DISTORTION FOR ABELIAN SUBGROUPS OF $\mathrm{Out}(F_n)$

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ABSTRACT. We prove that abelian subgroups of the outer automorphism group of a free group are quasi-isometrically embedded. Our proof uses recent developments in the theory of train track maps by Feighn-Handel. As an application, we prove the rank conjecture for $\mathrm{Out}(F_n)$.

1. Introduction

Given a finitely generated group G, a finitely generated subgroup H is *undistorted* if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding with respect to the word metrics on G and H for some (any) finite generating sets. A standard technique for showing that a subgroup is undistorted involves finding a space on which G acts nicely and constructing a height function on this space satisfying certain properties: elements which are large in the word metric on H should change the height function by a lot, elements of a fixed generating set for G should change the function by a uniformly bounded amount. In this paper, we use a couple of variations of this method.

Let R_n be the wedge of n circles and let F_n be its fundamental group, the free group of rank $n \geq 2$. The outer automorphism group of the free group, $\mathrm{Out}(F_n)$, is defined as the quotient of $\mathrm{Aut}(F_n)$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(F_n)$ draws parallels with the study of mapping class groups. Furthermore, many theorems concerning $\mathrm{Out}(F_n)$ and their proofs are inspired by analogous theorems and proofs in the context of mapping class groups. Both groups satisfy the Tits alternative [McC85, BFH00], both have finite virtual cohomological dimension [Har86, CV86], and both have Serre's property FA to name a few. Importantly, this approach to the study of $\mathrm{Out}(F_n)$ has yielded a classification of its elements in analogy with the Nielsen-Thurston classification of elements of the mapping class group [BH92], along with constructive ways for finding good representatives of these elements [FH14].

In [FLM01], the authors proved that infinite cyclic subgroups of the mapping class group are undistorted. Their proof also implies that higher rank abelian subgroups are undistorted. In [Ali02], Alibegović proved that infinite cyclic subgroups of $\mathrm{Out}(F_n)$ are undistorted. In contrast with the mapping class group setting, her proof does not directly apply to higher rank subgroups: the question of whether all abelian subgroups of $\mathrm{Out}(F_n)$ are undistorted has been left open. In this paper, we answer this in the affirmative.

Theorem 7.1. Abelian subgroups of $\mathrm{Out}(F_n)$ are undistorted.

This theorem has implications for various open problems in the study of $\mathrm{Out}(F_n)$. In [BM08], Behrstock and Minsky prove that the geometric rank of the mapping class group is equal to the maximal rank of an abelian subgroup of the mapping class group. As an application of Theorem 7.1, we prove the analogous result in the $\mathrm{Out}(F_n)$ setting.

Corollary 7.3. The geometric rank of $\mathrm{Out}(F_n)$ is $2n-3$, which is the maximal rank of an abelian subgroup of $\mathrm{Out}(F_n)$.

We remark that in principle, this could have been done earlier by using the techniques in [Ali02] to show that a specific maximal rank abelian subgroup is undistorted.

2010 *Mathematics Subject Classification.* Primary 20F65.

December 6, 2016.

The author is partially supported by the NSF grant of Mladen Bestvina (DMS-1607236) and also acknowledges support from U.S. National Science Foundation grants DMS-1246989 and DMS-0932078.

In the course of proving Theorem 7.1, we show that, up to finite index, only finitely many marked graphs are needed to get good representatives of every element of an abelian subgroup of $\mathrm{Out}(F_n)$. In the setting of mapping class groups, the analogous statement is that for a surface S and an abelian subgroup H of $\mathrm{MCG}(S)$ there is a Thurston decomposition of S into disjoint subsurfaces which is respected by every element of H. This can also be viewed as a version of the Kolchin Theorem of [BFH05] for abelian subgroups. We prove:

Proposition 5.2. For any abelian subgroup H of $\mathrm{Out}(F_n)$, there exists a finite index subgroup H' such that every $\phi \in H'$ can be realized as a CT on one of finitely many marked graphs.

The paper is outlined as follows:

In section 3 we prove that the translation length of an arbitrary element ϕ of $\mathrm{Out}(F_n)$ acting on Outer Space is the maximum of the logarithm of the expansion factors associated to the exponentially growing strata in a relative train track map for ϕ. This is the analog for $\mathrm{Out}(F_n)$ of Bers' result [Ber78] that the translation distance of a mapping class f acting on Teichmüller space endowed with the Teichmüller metric is the maximum of the logarithms of the dilatation constants for the pseudo-Anosov components in the Thurston decomposition of f. In section 4 we then use our result on translation distance to prove the main theorem in the special case where the abelian subgroup H has "enough" exponential data. More precisely, we will prove the result under the assumption that the collection of expansion factor homomorphisms determines an injective map $H \to \mathbb{Z}^N$.

In section 5 we prove Proposition 5.2 and then use this in section 6 to prove the main result in the case that H has "enough" polynomial data. This is the most technical part of the paper because we need to obtain significantly more control over the types of sub-paths that can occur in nice circuits in a marked graph than was previously available. The bulk of the work goes towards proving Proposition 6.1. This result provides a connection between the comparison homomorphisms introduced in [FH09] (which are only defined on subgroups of $\mathrm{Out}(F_n)$) and Alibegović's twisting function. We then use this connection to complete the proof of our main result in the polynomial case.

Finally, in section 7 we consolidate results from previous sections to prove Theorem 7.1. The methods used in sections 4 and 6 can be carried out with minimal modification in the general setting.

I would like to thank my advisor Mladen Bestvina for many hours of his time and for his patience. I would also like to thank Mark Feighn for his encouragement and support. Finally, I would also like to express my gratitude to Radhika Gupta for patiently listening to me go on about completely split paths for weeks on end and to MSRI for its hospitality and partial support.

2. Preliminaries

Identify F_n with $\pi_1(R_n, *)$ once and for all. A *marked graph* G is a finite graph of rank n with no valence one vertices equipped with a homotopy equivalence $\rho\colon R_n \to G$ called a *marking.* The marking identifies F_n with $\pi_1(G)$. As such, a homotopy equivalence $f\colon G \to G$ determines an (outer) automorphism ϕ of F_n. We say that $f\colon G \to G$ *represents* ϕ. All homotopy equivalences will be assumed to map vertices to vertices and the restriction to any edge will be assumed to be an immersion.

Let Γ be the universal cover of the marked graph G. A *path* in G (resp. Γ) is either an isometric immersion of a (possibly infinite) closed interval $\sigma\colon I \to G$ (resp. Γ) or a constant map $\sigma\colon I \to G$ (resp. Γ). If σ is a constant map, the path will be called *trivial.* If I is finite, then any map $\sigma\colon I \to G$ (resp. Γ) is homotopic rel endpoints to a unique path $[\sigma]$. We say that $[\sigma]$ is obtained by *tightening* σ. If $f\colon G \to G$ is a homotopy equivalence and σ is a path in G, we define $f_\#(\sigma)$ as $[f(\sigma)]$. If $\tilde{f}\colon \Gamma \to \Gamma$ is a lift of f, we define $\tilde{f}_\#$ similarly. If the domain of σ is finite, then the image has a natural decomposition into edges $E_1E_2\cdots E_k$ called the *edge path associated to* σ.

A *circuit* is an immersion $\sigma\colon S^1 \to G$. For any path or circuit, let $\overline{\sigma}$ be σ with its orientation reversed. A decomposition of a path or circuit into subpaths is a *splitting* for $f\colon G \to G$ and is denoted $\sigma = \ldots\sigma_1 \cdot \sigma_2 \ldots$ if $f^k_\#(\sigma) = \ldots f^k_\#(\sigma_1)f^k_\#(\sigma_2)\ldots$ for all $k \geq 1$.

Let G be a graph. An unordered pair of oriented edges $\{E_1, E_2\}$ is a *turn* if E_1 and E_2 have the same initial endpoint. As with paths, we denote by $\overline{E}$, the edge E with the opposite orientation. If σ is a path which contains $\ldots\overline{E}_1E_2\ldots$ or $\ldots\overline{E}_1E_2\ldots$ in its edge path, then we say σ *takes* the turn $\{E_1, E_2\}$. A *train* *track structure* on G is an equivalence relation on the set of edges of G such that $E_1 \sim E_2$ implies E_1 and E_2 have the same initial vertex. A turn $\{E_1, E_2\}$ is *legal* with respect to a train track structure if $E_1 \nsim E_2$. A path is legal if every turn crossed by the associated edge path is legal. The equivalence classes of this relation are called *gates*. A homotopy equivalence $f\colon G \to G$ induces a train track structure on G as follows. f determines a map Df on oriented edges in G by definining $Df(E)$ to be the first edge in the edge path $f(E)$. We then declare $E_1 \sim E_2$ if $D(f^k)(E_1) = D(f^k)(E_2)$ for some $k \geq 1$.

A *filtration* for a representative $f\colon G \to G$ of an outer automorphism ϕ is an increasing sequence of f-invariant subgraphs $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$. We let $H_i = \overline{G_i \setminus G_{i-1}}$ and call H_i the *i-th stratum*. A turn with one edge in H_i and the other in G_{i-1} is called *mixed* while a turn with both edges in H_i is called a *turn in H_i*. If $\sigma \subset G_i$ does not contain any illegal turns in H_i, then we say σ is *i-legal*.

We denote by M_i the submatrix of the transition matrix for f obtained by deleting all rows and columns except those labeled by edges in H_i. For the representatives that will be of interest to us, the transition matrices M_i will come in three flavors: M_i may be a zero matrix, it may be the 1×1 identity matrix, or it may be an irreducible matrix with Perron-Frobenius eigenvalue $\lambda_i > 1$. We will call H_i a *zero* (Z), *non-exponentially growing* (NEG), or *exponentially growing* (EG) stratum according to these possibilities. Any stratum which is not a zero stratum is called an *irreducible stratum*.

Definition 2.1 ([BH92]). We say that $f\colon G \to G$ is a *relative train track map* representing $\phi \in \mathrm{Out}(F_n)$ if for every exponentially growing stratum H_r, the following hold:

- **(RTT-i):** Df maps the set of oriented edges in H_r to itself; in particular all mixed turns are legal.
- **(RTT-ii):** If $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then so is $f_\#(\sigma)$.
- **(RTT-iii):** If $\sigma \subset G_r$ is r-legal, then $f_\#(\sigma)$ is r-legal.

