MATH 1210: HOMEWORK SOLUTIONS §3.3

1. $f(x) = x^3 - 6x^2 + 4$

$f'(x) = 3x^2 - 12x = 0 \iff x = 0$, or $3x - 12 = 0$

so $x = 0, 4$

$f''(x) = 6x - 12$

at $x = 0$, $f''(0) = -12 < 0$, so $x = 0$ is a local max

at $x = 4$, $f''(4) = 12 > 0$, so $x = 4$ is a local min

3. $f(\theta) = \sin(2\theta)$, $0 < \theta < \pi/4$

$f'(\theta) = 2\cos(2\theta) = 0$ when $\cos(2\theta) = 0$ which happens

when $2\theta = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$)

so $\theta = \pi/4, 3\pi/4, \ldots$ none of which are in $(0, \pi/4)$

$\therefore f$ has no mins or maxima on $(0, \pi/4)$

5. $\Psi(\theta) = \sin^2\theta$, $-\pi/2 < \theta < \pi/2$.

$\Psi'(\theta) = 2\sin\theta\cos\theta = 0$ when either $\sin\theta = 0$ or $\cos\theta = 0$

so $\Psi'(\theta) = 0$ when $\theta = \frac{k\pi}{2}$ for any $k \in \mathbb{Z}$. only $k = 0$ is in our interval. so $\theta = 0$ is the only critical point.

When $\theta < 0$, but close to 0, $\sin\theta < 0$, $\cos\theta > 0$, so $\Psi'(\theta) = 2(-)(+) < 0$. When $\theta > 0$, $\sin\theta > 0$, $\cos\theta > 0$, so $\Psi'(\theta) = 2(+)(+) > 0$, so $\theta = 0$ is a local minimum.

7. $f(x) = \frac{x}{x^2+4}$ $\quad f'(x) = \frac{x^2 + 4 - 2x^2}{(x^2+4)^2} = \frac{4 - x^2}{(x^2+4)^2}$

- denom is always > 0, so $f'(x)$ is defined everywhere.
- $f'(x) = 0$ when $4 - x^2 = 0$ i.e., when $x = \pm 2$.
- As $x \nearrow 2$, $4 - x^2 > 0$, as $x \searrow 2$, $4 - x^2 < 0$, so by first deriv. test f has local max at $x = 2$.
- Same argument shows f has a local max at $x = -2$.

9. $h(y) = y^2 + \frac{1}{y}$, $h'(y) = 2y - \frac{1}{y^2}$

- $h'(y) = 0 \iff 2y = \frac{1}{y^2} \iff y^3 = \frac{1}{2} \iff y = \frac{1}{\sqrt[3]{2}}$
- $h''(y) = 2 + \frac{1}{2y^3}$. At $y = \frac{1}{\sqrt[3]{2}}$, $h''\left(\frac{1}{\sqrt[3]{2}}\right) = 2 + \frac{1}{2\left(\frac{1}{\sqrt[3]{2}}\right)^3}$

$= 2 + \frac{1}{2/2} = 3$

- so $h''\left(\frac{1}{\sqrt[3]{2}}\right) > 0 \implies y = \frac{1}{\sqrt[3]{2}}$ is a local min.

11. $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$

- $f'(x) = 0 \iff x^2 = 1 \iff x = \pm 1$
- $f''(x) = 6x$. so $f''(-1) < 0$, $f''(1) > 0$

$\implies$ $x = -1$ is a local max
$x = 1$ is a local min

13. $H(x) = x^4 - 2x^3$, $H'(x) = 4x^3 - 6x^2$, $H''(x) = 12x^2 - 12x$
$= 12x(x-1)$

- $H'(x) = 0 \iff (4x-6)x^2 = 0 \iff x = 0, \frac{3}{2}$
- At $x = 3/2$, $H''(3/2) = 12(+)(+) > 0 \implies x = 3/2$ is a local min.
- Second deriv. test fails at $x = 0$, since $H''(0) = 0$.
- As $x \nearrow 0$, $H'(x) = x^2(4x-6) = (-)^2(4(-)-6) < 0$
As $x \searrow 0$, $H'(x) = x^2(4x-6) = (+)(4(+)-6) < 0$

small pos. #

so $x = 0$ is neither a min nor a max

15. $g(t) = \pi - (t-2)^{2/3}$ $\qquad g'(t) = -\frac{2}{3\sqrt[3]{t-2}}$

- $g'(t)$ is never zero, but $t = 2$ is a singular (hence critical) point.
- As $t \nearrow 2$, $g'(t) = \frac{-2}{3(-)} > 0$
As $t \searrow 2$, $g'(t) = \frac{-2}{3(+)} < 0$

$\implies$ so $t = 2$ is a local max by First Deriv. Test

17. $f(t) = t + \frac{1}{t}$ $\qquad f'(t) = 1 - \frac{1}{t^2}$ $\qquad f''(t) = \frac{1}{2t^3}$

