MATH 2270: Homework 8/Midterm 2 Practice Problems

due October 28, 2015

Instructions: Do the following problems on a separate sheet of paper. Show all of your work.

1. Determine whether or not the following matrix is invertible. *Do not try to invert it*

   (a)
   $$A = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

   Ans: The determinant of A is 9, so A is invertible.

   (b) Find the determinant of A^5. Ans: 6^5

2. Prove that if $\det(B^3) = 0$, then $\det B = 0$. Ans: Use the fact that $\det(B^3) = (\det B)^3$.

3. Answer each of the following yes/no questions, and then give an explanation of your reasoning.

   (a) Is $\mathbb{R}^3$ a subspace of $\mathbb{R}^4$? Ans: No

   (b) Is $\mathbb{P}_4$ a subspace of $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$? Ans: Yes

   (c) Is the function $T : \mathbb{P}_3 \to \mathbb{R}^4$ defined by $T(a_3t^3 + a_2t^2 + a_1t + a_0) = \begin{bmatrix} a_3 & a_2 & a_1 & a_0 \end{bmatrix}^T$ a vector space isomorphism? Ans: Yes

4. Show that $H = \{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is a subspace of $C(\mathbb{R})$. Ans: Show the 3 properties are satisfied. $0 \in H$, $f, g \in H$ implies $f + g \in H$, and $f \in H$, $c \in \mathbb{R}$ implies $cf \in H$.

5. Find a linear transformation $S : C(\mathbb{R}) \to \mathbb{R}$ whose kernel is equal to H from the previous problem. Ans: $S(f(t)) = f(0)$

6. Consider the following matrix:
   $$A = \begin{bmatrix} 5 & 1 & 6 & 2 & 0 \\ 3 & 3 & 6 & -1 & -18 \\ 8 & 4 & 12 & -5 & 18 \\ 2 & 1 & 3 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for $\operatorname{col} A$. Ans: $\{\begin{bmatrix} 5 & 3 & 8 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & 3 & 4 & 1 \end{bmatrix}^T, \begin{bmatrix} 2 & -1 & -5 & 0 \end{bmatrix}^T\}$

   (b) Find a basis for $\operatorname{row} A$. Ans: $\{\begin{bmatrix} 1 & 0 & 1 & 0 & 5 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 1 & 0 & -13 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 0 & 1 & -6 \end{bmatrix}^T\}$

   (c) Find a basis for $\ker A$. Ans: $\{\begin{bmatrix} -1 & -1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -5 & 13 & 0 & 6 & 1 \end{bmatrix}^T\}$

7. Consider the linear transformation $D : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p) = p' + 2p$. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ and let $\mathcal{C} = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$.

(a) Find the matrix for the linear transformation D with respect to the basis $\mathcal{B}$. Ans:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(b) Let $p(t) = -2t^3 + 3t^2 - 10t + 1$. Find $[p(t)]_{\mathcal{C}}$. Ans: $\begin{bmatrix} 11 & -13 & 5 & -2 \end{bmatrix}^T$

(c) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. *Hint: It's probably easier to find them both directly than it is to find one and then compute its inverse.* Ans:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) What is the dimension of the image of D? What is the dimension of the kernel of D? Ans: The dimension of the image is 4. The dimension of the kernel is 0.

(e) Find a basis for $\ker D$. Ans: Since the kernel is trivial, a basis for the trivial subspace is the empty set $\{\}$.

8. Is $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ an eigenvector of $\left[\begin{smallmatrix} 5 & 2 \\ 3 & 6 \end{smallmatrix}\right]$ If so, what is the associated eigenvalue? Ans: Yes. $\lambda = 3$

9. Is $\lambda = -3$ an eigenvalue of the matrix $\left[\begin{smallmatrix} -1 & 4 \\ 6 & 9 \end{smallmatrix}\right]$? Ans: Yes

10. Find a basis for the eigenspace of the matrix A corresponding to the eigenvalue $\lambda = 2$ where

$$A = \begin{bmatrix} 6 & -4 & -2 \\ 4 & -2 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Ans: $\{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T\}$