Research Statement

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My research interest is in geometric group theory, a field which aims to understand the large-scale geometry of finitely generated groups using topology, geometry, and dynamics. The main focus of my research has been the study of *finite rank free groups*, F_n for $n \geq 2$, and their automorphisms. In addition to being fundamental objects in group theory, free groups serve as a model for all hyperbolic groups: a broad class of groups that have been extensively studied since their advent in the work of Gromov [Gro87], who built on work of Dehn and Riemann. A general principle in mathematics holds that "to understand an object, one must understand its self-maps." For finite dimensional vector spaces, this leads to the study of linear groups; for surfaces, it leads to mapping class groups; and for free groups, it leads to $\mathrm{Out}(F_n)$, the group of outer autorphisms of a free group.

Much of the study of $\mathrm{Out}(F_n)$ draws parallels with the mapping class group, which is defined as the group of orientation preserving homeomorphisms of a finite-type surface up to isotopy. Furthermore, many theorems about $\mathrm{Out}(F_n)$ and their proofs are inspired by analogous theorems and proofs in the context of mapping class groups. Both groups are known to satisfy the Tits alternative [McC85, BFH00], both have finite virtual cohomological dimension [Har86, CV86], and both have Serre's property FA to name a few. In my doctoral work, I have focused on three projects that fall under the broad themes of better understanding the coarse geometry of $\mathrm{Out}(F_n)$ and further developing the analogy with mapping class groups of surfaces.

Abelian subgroups of $\mathrm{Out}(F_n)$: In my thesis, I prove that abelian subgroups of $\mathrm{Out}(F_n)$ are undistorted; roughly, this means that the intrinsic geometry of these subgroups agrees with the geometry of the ambient group. A corollary of this result is the geometric rank conjecture for $\mathrm{Out}(F_n)$. This project makes use of some recent developments in the theory of train tracks [FH11] and is inspired by the work of Alibegović [Ali02] and Farb-Lubotzky-Minsky [FLM01].

Loxodromic elements in the cyclic splitting complex: In joint work with R. Gupta, I am working to study $\mathrm{Out}(F_n)$ through its action on several geometric spaces. We have identified the elements of $\mathrm{Out}(F_n)$ that act with positive translation length on the cyclic splitting complex (denoted by $\mathcal{FZ}$), a Gromov-hyperbolic complex on which $\mathrm{Out}(F_n)$ acts. We can also classify the isometry type of outer automorphisms acting on $\mathcal{FZ}$.

Farrell-Jones conjecture for free-by-cyclic groups: In joint work with M. Bestvina and K. Fujiwara, we make use of topological methods to prove the K and L theoretic Farrell-Jones conjectures (FJC) for free-by-cyclic groups. We also provide a new proof of the unpublished result, originally due to Gautero-Lustig [GL07], that the mapping torus of a free group automorphism is hyperbolic relative to a certain collection of subgroups of polynomial growth.

While each of these projects has a different flavor, there are two common threads: they all involve $\mathrm{Out}(F_n)$ and each of them uses recent advances in the theory of train track maps, developed by Feighn-Handel [FH11], in a critical way. Specific plans for my research program are outlined below, but I expect that this new tool, called *completely split train tracks*, will continue to bear fruit and yield more results in the future.

1. Subgroup Distortion

Gromov first introduced the notion of a quasi-isometry in order to study finitely generated groups as geometric objects. One way we can understand the geometry of a group G is to study subgroups of G and to compare their intrinsic geometry with the geometry they inherit from G. A subgroup $H \leq G$ is said to be *undistorted* if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding. Results about distortion of particular classes of subgroups of a group G can be used to glean information about G. For example, they can be used to prove results on the Dehn function of G, the automaticity of G, the maximal rank of a (quasi-isometrically) embedded flat in G, and on embeddings of other groups in G. In my thesis, I prove

Theorem 1.1 ([Wig16]). *If $H \leq \mathrm{Out}(F_n)$ is abelian, then H is undistorted.*

The geometric rank of a group G is defined as the maximal rank of a quasi-isometrically embedded copy of $\mathbb{R}^k$ in G. As a corollary, we obtain:

Corollary 1.1 ([Wig16]). *The geometric rank of $\mathrm{Out}(F_n)$ is $2n-3$, which is the maximal rank of an abelian subgroup of $\mathrm{Out}(F_n)$.*

The proof of Theorem 1.1 has two main parts. The first involves a study of the action of $\mathrm{Out}(F_n)$ on *Culler and Vogtmann's outer space*, CV_n [CV86]. Outer space plays the role for $\mathrm{Out}(F_n)$ that Teichmüller space plays for the mapping class group. It is equipped with an $\mathrm{Out}(F_n)$-invariant (asymmetric) metric [FM11] defined in analogy with Thurston's metric on Teichmüller space. My proof relies on understanding translation lengths of outer automorphisms acting on CV_n.