Suppose that $u < r$, that H_u is irreducible, H_r is EG and each component of G_r is non-contractible, and that for each $u < i < r$, H_i is a zero stratum which is a component of G_{r-1} and each vertex of H_i has valence at least two in G_r. Then we say that H_i is *enveloped by H_r* and we define $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

A path or circuit σ in a representative $f\colon G \to G$ is called a *periodic Nielsen path* if $f_\#^k(\sigma) = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path*. A Nielsen path is *indivisible* if it cannot be written as a concatenation of non-trivial Nielsen paths. If w is a closed root-free Nielsen path and E_i is an edge such that $f(E_i) = E_i w^{d_i}$, then we say *E is a linear edge* and we call w the *axis* of E. If E_i, E_j are distinct linear edges with the same axis such that $d_i \neq d_j$ and $d_i, d_j > 0$, then we call a path of the form $E_i w^* \overline{E}_j$ an *exceptional path*. In the same scenario, if d_i and d_j have different signs, we call such a path a *quasi-exceptional path*. We say that x and y are *Nielsen equivalent* if there is a Nielsen path σ in G whose endpoints are x and y. We say that a periodic point $x \in G$ is *principal* if neither of the following conditions hold:

- x is not an endpoint of a non-trivial periodic Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same EG stratum.
- x is contained in a component C of periodic points that is topologically a circle and each point in C has exactly two periodic directions.

A relative train track map f is called *rotationless* if each principal periodic vertex is fixed and if each periodic direction based at a principal vertex is fixed. We remark that there is a closely related notion of an outer automorphism ϕ being rotationless. We will not need this definition, but will need the following relevant facts from [FH09]:

Theorem 2.2 ([FH09, Corollary 3.5]). *There exists $k > 0$ depending only on n, so that ϕ^k is rotationless for every $\phi \in \mathrm{Out}(F_n)$.*

Theorem 2.3 ([FH09, Corollary 3.14]). *For each abelian subgroup A of $\mathrm{Out}(F_n)$, the set of rotationless elements in A is a subgroup of finite index in A.*

For an EG stratum, H_r, we call a non-trivial path $\sigma \subset G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ a *connecting path for H_r*. Let E be an edge in an irreducible stratum, H_r and let σ be a maximal subpath of $f_\#^k(E)$ in a zero stratum for some $k \geq 1$. Then we say that σ is *taken*. A non-trivial path or circuit σ is called *completely split* if it has a splitting $\sigma = \tau_1 \cdot \tau_2 \cdots \tau_k$ where each of the τ_i's is a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a connecting path in a zero stratum which is both maximal and taken. We say that a relative train track map is *completely split* if $f(E)$ is completely split for every edge E in an irreducible stratum *and* if for every taken connecting path σ in a zero stratum, $f_\#(\sigma)$ is completely split.

Definition 2.4 ([FH11]). A relative train track map $f\colon G \to G$ and filtration $\mathcal{F}$ given by $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$ is said to be a CT if it satisfies the following properties.

(Rotationless): $f\colon G \to G$ is rotationless.

(Completely Split): $f\colon G \to G$ is completely split.

(Filtration): $\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.

(Vertices): The endpoints of all indivisible periodic (necessarily fixed) Nielsen paths are (necessarily principal) vertices. The terminal endpoint of each non-fixed NEG edge is principal (and hence fixed).

(Periodic Edges): Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge E_r in a fixed stratum H_r is not a loop then G_{r-1} is a core graph and both ends of E_r are contained in G_{r-1}.

(Zero Strata): If H_i is a zero stratum, then H_i is enveloped by an EG stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.

(Linear Edges): For each linear E_i there is a closed root-free Nielsen path w_i such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If E_i and E_j are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$.

(NEG Nielsen Paths): If the highest edges in an indivisible Nielsen path σ belong to an NEG stratum then there is a linear edge E_i with w_i as in (Linear Edges) and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \bar{E}_i$.

(EG Nielsen Paths): If H_r is EG and ρ is an indivisible Nielsen path of height r, then $f|G_r = \theta \circ f_{r-1} \circ f_r$ where :

(1) $f_r : G_r \to G^1$ is a composition of proper extended folds defined by iteratively folding ρ.
(2) $f_{r-1} : G^1 \to G^2$ is a composition of folds involving edges in G_{r-1}.
(3) $\theta : G^2 \to G_r$ is a homeomorphism.

We remark that several of the properties in Definition 2.4 use terms that have not been defined. We will not use these properties in the sequel. The main result for CTs is the following existence theorem:

Theorem 2.5 ([FH11, Theorem 4.28])**.** *Every rotationless $\phi \in \mathrm{Out}(F_n)$ is represented by a CT $f\colon G \to G$.*

For completely split paths and circuits, all cancellation under iteration of $f_\#$ is confined to the individual terms of the splitting. Moreover, $f_\#(\sigma)$ has a complete splitting which refines that of σ. Finally, just as with improved relative train track maps introduced in [BFH00], every circuit or path with endpoints at vertices eventually is completely split.

Culler and Vogtmann's outer space, CV_n, is defined as the space of homothety classes of free minimal actions of F_n on simplicial metric trees. Outer Space has a (non-symmetric) metric defined in analogy with the Teichmüller metric on Teichmüller space. The distance from T to T' is defined as the logarithm of the infimal Lipschitz constant among all F_n-equivariant maps $f\colon T \to T'$.

Let Γ be the universal cover of the marked graph G. Each non-trivial $c \in F_n$ acts by a *covering translation* $T_c\colon \Gamma \to \Gamma$ which is a hyperbolic isometry, and therefore has an *axis* which we denote by A_c. The projection of A_c to G is the circuit corresponding to the conjugacy class c. If E is a linear edge in a CT so that $f(E) = Ew^d$ as in (Linear Edges), then we say w is the axis of E.

The *space of lines* in Γ is denoted $\tilde{\mathcal{B}}(\Gamma)$ and is the set $((\partial\Gamma \times \partial\Gamma) \setminus \Delta)/\mathbb{Z}_2$ (where Δ denotes the diagonal and $\mathbb{Z}_2$ acts by interchanging the factors) equipped with the compact-open topology. The space of abstract lines is denoted by $\partial^2 F_n$ and defined by $((\partial F_n \times \partial F_n) \setminus \Delta)/\mathbb{Z}_2$. The action of F_n on ∂F_n (resp. $\partial\Gamma$) induces an action on $\partial^2 F_n$ (resp. $\tilde{\mathcal{B}}(\Gamma)$). The marking of G defines an F_n-equivariant homeomorphism between $\partial^2 F_n$ and $\tilde{\mathcal{B}}(\Gamma)$. The quotient of $\tilde{\mathcal{B}}(\Gamma)$ by the F_n action is the *space of lines* in G and is denoted $\mathcal{B}(G)$. The space of abstract lines in R_n is denoted by $\mathcal{B}$.

A *lamination*, Λ, is a closed set of lines in G, or equivalently, a closed F_n-invariant subset of $\tilde{\mathcal{B}}(\Gamma)$. The elements of a lamination are its *leaves*. Associated to each $\phi \in \mathrm{Out}(F_n)$ is a finite ϕ-invariant set of *attracting* *laminations*, denoted by $\mathcal{L}(\phi)$. In the coordinates given by a relative train track map $f: G \to G$ representing ϕ, the attracting laminations for ϕ are in bijection with the EG strata of G.

For each attracting lamination $\Lambda^+ \in \mathcal{L}(\phi)$, there is an associated *expansion factor homomorphism*, $PF_{\Lambda^+}\colon \mathrm{Stab}_{\mathrm{Out}(F_n)}(\Lambda^+) \to \mathbb{Z}$ which has been studied in [BFH00]. We briefly describe the essential features of PF_{Λ^+} here, but we direct the reader to [BFH00] for more details on lines, laminations, and expansion factor homomorphisms. For each $\psi \in \mathrm{Stab}(\Lambda^+)$, at most one of $\mathcal{L}(\psi)$ and $\mathcal{L}(\psi^{-1})$ can contain Λ^+. If neither $\mathcal{L}(\psi)$ nor $\mathcal{L}(\psi^{-1})$ contains Λ^+, then $PF_{\Lambda^+}(\psi) = 0$. Let $f: G \to G$ be a relative train track map representing ψ. If $\Lambda^+ \in \mathcal{L}(\psi)$ and H_r is the EG stratum of G associated to Λ^+ with corresponding PF eigenvalue λ_r, then $PF_{\Lambda^+}(\psi) = \log \lambda_r$. Conversely, if $\Lambda^+ \in \mathcal{L}(\psi^{-1})$, then $PF_{\Lambda^+}(\psi) = -\log \lambda_r$, where λ_r is the PF eigenvalue for the EG stratum of a RTT representative of ψ^{-1} which is associated to Λ^+. The image of PF_{Λ^+} is a discrete subset of $\mathbb{R}$ which we will frequently identify with $\mathbb{Z}$.

For $\phi \in \mathrm{Out}(F_n)$, each element $\Lambda^+ \in \mathcal{L}(\phi)$ has a *paired lamination* in $\mathcal{L}(\phi^{-1})$ which is denoted by Λ^-. The paired lamination is characterized by the fact that it has the same free factor support as Λ^+. That is, the minimal free factor carrying Λ^+ is the same as that which carries Λ^-. We denote the pair $\{\Lambda^+, \Lambda^-\}$ by $\Lambda^\pm$.

3. Translation Lengths in CV_n

In this section, we will compute the translation length for an arbitrary element of $\mathrm{Out}(F_n)$ acting on Outer Space. As is standard, for $\phi \in \mathrm{Out}(F_n)$ we define the translation length of ϕ on Outer Space as $\tau(\phi) = \lim_{n\to\infty} \frac{d(x, x\cdot\phi^n)}{n}$. It is straightforward to check that this is independent of $x \in \mathrm{CV}_n$. For the remainder of this section $\phi \in \mathrm{Out}(F_n)$ will be fixed, and $f: G \to G$ will be a relative train track map representing ϕ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$.

Lemma 3.1. *If H_r is an exponentially growing stratum of G, then there exists a metric ℓ on G such that $\ell(f_\#(E)) \geq \lambda_r \ell(E)$ for every edge $E \in H_r$, where λ_r is the Perron-Frobenius eigenvalue associated to H_r.*

Proof. Let M_r be the transition matrix for the exponentially growing stratum, H_r and let $\mathbf{v}$ be a left eigenvector for the PF eigenvalue λ_r with components $(\mathbf{v})_i$. Normalize $\mathbf{v}$ so that $\sum(\mathbf{v})_i = 1$. For $E_i \in H_r$ define $\ell(E_i) = (\mathbf{v})_i$. If $E \notin H_r$ define $\ell(E) = 1$. We now check the condition on the growth of edges in the EG stratum H_r.

If E is an edge in H_r, (RTT-iii) implies that $f(E)$ is r-legal. Now write $f_\#(E) = f(E)$ as an edge path, $f_\#(E) = E_1 E_2 \ldots E_j$, and we have

$$\ell(f(E)) = \ell(f_\#(E)) = \sum_{i=1}^{j} \ell(E_i) \geq \sum_{i=1}^{j} \ell(E_i \cap H_r) = \lambda_r \ell(E)$$

completing the proof of the lemma. □

We define the r-length ℓ_r of a path or circuit in G by ignoring the edges in other strata. Explicitly, $\ell_r(\sigma) = \ell(\sigma \cap H_r)$, where $\sigma \cap H_r$ is considered as a disjoint union of sub-paths of σ. Note that the definition of ℓ and the proof of the previous lemma show that $\ell_r(f_\#(E_i)) = \lambda_r \ell(E_i)$.

