Thus $c_1 = 2$, $c_2 = 3$, and $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The coordinate system on H determined by $\mathcal{B}$ is shown in Fig. 7.

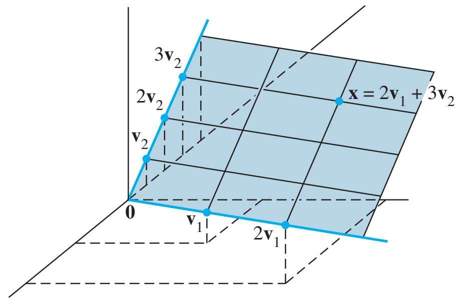


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

If a different basis for H were chosen, would the associated coordinate system also make H isomorphic to $\mathbb{R}^2$? Surely, this must be true. We shall prove it in the next section.

PRACTICE PROBLEMS

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.
   a. Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.
   b. Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.
   c. Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\,\mathbf{x}\,]_{\mathcal{B}}$.
   d. Find $[\,\mathbf{x}\,]_{\mathcal{B}}$, for the $\mathbf{x}$ given above.
2. The set $\mathcal{B} = \{1 + t, 1 + t^2, t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to $\mathcal{B}$.

4.4 EXERCISES

In Exercises 1–4, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$, $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$, $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

3. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

4. $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$, $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

In Exercises 5–8, find the coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

7. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$

8. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 9 and 10, find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

9. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

10. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

In Exercises 11 and 12, use an inverse matrix to find $[\,\mathbf{x}\,]_{\mathcal{B}}$ for the given $\mathbf{x}$ and $\mathcal{B}$.

11. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

12. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

13. The set $\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.

14. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.

In Exercises 15 and 16, mark each statement True or False. Justify each answer. Unless stated otherwise, $\mathcal{B}$ is a basis for a vector space V.

15. a. If $\mathbf{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is in $\mathbb{R}^n$.

b. If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$, for $\mathbf{x}$ in V.

c. The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

16. a. If $\mathcal{B}$ is the standard basis for $\mathbb{R}^n$, then the $\mathcal{B}$-coordinate vector of an $\mathbf{x}$ in $\mathbb{R}^n$ is $\mathbf{x}$ itself.

b. The correspondence $[\,\mathbf{x}\,]_{\mathcal{B}} \mapsto \mathbf{x}$ is called the coordinate mapping.

c. In some cases, a plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$.

17. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

18. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \dots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \dots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

19. Let S be a finite set in a vector space V with the property that every $\mathbf{x}$ in V has a unique representation as a linear combination of elements of S. Show that S is a basis of V.

20. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is a linearly dependent spanning set for a vector space V. Show that each $\mathbf{w}$ in V can be expressed in more than one way as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_4$. [*Hint:* Let $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_4\mathbf{v}_4$ be an arbitrary vector in V. Use the linear dependence of $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ to produce another representation of $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_4$.]

21. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \end{bmatrix} \right\}$. Since the coordinate mapping determined by $\mathcal{B}$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, this mapping must be implemented by some 2×2 matrix A. Find it. [*Hint:* Multiplication by A should transform a vector $\mathbf{x}$ into its coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$.]

22. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Produce a description of an $n \times n$ matrix A that implements the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$. (See Exercise 21.)

Exercises 23–26 concern a vector space V, a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, and the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$.

23. Show that the coordinate mapping is one-to-one. (*Hint:* Suppose $[\,\mathbf{u}\,]_{\mathcal{B}} = [\,\mathbf{w}\,]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{w}$ in V, and show that $\mathbf{u} = \mathbf{w}$.)

24. Show that the coordinate mapping is *onto* $\mathbb{R}^n$. That is, given any $\mathbf{y}$ in $\mathbb{R}^n$, with entries $y_1, \dots, y_n$, produce $\mathbf{u}$ in V such that $[\,\mathbf{u}\,]_{\mathcal{B}} = \mathbf{y}$.

25. Show that a subset $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in V is linearly independent if and only if the set of coordinate vectors $\{[\,\mathbf{u}_1\,]_{\mathcal{B}}, \dots, [\,\mathbf{u}_p\,]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$. *Hint:* Since the coordinate mapping is one-to-one, the following equations have the same solutions, $c_1, \dots, c_p$.

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0} \qquad \text{The zero vector in } V$$
$$[\,c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\,]_{\mathcal{B}} = [\,\mathbf{0}\,]_{\mathcal{B}} \qquad \text{The zero vector in } \mathbb{R}^n$$

26. Given vectors $\mathbf{u}_1, \dots, \mathbf{u}_p$, and $\mathbf{w}$ in V, show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_p$ if and only if $[\,\mathbf{w}\,]_{\mathcal{B}}$ is a linear combination of the coordinate vectors $[\,\mathbf{u}_1\,]_{\mathcal{B}}, \dots, [\,\mathbf{u}_p\,]_{\mathcal{B}}$.

In Exercises 27–30, use coordinate vectors to test the linear independence of the sets of polynomials. Explain your work.

