MATH2270: Midterm 3 Study Guide

All material from the course is fair game for the final exam. The following is a summary of all the material we have covered. Roughly, 30-40% of the material on the final will be focused on material covered since the third midterm. The rest of the exam will test material from earlier sections. That being said…

Connecting the Dots

On the final, I want to emphasize the connections and relationships between everything we've talked about this semester. This includes, but is not limited to:

- Matrices are linear transformations! Multiplication of matrices corresponds to composition of linear transformations.
- Diagonalizing an $n \times n$ matrix A corresponds to finding a basis of eigenvectors for the linear transformation determined by A, then writing the matrix of that linear transformation in your new basis.
- The connection between linear transformations on abstract vector spaces, $T\colon V \to V$ (abstract just means "not $\mathbb{R}^n$") and the matrices that represent these transformations. You can't write down a matrix until you choose a basis for V. A different choice of basis gives you a different matrix. The matrices are related by $A = PBP^{-1}$. Often choosing a "good basis" is helpful for understanding the linear transformation.
- The omnibus "Invertible matrix theorem"
- The usefulness of diagonalizing a matrix (or linear transformation), especially as it relates to applications (e.g., discrete dynamical systems, Markov chains, differential equations…).
- Being able to write a rigorous argument is important. There will be one question on the final exam that asks you to prove something. In order to receive full credit on this problem, you will need to convey your thoughts in a coherent manner, using complete sentences. The question will come straight from one of the Food for Thought assignments.

§1.1 Systems of Linear Equations

- Solving linear systems of equations by row reducing an augmented matrix.
- Determining the existence and uniqueness of solutions to a linear system.

§1.2 Row Reduction and Echelon Forms

- Row reduction of matrices to echelon or reduced echelon form.
- Identifying pivot positions in a matrix.
- Identifying basic and free variables in a linear system and giving a parametric description of the solution set.

§1.3 Vector Equations

- Be comfortable with performing basic vetor operations of addition and scalar multiplication.
- Writing a linear system of equations as a vector equation and vise versa.
- The definition of the span of a set of vectors in $\mathbb{R}^n$ and a geometric understanding of the span.
- The definition of a linear combination of vectors.

§1.4 The Matrix Equation $A\vec{x} = \vec{b}$

- Solving matrix equations.
- Translating between matrix equations, linear systems of equations, and vector equations.

§1.5 Solution Sets of Linear Systems

- Solutions to homogeneous and nonhomogeneous systems of equations.
- Describing the solution set of a linear system in parametric vector form.

§1.7 Linear Independence

- Definition of linearly independent/dependent sets.
- Determining whether a given set of vectors is linearly independent or not.
- A geometric description of linearly independent sets.

§1.8 Introduction to Linear Transformations

- Definition of a linear transformation.
- Determining whether a given function is a linear transformation.

§1.9 The Matrix of a Linear Transformation

- Finding the matrix of a linear transformation.
- 1-1 and onto linear transformations (and characterizations thereof).

§1.9 The Matrix of a Linear Transformation

- Finding the matrix of a linear transformation. For example, consider the function $T\colon \mathbb{R}^3 \to \mathbb{R}$ that maps each vector to the sum of its entries. Find the standard matrix for T.
- 1-1 and onto linear transformations (and characterizations thereof). You should know the connections between properties of a linear transformation and properties (number & location of pivots) of the matrix of that transformation.

§2.1 Matrix Operations

- You should be comfortable with the basic algebra of matrices, including addition, scalar multiplication, and matrix multiplication.
- A good resource for matrix algebra is FFT4. The solutions are now posted on the Canvas page for our course. I even posted two different solutions!!

- Understand the connection between matrix multiplication and composition of functions. Thinking about matrix multiplication as composition of functions can be a helpful way of approaching problems and questions about matrices or functions. For example, if I asked *"Why is the product of two invertible matrices an invertible matrix?,"* then a possible answer would be that the composition of invertible linear transformations is an invertible linear transformation! Going the other way, if the question were *"Why can't a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^5$ be onto?,"* then one answer would be that a linear transformation is onto if and only if the matrix of the linear transformation has a pivot in every column, and the matrix for T would be 3×5 so it couldn't possibly have a pivot in every column.

§2.2 The Inverse of a Matrix

- You should be able to write down the inverse of a 2×2 matrix.
- Use our algorithm for computing the inverse of a matrix.
- Check whether two matrices are inverses of one another.
- Understand the connection between inverses of matrices and the inverse of a linear transformation. Also, understand how to use A^{-1} to solve the matrix equation $A\vec{x} = \vec{b}$.