Lemma 3.2. *If σ is an r-legal reduced edge path in G and ℓ is the metric defined in Lemma 3.1, then $\ell_r(f_\#\sigma) = \lambda_r \ell_r(\sigma)$.*

Proof. We write $\sigma = a_1 b_1 a_2 \cdots b_j$ as a decomposition into maximal subpaths where $a_j \subset H_r$ and $b_j \subset G_{r-1}$ as in Lemma 5.8 of [BH92]. Applying the lemma, we conclude that $f_\#(\sigma) = f(a_1) \cdot f_\#(b_1) \cdot f(a_2) \cdot \ldots \cdot f_\#(b_j)$. Thus,

$$\ell_r(f_\#\sigma) = \sum_i \ell_r(f(a_i)) + \sum_i \ell_r(f_\#(b_i)) = \sum_i \ell_r(f(a_i)) = \sum_i \lambda_r \ell_r(a_i) = \lambda_r \ell_r(\sigma)$$ □

Theorem 3.3. *Let $\phi \in \mathrm{Out}(F_n)$ with $f: G \to G$ a RTT representative. For each EG stratum H_r of f, let λ_r be the associated PF eigenvalue. Then $\tau(\phi) = \max\{0, \log \lambda_r \mid H_r$ is an EG stratum$\}$.*

Proof. We first show that $\tau(\phi) \geq \log \lambda_r$ for every EG stratum H_r. Let $x = (G, \ell, \mathrm{id})$ where ℓ is the length function provided by Lemma 3.1. Recall [FM11] that the logarithm of the factor by which a candidate loop is stretched gives a lower bound on the distance between two points in CV_n. Let σ be an r-legal circuit contained in G_r of height r and let $C = \ell_r(\sigma)/\ell(\sigma)$. (RTT-iii) implies that $f^n_\#(\sigma)$ is r-legal for all n, so repeatedly applying Lemma 3.2, we have

$$\frac{\ell(f^n_\#\sigma)}{\ell(\sigma)} \geq \frac{\ell_r(f^n_\#\sigma)}{\ell(\sigma)} = \frac{\ell_r(f^n_\#\sigma)}{\ell_r(f^{n-1}_\#\sigma)}\frac{\ell_r(f^{n-1}_\#\sigma)}{\ell_r(f^{n-2}_\#\sigma)}\cdots\frac{\ell_r(f_\#\sigma)}{\ell_r(\sigma)}\frac{\ell_r(\sigma)}{\ell(\sigma)} \geq \lambda_r^n C$$

Rearranging the inequality, taking logarithms and using the result of [FM11] yields

$$\frac{d(x, x\cdot\phi^n)}{n} \geq \frac{\log(\lambda_r^n C)}{n} = \log\lambda_r + \frac{\log C}{n}$$

Taking the limit as $n \to \infty$, we have a lower bound on the translation length of ϕ.

For the reverse inequality, fix $\epsilon > 0$. We must find a point in outer space which is moved by no more than $\epsilon + \max\{0, \log\lambda_r\}$. The idea is to choose a point in the simplex of CV_n corresponding to a relative train track map for ϕ in which each stratum is much larger than the previous one. This way, the metric will see the growth in every EG stratum. Let $f\colon G \to G$ be a relative train track map as before, but assume that each NEG stratum consists of a single edge. This is justified, for example by choosing f to be a CT [FH11]. Let K be the maximum edge length of the image of any edge of G. Define a length function on G as follows:

$$\ell(E) = \begin{cases} (K/\epsilon)^r & \text{if } E \text{ is the unique edge in the NEG stratum } H_r \\ (K/\epsilon)^r & \text{if } E \text{ is an edge in the zero stratum } H_r \\ (K/\epsilon)^r\cdot v_i & \text{if } E_i \in H_r \text{ and } H_r \text{ is an EG stratum with } \vec{v} \text{ as above} \end{cases}$$

The logarithm of the maximum amount that any edge is stretched in a difference of markings map gives an upper bound on the Lipschitz distance between any two points. So we just check the factor by which every edge is stretched. Clearly the stretch factor for edges in fixed strata is 1. If E is the single edge in an NEG stratum, H_i, then

$$\frac{\ell(f(E))}{\ell(E)} \leq \frac{\ell(E) + K\max\{\ell(E') \mid E' \in G_{i-1}\}}{\ell(E)} = \frac{(K/\epsilon)^i + K(K/\epsilon)^{i-1}}{(K/\epsilon)^i} = 1+\epsilon$$

Similarly, if E is an edge in the zero stratum, H_i, then

$$\frac{\ell(f(E))}{\ell(E)} \leq \frac{K(K/\epsilon)^{i-1}}{(K/\epsilon)^i} = \epsilon$$

We will use the notation $\ell_r^{\downarrow}(\sigma)$ to denote the length of the intersection of σ with G_{r-1}. So for any path σ contained in G_r, we have $\ell(\sigma) = \ell_r(\sigma) + \ell_r^{\downarrow}(\sigma)$. Now, if E_i is an edge in the EG stratum, H_r, with normalized PF eigenvector $\mathbf{v}$ then

$$\frac{\ell(f(E_i))}{\ell(E_i)} = \frac{\ell_r(f(E_i)) + \ell_r^{\downarrow}(f(E_i))}{\ell(E_i)} = \lambda_r + \frac{\ell_r^{\downarrow}(f(E_i))}{\ell(E_i)} \leq \lambda_r + \frac{K(K/\epsilon)^{r-1}}{(K/\epsilon)^r(\mathbf{v})_i} = \lambda_r + \frac{\epsilon}{(\mathbf{v})_i}$$

Since the vector $\mathbf{v}$ is determined by f, after replacing ϵ we have that $\frac{\ell(f(E))}{\ell(E)} \leq \max\{\lambda_r, 1\} + \epsilon$ for every edge of G. Thus, the distance (G, ℓ, ρ) is moved by ϕ is less than $\max\{\log(\lambda_r), 0\} + \epsilon$ and the proof is complete. □

Now that we have computed the translation length of an arbitrary ϕ acting on outer space, we'll use this result to establish our main result in a special case.

4. The Exponential Case

In this section, we'll analyze the case that the abelian subgroup $H = \langle\phi_1, \ldots, \phi_k\rangle$ has enough exponential data so that the entire group is seen by the so called lambda map. More precisely, given an attracting lamination Λ^+ for an outer automorphism ϕ, let $PF_{\Lambda^+}\colon \mathrm{Stab}(\Lambda^+) \to \mathbb{Z}$ be the expansion factor homomorphism defined by Corollary 3.3.1 of [BFH00]. In [FH09, Corollary 3.14], the authors prove that every abelian subgroup of $\mathrm{Out}(F_n)$ has a finite index subgroup which is rotationless (meaning that every element of the subgroup is rotationless). Distortion is unaffected by passing to a finite index subgroup, so there is no loss in assuming that H is rotationless. Now let $\mathcal{L}(H) = \bigcup_{\phi\in H} \mathcal{L}(\phi)$ be the set of attracting laminations for elements of H. By [FH09, Lemma 4.4], $\mathcal{L}(H)$ is a finite set of H-invariant laminations. Define $PF_H\colon H \to \mathbb{Z}^{\#\mathcal{L}(H)}$ by taking the collection of expansion factor homomorphisms for attracting laminations of the subgroup H. In what follows, we will need to interchange PF_{Λ^+} for PF_{Λ^-} and for that we will need the following lemma.

Lemma 4.1. *If $\Lambda^+ \in \mathcal{L}(\phi)$ and $\Lambda^- \in \mathcal{L}(\phi^{-1})$ are paired laminations then $\frac{PF_{\Lambda^+}}{PF_{\Lambda^-}}$ is a constant map. That is, PF_{Λ^+} and PF_{Λ^-} differ by a multiplicative constant, and so determine the same homomorphism.*

Proof. First, Corollary 1.3(2) of [HM14] gives that $\mathrm{Stab}(\Lambda^+) = \mathrm{Stab}(\Lambda^-)$ (which we will henceforth refer to as $\mathrm{Stab}(\Lambda^\pm)$), so the ratio in the statement is always well defined. Now PF_{Λ^+} and PF_{Λ^-} each determine a homomorphism from $\mathrm{Stab}(\Lambda^\pm)$ to $\mathbb{R}$ and it suffices to show that these homomorphisms have the same kernel. Suppose $\psi \notin \ker PF_{\Lambda^+}$ so that by [BFH00, Corollary 3.3.1] either $\Lambda^+ \in \mathcal{L}(\psi)$ or $\Lambda^+ \in \mathcal{L}(\psi^{-1})$. After replacing ψ by ψ^{-1} if necessary, we may assume $\Lambda^+ \in \mathcal{L}(\psi)$. Now ψ has a paired lamination $\Lambda^-_\psi \in \mathcal{L}(\psi^{-1})$ which a priori could be different from Λ^-. But Corollary 1.3(1) of [HM14] says that in fact $\Lambda^-_\psi = \Lambda^-$ and therefore that $\Lambda^- \in \mathcal{L}(\psi^{-1})$. A final application of [BFH00, Corollary 3.3.1] gives that $\psi \notin \ker PF_{\Lambda^-}$. This concludes the proof. □

Theorem 4.2. *If PF_H is injective, then H is undistorted in* $\mathrm{Out}(F_n)$.

Proof. Let k be the rank of H and start by choosing laminations $\Lambda_1, \ldots, \Lambda_k \in \mathcal{L}(H)$ so the restriction of the function PF_H to the coordinates determined by $\Lambda_1, \ldots, \Lambda_k$ is still injective. First note that $\{\Lambda_1, \ldots, \Lambda_k\}$ cannot contain an attracting-repelling lamination pair by Lemma 4.1.

Next, pass to a finite index subgroup of H and choose generators ϕ_i so that after reordering the Λ_i's if necessary, each generator satisfies $PF_H(\phi_i) = (0, \ldots, 0, PF_{\Lambda_i}(\phi_i), 0, \ldots, 0)$. Let $* \in \mathrm{CV}_n$ be arbitrary and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k} \in H$. We complete the proof one orthant at a time by replacing some of the ϕ_i's by their inverses so that all the p_i's are non-negative. Next, after replacing some of the Λ_i's by their paired laminations (again using Lemma 4.1), we may assume that $PF_H(\psi)$ has all coordinates nonnegative.