The second ingredient in the proof of Theorem 1.1 involves a careful study of cancellation of paths in representatives of outer automorphisms as graph maps. This uses developments in the theory of train tracks for outer automorphisms (see [BH92, BFH00, BFH05, FH11, FH09, FH14]), which are a sort of normal form for elements of $\mathrm{Out}(F_n)$. The representatives with the best properties are called *completely split train track maps* (or CTs). I prove an analog of the Kolchin Theorem of [BFH05] for abelian subgroups. Namely,

Theorem 1.2 ([Wig16]). *For any abelian subgroup H of $\mathrm{Out}(F_n)$, there exists a finite index subgroup H' such that every $\phi \in H'$ can be realized as a CT on one of finitely many marked graphs.*

Research Proposal 1. Much research has been devoted to understanding the properties of the previously mentioned metric on outer space [HM11, MP16, AK11, AKP17a, QR17]. If g is an infinite order isometry of a metric space X, the *translation distance* of g is defined as $\tau(g) = \lim_{n\to\infty} d(x, gx)$ for some (any) basepoint $x \in X$. Translation distances for outer automorphisms acting on CV_n are well understood [Wad12, Wig16]. For elements that act with positive translation distance, the *minset of g* is defined as $m(g) = \{x \in X \mid d(x, gx) = \tau(g)\}$. It's known that fully irreducible automorphisms have non-empty minset, but these are not the only ones. Many examples of reducible automorphisms with non-empty minset can be constructed, but the following question is open:

Question 1.1. *Which elements of $\mathrm{Out}(F_n)$ have non-empty minset?*

Polynomial growth outer automorphisms (with infinite order) necessarily have empty minset. Each automorphism with exponential growth has a (non-empty) finite set of laminations, $\mathcal{L}(\phi)$. One possible approach to Question 1.1 is by using a leaf of an appropriate $\Lambda \in \mathcal{L}(\phi)$. In certain cases, running Whitehead's algorithm yields an honest train track map (an element of the minset). When Whitehead's algorithm fails to find a train track map, it may be possible to use iterative arguments to make conclusions about the limiting behavior of this procedure, which can in turn give information about whether or not ϕ has empty minset.

Research Proposal 2. There is a long history of quasi-isometric rigidity theorems in geometric group theory (see [BKMM12, GP91, KL97] et al). Theorem 1.1 is an important step toward obtaining a quasi-isometric rigidity theorem for $\mathrm{Out}(F_n)$ and such a result would constitute a major advance for $\mathrm{Out}(F_n)$. However, several of the ingredients necessary for such a proof in the $\mathrm{Out}(F_n)$ setting are missing, foremost of which is a systematic study of asymptotic cones of $\mathrm{Out}(F_n)$ along the lines of [BM08].

Question 1.2. *Do asymptotic cones of* $\mathrm{Out}(F_n)$ *contain embedded lines? Do they have a tree-graded structure?*

There is reason to be hopeful, despite the fact that of some of the tools used to answer these questions for mapping class groups are lacking in the $\mathrm{Out}(F_n)$ setting. For example, it follows from the fact that $\mathrm{Out}(F_n)$ is acylindrically hyperbolic that all asymptotic cones of $\mathrm{Out}(F_n)$ have cut-points [Sis16]. The perspective of acylindrically hyperbolic groups may provide an avenue to answer Question 1.2, allowing one to circumvent the aforementioned problems.

2. Curve complex analogs

Groups and spaces that are Gromov hyperbolic, a measure of large-scale negative curvature, play a central role in geometric group theory. One way to study the geometry of a group G is to understand the spaces on which G acts. A common theme, and one that has been explored extensively in the past twenty years, is that many groups which are not themselves Gromov hyperbolic nevertheless admit actions on Gromov hyperbolic spaces. From these actions, one can deduce ways in which G "is like a hyperbolic group." Examples of this phenomenon include the mapping class group acting on the curve complex, fundamental groups of $\mathrm{CAT}(0)$ cube complexes acting on the contact graph, and many others. $\mathrm{Out}(F_n)$ acts on several Gromov hyperbolic spaces, and this project has been focused at better understanding these actions.

