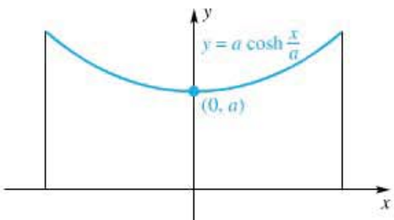


The catenary

Figure 5

An Inverted Catenary

**Applications: The Catenary** If a homogeneous flexible cable or chain is suspended between two fixed points at the same height, it forms a curve called a **catenary** (Figure 5). Furthermore (see Problem 53), a catenary can be placed in a coordinate system so that its equation takes the form

$$y = a\cosh\frac{x}{a}$$

**EXAMPLE 5** Find the length of the catenary $y = a\cosh(x/a)$ between $x = -a$ and $x = a$.

**SOLUTION** The desired length (see Section 5.4) is given by

$$\begin{aligned}
\int_{-a}^{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx &= \int_{-a}^{a}\sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\,dx \\
&= \int_{-a}^{a}\sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx \\
&= 2\int_{0}^{a}\cosh\left(\frac{x}{a}\right)dx \\
&= 2a\int_{0}^{a}\cosh\left(\frac{x}{a}\right)\left(\frac{1}{a}\,dx\right) \\
&= \left[2a\sinh\frac{x}{a}\right]_0^a \\
&= 2a\sinh 1 \approx 2.35a
\end{aligned}$$

■

## Concepts Review

**1.** sinh and cosh are defined by $\sinh x =$ _____ and $\cosh x =$ _____.

**2.** In *hyperbolic* trigonometry, the identity corresponding to $\sin^2 x + \cos^2 x = 1$ is _____.

**3.** Because of the identity in Question 2, the graph of the parametric equations $x = \cosh t$, $y = \sinh t$ is _____.

**4.** The graph of $y = a\cosh(x/a)$ is a curve called a _____; this curve is important as a model for _____.

## Problem Set 6.9

*In Problems 1–12, verify that the given equations are identities.*

**1.** $e^x = \cosh x + \sinh x$

**2.** $e^{2x} = \cosh 2x + \sinh 2x$

**3.** $e^{-x} = \cosh x - \sinh x$

**4.** $e^{-2x} = \cosh 2x - \sinh 2x$

**5.** $\sinh(x + y) = \sinh x\cosh y + \cosh x\sinh y$

**6.** $\sinh(x - y) = \sinh x\cosh y - \cosh x\sinh y$

**7.** $\cosh(x + y) = \cosh x\cosh y + \sinh x\sinh y$

**8.** $\cosh(x - y) = \cosh x\cosh y - \sinh x\sinh y$

**9.** $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x\tanh y}$

**10.** $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x\tanh y}$

**11.** $\sinh 2x = 2\sinh x\cosh x$

**12.** $\cosh 2x = \cosh^2 x + \sinh^2 x$

*In Problems 13–36, find $D_x y$.*

**13.** $y = \sinh^2 x$

**14.** $y = \cosh^2 x$

**15.** $y = 5\sinh^2 x$

**16.** $y = \cosh^3 x$

**17.** $y = \cosh(3x + 1)$

**18.** $y = \sinh(x^2 + x)$

**19.** $y = \ln(\sinh x)$

**20.** $y = \ln(\coth x)$

**21.** $y = x^2\cosh x$

**22.** $y = x^{-2}\sinh x$

**23.** $y = \cosh 3x\sinh x$

**24.** $y = \sinh x\cosh 4x$

**25.** $y = \tanh x\sinh 2x$

**26.** $y = \coth 4x\sinh x$

**27.** $y = \sinh^{-1}(x^2)$

**28.** $y = \cosh^{-1}(x^3)$

**29.** $y = \tanh^{-1}(2x - 3)$

**30.** $y = \coth^{-1}(x^5)$

**31.** $y = x\cosh^{-1}(3x)$

**32.** $y = x^2\sinh^{-1}(x^5)$

**33.** $y = \ln(\cosh^{-1} x)$

**34.** $y = \cosh^{-1}(\cos x)$

**35.** $y = \tanh(\cot x)$

**36.** $y = \coth^{-1}(\tanh x)$

**37.** Find the area of the region bounded by $y = \cosh 2x$, $y = 0$, $x = 0$, and $x = \ln 3$.

*In Problems 38–45, evaluate each integral.*

**38.** $\int \sinh(3x + 2)\, dx$

**39.** $\int x \cosh(\pi x^2 + 5)\, dx$

**40.** $\int \frac{\cosh \sqrt{z}}{\sqrt{z}}\, dz$

**41.** $\int \frac{\sinh(2z^{1/4})}{\sqrt[4]{z^3}}\, dz$

**42.** $\int e^x \sinh e^x\, dx$

**43.** $\int \cos x \sinh(\sin x)\, dx$

**44.** $\int \tanh x \ln(\cosh x)\, dx$

**45.** $\int x \coth x^2 \ln(\sinh x^2)\, dx$

**46.** Find the area of the region bounded by $y = \cosh 2x$, $y = 0$, $x = -\ln 5$, and $x = \ln 5$.

**47.** Find the area of the region bounded by $y = \sinh x$, $y = 0$, and $x = \ln 2$.

**48.** Find the area of the region bounded by $y = \tanh x$, $y = 0$, $x = -8$, and $x = 8$.

**49.** The region bounded by $y = \cosh x$, $y = 0$, $x = 0$, and $x = 1$ is revolved about the $x$-axis. Find the volume of the resulting solid. *Hint:* $\cosh^2 x = (1 + \cosh 2x)/2$.