27. $1 + 2t^3,\ 2 + t - 3t^2,\ -t + 2t^2 - t^3$

28. $1 - 2t^2 - t^3,\ t + 2t^3,\ 1 + t - 2t^2$

29. $(1 - t)^2,\ t - 2t^2 + t^3,\ (1 - t)^3$

30. $(2 - t)^3,\ (3 - t)^2,\ 1 + 6t - 5t^2 + t^3$

31. Use coordinate vectors to test whether the following sets of polynomials span $\mathbb{P}_2$. Justify your conclusions.

a. $1 - 3t + 5t^2, -3 + 5t - 7t^2, -4 + 5t - 6t^2, 1 - t^2$

b. $5t + t^2, 1 - 8t - 2t^2, -3 + 4t + 2t^2, 2 - 3t$

32. Let $\mathbf{p}_1(t) = 1 + t^2, \mathbf{p}_2(t) = t - 3t^2, \mathbf{p}_3(t) = 1 + t - 3t^2$.

a. Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

b. Consider the basis $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for $\mathbb{P}_2$. Find $\mathbf{q}$ in $\mathbb{P}_2$, given that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

In Exercises 33 and 34, determine whether the sets of polynomials form a basis for $\mathbb{P}_3$. Justify your conclusions.

33. [**M**] $3 + 7t, 5 + t - 2t^3, t - 2t^2, 1 + 16t - 6t^2 + 2t^3$

34. [**M**] $5 - 3t + 4t^2 + 2t^3, 9 + t + 8t^2 - 6t^3, 6 - 2t + 5t^2, t^3$

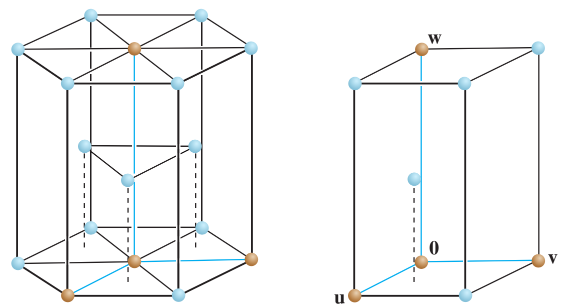
35. [**M**] Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Show that $\mathbf{x}$ is in H and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, for

$$\mathbf{v}_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

36. [**M**] Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Show that $\mathcal{B}$ is a basis for H and $\mathbf{x}$ is in H, and find the $\mathcal{B}$-coordinate vector of $\mathbf{x}$, for

$$\mathbf{v}_1 = \begin{bmatrix} -6 \\ 4 \\ -9 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 8 \\ -3 \\ 7 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -9 \\ 5 \\ -8 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 7 \\ -8 \\ 3 \end{bmatrix}$$

[**M**] Exercises 37 and 38 concern the crystal lattice for titanium, which has the hexagonal structure shown on the left in the accompanying figure. The vectors $\begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix}$ in $\mathbb{R}^3$ form a basis for the unit cell shown on the right. The numbers here are Ångstrom units ($1\text{ Å} = 10^{-8}$ cm). In alloys of titanium, some additional atoms may be in the unit cell at the *octahedral* and *tetrahedral* sites (so named because of the geometric objects formed by atoms at these locations).



The hexagonal close-packed lattice and its unit cell.

37. One of the octahedral sites is $\begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix}$, relative to the lattice basis. Determine the coordinates of this site relative to the standard basis of $\mathbb{R}^3$.

38. One of the tetrahedral sites is $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$. Determine the coordinates of this site relative to the standard basis of $\mathbb{R}^3$.

SOLUTIONS TO PRACTICE PROBLEMS

1. a. It is evident that the matrix $P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]$ is row-equivalent to the identity matrix. By the Invertible Matrix Theorem, $P_{\mathcal{B}}$ is invertible and its columns form a basis for $\mathbb{R}^3$.

b. From part (a), the change-of-coordinates matrix is $P_{\mathcal{B}} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$.

c. $\mathbf{x} = P_{\mathcal{B}}[\,\mathbf{x}\,]_{\mathcal{B}}$

d. To solve the equation in (c), it is probably easier to row reduce an augmented matrix than to compute $P_{\mathcal{B}}^{-1}$:

$$\underbrace{\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}}_{P_{\mathcal{B}} \qquad\qquad \mathbf{x}} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{I \qquad\qquad [\,\mathbf{x}\,]_{\mathcal{B}}}$$

Hence

$$[\,\mathbf{x}_{\mathcal{B}}\,] = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

2. The coordinates of $\mathbf{p}(t) = 6 + 3t - t^2$ with respect to $\mathcal{B}$ satisfy

$$c_1(1 + t) + c_2(1 + t^2) + c_3(t + t^2) = 6 + 3t - t^2$$

4.5 EXERCISES

For each subspace in Exercises 1–8, (a) find a basis for the subspace, and (b) state the dimension.