By Theorem 3.3, the translation length of ψ is the maximum of the Perron-Frobenius eigenvalues associated to the EG strata of a relative train track representative f of ψ. Some, but not necessarily all, of $\Lambda_1, \ldots, \Lambda_k$ are attracting laminations for ψ. Those Λ_i's which are in $\mathcal{L}(\psi)$ are associated to EG strata of f. For such a stratum, the logarithm of the PF eigenvalue is $PF_{\Lambda_i}(\psi)$ and the fact that PF_{Λ_i} is a homomorphism implies

$$PF_{\Lambda_i}(\psi) = PF_{\Lambda_i}(\phi_1^{p_1} \cdots \phi_k^{p_k}) = p_1 PF_{\Lambda_j}(\phi_1) + \ldots + p_k PF_{\Lambda_j}(\phi_k) = p_i PF_{\Lambda_i}(\phi_i)$$

Thus, the translation length of ψ acting on outer space is

$$\begin{aligned}
\tau(\psi) &= \max\{\log\lambda \mid \lambda \text{ is PF eigenvalue associated to an EG stratum of } \psi\} \\
&\geq \max\{PF_{\Lambda_i}(\psi) \mid \Lambda_i \text{ is in } \mathcal{L}(\psi) \text{ and } 1 \leq i \leq k\} \\
&= \max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\}
\end{aligned}$$

In the last equality, the maximum is taken over a larger set, but the only values added to the set were 0.

Let S be a symmetric (i.e., $S^{-1} = S$) generating set for $\mathrm{Out}(F_n)$ and let $D_1 = \max_{s\in S} d(*, * \cdot s)$. If we write ψ in terms of the generators $\psi = s_1 s_2 \cdots s_l$, then

$$\begin{aligned}
d(*, * \cdot \psi) &\leq d(*, * \cdot s_l) + d(* \cdot s_l, * \cdot s_{l-1}s_l) + \ldots + d(* \cdot (s_2 \ldots s_l), * \cdot (s_1 \ldots s_l)) \\
&= d(*, * \cdot s_l) + d(*, * \cdot s_{l-1} + \ldots + d(*, * \cdot s_1) \leq D_1 |\psi|_{\mathrm{Out}(F_n)}
\end{aligned}$$

Let $K_1 = \min\{PF_{\Lambda_i^\pm}(\phi_j^\pm) \mid 1 \leq i, j \leq k\}$. Rearranging this and combining these inequalities, we have

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_1} d(*, * \cdot \psi) \geq \frac{1}{D_1}\tau(\psi) \geq \frac{1}{D_1}\max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\} \geq \frac{K_1}{D_1}\max\{p_i\}$$

We have thus proved that the image of H under the injective homomorphism PF_H is undistorted in $\mathbb{Z}^k$. To conclude the proof, recall that any injective homomorphism between abelian groups is a quasi-isometric embedding. □

Now that we have established our result in the exponential setting, we move on to the polynomial case. First we prove a general result about CTs representing elements of abelian subgroups.

5. Abelian Subgroups are Virtually Finitely Filtered

In this section, we prove an analog of [BFH05, Theorem 1.1] for abelian subgroups. In that paper, the authors prove that any unipotent subgroup of $\mathrm{Out}(F_n)$ is contained in the subgroup $\mathcal{Q}$ of homotopy equivalences respecting a fixed filtration on a fixed graph G. They call such a subgroup "filtered". While generic abelian subgroups of $\mathrm{Out}(F_n)$ are *not* unipotent, we prove that they are virtually filtered. Namely, that such a subgroup is virtually contained in the union of finitely many $\mathcal{Q}$'s. First, we review the comparison homomorphisms introduced in [FH09].

5.1. **Comparison Homomorphisms.** Feighn and Handel defined certain homomorphisms to $\mathbb{Z}$ which measure the growth of linear edges and quasi-exceptional families in a CT representative. Though they can be given a canonical description in terms of principal lifts, we will only need their properties in coordinates given by a CT. Presently, we will define these homomorphisms and recall some basic facts about them. Complete details on comparison homomorphisms can be found in [FH09].

Comparison homomorphisms are defined in terms of principal sets for the subgroup H. The exact definition of a principal set is not important for us. We only need to know that a *principal set* $\mathcal{X}$ for an abelian subgroup H is a subset of ∂F_n which defines a lift $s\colon H \to \mathrm{Aut}(F_n)$ of H to the automorphism group. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two principal sets for H that define distinct lifts s_1 and s_2 to $\mathrm{Aut}(F_n)$. Suppose further that $\mathcal{X}_1 \cap \mathcal{X}_2$ contains the endpoints of an axis A_c. Since H is abelian, $s_1 \cdot s_2^{-1}\colon H \to \mathrm{Aut}(F_n)$ defined by $s_1 \cdot s_2^{-1}(\phi) = s_1(\phi) \cdot s_2(\phi)^{-1}$ is a homomorphism. It follows from [FH11, Lemma 4.14] that for any $\phi \in H$, $s_1(\phi) = s_2(\phi) i_c^k$ for some k, where $i_c\colon \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n)$ denotes conjugation by c. Therefore $s_1 \cdot s_2^{-1}$ defines homomorphism into $\langle i_c \rangle$, which we call the *comparison homomorphism* determined by $\mathcal{X}_1$ and $\mathcal{X}_2$. Generally, we will use the letter ω for comparison homomorphisms.

For a rotationless abelian subgroup H, there are only finitely many comparison homomorphisms [FH09, Lemma 4.3]. Let K be the number of distinct comparison homomorphisms and (as before) let N be the number of attracting laminations for H. The map $\Omega\colon H \to \mathbb{Z}^{N+K}$ defined as the product of the comparison homomorphisms and expansion factor homomorphisms is injective [FH09, Lemma 4.6]. An element $\phi \in H$ is called *generic* if every coordinate of $\Omega(\phi) \in \mathbb{Z}^{N+K}$ is non-zero. If ϕ is generic and $f\colon G \to G$ is a CT representing ϕ, then there is a correspondence between the comparison homomorphisms for H and the linear edges and quasi-exceptional families in G described in the introduction to §7 of [FH09] which we briefly describe now. There is a comparison homomorphism ω_{E_i} for each linear edge E_i in G. If $f(E_i) = E_i \cdot u^{d_i}$, then $\omega_{E_i}(\phi) = d_i$. There is also a comparison homomorphism for each quasi-exceptional family, $E_i u^* \overline{E}_j$ which is denoted by $\omega_{E_i u^* \overline{E}_j}$. If E_i is as before and $f(E_j) = E_j u^{d_j}$, then $\omega_{E_i u^* \overline{E}_j}$ and $\omega(\phi) = d_i - d_j$. We illustrate this correspondence with an example.

Example 5.1. Let $G = R_3$ be the rose with three petals labeled $a, b,$ and c. For $i, j \in \mathbb{Z}$, define $g_{i,j}\colon G \to G$ as follows:

$$g_{i,j}\colon \begin{array}{l} a \mapsto a \\ b \mapsto ba^i \\ c \mapsto ca^j \end{array}$$

Each $g_{i,j}$ determines an outer automorphism of F_3 which we denote by $\phi_{i,j}$. The automorphisms $\phi_{i,j}$ all lie in the rank two abelian subgroup $H = \langle \phi_{0,1}, \phi_{1,0} \rangle$. The subgroup H has three comparison homomorphisms which are easily understood in the coordinates of a CT for a generic element of H. The element $\phi_{2,1}$ is generic in H, and $g_{2,1}$ is a CT representing it. Two of the comparison homomorphisms manifest as ω_b and ω_c where $\omega_b(\phi_{i,j}) = i$ and $\omega_c(\phi_{i,j}) = j$. The third homomorphism is denoted by $\omega_{ba^*\overline{c}}$ and it measures how a path of the form $ba^*\overline{c}$ changes when $g_{i,j}$ is applied. Since $g_{i,j}(ba^*\overline{c}) = ba^{*+i-j}\overline{c}$, we have $\omega_{ba^*\overline{c}}(\phi_{i,j}) = i - j$.

In the sequel, we will rely heavily on this correspondence between the comparison homomorphisms of H and the linear edges and quasi-exceptional families in a CT for a generic element of H. We now prove the main result of this section.

Proposition 5.2. *For any abelian subgroup H of $\mathrm{Out}(F_n)$, there exists a finite index subgroup H' such that every $\phi \in H'$ can be realized as a CT on one of finitely many marked graphs.*

Most of the proof consists of restating and combining results of Feighn and Handel from [FH09]. We refer the reader to §6 of their paper for the relevant notation and most of the relevant results.

Proof. First replace H by a finite index rotationless subgroup [FH09, Corollary 3.14]. The proof is by induction on the rank of H. The base case follows directly from [FH09, Lemma 6.18]. Let $H = \langle \phi \rangle$ and let $f^{\pm}: G^{\pm} \to G^{\pm}$ be CT's for ϕ and ϕ^{-1} which are both generic in H. The definitions then guarantee that $\mathbf{i} = (i, i, \ldots, i)$ for $i > 0$ is both generic and admissible. Lemma 6.18 then says that $f^{\pm}_{\mathbf{i}}: G^{\pm} \to G^{\pm}$ is a CT representing $\phi^{\pm}_{\mathbf{i}} = \phi^{\pm i}$, so we are done.

Assume now that the claim holds for all abelian subgroups of rank less than k, and let $H = \langle \phi_1, \ldots, \phi_k \rangle$. The set of generic elements of H is the complement of a finite [FH09, Lemma 4.3] collection of hyperplanes. Every non-generic element, ϕ, lies in a rank $(k-1)$ abelian subgroup of H: the kernel of the corresponding comparison homomorphism. By induction and the fact that there are only finitely many hyperplanes, every non-generic element has a CT representative on one of finitely many marked graphs. We now add a single marked graph for each sector defined by the complement of the hyperplanes.

