Practice and Challenge Problems from §4.6, 5.1, 5.2, & 5.3

Not collected

I will post solutions to these problems approximately one week before the third midterm.

1. Is it possible that all solutions of a homogeneous system of 10 equations in 12 variables are multiples of one non-zero solution? Explain.

2. Let A be an $m \times n$ matrix. What is the relationship between $\dim(\mathrm{Row}A)$, $\dim(\mathrm{Nul}A)$, and the number of columns of A? What is the relationship between $\dim(\mathrm{Col}A)$, $\dim(\mathrm{Nul}A^T)$, and the number of rows of A? Write down two equations describing the relationships.

3. Using the previous exercise, explain why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

4. Give examples of the following:

   (a) A 2×2 matrix A with only one distinct eigenvalue.

   (b) A matrix A with a two-dimensional eigenspace.

   (c) A matrix A with all non-zero entries that has 0 as an eigenvalue.

5. Prove the following theorem which we will discuss on Friday.

Theorem: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial. (And therefore the same eigenvalues.)

HINTS:

- *Start by analyzing the matrix* $B - \lambda I$.
- *Use the facts that* $B = P^{-1}AP$ *and* $I = P^{-1}P$ *to write* $B - \lambda I$ *in terms of* A, λ, P, P^{-1}, *and* I.
- *Compute the characteristic polynomial of* B.
- *Remember that the determinant of the product is the product of the determinants. (In other words,* $det(AB) = detA detB$.*)*

6. Show that if A is a $n \times n$ matrix with the property that $A^2 = 0$, the only eigenvalue of A is 0. (*Hint: Let λ be an eigenvalue of A, with eigenvector $\mathbf{x}$. What is $A^2\mathbf{x}$?*)

7. Show that A and A^T have the same characteristic polynomial.

8. Let A be an $n \times n$ matrix with the property that each of its rows sums to the same number s. (For example, if $s = 4$, the matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ has this property.) Show that s is an eigenvalue of A. (*Hint: Find an eigenvector.*)