1. $\left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s, t \text{ in } \mathbb{R} \right\}$

2. $\left\{ \begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$

3. $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

4. $\left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q \text{ in } \mathbb{R} \right\}$

5. $\left\{ \begin{bmatrix} p-2q \\ 2p+5r \\ -2q+2r \\ -3p+6r \end{bmatrix} : p, q, r \text{ in } \mathbb{R} \right\}$

6. $\left\{ \begin{bmatrix} 3a-c \\ -b-3c \\ -7a+6b+5c \\ -3a+c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

7. $\{(a, b, c) : a - 3b + c = 0, b - 2c = 0, 2b - c = 0\}$

8. $\{(a, b, c, d) : a - 3b + c = 0\}$

9. Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

10. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix}$.

In Exercises 11 and 12, find the dimension of the subspace spanned by the given vectors.

11. $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$

Determine the dimensions of Nul A and Col A for the matrices shown in Exercises 13–18.

13. $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

16. $A = \begin{bmatrix} 3 & 2 \\ -6 & 5 \end{bmatrix}$

17. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In Exercises 19 and 20, V is a vector space. Mark each statement True or False. Justify each answer.

19. a. The number of pivot columns of a matrix equals the dimension of its column space.
 b. A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.
 c. The dimension of the vector space $\mathbb{P}_4$ is 4.
 d. If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.
 e. If a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

20. a. $\mathbb{R}^2$ is a two-dimensional subspace of $\mathbb{R}^3$.
 b. The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimension of Nul A.
 c. A vector space is infinite-dimensional if it is spanned by an infinite set.
 d. If $\dim V = n$ and if S spans V, then S is a basis of V.
 e. The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

21. The first four Hermite polynomials are $1, 2t, -2 + 4t^2$, and $-12t + 8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics.[2] Show that the first four Hermite polynomials form a basis of $\mathbb{P}_3$.

22. The first four Laguerre polynomials are $1, 1 - t, 2 - 4t + t^2$, and $6 - 18t + 9t^2 - t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

23. Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the Hermite polynomials in Exercise 21, and let $\mathbf{p}(t) = -1 + 8t^2 + 8t^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

24. Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ consisting of the first three Laguerre polynomials listed in Exercise 22, and let $\mathbf{p}(t) = 5 + 5t - 2t^2$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathcal{B}$.

25. Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

26. Let H be an n-dimensional subspace of an n-dimensional vector space V. Show that $H = V$.

27. Explain why the space $\mathbb{P}$ of all polynomials is an infinite-dimensional space.

[2] See *Introduction to Functional Analysis*, 2d ed., by A. E. Taylor and David C. Lay (New York: John Wiley & Sons, 1980), pp. 92–93. Other sets of polynomials are discussed there, too.

28. Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional space.

In Exercises 29 and 30, V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer. (These questions are more difficult than those in Exercises 19 and 20.)

29. a. If there exists a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ that spans V, then $\dim V \le p$.

b. If there exists a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \ge p$.

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

30. a. If there exists a linearly dependent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \le p$.

b. If every set of p elements in V fails to span V, then $\dim V > p$.

c. If $p \ge 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Exercises 31 and 32 concern finite-dimensional vector spaces V and W and a linear transformation $T : V \to W$.

31. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, by Exercise 35 in Section 4.2. Prove that $\dim T(H) \le \dim H$.

32. Let H be a nonzero subspace of V, and suppose T is a one-to-one (linear) mapping of V into W. Prove that $\dim T(H) = \dim H$. If T happens to be a one-to-one mapping of V *onto* W, then $\dim V = \dim W$. Isomorphic finite-dimensional vector spaces have the same dimension.

33. **[M]** According to Theorem 11, a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to create $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n\,]$, with $\mathbf{e}_1, \dots, \mathbf{e}_n$ the columns of the identity matrix; the pivot columns of A form a basis for $\mathbb{R}^n$.

a. Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

b. Explain why the method works in general: Why are the original vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ included in the basis found for Col A? Why is Col $A = \mathbb{R}^n$?

34. **[M]** Let $\mathcal{B} = \{1, \cos t, \cos^2 t, \dots, \cos^6 t\}$ and $\mathcal{C} = \{1, \cos t, \cos 2t, \dots, \cos 6t\}$. Assume the following trigonometric identities (see Exercise 37 in Section 4.1).

$$\begin{aligned}
\cos 2t &= -1 + 2\cos^2 t \\
\cos 3t &= -3\cos t + 4\cos^3 t \\
\cos 4t &= 1 - 8\cos^2 t + 8\cos^4 t \\
\cos 5t &= 5\cos t - 20\cos^3 t + 16\cos^5 t \\
\cos 6t &= -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t
\end{aligned}$$

Let H be the subspace of functions spanned by the functions in $\mathcal{B}$. Then $\mathcal{B}$ is a basis for H, by Exercise 38 in Section 4.3.

a. Write the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$, and use them to show that $\mathcal{C}$ is a linearly independent set in H.

b. Explain why $\mathcal{C}$ is a basis for H.

SOLUTIONS TO PRACTICE PROBLEMS

1. False. Consider the set $\{\mathbf{0}\}$.
2. True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. By Theorem 9, T is linearly dependent.

