SOLUTION Write the vectors in H as column vectors. Then an arbitrary vector in H has the form

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}$$

This calculation shows that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vectors indicated above. Thus H is a subspace of $\mathbb{R}^4$ by Theorem 1. ■

Example 11 illustrates a useful technique of expressing a subspace H as the set of linear combinations of some small collection of vectors. If $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, we can think of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in the spanning set as "handles" that allow us to hold on to the subspace H. Calculations with the infinitely many vectors in H are often reduced to operations with the finite number of vectors in the spanning set.

EXAMPLE 12 For what value(s) of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

SOLUTION This question is Practice Problem 2 in Section 1.3, written here with the term *subspace* rather than $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The solution there shows that $\mathbf{y}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h = 5$. That solution is worth reviewing now, along with Exercises 11–16 and 19–21 in Section 1.3. ■

Although many vector spaces in this chapter will be subspaces of $\mathbb{R}^n$, it is important to keep in mind that the abstract theory applies to other vector spaces as well. Vector spaces of functions arise in many applications, and they will receive more attention later.

PRACTICE PROBLEMS

1. Show that the set H of all points in $\mathbb{R}^2$ of the form $(3s, 2+5s)$ is not a vector space, by showing that it is not closed under scalar multiplication. (Find a specific vector $\mathbf{u}$ in H and a scalar c such that $c\mathbf{u}$ is not in H.)
2. Let $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V. Show that $\mathbf{v}_k$ is in W for $1 \le k \le p$. [*Hint:* First write an equation that shows that $\mathbf{v}_1$ is in W. Then adjust your notation for the general case.]

WEB

4.1 EXERCISES

1. Let V be the first quadrant in the xy-plane; that is, let
$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \ge 0, y \ge 0 \right\}$$
   a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u}+\mathbf{v}$ in V? Why?
   b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V. (This is enough to show that V is *not* a vector space.)
2. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \ge 0 \right\}$.
   a. If $\mathbf{u}$ is in W and c is any scalar, is $c\mathbf{u}$ in W? Why?

b. Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is not in W. This is enough to show that W is *not* a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

4. Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ *not* through the origin is not closed under vector addition.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

7. All polynomials of degree at most 3, with integers as coefficients.

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

9. Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\,\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

10. Let H be the set of all vectors of the form $\begin{bmatrix} 3t \\ 0 \\ -7t \end{bmatrix}$, where t is any real number. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)

11. Let W be the set of all vectors of the form $\begin{bmatrix} 2b + 3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

12. Let W be the set of all vectors of the form $\begin{bmatrix} 2s + 4t \\ 2s \\ 2s - 3t \\ 5t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)

13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

In Exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

15. $\begin{bmatrix} 2a + 3b \\ -1 \\ 2a - 5b \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix}$

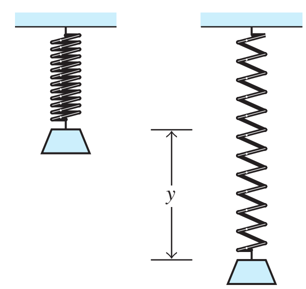
17. $\begin{bmatrix} 2a - b \\ 3b - c \\ 3c - a \\ 3b \end{bmatrix}$

18. $\begin{bmatrix} 4a + 3b \\ 0 \\ a + 3b + c \\ 3b - 2c \end{bmatrix}$

19. If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass–spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{5}$$

where ω is a constant that depends on the spring and the mass. (See the figure below.) Show that the set of all functions described in (5) (with ω fixed and c_1, c_2 arbitrary) is a vector space.



20. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

b. Show that $\{\mathbf{f} \text{ in } C[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a, b]$.

For fixed positive integers m and n, the set $M_{m \times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2 \times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2 \times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3 \times 4}$). Determine if H is a subspace of $M_{2 \times 4}$.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

b. A vector is an arrow in three-dimensional space.

c. A subset H of a vector space V is a subspace of V if the zero vector is in H.

d. A subspace is also a vector space.

e. Analog signals are used in the major control systems for the space shuttle, mentioned in the introduction to the chapter.

