LOXODROMIC ELEMENTS IN THE CYCLIC SPLITTING COMPLEX AND THEIR CENTRALIZERS

RADHIKA GUPTA AND DERRICK WIGGLESWORTH

Abstract. We show that an outer automorphism acts loxodromically on the cyclic splitting complex if and only if it has a filling lamination and no generic leaf of the lamination is carried by a vertex group of a cyclic splitting. This is the analog for the cyclic splitting complex of Handel-Mosher's theorem on loxodromics for the free splitting complex. We also show that such outer automorphisms have virtually cyclic centralizers.

1. Introduction

The study of the mapping class group of a closed orientable surface S has benefited greatly from its action on the curve complex, $\mathcal{C}(S)$, which was shown to be hyperbolic in [MM99]. Curve complexes have been used for bounded cohomology of subgroups of mapping class groups, rigidity results, and myriad other applications.

The outer automorphism group of a finite rank free group $\mathbb{F}$, denoted by $\mathrm{Out}(\mathbb{F})$, is defined as the quotient of $\mathrm{Aut}(\mathbb{F})$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(\mathbb{F})$ draws parallels with the study of mapping class groups. This analogy, however, is far from perfect; there are several $\mathrm{Out}(\mathbb{F})$-complexes that act as analogs for the curve complex. Among them are the free splitting complex $\mathcal{FS}$, the cyclic splitting complex $\mathcal{FZ}$, and the free factor complex $\mathcal{FF}$, all of which have been shown to be hyperbolic [HM13b, Man14, BF14]. Just as curve complexes have yielded useful information about mapping class groups, so too have these complexes furthered our understanding of $\mathrm{Out}(\mathbb{F})$.

The three hyperbolic $\mathrm{Out}(\mathbb{F})$-complexes mentioned above are related via Lipschitz maps, $\mathcal{FS} \to \mathcal{FZ} \to \mathcal{FF}$. The loxodromics for $\mathcal{FF}$ have been identified with the set of fully irreducible outer automorphisms [BF14]. In [HM14], the authors proved that an outer automorphism, ϕ, acts loxodromically on $\mathcal{FS}$ precisely when ϕ has a *filling lamination*, that is, some element of the finite set of laminations associated to ϕ (see [BFH00]) is not carried by a vertex group of any free splitting. In this paper, we focus our attention on the isometry type of outer automorphisms, considered as elements of $\mathrm{Isom}(\mathcal{FZ})$.

The cyclic splitting complex $\mathcal{FZ}$, introduced in [Man14], is defined as follows: vertices are one-edge splittings of $\mathbb{F}$ with edge stabilizer either trivial or $\mathbb{Z}$ and k-simplicies correspond to a collections of $k+1$ vertices, each of which is compatible with a k-edge $\mathcal{Z}$-splitting. In this paper, we determine precisely which outer automorphisms act loxodromically on $\mathcal{FZ}$.

In [BFH00], the authors associate to each $\phi \in \mathrm{Out}(\mathbb{F})$ a finite set of attracting laminations, denoted by $\mathcal{L}(\phi)$. We say that a lamination $\Lambda \in \mathcal{L}(\phi)$ is *$\mathcal{Z}$-filling* if no generic leaf of Λ is carried by a vertex group of a one-edge $\mathcal{Z}$-splitting; we say that ϕ has a $\mathcal{Z}$-filling lamination if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}$-filling. We prove

---

2010 *Mathematics Subject Classification.* Primary 20F65; Secondary 20F28, 20E05, 57M07.

October 28, 2017.

Both authors are partially supported by the U.S. National Science Foundation grant of Mladen Bestvina (DMS-1607236).

Theorem 1.1. *An outer automorphism, ϕ, acts loxodromically on the cyclic splitting complex if and only if it has a $\mathcal{Z}$-filling lamination. Furthermore, if ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then a power of ϕ fixes a point in $\mathcal{FZ}$.*

In [HW15], Horbez and Wade showed that every isometry of $\mathcal{FZ}$ is induced by an outer automorphism. Combining their result with [HM14, Theorem 1.1] and Theorem 1.1, this amounts to a classification of the isometries of $\mathcal{FZ}$.

Corollary 1.2 (Classification of isometries of $\mathcal{FZ}$)**.** *The following hold for all $\phi \in \mathrm{Isom}(\mathcal{FZ})$.*

(1) The action of ϕ on $\mathcal{FZ}$ is loxodromic if and only if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}$-filling.
(2) If the action of ϕ on $\mathcal{FZ}$ is not loxodromic, then it has bounded orbits (there are no parabolic isometries).

The proof of Theorem 1.1 relies on the description of the boundary of $\mathcal{FZ}$ due to Horbez [Hor14]; points in the boundary of $\mathcal{FZ}$ are equivalence classes of $\mathcal{Z}$-averse trees. The proof is carried out as follows. In Section 3, we extend the theory of folding paths to the boundary of Culler & Vogtmann's outer space, $\mathbb{PO}$, defining a folding path guided by ϕ which is entirely contained in $\partial\mathbb{PO}$. In Section 4, we show that the limit of the folding path thus constructed is $\mathcal{Z}$-averse. In Section 5, we show that an outer automorphism with a filling but not $\mathcal{Z}$-filling lamination fixes (up to taking a power) a point in $\mathcal{FZ}$ and conclude with a proof of Theorem 1.1.

The remainder of the paper is devoted to a study of the centralizers of automorphisms with filling laminations. We prove the following result:

Theorem 1.3. *An outer automorphism with a filling lamination has a virtually cyclic centralizer in* $\mathrm{Out}(\mathbb{F})$ *if and only if the lamination is $\mathcal{Z}$-filling.*

The key tools used to prove Theorem 1.3 are the completely split train tracks introduced in [FH11] and the disintegration theory for outer automorphisms developed in [FH09]. We first show (Proposition 7.3) that the disintegration of any outer automorphism ϕ, that has a $\mathcal{Z}$-filling lamination, is virtually cyclic. Then we show that Proposition 7.3 implies the centralizer of ϕ is also virtually cyclic. Conversely, in Proposition 7.10, we show that if ϕ has a filling lamination that is not $\mathcal{Z}$-filling, then ϕ commutes with an appropriately chosen partial conjugation.

The method used to prove Theorem 1.3 provides alternate (and simple) proof of the well-known fact due to Bestvina, Feighn and Handel that centralizers of fully irreducible outer automorphisms are virtually cyclic. In [BFH00], the stretch factor homomorphism is used to show that the stabilizer of the lamination of a fully irreducible outer automorphism is virtually cyclic, which implies that the centralizer is also virtually cyclic. In general, not much is known about the centralizers of outer automorphisms. In [RW15], Rodenhausen and Wade describe an algorithm to find the presentation of the centralizer of an outer automorphism that is a Dehn Twist. In [FH09], Feighn and Handel show that the disintegration of an outer automorphism $\mathcal{D}(\phi)$ is contained in the weak center of the centralizer of ϕ. Recently, Algom-Kfir and Pfaff showed [AP16] that centralizers of fully irreducible outer automorphisms with lone axes are isomorphic to $\mathbb{Z}$. We also mention a result of Kapovich and Lustig [KL11]: automorphisms whose limiting trees are free have virtually cyclic centralizers.

The main motivation for examining the centralizers of loxodromic elements of $\mathcal{FZ}$ (and $\mathcal{FS}$) is to understand which automorphisms have the potential to be WPD elements for the action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{FS}$ or $\mathcal{FZ}$.

Corollary 1.4. *Any outer automorphism that is loxodromic for the action of* $\mathrm{Out}(\mathbb{F})$ *on $\mathcal{FS}$ but elliptic for the action on $\mathcal{FZ}$ is not a WPD element for the action on $\mathcal{FS}$.*

The result that centralizers of loxodromic elements of $\mathcal{FZ}$ are virtually cyclic is a promising sign for the following conjecture:

Conjecture 1.5. The action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{FZ}$ is a WPD action. That is, every loxodromic element for the action satisfies WPD.

Acknowledgements: We would like to thank Mladen Bestvina for suggesting this project, sharing his intuition and for the many discussions we had with him. We are especially grateful to Carolyn Abbott for her sustained enthusiasm and interest this project. The first author would also like to thank Camille Horbez for his encouragement in the very early stage of this project. We also acknowledge Mathematical Sciences Research Institute (MSRI), Berkeley, California, where we first became interested in this project.

2. Preliminaries

2.1. **Isometries of metric spaces.** Let X be a Gromov hyperbolic metric space. We say that an infinite order isometry g of X is *loxodromic* if it acts with positive translation length on X: $\lim_{N\to\infty}\frac{d(x,g^N(x))}{N} > 0$ for some (any) $x \in X$. Every loxodromic element has exactly two limit points in the Gromov boundary of X.

2.2. **Outer space and its compactification.** Culler Vogtmann's *outer space*, $\mathbb{P}\mathcal{O}$, is defined in [CV86] as the space of simplicial, free, and minimal isometric actions of $\mathbb{F}$ on simplicial metric trees up to $\mathbb{F}$-equivariant homothety. We denote by $\mathcal{O}$ the *unprojectivized outer space*, in which the trees are considered up to isometry, rather than homothety. Each of these spaces is equipped with a natural (right) action of $\mathrm{Out}(\mathbb{F})$.

An $\mathbb{F}$-tree is an $\mathbb{R}$-tree with an isometric action of $\mathbb{F}$. An $\mathbb{F}$-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic, and tripod stabilizers are trivial. Outer space can be mapped into $\mathbb{R}^{\mathbb{F}}$ by the map $T \mapsto (\|g\|_T)_{g\in\mathbb{F}}$, where $\|g\|_T$ denotes the translation length of g in T. This was shown in [CM87] to be a continuous injection. The closure of $\mathbb{P}\mathcal{O}$ under the embedding into $\mathbb{P}\mathbb{R}^{\mathbb{F}}$ is compact and was identified in [BF94] and [CL95] with the space of all very small $\mathbb{F}$-trees. We denote by $\overline{\mathbb{P}\mathcal{O}}$ the closure of outer space and by $\partial\mathbb{P}\mathcal{O}$ its boundary.

2.3. **Free factor system.** A free factor system of $\mathbb{F}$ is a finite collection of conjugacy classes of proper free factors of $\mathbb{F}$ of the form $\mathcal{A} = \{[A_1], \dots, [A_k]\}$, where $k \geq 0$ and $[\cdot]$ denotes the conjugacy class of a subgroup, such that there exists a free factorization $\mathbb{F} = A_1 * \cdots * A_k * F_N$. We refer to the free factor F_N as the *cofactor* of $\mathcal{A}$ keeping in mind that it is not unique, even up to conjugacy.

The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose non-contractible connected components are denoted $C_1, \dots, C_k$. Let $[A_i]$ be the conjugacy class of a free factor of $\mathbb{F}$ determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ is a free factor system. We say $\mathcal{A}$ is *realized by* K and we denote it by $\mathcal{F}(K)$.

2.4. **Marked graphs.** We recall some basic definitions from [BH92]. Identify $\mathbb{F}$ with $\pi_1(\mathcal{R}, *)$ where $\mathcal{R}$ is a rose with $\mathfrak{n}$ petals, $\mathfrak{n}$ being the rank of $\mathbb{F}$. A *marked graph* G is a graph of rank $\mathfrak{n}$, all of whose vertices have valence at least two, equipped with a homotopy equivalence $m \colon \mathcal{R} \to G$ called a *marking*. The marking determines an identification of $\mathbb{F}$ with $\pi_1(G, m(*))$. A homotopy equivalence $f \colon G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element ϕ of $\mathrm{Out}(\mathbb{F})$. If f sends vertices to vertices and the restriction of f to edges is an immersion then we say that f is a *topological representative* of ϕ. All homotopy equivalences will be assumed to map vertices to vertices and the restriction to any edge will be assumed to be an immersion.