4.6 RANK

With the aid of vector space concepts, this section takes a look *inside* a matrix and reveals several interesting and useful relationships hidden in its rows and columns.

For instance, imagine placing 2000 random numbers into a 40×50 matrix A and then determining both the maximum number of linearly independent columns in A and the maximum number of linearly independent columns in A^T (rows in A). Remarkably, the two numbers are the same. As we'll soon see, their common value is the *rank* of the matrix. To explain why, we need to examine the subspace spanned by the rows of A.

NUMERICAL NOTE

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A, to be discussed in Section 7.4. This decomposition is also a reliable source of bases for Col A, Row A, Nul A, and Nul A^T.

WEB

PRACTICE PROBLEMS

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.
2. Find bases for Col A and Row A.
3. What is the next step to perform to find a basis for Nul A?
4. How many pivot columns are in a row echelon form of A^T?

4.6 EXERCISES

In Exercises 1–4, assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

1. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$,
$B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$,
$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{bmatrix}$,
$B = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix}$,
$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

5. If a 4×7 matrix A has rank 3, find $\dim \text{Nul}\, A$, $\dim \text{Row}\, A$, and $\text{rank}\, A^T$.

6. If a 7×5 matrix A has rank 2, find $\dim \text{Nul}\, A$, $\dim \text{Row}\, A$, and $\text{rank}\, A^T$.

7. Suppose a 4×7 matrix A has four pivot columns. Is $\text{Col}\, A = \mathbb{R}^4$? Is $\text{Nul}\, A = \mathbb{R}^3$? Explain your answers.

8. Suppose a 6×8 matrix A has four pivot columns. What is $\dim \text{Nul}\, A$? Is $\text{Col}\, A = \mathbb{R}^4$? Why or why not?

9. If the null space of a 4×6 matrix A is 3-dimensional, what is the dimension of the column space of A? Is $\text{Col}\, A = \mathbb{R}^3$? Why or why not?

10. If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?

11. If the null space of an 8×5 matrix A is 3-dimensional, what is the dimension of the row space of A?

12. If the null space of a 5×4 matrix A is 2-dimensional, what is the dimension of the row space of A?

13. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

14. If A is a 5×4 matrix, what is the largest possible dimension of the row space of A? If A is a 4×5 matrix, what is the largest possible dimension of the row space of A? Explain.

15. If A is a 3×7 matrix, what is the smallest possible dimension of $\text{Nul}\, A$?

16. If A is a 7×5 matrix, what is the smallest possible dimension of $\text{Nul}\, A$?

In Exercises 17 and 18, A is an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

17. a. The row space of A is the same as the column space of A^T.

b. If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for $\text{Row}\, A$.

c. The dimensions of the row space and the column space of A are the same, even if A is not square.

d. The sum of the dimensions of the row space and the null space of A equals the number of rows in A.

e. On a computer, row operations can change the apparent rank of a matrix.

18. a. If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

b. Row operations preserve the linear dependence relations among the rows of A.

c. The dimension of the null space of A is the number of columns of A that are *not* pivot columns.

d. The row space of A^T is the same as the column space of A.

e. If A and B are row equivalent, then their row spaces are the same.

19. Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Explain.

20. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain.

21. Suppose a nonhomogeneous system of nine linear equations in ten unknowns has a solution for all possible constants on the right sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are *not* multiples of each other? Discuss.

22. Is is possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

23. A homogeneous system of twelve linear equations in eight unknowns has two fixed solutions that are not multiples of each other, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than twelve homogeneous linear equations? If so, how many? Discuss.

24. Is it possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right-hand side of constants? Is it possible for such a system to have a unique solution for every right-hand side? Explain.

25. A scientist solves a nonhomogeneous system of ten linear equations in twelve unknowns and finds that three of the unknowns are free variables. Can the scientist be certain that, if the right sides of the equations are changed, the new nonhomogeneous system will have a solution? Discuss.

26. In statistical theory, a common requirement is that a matrix be of *full rank*. That is, the rank should be as large as possible. Explain why an $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly independent.

Exercises 27–29 concern an $m \times n$ matrix A and what are often called the *fundamental subspaces* determined by A.

27. Which of the subspaces $\text{Row}\, A$, $\text{Col}\, A$, $\text{Nul}\, A$, $\text{Row}\, A^T$, $\text{Col}\, A^T$, and $\text{Nul}\, A^T$ are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are in this list?

28. Justify the following equalities:

a. $\dim \text{Row}\, A + \dim \text{Nul}\, A = n$ Number of columns of A

b. $\dim \text{Col}\, A + \dim \text{Nul}\, A^T = m$ Number of rows of A

29. Use Exercise 28 to explain why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. Suppose A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$. What has to be true about the two numbers rank $[\,A \quad \mathbf{b}\,]$ and rank A in order for the equation $A\mathbf{x} = \mathbf{b}$ to be consistent?

