mathbf{b} = A\mathbf{x}$. Then use a matrix program to compute the solution \mathbf{x}_1 of $A\mathbf{x} = \mathbf{b}$. To how many digits do \mathbf{x} and \mathbf{x}_1 agree? Find out the number of digits the matrix program stores accurately, and report how many digits of accuracy are lost when \mathbf{x}_1 is used in place of the exact solution \mathbf{x} .
43. [M] Repeat Exercise 42 for the matrix in Exercise 10.
44. [M] Solve an equation $A\mathbf{x} = \mathbf{b}$ for a suitable \mathbf{b} to find the last column of the inverse of the *fifth-order Hilbert matrix*

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}$$

How many digits in each entry of \mathbf{x} do you expect to be correct? Explain. [Note: The exact solution is (630, -12600, 56700, -88200, 44100).]

45. [M] Some matrix programs, such as MATLAB, have a command to create Hilbert matrices of various sizes. If possible, use an inverse command to compute the inverse of a twelfth-order or larger Hilbert matrix, A . Compute AA^{-1} . Report what you find.

The exercises that follow give practice with matrix algebra and illustrate typical calculations found in applications.

PRACTICE PROBLEMS

1. Show that $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible and find its inverse.
2. Compute $X^T X$, where X is partitioned as $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$.

2.4 EXERCISES

In Exercises 1–9, assume that the matrices are partitioned conformably for block multiplication. Compute the products shown in Exercises 1–4.

$$\begin{array}{ll} 1. \begin{bmatrix} I & 0 \\ E & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & 2. \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \\ 3. \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & 4. \begin{bmatrix} I & 0 \\ -E & I \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \end{array}$$

In Exercises 5–8, find formulas for X , Y , and Z in terms of A , B , and C , and justify your calculations. In some cases, you may need to make assumptions about the size of a matrix in order to produce a formula. [Hint: Compute the product on the left, and set it equal to the right side.]

$$\begin{array}{l} 5. \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix} \\ 6. \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ 7. \begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix} \begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ 8. \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}$$

9. Suppose B_{11} is an invertible matrix. Find matrices A_{21} and A_{31} (in terms of the blocks of B) such that the product below has the form indicated. Also, compute C_{22} (in terms of the blocks of B). [Hint: Compute the product on the left, and set it equal to the right side.]

$$\begin{bmatrix} I & 0 & 0 \\ A_{21} & I & 0 \\ A_{31} & 0 & I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \\ 0 & C_{32} \end{bmatrix}$$

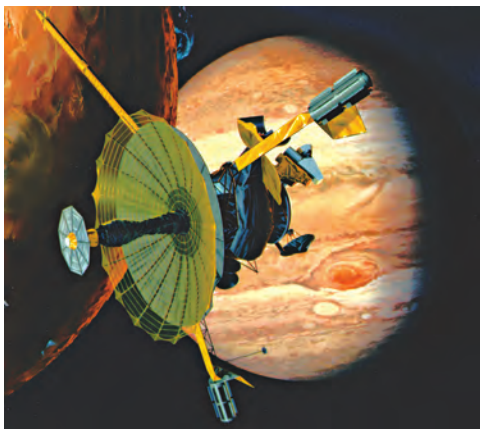
10. The inverse of

$$\begin{bmatrix} I & 0 & 0 \\ A & I & 0 \\ B & D & I \end{bmatrix} \text{ is } \begin{bmatrix} I & 0 & 0 \\ P & I & 0 \\ Q & R & I \end{bmatrix}.$$

Find P , Q , and R .