- $f'(t) = 0 \iff 1 = \frac{1}{t^2} \iff t = \pm 1$
- At $t=-1$, $f''(-1) < 0$, so f has local max
  " $t = 1$ " > 0, " " " " min

19. $\Lambda(\theta) = \dfrac{\cos\theta}{1+\sin\theta}$, $\theta \in (0, 2\pi)$,

$$\Lambda'(\theta) = \frac{(1+\sin\theta)(-\sin\theta) - \cos\theta(\cos\theta)}{(1+\sin\theta)^2} = \frac{-\sin\theta - 1}{(1+\sin\theta)^2} = \frac{-1}{1+\sin\theta}$$

$\Lambda'(\theta)$ is never zero on $(0, 2\pi)$, so no critical points

21. $f(x) = \sin^2(2x)$ on $[0,2]$ $\qquad f'(x) = 4\sin(2x)\cos(2x)$

- $f'(x) = 0 \iff \sin(2x) = 0$ or $\cos(2x) = 0$
  $\iff 2x = 0, \pi, \ldots$ or $2x = \frac{\pi}{2}, 3\pi/2, \ldots$
  $\iff x = 0, \frac{\pi}{2}$ or $x = \frac{\pi}{4}, 3\pi/4$

$\frac{3\pi}{4} > 2 > \pi/2$, so the only critical points are $0, \pi/4, \pi/2, 2$.

- $f(x) \geq 0$ for all x, and $f(0) = 0$, so 0 is the min of f

$f(\pi/4) = \sin^2(\pi/2) = 1$ is the max of f, since $0 \leq \sin^2 x \leq \underline{1}$

$f(\pi/2) = \sin^2 \pi = 0$ is a min

$x = 2$ is neither max nor min.

23. $g(x) = \dfrac{x^2}{x^3+32}$ on $[0,\infty)$,

$$g'(x) = \frac{(x^3+32)2x - 3x^4}{(x^3+32)^2} = \frac{64x - x^4}{(x^3+32)^2}$$

- denom ≥ 0, actually can't be zero because $x \in [0,\infty)$
- $g'(x)=0 \iff 64x(1-x^3)=0 \iff x = 0, 1$
- critical points are $x=0, 1$
- $g(0)=0$, $g(1) = \frac{1}{33}$.
- we do <u>not</u> know at this point that $x=0$ is the min and $x=1$ is the max., b/c $[0,\infty)$ is not a closed interval.
- Note $\lim_{x\to\infty} g(x) = 0$ and $g(x) \geq 0 \quad \forall x \in [0,\infty)$
- Since $g(x) \geq 0$, $x=0$ is the min b/c $g(0)=0$
- Since $g'(x)$ exists for all $x \geq 0$, there are no other stationary pts or cusps, so $(1, \frac{1}{33})$ is the max.

25. $F(x) = 6\sqrt{x} - 4x$ on $[0,4]$ $\qquad F'(x) = \dfrac{3}{\sqrt{x}} - 4$

$F'(x) = 0 \iff 3 = 4\sqrt{x} \iff x = \frac{9}{16}$

critical points: $x = 0, \frac{9}{16}, 4$

$F(0) = 0$, $F(4) = -4$, $F(\frac{9}{16}) = \dfrac{6\cdot 3}{4} - \dfrac{4\cdot 9}{16} = \dfrac{18-9}{4} = \dfrac{9}{4}$

- By max-min existence theorem, max is at $x = \frac{9}{16}$,
min is at $x = -4$.

27. $f(x) = \frac{64}{\sin x} + \frac{27}{\cos x}$ on $(0, 2\pi)$

- $f'(x) = \frac{-64}{\sin^2 x} \cdot \cos x + \frac{27}{\cos^2 x} \sin x$
- $f'(x) = 0 \iff 27 \sin^3 x = 64 \cos^3 x \iff 3 \sin x = 4 \cos x$

$\iff \tan x = 4/3 \iff x = \tan^{-1}(4/3)$.

- f has no max since denom. $\to 0$, while numerator stays constant.
- $f(\tan^{-1}(4/3)) = 125$ is a min.

31. $f'(x) = x^3(1-x)^2 = 0 \iff x = 0, x = 1$

- As $x \nearrow 0$, $f'(x) = (-)^3(+)^2 < 0$ $\Rightarrow x=0$ is local min.
  As $x \searrow 0$, $f'(x) = (+)^3(+)^2 > 0$
- As $x \nearrow 1$, $f'(x) \sim (+)^3(+)^2 > 0$ $\Rightarrow x=1$ is not a max. or min.
  As $x \searrow 1$, $f'(x) \sim (+)^3(-)^2 > 0$

33. $f'(x) = (x-1)^2(x-2)^2(x-3)(x-4)$

- $f'(x) = 0 \iff x = 1, 2, 3, 4$
- 2nd deriv. is too complicated
- use first deriv. test
  - $x=1$ neither max nor min
  - $x=2$ " " " "
  - $x=3$ is local max
  - $x=4$ is local min

35. $f'(x) = (x-A)^2(x-B)^2$

no minima or maxima.