MATH 2270: Midterm 2 Practice Problems

Here are some practice problems for the first exam. This is not meant to mimic the length of the exam.

1. Inventions:

   (a) Invent a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ that is onto. You can describe the linear transformation T by giving explicit formulas for $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. As long as $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$ span $\mathbb{R}^2$, the linear transformation will be onto. An example would be $T(\vec{e}_1) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], T(\vec{e}_2) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and $T(\vec{e}_3) = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

   (b) Invent a square matrix A such that $\operatorname{rank}(A) = 2$ and $\det A = 0$.

   $$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

   (c) Invent a 2×2 matrix A such that $A^T \neq A$ and $\operatorname{col} A = \operatorname{row} A^T$. $A = \begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}$

   (d) Invent a 2×2 matrix A such that $A \neq I_2$, but $A^2 = I_2$. $A = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

   (e) Invent a matrix A such that the nullspace of A is $V = \left\{ \begin{bmatrix}a\\b\\c\end{bmatrix} \in \mathbb{R}^3 \middle| 3a - b + c = 0. \right\}$

   $A = \begin{bmatrix}3 & -1 & 1\end{bmatrix}$

   (f) Invent a 2×2 matrix A such that $A\begin{bmatrix}3\\1\end{bmatrix} = \vec{e}_1$ and $A\begin{bmatrix}-1\\0\end{bmatrix} = \vec{e}_2$. $A = \begin{bmatrix}0 & 1\\-1 & 3\end{bmatrix}$

   (g) Invent three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$ so that the subspace of $\mathbb{R}^4$ spanned by these vectors is 2-dimensional. $\left[\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\3\\4\\5\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\4\\6\\8\end{smallmatrix}\right]$

   (h) Invent a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of $\mathbb{R}^3$ such that the $\mathcal{B}$-coordinates of the vector $\vec{x} = \left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$ are $[\vec{x}]_{\mathcal{B}} = \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]$. $\mathcal{B} = \left\{ \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] \right\}$

   (i) Invent a matrix A such that $\det(A) = 3$ and $\det(2A) = 24$. $A = \begin{bmatrix}1 & 0 & 0\\0 & 3 & 0\\0 & 0 & 1\end{bmatrix}$

2. Answer each of the following true/false questions, and then give an explanation of your reasoning.

   (a) $\mathbb{R}^3$ a subspace of $\mathbb{R}^4$. False

   (b) A linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ can be one-to-one but not onto. False

   (c) For any $n \times n$ matrix A, $\det(-A) = -\det(A)$. False

   (d) If the columns of a 4×5 matrix span $\mathbb{R}^4$, then the columns are linearly independent. False

(e) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are vectors in $\mathbb{R}^n$, then $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the same as the column space of the matrix $[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p]$. True

(f) The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. True

(g) Every line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$. False

(h) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly independent set of p vectors in $\mathbb{R}^n$, then $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a p-dimensional subspace of $\mathbb{R}^n$. True

(i) For any two $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$. False

(j) For any two $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$. True

(k) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then H is an n-dimensional subspace. False

(l) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$ and $\vec{x} \in H$, then the $\mathcal{B}$-coordinates of $\vec{x}$, which we write as $[\vec{x}]_{\mathcal{B}}$ is a vectors with n entries. False

3. Solve the matrix equation $AB = BC$ for A, assuming that $A, B,$ and C are square and invertible. $A = BCB^{-1}$

4. Consider the following matrices

$$A = \begin{bmatrix} 4 & -1 & 2 & 8 \\ 4 & 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & 0 \\ -7 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -8 & 5 \\ 12 & -3 \\ -4 & -1 \end{bmatrix}$$

Which matrices correspond to one-to-one transformations? Which ones correspond to onto transformations? Explain. (*You don't need to row-reduce the matrices if you don't want to, but give a brief reason for each matrix.*) The linear transformation determined by matrix A is onto but not one-to-one. The linear transformation determined by matrix B is neither onto nor one-to-one. The linear transformation determined by matrix C is one-to-one but not onto.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\vec{e_1}) = \vec{e_2}$ and $T(\vec{e_2}) = -\vec{e_1}$, where $\vec{e_1}$ and $\vec{e_2}$ are columns of the 2×2 identity matrix.

(a) Find the standard matrix A of T. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) Plot the unit square on the left. *Reminder: The unit square is the square with vertices* $(0, 0), (0, 1), (1, 0), (1, 1)$. On the right, plot the image of the unit square under the transformation T. We did this in class. The unit square is rotated counter-clockwise about the origin by an angle of $90°$.

(c) Describe in words what the transformation T does to $\mathbb{R}^2$. It rotates counter-clockwise about the origin by an angle of $90°$.

(d) Is T one-to-one? Is T onto? Explain. T is one-to-one and onto.

6. Determine whether or not the following matrix is invertible. *Do not try to invert it*

(a)

$$A = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

Ans: The determinant of A is 9, so A is invertible.

(b) Find the determinant of A^5. Ans: 6^5

7. Suppose a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\vec{u}) = T(\vec{v})$ for some pair of distinct vectors $\vec{u}$ and $\vec{v}$. Can T be onto? Why or why not? T cannot be onto. The assumption means that T is not one-to-one, so the matrix for T has a free variable. Therefore, the matrix for T does not have a pivot in every row, and cannot be onto. A word of caution: Our invertible matrix theorem says that the linear transformation determined by an $n \times n$ matrix is onto if and only if it is one-to-one. DO NOT make the mistake of thinking that all functions (or even all linear transformations) are one-to-one if and only if they are onto.

8. Prove that if $\det(B^3) = 0$, then $\det B = 0$. Ans: Use the fact that $\det(B^3) = (\det B)^3$.

9. Consider the following matrix:

$$A = \begin{bmatrix} 5 & 1 & 6 & 2 & 0 \\ 3 & 3 & 6 & -1 & -18 \\ 8 & 4 & 12 & -5 & 18 \\ 2 & 1 & 3 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\operatorname{col} A$. Ans: $\{\begin{bmatrix} 5 & 3 & 8 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & 3 & 4 & 1 \end{bmatrix}^T, \begin{bmatrix} 2 & -1 & -5 & 0 \end{bmatrix}^T\}$

(b) Find a basis for $\operatorname{row} A$. Ans: $\{\begin{bmatrix} 1 & 0 & 1 & 0 & 5 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 1 & 0 & -13 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 0 & 1 & -6 \end{bmatrix}^T\}$

(c) Find a basis for $\ker A$. Ans: $\{\begin{bmatrix} -1 & -1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} -5 & 13 & 0 & 6 & 1 \end{bmatrix}^T\}$

10. Find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

11. Is it possible for a 5×5 matrix to be invertible if the columns of A do not span $\mathbb{R}^5$. No.