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. If $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, with A_1 and A_2 the same sizes as B_1 and B_2 , respectively, then $A + B = \begin{bmatrix} A_1 + B_1 & A_2 + B_2 \end{bmatrix}$.
b. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then the partitions of A and B are conformable for block multiplication.
12. a. If A_1, A_2, B_1 , and B_2 are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, then the product BA is defined, but AB is not.
b. If $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then the transpose of A is $A^T = \begin{bmatrix} P^T & Q^T \\ R^T & S^T \end{bmatrix}$.
13. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B and C are square. Show that A is invertible if and only if both B and C are invertible.
14. Show that the block upper triangular matrix A in Example 5 is invertible if and only if both A_{11} and A_{22} are invertible. [Hint: If A_{11} and A_{22} are invertible, the formula for A^{-1} given in Example 5 actually works as the inverse of A .] This fact about A is an important part of several computer algorithms that estimate eigenvalues of matrices. Eigenvalues are discussed in Chapter 5.
15. When a deep space probe is launched, corrections may be necessary to place the probe on a precisely calculated trajectory. Radio telemetry provides a stream of vectors, $\mathbf{x}_1, \dots, \mathbf{x}_k$, giving information at different times about how the probe's position compares with its planned trajectory. Let X_k be the matrix $[\mathbf{x}_1 \cdots \mathbf{x}_k]$. The matrix $G_k = X_k X_k^T$ is computed as the radar data are analyzed. When \mathbf{x}_{k+1} arrives, a new G_{k+1} must be computed. Since the data vectors arrive at high speed, the computational burden could be severe. But partitioned matrix multiplication helps tremendously. Compute the column–row expansions of G_k and G_{k+1} , and describe what must be computed in order to update G_k to form G_{k+1} .



The probe Galileo was launched October 18, 1989, and arrived near Jupiter in early December 1995.

16. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. If A_{11} is invertible, then the matrix $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the **Schur complement** of A_{11} . Likewise, if A_{22} is invertible, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is called the Schur complement of A_{22} . Suppose A_{11} is invertible. Find X and Y such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \quad (7)$$

17. Suppose the block matrix A on the left side of (7) is invertible and A_{11} is invertible. Show that the Schur complement S of A_{11} is invertible. [Hint: The outside factors on the right side of (7) are always invertible. Verify this.] When A and A_{11} are both invertible, (7) leads to a formula for A^{-1} , using S^{-1} , A_{11}^{-1} , and the other entries in A .

18. Let X be an $m \times n$ data matrix such that $X^T X$ is invertible, and let $M = I_m - X(X^T X)^{-1}X^T$. Add a column \mathbf{x}_0 to the data and form

$$W = \begin{bmatrix} X & \mathbf{x}_0 \end{bmatrix}$$

Compute $W^T W$. The (1, 1)-entry is $X^T X$. Show that the Schur complement (Exercise 16) of $X^T X$ can be written in the form $\mathbf{x}_0^T M \mathbf{x}_0$. It can be shown that the quantity $(\mathbf{x}_0^T M \mathbf{x}_0)^{-1}$ is the (2, 2)-entry in $(W^T W)^{-1}$. This entry has a useful statistical interpretation, under appropriate hypotheses.

In the study of engineering control of physical systems, a standard set of differential equations is transformed by Laplace transforms into the following system of linear equations:

$$\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix} \quad (8)$$

where A is $n \times n$, B is $n \times m$, C is $m \times n$, and s is a variable. The vector \mathbf{u} in \mathbb{R}^m is the “input” to the system, \mathbf{y} in \mathbb{R}^m is the “output,” and \mathbf{x} in \mathbb{R}^n is the “state” vector. (Actually, the vectors \mathbf{x} , \mathbf{u} , and \mathbf{y} are functions of s , but this does not affect the algebraic calculations in Exercises 19 and 20.)

19. Assume $A - sI_n$ is invertible and view (8) as a system of two matrix equations. Solve the top equation for \mathbf{x} and substitute into the bottom equation. The result is an equation of the form $W(s)\mathbf{u} = \mathbf{y}$, where $W(s)$ is a matrix that depends on s . $W(s)$ is called the *transfer function* of the system because it transforms the input \mathbf{u} into the output \mathbf{y} . Find $W(s)$ and describe how it is related to the partitioned *system matrix* on the left side of (8). See Exercise 16.

20. Suppose the transfer function $W(s)$ in Exercise 19 is invertible for some s . It can be shown that the inverse transfer function $W(s)^{-1}$, which transforms outputs into inputs, is the Schur complement of $A - BC - sI_n$ for the matrix below. Find this Schur complement. See Exercise 16.

$$\begin{bmatrix} A - BC - sI_n & B \\ -C & I_m \end{bmatrix}$$

21. a. Verify that $A^2 = I$ when $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.
b. Use partitioned matrices to show that $M^2 = I$ when

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

22. Generalize the idea of Exercise 21 by constructing a 6×6 matrix $M = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ C & 0 & D \end{bmatrix}$ such that $M^2 = I$. Make C a nonzero 2×2 matrix. Show that your construction works.