2.5. **Paths, circuits, and tightening.** Let Γ be either a marked graph or an $\mathbb{F}$-tree. A *path* in Γ is either an isometric immersion of a (possibly infinite) closed interval $\sigma\colon I \to \Gamma$ or a constant map $\sigma\colon I \to \Gamma$. If σ is a constant map, the path will be called *trivial*. If I is finite, then any map $\sigma\colon I \to \Gamma$ is homotopic rel endpoints to a unique path $[\sigma]$. We say that $[\sigma]$ is obtained by *tightening* σ. If $f\colon \Gamma \to \Gamma$ is continuous and σ is a path in Γ, we define $f_\#(\sigma)$ as $[f(\sigma)]$. If the domain of σ is finite and Γ is either a graph or a simplicial tree, then the image has a natural decomposition into edges $E_1E_2\cdots E_k$ called the *edge path associated to* σ. If Γ is a tree, we may use $[x, x']$ to denote the unique geodesic path connecting x and x'.

A *circuit* is an immersion $\sigma\colon S^1 \to \Gamma$. For any path or circuit, let $\overline{\sigma}$ be σ with its orientation reversed. A decomposition of a path or circuit into subpaths is a *splitting* for $f\colon \Gamma \to \Gamma$ and is denoted $\sigma = \ldots\sigma_1 \cdot \sigma_2 \ldots$ if $f^k_\#(\sigma) = \ldots f^k_\#(\sigma_1)f^k_\#(\sigma_2)\ldots$ for all $k \geq 1$.

2.6. **Turns, directions, and train track structures.** Let Γ be a $\mathbb{F}$-tree. A direction d based at $p \in \Gamma$ is a component of $\Gamma - \{p\}$. A *turn* is an unordered pair of directions based at the same point. If Γ is a graph, then a direction based at $p \in \Gamma$ is an $\mathbb{F}$-orbit of directions in its universal cover based at lifts of p. In the case that Γ is a graph or a simplicial tree, and p is a vertex, we may identify directions at p with edges emanating from p. A *train track structure* on Γ is an equivalence relation on the set of directions at each point $p \in \Gamma$. The classes of this relation are called *gates*. A turn (d, d') is *legal* if d and d' do not belong to the same gate. A path is legal if it only crosses legal turns.

A train track structure on a graph is defined by passing to its universal cover. If Γ is a graph and $f\colon G \to G$ is a homotopy equivalence, then f induces a train track structure on Γ as follows. The map f determines a map Df on the directions in G by definining $Df(E)$ to be the first edge in the edge path $f(E)$. We then declare $E_1 \sim E_2$ if $D(f^k)(E_1) = D(f^k)(E_2)$ for some $k \geq 1$.

2.7. **Optimal morphisms and train track maps.** Given two $\mathbb{F}$-trees Γ and Γ', an $\mathbb{F}$-equivariant map $f\colon \Gamma \to \Gamma'$ is called a *morphism* if every segment of Γ can be subdivided into finitely many subintervals such that f is an isometry when restricted to each subinterval. Just as with graphs, a morphism between $\mathbb{F}$-trees induces a train track structure on the domain, Γ. A morphism is called *optimal* if there are at least two gates at each point of Γ.

A morphism is called a *train track map* if f is an embedding on each edge and legal turns are sent to legal turns. When Γ is a graph, train track maps are defined by passing to the universal cover. For more details on train track maps, the reader is referred to [BF10, BH92].

2.8. **Relative train track maps and CTs.** A *filtration* for a topological representative $f\colon G \to G$ of an outer automorphism ϕ, where G is a marked graph, is an increasing sequence of f-invariant subgraphs $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$. We let $H_i = \overline{G_i \setminus G_{i-1}}$ and call H_i the *i-th stratum*. A turn with one edge in H_i and the other in G_{i-1} is called *mixed* while a turn with both edges in H_i is called a *turn in* H_i. If $\sigma \subset G_i$ does not contain any illegal turns in H_i, then we say σ is *i-legal*.

We denote by M_i the submatrix of the transition matrix for f obtained by deleting all rows and columns except those labeled by edges in H_i. For the topological representatives that will be of interest to us, the transition matrices M_i will come in three flavors: M_i may be a zero matrix, it may be the 1×1 identity matrix, or it may be an irreducible matrix with Perron-Frobenius eigenvalue $\lambda_i > 1$. We will call H_i a *zero* (Z), *non-exponentially growing* (NEG), or *exponentially growing* (EG) stratum, respectively. Any stratum which is not a zero stratum is called an *irreducible stratum*.

Definition 2.1 ([BH92])**.** We say that $f\colon G \to G$ is a *relative train track map* representing $\phi \in \mathrm{Out}(F_n)$ if for every exponentially growing stratum H_r, the following hold:

(RTT-i): Df maps the set of oriented edges in H_r to itself; in particular all mixed turns are legal.
(RTT-ii): If $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then so is $f_\#(\sigma)$.
(RTT-iii): If $\sigma \subset G_r$ is r-legal, then $f_\#(\sigma)$ is r-legal.

Suppose that $u < r$, that H_u is irreducible, H_r is EG and each component of G_r is non-contractible, and that for each $u < i < r$, H_i is a zero stratum which is a component of G_{r-1} and each vertex of H_i has valence at least two in G_r. Then we say that H_i is *enveloped by* H_r and we define $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

A path or circuit σ in a representative $f\colon G \to G$ is called a *periodic Nielsen path* if $f_\#^k(\sigma) = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path.* A Nielsen path is *indivisible*, denoted INP, if it cannot be written as a concatenation of non-trivial Nielsen paths. If w is a closed root-free Nielsen path and E_i is an edge such that $f(E_i) = E_i w^{d_i}$, then we say E *is a linear edge* and we call w the *axis* of E. If E_i, E_j are distinct linear edges with the same axis such that $d_i \neq d_j$ and $d_i, d_j > 0$, then we call a path of the form $E_i w^* \overline{E}_j$ an *exceptional path.* In the same scenario, if d_i and d_j have different signs, we call such a path a *quasi-exceptional path.* We say that x and y are *Nielsen equivalent* if there is a Nielsen path σ in G whose endpoints are x and y. We say that a periodic point $x \in G$ is *principal* if neither of the following conditions hold:

- x is an endpoint of a non-trivial periodic Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same EG stratum.
- x is contained in a component C of periodic points that is topologically a circle and each point in C has exactly two periodic directions.

A relative train track map f is called *rotationless* if each principal periodic vertex is fixed and if each periodic direction based at a principal vertex is fixed. We remark that there is a closely related notion, whose definition we will omit, of an outer automorphism ϕ being rotationless. We will simply rely on the following fact from [FH09], which (combined with the definition of a CT) provides a connection between these two notions:

Proposition 2.2 ([FH09, Corollary 3.5])**.** *There exists $k > 0$ depending only on $\mathfrak{n}$, so that ϕ^k is rotationless for every $\phi \in \mathrm{Out}(\mathbb{F})$.*

For an EG stratum, H_r, we call a non-trivial path $\sigma \subset G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ a *connecting path for H_r*. Let E be an edge in an irreducible stratum, H_r and let σ be a maximal subpath of $f_\#^k(E)$ in a zero stratum for some $k \geq 1$. Then we say that σ is *taken.* A non-trivial path or circuit σ is called *completely split* if it has a splitting $\sigma = \tau_1 \cdot \tau_2 \cdots \tau_k$ where each of the τ_i's is a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a connecting path in a zero stratum which is both maximal and taken. We say that a relative train track map is *completely split* if $f(E)$ is completely split for every edge E in an irreducible stratum *and* if for every taken connecting path σ in a zero stratum, $f_\#(\sigma)$ is completely split.

Definition 2.3 ([FH11])**.** A relative train track map $f\colon G \to G$ and filtration $\mathcal{F}$ given by $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$ is said to be a CT if it satisfies the following properties.

(Rotationless): $f\colon G \to G$ is rotationless.
(Completely Split): $f\colon G \to G$ is completely split.
(Filtration): $\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.
(Vertices): The endpoints of all indivisible periodic (necessarily fixed) Nielsen paths are (necessarily principal) vertices. The terminal endpoint of each non-fixed NEG edge is principal (and hence fixed).

(Periodic Edges): Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge E_r in a fixed stratum H_r is not a loop then G_{r-1} is a core graph and both ends of E_r are contained in G_{r-1}.

(Zero Strata): If H_i is a zero stratum, then H_i is enveloped by an EG stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.

(Linear Edges): For each linear E_i there is a closed root-free Nielsen path w_i such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If E_i and E_j are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$.

(NEG Nielsen Paths): If the highest edges in an indivisible Nielsen path σ belong to an NEG stratum then there is a linear edge E_i with w_i as in (Linear Edges) and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \bar{E}_i$.

(EG Nielsen Paths): If H_r is EG and ρ is an indivisible Nielsen path of height r, then $f|G_r = \theta \circ f_{r-1} \circ f_r$ where :

(1) $f_r : G_r \to G^1$ is a composition of proper extended folds defined by iteratively folding ρ.
(2) $f_{r-1} : G^1 \to G^2$ is a composition of folds involving edges in G_{r-1}.
(3) $\theta : G^2 \to G_r$ is a homeomorphism.

We remark that several of the properties in Definition 2.3 use terms that have not been defined. We will not use these properties in the sequel. The main result for CTs is the following existence theorem:

Theorem 2.4 ([FH11, Theorem 4.28])**.** *Given a rotationless $\phi \in \mathrm{Out}(F_n)$ and a nested sequence of ϕ-invariant free factor systems, there is a CT representing ϕ such that each of the free factor systems is realized by some filtration element.*

For completely split paths and circuits, all cancellation under iteration of $f_\#$ is confined to the individual terms of the splitting. Moreover, $f_\#(\sigma)$ has a complete splitting which refines that of σ. Finally, just as with improved relative train track maps introduced in [BFH00], every circuit or path with endpoints at vertices eventually is completely split.

2.9. **Lines and Laminations.** We briefly recall some definitions, but the reader is directed to [BFH00] for details. The *space of abstract lines*, $\widetilde{\mathcal{B}} = (\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$ is the set of unordered distinct pairs of points in the boundary of $\mathbb{F}$; $\widetilde{\mathcal{B}}$ is equipped with the topology of cylinder sets. The quotient of $\widetilde{\mathcal{B}}$ by the natural $\mathbb{F}$ action is *the space of lines in* $\mathcal{R}$ and is called $\mathcal{B}$. It is given the quotient topology, which satisfies none of the separation axioms.

A closed subset Λ of $\mathcal{B}$ is an *attracting lamination for* ϕ if it is the closure of a single point β that is *bireccurrent* (every finite subpath σ of β occurs infinitely many times as an unoriented subpath of each end of β), has an *attracting neigborhood* (there is some open $U \ni \beta$ so that $\phi^k(\gamma) \to \beta$ for all $\gamma \in U$), and which is not carried by a rank one ϕ-periodic free factor. The collection of lines in Λ satisfying the above properties are called the *generic leaves* of Λ.

A finitely generated subgroup A of $\mathbb{F}$ determines a subset of the boundary of $\mathbb{F}$ called $\partial A \subset \partial\mathbb{F}$. We say that A *carries* the lamination Λ if there is some lift $\widetilde{\beta}$ of a generic leaf of Λ whose endpoints are in ∂A.