24. a. A vector is any element of a vector space.

b. If $\mathbf{u}$ is a vector in a vector space V, then $(-1)\mathbf{u}$ is the same as the negative of $\mathbf{u}$.

c. A vector space is also a subspace.

d. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

e. A subset H of a vector space V is a subspace of V if the following conditions are satisfied: (i) the zero vector of V is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

Exercises 25–29 show how the axioms for a vector space V can be used to prove the elementary properties described after the definition of a vector space. Fill in the blanks with the appropriate axiom numbers. Because of Axiom 2, Axioms 4 and 5 imply, respectively, that $\mathbf{0}+\mathbf{u}=\mathbf{u}$ and $-\mathbf{u}+\mathbf{u}=\mathbf{0}$ for all $\mathbf{u}$.

25. Complete the following proof that the zero vector is unique. Suppose that $\mathbf{w}$ in V has the property that $\mathbf{u}+\mathbf{w}=\mathbf{w}+\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}$ in V. In particular, $\mathbf{0}+\mathbf{w}=\mathbf{0}$. But $\mathbf{0}+\mathbf{w}=\mathbf{w}$, by Axiom ______. Hence $\mathbf{w}=\mathbf{0}+\mathbf{w}=\mathbf{0}$.

26. Complete the following proof that $-\mathbf{u}$ is the *unique vector* in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$. Suppose that $\mathbf{w}$ satisfies $\mathbf{u}+\mathbf{w}=\mathbf{0}$. Adding $-\mathbf{u}$ to both sides, we have

$(-\mathbf{u})+[\mathbf{u}+\mathbf{w}]=(-\mathbf{u})+\mathbf{0}$

$[(-\mathbf{u})+\mathbf{u}]+\mathbf{w}=(-\mathbf{u})+\mathbf{0}$ by Axiom ______ (a)

$\mathbf{0}+\mathbf{w}=(-\mathbf{u})+\mathbf{0}$ by Axiom ______ (b)

$\mathbf{w}=-\mathbf{u}$ by Axiom ______ (c)

27. Fill in the missing axiom numbers in the following proof that $0\mathbf{u}=\mathbf{0}$ for every $\mathbf{u}$ in V.

$0\mathbf{u}=(0+0)\mathbf{u}=0\mathbf{u}+0\mathbf{u}$ by Axiom ______ (a)

Add the negative of $0\mathbf{u}$ to both sides:

$0\mathbf{u}+(-0\mathbf{u})=[0\mathbf{u}+0\mathbf{u}]+(-0\mathbf{u})$

$0\mathbf{u}+(-0\mathbf{u})=0\mathbf{u}+[0\mathbf{u}+(-0\mathbf{u})]$ by Axiom ______ (b)

$\mathbf{0}=0\mathbf{u}+\mathbf{0}$ by Axiom ______ (c)

$\mathbf{0}=0\mathbf{u}$ by Axiom ______ (d)

28. Fill in the missing axiom numbers in the following proof that $c\mathbf{0}=\mathbf{0}$ for every scalar c.

$c\mathbf{0}=c(\mathbf{0}+\mathbf{0})$ by Axiom ______ (a)

$=c\mathbf{0}+c\mathbf{0}$ by Axiom ______ (b)

Add the negative of $c\mathbf{0}$ to both sides:

$c\mathbf{0}+(-c\mathbf{0})=[c\mathbf{0}+c\mathbf{0}]+(-c\mathbf{0})$

$c\mathbf{0}+(-c\mathbf{0})=c\mathbf{0}+[c\mathbf{0}+(-c\mathbf{0})]$ by Axiom ______ (c)

$\mathbf{0}=c\mathbf{0}+\mathbf{0}$ by Axiom ______ (d)