23. Use partitioned matrices to prove by induction that the product of two lower triangular matrices is also lower triangular. [Hint: A $(k+1) \times (k+1)$ matrix A_1 can be written in the form below, where a is a scalar, \mathbf{v} is in \mathbb{R}^k , and A is a $k \times k$ lower triangular matrix. See the *Study Guide* for help with induction.]

$$A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$$

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The Principle of Induction 2-19

24. Use partitioned matrices to prove by induction that for $n = 2, 3, \dots$, the $n \times n$ matrix A shown below is invertible and B is its inverse.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \end{bmatrix}$$

For the induction step, assume A and B are $(k+1) \times (k+1)$ matrices, and partition A and B in a form similar to that displayed in Exercise 23.

25. Without using row reduction, find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

26. [M] For block operations, it may be necessary to access or enter submatrices of a large matrix. Describe the functions or commands of a matrix program that accomplish the following tasks. Suppose A is a 20×30 matrix.

- Display the submatrix of A from rows 5 to 10 and columns 15 to 20.
- Insert a 5×10 matrix B into A , beginning at row 5 and column 10.

- Create a 50×50 matrix of the form $C = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$.

[Note: It may not be necessary to specify the zero blocks in C .]

27. [M] Suppose memory or size restrictions prevent a matrix program from working with matrices having more than 32 rows and 32 columns, and suppose some project involves 50×50 matrices A and B . Describe the commands or operations of the matrix program that accomplish the following tasks.

- Compute $A + B$.
- Compute AB .
- Solve $A\mathbf{x} = \mathbf{b}$ for some vector \mathbf{b} in \mathbb{R}^{50} , assuming that A can be partitioned into a 2×2 block matrix $[A_{ij}]$, with A_{11} an invertible 20×20 matrix, A_{22} an invertible 30×30 matrix, and A_{12} a zero matrix. [Hint: Describe appropriate smaller systems to solve, without using any matrix inverses.]

SOLUTIONS TO PRACTICE PROBLEMS

1. If $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible, its inverse has the form $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$. Verify that

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} W & X \\ AW + Y & AX + Z \end{bmatrix}$$

So W, X, Y, Z must satisfy $W = I$, $X = 0$, $AW + Y = 0$, and $AX + Z = I$. It follows that $Y = -A$ and $Z = I$. Hence

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The product in the reverse order is also the identity, so the block matrix is invertible, and its inverse is $\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}$. (You could also appeal to the Invertible Matrix Theorem.)

2. $X^T X = \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{bmatrix}$. The partitions of X^T and X are automatically conformable for block multiplication because the columns of X^T are the rows of X . This partition of $X^T X$ is used in several computer algorithms for matrix computations.

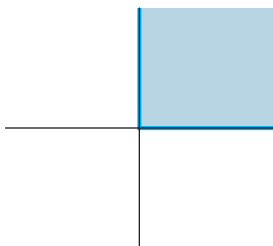
2.5 MATRIX FACTORIZATIONS

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices. Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data. In the language of computer science, the expression of A as a product amounts to a *preprocessing* of the data in A , organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.

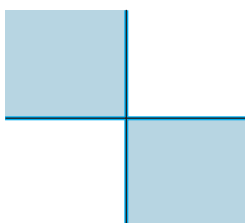
2.8 EXERCISES

Exercises 1–4 display sets in \mathbb{R}^2 . Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of \mathbb{R}^2 . (For instance, find two vectors in H whose sum is *not* in H , or find a vector in H with a scalar multiple that is not in H . Draw a picture.)

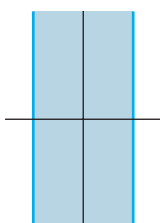
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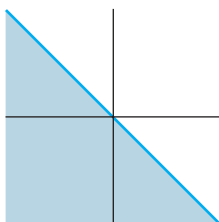
2.



3.



4.



5. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$. Determine if \mathbf{w} is in the subspace of \mathbb{R}^3 generated by \mathbf{v}_1 and \mathbf{v}_2 .

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 5 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 2 \end{bmatrix}$. Determine if \mathbf{u} is in the subspace of \mathbb{R}^4 generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix},$$

$$\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}, \quad \text{and} \quad A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3].$$

- How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
- How many vectors are in $\text{Col } A$?
- Is \mathbf{p} in $\text{Col } A$? Why or why not?

8. Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix},$$

and $\mathbf{p} = \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}$. Determine if \mathbf{p} is in $\text{Col } A$, where $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.