2.10. **Bounded backtracking constant (BBT).** Let $f\colon T \to T'$ be a continuous map between two $\mathbb{R}$-trees T and T'. We say that f has bounded backtracking if the f image of any path $[p, q]$ is contained in a C-neighborhood of $[f(p), f(q)]$. The smallest such C is called the *bounded backtracking constant* of f, denoted $\mathrm{BBT}(f)$.

2.11. **Folding paths.** Let T and T' be two simplicial $\mathbb{F}$-trees in $\overline{\mathcal{O}}$ such that the set of point stabilizers of T and T' are the same. In [GL07b, Section 3], Guirardel and Levitt construct a *canonical optimal folding path* $(T_t)_{t\in\mathbb{R}^+}$ guided by an optimal morphism $f\colon T \to T'$. The tree T_t is constructed as follows. Given $a, b \in T$ with $f(a) = f(b)$, the *identification time* of a and b is defined as $\tau(a, b) = \sup_{x\in[a,b]} d_{T'}(f(x), f(a))$. Define $L := \frac{1}{2}\operatorname{BBT}(f)$. For each $t \in [0, L]$, one defines an equivalence relation $\sim_t$ by $a \sim_t b$ if $f(a) = f(b)$ and $\tau(a, b) < t$. The tree T_t is then a quotient of T by the equivalence relation $\sim_t$. The authors prove that for each $t \in [0, L]$, T_t is an $\mathbb{R}$-tree. The collection of trees $(T_t)_{t\in[0,L]}$ comes equipped with $\mathbb{F}$-equivariant morphisms $f_{s,t}\colon T_t \to T_s$ for all $t < s$ and these maps satisfy the semi-flow property: for all $r < s < t$, we have $f_{t,s} \circ f_{s,r} = f_{t,r}$. Moreover $T_L = T'$ and $f_{L,0} = f$. The set of data $(T_t)_{t\in[0,L]}$, $(f_{s,t}\colon T_t \to T_s)_{t<s\in[0,L]}$ is called the *connection data*.

2.12. **Transverse families and transverse coverings.** A subtree Y of a tree T is called *closed* [Gui04, Definition 2.4] if $Y \cap \sigma$ is either empty or a path in T for all paths $\sigma \subset T$; recall that paths are defined on closed intervals. A *transverse family* [Gui04, Definition 4.6] of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of non-degenerate closed subtrees of T such that any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point. If every path in T intersects only finitely many subtrees in $\mathcal{Y}$, then the transverse family is called a *transverse covering*.

2.13. **Mixing and indecomposable trees.** A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is *mixing* if for all finite subarcs $I, J \subset T$, there exist $g_1, \ldots, g_k \in \mathbb{F}$ such that $J \subseteq g_1I \cup g_2I \cup \cdots \cup g_kI$ and $g_iI \cap g_{i+1}I \neq \emptyset$ for all $i \in \{1, \ldots, k-1\}$. A tree $T \in \overline{\mathbb{P}\mathcal{O}}$ is called *indecomposable* [Gui08] if it is mixing and $g_i \cap g_{i+1}I$ is a non-degenerate arc for each $i \in \{1, \ldots, k-1\}$. An $\mathbb{F}$-tree is indecomposable if and only if it has no transverse family containing a proper subtree.

2.14. **Cyclic splitting complex and $\mathcal{Z}$-averse trees.** Let $\mathcal{Z}$ be the set of cyclic subgroups of $\mathbb{F}$. A *$\mathcal{Z}$-splitting* is a minimal, simplicial $\mathbb{F}$-tree whose edge stabilizers belong to the set $\mathcal{Z}$; it is a *one-edge splitting* if there is one $\mathbb{F}$ orbit of edges. A *cyclic splitting* is a one-edge $\mathcal{Z}$-splitting whose edge stabilizer is infinite cyclic. Two $\mathcal{Z}$-splittings are *equivalent* if they are $\mathbb{F}$-equivariantly homeomorphic. Given two $\mathcal{Z}$-splittings T and T', T is a *refinement* of T' if there is a collapse map from T to T', that is, T' is obtained from T by equivariantly collapsing a set of edges of T. Two $\mathcal{Z}$-splittings are *compatible* if there is a common refinement. A tree T is *$\mathcal{Z}$-incompatible* if the set of $\mathcal{Z}$-splittings that are compatible with T is empty. The cyclic splitting complex $\mathcal{FZ}$ is the simplicial complex whose vertices are equivalence classes of one-edge $\mathcal{Z}$-splittings and whose k-simplicies are collections of $k+1$ pairwise compatible one-edge $\mathcal{Z}$-splittings. In [Man14], Mann showed that $\mathcal{FZ}$ is a δ-hyperbolic space.

In [Hor14], Horbez, characterized the boundary of the cyclic splitting complex as the set of $\mathcal{Z}$-*averse* trees. A tree in $\overline{\mathbb{P}\mathcal{O}}$ is called $\mathcal{Z}$-averse if it is not compatible with any $\mathbb{F}$-tree in $\mathbb{P}\mathcal{O}$ that is itself compatible with a $\mathcal{Z}$-splitting. Let $\mathcal{X}(\mathbb{F})$ denote the set of $\mathcal{Z}$-averse trees.

3. Folding sequence construction

Throughout this section, ϕ will be an outer automorphism with a $\mathcal{Z}$-filling lamination Λ^+_ϕ. Our first goal is to extract from ϕ a folding path converging to a tree in $\partial\mathbb{P}\mathcal{O}$ which "witnesses" the lamination Λ^+_ϕ. As the assumption on ϕ implies that it is fully irreducible relative to some free factor system $\mathcal{A}$, we let $f\colon T \to T$ be the universal cover of a relative train track representative of ϕ realizing the invariant free factor system $\mathcal{A}$. Let $G = T/\mathbb{F}$ be the quotient graph, which comes with a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_r = G$ such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$ and H_r is an EG stratum with Perron-Frobenius eigenvalue λ_ϕ. Let T_r (resp. T_{r-1}) denote the full preimage of H_r (resp. G_{r-1})

under the quotient map $T \to G$. We will henceforth consider T as a point in unprojectivized outer space $\mathcal{O}$, whereby f may be thought of as an $\mathbb{F}$-equivariant map $T \to T \cdot \phi$.

Let T_0' be the tree obtained from T by equivariantly collapsing the $\mathcal{A}$-minimal subtree. Our present aim is to construct a folding path ending at $T_\phi^+ := \lim_{n\to\infty} T_0'\phi^n/\lambda_\phi^n$. To accomplish this, we will construct simplicial trees T_0, T_1 and define an optimal morphism $f_0 \colon T_0 \to T_1$. From this we will obtain a periodic canonical optimal folding path $(f_t)_{t\in[0,L]}$ which will end at T_ϕ^+. It is worth noting that the natural map $f_0' \colon T_0' \to T_0'\phi$ induced by f is neither optimal nor a morphism as there may be non-degenerate intervals which are mapped to points.

Constructing T_0. The following is based on the construction in the proof of [BH92, Lemma 5.10]. Define a measure μ on T with support contained in the set $\{x \in T_r : f^k(x) \in T_r \text{ for all } k \geq 0\}$ as follows: choose a Perron Frobenius (PF) eigenvector $\vec{v}$ corresponding to the PF eigenvalue λ_ϕ. For an edge e in T_r, let $\mu(e) = v_e$ where v_e is the component of $\vec{v}$ corresponding to e. Define $\mu(e) = 0$ for all edges $e \in T_{r-1}$. Let V be the set of vertices of T and let $V_m := \{x \in T : f^m(x) \in V\}$. Subdividing T at V_m divides each edge into segments that map to edge paths under f^m. If a is such a segment then define $\mu(a) = \mu(f^m(a))/\lambda_\phi^m$. The definition of μ together with the fact that relative train track maps take r-legal paths to r-legal paths implies:

Lemma 3.1. *If $[x, y]$ is an r-legal path in T, then $\mu(f_\#([x, y])) = \lambda_\phi \mu([x, y])$. If $[x, y]$ contains an initial or terminal segment of some edge in T_r, then $\mu([x, y]) > 0$.*

The measure μ defines a pseudometric d_μ on T. Collapsing the sets of μ-measure zero to make d_μ into a metric, we obtain a tree T_0.

Lemma 3.2. *T_0 is simplicial.*

Proof. We will show that the $\mathbb{F}$-orbit of any point in T_0 must be discrete. Let $x \in T_0$ and choose a point $\tilde{x} \in p^{-1}(x)$. The $\mathbb{F}$-orbit of $\tilde{x}$ in T is discrete, and to understand the orbit of x, we need only understand $\mu([\tilde{x}, g\tilde{x}])$ for $g \in \mathbb{F}$. If $[\tilde{x}, g\tilde{x}]$ contains no edges in T_r, then $\mu([\tilde{x}, g\tilde{x}]) = 0$, in which case $g \in \mathrm{Stab}(x)$. Otherwise, the segment contains an edge in T_r, and hence has positive μ-measure. Since there are only finitely many $\mathbb{F}$-orbits of edges in T_r, there is a lower bound on the μ-measure of $[\tilde{x}, g\tilde{x}]$. Hence, there is a lower bound on $d_{T_0}(x, gx)$. This concludes the proof. ☐

Defining $f_0 \colon T_0 \to T_1$. Let T_1 be the tree $\lambda_\phi^{-1} T_0 \cdot \phi$: the leading coefficient indicates that the metric has been scaled by λ_ϕ^{-1}. The relative train track map $f \colon T \to T \cdot \phi$ naturally induces a map $f_0 \colon T_0 \to T_1$. For each $x \in T_0$, its pre-image $p^{-1}(x)$ is a connected subtree of T with μ-measure zero. The definition of μ guarantees that the f-image of this set is also connected and has μ-measure zero. Therefore $p \circ f \circ p^{-1}(x)$ is a single point in $T_0 \cdot \phi$, which is identified with T_1 and we define $f_0 := p \circ f \circ p^{-1}$.

Lemma 3.3. *f_0 is an optimal morphism.*

Proof. We first show that f_0 is a morphism, which will follow from the definition of μ and properties of relative train track maps. Given a non-degenerate segment $[x, x']$ in T_0, choose $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(x')$. The intersection of $[\tilde{x}, \tilde{x}']$ with the vertices of T is a finite set $\{\tilde{x}_1, \ldots, \tilde{x}_{k-1}\}$. Let $\tilde{x}_0 := \tilde{x}$ and $\tilde{x}_k := \tilde{x}'$. Taking the p-image of $\tilde{x}_i$ for $i \in \{0, \ldots, k\}$ yields a subdivision of $[x, x']$ into finitely many subsegments $[x_i, x_{i+1}]$, some of which may be degenerate. We will ignore the degenerate subdivisions: they occur as the projections of edges in T_{r-1} (all of which have μ-measure zero).

We claim that f_0 is an isometry in restriction to each of these subsegments. Indeed, let $e = [\tilde{x}_i, \tilde{x}_{i+1}]$ be an edge in T. Assume without loss of generality that $x_i \neq x_{i+1}$ so that $\mu(e) \neq 0$ and

e is therefore an edge in T_r. It is an immediate consequence of Lemma 3.1 that for each $y \in e$, $\mu([f(\tilde{x}_i), f(y)]) = \lambda_\phi \mu([\tilde{x}_i, y])$ and hence that f_0 is an isometry in restriction to $[x_i, x_{i+1}]$.