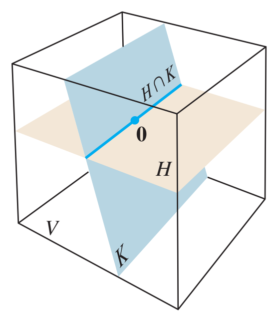
$\mathbf{0}=c\mathbf{0}$ by Axiom ______ (e)

29. Prove that $(-1)\mathbf{u}=-\mathbf{u}$. [*Hint:* Show that $\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$. Use some axioms and the results of Exercises 27 and 26.]

30. Suppose $c\mathbf{u}=\mathbf{0}$ for some nonzero scalar c. Show that $\mathbf{u}=\mathbf{0}$. Mention the axioms or properties you use.

31. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$. Explain why H also contains Span $\{\mathbf{u},\mathbf{v}\}$. This shows that Span $\{\mathbf{u},\mathbf{v}\}$ is the smallest subspace of V that contains both $\mathbf{u}$ and $\mathbf{v}$.

32. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H\cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K. Show that $H\cap K$ is a subspace of V. (See the figure.) Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.



33. Given subspaces H and K of a vector space V, the **sum** of H and K, written as $H+K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

$$H+K=\{\mathbf{w}:\mathbf{w}=\mathbf{u}+\mathbf{v} \text{ for some } \mathbf{u} \text{ in } H \text{ and some } \mathbf{v} \text{ in } K\}$$

a. Show that $H+K$ is a subspace of V.

b. Show that H is a subspace of $H+K$ and K is a subspace of $H+K$.

34. Suppose $\mathbf{u}_1,\ldots,\mathbf{u}_p$ and $\mathbf{v}_1,\ldots,\mathbf{v}_q$ are vectors in a vector space V, and let

$H=\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ and $K=\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_q\}$

Show that $H+K=\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$.

35. [**M**] Show that $\mathbf{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, where

$$\mathbf{w} = \begin{bmatrix} 9 \\ -4 \\ -4 \\ 7 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 8 \\ -4 \\ -3 \\ 9 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \\ -2 \\ -8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -7 \\ 6 \\ -5 \\ -18 \end{bmatrix}$$

36. [**M**] Determine if $\mathbf{y}$ is in the subspace of $\mathbb{R}^4$ spanned by the columns of A, where

$$\mathbf{y} = \begin{bmatrix} -4 \\ -8 \\ 6 \\ -5 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -5 & -9 \\ 8 & 7 & -6 \\ -5 & -8 & 3 \\ 2 & -2 & -9 \end{bmatrix}$$

37. [**M**] The vector space $H = \text{Span}\,\{1, \cos^2 t, \cos^4 t, \cos^6 t\}$ contains at least two interesting functions that will be used in a later exercise:

$$\mathbf{f}(t) = 1 - 8\cos^2 t + 8\cos^4 t$$
$$\mathbf{g}(t) = -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t$$

Study the graph of $\mathbf{f}$ for $0 \le t \le 2\pi$, and guess a simple formula for $\mathbf{f}(t)$. Verify your conjecture by graphing the difference between $1 + \mathbf{f}(t)$ and your formula for $\mathbf{f}(t)$. (Hopefully, you will see the constant function 1.) Repeat for $\mathbf{g}$.

38. [**M**] Repeat Exercise 37 for the functions

$$\mathbf{f}(t) = 3\sin t - 4\sin^3 t$$
$$\mathbf{g}(t) = 1 - 8\sin^2 t + 8\sin^4 t$$
$$\mathbf{h}(t) = 5\sin t - 20\sin^3 t + 16\sin^5 t$$

in the vector space Span $\{1, \sin t, \sin^2 t, \ldots, \sin^5 t\}$.

