EXAMPLE 10 A man standing on top of a vertical cliff is 200 feet above a lake. As he watches, a motorboat moves directly away from the foot of the cliff at a rate of 25 feet per second. How fast is the angle of depression of his line of sight changing when the boat is 150 feet from the foot of the cliff?

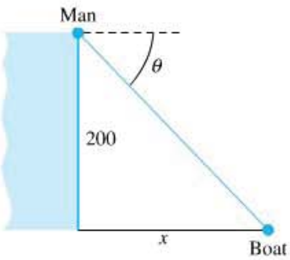


Figure 8

SOLUTION The essential details are shown in Figure 8. Note that θ, the angle of depression, is

$$\theta = \tan^{-1}\left(\frac{200}{x}\right)$$

Thus,

$$\frac{d\theta}{dt} = \frac{1}{1 + (200/x)^2} \cdot \frac{-200}{x^2} \cdot \frac{dx}{dt} = \frac{-200}{x^2 + 40{,}000} \cdot \frac{dx}{dt}$$

When we substitute $x = 150$ and $dx/dt = 25$, we obtain $d\theta/dt = -0.08$ radian per second. ■

Manipulating the Integrand Before you make a substitution, you may find it helpful to rewrite the integrand in a more convenient form. Integrals with quadratic expressions in the denominator can often be reduced to standard forms by *completing the square*. Recall that $x^2 + bx$ becomes a perfect square by the addition of $(b/2)^2$.

EXAMPLE 11 Evaluate $\displaystyle\int \frac{7}{x^2 - 6x + 25}\,dx$.

SOLUTION

$$\begin{aligned}\int \frac{7}{x^2 - 6x + 25}\,dx &= \int \frac{7}{x^2 - 6x + 9 + 16}\,dx \\ &= 7\int \frac{1}{(x-3)^2 + 4^2}\,dx \\ &= \frac{7}{4}\tan^{-1}\left(\frac{x-3}{4}\right) + C\end{aligned}$$

We made the mental substitution $u = x - 3$ at the final stage. ■

Concepts Review

1. To obtain an inverse for the sine function, we restrict its domain to _____. The resulting inverse function is denoted by $\sin^{-1}$ or by _____.

2. To obtain an inverse for the tangent function, we restrict the domain to _____. The resulting inverse function is denoted by $\tan^{-1}$ or by _____.

3. $D_x \sin(\arcsin x) =$ _____.

4. Since $D_x \arctan x = 1/(1 + x^2)$, it follows that $4\int_0^1 1/(1 + x^2)\,dx =$ _____.

Problem Set 6.8

In Problems 1–10, find the exact value without using a calculator.

1. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

3. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

4. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

5. $\arctan\left(\sqrt{3}\right)$

6. $\operatorname{arcsec}(2)$

7. $\arcsin\left(-\frac{1}{2}\right)$

8. $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

9. $\sin(\sin^{-1} 0.4567)$

10. $\cos(\sin^{-1} 0.56)$

In Problems 11–18, approximate each value.

11. $\sin^{-1}(0.1113)$ **12.** $\arccos(0.6341)$

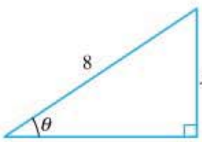
13. $\cos(\text{arccot } 3.212)$ **14.** $\sec(\arccos 0.5111)$

15. $\sec^{-1}(-2.222)$ **16.** $\tan^{-1}(-60.11)$

17. $\cos(\sin(\tan^{-1} 2.001))$ **18.** $\sin^2(\ln(\cos 0.5555))$

In Problems 19–24, express θ in terms of x using the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, *and* $\sec^{-1}$.

19.



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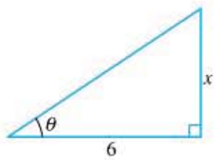

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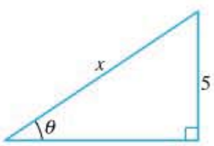

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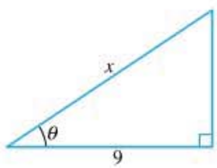

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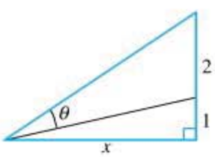

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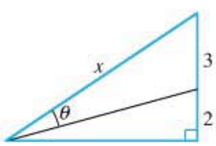


In Problems 25–28, find each value without using a calculator (see Example 4).

25. $\cos\left[2\sin^{-1}\left(-\frac{2}{3}\right)\right]$ **26.** $\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]$

27. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{5}{13}\right)\right]$

28. $\cos\left[\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right]$

In Problems 29–32, show that each equation is an identity.

