Note in the table that as the step size h is *halved* the error is approximately *halved*. The error at a given point is therefore roughly proportional to the step size h. We found a similar result with numerical integration in Section 4.6. There we saw that the error for the left or right Riemann Sum Rule is proportional to $h = 1/n$ and that the error for the Trapezoidal Rule is proportional to $h^2 = 1/n^2$. The Parabolic Rule is even better, having an error proportional to $h^4 = 1/n^4$. This raises the question of whether there are better methods for approximating the solution of $y' = f(x, y)$, $y(x_0) = y_0$. In fact, there are a number of methods that are better then Euler's Method, in the sense that the error is proportional to a higher power of h. These methods are conceptually similar to Euler's Method: they are "step methods," that is, they begin with the initial condition and successively approximate the solution at each of a number of steps to the right. One method, the **Fourth-Order Runge-Kutta Method**, has an error that is proportional to $h^4 = 1/n^4$.

Concepts Review

1. For the differential equation $y' = f(x, y)$, a plot of line segments whose slopes equal $f(x, y)$ is called a _____.

2. The basis for Euler's Method is that the _____ to the solution at x_0 will be a good approximation to the solution over the interval $[x_0, x_0 + h]$.

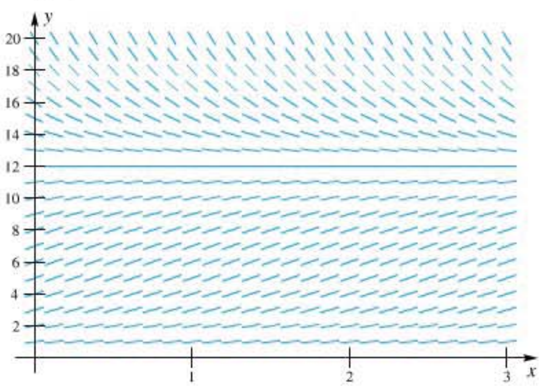
3. The recursive formula for the approximation to the solution of a differential equation using Euler's Method is $y_n =$ _____.

4. If the solution of a differential equation is concave up, then Euler's Method will _____ (underestimate or overestimate) the solution.

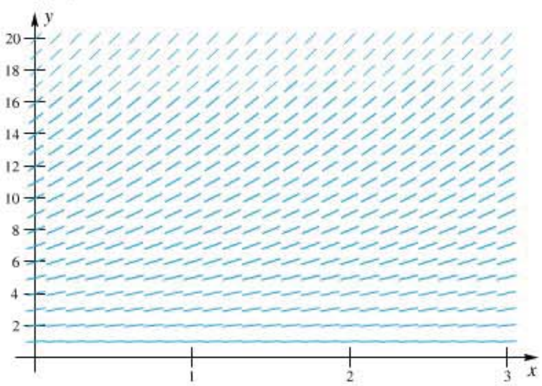
Problem Set 6.7

In Problems 1–4, a slope field is given for a differential equation of the form $y' = f(x, y)$. Use the slope field to sketch the solution that satisfies the given initial condition. In each case, find $\lim_{x\to\infty} y(x)$ and approximate $y(2)$.