We now address the optimality of f_0. There are three types of points to consider: points in the interior of an edge, vertices with trivial stabilizer, and vertices with non-trivial stabilizer. We have already established that f_0 is an isometry in restriction to edges, so there are two gates at each $x \in T_0$ contained in the interior of an edge. If $x \in T_0$ is a vertex with trivial stabilizer, then $p^{-1}(x)$ is a vertex (Lemma 3.1) contained in $T_r \smallsetminus T_{r-1}$. As f is a relative train track map, there are at least two gates at $p^{-1}(x)$ and each is necessarily contained in T_r. A short path in T containing $p^{-1}(x)$ entering through the first gate and leaving through the second will be legal. Lemma 3.1 again gives that f_0 is an isometry in restriction to such a path, so there are at least two gates at x.

Now let $x \in T_0$ be a vertex with non-trivial stabilizer. Then $p^{-1}(x)$ is a subtree which is the inverse image of a component of G_{r-1} under the quotient map $T \to G$. Let $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ be distinct vertices in $T_r \cap T_{r-1}$ and let d (resp. d') be a direction based at $\tilde{x}$ (resp. $\tilde{x}'$) corresponding to an edge e (resp. e') in T_r. Lemma 3.1 provides that d and d' determine distinct directions at x. As mixed turns are legal, the path $\overline{e} \cup [\tilde{x}, \tilde{x}'] \cup e'$ in T is r-legal. A final application of Lemma 3.1 gives that the restriction of f_0 to the p-image of this path is an isometry, and hence that there are at least two directions at x. □

The reader will note that we have actually proved

Lemma 3.4. *f_0 is a train track map.*

Next, we use this map to construct a folding path starting at T_0. This folding path will converge in $\partial\mathbb{P}\mathcal{O}$ to a tree T_L. We then prove that T_L is in fact the tree T_ϕ^+ defined above.

Folding T_0. Applying the canonical folding path construction, we obtain a folding path $(T_t)_{t\in[0,L_0]}$ guided by $f_0\colon T_0 \to T_1$ which begins at T_0 and ends at T_1, where $L_0 = \frac{1}{2}\operatorname{BBT}(f_0)$. Adapting a construction of Handel-Mosher [HM11, Section 7.1], we now extend this to a *periodic fold path guided by* f_0. For each $i \in \mathbb{N}$, let $T_i = \lambda_\phi^{-i} T_0 \cdot \phi^i$, whence we have optimal morphisms $f_i\colon T_i \to T_{i+1}$ satisfying $\operatorname{BBT}(f_i) = \lambda_\phi^{-i}\operatorname{BBT}(f_0)$. For each i, inductively define $L_i := L_{i-1} + \frac{1}{2}\operatorname{BBT}(f_i)$ and extend the folding path (which has so far been defined on $[0, L_{i-1}]$) using f_i to a folding path $(T_t)_{t\in[0,L_i]}$. Define $L := \lim_{i\to\infty} L_i$, which is finite as $\operatorname{BBT}(f_i)$ is a geometric sequence. We have thus defined the trees $(T_t)_{t\in[0,L)}$.

We now describe the maps $f_{t,s}$ for $s, t \in [0, L)$ with $s < t$. Indeed, given s, t, there is a natural choice of a map $f_{t,s}\colon T_s \to T_t$. Suppose $s \in [L_i, L_{i+1})$ and $t \in [L_j, L_{j+1})$. Then

$$f_{t,s} := f_{t,L_j} \circ f_{j-1} \circ f_{j-2} \circ \ldots \circ f_{i+1} \circ f_{L_{i+1},s}$$

The semi-flow property for the connection data follows from the definitions. Though our setting differs slightly from that of [BF14], Proposition 2.2 (5) can still be applied to give that each tree T_t has a well defined train track structure.

Along with the connection data, the fold path $(T_t)_{t\in[0,L)}$ forms a directed system in the category of $\mathbb{F}$-equivariant metric spaces and distance non-increasing maps. As direct limits exist in this category, let $T_L := \varinjlim T_t$ and let $f_{L,t}$ be the direct limit maps. The proof of the following proposition is contained in Section 7.3 of [HM11], though it is not stated in this way. While Handel-Mosher deal with trees in $\mathcal{O}$ rather than $\partial\mathbb{P}\mathcal{O}$, the reader will easily verify that their proof goes through directly in our setting.

Proposition 3.5 ([HM11])**.** *T_L is a non-trivial, minimal, $\mathbb{R}$-tree. Moreover T_t converges to T_L in the length function topology.*

We have now described two trees in the boundary of outer space: $T_\phi^+ = \lim_{n\to\infty} T_0'\phi^n$ and T_L. We observe that both T_0 and T_0' are points in the relative outer space $\mathcal{O}(\mathbb{F},\mathcal{A})$, which inherits the subspace topology from $\overline{\mathcal{O}}$. Moreover, ϕ is fully irreducible relative to $\mathcal{A}$, and as such, it acts with north-south dynamics on $\mathcal{O}(\mathbb{F},\mathcal{A})$ [Gup16]. Recall that for each $i \in \mathbb{N}$, $T_{L_i} = T_0 \cdot \phi^i/\lambda_\phi^i$, and that $L_i \to L$. As T_L is the limit of the fold path $(T_t)_{t\in[0,L)}$, we conclude

Lemma 3.6. $T_L = T_\phi^+$.

4. Stable tree is $\mathcal{Z}$-averse

Our present aim is to understand T_ϕ^+; we would like to show that it is $\mathcal{Z}$-averse. In this section, we will use the leaves of the topmost lamination Λ_ϕ^+ to construct a transverse covering of T_ϕ^+, then use the transverse covering to achieve our goal.

Definition 4.1 (Transverse family)**.** Let $I = [x, y]$ be a non-degenerate arc in T_ϕ^+ which is a segment of a leaf of Λ_ϕ^+. Define Y_I as the union of all arcs J such that there exists $g_1, \ldots, g_m \in \mathbb{F}$ with $J \subseteq g_1 I \cup \cdots \cup g_m I$ and such that $g_i I \cap g_{i+1} I$ is non-degenerate for each $i \in \{1, \ldots, m-1\}$. The collection $\mathcal{Y} = \{gY_I\}_{g\in\mathbb{F}}$ is a transverse family in T_ϕ^+ since, by definition, distinct $\mathbb{F}$-translates of Y_I intersect in a point or not at all. This construction is due to Guirardel-Levitt.

Lemma 4.2. *With notation as above, Y_I is an indecomposable tree. Moreover, $\mathcal{Y} = \{gY_I\}_{g\in\mathbb{F}}$ is a transverse covering of T_ϕ^+.*

Proof. We first show that Y_I is indecomposable. It is enough to show that every arc $J \subseteq Y_I$ can be covered by finitely many translates with non-degenerate overlap of the fixed arc I, and conversely that I can be covered by finitely many translates of J with non-degenerate overlap. Indeed, let $J = [x', y']$ be a non-degenerate arc in T_ϕ^+ and recall that $I = [x, y]$. The definition of Y_I guarantees that J can be covered by finitely many translates of I, so we are left to show the converse.

The construction in Section 3 provides an optimal folding path $(T_t)_{t\in[0,L]}$, and optimal morphisms $f_{s,t}\colon T_t \to T_s$ for all $s, t \in [0, L]$ with $s > t$ which satisfy the semi-flow property. Since (T_t) is a folding path, for any z in $T_L = T_\phi^+$, the set $f_{L,0}^{-1}(z)$ is a discrete set of points in T_0. Let $x_0 \in f_{L,0}^{-1}(x)$ and $y_0 \in f_{L,0}^{-1}(y)$ be points in T_0 chosen so that (x_0, y_0) contains no points in $f_{L,0}^{-1}(x) \cup f_{L,0}^{-1}(y)$ and define $I_0 = [x_0, y_0]$. Define J_0 by choosing $x_0' \in f_{L,0}^{-1}(x')$ and $y_0' \in f_{L,0}^{-1}(y')$ similarly. Define the arc I_t (resp. J_t) in T_t by $I_t := [f_{t,0}(I_0)]$ (resp. $J_t := [f_{t,0}(J_0)]$). The definitions of I_0 and J_0 guarantee that $[f_{L,0}(I_0)] = I$ and similarly for J_0. The semiflow property of the maps $f_{s,t}$ gives that for all $s, t \in [0, L]$ with $s > t$, we have $[f_{s,t}(I_t)] = I_s$.

Now choose t large enough so that I_t is a crosses every turn taken by a leaf of Λ_ϕ^+; this is possible because I is itself a leaf segment. By enlarging t if necessary, we may arrange that J_t also crosses every turn taken by a leaf. While J_t may have illegal turns, I_t can nonetheless be covered by finitely many translates of the interval J_t with non-degenerate overlaps. This is because every turn in I_t can be covered by a legal turn in J_t. Thus finitely many translates of $[f_{L,t}(J_t)] = J$ cover $[f_{L,t}(I_t)] = I$ with non-degenerate overlaps.

We now show that $\mathcal{Y}$ is in fact a transverse covering. As before, by choosing t sufficiently large, we may assume that I_t crosses every turn taken by a leaf. Let J be an arc in T_ϕ^+ with pre-image J_t in T_t, which is necessarily a concatenation of finitely many leaf segments. As I_t crosses every turn taken by a leaf, each of these leaf segments can be covered by finitely many translates of I_t and we have a covering of J_t by finitely many translates of I_t (with degenerate overlaps at illegal turns). Using $f_{L,t}$, we conclude that finitely many translates of $I = [f_{L,t}(I_t)]$ cover $J = [f_{L,t}(I_t)]$. □

Lemma 4.3. *T_ϕ^+ is mixing.*

Proof. Since T_ϕ^+ has a transverse covering by translates of an indecomposable tree Y_I, T_ϕ^+ is mixing. □

Lemma 4.4. *If $T \in \overline{\mathcal{O}}$ is mixing, then either T is indecomposable, or T splits as a graph of actions with one orbit of subtrees.*

Proof. Follows from [Rey12, Lemma 5.5]. Alternatively, the proof is straightforward from the definitions. □

For convenience of the reader, we recall the following essential fact:

Proposition 4.5 ([Hor14, Proposition 4.3])**.** *If $T \in \overline{\mathcal{O}}$ is mixing, then T is $\mathcal{Z}$-averse if and only if T is $\mathcal{Z}$-incompatible.*

Proposition 4.6. *T_ϕ^+ is $\mathcal{Z}$-averse.*

Proof. If the transverse covering $\mathcal{Y}$ is trivial, so that $T_\phi^+ = Y_I$, then T_ϕ^+ is indecomposable and hence $\mathcal{Z}$-averse. So suppose that $\mathcal{Y}$ is non-trivial and assume, for a contradiction, that T_ϕ^+ is not $\mathcal{Z}$-averse. As T_ϕ^+ is mixing, Proposition 4.5 implies that it is compatible with a $\mathcal{Z}$-splitting S. Lemma 1.18 of [Gui08] states that for a subgroup H of $\mathbb{F}$, if the H-minimal subtree T_H of T_ϕ^+ is indecomposable, then H is elliptic in S. Since Y_I is indecomposable (Lemma 4.2), letting $H = \mathrm{Stab}(Y_I)$, we conclude that $\mathrm{Stab}(Y_I)$ is contained in a vertex group of the cyclic splitting S. Hence, Λ_ϕ^+ is carried by a vertex group S, contradicting the assumption that it is $\mathcal{Z}$-filling. □

5. Filling but not $\mathcal{Z}$-filling laminations

In this section, we endeavor to study filling laminations which are not $\mathcal{Z}$-filling. We then use this understanding to establish the following proposition, which is a restatement of the second claim in Theorem 1.1. This section concludes with a proof of the first statement in Theorem 1.1.

