Also, statement (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d). Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete. ■

SG   Expanded Table
       for the IMT 2–39

NUMERICAL NOTES

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$

is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

WEB

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A, to be discussed in Section 7.4.

PRACTICE PROBLEMS

1. Determine the dimension of the subspace H of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (First, find a basis for H.)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix}$$

2. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ .2 \end{bmatrix}, \begin{bmatrix} .2 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$. If $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, what is $\mathbf{x}$?

3. Could $\mathbb{R}^3$ possibly contain a four-dimensional subspace? Explain.

2.9 EXERCISES

In Exercises 1 and 2, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$. Illustrate your answer with a figure, as in the solution of Practice Problem 2.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

In Exercises 3–6, the vector $\mathbf{x}$ is in a subspace H with a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.
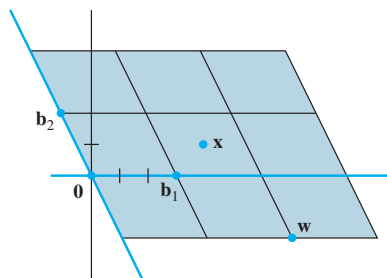
3. $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

4. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$

5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$

7. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{w}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{B}$. Confirm your estimate of $[\mathbf{x}]_\mathcal{B}$ by using it and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{x}$.

8. Let $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Use the figure to estimate $[\mathbf{x}]_\mathcal{B}$, $[\mathbf{y}]_\mathcal{B}$, and $[\mathbf{z}]_\mathcal{B}$. Confirm your estimates of $[\mathbf{y}]_\mathcal{B}$ and $[\mathbf{z}]_\mathcal{B}$ by using them and $\{\mathbf{b}_1, \mathbf{b}_2\}$ to compute $\mathbf{y}$ and $\mathbf{z}$.
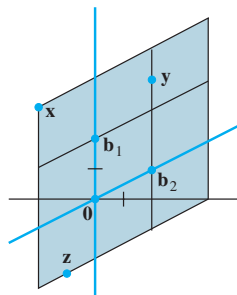
Exercises 9–12 display a matrix A and an echelon form of A. Find bases for Col A and Nul A, and then state the dimensions of these subspaces.

9. $A = \begin{bmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -6 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & 4 & -5 & 2 & -3 \\ 3 & 6 & -8 & 3 & -5 \\ 0 & 0 & 9 & 0 & 9 \\ -3 & -6 & -7 & -3 & -10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -5 & 1 & -4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & -4 & 4 & 6 \\ 5 & 1 & -9 & 2 & 10 \\ 4 & 6 & -9 & 12 & 15 \\ 3 & 4 & -5 & 8 & 9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 13 and 14, find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

13. $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ -1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -7 \\ 6 \\ -9 \end{bmatrix}$

15. Suppose a 4×6 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

16. Suppose a 4×7 matrix A has three pivot columns. Is Col $A = \mathbb{R}^3$? What is the dimension of Nul A? Explain your answers.

In Exercises 17 and 18, mark each statement True or False. Justify each answer. Here A is an $m \times n$ matrix.

17. a. If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for a subspace H and if $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then $c_1, \ldots, c_p$ are the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$.

   b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

   c. The dimension of Col A is the number of pivot columns in A.

   d. The dimensions of Col A and Nul A add up to the number of columns in A.

   e. If a set of p vectors spans a p-dimensional subspace H of $\mathbb{R}^n$, then these vectors form a basis for H.

18. a. If $\mathcal{B}$ is a basis for a subspace H, then each vector in H can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.

   b. The dimension of Nul A is the number of variables in the equation $A\mathbf{x} = \mathbf{0}$.

   c. The dimension of the column space of A is rank A.

d. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then the correspondence $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ makes H look and act the same as $\mathbb{R}^p$.

e. If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.

In Exercises 19–24, justify each answer or construction.

19. If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?

20. What is the rank of a 6×8 matrix whose null space is three-dimensional?

21. If the rank of a 9×8 matrix A is 7, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

22. Show that a set $\{\mathbf{v}_1, \dots, \mathbf{v}_5\}$ in $\mathbb{R}^n$ is linearly dependent if $\dim \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_5\} = 4$.

23. If possible, construct a 3×5 matrix A such that $\dim \text{Nul } A = 3$ and $\dim \text{Col } A = 2$.

24. Construct a 3×4 matrix with rank 1.

25. Let A be an $n \times p$ matrix whose column space is p-dimensional. Explain why the columns of A must be linearly independent.

26. Suppose columns 1, 3, 4, 5, and 7 of a matrix A are linearly independent (but are not necessarily pivot columns) and the rank of A is 5. Explain why the five columns mentioned must be a basis for the column space of A.