§2.3 Characterizations of Invertible Matrices

- Determine whether a matrix (or linear transformation) is invertible.
- I want most (not necessarily all) of the equivalences in the *Invertible Matrix Theorem* to be natural. I want everyone to be able to translate between all the different languages we know: matrix equation, system of linear equations, vector equation, linear combinations, span of a set of vectors, linear transformations, etc. The more fluent you are in doing this translation, the better prepared you will be when we start using all of this to do really awesome stuff!!

§2.4 Partitioned Matrices (omitted)

- This section was assigned as homework and will not be tested. That doesn't mean you shouldn't know this material. It's likely that we will use it at some point later in the semester.

§2.8 Subspaces of $\mathbb{R}^n$

- You should know the definition of a subspace.
- You should have an *intuition* for subspaces of $\mathbb{R}^n$.
- Know the fundamental subspaces associated to a matrix A: they are $\operatorname{null} A$ and $\operatorname{col} A$. If A is an $m \times n$ matrix, you should know in which space ($\mathbb{R}^m$ or $\mathbb{R}^n$) each of these live.

§2.9 Dimension and Rank This section is subsumed by material in Ch4.

- Know the definition of dimension of a subspace, and be able to compute it. Basically, the only way to find the dimension of a subspace is to find a basis for the subspace. The dimension is then the number of elements in that basis.
- Know what the rank of a matrix A is.

- If you are given a matrix, A, you should be able to find a basis for A and a basis for $\operatorname{col} A$. You should also be able to find the dimension of each of these subspaces.
- Know the "rank-nullity theorem" This says that if A is an $m \times n$ matrix, then

$$\dim \operatorname{col} A + \dim \operatorname{null} A = m$$

  You should also be able to interpret this as a statement about linear transformations.
- If $\vec{x}$ is a vector in the subspace H and $\mathcal{B}$ is a basis for H, then you should be able to compute the $\mathcal{B}$-coordinates of $\vec{x}$. That is, compute $[\vec{x}]_{\mathcal{B}}$.

§3.1 & §3.2 Determinants

- Be able to compute determinants using cofactor expansions. You should choose the row/column to expand in strategically.
- Know the defining properties of the determinant:
  1. The determinant is linear in each row
  2. The determinant is alternating (if you swap two rows, the determinant changes sign).
  3. $\det I_n = 1$.
- Know the other basic properties of the determinant:
  1. $\det(AB) = \det(A)\det(B)$
  2. $\det A^T = \det A$
  3. A is invertible if and only if $\det A \neq 0$.

§3.3 Cramer's Rule, Volume, and Linear Transformations

- Understand how the volume of a region in $\mathbb{R}^n$ changes under the application of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. Specifically, $\operatorname{vol}(T(R)) = \operatorname{vol}(R) \cdot |\det(T)|$.
- If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix for T, what does $\det A$ tell you about T?

§4.1 Vector Spaces and Subspaces

- Know and have familiarity with vector spaces besides $\mathbb{R}^n$. Some examples: the vector space of polynomials of degree less than or equal to n, $\mathbb{P}_n$; set of all polynomials, $\mathbb{P}$; the vector space of $m \times n$ matrices, $M_{m\times n}$; the vector space of infinite sequences, $\mathbb{R}^\infty$; the vector space of all functions $\mathcal{F}([a,b]) = \{f\colon [a,b] \to \mathbb{R}\}$; the vector space of continuous functions $\mathcal{C}([a,b]) = \{f\colon [a,b] \to \mathbb{R} \mid f \text{ is cont.}\}$; the vector space of k-times differentiable functions $\mathcal{C}^k([a,b]) = \{f\colon [a,b] \to \mathbb{R} \mid f \text{ is differentiable } k \text{ times}\}$.
- You should be able to use your intuition from $\mathbb{R}^n$ to determine whether or not a given set is a subspace of a vector space V. You should be able to check, using the definition, if a given set is a subspace.

§4.2 Linear Transformations, Kernels & Ranges

- Find the kernel of a linear transformation
- Describe the range of a linear transformation explicitly.

- Use your intuition from $\mathbb{R}^n$ to understand linear transformations.

§4.3 Linear (In)Dependence & Bases

- Be able to answer linear algebra questions in general vector spaces
  - Is this set linearly independent?
  - Is this set a basis?
  - Find a basis for this subspace.
- Know the definitions (as always) of linear (in)dependence, basis, and other relevant terms.

§4.4 Coordinate Systems

- Understand coordinate mappings.
- Be able to use them to translate questions/computations in general vector spaces into questions/computations in $\mathbb{R}^n$.