Proposition 5.1. *Let ϕ be an automorphism with a filling lamination Λ_ϕ^+ that is not $\mathcal{Z}$-filling, so that Λ_ϕ^+ is carried by a vertex group of a cyclic splitting S. Then there is a cyclic splitting S' that is fixed by a power of ϕ.*

The splitting S' is canonical in the sense that the vertex group which carries Λ_ϕ^+ is as small as possible. The proof of Proposition 5.1 will require an excursion into the theory of JSJ-decompositions; the reader is referred to [FP06] for details about JSJ theory.

We say a lamination is *elliptic* in an $\mathbb{F}$-tree T if it is is carried by a vertex stabilizer of T. Let $\mathfrak{S}$ be the set of all one-edge $\mathcal{Z}$-splittings in which the lamination Λ_ϕ^+ is elliptic. Since Λ_ϕ^+ is filling, the set $\mathfrak{S}$ does not contain any free splittings.

Definition 5.2 (Types of pairs of splittings [RS97])**.** Let $S = A *_C B$ (or $A*_C$) and $S' = A' *_{C'} B'$ (or $A'*_{C'}$) be one-edge cyclic splittings with corresponding Bass-Serre trees T and T'. We say S is *hyperbolic* with respect to S' if there is an element $c \in C$ that acts hyperbolically on T'. We say S is *elliptic* with respect to S' if C is fixes a point of T'. We say this pair is *hyperbolic-hyperbolic* if each splitting is hyperbolic with respect to the other. We define elliptic-elliptic, hyperbolic-elliptic and elliptic-hyperbolic splittings similarly.

Lemma 5.3. *With notation as above, suppose that $S, S' \in \mathfrak{S}$, and assume without loss that Λ_ϕ^+ is carried by the vertex groups A and A'. Then Λ_ϕ^+ is elliptic in the minimal subtree of A in T', denoted T'_A and in the minimal subtree of A' in T, denoted $T_{A'}$.*

Proof. Since A and A' both carry Λ_ϕ^+, their intersection $A \cap A'$ also carries Λ_ϕ^+. The vertex stabilizers of $T_{A'}$ are precisely the intersection of vertex stabilizers of T with A', namely the conjugates of $A \cap A'$. Thus Λ_ϕ^+ is carried by a vertex group of $T_{A'}$. □

Lemma 5.4. *With notation as above, suppose that S, S' are one-edge cyclic splittings in $\mathfrak{S}$. Then S and S' are either hyperbolic-hyperbolic or elliptic-elliptic.*

Proof. The following is based on the proof of [FP06, Proposition 2.2]. We will address the case that both the splittings are free products with amalgamations; when one or both are HNN extensions, the proof is similar. Toward a contradiction, suppose C is hyperbolic in T' and C' is elliptic in T. Without loss of generality, we may assume that C' fixes the vertex stabilized by A in T. Suppose first that both A' and B' fix vertices in T. The two subgroups cannot fix the same vertex because they generate $\mathbb{F}$. On the other hand, if the vertices are distinct, then C' fixes an edge in T. Hence C' must be a finite index subgroup of C, in contradiction to the assumption that C is hyperbolic in T'. Thus, one of A' or B' does not fix a vertex in T.

Assume without loss that A' does not fix a vertex of T. The minimal subtree of A' in T, $T_{A'}$, gives a minimal splitting of A' over an infinite index subgroup of C (i.e., a free splitting). As C' is elliptic in T, it is also elliptic in $T_{A'}$. Blowing up the vertex stabilized by A' in T' to the free splitting of A' just obtained, we get a free splitting of $\mathbb{F}$. Lemma 5.3 implies that Λ_ϕ^+ is elliptic in this free splitting, which is a contradiction. □

In [FP06], the existence of a JSJ decompositions for splittings with slender edge groups ([FP06, Theorem 5.13]) is established via an iterative process: one starts with a pair of splittings, and produces a new splitting which is a common refinement (in the case of an elliptic-elliptic pair) [FP06, Proposition 5.10], or an enclosing subgroup (in the case of a hyperbolic-hyperbolic pair) [FP06, Proposition 5.8]. One then repeats this process for all the splittings under consideration, and uses an accessibility result due to Bestvina-Feighn [BF91] to conclude that the process stops after finitely many iterations. In order to use Fujiwara-Papasoglu's techniques, we need only ensure that if two splittings to belong to the set $\mathfrak{S}$, then the splittings created in this process also belong to $\mathfrak{S}$. By examining the construction of an enclosing subgroup for a pair of hyperbolic-hyperbolic splittings (Proposition 4.7) and using Lemma 5.3, we see that the enclosing graph decomposition of $\mathbb{F}$ for this pair of splittings indeed belongs to $\mathfrak{S}$. Similarly, by the construction of refinement for two elliptic-elliptic splittings and Lemma 5.3, we see that the refined splitting is also contained in $\mathfrak{S}$. This discussion implies that a JSJ decomposition exists for cyclic splittings of $\mathbb{F}$ in which Λ_ϕ^+ is elliptic.

We conclude our foray into JSJ decompositions by using the theory of deformation spaces [For02, GL07a] to show that the set of JSJ splittings of $\mathbb{F}$ in which Λ_ϕ^+ is elliptic is finite. By passing to a power, we will then obtain a ϕ-invariant splitting in $\mathfrak{S}$.

Definition 5.5 (Slide moves [GL07a, Section 7])**.** Let $e = vw$ and $f = vu$ be adjacent edges in an $\mathbb{F}$-tree T such that the vertex stabilizer of f, denoted G_f, is contained in G_e. Assume that e and f are not in the same orbit as non-oriented edges. Define a new tree T' with the same vertex set as T and replacing f by an edge $f' = wu$ equivariantly. Then we say f *slides* across e. Often, a slide move is described on the quotient of T by $\mathbb{F}$.

Definition 5.6 ([GL07a, For02])**.** The deformation space $\mathcal{D}$ containing a tree T is the set of all trees T' such that there are equivariant maps from T to T' and from T' to T, up to equivariant isometry. A deformation space $\mathcal{D}$ for $\mathbb{F}$ is *non-ascending* if it is irreducible, and no T in $\mathcal{D}$ is such that $T/\mathbb{F}$ contains a strict ascending loop.

Definition 5.7 ([For02])**.** A tree T is reduced if no inclusion of an edge group into either of its vertex group is an isomorphism.

Theorem 5.8 ([GL07a, Theorem 7.2])**.** *Let $\mathcal{D}$ be a non-ascending deformation space. Any two reduced simplicial trees $T, T' \in \mathcal{D}$ may be connected by a finite sequence of slides.*

Lemma 5.9. *There are only finitely many slide moves that can be performed on a reduced cyclic splitting S.*

Proof. First suppose that the splitting $S/\mathbb{F}$ does not have any loops or circuits. Then it is clear that only finitely many slide moves can be performed on S. If S has a loop, then we can slide an edge f along the loop e only once. Indeed, we have $G_f \subseteq G_e$ and after sliding we have $G_{f'} \subseteq tG_et^{-1}$, where t is the stable letter corresponding to the loop. Since $G_e \cong \mathbb{Z}$ and $G_e \cap tG_et^{-1} = 1$, $G_{f'} \nsubseteq G_e$ which prevents sliding of f' over e. The proof in the case of a circuit is similar. □

Proof of Proposition 5.1. By assumption, there exists a one-edge cyclic splitting S such that Λ^+_ϕ is elliptic in S. The existence of JSJ decomposition for splittings in $\mathfrak{S}$ implies that the deformation space $\mathcal{D}$ for cyclic splittings in $\mathfrak{S}$ is non-empty. Theorem 5.9 and Lemma 5.10 together imply that the set of reduced trees in $\mathcal{D}$ is finite. As the set of reduced trees in $\mathcal{D}$ is ϕ-invariant, passing to a power yields a reduced cyclic splitting S' in $\mathcal{D}$ which is fixed by ϕ^k. □

Proof of Theorem 1.1 (Loxodromic). Applying Proposition 4.6 to each of ϕ and ϕ^{-1}, we conclude that T^+_ϕ and T^-_ϕ are both $\mathcal{Z}$-averse. We now argue that these trees are distinct. We denote the dual lamination of a tree T by $L(T)$ [CHL08]. Since the attracting lamination Λ^+_ϕ and the repelling lamination Λ^-_ϕ are different, and $\Lambda^\mp_\phi \subseteq L(T^\pm_\phi)$ and $\Lambda^\pm_\phi \nsubseteq L(T^\pm_\phi)$, we have that T^+_ϕ and T^-_ϕ are distinct points in the Gromov boundary of $\mathcal{FZ}$ that are fixed by ϕ.

Thus, ϕ fixes two distinct $\mathcal{Z}$-averse trees in the boundary of $\mathcal{FZ}$. Furthermore, we saw in Section 3 that the $\mathcal{Z}$-splitting T'_0 in $\mathcal{FZ}$ converges to T^+_ϕ (resp. T^-_ϕ) under forward (resp. backward) iterates of ϕ. Thus ϕ acts loxodromically on $\mathcal{FZ}$.

We now prove the converse: if ϕ acts loxodromically on $\mathcal{FZ}$, then ϕ has a $\mathcal{Z}$-filling lamination. Indeed, if ϕ acts loxodromically on $\mathcal{FZ}$, then ϕ necessarily act loxodromically on $\mathcal{FS}$, and thus has a filling lamination Λ^+_ϕ. If the lamination is not $\mathcal{Z}$-filling, then Proposition 5.1 implies that ϕ fixes a point in $\mathcal{FZ}$, contradicting our assumption on ϕ. Thus, Λ^+_ϕ is $\mathcal{Z}$-filling. □

6. Examples

This section will provide several examples exhibiting the range of behaviors of outer automorphisms acting on $\mathcal{FZ}$. We begin with an automorphism that acts loxodromically on $\mathcal{FZ}$.

Example 6.1 (Loxodromic element)**.** Let ϕ be a rotationless automorphism with a CT representative $f\colon G \to G$ satisfying the following properties:

- f has exactly two strata, each of which is EG and non-geometric
- the lamination corresponding to the top stratum of f is filling

An explicit example satisfying these properties can be constructed using the sage-train-tracks package written by T. Coulbois. The fact that the top lamination is filling guarantees that ϕ acts loxodromically on $\mathcal{FS}$. As both strata are non-geometric, [HM13a, Fact 1.42(1a)] guarantees that ϕ does not fix the conjugacy class of any element of $\mathbb{F}$, and therefore cannot possibly fix a cyclic splitting. Theorem 1.2 implies that ϕ acts loxodromically.

Example 6.2 (Bounded orbit without fixed point)**.** Building on Example 6.1, we can construct an automorphism ψ which acts on $\mathcal{FZ}$ with bounded orbits but without a fixed point. Let ψ be a three stratum automorphism obtained from f by creating a duplicate of H_2. Explicitly, ψ has a CT representative $f'\colon G' \to G'$ defined as follows. The graph G' is obtained by taking two copies of G and identifying them along G_1. Each edge E of G' is naturally identified with an edge of G, and $f'(E)$ is defined via this identification. Moreover, the marking of G naturally gives a marking of G' (by a larger free group). That f' is a CT is evident from the fact that f is a CT.