27. Suppose vectors $\mathbf{b}_1, \dots, \mathbf{b}_p$ span a subspace W, and let $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ be any set in W containing more than p vectors. Fill in the details of the following argument to show that $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ must be linearly dependent. First, let $B = [\, \mathbf{b}_1 \ \ \cdots \ \ \mathbf{b}_p \,]$ and $A = [\, \mathbf{a}_1 \ \ \cdots \ \ \mathbf{a}_q \,]$.

a. Explain why for each vector $\mathbf{a}_j$, there exists a vector $\mathbf{c}_j$ in $\mathbb{R}^p$ such that $\mathbf{a}_j = B\mathbf{c}_j$.

b. Let $C = [\, \mathbf{c}_1 \ \ \cdots \ \ \mathbf{c}_q \,]$. Explain why there is a nonzero vector $\mathbf{u}$ such that $C\mathbf{u} = \mathbf{0}$.

c. Use B and C to show that $A\mathbf{u} = \mathbf{0}$. This shows that the columns of A are linearly dependent.

28. Use Exercise 27 to show that if $\mathcal{A}$ and $\mathcal{B}$ are bases for a subspace W of $\mathbb{R}^n$, then $\mathcal{A}$ cannot contain more vectors than $\mathcal{B}$, and, conversely, $\mathcal{B}$ cannot contain more vectors than $\mathcal{A}$.

29. **[M]** Let $H = \text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, when

$$\mathbf{v}_1 = \begin{bmatrix} 15 \\ -5 \\ 12 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 14 \\ -10 \\ 13 \\ 17 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 16 \\ 0 \\ 11 \\ -3 \end{bmatrix}$$

30. **[M]** Let $H = \text{Span} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that $\mathcal{B}$ is a basis for H and $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, when

$$\mathbf{v}_1 = \begin{bmatrix} -6 \\ 3 \\ -9 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 0 \\ 7 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -9 \\ 4 \\ -8 \\ 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 11 \\ -2 \\ 17 \\ -8 \end{bmatrix}$$

SG   Mastering: Dimension and Rank 2–41
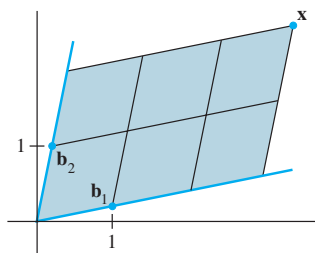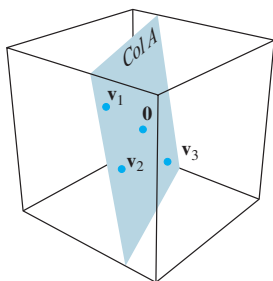
SOLUTIONS TO PRACTICE PROBLEMS

1. Construct $A = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$ so that the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the column space of A. A basis for this space is provided by the pivot columns of A.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two columns of A are pivot columns and form a basis for H. Thus $\dim H = 2$.

2. If $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x}$ is formed from a linear combination of the basis vectors using weights 3 and 2:

$$\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3\begin{bmatrix} 1 \\ .2 \end{bmatrix} + 2\begin{bmatrix} .2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 2.6 \end{bmatrix}$$

The basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ determines a *coordinate system* for $\mathbb{R}^2$, illustrated by the grid in the figure. Note how $\mathbf{x}$ is 3 units in the $\mathbf{b}_1$-direction and 2 units in the $\mathbf{b}_2$-direction.

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there. We have

$$\det A = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

This 3×3 determinant was computed in Example 1 and found to equal -2. Hence $\det A = 3 \cdot 2 \cdot (-2) = -12$. ∎

The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem.

THEOREM 2 If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and 25! is approximately 1.5×10^{25}.

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the 2×2 case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for $n \times n$ matrices.

PRACTICE PROBLEM

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

9. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$

10. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:

Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. *Warning: This trick does not generalize in any reasonable way to 4×4 or larger matrices.*

15. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

18. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

21. $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$

22. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}, \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$

24. $\begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$

27. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

35. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39. a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

   b. The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

40. a. The cofactor expansion of $\det A$ down a column is the negative of the cofactor expansion along a row.

b. The determinant of a triangular matrix is the sum of the entries on the main diagonal.

41. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\,\mathbf{u} \quad \mathbf{v}\,]$. How do they compare? Replace the first entry of $\mathbf{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

42. Let $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, c are positive (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrices $[\,\mathbf{u} \quad \mathbf{v}\,]$ and $[\,\mathbf{v} \quad \mathbf{u}\,]$. Draw a picture and explain what you find.

43. **[M]** Is it true that $\det(A + B) = \det A + \det B$? To find out, generate random 5×5 matrices A and B, and compute $\det(A + B) - \det A - \det B$. (Refer to Exercise 37 in Section 2.1.) Repeat the calculations for three other pairs of $n \times n$ matrices, for various values of n. Report your results.