§4.5/4.6 Dimension & the Rank-Nullity Theorem

- As always, know the definition of dimension of a vector space or subspace.
- Know the rank-nullity theorem: If $T\colon V \to W$ is a linear transformation, then $\dim(\ker T) + \dim(\Im T) = \dim(V)$.
- Understand the rank-nullity theorem using your intuition from $\mathbb{R}^n$. Can you draw a picture?

§4.7 Change of Basis

- If V is a vector space, and $\mathcal{B}, \mathcal{C}$ are bases for V, in what sense are the associated coordinate mappings the same? In what sense are they different?
- Be able to compute change of coordinate matrices.
- Understand what changing coordinates does!

§5.1 Eigenvectors & Eigenvalues

- Know the definitions... eigenvector, eigenvalue
- Be able to answer questions like:
  - Is $\lambda = 7$ an eigenvalue of A?
  - Is $\vec{v}$ an eigenvector for A?
  - Write down a 3×3 matrix that has $\left[\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right]$ as an eigenvector.
  - Write down a 3×3 matrix that has $\lambda = 2$ as an eigenvalue.
- Compute a basis for an eigenspace of A.

§5.2 The Characteristic Equation

- Compute the characteristic polynomial of a matrix.
- Find the eigenvalues of a matrix (or linear transformation).

§5.3/5.4 Eigenvectors in relation to Linear Transformations/Diagonalization

- Be able to diagonalize a matrix. That is, write $A = PDP^{-1}$. If the matrix is not diagonalizable, you should be able to determine that.
- Given a diagonalizable matrix, A, be able to find a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
- Understand the connection between diagonalizing a matrix, and viewing a linear transformation in a "well chosen" basis.
- Given a linear transformation, $T\colon V \to V$, be able to find a basis of V (if one exists) for which the matrix of T with respect to $\mathcal{B}$ is diagonal.
- Given a linear transformation, $T\colon V \to V$, know the relationship between $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ for bases $\mathcal{B}, \mathcal{C}$ of V.

§4.9/5.6 Markov Chains & Discrete Dynamical Systems

- Apply the theory of eigenvectors and eigenvalues to determine the long term or steady-state behavior of a Markov chain or discrete dynamical system.
- Plot trajectories for discrete dynamical systems.
- Determine if the origin is an attractor/repeller/saddle point for a discrete dynamical system.
- Use the above analysis to describe the long-term behavior of such a system.

§5.7 Applications to Differential Equations

- Use the process of diagonalization to decouple a dynamical system determined by a system of first order differential equations.
- Determine if the origin is an attractor/repeller/saddle point for a system of differential equations.

§6.1 Inner Products, Length, & Orthogonality

- Dot products, norm, distance, orthogonality for vectors in $\mathbb{R}^n$.
- Know and be able to use that $(\operatorname{row} A)^\perp = \operatorname{null} A$ and $(\operatorname{col} A)^\perp = \operatorname{null} A^T$.

§6.2 Orthogonal Sets

- Be able to compute orthogonal projections onto a subspace using an orthogonal basis for that subspace.
- This is the section where we learned that orthonormal matrices "preserve dot products" (see Theorem 7 from the book).
- This is also where we learned that if U is a square matrix with orthonormal columns, then $U^T = U^{-1}$.

§6.3 Orthogonal Projection

- Be able to compute orthogonal projections onto a subspace using an orthogonal basis for that subspace.

§6.4 Gram-Schmidt Algorithm

- Know what Gram-Schmidt's Algorithm does (i.e., what is the input, and what is the output).
- Have some thoughts about why Gram-Schmidt's Algorithm may be useful.
- Be able to use Gram-Schmidt to take a given basis for a subspace W and produce an orthogonal basis for W.

§6.5 & 6.6 Least Squares and Applications

- Be able to explain the setup of a "least-squares problem," and the geometric idea behind the solution.
- As a practical matter, you should be able to set up and solve least-squares problems using the methods described in class and this section.

§7.1 Diagonalization of Symmetric Matrices

- You should know that all symmetric matrices are diagonalizable.
- An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.
- Know that if A is symmetric and $A = PDP^{-1}$ for an orthonormal matrix P, then $P^{-1} = P^T$.

§7.2 Quadratic Forms

- Go back and forth between the matrix of a quadratic form and the "formula for the form."
- Determine if a quadratic form is positive definite, negative definite, or indefinite using the eigenvalues of the matrix for Q.

§7.4 Singular Value Decomposition

- Let A be an $m \times n$ matrix, and let $A = U\Sigma V^T$ be a singular value decomposition of A. Write a paragraph (that means use complete sentences) explaining the meaning of the matrix factorization $A = U\Sigma V^T$ as it relates to the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ determined by the matrix A.