There are three laminations in $\mathcal{L}(\psi)$, none of which is filling. Since the top lamination in $\mathcal{L}(\phi)$ (where ϕ is as in Example 6.1) is filling, we know that $\mathcal{L}(\psi)$ must fill. Thus, ψ acts on $\mathcal{FS}$ with bounded orbits. As before, [HM09, Fact 1.42 (1a)] implies that ψ is atoroidal: each stratum, H_i, may have an INP, ρ_i, but none of these INPs can be closed loops, and they cannot be concatenated to form a closed loop. Therefore does not fix any cyclic splitting and ψ must act on $\mathcal{FZ}$ with bounded orbits, but no fixed point.

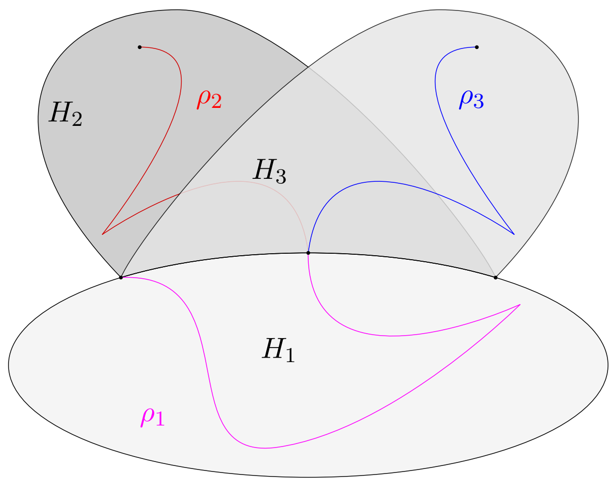

FIGURE 1. A CT representative for the automorphism in Example 6.2, which acts with bounded orbits but no fixed point

Example 6.3 (Loxodromic element)**.** Consider the outer automorphism $\phi\colon F_4 \to F_4$ given by

$$\phi(a) = ab, \phi(b) = bcab, \phi(c) = d, \phi(d) = cd.$$

In [Rey12], it is shown that the stable tree for ϕ is indecomposable and hence $\mathcal{Z}$-averse. Therefore ϕ acts loxodromically on $\mathcal{FZ}$.

Example 6.4 (Fixed point)**.** Let $\Sigma_{2,1}$ be the surface of genus two with one puncture. There is a unique free homotopy class of separating curve, and it divides $\Sigma_{2,1}$ into two subsurfaces: a once punctured torus and a twice punctured torus. Placing a pseudo-Anosov on each of these subsurfaces and taking the outer automorphism induced by this mapping class yields an element of $\mathrm{Out}(\mathbb{F})$ that acts loxodromically on $\mathcal{FS}$, but fixes a point in $\mathcal{FZ}$.

7. Virtually Cyclic Centralizers

In this section, we investigate centralizers of automorphisms acting loxodromically on $\mathcal{FZ}$. To do this, we use the machinery of completely split train tracks, and the "disintegration" procedure of [FH09], which takes a rotationless outer automorphism and returns an abelian subgroup of $\mathrm{Out}(\mathbb{F})$. The main result is:

Theorem 1.3. *An outer automorphism with a filling lamination has a virtually cyclic centralizer in* $\mathrm{Out}(\mathbb{F})$ *if and only if the lamination is* $\mathcal{Z}$*-filling.*

We begin with a terse review of disintegration for outer automorphisms.

7.1. **Disintegration and rotationless abelian subgroups in** $\mathrm{Out}(\mathbb{F})$**.** Given a mapping class f in Thurston normal form, there is a straightforward way of making a subgroup of the mapping class group, called *the disintegration of* f, by "doing one piece at a time." The subgroup is easily seen to be abelian as each pair of generators can be realized as homeomorphisms with disjoint supports. The process of disintegration in $\mathrm{Out}(\mathbb{F})$ is analogous, but more difficult.

The reader is warned that we will only review those ingredients from [FH09] that will be used directly; the reader is directed there, specifically to §6, for complete details. Given a rotationless outer automorphism ϕ, one can form an abelian subgroup called $\mathcal{D}(\phi)$. The process of disintegrating ϕ begins by creating a finite graph, B, which records the interactions between different strata in a CT representing ϕ. As a first approximation, the components of B correspond to generators of $\mathcal{D}(\phi)$. However, there may be additional relations between strata that are unseen by B, so the number of components of B only gives an upper bound to the rank of $\mathcal{D}(\phi)$.

Let $f\colon G \to G$ be a CT representing the rotationless outer automorphism ϕ. While the construction of $\mathcal{D}(\phi)$ does depend on f, using different representatives will produce subgroups that are commensurable. We will need to consider a weakening of the complete splitting of paths and circuits in f. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element.

Definition 7.1. Define a finite directed graph B as follows. There is one vertex v_i^B for each nonfixed irreducible stratum H_i. If H_i is NEG, then a v_i^B-path is defined as the unique edge in H_i; if H_i is EG, then a v_i^B-path is either an edge in H_i or a taken connecting path in a zero stratum contained in H_i^z. There is a directed edge from v_i^B to v_j^B if there exists a v_i^B-path κ_i such that some term in the QE-splitting of $f_\#(\kappa_i)$ is an edge in H_j. The components of B are labeled $B_1, \dots, B_K$. For each B_s, define X_s to be the minimal subgraph of G that contains H_i for each NEG stratum with $v_i^B \in B_s$ and contains H_i^z for each EG stratum with $v_i^B \in B_s$. We say that $X_1, \dots, X_K$ are the *almost invariant subgraphs* associated to $f\colon G \to G$.

The reader should note that the number of components of B is left unchanged if an iterate of $f_\#$ is used in the definition, rather than $f_\#$ itself. In the sequel, we will frequently make statements about B using an iterate of $f_\#$.

For each K-tuple $\vec{a} = (a_1, \dots, a_K)$ of non-negative integers, define

$$f_{\vec{a}}(E) = \begin{cases} f_\#^{a_i}(E) & \text{if } E \in X_i \\ E & \text{if } E \text{ is fixed by } f \end{cases}$$

It turns out that $f_{\vec{a}}$ is always a homotopy equivalence of G [FH09, Lemma 6.7], but in general $\langle f_{\vec{a}} \mid \vec{a} \text{ is a non-negative tuple}\rangle$ is not abelian. To obtain an abelian subgroup, one has to pass to a certain subset of tuples which take into account interactions between the almost invariant subgraphs that are unseen by B. The reader is referred to [FH09, Example 6.9] for an example.

Definition 7.2. A K-tuple $(a_1, \dots, a_K)$ is called *admissible* if for all axes μ, whenever

- X_s contains a linear edge E_i with axis μ and exponent d_i,
- X_t contains a linear edge E_j with axis μ and exponent d_j,
- there is a vertex v^B of B and a v^B-path $\kappa \subseteq X_r$ such that some element in the quasi-exceptional family $E_i\overline{E}_j$ is a term in the QE-splitting of $f_\#(\kappa)$,

then $a_r(d_i - d_j) = a_s d_i - a_t d_j$.

The disintegration of ϕ is then defined as

$$\mathcal{D}(\phi) = \langle f_{\vec{a}} \mid \vec{a} \text{ is admissible}\rangle$$

If an abelian subgroup H is generated by rotationless automorphisms, then all elements of H are rotationless [FH09, Corollary 3.13]. In this case, H is said to be rotationless. Rotationless abelian subgroups of $\mathrm{Out}(\mathbb{F})$ come equipped with a finite collection of (nontrivial) homomorphisms to $\mathbb{Z}$. Combining these, one obtains a homomorphism $\Omega\colon H \to \mathbb{Z}^N$ that is injective [FH09, Lemma 4.6]. An element $\psi \in H$ is said to be *generic* if all coordinates of $\Omega(\psi)$ are nonzero.

Every abelian subgroup of $\mathrm{Out}(\mathbb{F})$ has finitely many attracting laminations: if H is abelian and $\mathcal{L}(H) := \bigcup_{\phi\in H}\mathcal{L}(\phi)$, then $|\mathcal{L}(H)| < \infty$. For the purposes of this section, we require only two facts concerning Ω. First, there is one coordinate of the homomorphism Ω corresponding to each element of the finite set $\mathcal{L}(H)$ (there are other coordinates, which we will not need, corresponding to the so called "comparison homomorphisms"). Second is the fact that the coordinate of $\Omega(\psi)$ corresponding to $\Lambda \in \mathcal{L}(H)$ is positive if and only if $\Lambda \in \mathcal{L}(\psi)$.

7.2. **From disintegrations to centralizers.** In this subsection, we explain how to deduce Theorem 1.3 from the following proposition concerning the disintegration of elements acting loxodromically on $\mathcal{FZ}$. The proof of Proposition 7.3 is postponed until the next subsection.

Proposition 7.3. *If ϕ is rotationless and has a $\mathcal{Z}$-filling lamination, then $\mathcal{D}(\phi)$ is virtually cyclic.*

Proof of Theorem 1.3. Suppose $\psi \in C(\phi)$ has infinite order, and assume that $\langle\phi, \psi\rangle \simeq \mathbb{Z}^2$. If no such element exists, then $C(\phi)$ is virtually cyclic, as there is a bound on the order of a finite subgroup of $\mathrm{Out}(\mathbb{F})$ [Cul84]. Now let H_R be the finite index subgroup of $\langle\phi, \psi\rangle$ consisting of rotationless elements [FH09, Corollary 3.14] and let ψ' be a generic element of this subgroup. If the coordinate of $\Omega(\psi)$ corresponding to the $\mathcal{Z}$-filling lamination Λ^+_ϕ is negative, then replace ψ' by $(\psi')^{-1}$, which is also rotationless as Ω is a homomorphism. Since $\Lambda^+_\phi \in \mathcal{L}(\psi')$ is $\mathcal{Z}$-filling, Theorem 1.1 implies that ψ' acts loxodromically on $\mathcal{FZ}$. Since ψ' is generic in H_R, [FH09, Theorem 7.2] says that $\mathcal{D}(\psi') \cap \langle\phi, \psi\rangle$ has finite index in $\langle\phi, \psi\rangle$. This contradicts Proposition 7.3, which says that the disintigration of ψ' is virtually cyclic. □

7.3. **The proof of Proposition 7.3.** The idea of the proof is as follows. We noted above that the number of components in B only gives an upper bound to the rank of $\mathcal{D}(\phi)$; it may happen that there are interactions between the strata of f that are unseen by B (Definition 7.2). We will obtain precise information about the structure of B; it consists of one main component (B_1), and several components consisting of a single point ($B_2, \dots, B_K$). We will then show that the admissibility condition provides sufficiently many constraints so that choosing a_1 determines $a_2, \dots, a_K$. Thus, the set of admissible tuples consists of a line in $\mathbb{Z}^K$.

Let $f\colon G \to G$ be a CT representing ϕ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$. Let $\Lambda^+_\phi \in \mathcal{L}(\phi)$ be filling and let $\ell \in \Lambda^+_\phi$ be a generic leaf. As Λ^+_ϕ is filling, the corresponding EG stratum is necessarily the top stratum, H_M. We will understand the graph B by studying the realization of ℓ in G. The results of [BFH00, §3.1], together with Lemma 4.25 of [FH11] give that the realization of ℓ in G is completely split, and this splitting is unique. Thus, we may consider the QE-splitting of ℓ.

We begin with a lemma that allows the structure of INPs and quasi-exceptional paths to be understood inductively.

