MATH 2270: Final Exam Practice Problems

The following are practice problems for the final exam covering material that we have discussed since the third midterm. These problems *do not* cover all of the material that is fair game on the final. A good place to find practice problems for earlier material is: Midterm study guides, Midterms, Food for Thought assignments.

1. Decide whether each of the following statements is true or False. Write a sentence justifying your answer.
   (a) If $\vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$ and $\vec{v} \cdot \vec{w} = 0$, then either $\vec{v} = 0$ or $\vec{w} = 0$. False
   (b) Every orthogonal set in $\mathbb{R}^n$ is linearly independent. True
   (c) If the columns of a square matrix U form an orthogonal set, then $U^{-1} = U^T$. False
   (d) Let U be a matrix with orthonormal columns $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$. If $\vec{v} \cdot \vec{u}_i = 0$ for $i = 1, 2, \ldots, p$, then $\vec{v} \in \operatorname{nul} U^T$. True
   (e) If $\vec{v}$ is in a subspace W, then the orthogonal projection of $\vec{v}$ onto W is the vector $\vec{v}$. True
   (f) If $\vec{x}$ is not in a subspace W, then $\operatorname{proj}_W \vec{x}$ is not zero. False
   (g) Any solution of $A^T A\vec{x} = A^T\vec{b}$ is a least squares-solution of $A\vec{x} = \vec{b}$. True
   (h) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric. True
   (i) The principal axes of a quadratic form $\vec{x}^T A\vec{x}$ are eigenvectors of A. True
   (j) If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\vec{x}^T A\vec{x}$ is positive definite. True
   (k) If A is a 2×2 symmetric matrix, then the set of $\vec{x}$ such that $\vec{x}^T A\vec{x} = c$ (for a constant c) corresponds to either a circle, an ellipse, or a hyperbola. True

2. Inventions:
   (a) Invent a basis for the orthogonal complement of $W = \operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$. Possible answer: $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$
   (b) Invent a vector v such that $\operatorname{proj}_W \vec{v} = \vec{0}$, where W is the subspace from part (a). Possible answer: $\begin{bmatrix}1\\0\\-1\end{bmatrix}$
   (c) Invent a matrix U such that $U^T U = I$. Possible answer:
   $$U = \begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\ \sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}$$
   (d) Consider the vector space $C[0,1]$ of continuous functions $f\colon [0,1] \to \mathbb{R}$ with the inner product defined by $\langle f(t), g(t)\rangle = \int_0^1 fg dt$. Invent two non-zero functions (i.e., vectors) that are orthogonal. *Hint: One approach might be to start with any functions at all and use projections to modify one of them to make the pair orthogonal.* Possible answer: $f(t) = t$ and $g(t) = 3t - 2$

(e) Invent a square matrix A that is orthogonally diagonalizable. Possible answer:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(f) Invent a symmetric square matrix A so that the quadratic form $\vec{x}^T A\vec{x}$ is indefinite. Possible answer:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

3. Let $v = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ and $w = \left[\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right]$. Find a basis for the orthogonal complement of $\text{span}\{v, w\}$. Possible answer: $\left[\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right]$

4. Compute the orthogonal projection of $\left[\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right]$ onto $W = \text{span}\{v, w\}$ where $v = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ and $w = \left[\begin{smallmatrix} 2 \\ -1 \\ 0 \end{smallmatrix}\right]$. Answer: $10/14\vec{v} + 4/5\vec{w}$

5. §6.2 # 9

6. §6.3 # 9, 10

7. §6.4 # 3, 4

8. §6.5 # 10

9. §6.6 # 2, 4

10. §6.7 # 25 (*There will be a question where you have to use an inner product defined by an integral on a vector space of functions.*)

11. §7.1 # 18

12. Consider the quadratic form $\vec{x}^T A\vec{x}$ where $A = \left[\begin{smallmatrix} -3 & 4 \\ 4 & 3 \end{smallmatrix}\right]$. Determine whether A is positive definite, negative definite, indefinite, or none of the above. Answer: The quadratic form is indefinite.

13. §7.4 # 10

14. Let A be an $m \times n$ matrix, and let $A = U\Sigma V^T$ be a singular value decomposition of A. Write a paragraph (that means use complete sentences) explaining the meaning of the matrix factorization $A = U\Sigma V^T$ as it relates to the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ determined by the matrix A.