Let ϕ be generic and let $f: G \to G$ be a CT representative. Let $\mathcal{D}(\phi)$ be the disintegration of ϕ as defined in [FH09] and recall that $\mathcal{D}(\phi) \cap H$ is finite index in H [FH09, Theorem 7.2]. Let Γ be the semigroup of generic elements of $\mathcal{D}(\phi) \cap H$ that lie in the same sector of H as ϕ (i.e., for every $\gamma \in \Gamma$ and every coordinate ω of Ω, the signs of $\omega(\gamma)$ and $\omega(\phi)$ agree). The claim is that every element of Γ can be realized as a CT on the marked graph G and we will show this by explicitly reconstructing the generic tuple $\mathbf{a}$ such that $\gamma = [f_{\mathbf{a}}]$. Fix $\gamma \in \Gamma$ and let $\phi_{\mathbf{a}_1}, \ldots, \phi_{\mathbf{a}_k}$ be a generating set for H with $\mathbf{a}_i$ generic [FH09, Corollary 6.20]. Write γ as a word in the generators, $\gamma = \phi^{j_1}_{\mathbf{a}_1} \cdots \phi^{j_k}_{\mathbf{a}_k}$ and define $\mathbf{a} = j_1\mathbf{a}_1 + \ldots + j_k\mathbf{a}_k$. Since the admissibility condition is a set of homogeneous linear equations which must be preserved under taking linear combinations, as long as every coordinate of $\mathbf{a}$ is non-negative, $\mathbf{a}$ must be admissible. To see that every coordinate of $\mathbf{a}$ is in fact positive, let ω be a coordinate of Ω^{ϕ}. Using the fact that ω is a homomorphism to $\mathbb{Z}$ and repeatedly applying [FH09, Lemma 7.5] to the $\phi_{\mathbf{a}_i}$'s, we have

$$\begin{aligned}
\omega(\gamma) &= j_1\omega(\phi_{\mathbf{a}_1}) + j_2\omega(\phi_{\mathbf{a}_2}) + \ldots + j_k\omega(\phi_{\mathbf{a}_k}) \\
&= j_1(\mathbf{a}_1)_s\omega(\phi) + j_2(\mathbf{a}_2)_s\omega(\phi) + \ldots + j_k(\mathbf{a}_k)_s\omega(\phi) \\
&= (j_1\mathbf{a}_1 + j_2\mathbf{a}_2 + \ldots + j_k\mathbf{a}_k)_s\omega(\phi) \\
&= (\mathbf{a})_s\omega(\phi)
\end{aligned}$$

where $(\mathbf{a})_s$ denotes the s-th coordinate of the vector $\mathbf{a}$. Since γ and ϕ were assumed to be generic and to lie in the same sector, we conclude that every coordinate of $\mathbf{a}$ is positive. The injectivity Ω^{ϕ} [FH09, Lemma 7.4] then implies that $\gamma = [f_{\mathbf{a}}]$. That $\mathbf{a}$ is in fact generic follows from the fact, which is directly implied by the definitions, that if $\mathbf{a}$ is a generic tuple, then $\phi_{\mathbf{a}}$ is a generic element of H. Finally, we apply [FH09, Lemma 6.18] to conclude that $f_{\mathbf{a}}: G \to G$ is a CT. Thus, every element of Γ has a CT representative on the marked, filtered graph G. Repeating this argument in each of the finitely many sectors and passing to the intersection of all the finite index subgroups obtained this way yields a finite index subgroup H' and finitely many marked graphs, so that every generic element of H' can be realized as a CT on one of the marked graphs. The non-generic elements were already dealt with using the inductive hypothesis, so the proof is complete. ☐

6. The Polynomial Case

In [Ali02], the author introduced a function that measures the twisting of conjugacy classes about an axis in F_n and used this function to prove that cyclic subgroups of UPG are undistorted. In order to use the comparison homomorphisms in conjunction with this twisting function, we need to establish a result about the possible terms occuring in completely split circuits. After establishing this connection, we use it to prove (Theorem 6.12) the main result under the assumption that H has "enough" polynomial data.

In the last section, we saw the correspondence between comparison homomorphisms and certain types of paths in a CT. In order to use the twisting function from [Ali02], our goal is to find circuits in G with single linear edges or quasi-exceptional families as subpaths, and moreover to do so in such a way that we can control cancellation at the ends of these subpaths under iteration of f. This is the most technical section of the paper, and the one that most heavily relies on the use of CTs. The main result is Proposition 6.1.

6.1. **Completely Split Circuits.** One of the main features of train track maps is that they allow one to understand how cancellation occurs when tightening $f^k(\sigma)$ to $f^k_\#(\sigma)$. In previous incarnations of train track maps, this cancellation was understood inductively based on the height of the path σ. One of the main advantages of completely split train track maps is that the way cancellation can occur is now understood directly, rather than inductively.

Given a CT $f\colon G \to G$ representing ϕ, the set of allowed terms in completely split paths would be finite were it not for the following two situations: a linear edge $E \mapsto Eu$ gives rise to an infinite family of INPs of the form $Eu^*\overline{E}$, and two linear edges with the same axis $E_1 \mapsto E_1u^{d_1}$, $E_2 \mapsto E_2u^{d_2}$ (with d_1 and d_2 having the same sign) give rise to an infinite family of exceptional paths of the form $E_1u^*\overline{E}_2$. To see that these are the only two subtleties, one only needs to know that there is at most one INP of height r for each EG stratum H_r. This is precisely [FH09, Corollary 4.19].

To connect Feighn-Handel's comparison homomorphisms to Alibegović's twisting function, we would like to show that every linear edge and exceptional family occurs as a term in the complete splitting of some completely split circuit. We will in fact show something stronger:

Proposition 6.1. *There is a completely split circuit σ containing every allowable term in its complete splitting. That is the complete splitting of σ contains at least one instance of every*

- *edge in an irreducible stratum (fixed, NEG, or EG)*
- *maximal, taken connecting subpath in a zero stratum*
- *infinite family of INPs $Eu^*\overline{E}$*
- *infinite family of exceptional paths $E_1u^*\overline{E}_2$*

The proof of this proposition will require a careful study of completely split paths. With that aim, we define a directed graph that encodes the complete splittings of such paths. Given a CT $f\colon G \to G$ representing ϕ define a di-graph $\mathcal{CSP}(f)$ (or just $\mathcal{CSP}$ when f is clear) whose vertices are oriented allowed terms in completely split paths. More precisely, there are two vertices for each edge in an irreducible stratum: one labeled by E and one labeled by $\overline{E}$ (which we will refer to at τ_E and $\tau_{\overline{E}}$). There are two vertices for each maximal taken connecting path in a zero stratum: one for σ and one for $\overline{\sigma}$ (which will be referred to as τ_σ and $\tau_{\overline{\sigma}}$). Similarly, there are two vertices for each family of exceptional paths, two vertices for each INP of EG height, and *one* vertex for each infinite family of NEG Nielsen paths. There is only one vertex for each family of indivisible Nielsen path σ whose height is NEG because σ and $\overline{\sigma}$ determine the same initial direction. There is an edge connecting two vertices τ_σ and $\tau_{\sigma'}$ in $\mathcal{CSP}(f)$ if the path $\sigma\sigma'$ is completely split with splitting given by $\sigma \cdot \sigma'$. This is equivalent to the turn $(\overline{\sigma}, \sigma')$ being legal by the uniqueness of complete splittings [FH11, Lemma 4.11].

Any completely split path (resp. circuit) σ with endpoints at vertices in G defines a directed edge path (resp. directed loop) in $\mathcal{CSP}(f)$ given by reading off the terms in the complete splitting of σ. Conversely, a directed path or loop in $\mathcal{CSP}(f)$ yields a not quite well defined path or circuit σ in G which is necessarily completely split. The only ambiguity lies in how to define σ when the path in $\mathcal{CSP}(f)$ passes through a vertex labeled by a Nielsen path of NEG height or a quasi-exceptional family.

Example 6.2. Consider the rose R_2 consisting of two edges a and b with the identity marking. Let $f\colon R_2 \to R_2$ be defined by $a \mapsto ab$, $b \mapsto bab$. This is a CT representing a fully irreducible outer automorphism. There is one indivisible Nielsen path $\sigma = ab\overline{a}\overline{b}$. The graph $\mathcal{CSP}(f)$ is shown in Figure 1. The blue edges represent the fact that each of the paths $\overline{b} \cdot \overline{b}$, $\overline{b} \cdot \overline{a}$, $\overline{b} \cdot \sigma$, and $\overline{b} \cdot a$ is completely split.

Remark 6.3. A basic observation about the graph $\mathcal{CSP}$ is that every vertex τ_σ has at least one incoming and at least one outgoing edge. While this is really just a consequence of the fact that every vertex in a CT has at least two gates, a bit of care is needed to justify this formally. Indeed, let v be the initial endpoint

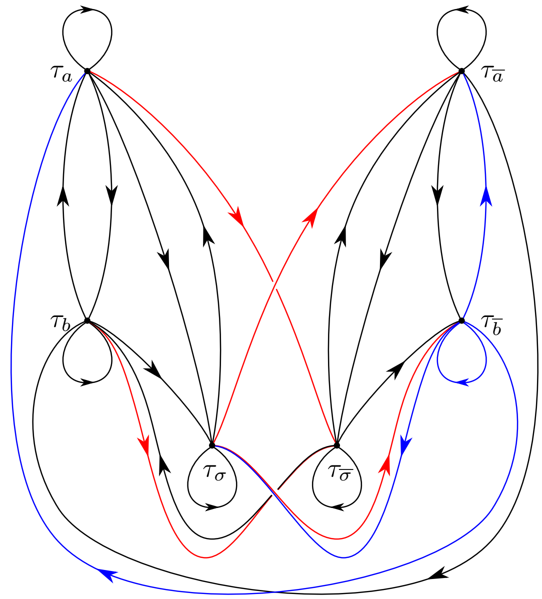

FIGURE 1. The graph of $\mathcal{CSP}(f)$ for Example 6.2

of σ. If there is some legal turn (E, σ) at v where E is an edge in an irreducible stratum, then $\overline{E} \cdot \sigma$ is completely split so there is an edge in $\mathcal{CSP}$ from $\tau_{\overline{E}}$ to τ_σ. The other possibility is that the only legal turns $(\_\_, \sigma)$ at v consist of an edge in a zero stratum H_i. In this case, (Zero Strata) guarantees that v is contained in the EG stratum H_r which envelops H_i and that the link of v is contained in $H_i \cup H_r$. In particular, there are a limited number of possibilities for σ; σ may be a taken connecting subpath in H_i, an edge in H_r, or an EG INP of height r. In the first two cases, σ is a term in the complete splitting of $f_\#^k(E)$ for some edge E. By increasing k if necessary, we can guarantee that σ is not the first or last term in this splitting. Therefore, there is a directed edge in $\mathcal{CSP}$ with terminal endpoint τ_σ. In the case that σ is an INP, σ has a first edge E_0 which is necessarily of EG height. We have already established that there is a directed edge in $\mathcal{CSP}$ pointed to τ_{E_0}, so we just observe that any vertex in $\mathcal{CSP}$ with a directed edge ending at E_0 will also have a directed edge terminating at τ_σ. The same argument shows that there is an edge in $\mathcal{CSP}$ emanating from τ_σ.

The statement of Proposition 6.1 can now be rephrased as a statement about the graph $\mathcal{CSP}$. Namely, that there is a directed loop in $\mathcal{CSP}$ which passes through every vertex.

We will need some basic terminology from the study of directed graphs. We say a di-graph Γ is *strongly connected* if every vertex can be connected to every other vertex in Γ by a directed edge path. In any di-graph, we may define an equivalence relation on the vertices by declaring $v \sim w$ if there is a directed edge path from v to w and vice versa (we are required to allow the trivial edge path so that $v \sim v$). of Γ. The equivalence classes of this relation partition the vertices of Γ into *strongly connected components*.

We will prove that $\mathcal{CSP}(f)$ is connected and has one strongly connected component. From this, Proposition 6.1 follows directly. The proof proceeds by induction on the core filtration of G, which is the filtration obtained from the given one by considering only the filtration elements which are their own cores. Because the base case is in fact more difficult than the inductive step, we state it as a lemma.