The free splitting complex $\mathcal{FS}$, the cyclic splitting complex $\mathcal{FZ}$, and the free factor complex $\mathcal{FF}$ are all Gromov hyperbolic spaces on which $\mathrm{Out}(F_n)$ acts by isometries. These complexes have been used to understand subgroups of $\mathrm{Out}(F_n)$, obtain results about bounded cohomology, to develop analogs of convex cocompact subgroups, and to better understand the large-scale geometry of the group.

The isometries of a Gromov hyperbolic space X come in three flavors: bounded (those for which every orbit is bounded), parabolic (those which fix a unique point in the boundary of X), and loxodromic. Loxodromic isometries are the most dynamically interesting in that fix exactly two points in the boundary of X and act with north-south dynamics on $\overline{X}$. As such, they serve as a basis for many geometric group-theoretic arguments: they can be used as the basis for ping-pong type arguments, to establish an H^2_b-alternative for subgroups of G, or to define analogs of convex cocompact subgroups. In [Man14], Mann defined the cyclic splitting complex and proved its hyperbolicity. He then asked: Which elements of $\mathrm{Out}(F_n)$ act loxodromically on $\mathcal{FZ}$?

R. Gupta and I provide an answer to this question. Each outer automorphism ϕ has a finite set of *laminations*, $\mathcal{L}(\phi)$. A lamination (or set of laminations) is said to fill if it is not contained in a proper free factor of F_n. We prove

Theorem 2.1 ([GW17]). *If $\phi \in \mathrm{Out}(F_n)$, then ϕ acts loxodromically on $\mathcal{FZ}$ if and only if one of the following equivalent conditions is satisfied:*
- *ϕ has a filling lamination Λ and no generic leaf of Λ is carried by a vertex group of a cyclic splitting.*
- *ϕ has a filling lamination Λ and ϕ has virtually cyclic centralizer.*

The idea of our proof is to use the boundary of $\mathcal{FZ}$, which was recently identified by Horbez as the set of $\mathcal{Z}$-averse trees [Hor14]. There is a coarsely well defined $\mathrm{Out}(F_n)$ equivariant map from *outer space*, CV_n, to the cyclic splitting complex. Starting with a particular tree T in outer space, one would like to iterate the automorphism ϕ and then argue that the projection of $T_\infty = \lim_{k\to\infty} \phi^k \cdot T$ to $\mathcal{FZ}$ is $\mathcal{Z}$-averse. Because this approach sometimes yields "the wrong tree," we develop folding paths in the boundary of outer space as a way to gain access to "the right tree." After getting some information about the tree of interest, we use the theory of JSJ-decompositions to prove that if the tree is not $\mathcal{Z}$-averse, then a power of ϕ must fix a cyclic splitting.

To prove the second equivalence in Theorem 2.1, we use *completely split train track maps* to execute a careful analysis of the leaves of the topmost lamination. We then show that the *disintegration of* ϕ, $\mathcal{D}(\phi)$, (an abelian subgroup associated to the outer automorphism [FH09]) must be virtually cyclic. In general, this would not be sufficient to conclude that centralizers are virtually cyclic; the general problem of understanding centralizers of elements of $\mathrm{Out}(F_n)$ is considered difficult, and only sporadic progress has been made [AKP17b, RW15]. In this case, however, we are able to use the action on $\mathcal{FZ}$ make conclusions about the centralizers of loxodromic elements.

Research Proposal. A group G is said to be *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a Gromov hyperbolic space. This astonishingly large class of groups includes: hyperbolic and relatively hyperbolic groups, most mapping class groups, $\mathrm{Out}(F_n)$ for $n \geq 2$, right angled Artin groups, and one-relator groups, among many others. Furthermore, many results are known for the class of acylindrically hyperbolic groups: simplicity of the reduced C^*-algebra, results about cut points in the asymptotic cone, and results about random walks on G, and others [Osi16]. While $\mathrm{Out}(F_n)$ is known to be acylindrically hyperbolic, there are no known natural witnesses to this fact. The following questions are open.