Lemma 7.4. *Let H_r be a non-fixed irreducible stratum and let ρ be a path of height $s \geq r$ which is either an INP or a quasi-exceptional path. Assume further that ρ intersects H_r non-trivially. Then one of the following holds:*

- *H_r and H_s are NEG linear strata with the same axis, each consisting of a single edge E_r (resp. E_s), and $\rho = E_s w^* \overline{E}_r$, where w is a closed, root-free Nielsen path of height $< s$.*
- *ρ can be written as a concatenation $\rho = \beta_0\rho_1\beta_1\rho_2\beta_2 \dots \rho_j\beta_j$, where each ρ_i is an INP of height r and each β_i is a path contained in $G - \mathrm{int}(H_r)$ (some of the β_i's may be trivial).*

Proof. The proof proceeds by strong induction on the height s of the path ρ. In the base case, $s = r$, and ρ is either an INP of height r or a quasi-exceptional path of the form described. The inductive step breaks into cases according whether H_s is an EG stratum, or an NEG stratum.

If H_s is an EG stratum, then ρ must be an INP, as there are no exceptional paths of EG height. In this case, [FH11, Lemma 4.24 (2)] provides a decomposition of ρ into subpaths of height s and maximal subpaths of height $< s$, and each of the subpaths of height $< s$ is a Nielsen path. The inductive hypothesis then guarantees that each of these Nielsen paths has the desired form. By breaking apart and combining these terms appropriately, we conclude that ρ does as well.

Suppose now that H_s is an NEG stratum and let E_s be the unique edge in H_s. Using (NEG Nielsen Paths), we see that E_s must be a linear edge, and therefore that ρ is either $E_s w^k \overline{E_s}$ or $E_s w^k \overline{E'}$, where E' is another linear edge with the same axis and w is a closed root free Nielsen path of height $< s$. If H_r is NEG linear, and $E' = E_r$, then the first conclusion holds. Otherwise, we may apply the inductive hypothesis to w to obtain a decomposition as desired. This completes the proof. □

We now begin our study of the graph B. We call the component of B containing v_M^B, the vertex corresponding to the topmost stratum of f, the *main component*.

Lemma 7.5. *All nonlinear NEG strata are in the main component of B.*

Proof. Let H_r be a nonlinear NEG stratum, with single edge E_r. It is enough to show that the single edge E_r occurs as a term in the QE-splitting of ℓ, as this implies that there is an edge in B connecting v_M^B to v_r^B. As ℓ is filling, we know that its realization in G must cross E_r. If the corresponding term in the QE-splitting of ℓ is the single edge E, then we are done. The only other possibility is that the corresponding term is an INP or a quasi-exceptional path of some height $s \geq r$. An application of Lemma 7.4 shows that this is impossible, as it implies the existence of an INP of height r or a quasi-exceptional path of the form $E_r w^* \overline{E}'$, contradicting (NEG Nielsen Paths). □

Lemma 7.6. *All EG strata are in the main component of B.*

Proof. Let H_r be an EG stratum. As before, it is enough to show that some (every) edge of H_r occurs as a term in the QE-splitting of ℓ. There are three types of pieces in a QE-splitting that can cross H_r: a single edge in H_r, an INP of height $\geq r$, or a quasi-exceptional path. In the first case, we are done, so suppose that every time ℓ crosses H_r, the corresponding term in its QE-splitting is an INP or a quasi-exceptional path.

We may therefore write ℓ as a concatenation $\ell = \dots \gamma_1\sigma_1\gamma_2\sigma_2 \dots$ where each σ_i is a single term in the QE-splitting of ℓ which intersects $\mathrm{int}(H_r)$, and each γ_i is a maximal concatenation of terms in the QE-splitting of ℓ which does not intersect $\mathrm{int}(H_r)$ (some γ_i's may be trivial). By assumption, each σ_i is an INP or a QEP. Applying Lemma 7.4 to each of the σ_i's, then combining and breaking apart the terms appropriately, we see that ℓ can be written as a concatenation $\ell = \dots \gamma_1\rho_1\gamma_2\rho_2 \dots$ where each ρ_i is the unique INP of height r or its inverse. Call this INP ρ.

The key to proving Proposition 7.3 is using the information we have about ℓ to find a $\mathcal{Z}$-splitting in which ℓ is carried by a vertex group, thus contradicting our assumption. We now modify G to produce a 2-complex, G'', whose fundamental group is identified with $\mathbb{F}$. First assume H_r is non-geometric, so that ρ has distinct endpoints, v_0 and v_1. Let G' be the graph obtained from G by replacing each vertex v_i for $i \in \{0,1\}$ with two vertices, v_i^u and v_i^d (u and d stand for "up" and "down"), which are to be connected by an edge E_i. For each edge E of G that is incident to v_i, connect it in G' to the new vertices as follows: if $E \in H_r$, then E is connected to v_i^d, and if $E \notin H_r$, then E is connected to v_i^u. G' deformation retracts onto G by collapsing the new edges, and this retraction identifies $\pi_1(G')$ with $\mathbb{F}$ via the marking of G. Let $R = [0,1] \times [0,1]$ be a rectangle and define G'' by gluing $\{i\} \times [0,1]$ homeomorphically onto E_i for $i \in \{0,1\}$, then gluing $[0,1] \times \{0\}$ homeomorphically to the INP ρ. As only three sides of the rectangle have been glued, G'' deformation retracts onto G', and its fundamental group is again identified with $\mathbb{F}$.

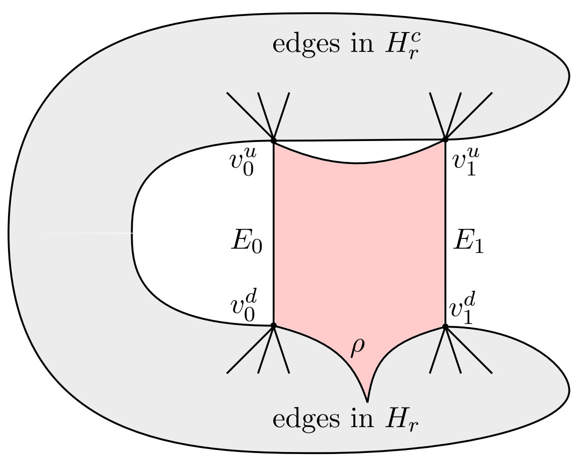

FIGURE 2. G'' when H_r is a non-geometric EG stratum

The construction of G'' differs only slightly if H_r is geometric. In this case, ρ is a closed loop based at v_0 and we blow up v_0 to two vertices, v_0^u and v_0^d, that are connected by an edge E_0. Instead of gluing in a rectangle, we glue in a cylinder $R = S^1 \times [0,1]$; $\{p\} \times [0,1]$ is glued homeomorphically to E_0 where p is a point in S^1, and $S^1 \times \{0\}$ is glued homeomorphically to ρ.

Recall that in G, the leaf ℓ can be written as a concatenation $\ell = \dots \gamma_1\rho_1\gamma_2\rho_2 \dots$ where each ρ_i is either ρ or $\overline{\rho}$. Thus we can realize ℓ in G' as $\ell = \dots \gamma_1\rho_1'\gamma_2\rho_2' \dots$ where each ρ_i' is either $E_0\rho\overline{E}_1$ or $E_1\overline{\rho}\overline{E}_0$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, after performing a (proper!) homotopy to the image of ℓ, we can arrange that it never intersects the interior of R, nor the vertical sides of R. Cutting R along its centerline yields a $\mathcal{Z}$-splitting S of $\mathbb{F}$, and ℓ is carried by a vertex group of this splitting. If H_r is non-geometric, then S is a free splitting and if H_r is geometric, then S is a cyclic splitting.

In either case, if S is non-trivial, then we get a contradiction as our lamination is assumed to be $\mathcal{Z}$-filling. ☐

Claim 7.7. *The splitting S is non-trivial.*

Proof of Claim 7.7. We first handle the case that H_r is geometric. We have described a one-edge cyclic splitting S which was obtained as follows: cut G' along the edge E_0 to get a free splitting of $\mathbb{F}$, then fold $\langle w \rangle$, where w is the conjugacy class of the INP ρ. If $G' - E_0$ is connected, then the free splitting is an HNN extension, and there is no danger of S being trivial as $\mathrm{rk}(\mathbb{F}) \geq 3$. On the other hand, if $G' - E_0$ is disconnected, then let $G^{d'}$ and $G^{u'}$ be the components of $G' - E_0$ containing v_0^d and v_0^u respectively. The free splitting which is folded to get S is precisely $\pi_1(G^{d'}) * \pi_1(G^{u'})$. In this case, $G^{d'}$ is necessarily a component of G_r and [FH11, Proposition 2.20 (2)] together with (Filtration) imply that this component is a core graph. As H_r is EG, the rank of $\pi_1(G^{d'})$ is at least two and the splitting S is therefore non-trivial. To see that $\mathrm{rk}(\pi_1(G^{u'})) \geq 1$, we need only recall that ℓ is not periodic and is carried by $\pi_1(G^{c'}) * \langle w \rangle$.

In the case that H_r is non-geometric, the splitting obtained above is a free splitting. If $G' - \{E_0, E_1\}$ is connected, then the free splitting is an HNN extension, and as before S is non-trivial. If $G' - \{E_0, E_1\}$ is disconnected, then the component containing v_0^d (and by necessity v_1^d), denoted $G^{d'}$, corresponds to a vertex group of S. By the same reasoning as in the previous case, we get that $\pi_1(G^{d'})$ is non-trivial. As before, the other vertex group of S carries the leaf ℓ and hence S is a non-trivial free splitting. ☐

Before addressing the NEG linear strata and concluding the proof of Proposition 7.3, we present a final lemma concerning the structure of B.

Lemma 7.8. *Assume H_r is a linear NEG stratum consisting of an edge E_r. If v_r^B is not in the main component of B, then the component of B containing v_r^B is a single point.*

Proof. This follows directly from the definition of B, together with Lemmas 7.5 and 7.6. If H_r is a linear NEG stratum, then the definition of B implies that v_r^B has no outgoing edges. For any edge in B whose terminal vertex is v_r^B, its initial vertex necessarily corresponds to a non-linear NEG stratum or an EG stratum, and hence is in the main component of B. ☐

When dealing with an NEG linear stratum, we would like to carry out a similar strategy to the EG case: blow up the terminal vertex, v_0, to an edge and glue in a cylinder, thereby producing a cyclic splitting in which ℓ is carried by a vertex group. The main difficulty in implementing this comes from other linear edges with the same axis; for each such edge, one has to decide whether to glue it in G' to v_0^d or v_0^u.

Let μ be an axis with corresponding unoriented root-free conjugacy class w. Let $\mathcal{E}_\mu$ be the set of linear edges in G with axis μ. Define a relation on $\mathcal{E}_\mu$ by declaring $E \sim_R E'$ if the quasi-exceptional path $Ew^*\overline{E}'$ occurs as a term in the QE-splitting of ℓ or if both E and E' occur as terms in the QE-splitting of ℓ. Then let $\sim$ be the equivalence relation generated by $\sim_R$. Note that all edges in $\mathcal{E}_\mu$ which occur as terms in the QE-splitting of ℓ are in the same equivalence class.

Lemma 7.9. *There is only one equivalence class of $\sim$. Moreover, at least one edge in $\mathcal{E}_\mu$ occurs as a term in the QE-splitting of ℓ.*

Proof. Suppose for a contradiction that there is more than one equivalence class of $\sim$ and $[E]$ be an equivalence class for which no edge in $[E]$ occurs as a term in the QE-splitting of ℓ. Now build G' as in the proof of Lemma 7.6. Let v_0 be the terminal vertex of the edges in $\mathcal{E}_\mu$ (they all have the same terminal vertex), and define G' by blowing up v_0 into two vertices, v_0^u and v_0^d, which are connected by an edge E_0. The terminal vertex of each edge of $[E]$ is to be glued in G' to v_0^u, while all other edges in G that are incident to v_0 are glued to v_0^d. Define G'' as before, gluing the bottom of a cylinder R along the closed loop w, and gluing the vertical interval above v_0 homeomorphically to the edge E_0.