Lemma 6.4. *If $f\colon G \to G$ is a CT representing a fully irreducible automorphism, then $\mathcal{CSP}(f)$ is connected and strongly connected.*

Proof. Under these assumptions, there are two types of vertices in $\mathcal{CSP}(f)$: those labeled by edges, and those labeled by INPs. We denote by $\mathcal{CSP}_e$ the subgraph consisting of only the vertices which are labeled by edges. Recall that τ_E denotes the vertex in $\mathcal{CSP}$ corresponding to the edge E. If the leaves of the attracting lamination are non-orientable, then we can produce a path in $\mathcal{CSP}_e$ starting at τ_E, then passing through every other vertex in $\mathcal{CSP}_e$, and finally returning to τ_E by looking at a long segment of a leaf of the attracting lamination. More precisely, (Completely Split) says that $f^k(E)$ is a completely split path for all $k \geq 0$ and the fact that f is a train track map says that this complete splitting contains no INPs. Moreover, irreducibility of the transition matrix and non-orientability of the lamination implies that for sufficiently large k this path not only contains every edge in G (with both orientations), but contains the edge E followed by every other edge in G with both of its orientations, and then the edge E again. Such a path in G exactly shows that $\mathcal{CSP}_e$ is connected and strongly connected.

We isolate the following remark for future reference.

Remark 6.5. If there is an indivisible Nielsen path σ in G, write its edge path $\sigma = E_1E_2 \dots E_k$ (recall that all INPs in a CT have endpoints at vertices). If $\tau_{\sigma'}$ is any vertex in $\mathcal{CSP}$ with a directed edge pointing to τ_{E_1}, then $\sigma' \cdot \sigma$ is completely split since the turn $(\overline{\sigma}', \sigma)$ must be legal. Hence there is also a directed edge in $\mathcal{CSP}$ from $\tau_{\sigma'}$ to τ_σ. The same argument shows that there is an edge in $\mathcal{CSP}$ from τ_σ to some vertex $\tau' \neq \tau_\sigma$.

Since $\mathcal{CSP}_e$ is strongly connected, and the remark implies that each vertex τ_σ (for σ an INP in G) has directed edges coming from and going back into $\mathcal{CSP}_e$, we conclude that $\mathcal{CSP}$ is strongly connected in the case that leaves of the attracting lamination are non-orientable.

Now choose an orientation on the attracting lamination Λ. If we imagine an ant following the path in G determined by a leaf of Λ, then at each vertex v we see the ant arrive along certain edges and leave along others. Let E be an edge with initial vertex v so that E determines a gate $[E]$ at v. We say that $[E]$ is a *departure gate* at v if E occurs in some (any) oriented leaf λ. Similarly, we say the gate $[E]$ is an *arrival gate* at v if the edge $\overline{E}$ occurs in λ. Some gates may be both arrival and departure gates.

Suppose now that there is some vertex v in G that has at least two arrival gates and some vertex w that has at least two departure gates. As before, we will produce a path in $\mathcal{CSP}_e$ that shows this subgraph has one strongly connected component. Start at any edge in G and follow a leaf λ of the lamination until you have crossed every edge with its forward orientation. Continue following the leaf until you arrive at v, say through the gate $[\overline{E}]$. Since v has two arrival gates, there is some edge E' which occurs in λ with the given orientation and whose terminal vertex is v ($[\overline{E'}]$ is a second arrival gate). Now turn onto $\overline{E'}$. Since $[\overline{E}]$ and $[\overline{E'}]$ are distinct gates, this turn is legal. Follow $\overline{\lambda}$ going backwards until you have crossed every edge of G (now in the opposite direction). Finally, continue following $\overline{\lambda}$ until you arrive at w, where there are now two arrival gates because you are going backwards. Use the second arrival gate to turn around a second time, and follow λ (now in the forwards direction again) until you cross the edge you started with. By construction, this path in G is completely split and every term in its complete splitting is a single edge. The associated path in $\mathcal{CSP}_e$ passes through every vertex and then returns to the starting vertex, so $\mathcal{CSP}_e$ is strongly connected. In the presence of an INP, Remark 6.5 completes the proof of the lemma under the current assumptions.

We have now reduced to the case where the lamination is orientable *and* either every vertex has only one departure gate or every vertex has only one arrival gate. The critical case is the latter of the two, and we would like to conclude in this situation that there is an INP. Example 6.2 illustrates this scenario. Some edges are colored red to illustrate the fact that in order to turn around and get from the vertices labeled by a and b to those labeled by $\overline{a}$ and $\overline{b}$, one must use an INP. The existence of an INP in this situation is provided by the following lemma.

Lemma 6.6. *Assume $f\colon G \to G$ is a CT representing a fully irreducible rotationless automorphism. Suppose that the attracting lamination is orientable and that every vertex has exactly one arrival gate. Then G has an INP, σ, and the initial edges of σ and $\overline{\sigma}$ are oriented consistently with the orientation of the lamination.*

We postpone the proof of this lemma and explain how to conclude our argument. If every vertex has one arrival gate, then we apply the lemma to conclude that there must be an INP. Since INPs have exactly one illegal turn, using the previous argument, we can turn around once. Now if we are again in a situation where there is only one arrival gate, then we can apply the lemma a second time (this time with the orientation of Λ reversed) to obtain the existence of a second INP, allowing us to turn around a second time. $\square$

We remark that since there is at most one INP in each EG stratum of a CT, Lemma 6.6 implies that if the lamination is orientable, then some vertex of G must have at least 3 gates.

Proof of Lemma 6.6. There is a vertex of G that is fixed by f since [FH11, Lemma 3.19] guarantees that every EG stratum contains at least one principal vertex and principal vertices are fixed by (Rotationless). Choose such a vertex v and let $\tilde{v} \in \Gamma$ be a lift of v to the universal cover Γ of G. Let g be the unique arrival gate at $\tilde{v}$. Lift f to a map $\tilde{f}\colon \Gamma \to \Gamma$ fixing $\tilde{v}$. Let T be the infinite subtree of Γ consisting of all embedded rays $\gamma\colon [0,\infty) \to \Gamma$ starting at $\tilde{v}$ and leaving every vertex through its unique arrival gate. That is $\gamma(0) = \tilde{v}$ and whenever $\gamma(t)$ is a vertex, $D\gamma(t)$ should be the unique arrival gate at $\gamma(t)$. Refer to Figure 2 for the tree T for Example 6.2.

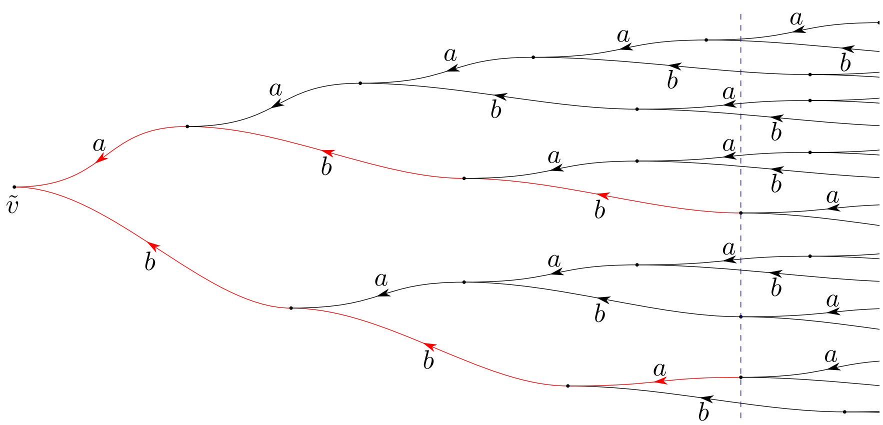

FIGURE 2. The tree T for Example 6.2. The red path connects two vertices of the same height.

First, we claim that $\tilde{f}(T) \subset T$. To see this, notice that since f is a topological representative, it suffices to show that $\tilde{f}(p) \in T$ for every vertex p of T. Notice that vertices p of T are characterized by two things: first $[\tilde{v}, p]$ is legal, and second, for every edge E in the edge path of $[\tilde{v}, p]$, the gate $[E]$ is the unique arrival gate at the initial endpoint of E. Now $[\tilde{v}, \tilde{f}(p)] = \tilde{f}([\tilde{v}, p])$ is legal because f is a train track map. Moreover, every edge E in the edge path of $[\tilde{v}, p]$ occurs (with orientation) in a leaf $\overline{\lambda}$ of the lamination. Since $\tilde{f}$ takes leaves to leaves preserving orientation, the same is true for $\tilde{f}([\tilde{v}, p])$. The gate determined by every edge in the edge path of $\overline{\lambda}$ is the unique arrival gate at that vertex. Thus, for every edge E in the edge path of $\tilde{f}([\tilde{v}, p])$, $[E]$ is the unique arrival gate at that vertex, which means that $\tilde{f}(p) \in T$.

Endow G with a metric using the left PF eigenvector of the transition matrix so that for every edge of G, we have $\ell(f(E)) = \nu\,\ell(E)$ where ν is the PF eigenvalue of the transition matrix. Lift the metric on G to a metric on Γ and define a height function on the tree T by measuring the distance to $\tilde{v}$: $h(p) = d(p, \tilde{v})$. Since legal paths are stretched by exactly ν, we have that for any $p \in T$, $h(\tilde{f}(p)) = \nu h(p)$.

Now let w and w' be two distinct lifts of v with the same height, $h(w) = h(w')$. To see that this is possible, just take α and β to be two distinct ($\langle \alpha, \beta \rangle \simeq F_2$) circuits in G based at v which are obtained by following a leaf of the lamination. The initial vertices of the lifts of $\alpha\beta$ and $\beta\alpha$ which end at $\tilde{v}$ are distinct lifts of v which are contained in T, and have the same height.

Let τ be the unique embedded segment connecting w to w' in T. By [FH11, Lemma 4.25], $\tilde{f}^k_\#(\tau)$ is completely split for all sufficiently large k. Moreover, the endpoints of $\tilde{f}^k_\#(\tau)$ are distinct since the restriction of $\tilde{f}$ to the lifts of v is injective. This is simply because $\tilde{f}\colon (\Gamma, \tilde{v}) \to (\Gamma, \tilde{v})$ represents an automorphism of F_n and lifts of v correspond to elements of F_n. Now observe that the endpoints $\tilde{f}^k_\#(\tau)$ have the same height and for any pair of distinct vertices with the same height, the unique embedded segment connecting them must contain an illegal turn. This follows from the definition of T and the assumption that every vertex has a unique arrival gate. Therefore, the completely split path $\tilde{f}^k_\#(\tau)$ contains an illegal turn. In particular, it must have an INP in its complete splitting. That the initial edges of σ and $\overline{\sigma}$ are oriented consistently with the orientation on λ is evident from the construction. □

The key to the inductive step is provided by the "moving up through the filtration" lemma from [FH09] which explicitly describes how the graph G can change when moving from one element of the core filtration to the next. Recall the *core filtration* of G is the filtration $G_0 \subseteq G_{l_1} \subseteq \ldots \subseteq G_{l_k} = G_m = G$ obtained by restricting to those filtration elements which are their own cores. For each G_{l_i}, the i-th stratum of the core filtration is defined to be $H^c_{l_i} = \bigcup_{j=l_{i-1}+1}^{l_i} H_j$. Finally, we let $\Delta\chi^-_i = \chi(G_{l_{i-1}}) - \chi(G_{l_i})$ denote the negative of the change in Euler characteristic.