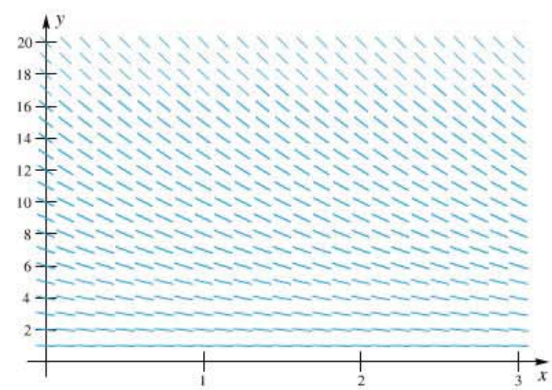
1. $y(0) = 5$



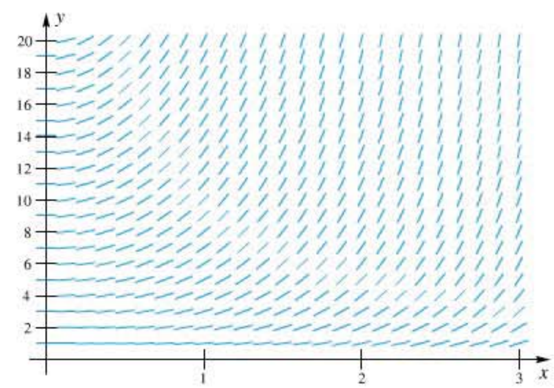
2. $y(0) = 6$



3. $y(0) = 16$

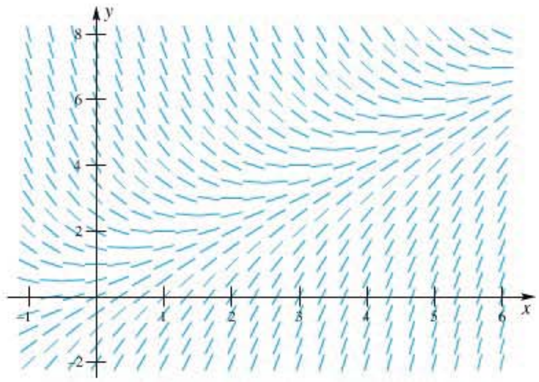


4. $y(1) = 3$

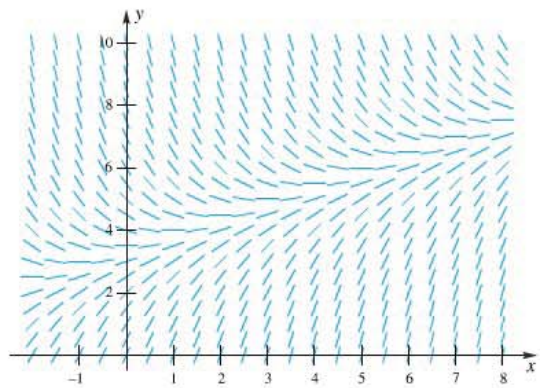


In Problems 5 and 6, a slope field is given for a differential equation of the form $y' = f(x, y)$. In both cases, every solution has the *same oblique asymptote (see Section 3.5). Sketch the solution that satisfies the given initial condition, and find the equation of the oblique asymptote.*

5. $y(0) = 6$



6. $y(0) = 8$



CAS *In Problems 7–10, plot a slope field for each differential equation. Use the method of separation of variables (Section 3.9) or an integrating factor (Section 6.6) to find a particular solution of the differential equation that satisfies the given initial condition, and plot the particular solution.*

7. $y' = \frac{1}{2}y;\ y(0) = \frac{1}{2}$

8. $y' = -y;\ y(0) = 4$

9. $y' = x - y + 2;\ y(0) = 4$

10. $y' = 2x - y + \frac{3}{2};\ y(0) = 3$

C *In Problems 11–16, use Euler's Method with $h = 0.2$ to approximate the solution over the indicated interval.*

11. $y' = 2y,\ y(0) = 3,\ [0, 1]$

12. $y' = -y,\ y(0) = 2,\ [0, 1]$

13. $y' = x,\ y(0) = 0,\ [0, 1]$

14. $y' = x^2,\ y(0) = 0,\ [0, 1]$

15. $y' = xy,\ y(1) = 1,\ [1, 2]$

16. $y' = -2xy,\ y(1) = 2,\ [1, 2]$

17. Apply Euler's Method to the equation $y' = y,\ y(0) = 1$ with an arbitrary step size $h = 1/N$ where N is a positive integer.

(a) Derive the relationship $y_n = y_0(1 + h)^n$.

(b) Explain why y_N is an approximation to e.

18. Suppose that the function $f(x, y)$ depends only on x. The differential equation $y' = f(x, y)$ can then be written as

$$y' = f(x), \qquad y(x_0) = 0$$

Explain how to apply Euler's Method to this differential equation if $y_0 = 0$.

19. Consider the differential equation $y' = f(x),\ y(x_0) = 0$ of Problem 18. For this problem, let $f(x) = \sin x^2$, $x_0 = 0$, and $h = 0.1$.

(a) Integrate both sides of the equation from x_0 to $x_1 = x_0 + h$. To approximate the integral, use a Riemann sum with a single interval, evaluating the integrand at the left end point.

(b) Integrate both sides from x_0 to $x_2 = x_0 + 2h$. Again, to approximate the integral use a left end point Riemann sum, but with two intervals.

(c) Continue the process described in parts (a) and (b) until $x_n = 1$. Use a left end point Riemann sum with ten intervals to approximate the integral.

(d) Describe how this method is related to Euler's Method.

20. Repeat parts (a) through (c) of Problem 19 for the differential equation $y' = \sqrt{x + 1},\ y(0) = 0$.

21. (Improved Euler Method) Consider the change Δy in the solution between x_0 and x_1. One approximation is obtained from Euler's Method: $\dfrac{\Delta y}{\Delta x} = \dfrac{y(x_1) - y_0}{h} \approx \dfrac{\hat{y}_1 - y_0}{h} = f(x_0, y_0)$. (Here we have used $\hat{y}_1$ to indicate Euler's approximation to the solu-tion at x_1.) Another approximation is obtained by finding an approximation to the slope of the solution at x_1:

$$\frac{\Delta y}{\Delta x} = \frac{y(x_1) - y_0}{h} \approx f(x_1, y_1) \approx f(x_1, \hat{y}_1)$$

(a) Average these two solutions to get a single approximation for $\Delta y/\Delta x$.

(b) Solve for $y_1 = y(x_1)$ to obtain

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, \hat{y}_1)]$$

(c) This is the first step in the Improved Euler Method. Additional steps follow the same pattern. Fill in the blanks for the following three-step algorithm that yields the Improved Euler Method:

1. Set $x_n =$ ________________
2. Set $\hat{y}_n =$ ____________________
3. Set $y_n =$ _________________________

C *For Problems 22–27, use the Improved Euler Method with $h = 0.2$ on the equations in Problems 11–16. Compare your answer with those obtained using Euler's Method.*

CAS **28.** Apply the Improved Euler Method to the equation $y' = y,\ y(0) = 1$, with $h = 0.2, 0.1, 0.05, 0.01, 0.005$ to approximate the solution on the interval $[0, 1]$. (Note that the exact solution is $y = e^x$, so $y(1) = e$.) Compute the error in approximating $y(1)$ (see Example 3 and the subsequent discussion) and fill in the following table. For the Improved Euler Method, is the error proportional to h, h^2, or some other power of h?