Rank 1 matrices are important in some computer algorithms and several theoretical contexts, including the singular value decomposition in Chapter 7. It can be shown that an $m \times n$ matrix A has rank 1 if and only if it is an outer product; that is, $A = \mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}$ in $\mathbb{R}^m$ and $\mathbf{v}$ in $\mathbb{R}^n$. Exercises 31–33 suggest why this property is true.

31. Verify that rank $\mathbf{u}\mathbf{v}^T \le 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

32. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\mathbf{v}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = \mathbf{u}\mathbf{v}^T$.

33. Let A be any 2×3 matrix such that rank $A = 1$, let $\mathbf{u}$ be the first column of A, and suppose $\mathbf{u} \ne \mathbf{0}$. Explain why there is a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $A = \mathbf{u}\mathbf{v}^T$. How could this construction be modified if the first column of A were zero?

34. Let A be an $m \times n$ matrix of rank $r > 0$ and let U be an echelon form of A. Explain why there exists an invertible matrix E such that $A = EU$, and use this factorization to write A as the sum of r rank 1 matrices. [*Hint:* See Theorem 10 in Section 2.4.]

35. [**M**] Let $A = \begin{bmatrix} 7 & -9 & -4 & 5 & 3 & -3 & -7 \\ -4 & 6 & 7 & -2 & -6 & -5 & 5 \\ 5 & -7 & -6 & 5 & -6 & 2 & 8 \\ -3 & 5 & 8 & -1 & -7 & -4 & 8 \\ 6 & -8 & -5 & 4 & 4 & 9 & 3 \end{bmatrix}$.

a. Construct matrices C and N whose columns are bases for Col A and Nul A, respectively, and construct a matrix R whose *rows* form a basis for Row A.

b. Construct a matrix M whose columns form a basis for Nul A^T, form the matrices $S = [\,R^T \quad N\,]$ and $T = [\,C \quad M\,]$, and explain why S and T should be square. Verify that both S and T are invertible.

36. [**M**] Repeat Exercise 35 for a random integer-valued 6×7 matrix A whose rank is at most 4. One way to make A is to create a random integer-valued 6×4 matrix J and a random integer-valued 4×7 matrix K, and set $A = JK$. (See Supplementary Exercise 12 at the end of the chapter; and see the *Study Guide* for matrix-generating programs.)

37. [**M**] Let A be the matrix in Exercise 35. Construct a matrix C whose columns are the pivot columns of A, and construct a matrix R whose rows are the nonzero rows of the reduced echelon form of A. Compute CR, and discuss what you see.

38. [**M**] Repeat Exercise 37 for three random integer-valued 5×7 matrices A whose ranks are 5, 4, and 3. Make a conjecture about how CR is related to A for any matrix A. Prove your conjecture.

SOLUTIONS TO PRACTICE PROBLEMS

1. A has two pivot columns, so rank $A = 2$. Since A has 5 columns altogether, dim Nul $A = 5 - 2 = 3$.

2. The pivot columns of A are the first two columns. So a basis for Col A is

$$\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

The nonzero rows of B form a basis for Row A, namely, $\{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$. In this particular example, it happens that any two rows of A form a basis for the row space, because the row space is two-dimensional and none of the rows of A is a multiple of another row. In general, the nonzero rows of an echelon form of A should be used as a basis for Row A, not the rows of A itself.

3. For Nul A, the next step is to perform row operations on B to obtain the reduced echelon form of A.

4. Rank A^T = rank A, by the Rank Theorem, because Col A^T = Row A. So A^T has two pivot positions.

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EXAMPLE 3 Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

SOLUTION

a. Notice that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is needed rather than $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, and compute

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \vdots & \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} = \left[\begin{array}{rr:rr} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{rr:rr} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

b. By part (a) and property (6) above (with $\mathcal{B}$ and $\mathcal{C}$ interchanged),

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (\underset{\mathcal{B}\leftarrow\mathcal{C}}{P})^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \quad \blacksquare$$

Another description of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ uses the change-of-coordinate matrices $P_{\mathcal{B}}$ and $P_{\mathcal{C}}$ that convert $\mathcal{B}$-coordinates and $\mathcal{C}$-coordinates, respectively, into standard coordinates. Recall that for each $\mathbf{x}$ in $\mathbb{R}^n$,

$$P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}, \quad P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x}$$

Thus

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$$

In $\mathbb{R}^n$, the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ may be computed as $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. Actually, for matrices larger than 2×2, an algorithm analogous to the one in Example 3 is faster than computing $P_{\mathcal{C}}^{-1}$ and then $P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$. See Exercise 12 in Section 2.2.

PRACTICE PROBLEMS

1. Let $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathcal{G} = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space V, and let P be a matrix whose columns are $[\,\mathbf{f}_1\,]_{\mathcal{G}}$ and $[\,\mathbf{f}_2\,]_{\mathcal{G}}$. Which of the following equations is satisfied by P for all $\mathbf{v}$ in V?