The definition of $\sim$ guarantees that ℓ is carried by a vertex group of the cyclic splitting determined by cutting along the centerline of R. Indeed, whenever ℓ crosses an edge from $[E]$, the corresponding term in the QE-splitting is either an INP or a quasi-exceptional path $E'w^*\overline{E}''$, where $E', E'' \in [E]$. Repeatedly applying Lemma 7.4 to each of these terms, then rearranging and combining terms appropriately, we see that ℓ can be written in G as a concatenation $\ell = \ldots \gamma_1\rho_1\gamma_2\rho_2 \ldots$ where each ρ_i is either $E'w^*\overline{E'}$ or $E'w^*\overline{E''}$ with $E', E'' \in [E]$. Thus we can realize ℓ in G' as $\ell = \ldots \gamma_1\rho_1'\gamma_2\rho_2' \ldots$ where each ρ_i' is $E'E_0w^*\overline{E}_0\overline{E'}$ or $E'E_0w^*\overline{E}_0\overline{E''}$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, we have again produced a cyclic splitting in which ℓ is carried by a vertex group.

We now argue that the splitting is non-trivial. There is a free splitting S which comes from cutting the edge E_0 in G', which cannot be a self loop. The cyclic splitting of interest S' is obtained from S by folding w across the single edge. If $G' - E_0$ is connected, then S' is an HNN extension with edge group $\langle [w] \rangle$. As $\mathrm{rk}(\mathbb{F}) \geq 3$, the vertex group has rank at least two and we are done. Now suppose E_0 is separating so that $G' - E_0$ consists of two components. Let G'^u be the component containing the vertex v_0^u and let G'^d be the other component. The vertex groups of the splitting S' are $\pi_1(G^d)$ and $\pi_1(G^u) * \langle [w] \rangle$. The fact that v is a principal vertex guarantees that $\pi_1(G^d) \ncong \mathbb{Z}$, and the fact that G is a finite graph without valence one vertices ensures that $\pi_1(G^u)$ is non-trivial.

The proof of the second statement is exactly the same as that of the first. $\square$

Finally, we finish the proof of Proposition 7.3. As before, B_1 is the main component of B, with corresponding almost invariant subgraph X_1. All other components $B_2, \ldots, B_K$ are single points, and each almost invariant subgraph X_i consist of a single linear edge. Let $(a_1, \ldots, a_K)$ be a K-tuple and suppose that a_1 has been chosen. We claim that imposing the admissibility condition determines all other a_i's.

Suppose first that E_i, E_j are linear edges with the same axis, μ, such that $E_i \in X_1$, $E_j \in X_k$, and $E_i \sim_R E_j$. Let d_i and d_j be the exponents of E_i and E_j respectively. Applying the definition of admissibility with $s = r = 1$, $t = k$, and κ a v^B path such that $f_\#(\kappa)$ contains a quasi-exceptional path of the form $E_iw^*\overline{E}_j$ in its QE-splitting (such a κ must exist as a quasi-exceptional path of this type occurs in the QE-splitting of ℓ), we obtain the relation $a_1(d_i - d_j) = a_1d_i - a_kd_j$. Thus a_k is determined by a_1.

Now suppose E_i and E_j are as above, but rather than being related by $\sim_R$, we only have that $E_i \sim E_j$. There is a finite chain of $\sim_R$-relations to get from E_i to E_j. At each stage in this chain, the definition of admissibility (applied with $r = 1$ and κ chosen appropriately) will impose a relation that determines the next coordinate from the previous ones. Ultimately, this determines a_k.

We have thus shown that an admissible tuple is completely determined by choosing a_1, and therefore that the set of admissible tuples forms a line in $\mathbb{Z}^K$. Therefore $\mathcal{D}(\phi)$ is virtually cyclic.

7.4. A Converse.

Proposition 7.10. *If ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then the centralizer of ϕ in $\mathrm{Out}(\mathbb{F})$ is not virtually cyclic.*

Proof. Since ϕ has filling lamination which is not $\mathcal{Z}$-filling, it follows by Proposition 5.1 that ϕ fixes a one-edge cyclic splitting S.

Suppose $S/\mathbb{F}$ is a free product with amalgamation with vertex stabilizers $\langle A, w\rangle$ and B and edge group $\langle w\rangle \subset B$. Consider the Dehn twist D_w given by S as follows: D_w acts as identity on B and conjugation by w on A. The automorphism D_w has infinite order. We claim that D_w and ϕ commute. Indeed, consider a generating set $\{a_1, \dots, a_k, b_1, \dots b_m\}$ for $\mathbb{F}$ such that the a_i's generate A and the b_i's generate B. Without loss of generality, we may assume $\phi(B) = B$ and $\phi(\langle A, w\rangle) = \langle A, w\rangle^b$ for some element $b \in B$. Since D_w is identity on B and $\phi(B) = B$, we have $\phi(D_w(b_i)) = D_w(\phi(b_i))$ for all generators b_i. Since $D_w(a_i) = wa_i\overline{w}$, $\phi(w) = w$ and $\phi(\langle A, w\rangle) = \langle A, w\rangle^b$, we have $D_w(\phi(a_i)) = \phi(D_w(a_i))$ for all generators a_i. Thus D_w and ϕ commute.

We now address the case that $S/\mathbb{F}$ is an HNN extension. Assume $S/\mathbb{F}$ has stable letter t, edge group $\langle w\rangle$ and vertex group $\langle A, \bar{t}wt\rangle$. Consider a basis of $\mathbb{F}$ given by $\{a_1, a_2, \dots, a_k, t\}$, where the a_i's generate A. Consider the Dehn twist D_w determined by S such that D_w is identity on A and sends t to wt. The automorphism D_w has infinite order. Since $\langle A, \bar{t}wt\rangle$ is ϕ-invariant, for every generator a_i, $\phi(a_i)$ is a word in the a_is and $\bar{t}wt$. Since D_w is identity on A and fixes $\bar{t}wt$, we get $\phi(D_w(a_i)) = D_w(\phi(a_i))$. Again, since $\langle A, \bar{t}wt\rangle$ is ϕ-invariant, $\phi(t)$ is equal to $w^m d\alpha$, where α is some word in $\langle A, \bar{t}wt\rangle$ and $m \in \mathbb{Z}$. On one hand, $\phi(D_w(t)) = \phi(wt) = \phi(w)\phi(t) = ww^m t\alpha$ and on the other hand, $D_w(\phi(t)) = D_w(w^m t\alpha) = w^m D_w(t) D_w(\alpha) = w^m wt\alpha$. Thus D_w and ϕ commute.

Thus when ϕ fixes a cyclic splitting, then an infinite order element other than a power of ϕ exists in the centralizer of ϕ. $\square$

References

[AP16] Yael Algom-Kfir and Catherine Pfaff. Normalizers and centralizers of cyclic subgroups generated by lone axis fully irreducible outer automorphisms. *arXiv 1603.07206*, March 2016.

[BF91] Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

[BF94] Mladen Bestvina and Mark Feighn. Outer limits. *preprint, available at http://andromeda.rutgers.edu/ feighn/papers/outer.pdf*, 1994.

[BF10] Mladen Bestvina and Mark Feighn. A hyperbolic $\mathrm{Out}(F_n)$-complex. *Groups Geom. Dyn.*, 4(1):31–58, 2010.

[BF14] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *Adv. Math.*, 256:104–155, 2014.

[BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\mathrm{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[CHL08] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. $\mathbb{R}$-trees and laminations for free groups. II. The dual lamination of an $\mathbb{R}$-tree. *J. Lond. Math. Soc. (2)*, 78(3):737–754, 2008.

[CL95] Marshall M. Cohen and Martin Lustig. Very small group actions on $\mathbf{R}$-trees and Dehn twist automorphisms. *Topology*, 34(3):575–617, 1995.

[CM87] Marc Culler and John W. Morgan. Group actions on $\mathbf{R}$-trees. *Proc. London Math. Soc. (3)*, 55(3):571–604, 1987.

[Cul84] Marc Culler. Finite groups of outer automorphisms of a free group. In *Contributions to group theory*, volume 33 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 1984.

[CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[FH09] Mark Feighn and Michael Handel. Abelian subgroups of $\mathrm{Out}(F_n)$. *Geom. Topol.*, 13(3):1657–1727, 2009.

[FH11] Mark Feighn and Michael Handel. The recognition theorem for $\mathrm{Out}(F_n)$. *Groups Geom. Dyn.*, 5(1):39–106, 2011.

[For02] Max Forester. Deformation and rigidity of simplicial group actions on trees. *Geom. Topol.*, 6:219–267, 2002.

[FP06] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.

[GL07a] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135–181, 2007.

[GL07b] Vincent Guirardel and Gilbert Levitt. The outer space of a free product. *Proc. Lond. Math. Soc. (3)*, 94(3):695–714, 2007.

[Gui04] Vincent Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geom. Topol.*, 8:1427–1470 (electronic), 2004.

[Gui08] Vincent Guirardel. Actions of finitely generated groups on $\mathbb{R}$-trees. *Ann. Inst. Fourier (Grenoble)*, 58(1):159–211, 2008.

[Gup16] Radhika Gupta. Loxodromic elements for the relative free factor complex. *arXiv:1612.04261*, December 2016.

[HM09] Michael Handel and Lee Mosher. Subgroup classification in Out(F_n). *arXiv:0908.1255*, August 2009.

[HM11] Michael Handel and Lee Mosher. Axes in outer space. *Mem. Amer. Math. Soc.*, 213(1004):vi+104, 2011.

[HM13a] M. Handel and L. Mosher. Subgroup decomposition in Out(F_n), Part I: Geometric Models. *ArXiv e-prints*, February 2013.

[HM13b] Michael Handel and Lee Mosher. The free splitting complex of a free group, I: hyperbolicity. *Geom. Topol.*, 17(3):1581–1672, 2013.

[HM14] Michael Handel and Lee Mosher. The free splitting complex of a free group II: Loxodromic outer automorphisms. *arXiv:1402.1886*, February 2014.

[Hor14] Camille Horbez. Hyperbolic graphs for free products, and the Gromov boundary of the graph of cyclic splittings. *arXiv:1408.0544*, August 2014.

[HW15] Camille Horbez and Richard D. Wade. Automorphisms of graphs of cyclic splittings of free groups. *Geometriae Dedicata*, 178(1):171–187, Oct 2015.

[KL11] Ilya Kapovich and Martin Lustig. Stabilizers of $\mathbb{R}$-trees with free isometric actions of F_N. *J. Group Theory*, 14(5):673–694, 2011.

[Man14] Brian Mann. Hyperbolicity of the cyclic splitting graph. *Geom. Dedicata*, 173:271–280, 2014.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[Rey12] Patrick Reynolds. Reducing systems for very small trees. *arXiv:1211.3378*, November 2012.

[RS97] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[RW15] Moritz Rodenhausen and Richard D. Wade. Centralisers of dehn twist automorphisms of free groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 159(1):89114, 2015.

R. Gupta, Department of Mathematics, Technion,
Haifa, Israel, 32000
http://www.math.utah.edu/~gupta/
E-mail address: radhikagup@technion.ac.il

D. Wigglesworth, Department of Mathematics, University of Utah, 155 S. 1400 E.
Salt Lake City, UT 84112, U.S.A.
http://www.math.utah.edu/~dwiggles/
E-mail address: dwiggles@math.utah.edu