9. With A and \mathbf{p} as in Exercise 7, determine if \mathbf{p} is in $\text{Nul } A$.

10. With $\mathbf{u} = \begin{bmatrix} -5 \\ 5 \\ 3 \end{bmatrix}$ and A as in Exercise 8, determine if \mathbf{u} is in $\text{Nul } A$.

In Exercises 11 and 12, give integers p and q such that $\text{Nul } A$ is a subspace of \mathbb{R}^p and $\text{Col } A$ is a subspace of \mathbb{R}^q .

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \\ 3 & 3 & 4 \end{bmatrix}$

13. For A as in Exercise 11, find a nonzero vector in $\text{Nul } A$ and a nonzero vector in $\text{Col } A$.
14. For A as in Exercise 12, find a nonzero vector in $\text{Nul } A$ and a nonzero vector in $\text{Col } A$.

Determine which sets in Exercises 15–20 are bases for \mathbb{R}^2 or \mathbb{R}^3 . Justify each answer.

15. $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

16. $\begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

17. $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$

19. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A subspace of \mathbb{R}^n is any set H such that (i) the zero vector is in H , (ii) \mathbf{u} , \mathbf{v} , and $\mathbf{u} + \mathbf{v}$ are in H , and (iii) c is a scalar and $c\mathbf{u}$ is in H .
 b. If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the same as the column space of the matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$.
 c. The set of all solutions of a system of m homogeneous equations in n unknowns is a subspace of \mathbb{R}^n .
 d. The columns of an invertible $n \times n$ matrix form a basis for \mathbb{R}^n .
 e. Row operations do not affect linear dependence relations among the columns of a matrix.
22. a. A subset H of \mathbb{R}^n is a subspace if the zero vector is in H .
 b. If B is an echelon form of a matrix A , then the pivot columns of B form a basis for $\text{Col } A$.
 c. Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ in \mathbb{R}^n , the set of all linear combinations of these vectors is a subspace of \mathbb{R}^n .
 d. Let H be a subspace of \mathbb{R}^n . If \mathbf{x} is in H , and \mathbf{y} is in \mathbb{R}^n , then $\mathbf{x} + \mathbf{y}$ is in H .
 e. The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

Exercises 23–26 display a matrix A and an echelon form of A . Find a basis for $\text{Col } A$ and a basis for $\text{Nul } A$.

$$23. A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$24. A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$25. A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

27. Construct a 3×3 matrix A and a nonzero vector \mathbf{b} such that \mathbf{b} is in $\text{Col } A$, but \mathbf{b} is not the same as any one of the columns of A .
 28. Construct a 3×3 matrix A and a vector \mathbf{b} such that \mathbf{b} is *not* in $\text{Col } A$.
 29. Construct a nonzero 3×3 matrix A and a nonzero vector \mathbf{b} such that \mathbf{b} is in $\text{Nul } A$.
 30. Suppose the columns of a matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is a basis for $\text{Col } A$.

In Exercises 31–36, respond as comprehensively as possible, and justify your answer.

31. Suppose F is a 5×5 matrix whose column space is not equal to \mathbb{R}^5 . What can be said about $\text{Nul } F$?
 32. If B is a 7×7 matrix and $\text{Col } B = \mathbb{R}^7$, what can be said about solutions of equations of the form $B\mathbf{x} = \mathbf{b}$ for \mathbf{b} in \mathbb{R}^7 ?
 33. If C is a 6×6 matrix and $\text{Nul } C$ is the zero subspace, what can be said about solutions of equations of the form $C\mathbf{x} = \mathbf{b}$ for \mathbf{b} in \mathbb{R}^6 ?
 34. What can be said about the shape of an $m \times n$ matrix A when the columns of A form a basis for \mathbb{R}^m ?
 35. If B is a 5×5 matrix and $\text{Nul } B$ is *not* the zero subspace, what can be said about $\text{Col } B$?
 36. What can be said about $\text{Nul } C$ when C is a 6×4 matrix with linearly independent columns?

[M] In Exercises 37 and 38, construct bases for the column space and the null space of the given matrix A . Justify your work.

$$37. A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$$

$$38. A = \begin{bmatrix} 5 & 3 & 2 & -6 & -8 \\ 4 & 1 & 3 & -8 & -7 \\ 5 & 1 & 4 & 5 & 19 \\ -7 & -5 & -2 & 8 & 5 \end{bmatrix}$$

WEB Column Space and Null Space

WEB A Basis for $\text{Col } A$

SG

Expanded Table
for the IMT 2-39

Also, statement (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d). Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete. ■

NUMERICAL NOTES

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A , to be discussed in Section 7.4.