**50.** The region bounded by $y = \sinh x$, $y = 0$, $x = 0$, and $x = \ln 10$ is revolved about the $x$-axis. Find the volume of the resulting solid.

**51.** The curve $y = \cosh x, 0 \le x \le 1$, is revolved about the $x$-axis. Find the area of the resulting surface.

**52.** The curve $y = \sinh x, 0 \le x \le 1$, is revolved about the $x$-axis. Find the area of the resulting surface.

**53.** To derive the equation of a hanging cable (catenary), we consider the section $AP$ from the lowest point $A$ to a general point $P(x, y)$ (see Figure 6) and imagine the rest of the cable to have been removed.

The forces acting on the cable are

1. $H$ = horizontal tension pulling at $A$;
2. $T$ = tangential tension pulling at $P$;
3. $W = \delta s$ = weight of $s$ feet of cable of density $\delta$ pounds per foot.

To be in equilibrium, the horizontal and vertical components of $T$ must just balance $H$ and $W$, respectively. Thus, $T \cos \phi = H$ and $T \sin \phi = W = \delta s$, and so

$$\frac{T \sin \phi}{T \cos \phi} = \tan \phi = \frac{\delta s}{H}$$

But since $\tan \phi = dy/dx$, we get

$$\frac{dy}{dx} = \frac{\delta s}{H}$$

and therefore

$$\frac{d^2y}{dx^2} = \frac{\delta}{H}\frac{ds}{dx} = \frac{\delta}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Now show that $y = a \cosh(x/a) + C$ satisfies this differential equation with $a = H/\delta$.

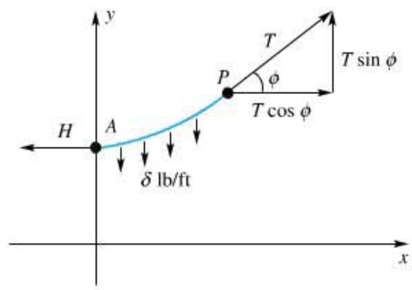


Figure 6

C **54.** Call the graph of $y = b - a \cosh(x/a)$ an inverted catenary and imagine it to be an arch sitting on the $x$-axis. Show that if the width of this arch along the $x$-axis is $2a$ then each of the following is true.

(a) $b = a \cosh 1 \approx 1.54308a$.

(b) The height of the arch is approximately $0.54308a$.

(c) The height of an arch of width 48 is approximately 13.

C **55.** A farmer built a large hayshed of length 100 feet and width 48 feet. A cross section has the shape of an inverted catenary (see Problem 54) with equation $y = 37 - 24\cosh(x/24)$.

(a) Draw a picture of this shed.

(b) Find the volume of the shed.

(c) Find the surface area of the roof of the shed.

**56.** Show that $A = t/2$, where $A$ denotes the area in Figure 2 of this section. *Hint:* At some point you will need to use Formula 44 from the back of the book.

**57.** Demonstrate that for every real number $r$:

(a) $(\sinh x + \cosh x)^r = \sinh rx + \cosh rx$

(b) $(\cosh x - \sinh x)^r = \cosh rx - \sinh rx$

(c) $(\cos x + i \sin x)^r = \cos rx + i \sin rx$

(d) $(\cos x - i \sin x)^r = \cos rx - i \sin rx$

**58.** The **gudermannian** of $t$ is defined by

$$\mathrm{gd}(t) = \tan^{-1}(\sinh t)$$

Show that

(a) gd is odd and increasing with an inflection point at the origin;

(b) $\mathrm{gd}(t) = \sin^{-1}(\tanh t) = \int_0^t \operatorname{sech} u\, du$.

**59.** Show that the area under the curve $y = \cosh t$, $0 \le t \le x$, is numerically equal to its arc length.

**60.** Find the equation of the Gateway Arch in St. Louis, Missouri, given that it is an inverted catenary (see Problem 54). Assume that it stands on the $x$-axis, that it is symmetric with respect to the $y$-axis, and that it is 630 feet wide at the base and 630 feet high at the center.

GC **61.** Draw the graphs of $y = \sinh x$, $y = \ln\left(x + \sqrt{x^2 + 1}\right)$, and $y = x$ using the same axes and scaled so that $-3 \le x \le 3$ and $-3 \le y \le 3$. What does this demonstrate?