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Loxodromics for the cyclic splitting complex

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Theme in GGT:

Many groups we like are not hyperbolic: By making them act on a hyperbolic space, we can find ways that they are "like a hyperbolic group."

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- $\mathrm{MCG}(S) \curvearrowright \mathcal{C}(S)$, the curve complex.
- A relatively hyperbolic group acting on its coned-off Cayley graph.
- $\mathrm{CAT}(0)$ cube complex group acting on its contact graph.

Definition

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\,\mathrm{Inn}(F_n)$$

Theorem [Bestvina-Feighn]

$\mathrm{Out}(F_n)$ is **acylindrically hyperbolic**: $\mathrm{Out}(F_n)$ admits an acylindrical action on a Gromov hyperbolic metric space.

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Motivating Question

Find a natural, concrete example of such a space.

- A **one-edge free splitting** of F_n is a decomposition $F_n = A * B$ where each of A and B is nontrivial.
- A **one-edge $\mathcal{Z}$-splitting** of F_n is a decomposition $F_n = A *_C B$ where C is either trivial or $\mathbb{Z}$ and neither A nor B is trivial or equal to C.

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- A and B are called **vertex groups** of the splittings.
- A **free factor** of F_n is a subgroup A such that there exists a B with $F_n = A * B$.

Some Hyperbolic $\mathrm{Out}(F_n)$ graphs
Definitions

- The free splitting graph, $\mathcal{FS}_n$ is

$$\mathcal{FS}_n = \begin{cases} \text{vertices:} & \text{one-edge free splittings} \\ \text{edges:} & \text{compatibility} \end{cases}$$

- The cyclic splitting graph, $\mathcal{FZ}_n$ is

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- The free factor graph, $\mathcal{FF}_n$ is

$$\mathcal{FF}_n = \begin{cases} \text{vertices:} & \text{free factors} \\ \text{edges:} & \text{inclusion} \end{cases}$$

Definition

Let $g \in \mathrm{Isom}(X)$, for X a metric space. The **translation distance** of g is

$$\tau(g) = \lim_{n\to\infty} \frac{1}{n} d(x, g^n \cdot x)$$

If X is hyperbolic and $\tau(g) > 0$, we say g is **loxodromic**.

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Loxodromic elements have nice dynamical properties:

- g fixes exactly two points in $\overline{X} = X \cup \partial X$.
- g acts with north-south dynamics on $\overline{X}$.
- g has a (quasi) “axis” along which it acts by translation.
- independent loxodromics can be used to play ping pong.

Loxodromics for some $\mathrm{Out}(F_n)$ graphs

- Associated to each $\phi \in Out(F_n)$ is a finite set of **attracting laminations**, $\mathcal{L}(\phi)$, that are analogs of the finite set of measured laminations associated to a reducible mapping class.
- A lamination $\Lambda \in \mathcal{L}(\phi)$ is **filling** if it isn't contained in (carried by) a vertex group of any free splitting.

Theorem [Handel-Mosher '14]

An element $\phi \in \mathrm{Out}(F_n)$ acts loxodromically on the free splitting complex if and only if ϕ has a filling lamination.

Loxodromics for some $\mathrm{Out}(F_n)$ graphs

- Associated to each $\phi \in Out(F_n)$ is a finite set of **attracting laminations**, $\mathcal{L}(\phi)$, that are analogs of the finite set of measured laminations associated to a reducible mapping class.
- A lamination $\Lambda \in \mathcal{L}(\phi)$ is $\mathcal{Z}$-filling if it isn't contained in (carried by) a vertex group of any $\mathcal{Z}$-splitting.

Theorem [Gupta-W. '17]

An element $\phi \in \mathrm{Out}(F_n)$ acts loxodromically on the cyclic splitting complex if and only if ϕ has a $\mathcal{Z}$-filling lamination.

Back to Acylindrical Hyperbolicity

Questions:

- Is the action $\mathrm{Out}(F_n) \curvearrowright \mathcal{FZ}_n$ acylindrical?
- Is the action $\mathrm{Out}(F_n) \curvearrowright \mathcal{FF}_n$ acylindrical?

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WPD

WPD is a still weaker version of proper discontinuity for a group action. Roughly speaking, WPD is "acylindricity in the directions of axes of loxodromics."

- Does the action $\mathrm{Out}(F_n) \curvearrowright \mathcal{FZ}_n$ satisfy weak proper discontinuity?

Promising Results

Theorem [Gupta-W., '17]

If $\phi \in \mathrm{Out}(F_n)$ has a filling lamination Λ, then ϕ has virtually cyclic centralizer if and only if Λ is $\mathcal{Z}$-filling.

In particular, the Lipschitz map

$$\mathcal{FS}_n \quad \longrightarrow \quad \mathcal{FZ}_n$$

kills the axes of precisely those loxodromics whose stabilizer is large.

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Conjecture

The action of $\mathrm{Out}(F_n)$ on $\mathcal{FZ}_n$ satisfies WPD.

A Few Words on Proofs

- The boundary of $\mathcal{FZ}_n$ consists of equivalence classes of $\mathcal{Z}$-averse trees [Horbez '16].
- One wants to take an automorphism, ϕ, that has a $\mathcal{Z}$-filling lamination and a $\mathcal{Z}$-splitting, S, then start iterating: $\phi^n \cdot S$. We then want to show that the limiting tree is $\mathcal{Z}$-averse.

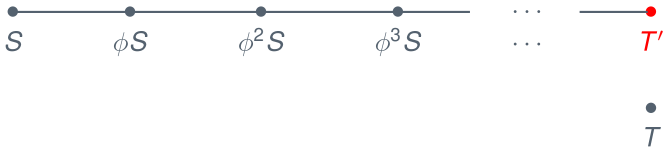
$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ S & \phi S & \phi^2 S & \phi^3 S & \cdots & T \end{array}$$

A Few Words on Proofs

- There is one exactly “good representative” in each equivalence class of $\mathcal{Z}$-averse trees; it’s a tree that is mixing.
- A standard way to study trees in the boundary of outer space is using folding paths, so we want to construct a folding path that ends at the good limiting tree.

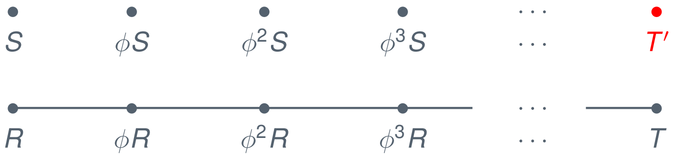
A Few Words on Proofs

- For certain automorphisms, the tree obtained from iterating ϕ will not be the mixing representative in it's class.
- To get from T' to T one has to collapse the complement of a Cantor set. . . in an $\mathbb{R}$-tree.



A Few Words on Proofs

- To overcome this, we collapse first to get a simplicial tree in the boundary of outer space.
- Then develop folding paths for trees in the boundary, and study the limiting mixing tree via those folding paths.
- Show T is $\mathcal{Z}$-averse.



A Word on Proofs

Keywords:

- The boundary of $\mathcal{FZ}_n$: equivalence classes of $\mathcal{Z}$-averse trees.
- Folding paths for simplicial trees in the boundary of outer space.
- JSJ decompositions and deformation spaces of trees.
- Completely split train tracks

Thank you!