WEB

PRACTICE PROBLEMS

1. Determine the dimension of the subspace H of \mathbb{R}^3 spanned by the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 . (First, find a basis for H .)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}$$

2. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ .2 \end{bmatrix}, \begin{bmatrix} .2 \\ 1 \end{bmatrix} \right\}$$

for \mathbb{R}^2 . If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is \mathbf{x} ?

3. Could \mathbb{R}^3 possibly contain a four-dimensional subspace? Explain.

2.9 EXERCISES

In Exercises 1 and 2, find the vector \mathbf{x} determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis \mathcal{B} . Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

In Exercises 3–6, the vector \mathbf{x} is in a subspace H with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the \mathcal{B} -coordinate vector of \mathbf{x} .

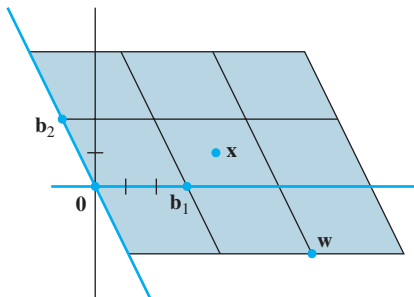
3. $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

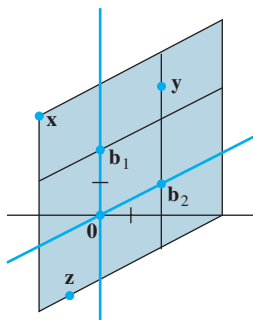
5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$

7. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{w}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}}$. Confirm your estimate of $[\mathbf{x}]_{\mathcal{B}}$ by using it and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute \mathbf{x} .



8. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_{\mathcal{B}}$, $[\mathbf{y}]_{\mathcal{B}}$, and $[\mathbf{z}]_{\mathcal{B}}$. Confirm your estimates of $[\mathbf{y}]_{\mathcal{B}}$ and $[\mathbf{z}]_{\mathcal{B}}$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute \mathbf{y} and \mathbf{z} .



Exercises 9–12 display a matrix A and an echelon form of A . Find bases for $\text{Col } A$ and $\text{Nul } A$, and then state the dimensions of these subspaces.

$$9. A = \begin{bmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -6 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$11. A = \begin{bmatrix} 2 & 4 & -5 & 2 & -3 \\ 3 & 6 & -8 & 3 & -5 \\ 0 & 0 & 9 & 0 & 9 \\ -3 & -6 & -7 & -3 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 1 & -4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$12. A = \begin{bmatrix} 1 & 2 & -4 & 4 & 6 \\ 5 & 1 & -9 & 2 & 10 \\ 4 & 6 & -9 & 12 & 15 \\ 3 & 4 & -5 & 8 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In Exercises 13 and 14, find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$13. \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$$

$$14. \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 6 \\ -9 \end{bmatrix}$$

15. Suppose a 4×6 matrix A has four pivot columns. Is $\text{Col } A = \mathbb{R}^4$? Is $\text{Nul } A = \mathbb{R}^2$? Explain your answers.

16. Suppose a 4×7 matrix A has three pivot columns. Is $\text{Col } A = \mathbb{R}^3$? What is the dimension of $\text{Nul } A$? Explain your answers.

In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here A is an $m \times n$ matrix.

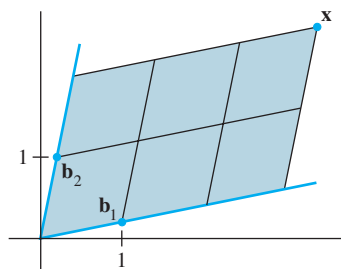
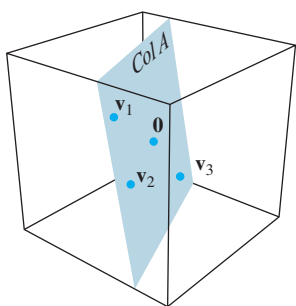
17. a. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$, then c_1, \dots, c_p are the coordinates of \mathbf{x} relative to the basis \mathcal{B} .
 b. Each line in \mathbb{R}^n is a one-dimensional subspace of \mathbb{R}^n .
 c. The dimension of $\text{Col } A$ is the number of pivot columns in A .
 d. The dimensions of $\text{Col } A$ and $\text{Nul } A$ add up to the number of columns in A .
 e. If a set of p vectors spans a p -dimensional subspace H of \mathbb{R}^n , then these vectors form a basis for H .
18. a. If \mathcal{B} is a basis for a subspace H , then each vector in H can be written in only one way as a linear combination of the vectors in \mathcal{B} .
 b. The dimension of $\text{Nul } A$ is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.
 c. The dimension of the column space of A is $\text{rank } A$.