Question 2.1. *Is the action of* $\mathrm{Out}(F_n)$ *on* $\mathcal{FZ}$ *acylindrical? How about the action on* $\mathcal{FF}$*?*

In a project with C. Abbott and R. Gupta, I am working to resolve this question. In [BF02], the authors define a generalization of proper discontinuity that they call *weak proper discontinuity* (or *WPD*) for a group action on a hyperbolic space, $G \curvearrowright X$. Roughly speaking, an action that is uniformly WPD is acylindrically hyperbolic; in this way, studying WPD actions can be seen as a stepping stone toward resolving Question 2.1. The authors prove that the mapping class group action on the curve complex, which was proved to be hyperbolic in the seminal work of Masur-Minsky [MM99], satisfies WPD. As an application, they use this to obtain results about second bounded cohomology for subgroups of the mapping class group. In his thesis, Mann asked

Question 2.2 ([Man14]). *Does the action of* $\mathrm{Out}(F_n)$ *on* $\mathcal{FZ}$ *satisfy WPD?*

In continuing work with C. Abbott and R. Gupta, we will attempt to answer this question in the affirmative. In [BBF15], the authors introduce a still weaker notion, called WWPD, which implies weak proper discontinuity in the case that all of the loxodromic elements have virtually cyclic centralizers. Accordingly, the aforementioned result (Theorem 2.1) is an important first step toward resolving Mann's question. While the $\mathrm{Out}(F_n)$ action on the free splitting complex does not satisfy WWPD, restricting the action appropriately can yield WWPD actions [HM14]. My coauthors and I plan to use currents to give an alternative proof of this result following an outline in [Man14], which is based on arguments in [BF02]. We will then use similar techniques to attack Question 2.2.

3. Farrell-Jones Conjecture

Given a torsion-free discrete group G, the *K and L theoretic Farrell-Jones conjectures for G with integral coefficients* (or FJC, for short) are statements that connects the topology of spaces on which G acts (namely, the classifying space of G with respect to the family of virtually cyclic subgroups) to the structure of the group ring $\mathbb{Z}G$. The Farrell-Jones conjecture for such a group has significant consequences; in topology, it has strong implications for the homeomorphism types of manifolds whose fundamental group is G. As an example, if one takes G to be the trivial group (for which the FJC is known), then Farrell-Jones conjecture for G implies the Poincaré conjecture in dimensions ≥ 5. More generally, when G is non-trivial, the conjecture implies that any two closed aspherical n-manifolds with isomorphic fundamental group are in fact homeomorphic; this is known as the Borel conjecture. The Farrell-Jones conjecture for a group G also implies Serre's conjecture for G: if G is of type FP, then G is of type FF. Also notable is the fact that Farrell-Jones implies the rational Novikov Conjecture.

In a forthcoming paper, M. Bestvina, K. Fujiwara and I prove:

Theorem 3.1 ([BFW17])**.** *If G is a free-by-cyclic group, then G satisfies the Farrell-Jones Conjecture.*

Our proof makes use of recently discovered topological criteria for a group to satisfy FJC. Specifically, a theorem of Bartels [Bar17] establishes the Farrell-Jones conjecture for groups that are relatively hyperbolic, provided that the peripheral subgroups also satisfy FJC. This was used to prove FJC for mapping class groups [BB16]. The second ingredient in our proof is another topological criterion: if a group G acts acylindrically on a simplicial tree and point stabilizers satisfy FJC, then G satisfies FJC [Kno17]. The final ingredient in Theorem 3.1 is the following unpublished result, originally due to Gautero-Lustig.

Theorem 3.2 ([GL07])**.** *Let $\phi \in \mathrm{Aut}(F_n)$ be an automorphism with exponential growth and let $G_\phi = \langle F_n, t \mid tgt^{-1} = \phi(g), g \in F_n\rangle$ be the mapping torus of ϕ. Then G_ϕ is relatively hyperbolic.*

Using completely split train tracks, my coauthors and I provide a new and simplified proof of this result.

Research Proposal. We expect that the topological criteria used to establish Theorem 3.1 will pave the way for the establishment of the Farrell-Jones Conjecture for broad classes of groups. I am currently exploring this possibility:

Question 3.1. *Can our techniques be used to prove FJC for mapping tori of automorphisms of hyperbolic groups? How about graphs of groups in which the vertex groups are free?*

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