29. $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$

30. $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1 + x^2}}$

31. $\cos(2\sin^{-1} x) = 1 - 2x^2$

32. $\tan(2\tan^{-1} x) = \dfrac{2x}{1 - x^2}$

33. Find each limit.

(a) $\lim_{x\to\infty} \tan^{-1} x$ (b) $\lim_{x\to-\infty} \tan^{-1} x$

34. Find each limit.

(a) $\lim_{x\to\infty} \sec^{-1} x$ (b) $\lim_{x\to-\infty} \sec^{-1} x$

35. Find each limit.

(a) $\lim_{x\to 1^-} \sin^{-1} x$ (b) $\lim_{x\to -1^+} \sin^{-1} x$

36. Does $\lim_{x\to 1} \sin^{-1} x$ exist? Explain.

37. Describe what happens to the slope of the tangent line to the graph of $y = \sin^{-1} x$ at the point c if c approaches 1 from the left.

38. Sketch the graph of $y = \cot^{-1} x$, assuming that it has been obtained by restricting the domain of the cotangent to $(0, \pi)$.

In Problems 39–54, find dy/dx.

39. $y = \ln(2 + \sin x)$ **40.** $y = e^{\tan x}$

41. $y = \ln(\sec x + \tan x)$ **42.** $y = -\ln(\csc x + \cot x)$

43. $y = \sin^{-1}(2x^2)$ **44.** $y = \arccos(e^x)$

45. $y = x^3 \tan^{-1}(e^x)$ **46.** $y = e^x \arcsin x^2$

47. $y = (\tan^{-1} x)^3$ **48.** $y = \tan(\cos^{-1} x)$

49. $y = \sec^{-1}(x^3)$ **50.** $y = (\sec^{-1} x)^3$

51. $y = (1 + \sin^{-1} x)^3$ **52.** $y = \sin^{-1}\left(\dfrac{1}{x^2 + 4}\right)$

53. $y = \tan^{-1}(\ln x^2)$ **54.** $y = x\,\text{arcsec}(x^2 + 1)$

In Problems 55–72, evaluate each integral.

55. $\int \cos 3x\,dx$ **56.** $\int x\sin(x^2)\,dx$

57. $\int \sin 2x \cos 2x\,dx$ **58.** $\int \tan x\,dx = \int \dfrac{\sin x}{\cos x}\,dx$

59. $\int_0^1 e^{2x}\cos(e^{2x})\,dx$ **60.** $\int_0^{\pi/2} \sin^2 x \cos x\,dx$

61. $\int_0^{\sqrt{2}/2} \dfrac{1}{\sqrt{1 - x^2}}\,dx$ **62.** $\int_{\sqrt{2}}^2 \dfrac{dx}{x\sqrt{x^2 - 1}}$

63. $\int_{-1}^1 \dfrac{1}{1 + x^2}\,dx$ **64.** $\int_0^{\pi/2} \dfrac{\sin\theta}{1 + \cos^2\theta}\,d\theta$

65. $\int \dfrac{1}{1 + 4x^2}\,dx$ **66.** $\int \dfrac{e^x}{1 + e^{2x}}\,dx$

67. $\int \dfrac{1}{\sqrt{12 - 9x^2}}\,dx$ **68.** $\int \dfrac{x}{\sqrt{12 - 9x^2}}\,dx$

69. $\int \dfrac{1}{x^2 - 6x + 13}\,dx$ **70.** $\int \dfrac{1}{2x^2 + 8x + 25}\,dx$

71. $\int \dfrac{1}{x\sqrt{4x^2 - 9}}\,dx$ **72.** $\int \dfrac{x + 1}{\sqrt{4 - 9x^2}}\,dx$

C **73.** A picture 5 feet in height is hung on a wall so that its bottom is 8 feet from the floor, as shown in Figure 9. A viewer with eye level at 5.4 feet stands b feet from the wall. Express θ, the vertical angle subtended by the picture at her eye, in terms of b, and then find θ if $b = 12.9$ feet.

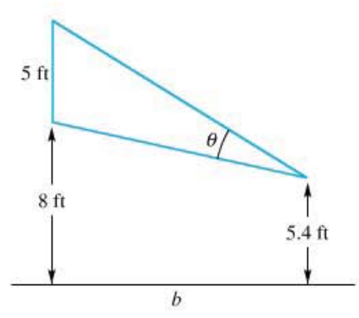


Figure 9

74. Find formulas for $f^{-1}(x)$ for each of the following functions f, first indicating how you would restrict the domain so that f has an inverse. For example, if $f(x) = 3\sin 2x$ and we restrict the domain to $-\pi/4 \le x \le \pi/4$, then $f^{-1}(x) = \frac{1}{2}\sin^{-1}(x/3)$.

