MATH 2270: Final Exam Practice Problems

The following are practice problems for the final exam. Don't feel you need to do all of these problems to prepare for the final. These problems are just a representation of the material that is fair game on the final.

1. §1.5 # 17

2. §1.7 # 6, 27, 28

3. Let T be the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by counterclockwise rotation about the origin by $\pi/4$. Find the matrix for T with respect to the standard basis of $\mathbb{R}^2$. Answer: $\begin{bmatrix} \sqrt{2}2 & -\sqrt{2}2 \\ \sqrt{2}2 & \sqrt{2}2 \end{bmatrix}$

4. Invent 2×2 matrices A and B such that $AB \neq BA$. Answer: Write down a two random matrices with no zero entries. They almost surely won't commute.

5. Compute the inverse of a 3×3 matrix. (e.g., §2.2 #31, 33, 34)

6. Write down the inverse of a 2×2 matrix. Answer: There's a formula...

7. Find an example of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ which is 1-1 but not onto. If no such transformation exists, state why. Answer: No such transformation exists by the Invertible Matrix Theorem.

8. If an $n \times n$ matrix A cannot be row reduced to the identity, what can you say about the linear transformation $x \mapsto Ax$? Is it 1-1? onto? invertible? Answer: See the Invertible Matrix Theorem

9. Find a basis for the null space, column space, and row space of the following matrix:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 7 \\ -5 & -9 & -20 \\ 2 & 3 & 11 \\ 5 & 11 & 10 \end{bmatrix}$$
(Note: This computation is a little too difficult to do by hand for the exam, but you should do it for practice, and the question is certainly fair game.) Answer: Row reduce and find pivots. The pivot columns of A form a basis for $\operatorname{col} A$. The pivot rows of the row reduced matrix form a basis for $\operatorname{row} A$. To find a basis for $\operatorname{null} A$, write the solutions to $A\vec{x} = \vec{0}$ in parametric vector form.

10. §2.8 # 26

11. §2.9 # 13, 19

12. Let A be a 6×9 matrix with 4 pivot columns. What is $\dim \operatorname{null} A$? How about $\dim \operatorname{col} A$? What is $\dim \operatorname{row} A$? Answer: 5, 4, 4

13. Let A be an 5×5 matrix with $\det A = 12$. Consider the matrix B obtained from A via following sequence of row operations:

(a) Swap row 2 with row 4.

(b) Multiply row 3 by $\frac{1}{3}$.

(c) Subtract twice row 4 from row 5.

(d) Multiply row 1 by 4.

What is the determinant of B? Answer: -16

14. If A is a square matrix with integer entries and $\det A = 1$, why are the entries of A^{-1} also integers? Similarly, if B is invertible and has rational entries (with no assumptions about the determinant), then why are the entries of B^{-1} also rational numbers? Answer: This follows directly from Cramer's Rule, which states that the ij-entry of A^{-1} is $\det(A_j(\vec{e}_i)/\det(A)$ where $A_j(\vec{e}_i)$ is the matrix obtained from A by replacing the j-th column with the basis vector $\vec{e}_i$. The point is that if $\det(A)$ is one, then the denominator is one and the entries of the inverse are just the determinant of an integer matrix, which will always be an integer. A similar, but slightly modified argument works for the case that the entries of A are rational.

15. Let R be a region in the plane whose area is 7 and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $x \mapsto Ax$ where $A = \left[\begin{smallmatrix} 9 & 3 \\ 5 & 2 \end{smallmatrix}\right]$. What is the area of the region $T(R)$? Answer: 21

16. §4.1 # 5-9, 20, 22

17. §4.2 # 31, 32, 33, 34

18. Show that $\{\sin t, \cos t, \sin t \cos t\}$ is a linearly dependent set in $C(\mathbb{R})$. *Hint: Start by assuming*

$$c_1 \cdot \sin t + c_2 \cdot \cos t + c_3 \cdot \sin t \cos t = 0$$

This equation must hold for all t, so choose several specific values of t until you get a system of enough equations to determine that all the c_i's must be zero. Answer: Following the hint, plug in $t = 0$ to get that $c_2 = 0$, then $t = \pi/2$ to get $c_1 = 0$. Finally, plug in $t = \pi/4$ to get that $c_3 = 0$. Just about any three choices of t will work, but these three are probably the "easiest".

19. Let $D\colon \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by taking the derivative. Find the matrix for D with respect to the standard basis $\{1, t, t^2, t^3\}$. *Remark: You should be able to do more general problems where you find the matrix of a linear transformation on an abstract vector space with respect to some given basis.* Answer: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

20. Problem # 4 from Midterm 2.

21. §4.4 # 32

22. §4.5 # 23

23. §4.6 # 24

24. §4.7 # 14

25. §5.1 # 14-16

26. Consider the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Without doing any calculations, write down the roots of the characteristic polynomial of A (with multiplicities) and a basis of eigenvectors. (*I've done this several times in class, and won't require you to do this on the exam, but it's very useful to be able to guess eigenvectors and being able to do so might help you on the exam.*) Answer: By inspection, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector because all row sums are equal. The eigenvalue is 6. Since the rank of A is 1, the other two roots of the characteristic polynomial must be 0. Two possible linearly independent eigenvectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. The factored characteristic polynomial is $\lambda^2(\lambda - 6)$.

27. §5.2 # 10, 23

28. §5.3 # 11, 17

29. §5.4 # 5(c), 6(c), 7, 9(c), 10(b) (*These all have the same flavor. You should feel comfortable doing problems of this type.*)

30. You should also be able to diagonalize a linear transformation on an abstract vector space.

31. Discrete Dynamical Systems & Linear Systems of Differential Equations. The last midterm and the practice problems for that test are good places to find problems on these topics.

32. Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find a basis for the orthogonal complement of $\text{span}\{v, w\}$. Answer: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

33. Compute the orthogonal projection of $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ onto $W = \text{span}\{v, w\}$ where $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Answer: $\frac{1}{35}\begin{bmatrix} 81 \\ 22 \\ 75 \end{bmatrix}$

34. §6.2 # 9

35. §6.4 # 9, 10

36. §6.5 # 10

37. §6.6 # 2, 4

38. §6.7 # 25 (*There will be a question where you have to use an inner product defined by an integral on a vector space of functions.*)

39. §7.1 # 18

40. §7.4 # 10, 13