SOLUTIONS TO PRACTICE PROBLEMS

1. Take any $\mathbf{u}$ in H—say, $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$—and take any $c \neq 1$—say, $c = 2$. Then $c\mathbf{u} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$. If this is in H, then there is some s such that

$$\begin{bmatrix} 3s \\ 2 + 5s \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

That is, $s = 2$ and $s = 12/5$, which is impossible. So $2\mathbf{u}$ is not in H and H is not a vector space.

2. $\mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. This expresses $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$, so $\mathbf{v}_1$ is in W. In general, $\mathbf{v}_k$ is in W because

$$\mathbf{v}_k = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{k-1} + 1\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_p$$

4.2 NULL SPACES, COLUMN SPACES, AND LINEAR TRANSFORMATIONS

In applications of linear algebra, subspaces of $\mathbb{R}^n$ usually arise in one of two ways: (1) as the set of all solutions to a system of homogeneous linear equations or (2) as the set of all linear combinations of certain specified vectors. In this section, we compare and contrast these two descriptions of subspaces, allowing us to practice using the concept of a subspace. Actually, as you will soon discover, we have been working with subspaces ever since Section 1.3. The main new feature here is the terminology. The section concludes with a discussion of the kernel and range of a linear transformation.

The Null Space of a Matrix

Consider the following system of homogeneous equations:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned} \tag{1}$$

Typically, such a linear transformation is described in terms of one or more derivatives of a function. To explain this in any detail would take us too far afield at this point. So we consider only two examples. The first explains why the operation of differentiation is a linear transformation.

EXAMPLE 8 (Calculus required) Let V be the vector space of all real-valued functions f defined on an interval $[a,b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a,b]$. Let W be the vector space $C[a,b]$ of all continuous functions on $[a,b]$, and let $D : V \to W$ be the transformation that changes f in V into its derivative f'. In calculus, two simple differentiation rules are

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(cf) = cD(f)$$

That is, D is a linear transformation. It can be shown that the kernel of D is the set of constant functions on $[a,b]$ and the range of D is the set W of all continuous functions on $[a,b]$. ■

EXAMPLE 9 (Calculus required) The differential equation

$$y'' + \omega^2 y = 0 \tag{4}$$

where ω is a constant, is used to describe a variety of physical systems, such as the vibration of a weighted spring, the movement of a pendulum, and the voltage in an inductance-capacitance electrical circuit. The set of solutions of (4) is precisely the kernel of the linear transformation that maps a function $y = f(t)$ into the function $f''(t) + \omega^2 f(t)$. Finding an explicit description of this vector space is a problem in differential equations. The solution set turns out to be the space described in Exercise 19 in Section 4.1. ■

PRACTICE PROBLEMS

1. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a - 3b - c = 0 \right\}$. Show in two different ways that W is a subspace of $\mathbb{R}^3$. (Use two theorems.)
2. Let $A = \begin{bmatrix} 7 & -3 & 5 \\ -4 & 1 & -5 \\ -5 & 2 & -4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$. Suppose you know that the equations $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = \mathbf{w}$ are both consistent. What can you say about the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$?

4.2 EXERCISES

1. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

2. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}.$$

In Exercises 3–6, find an explicit description of Nul A, by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–14, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

7. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$

8. $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t \right\}$

9. $\left\{ \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} : \begin{array}{l} p - 3q = 4s \\ 2p = s + 5r \end{array} \right\}$

10. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} 3a + b = c \\ a + b + 2c = 2d \end{array} \right\}$

11. $\left\{ \begin{bmatrix} s - 2t \\ 3 + 3s \\ 3s + t \\ 2s \end{bmatrix} : s, t \text{ real} \right\}$

12. $\left\{ \begin{bmatrix} 3p - 5q \\ 4q \\ p \\ q + 1 \end{bmatrix} : p, q \text{ real} \right\}$

13. $\left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \text{ real} \right\}$

14. $\left\{ \begin{bmatrix} -s + 3t \\ s - 2t \\ 5s - t \end{bmatrix} : s, t \text{ real} \right\}$

In Exercises 15 and 16, find A such that the given set is Col A.