(i) $[\,\mathbf{v}\,]_{\mathcal{F}} = P[\,\mathbf{v}\,]_{\mathcal{G}}$ (ii) $[\,\mathbf{v}\,]_{\mathcal{G}} = P[\,\mathbf{v}\,]_{\mathcal{F}}$

2. Let $\mathcal{B}$ and $\mathcal{C}$ be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

4.7 EXERCISES

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -2\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 3\mathbf{c}_1 - 6\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$ for $\mathbf{x} = 2\mathbf{b}_1 + 3\mathbf{b}_2$.

3. Let $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for V, and let P be a matrix whose columns are $[\mathbf{u}_1]_{\mathcal{W}}$ and $[\mathbf{u}_2]_{\mathcal{W}}$. Which of the following equations is satisfied by P for all $\mathbf{x}$ in V?

(i) $[\mathbf{x}]_{\mathcal{U}} = P[\mathbf{x}]_{\mathcal{W}}$ (ii) $[\mathbf{x}]_{\mathcal{W}} = P[\mathbf{x}]_{\mathcal{U}}$

4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be bases for V, and let $P = [\,[\mathbf{d}_1]_{\mathcal{A}} \quad [\mathbf{d}_2]_{\mathcal{A}} \quad [\mathbf{d}_3]_{\mathcal{A}}\,]$. Which of the following equations is satisfied by P for all $\mathbf{x}$ in V?

(i) $[\mathbf{x}]_{\mathcal{A}} = P[\mathbf{x}]_{\mathcal{D}}$ (ii) $[\mathbf{x}]_{\mathcal{D}} = P[\mathbf{x}]_{\mathcal{A}}$

5. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

a. Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

b. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.

a. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

b. Find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

In Exercises 7–10, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. In each exercise, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

7. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

8. $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

9. $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

10. $\mathbf{b}_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

In Exercises 11 and 12, $\mathcal{B}$ and $\mathcal{C}$ are bases for a vector space V. Mark each statement True or False. Justify each answer.

11. a. The columns of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

b. If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the *standard* basis for V, then $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4.

12. a. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent.

b. If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of $[\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2\,]$ to $[\,I \quad P\,]$ produces a matrix P that satisfies $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x}$ in V.

13. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.

14. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Exercises 15 and 16 provide a proof of Theorem 15. Fill in a justification for each step.

15. Given $\mathbf{v}$ in V, there exist scalars $x_1, \dots, x_n$, such that

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

because (a) _____. Apply the coordinate mapping determined by the basis $\mathcal{C}$, and obtain

$$[\mathbf{v}]_{\mathcal{C}} = x_1[\mathbf{b}_1]_{\mathcal{C}} + x_2[\mathbf{b}_2]_{\mathcal{C}} + \cdots + x_n[\mathbf{b}_n]_{\mathcal{C}}$$

because (b) _____. This equation may be written in the form

$$[\mathbf{v}]_{\mathcal{C}} = \left[\,[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}\,\right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{8}$$

by the definition of (c) _____. This shows that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ shown in (5) satisfies $[\mathbf{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{v}]_{\mathcal{B}}$ for each $\mathbf{v}$ in V, because the vector on the right side of (8) is (d) _____.

16. Suppose Q is any matrix such that

$$[\mathbf{v}]_{\mathcal{C}} = Q[\mathbf{v}]_{\mathcal{B}} \quad \text{for each } \mathbf{v} \text{ in } V \tag{9}$$

Set $\mathbf{v} = \mathbf{b}_1$ in (9). Then (9) shows that $[\mathbf{b}_1]_{\mathcal{C}}$ is the first column of Q because (a) _____. Similarly, for $k = 2, \dots, n$, the kth column of Q is (b) _____ because (c) _____. This shows that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ defined by (5) in Theorem 15 is the only matrix that satisfies condition (4).

17. [**M**] Let $\mathcal{B} = \{\mathbf{x}_0, \dots, \mathbf{x}_6\}$ and $C = \{\mathbf{y}_0, \dots, \mathbf{y}_6\}$, where $\mathbf{x}_k$ is the function $\cos^k t$ and $\mathbf{y}_k$ is the function $\cos kt$. Exercise 34 in Section 4.5 showed that both $\mathcal{B}$ and $\mathcal{C}$ are bases for the vector space $H = \text{Span}\,\{\mathbf{x}_0, \dots, \mathbf{x}_6\}$.

a. Set $P = \left[\,[\mathbf{y}_0]_{\mathcal{B}} \quad \cdots \quad [\mathbf{y}_6]_{\mathcal{B}}\,\right]$, and calculate P^{-1}.

b. Explain why the columns of P^{-1} are the $\mathcal{C}$-coordinate vectors of $\mathbf{x}_0, \dots, \mathbf{x}_6$. Then use these coordinate vectors to write trigonometric identities that express powers of $\cos t$ in terms of the functions in $\mathcal{C}$.

See the *Study Guide*.