44. **[M]** Is it true that $\det AB = (\det A)(\det B)$? Experiment with four pairs of random matrices as in Exercise 43, and make a conjecture.

45. **[M]** Construct a random 4×4 matrix A with integer entries between -9 and 9, and compare $\det A$ with $\det A^T$, $\det(-A)$, $\det(2A)$, and $\det(10A)$. Repeat with two other random 4×4 integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random 5×5 and 6×6 integer matrices. Modify your conjectures, if necessary, and report your results.

46. **[M]** How is $\det A^{-1}$ related to $\det A$? Experiment with random $n \times n$ integer matrices for $n = 4, 5$, and 6, and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

SOLUTION TO PRACTICE PROBLEM

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2 \cdot (-1)^{2+1}(-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The $(-1)^{2+1}$ in the next-to-last calculation came from the $(2, 1)$-position of the -5 in the 3×3 determinant.

3.2 PROPERTIES OF DETERMINANTS

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

THEOREM 3   **Row Operations**

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

The following examples show how to use Theorem 3 to find determinants efficiently.

2. Use a determinant to decide if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

3.2 EXERCISES

Each equation in Exercises 1–4 illustrates a property of determinants. State the property.

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}$

Find the determinants in Exercises 5–10 by row reduction to echelon form.

5. $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants in Exercises 11–14.

11. $\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$

14. $\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$

Find the determinants in Exercises 15–20, where

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$

17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

18. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

19. $\begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{vmatrix}$

20. $\begin{vmatrix} a + d & b + e & c + f \\ d & e & f \\ g & h & i \end{vmatrix}$

In Exercises 21–23, use determinants to find out if the matrix is invertible.

21. $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

In Exercises 24–26, use determinants to decide if the set of vectors is linearly independent.

24. $\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$

25. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}$

In Exercises 27 and 28, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

27. a. A row replacement operation does not affect the determinant of a matrix.

b. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.

   c. If the columns of A are linearly dependent, then $\det A = 0$.

   d. $\det(A + B) = \det A + \det B$.

28. a. If two row interchanges are made in succession, then the new determinant equals the old determinant.

   b. The determinant of A is the product of the diagonal entries in A.

   c. If $\det A$ is zero, then two rows or two columns are the same, or a row or a column is zero.

   d. $\det A^T = (-1)\det A$.

29. Compute $\det B^5$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

30. Use Theorem 3 (but not Theorem 4) to show that if two rows of a square matrix A are equal, then $\det A = 0$. The same is true for two columns. Why?

In Exercises 31–36, mention an appropriate theorem in your explanation.

31. Show that if A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

32. Find a formula for $\det(rA)$ when A is an $n \times n$ matrix.

33. Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det AB = \det BA$.

34. Let A and P be square matrices, with P invertible. Show that $\det(PAP^{-1}) = \det A$.

35. Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

36. Suppose that A is a square matrix such that $\det A^4 = 0$. Explain why A cannot be invertible.

Verify that $\det AB = (\det A)(\det B)$ for the matrices in Exercises 37 and 38. (Do not use Theorem 6.)

37. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

38. $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$

39. Let A and B be 3×3 matrices, with $\det A = 4$ and $\det B = -3$. Use properties of determinants (in the text and in the exercises above) to compute:

   a. $\det AB$ b. $\det 5A$ c. $\det B^T$

   d. $\det A^{-1}$ e. $\det A^3$

40. Let A and B be 4×4 matrices, with $\det A = -1$ and $\det B = 2$. Compute:

   a. $\det AB$ b. $\det B^5$ c. $\det 2A$

   d. $\det A^T A$ e. $\det B^{-1} AB$

41. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad C = \begin{bmatrix} e & f \\ c & d \end{bmatrix}$$

42. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

43. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix},$$

$$B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

Note, however, that A is *not* the same as $B + C$.

44. Right-multiplication by an elementary matrix E affects the *columns* of A in the same way that left-multiplication affects the *rows*. Use Theorems 5 and 3 and the obvious fact that E^T is another elementary matrix to show that

$$\det AE = (\det E)(\det A)$$

Do not use Theorem 6.

45. [M] Compute $\det A^T A$ and $\det AA^T$ for several random 4×5 matrices and several random 5×6 matrices. What can you say about $A^T A$ and AA^T when A has more columns than rows?

46. [M] If $\det A$ is close to zero, is the matrix A nearly singular? Experiment with the nearly singular 4×4 matrix A in Exercise 9 of Section 2.3. Compute the determinants of A, $10A$, and $0.1A$. In contrast, compute the condition numbers of these matrices. Repeat these calculations when A is the 4×4 identity matrix. Discuss your results.

SOLUTIONS TO PRACTICE PROBLEMS

1. Perform row replacements to create zeros in the first column and then create a row of zeros.

$$\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$