15. $\left\{ \begin{bmatrix} 2s + t \\ r - s + 2t \\ 3r + s \\ 2r - s - t \end{bmatrix} : r, s, t \text{ real} \right\}$

16. $\left\{ \begin{bmatrix} b - c \\ 2b + 3d \\ b + 3c - 3d \\ c + d \end{bmatrix} : b, c, d \text{ real} \right\}$

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

17. $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$

18. $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -3 & 2 & 0 & -5 \end{bmatrix}$

21. With A as in Exercise 17, find a nonzero vector in Nul A and a nonzero vector in Col A.

22. With A as in Exercise 18, find a nonzero vector in Nul A and a nonzero vector in Col A.

23. Let $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

24. Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

In Exercises 25 and 26, A denotes an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

25. a. The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

b. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

c. The column space of A is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

d. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then Col A is $\mathbb{R}^m$.

e. The kernel of a linear transformation is a vector space.

f. Col A is the set of all vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$.

26. a. A null space is a vector space.

b. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

c. Col A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

d. Nul A is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

e. The range of a linear transformation is a vector space.

f. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

27. It can be shown that a solution of the system below is $x_1 = 3$, $x_2 = 2$, and $x_3 = -1$. Use this fact and the theory from this section to explain why another solution is $x_1 = 30$, $x_2 = 20$, and $x_3 = -10$. (Observe how the solutions are related, but make no other calculations.)

$$\begin{aligned} x_1 - 3x_2 - 3x_3 &= 0 \\ -2x_1 + 4x_2 + 2x_3 &= 0 \\ -x_1 + 5x_2 + 7x_3 &= 0 \end{aligned}$$

28. Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

It can be shown that the first system has a solution. Use this fact and the theory from this section to explain why the second system must also have a solution. (Make no row operations.)

29. Prove Theorem 3 as follows: Given an $m \times n$ matrix A, an element in Col A has the form $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. Let $A\mathbf{x}$ and $A\mathbf{w}$ represent any two vectors in Col A.

a. Explain why the zero vector is in Col A.

b. Show that the vector $A\mathbf{x} + A\mathbf{w}$ is in Col A.

c. Given a scalar c, show that $c(A\mathbf{x})$ is in Col A.

30. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint:* Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}, \mathbf{w}$ in V.]

31. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. For instance, if $\mathbf{p}(t) = 3 + 5t + 7t^2$, then $T(\mathbf{p}) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.

a. Show that T is a linear transformation. [*Hint:* For arbitrary polynomials $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_2$, compute $T(\mathbf{p} + \mathbf{q})$ and $T(c\mathbf{p})$.]

b. Find a polynomial $\mathbf{p}$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

32. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbb{P}_2$ that span the kernel of T, and describe the range of T.

33. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a. Show that T is a linear transformation.

b. Let B be any element of $M_{2\times 2}$ such that $B^T = B$. Find an A in $M_{2\times 2}$ such that $T(A) = B$.

c. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.

d. Describe the kernel of T.

34. (*Calculus required*) Define $T : C[0, 1] \to C[0, 1]$ as follows: For $\mathbf{f}$ in $C[0, 1]$, let $T(\mathbf{f})$ be the antiderivative $\mathbf{F}$ of $\mathbf{f}$ such that $\mathbf{F}(0) = 0$. Show that T is a linear transformation, and describe the kernel of T. (See the notation in Exercise 20 of Section 4.1.)

35. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\mathbf{x})$, where $\mathbf{x}$ is in U. Show that $T(U)$ is a subspace of W.

36. Given $T : V \to W$ as in Exercise 35, and given a subspace Z of W, let U be the set of all $\mathbf{x}$ in V such that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V.

