MATH 2270: Midterm 2 Practice Problems

Here are some practice problems for the first exam. This is not meant to mimic the length of the exam.

1. Inventions:

   (a) Invent a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ that is onto. You can describe the linear transformation T by giving explicit formulas for $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$.

   (b) Invent a square matrix A such that $\operatorname{rank}(A) = 2$ and $\det A = 0$.

   (c) Invent a 2×2 matrix A such that $A^T \neq A$ and $\operatorname{col} A = \operatorname{row} A^T$.

   (d) Invent a 2×2 matrix A such that $A \neq I_2$, but $A^2 = I_2$.

   (e) Invent a matrix A such that the nullspace of A is $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \middle| 3a - b + c = 0. \right\}$

   (f) Invent a 2×2 matrix A such that $A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \vec{e}_1$ and $A \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \vec{e}_2$.

   (g) Invent three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$ so that the subspace of $\mathbb{R}^4$ spanned by these vectors is 2-dimensional.

   (h) Invent a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of $\mathbb{R}^3$ such that the $\mathcal{B}$-coordinates of the vector $\vec{x} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ are $[\vec{x}]_{\mathcal{B}} = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$.

   (i) Invent a matrix A such that $\det(A) = 3$ and $\det(2A) = 24$.

2. Answer each of the following true/false questions, and then give an explanation of your reasoning.

   (a) $\mathbb{R}^3$ a subspace of $\mathbb{R}^4$.

   (b) A linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ can be one-to-one but not onto.

   (c) For any $n \times n$ matrix A, $\det(-A) = -\det(A)$.

   (d) If the columns of a 4×5 matrix span $\mathbb{R}^4$, then the columns are linearly independent.

   (e) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are vectors in $\mathbb{R}^n$, then $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the same as the column space of the matrix $[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p]$.

   (f) The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

   (g) Every line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

   (h) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a linearly independent set of p vectors in $\mathbb{R}^n$, then $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a p-dimensional subspace of $\mathbb{R}^n$.

   (i) For any two $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$.

   (j) For any two $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

   (k) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$, then H is an n-dimensional subspace.

(l) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is a basis for a subspace H of $\mathbb{R}^n$ and $\vec{x} \in H$, then the $\mathcal{B}$-coordinates of $\vec{x}$, which we write as $[\vec{x}]_{\mathcal{B}}$ is a vectors with n entries.

3. Solve the matrix equation $AB = BC$ for A, assuming that A, B, and C are square and invertible.

4. Consider the following matrices

$$A = \begin{bmatrix} 4 & -1 & 2 & 8 \\ 4 & 3 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & 0 \\ -7 & 3 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -8 & 5 \\ 12 & -3 \\ -4 & -1 \end{bmatrix}$$

Which matrices correspond to one-to-one transformations? Which ones correspond to onto transformations? Explain. (*You don't need to row-reduce the matrices if you don't want to, but give a brief reason for each matrix.*)

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(\vec{e_1}) = \vec{e_2}$ and $T(\vec{e_2}) = -\vec{e_1}$, where $\vec{e_1}$ and $\vec{e_2}$ are columns of the 2×2 identity matrix.

   (a) Find the standard matrix A of T.

   (b) Plot the unit square on the left. *Reminder: The unit square is the square with vertices* $(0,0), (0,1), (1,0), (1,1)$. On the right, plot the image of the unit square under the transformation T.

   (c) Describe in words what the transformation T does to $\mathbb{R}^2$.

   (d) Is T one-to-one? Is T onto? Explain.

6. Determine whether or not the following matrix is invertible. *Do not try to invert it*

   (a)

$$A = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

   (b) Find the determinant of A^5.

7. Suppose a linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\vec{u}) = T(\vec{v})$ for some pair of distinct vectors $\vec{u}$ and $\vec{v}$. Can T be onto? Why or why not?

8. Prove that if $\det(B^3) = 0$, then $\det B = 0$.

9. Consider the following matrix:

$$A = \begin{bmatrix} 5 & 1 & 6 & 2 & 0 \\ 3 & 3 & 6 & -1 & -18 \\ 8 & 4 & 12 & -5 & 18 \\ 2 & 1 & 3 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & -13 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for $\operatorname{col} A$.

(b) Find a basis for row A.

(c) Find a basis for $\ker A$.

10. Find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

11. Is it possible for a 5×5 matrix to be invertible if the columns of A do not span $\mathbb{R}^5$.