18. [**M**] (*Calculus required*)[3] Recall from calculus that integrals such as

$$\int (5\cos^3 t - 6\cos^4 t + 5\cos^5 t - 12\cos^6 t)\, dt \tag{10}$$

are tedious to compute. (The usual method is to apply integration by parts repeatedly and use the half-angle formula.) Use the matrix P or P^{-1} from Exercise 17 to transform (10); then compute the integral.

[3] The idea for Exercises 17 and 18 and five related exercises in earlier sections came from a paper by Jack W. Rogers, Jr., of Auburn University, presented at a meeting of the International Linear Algebra Society, August 1995. See "Applications of Linear Algebra in Calculus," *American Mathematical Monthly* **104** (1), 1997.

19. [**M**] Let

$$P = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}$$

a. Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ such that P is the change-of-coordinates matrix from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. [*Hint:* What do the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ represent?]

b. Find a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbb{R}^3$ such that P is the change-of-coordinates matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

20. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for a two-dimensional vector space.

a. Write an equation that relates the matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}$, and $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$. Justify your result.

b. [**M**] Use a matrix program either to help you find the equation or to check the equation you write. Work with three bases for $\mathbb{R}^2$. (See Exercises 7–10.)

SOLUTIONS TO PRACTICE PROBLEMS

1. Since the columns of P are $\mathcal{G}$-coordinate vectors, a vector of the form $P\mathbf{x}$ must be a $\mathcal{G}$-coordinate vector. Thus P satisfies equation (ii).

2. The coordinate vectors found in Example 1 show that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$$

Hence

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = (\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1} = \frac{1}{10}\begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} .1 & .6 \\ -.1 & .4 \end{bmatrix}$$

4.8 APPLICATIONS TO DIFFERENCE EQUATIONS

Now that powerful computers are widely available, more and more scientific and engineering problems are being treated in a way that uses discrete, or digital, data rather than continuous data. Difference equations are often the appropriate tool to analyze such data. Even when a differential equation is used to model a continuous process, a numerical solution is often produced from a related difference equation.

This section highlights some fundamental properties of linear difference equations that are best explained using linear algebra.

Discrete-Time Signals

The vector space $\mathbb{S}$ of discrete-time signals was introduced in Section 4.1. A **signal** in $\mathbb{S}$ is a function defined only on the integers and is visualized as a sequence of numbers, say, $\{y_k\}$. Figure 1 shows three typical signals whose general terms are $(.7)^k$, 1^k, and $(-1)^k$, respectively.

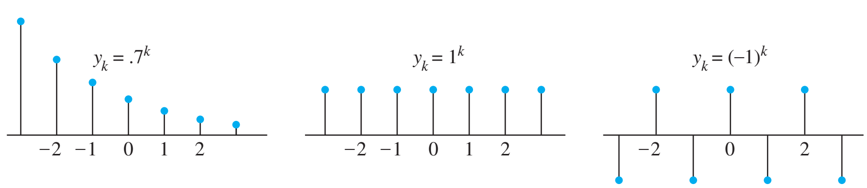


FIGURE 1 Three signals in $\mathbb{S}$.

Eigenvectors and Difference Equations

This section concludes by showing how to construct solutions of the first-order difference equation discussed in the chapter introductory example:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2, \ldots) \tag{8}$$

If A is an $n \times n$ matrix, then (8) is a *recursive* description of a sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$. A **solution** of (8) is an explicit description of $\{\mathbf{x}_k\}$ whose formula for each $\mathbf{x}_k$ does not depend directly on A or on the preceding terms in the sequence other than the initial term $\mathbf{x}_0$.

The simplest way to build a solution of (8) is to take an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue λ and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \quad (k = 1, 2, \ldots) \tag{9}$$

This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1}\mathbf{x}_0 = \mathbf{x}_{k+1}$$

Linear combinations of solutions in the form of equation (9) are solutions, too! See Exercise 33.

PRACTICE PROBLEMS

1. Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?
2. If $\mathbf{x}$ is an eigenvector of A corresponding to λ, what is $A^3\mathbf{x}$?
3. Suppose that $\mathbf{b}_1$ and $\mathbf{b}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, respectively, and suppose that $\mathbf{b}_3$ and $\mathbf{b}_4$ are linearly independent eigenvectors corresponding to a third distinct eigenvalue λ_3. Does it necessarily follow that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a linearly independent set? [*Hint*: Consider the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + (c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = \mathbf{0}$.]

5.1 EXERCISES

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? Why or why not?
2. Is $\lambda = -3$ an eigenvalue of $\begin{bmatrix} -1 & 4 \\ 6 & 9 \end{bmatrix}$? Why or why not?
3. Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix}$? If so, find the eigenvalue.
4. Is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$? If so, find the eigenvalue.
5. Is $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$? If so, find the eigenvalue.
6. Is $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue.
7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.
8. Is $\lambda = 1$ an eigenvalue of $\begin{bmatrix} 4 & -2 & 3 \\ 0 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$? If so, find one corresponding eigenvector.

