MATH 2270: Final Exam Practice Problems

The following are practice problems for the final exam. Don't feel you need to do all of these problems to prepare for the final. These problems are just a representation of the material that is fair game on the final.

1. §1.5 # 17
2. §1.7 # 6, 27, 28
3. Let T be the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by counterclockwise rotation about the origin by $\pi/4$. Find the matrix for T with respect to the standard basis of $\mathbb{R}^2$.
4. Invent 2×2 matrices A and B such that $AB \neq BA$.
5. Compute the inverse of a 3×3 matrix. (e.g., §2.2 #31, 33, 34)
6. Write down the inverse of a 2×2 matrix.
7. Find an example of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ which is 1-1 but not onto. If no such transformation exists, state why.
8. If an $n \times n$ matrix A cannot be row reduced to the identity, what can you say about the linear transformation $x \mapsto Ax$? Is it 1-1? onto? invertible?
9. Find a basis for the null space, column space, and row space of the following matrix:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 7 \\ -5 & -9 & -20 \\ 2 & 3 & 11 \\ 5 & 11 & 10 \end{bmatrix}$$
(Note: This computation is a little too difficult to do by hand for the exam, but you should do it for practice, and the question is certainly fair game.)
10. §2.8 # 26
11. §2.9 # 13, 19
12. Let A be a 6×9 matrix with 4 pivot columns. What is $\dim A$? How about $\dim \operatorname{col} A$? What is $\dim \operatorname{row} A$?
13. Let A be an 5×5 matrix with $\det A = 12$. Consider the matrix B obtained from A via following sequence of row operations:
    (a) Swap row 2 with row 4.
    (b) Multiply row 3 by $\frac{1}{3}$.
    (c) Subtract twice row 4 from row 5.
    (d) Multiply row 1 by 4.

What is the determinant of B?

14. If A is a square matrix with integer entries and $\det A = 1$, why are the entries of A^{-1} also integers? Similarly, if B is invertible and has rational entries (with no assumptions about the determinant), then why are the entries of B^{-1} also rational numbers?

15. Let R be a region in the plane whose area is 7 and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $x \mapsto Ax$ where $A = \left[\begin{smallmatrix} 9 & 3 \\ 5 & 2 \end{smallmatrix}\right]$. What is the area of the region $T(R)$?

16. §4.1 # 5-9, 20, 22

17. §4.2 # 31, 32, 33, 34

18. Show that $\{\sin t, \cos t, \sin t \cos t\}$ is a linearly dependent set in $C(\mathbb{R})$. *Hint: Start by assuming*

$$c_1 \cdot \sin t + c_2 \cdot \cos t + c_3 \cdot \sin t \cos t = 0$$

This equation must hold for all t, so choose several specific values of t until you get a system of enough equations to determine that all the c_i's must be zero.

19. Let $D\colon \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by taking the derivative. Find the matrix for D with respect to the standard basis $\{1, t, t^2, t^3\}$. *Remark: You should be able to do more general problems where you find the matrix of a linear transformation on an abstract vector space with respect to some given basis.*

20. Problem # 4 from Midterm 2.

21. §4.4 # 32

22. §4.5 # 23

23. §4.6 # 24

24. §4.7 # 14

25. §5.1 # 14-16

26. Consider the matrix $A = \left[\begin{smallmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{smallmatrix}\right]$. Without doing any calculations, write down the roots of the characteristic polynomial of A (with multiplicities) and a basis of eigenvectors. (*I've done this several times in class, and won't require you to do this on the exam, but it's very useful to be able to guess eigenvectors and being able to do so might help you on the exam.*)

27. §5.2 # 10, 23

28. §5.3 # 11, 17

29. §5.4 # 5(c), 6(c), 7, 9(c), 10(b) (*These all have the same flavor. You should feel comfortable doing problems of this type.*)

30. You should also be able to diagonalize a linear transformation on an abstract vector space.

31. Discrete Dynamical Systems & Linear Systems of Differential Equations. The last midterm and the practice problems for that test are good places to find problems on these topics.

32. Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find a basis for the orthogonal complement of $\mathrm{span}\{v, w\}$.

33. Compute the orthogonal projection of $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ onto $W = \mathrm{span}\{v, w\}$ where $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

34. §6.2 # 9

35. §6.4 # 9, 10

36. §6.5 # 10

37. §6.6 # 2, 4

38. §6.7 # 25 (*There will be a question where you have to use an inner product defined by an integral on a vector space of functions.*)

39. §7.1 # 18

40. §7.4 # 10, 13