Problem Set 6.3

C 1. Use your calculator to calculate each of the following. *Note:* On some calculators there is an e^x button. On others you must press the INV (or 2nd) and ln x buttons.

(a) e^3 (b) $e^{2.1}$

(c) $e^{\sqrt{2}}$ (d) $e^{\cos(\ln 4)}$

C 2. Calculate the following and explain why your answers are not surprising.

(a) $e^{3\ln 2}$ (b) $e^{(\ln 64)/2}$

In Problems 3–10, simplify the given expression.

3. $e^{3\ln x}$ **4.** $e^{-2\ln x}$

5. $\ln e^{\cos x}$ **6.** $\ln e^{-2x-3}$

7. $\ln(x^3e^{-3x})$ **8.** $e^{x-\ln x}$

9. $e^{\ln 3+2\ln x}$ **10.** $e^{\ln x^2-y\ln x}$

In Problems 11–22, find D_xy (see Examples 1 and 2).

11. $y = e^{x+2}$ **12.** $y = e^{2x^2-x}$

13. $y = e^{\sqrt{x+2}}$ **14.** $y = e^{-1/x^2}$

15. $y = e^{2\ln x}$ **16.** $y = e^{x/\ln x}$

17. $y = x^3e^x$ **18.** $y = e^{x^3\ln x}$

19. $y = \sqrt{e^{x^2}} + e^{\sqrt{x^2}}$ **20.** $y = e^{1/x^2} + 1/e^{x^2}$

21. $e^{xy} + xy = 2$ *Hint:* Use implicit differentiation.

22. $e^{x+y} = 4 + x + y$

23. Use your knowledge of the graph of $y = e^x$ to sketch the graphs of (a) $y = -e^x$ and (b) $y = e^{-x}$.

24. Explain why $a < b \Rightarrow e^{-a} > e^{-b}$.

In Problems 25–36, first find the domain of the given function f and then find where it is increasing and decreasing, and also where it is concave upward and downward. Identify all extreme values and points of inflection. Then sketch the graph of $y = f(x)$.

25. $f(x) = e^{2x}$ **26.** $f(x) = e^{-x/2}$

27. $f(x) = xe^{-x}$ **28.** $f(x) = e^x + x$

29. $f(x) = \ln(x^2 + 1)$ **30.** $f(x) = \ln(2x - 1)$

31. $f(x) = \ln(1 + e^x)$ **32.** $f(x) = e^{1-x^2}$

33. $f(x) = e^{-(x-2)^2}$ **34.** $f(x) = e^x - e^{-x}$

35. $f(x) = \int_0^x e^{-t^2}\,dt$ **36.** $f(x) = \int_0^x te^{-t}\,dt$

In Problems 37–44, find each integral.

37. $\int e^{3x+1}\,dx$ **38.** $\int xe^{x^2-3}\,dx$

39. $\int (x + 3)e^{x^2+6x}\,dx$ **40.** $\int \frac{e^x}{e^x - 1}\,dx$

41. $\int \frac{e^{-1/x}}{x^2}\,dx$ **42.** $\int e^{x+e^x}\,dx$

43. $\int_0^1 e^{2x+3}\,dx$ **44.** $\int_1^2 \frac{e^{3/x}}{x^2}\,dx$

45. Find the volume of the solid generated by revolving the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = \ln 3$ about the x-axis.

46. The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$ is revolved about the y-axis. Find the volume of the resulting solid.

47. Find the area of the region bounded by the curve $y = e^{-x}$ and the line through the points $(0, 1)$ and $(1, 1/e)$.

48. Show that $f(x) = \frac{x}{e^x - 1} - \ln(1 - e^{-x})$ is decreasing for $x > 0$.

C **49. Stirling's Formula** says that for large n we can approximate $n! = 1\cdot2\cdot3\cdots n$ by

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

(a) Calculate 10! exactly and then approximately using the above formula.

(b) Approximate 60!.

C **50.** It will be shown later (Section 9.9) that for small x

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Use this result to approximate $e^{0.3}$ and compare your result with what you get by calculating it directly. (Computers and calculators use sums like this to approximate e^x.)

51. Find the length of the curve given parametrically by $x = e^t \sin t$, $y = e^t \cos t$, $0 \le t \le \pi$.

C **52.** If customers arrive at a check-out counter at the average rate of k per minute, then (see books on probability theory) the probability that exactly n customers will arrive in a period of x minutes is given by the formula

$$P_n(x) = \frac{(kx)^n e^{-kx}}{n!}, n = 0, 1, 2, \ldots$$

Find the probability that exactly 8 customers will arrive during a 30-minute period if the average arrival rate for this check-out counter is 1 customer every 4 minutes.

