MATH2270: Midterm 2 Study Guide

The following is an overview of the material that will be covered on the second exam.

§1.9 The Matrix of a Linear Transformation

- Finding the matrix of a linear transformation. For example, consider the function $T: \mathbb{R}^3 \to \mathbb{R}$ that maps each vector to the sum of its entries. Find the standard matrix for T.
- 1-1 and onto linear transformations (and characterizations thereof). You should know the connections between properties of a linear transformation and properties (number & location of pivots) of the matrix of that transformation.

§2.1 Matrix Operations

- You should be comfortable with the basic algebra of matrices, including addition, scalar multiplication, and matrix multiplication.
- A good resource for matrix algebra is FFT4. The solutions are now posted on the Canvas page for our course. I even posted two different solutions!!
- Understand the connection between matrix multiplication and composition of functions. Thinking about matrix multiplication as composition of functions can be a helpful way of approaching problems and questions about matrices or functions. For example, if I asked *"Why is the product of two invertible matrices an invertible matrix?,"* then a possible answer would be that the composition of invertible linear transformations is an invertible linear transformation! Going the other way, if the question were *"Why can't a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^5$ be onto?,"* then one answer would be that a linear transformation is onto if and only if the matrix of the linear transformation has a pivot in every column, and the matrix for T would be 3×5 so it couldn't possibly have a pivot in every column.

§2.2 The Inverse of a Matrix

- You should be able to write down the inverse of a 2×2 matrix.
- Use our algorithm for computing the inverse of a matrix.
- Check whether two matrices are inverses of one another.
- Understand the connection between inverses of matrices and the inverse of a linear transformation. Also, understand how to use A^{-1} to solve the matrix equation $A\vec{x} = \vec{b}$.

§2.3 Characterizations of Invertible Matrices

- Determine whether a matrix (or linear transformation) is invertible.
- I want most (not necessarily all) of the equivalences in the *Invertible Matrix Theorem* to be natural. I want everyone to be able to translate between all the different languages we know: matrix equation, system of linear equations, vector equation, linear combinations, span of a set of vectors, linear transformations, etc. The more fluent you are in doing this translation, the better prepared you will be when we start using all of this to do really awesome stuff!!

§2.4 Partitioned Matrices (omitted)

- This section was assigned as homework and will not be tested. That doesn't mean you shouldn't know this material. It's likely that we will use it at some point later in the semester.

§2.8 Subspaces of $\mathbb{R}^n$

- You should know the definition of a subspace.
- You should have an *intuition* for subspaces of $\mathbb{R}^n$.
- Know the fundamental subspaces associated to a matrix A: they are $\operatorname{null} A$ and $\operatorname{col} A$. If A is an $m \times n$ matrix, you should know in which space ($\mathbb{R}^m$ or $\mathbb{R}^n$) each of these live.

§2.9 Dimension and Rank This section is subsumed by material in Ch $\dot{4}$.

- Know the definition of dimension of a subspace, and be able to compute it. Basically, the only way to find the dimension of a subspace is to find a basis for the subspace. The dimension is then the number of elements in that basis.
- Know what the rank of a matrix A is.
- If you are given a matrix, A, you should be able to find a basis for A and a basis for $\operatorname{col} A$. You should also be able to find the dimension of each of these subspaces.
- Know the "rank-nullity theorem" This says that if A is an $m \times n$ matrix, then

$$\dim \operatorname{col} A + \dim \operatorname{null} A = m$$

  You should also be able to interpret this as a statement about linear transformations.
- If $\vec{x}$ is a vector in the subspace H and $\mathcal{B}$ is a basis for H, then you should be able to compute the $\mathcal{B}$-coordinates of $\vec{x}$. That is, compute $[\vec{x}]_{\mathcal{B}}$.

§3.1 & §3.2 Determinants

- Be able to compute determinants using cofactor expansions. You should choose the row/column to expand in strategically.
- Know the defining properties of the determinant:
  1. The determinant is linear in each row
  2. The determinant is alternating (if you swap two rows, the determinant changes sign).
  3. $\det I_n = 1$.
- Know the other basic properties of the determinant:
  1. $\det(AB) = \det(A)\det(B)$
  2. $\det A^T = \det A$
  3. A is invertible if and only if $\det A \neq 0$.

§3.3 Cramer's Rule, Volume, and Linear Transformations

- Understand how the volume of a region in $\mathbb{R}^n$ changes under the application of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. Specifically, $\operatorname{vol}(T(R)) = \operatorname{vol}(R) \cdot |\det(T)|$.

- If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and A is the standard matrix for T, what does $\det A$ tell you about T?

Basically, the point of Chapter 2 is to further develop the theory of matrices so that we can use it later on. At a fundamental level, this is piling new terminology and definitions on to what we already have, so in some sense "we aren't doing anything new." The sense in which that is true, and a good way to approach computational problems from Ch. 2, is to trace back through the definitions, rephrasing the question several times along the way. Eventually the question will boil down to "row reduce this matrix." After you do that, an important final step is to take the result you get from row reduction and reinterpret it to answer the original question. Below is an example.

Problem 1. *Find the dimension of* $\operatorname{null} A$*, for a given matrix* A*.*

1. The definition of the dimension of a subspace is the number of vectors in any basis for that subspace, so I should find a basis for $\operatorname{null} A$.
2. The definition of $\operatorname{null} A$ is the set of vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$, so I need to find all of the solutions to that matrix equation.
3. To do that, I should augment A with a column of zeros, then row reduce the matrix.
4. I have a very good way of describing all the vectors in $\operatorname{null} A$: I can use parametric vector form to write all the solutions to $A\vec{x} = \vec{0}$. When I do that, the vectors that I use will automatically be linearly independent, and they must also span $\operatorname{null} A$. This is the definition of a basis! So the vectors in my parametric vector form for the solution to this equation form a basis for $\operatorname{null} A$. The number of vectors is the dimension of $\operatorname{null} A$.