Lemma 6.7 ([FH09, Lemma 8.3])**.** (1) *If $H^c_{l_i}$ does not contain any EG strata then one of the following holds.*
  (a) *$l_i = l_{i-1} + 1$ and the unique edge in $H^c_{l_i}$ is a fixed loop that is disjoint from $G_{l_{i-1}}$.*
  (b) *$l_i = l_{i-1} + 1$ and both endpoints of the unique edge in $H^c_{l_i}$ are contained in $G_{l_{i-1}}$.*
  (c) *$l_i = l_{i-1} + 2$ and the two edges in $H^c_{l_i}$ are nonfixed and have a common initial endpoint that is not in $H_{l_{i-1}}$ and terminal endpoints in $G_{l_{i-1}}$.*

  In case 1a, $\Delta_i\chi^- = 0$; in cases 1b and 1c, $\Delta_i\chi^- = 1$.

(2) *If $H^c_{l_i}$ contains an EG stratum, then H_{l_i} is the unique EG stratum in $H^c_{l_i}$ and there exists $l_{i-1} \leq u_i < l_i$ such that both of the following hold.*
  (a) *For $l_{i_1} < j \leq u_i$, H_j is a single nonfixed edge E_j whose terminal vertex is in $G_{l_{i-1}}$ and whose initial vertex has valence one in G_{u_i}. In particular, G_{u_i} deformation retracts to $G_{l_{i-1}}$ and $\chi(G_{u_i}) = \chi(G_{l_{i-1}})$.*
  (b) *For $u_i < j < l_i$, H_j is a zero stratum. In other words, the closure of $G_{l_i} \setminus G_{u_i}$ is the extended EG stratum $H^z_{l_i}$.*

  If some component of $H^c_{l_i}$ is disjoint from G_{u_i} then $H^c_{l_i} = H_{l_i}$ is a component of G_{l_i} and $\Delta_i\chi^- \geq 1$; otherwise $\Delta_i\chi^- \geq 2$.

As we move up through the core filtration, we imagine adding new vertices to $\mathcal{CSP}$ and adding new edges connecting these vertices to each other and to the vertices already present. Thus, we define $\mathcal{CSP}_{l_i}$ to be the subgraph of $\mathcal{CSP}$ consisting of vertices labeled by allowable terms in G_{l_i}. Here we use the fact that the restriction of f to each connected component of an element of the core filtration is a CT.

The problem with proving that $\mathcal{CSP}$ is strongly connected by induction on the core filtration is that $\mathcal{CSP}_{l_i}$ may have multiple connected components. This only happens, however, if G_{l_i} has more than one connected component in which case $\mathcal{CSP}_{l_i}$ will have multiple connected components. If any component of G_{l_i} is a topological circle (necessarily consisting of a single fixed edge E), then $\mathcal{CSP}_{l_i}$ will have two connected components for this circle.

Lemma 6.8. *For every $1 \leq i \leq k$, the number of strongly connected components of $\mathcal{CSP}_{l_i}(f)$ is equal to*

$$2 \cdot \#\{\textit{connected components of } G_{l_i} \textit{that are circles}\} + \#\{\textit{connected components of } G_{l_i} \textit{that are not circles}\}$$

The following proof is in no way difficult. It only requires a careful analysis of the many possible cases. The only case where there is any real work is in case 2 of Lemma 6.7.

Proof. Lemma 6.4 establishes the base case when H^c_1 is exponentially growing. If H^c_1 is a circle, then $\mathcal{CSP}_1$ has exactly two vertices, each with a self loop, so the lemma clearly holds. We now proceed to the inductive step, which is case-by-case analysis based on Lemma 6.7. We set some notation to be used throughout: E will be an edge with initial vertex v and terminal vertex w (it's possible that $v = w$). We denote by $G^v_{l_i}$ the component of G_{l_i} containing v and similarly for w. Let $\mathcal{CSP}^v_{l_i}$ be the component(s) of $\mathcal{CSP}_{l_i}$ containing paths which pass through v. In the case that $G^v_{l_i}$ is a topological circle, there will be two such components.

In case 1a of Lemma 6.7, $\mathcal{CSP}_{l_i}$ is obtained from $\mathcal{CSP}_{l_{i-1}}$ by adding two new vertices: τ_E and $\tau_{\overline{E}}$. Each new vertex has a self loop, and no other new edges are added. So the number of connected components of $\mathcal{CSP}$ increases by two. Each component is strongly connected.

In case 1b, there are several subcases according to the various possibilities for the edge E, and the topological types of $G^v_{l_{i-1}}$ and $G^w_{l_{i-1}}$. First, suppose that E is a fixed edge. Then $\mathcal{CSP}_{l_i}$ is obtained from

$\mathcal{CSP}_{l_{i-1}}$ by adding two new vertices. There are no new INPs since the restriction of f to each component of G_{l_i} is a CT and any INP is of the form provided by (NEG Nielsen Paths) or (EG Nielsen Paths). As in Remark 6.3, the vertex τ_E has an incoming edge with initial endpoint τ and an outgoing edge with terminal endpoint τ'. Moreover, $\tau \in \mathcal{CSP}^v_{l_{i-1}}$ and $\tau' \in \mathcal{CSP}^w_{l_{i-1}}$. We then have a directed edge from $\sigma \in \mathcal{CSP}^w_{l_{i-1}}$ to $\tau_{\overline{E}}$ and a directed edge from $\tau_{\overline{E}}$ to $\sigma' \in \mathcal{CSP}^v_{l_{i-1}}$. Hence, there are directed paths in $\mathcal{CSP}_{l_i}$ connecting the two strongly connected subgraphs $\mathcal{CSP}^v_{l_{i-1}}$ and $\mathcal{CSP}^w_{l_{i-1}}$ to each other, and passing through all new vertices. Therefore, there is one strongly connected component of $\mathcal{CSP}_{l_i}$ corresponding to the component of G_{l_i} containing v (and w). This component cannot be a circle, since it contains at least two edges. In the case that $G^v_{l_{i-1}}$ (resp. $G^w_{l_{i-1}}$) is a topological circle, we remark that there are incoming (resp. outgoing) edges in $\mathcal{CSP}^v_{l_i}$ (resp. $\mathcal{CSP}^w_{l_i}$) to τ_E from each of the components of $\mathcal{CSP}^v_{l_{i-1}}$ (resp. $\mathcal{CSP}^w_{l_{i-1}}$). See Figure 3.

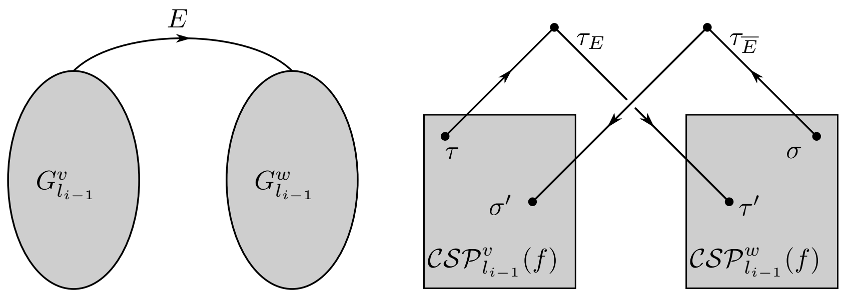

FIGURE 3. A possibility for G_{l_i} and the graph $\mathcal{CSP}_{l_i}$ when $H^c_{l_i}$ is a single NEG edge

Suppose now that E is a non-fixed NEG edge. There are two new vertices in $\mathcal{CSP}_{l_i}$ labeled τ_E and $\tau_{\overline{E}}$. The argument given in the previous paragraph goes through once we notice that if $v \neq w$, then $G^w_{l_{i-1}}$ cannot be a circle since this would imply that w is not a principal vertex in G_{l_i} (see first bullet point in the definition) contradicting the fact that $f|_{G_{l_i}}$ is a CT ((Vertices) is not satisfied).

If E is a non-linear edge, then we are done. If E is linear, then there will be other new vertices in $\mathcal{CSP}_{l_i}$. There will be a new vertex for the family of NEG Nielsen paths $Eu^*\overline{E}$. The fact that we have concluded the inductive step for the vertex τ_E along with remark 6.5 shows that this new vertex is in the same strongly connected component as τ_E. There will also be two vertices for each family of exceptional paths $Eu^*\overline{E'}$. For the exact same reasons, these vertices are also in this strongly connected component. This concludes the proof in case 1b of Lemma 6.7.

The arguments given thus far apply directly to case 1c of Lemma 6.7. We remark that in this case, neither of the components of $G_{l_{i-1}}$ containing the terminal endpoints of the new edges can be circles for the same reason as before.

The most complicated way that G (and hence $\mathcal{CSP}$) can change is when $H^c_{l_i}$ contains an EG stratum. In case 2 of Lemma 6.7, if some component of $H^c_{l_i}$ is disjoint from G_{u_i}, then $H^c_{l_i}$ is a component of G_{l_i} and the restriction of f to this component is a fully irreducible. In particular, $\mathcal{CSP}_{l_i}$ has one more strongly connected component than $\mathcal{CSP}_{l_{i-1}}$ by Lemma 6.4.

Though case 2 of Lemma 6.7 describes G_{l_i} as being built from $G_{l_{i-1}}$ in three stages from bottom to top, somehow it is easier to prove $\mathcal{CSP}_{l_i}$ has the correct number of connected components by going from top to bottom.

By looking at a long segment of a leaf of the attracting lamination for H_{l_i}, we can see as in Lemma 6.4 that the vertices in $\mathcal{CSP}_{l_i}$ labeled by edges in the EG stratum H_{l_i} are in at most two different strongly connected components. In fact, we can show that these vertices are all in the same strongly connected component. Since we are working under the assumption that no component of $H^c_{l_i}$ is disjoint from G_{u_i}, we can use one of the components of G_{u_i} to turn around on a leaf of the lamination. Indeed, choose some component G^1 of G_{u_i} which intersects H_{l_i}. Let E be an EG edge in H_{l_i} with terminal vertex $w \in G^1$. Note that if G^1 deformation retracts onto a circle with vertex v, then some EG edge in H_{l_i} must be incident to v, since otherwise $f|_{G_{l_i}}$ would not be a CT. Thus, by replacing E if necessary, we may assume in this situation that w is on the circle. Using the inductive hypothesis and the fact that mixed turns are legal, we can connect the vertex τ_E to the vertex $\tau_{\overline{E}}$ in $\mathcal{CSP}_{l_i}$. Then we can follow a leaf of the lamination going backwards until we return to w, say along E'. If $E = E'$, then the leaves of the lamination were non-orientable in the first place, and all the vertices labeled by edges in H_{l_i} are in the same strongly connected component of $\mathcal{CSP}_{l_i}$. Otherwise, apply the inductive hypothesis again and use the fact that mixed turns are legal to get a path from $\tau_{E'}$ to $\tau_{\overline{E}'}$. This shows all vertices labeled by edges in H_{l_i} are in the same strongly connected component of $\mathcal{CSP}_{l_i}$. We will henceforth denote the strongly connected component of $\mathcal{CSP}_{l_i}$ which contains all these vertices by $\mathcal{CSP}_{l_i}^{EG}$.