In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

9. $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1, 3$

10. $A = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}, \lambda = -5$

11. $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}, \lambda = -1, 7$

12. $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}, \lambda = 3, 7$

13. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

14. $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}, \lambda = 3$

15. $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \lambda = -5$

16. $A = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}, \lambda = 4$

Find the eigenvalues of the matrices in Exercises 17 and 18.

17. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

19. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue, with no calculation. Justify your answer.

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Justify your answer.

In Exercises 21 and 22, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer

.

21. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

b. A matrix A is not invertible if and only if 0 is an eigenvalue of A.

c. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

e. To find the eigenvalues of A, reduce A to echelon form.

22. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

b. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

c. A steady-state vector for a stochastic matrix is actually an eigenvector.

d. The eigenvalues of a matrix are on its main diagonal.

e. An eigenspace of A is a null space of a certain matrix.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

24. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

25. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

26. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

28. Use Exercise 27 to complete the proof of Theorem 1 for the case in which A is lower triangular.

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Find an eigenvector.]

30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Use Exercises 27 and 29.]

In Exercises 31 and 32, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.

31. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

32. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

33. Let $\mathbf{u}$ and $\mathbf{v}$ be eigenvectors of a matrix A, with corresponding eigenvalues λ and μ, and let c_1 and c_2 be scalars. Define

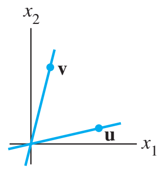
$$\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v} \quad (k = 0, 1, 2, \ldots)$$

a. What is $\mathbf{x}_{k+1}$, by definition?

b. Compute $A\mathbf{x}_k$ from the formula for $\mathbf{x}_k$, and show that $A\mathbf{x}_k = \mathbf{x}_{k+1}$. This calculation will prove that the sequence $\{\mathbf{x}_k\}$ defined above satisfies the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$.

34. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \ldots)$ if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. [*Hint:* How might you relate $\mathbf{x}_0$ to eigenvectors of A?]

35. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown in the figure, and suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.



36. Repeat Exercise 35, assuming $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A that correspond to eigenvalues -1 and 3, respectively.

[**M**] In Exercises 37–40, use a matrix program to find the eigenvalues of the matrix. Then use the method of Example 4 with a row reduction routine to produce a basis for each eigenspace.

37. $\begin{bmatrix} 12 & 1 & 4 \\ 2 & 11 & 4 \\ 1 & 3 & 7 \end{bmatrix}$ **38.** $\begin{bmatrix} 5 & -2 & 2 & -4 \\ 7 & -4 & 2 & -4 \\ 4 & -4 & 2 & 0 \\ 3 & -1 & 1 & -3 \end{bmatrix}$

39. $\begin{bmatrix} 12 & -90 & 30 & 30 & 30 \\ 8 & -49 & 15 & 15 & 15 \\ 16 & -52 & 12 & 0 & 20 \\ 0 & -30 & 10 & 22 & 10 \\ 8 & -41 & 15 & 15 & 7 \end{bmatrix}$

40. $\begin{bmatrix} -23 & 57 & -9 & -15 & -59 \\ -10 & 12 & -10 & 2 & -22 \\ 11 & 5 & -3 & -19 & -15 \\ -27 & 31 & -27 & 25 & -37 \\ -5 & -15 & -5 & 1 & 31 \end{bmatrix}$

SOLUTIONS TO PRACTICE PROBLEMS

1. The number 5 is an eigenvalue of A if and only if the equation $(A - 5I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Form

$$A - 5I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

and row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

At this point, it is clear that the homogeneous system has no free variables. Thus $A - 5I$ is an invertible matrix, which means that 5 is *not* an eigenvalue of A.

2. If $\mathbf{x}$ is an eigenvector of A corresponding to λ, then $A\mathbf{x} = \lambda\mathbf{x}$ and so

$$A^2\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

Again, $A^3\mathbf{x} = A(A^2\mathbf{x}) = A(\lambda^2\mathbf{x}) = \lambda^2 A\mathbf{x} = \lambda^3\mathbf{x}$. The general pattern, $A^k\mathbf{x} = \lambda^k\mathbf{x}$, is proved by induction.

3. Yes. Suppose $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 + c_4\mathbf{b}_4 = \mathbf{0}$. Since any linear combination of eigenvectors from the same eigenvalue is again an eigenvector for that eigenvalue, $c_3\mathbf{b}_3 + c_4\mathbf{b}_4$ is an eigenvector for λ_3. By Theorem 2, the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $c_3\mathbf{b}_3 + c_4\mathbf{b}_4$ are linearly independent, so

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + (c_3\mathbf{b}_3 + c_4\mathbf{b}_4) = \mathbf{0}$$

implies $c_1 = c_2 = 0$. But then, c_3 and c_4 must also be zero since $\mathbf{b}_3$ and $\mathbf{b}_4$ are linearly independent. Hence all the coefficients in the original equation must be zero, and the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, and $\mathbf{b}_4$ are linearly independent.

5.2 THE CHARACTERISTIC EQUATION

Useful information about the eigenvalues of a square matrix A is encoded in a special scalar equation called the characteristic equation of A. A simple example will lead to the general case.