37. [**M**] Determine whether $\mathbf{w}$ is in the column space of A, the null space of A, or both, where

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

38. [**M**] Determine whether $\mathbf{w}$ is in the column space of A, the null space of A, or both, where

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix}$$

39. [**M**] Let $\mathbf{a}_1, \dots, \mathbf{a}_5$ denote the columns of the matrix A, where

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}, \quad B = [\, \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_4 \,]$$

a. Explain why $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B.

b. Find a set of vectors that spans Nul A.

c. Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Explain why T is neither one-to-one nor onto.

40. [**M**] Let $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ and $K = \text{Span}\,\{\mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ -12 \\ -28 \end{bmatrix}.$$

Then H and K are subspaces of $\mathbb{R}^3$. In fact, H and K are planes in $\mathbb{R}^3$ through the origin, and they intersect in a line through $\mathbf{0}$. Find a nonzero vector $\mathbf{w}$ that generates that line. [*Hint:* $\mathbf{w}$ can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and also as $c_3\mathbf{v}_3 + c_4\mathbf{v}_4$. To build $\mathbf{w}$, solve the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_3\mathbf{v}_3 + c_4\mathbf{v}_4$ for the unknown c_j's.]

SG Mastering: Vector Space, Subspace, Col A, and Nul A 4–6

SOLUTIONS TO PRACTICE PROBLEMS

1. *First method:* W is a subspace of $\mathbb{R}^3$ by Theorem 2 because W is the set of all solutions to a system of homogeneous linear equations (where the system has only one equation). Equivalently, W is the null space of the 1×3 matrix $A = [\, 1 \quad -3 \quad -1 \,]$.

the spanning property.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ | A basis for $\mathbb{R}^3$ | Spans $\mathbb{R}^3$ but is linearly dependent ■

PRACTICE PROBLEMS

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-3\\4\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}6\\2\\-1\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}2\\-2\\3\end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix}-4\\-8\\9\end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \text{ in } \mathbb{R}\right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

SG Mastering: Basis 4–9

4.3 EXERCISES

Determine whether the sets in Exercises 1–8 are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$

2. $\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}$

3. $\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}3\\1\\-4\end{bmatrix},\begin{bmatrix}-2\\-1\\1\end{bmatrix}$

4. $\begin{bmatrix}2\\-1\\1\end{bmatrix},\begin{bmatrix}2\\-3\\2\end{bmatrix},\begin{bmatrix}-8\\5\\4\end{bmatrix}$

5. $\begin{bmatrix}3\\-3\\0\end{bmatrix},\begin{bmatrix}-3\\7\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\-3\\5\end{bmatrix}$

6. $\begin{bmatrix}1\\2\\-4\end{bmatrix},\begin{bmatrix}-4\\3\\6\end{bmatrix}$

7. $\begin{bmatrix}-2\\3\\0\end{bmatrix},\begin{bmatrix}6\\-1\\5\end{bmatrix}$

8. $\begin{bmatrix}1\\-2\\3\end{bmatrix},\begin{bmatrix}0\\3\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\5\end{bmatrix},\begin{bmatrix}0\\0\\-1\end{bmatrix}$

Find bases for the null spaces of the matrices given in Exercises 9 and 10. Refer to the remarks that follow Example 3 in Section 4.2.

9. $\begin{bmatrix}1&0&-2&-2\\0&1&1&4\\3&-1&-7&3\end{bmatrix}$

10. $\begin{bmatrix}1&1&-2&1&5\\0&1&0&-1&-2\\0&0&-8&0&16\end{bmatrix}$

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$. [*Hint:* Think of the equation as a "system" of homogeneous equations.]

12. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A and Col A.

13. $A = \begin{bmatrix}-2&4&-2&-4\\2&-6&-3&1\\-3&8&2&-3\end{bmatrix}$, $B = \begin{bmatrix}1&0&6&5\\0&2&5&3\\0&0&0&0\end{bmatrix}$