MATH2270: Midterm 1 Practice Problems

Here are some practice problems for the first exam. This is not meant to mimic the length of the exam.

1. Inventions:
   - Give an example of a linearly independent set of vectors in $\mathbb{R}^3$ that contains as many vectors as possible.
   - Give an example of a 3×3 matrix A in reduced row echelon form such that the first and third columns of A are pivot columns, and $A \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
   - Give an example of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ such that $\text{span}\{\vec{v}, \vec{w}\}$ is a line.
2. True/False: Determine if each statement is true or false.
   - A free variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.
   - A consistent linear system always has infinitely many solutions.
   - If A is a 2×3 matrix, then the linear transformation $x \mapsto Ax$ has domain $\mathbb{R}^3$.
   - If A is a 2×3 matrix, then the linear transformation $x \mapsto Ax$ cannot be onto.
   - If the columns of a matrix A are linearly dependent, then the equation $Ax = 0$ has only one solution.
   - If the columns of a matrix A are linearly dependent, the linear transformation $x \mapsto Ax$ is one-to-one.
3. For what values of h and k is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

4. Give a parametric description of the solutions to the equation $A\vec{x} = \vec{0}$ where A is the matrix shown below:

$$\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Determine if the vector $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is in $\text{span}\{v_1, v_2\}$ where

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \vec{v_2} = \begin{bmatrix} 5 \\ 9 \\ 7 \end{bmatrix}$$

   If the answer is yes, then write $\vec{b}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

6. If $\vec{b}$ is in the span of the vectors $\vec{v_1}, \ldots, \vec{v_k}$, what can you say about solutions to the matrix equation $A\vec{x} = \vec{b}$ where A is the matrix whose columns are $\vec{v_1}, \ldots, \vec{v_k}$ (i.e., $A = [\vec{v_1}\ \vec{v_2}\ \ldots\ \vec{v_k}]$)?

7. Is the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ linearly independent? Why or why not?

8. Is the set of vectors $\{\text{span}\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}\}$ linearly independent? Why or why not?

9. Determine if the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, whose standard matrix is A, is 1-1. Is it onto?

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

10. Suppose $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $S\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 3 \\ -2 \\ -1 \end{smallmatrix}\right]$ and $S\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 2 \\ 2 \end{smallmatrix}\right]$.

   (a) Find $S\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right)$.

   (b) Find the standard matrix for S.

11. If the columns of a 6×4 matrix A are linearly dependent, then what is the maximum number of pivot positions of A?