53. Let $f(x) = \frac{\ln x}{1 + (\ln x)^2}$ for x in $(0, \infty)$. Find

(a) $\lim_{x\to0^+} f(x)$ and $\lim_{x\to\infty} f(x)$;

(b) the maximum and minimum values of $f(x)$;

(c) $F'(\sqrt{e})$ if $F(x) = \int_1^{x^2} f(t)\,dt$.

54. Let R be the region bounded by $x = 0$, $y = e^x$, and the tangent line to $y = e^x$ that goes through the origin. Find

(a) the area of R;

(b) the volume of the solid obtained when R is revolved about the x-axis.

GC *Use a graphing calculator or a CAS to do Problems 55–60.*

55. Evaluate.

(a) $\int_{-3}^{3} \exp(-1/x^2)\,dx$ (b) $\int_0^{8\pi} e^{-0.1x}\sin x\,dx$

56. Evaluate.

(a) $\lim_{x\to0}(1 + x)^{1/x}$ (b) $\lim_{x\to0}(1 + x)^{-1/x}$

57. Find the area of the region between the graphs of $y = f(x) = \exp(-x^2)$ and $y = f''(x)$ on $[-3, 3]$.

EXPL **58.** Draw the graphs of $y = x^p e^{-x}$ for various positive values of p using the same axes. Make conjectures about

(a) $\lim_{x\to\infty} x^p e^{-x}$,

(b) the x-coordinate of the maximum point for $f(x) = x^p e^{-x}$.

59. Describe the behavior of $\ln(x^2 + e^{-x})$ for large negative x. For large positive x.

60. Draw the graphs of f and f', where $f(x) = 1/(1 + e^{1/x})$. Then determine each of the following:

(a) $\lim_{x\to 0^+} f(x)$ (b) $\lim_{x\to 0^-} f(x)$

(c) $\lim_{x\to \pm\infty} f(x)$ (d) $\lim_{x\to 0} f'(x)$

(e) The maximum and minimum values of f (if they exist).

Answers to Concepts Review: **1.** increasing; exp **2.** $\ln e = 1$; 2.72 **3.** x; x **4.** e^x; $e^x + C$

6.4 General Exponential and Logarithmic Functions

We defined $e^{\sqrt{2}}$, e^{π}, and all other irrational powers of e in the previous section. But what about $2^{\sqrt{2}}$, π^{π}, π^{e}, and similar irrational powers of other numbers? In fact, we want to give meaning to a^x for $a > 0$ and x any real number. Now, if $r = p/q$ is a rational number, then $a^r = (\sqrt[q]{a})^p$. But we also know that

$$a^r = \exp(\ln a^r) = \exp(r \ln a) = e^{r \ln a}$$

This suggests the definition of the **exponential function to the base a.**

Definition

For $a > 0$ and any real number x,

$$a^x = e^{x \ln a}$$

Of course, this definition will be appropriate only if the usual properties of exponents are valid for it, a matter we take up shortly. To shore up our confidence in the definition, we use it to calculate 3^2 (with a little help from our calculator):

$$3^2 = e^{2 \ln 3} \approx e^{2(1.0986123)} \approx 9.000000$$

Your calculator may give a result that differs slightly from 9. Calculators use approximations for e^x and $\ln x$, and they round to a fixed number of decimal places (usually about 8).

Now we can fill a small gap in the properties of the natural logarithm left over from Section 6.1.

$$\ln(a^x) = \ln(e^{x \ln a}) = x \ln a$$

Thus, Property (iv) of Theorem 6.1A holds for all real x, not just rational x as claimed there. We will need this fact in the proof of Theorem A below.

What is 2^{π}?

In algebra, 2^n is first defined for positive integers n. Thus, $2^1 = 2$ and $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$. Next, we define 2^n for zero,

$$2^0 = 1$$

and for negative integers:

$$2^{-n} = 1/2^n \quad \text{if} \quad n > 0$$

This means that $2^{-3} = 1/2^3 = 1/8$. Finally, we used root functions to define 2^r for rational numbers r. Thus,

$$2^{7/3} = \sqrt[3]{2^7}$$

Calculus is required to extend the definition of 2^x to the set of real numbers. One way to define 2^{π} would be to say that it is the limit of the sequence

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, \ldots$$

The definition we use is

$$2^{\pi} = e^{\pi \ln 2}$$

This definition involves calculus, because our definition of the natural log function involved the definite integral.

Properties of a^x Theorem A summarizes the familiar properties of exponents, which can all be proved now in a completely rigorous manner. Theorem B shows us how to differentiate and integrate a^x.

Theorem A Properties of Exponents

If $a > 0$, $b > 0$, and x and y are real numbers, then

(i) $a^x a^y = a^{x+y}$; (ii) $\dfrac{a^x}{a^y} = a^{x-y}$;

(iii) $(a^x)^y = a^{xy}$; (iv) $(ab)^x = a^x b^x$;

(v) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.