- d. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H of \mathbb{R}^n , then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ makes H look and act the same as \mathbb{R}^p .
- e. If H is a p -dimensional subspace of \mathbb{R}^n , then a linearly independent set of p vectors in H is a basis for H .

In Exercises 19–24, justify each answer or construction.

19. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A ?
20. What is the rank of a 6×8 matrix whose null space is three-dimensional?
21. If the rank of a 9×8 matrix A is 7, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?
22. Show that a set $\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$ in \mathbb{R}^n is linearly dependent if $\dim \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_5\} = 4$.
23. If possible, construct a 3×5 matrix A such that $\dim \text{Nul } A = 3$ and $\dim \text{Col } A = 2$.
24. Construct a 3×4 matrix with rank 1.
25. Let A be an $n \times p$ matrix whose column space is p -dimensional. Explain why the columns of A must be linearly independent.
26. Suppose columns 1, 3, 4, 5, and 7 of a matrix A are linearly independent (but are not necessarily pivot columns) and the rank of A is 5. Explain why the five columns mentioned must be a basis for the column space of A .
27. Suppose vectors $\mathbf{b}_1, \dots, \mathbf{b}_p$ span a subspace W , and let $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ be any set in W containing more than p vectors. Fill in the details of the following argument to show that $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ must be linearly dependent. First, let $B = [\mathbf{b}_1 \ \dots \ \mathbf{b}_p]$ and $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_q]$.
- a. Explain why for each vector \mathbf{a}_j , there exists a vector \mathbf{c}_j in \mathbb{R}^p such that $\mathbf{a}_j = B\mathbf{c}_j$.
- b. Let $C = [\mathbf{c}_1 \ \dots \ \mathbf{c}_q]$. Explain why there is a nonzero vector \mathbf{u} such that $C\mathbf{u} = \mathbf{0}$.
- c. Use B and C to show that $A\mathbf{u} = \mathbf{0}$. This shows that the columns of A are linearly dependent.
28. Use Exercise 27 to show that if \mathcal{A} and \mathcal{B} are bases for a subspace W of \mathbb{R}^n , then \mathcal{A} cannot contain more vectors than \mathcal{B} , and, conversely, \mathcal{B} cannot contain more vectors than \mathcal{A} .
29. [M] Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that \mathbf{x} is in H , and find the \mathcal{B} -coordinate vector of \mathbf{x} , when
- $$\mathbf{v}_1 = \begin{bmatrix} 15 \\ -5 \\ 12 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 14 \\ -10 \\ 13 \\ 17 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 16 \\ 0 \\ 11 \\ -3 \end{bmatrix}$$
30. [M] Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that \mathcal{B} is a basis for H and \mathbf{x} is in H , and find the \mathcal{B} -coordinate vector of \mathbf{x} , when
- $$\mathbf{v}_1 = \begin{bmatrix} -6 \\ 3 \\ -9 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 0 \\ 7 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -9 \\ 4 \\ -8 \\ 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 11 \\ -2 \\ 17 \\ -8 \end{bmatrix}$$

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SOLUTIONS TO PRACTICE PROBLEMS

1. Construct $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ so that the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the column space of A . A basis for this space is provided by the pivot columns of A .

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A are pivot columns and form a basis for H . Thus $\dim H = 2$.

2. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then \mathbf{x} is formed from a linear combination of the basis vectors using weights 3 and 2:

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3 \begin{bmatrix} 1 \\ .2 \end{bmatrix} + 2 \begin{bmatrix} .2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 2.6 \end{bmatrix}$$

The basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ determines a *coordinate system* for \mathbb{R}^2 , illustrated by the grid in the figure. Note how \mathbf{x} is 3 units in the \mathbf{b}_1 -direction and 2 units in the \mathbf{b}_2 -direction.