(a) $f(x) = 3\cos 2x$ (b) $f(x) = 2\sin 3x$

(c) $f(x) = \frac{1}{2}\tan x$ (d) $f(x) = \sin \dfrac{1}{x}$

75. By repeated use of the addition formula

$$\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$$

show that

$$\frac{\pi}{4} = 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{5}{99}\right)$$

76. Verify that

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

a result discovered by John Machin in 1706 and used by him to calculate the first 100 decimal places of π.

77. Without using calculus, find a formula for the area of the shaded region in Figure 10 in terms of a and b. Note that the center of the larger circle is on the rim of the smaller.

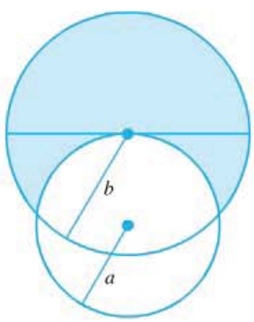


Figure 10

GC **78.** Draw the graphs of

$$y = \arcsin x \qquad \text{and} \qquad y = \arctan\left(x/\sqrt{1 - x^2}\right)$$

using the same axes. Make a conjecture. Prove it.

GC **79.** Draw the graph of $y = \pi/2 - \arcsin x$. Make a conjecture. Prove it.

GC **80.** Draw the graph of $y = \sin(\arcsin x)$ on $[-1, 1]$. Then draw the graph of $y = \arcsin(\sin x)$ on $[-2\pi, 2\pi]$. Explain the differences that you observe.

81. Show that

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, \qquad a > 0$$

by writing $a^2 - x^2 = a^2[1 - (x/a)^2]$ and making the substitution $u = x/a$.

82. Show the result in Problem 81 by differentiating the right side to get the integrand.

83. Show that

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad a \ne 0$$

84. Show that

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|x|}{a} + C, \qquad a > 0$$

85. Show, by differentiating the right side, that

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C, \qquad a > 0$$

86. Use the result of Problem 85 to show that

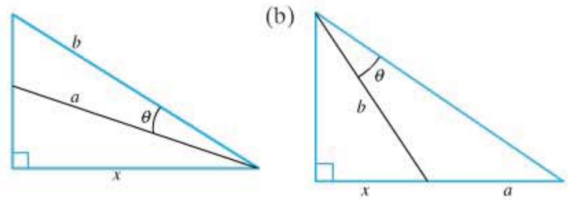
$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{2}$$

Why is this result expected?

87. The lower edge of a wall hanging, 10 feet in height, is 2 feet above the observer's eye level. Find the ideal distance b to stand from the wall for viewing the hanging; that is, find b that maximizes the angle subtended at the viewer's eye. (See Problem 73.)

88. Express $d\theta/dt$ in terms of x, dx/dt, and the constants a and b.

(a) (b)



89. The structural steel work of a new office building is finished. Across the street, 60 feet from the ground floor of the freight elevator shaft in the building, a spectator is standing and watching the freight elevator ascend at a constant rate of 15 feet per second. How fast is the angle of elevation of the spectator's line of sight to the elevator increasing 6 seconds after his line of sight passes the horizontal?

90. An airplane is flying at a constant altitude of 2 miles and a constant speed of 600 miles per hour on a straight course that will take it directly over an observer on the ground. How fast is the angle of elevation of the observer's line of sight increasing when the distance from her to the plane is 3 miles? Give your result in radians per minute.

91. A revolving beacon light is located on an island and is 2 miles away from the nearest point P of the straight shoreline of the mainland. The beacon throws a spot of light that moves along the shoreline as the beacon revolves. If the speed of the spot of light on the shoreline is 5π miles per minute when the spot is 1 mile from P, how fast is the beacon revolving?

92. A man on a dock is pulling in a rope attached to a rowboat at a rate of 5 feet per second. If the man's hands are 8 feet higher than the point where the rope is attached to the boat, how fast is the angle of depression of the rope changing when there are still 17 feet of rope out?

C **93.** A visitor from outer space is approaching the earth (radius = 6376 kilometers) at 2 kilometers per second. How fast is the angle θ subtended by the earth at her eye increasing when she is 3000 kilometers from the surface?

Answers to Concepts Review: **1.** $[-\pi/2, \pi/2]$; arcsin **2.** $(-\pi/2, \pi/2)$; arctan **3.** 1 **4.** π