If there is an INP σ of height H_{l_i}, its first and last edges are necessarily in H_{l_i}. Remark 6.5 then implies that τ_σ and $\tau_{\overline{\sigma}}$ are in $\mathcal{CSP}_{l_i}^{EG}$. Recall that the only allowable terms in complete splittings which intersect zero strata are connecting paths which are both maximal and *taken*. In particular, each vertex in $\mathcal{CSP}_{l_i}$ corresponding to such a connecting path is in the aforementioned strongly connected component, $\mathcal{CSP}_{l_i}^{EG}$.

Now let E be an NEG edge in $H_{l_i}^c$ with terminal vertex w. There is necessarily an outgoing edge from τ_E into $\mathcal{CSP}_{l_{i-1}}^w$ and an incoming edge to τ_E from $\mathcal{CSP}_{l_i}^{EG}$. If the graph $G_{l_{i-1}}^w$ is not a topological circle, then the corresponding component $\mathcal{CSP}_{l_{i-1}}^w$ is already strongly connected and there is a directed edge from this graph back to $\tau_{\overline{E}}$ and from there back into $\mathcal{CSP}_{l_i}^{EG}$. Thus, this subgraph is contained in the strongly connected component $\mathcal{CSP}_{l_i}^{EG}$. On the other hand, if $G_{l_{i-1}}^w$ is a topological circle, then there is a directed edge from τ_E back into $\mathcal{CSP}_{l_i}^{EG}$ because mixed turns are legal, and as before, some edge in H_{l_i} must be incident to w. Thus all the vertices in $\mathcal{CSP}_{l_i}$ labeled by NEG edges are in the strongly connected component $\mathcal{CSP}_{l_i}^{EG}$, as are all vertices in $\mathcal{CSP}_{l_{i-1}}^w$ for w as above.

The same argument and the inductive hypothesis shows that for any component of $G_{l_{i-1}}$ which intersects H_{l_i}, the corresponding strongly connected component(s) of $\mathcal{CSP}_{l_{i-1}}$ are also in $\mathcal{CSP}_{l_i}^{EG}$. The only thing remaining is to deal with NEG Nielsen paths and families of exceptional paths. Both of these are handled by Remark 6.5 and the fact that we have already established that $\mathcal{CSP}_{l_i}^{EG}$ contains all vertices of the form τ_E or $\tau_{\overline{E}}$ for NEG edges in $H_{l_i}^c$. We have shown that every vertex of a strongly connected component of $\mathcal{CSP}_{l_{i-1}}$ coming from a component of $G_{l_{i-1}}$ which intersects $H_{l_i}^c$ is in the strongly connected component $\mathcal{CSP}_{l_i}^{EG}$. In particular, there is only one strongly connected component of $\mathcal{CSP}_{l_i}$ for the component of G_{l_i} which contains edges in $H_{l_i}^c$. This completes the proof of the proposition. □

In the proof of Theorem 6.12, we will need to consider a weakening of the complete splitting of paths and circuits. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element. Given a CT $f\colon G \to G$, we define the graph $\mathcal{CSP}^{QE}(f)$ by adding two vertices to $\mathcal{CSP}(f)$ for each QE-family (one for $E_i u^* \overline{E}_j$ and one for $E_j u^* \overline{E}_i$). For every vertex τ_σ with a directed edge terminating at τ_{E_i} add an edge from τ_σ to $\tau_{E_i u^* \overline{E}_j}$ and similarly for every edge emanating from $\tau_{\overline{E}_j}$, add an edge to the same vertex beginning at $\tau_{E_i u^* \overline{E}_j}$. Do the same for the vertex $\tau_{E_j u^* \overline{E}_i}$. As before, every completely split path σ gives rise to a directed edge path in $\mathcal{CSP}^{QE}$ corresponding to its QE-splitting. It follows immediately from the definition and Proposition 6.1 that

Corollary 6.9. *There is a completely split circuit σ containing every allowable term in its QE-splitting.*

We are now ready to prove our main result in the polynomial case.

6.2. **Polynomial Subgroups are Undistorted.** In this subsection, we will complete the proof of our main result in the polynomial case. We first recall the height function defined by Alibegović in [Ali02]. Given two conjugacy classes $[u], [w]$ of elements of F_n, define the *twisting of* $[w]$ *about* $[u]$ as

$$\mathrm{tw}_u(w) = \max\{k \mid w = au^k b \text{ where } u, w \text{ are a cyclically reduced conjugates of } [u], [w]\}$$

Then define the twisting of $[w]$ by $\mathrm{tw}(w) = \max\{\mathrm{tw}_u(w) \mid u \in F_n\}$. Alibegović proved the following lemma using bounded cancellation, which we restate for convenience. A critical point is that D_2 is independent of w.

Lemma 6.10 ([Ali02, Lemma 2.4])**.** *There is a constant D_2 such that $\mathrm{tw}(s(w)) \leq \mathrm{tw}(w) + D_2$ for all conjugacy classes w and all $s \in S$, our symmetric finite generating set of $\mathrm{Out}(F_n)$.*

Since we typically work with train tracks, we have a similar notion of twisting adapted to that setting. Let τ be a path or circuit in a graph G and let σ be a circuit in G. Define the *twisting of τ about σ* as

$$\mathrm{tw}_\sigma(\tau) = \max\{k \mid \tau = \alpha\sigma^k\beta \text{ where the path } \alpha\sigma^k\beta \text{ is immersed}\}$$

Then define $\mathrm{tw}(\tau) = \max\{\mathrm{tw}_\sigma(\tau) \mid \sigma \text{ is a circuit}\}$. The bounded cancellation lemma of [Coo87] directly implies

Lemma 6.11. *If $\rho\colon R_n \to G$ and $[w]$ is a conjugacy class in $F_n = \pi_1(R_n)$, then $\mathrm{tw}(\rho(w)) \geq \mathrm{tw}(w) - 2C_\rho$.*

We are now ready to prove non-distortion for polynomial abelian subgroups. Recall the map $\Omega\colon H \to \mathbb{Z}^{N+K}$ was defined by taking the product of comparison and expansion factor homomorphisms. In the following theorem, we will denote the restriction of this map to the last K coordinates (those corresponding to comparison homomorphisms) by Ω_{comp}.

Theorem 6.12. *Let H be a rotationless abelian subgroup of $\mathrm{Out}(F_n)$ and assume that the map from H into the collection of comparison factor homomorphisms $\Omega_{comp}\colon H \to \mathbb{Z}^K$ is injective. Then H is undistorted.*

Proof. The first step is to note that it suffices to prove the generic elements of H are uniformly undistorted. This is just because the set of non-generic elements of H is a finite collection of hyperplanes, so there is a uniform bound on the distance from a point in one of these hyperplanes to a generic point.

We set up some constants now for later use. This is just to emphasize that they depend only on the subgroup we are given and the data we have been handed thus far. Let $\mathcal{G}$ be the finite set of marked graphs provided by Proposition 5.2 and define K_2 as the maximum of $BCC(\rho_G)$ and $BCC(\rho_G^{-1})$ as G varies over the finitely many marked graphs in $\mathcal{G}$. Lemma 6.11 then implies that $\mathrm{tw}(\rho(w)) \geq \mathrm{tw}(w) - K_2$ for any conjugacy class w and any of the finitely many marked graphs in $\mathcal{G}$. Let D_2 be the constant from Lemma 6.10.

Fix a minimal generating set $\phi_1, \ldots, \phi_k$ for H and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k}$ be generic in H. Let $f\colon G \to G$ be a CT representing ψ with G chosen from $\mathcal{G}$ and let ω be the comparison homomorphism for which $\omega(\psi)$ is the largest. The key point is that given ψ, Corollary 6.9 will provide a split circuit σ for which the twisting will grow by $|\omega(\psi)|$ under application of the map f.

Indeed, let σ be the circuit provided by Corollary 6.9. As we discussed in section 5.1, there is a correspondence between the comparison homomorphisms for H and the set of linear edges and quasi-exceptional families in G. Assume first that ω corresponds to the linear edge E with axis u, so that by definition $f(E) = E \cdot u^{\omega(\psi)}$. Since the splitting of $f_\#(\sigma)$ refines that of σ and E is a term in the complete splitting of σ, $f_\#(\sigma)$ not only contains the path $E \cdot u^{\omega(\psi)}$, but in fact splits at the ends of this subpath. Under iteration, we see that $f_\#^t(\sigma)$ contains the path $E \cdot u^{t\omega(\psi)}$, and therefore $\mathrm{tw}(f_\#^t(\sigma)) \geq t|\omega(\psi)|$. This isn't quite good enough for our purposes, so we will argue further to conclude that for some t_0,

$$\mathrm{tw}(f_\#^{t_0}(\sigma)) - \mathrm{tw}(f_\#^{t_0-1}(\sigma)) \geq |\omega(\psi)| \tag{1}$$

Suppose for a contradiction that no such t exists. Then for every t, we have $\mathrm{tw}(f_\#^t(\sigma)) - \mathrm{tw}(f_\#^{t-1}(\sigma)) \leq |\omega(\psi)| - 1$. Using a telescoping sum and repeatedly applying this assumption, we obtain $\mathrm{tw}(f_\#^t(\sigma)) - \mathrm{tw}(\sigma) \leq t|\omega(\psi)| - t$. Combining and rearranging inequalities, this implies

$$\mathrm{tw}(\sigma) \geq \mathrm{tw}(f_\#^t(\sigma)) + t - t|\omega(\psi)| \geq t|\omega(\psi)| + t - t|\omega(\psi)| = t$$

for all t, a contradiction. This establishes the existence of t_0 satisfying equation (1).

The above argument works without modification in the case that ω corresponds to a family of quasi-exceptional paths. We now address the minor adjustment needed in the case that ω corresponds to a family of exceptional paths, $E_i u^* \overline{E}_j$. Let $f(E_i) = E_i u^{d_i}$ and $f(E_j) = E_j u^{d_j}$. Since σ contains both $E_i u^* \overline{E}_j$ and $E_j u^* \overline{E}_i$ in its complete splitting, we may assume without loss that $d_i > d_j$. The only problem is that the exponent of u in the term $E_i u^* \overline{E}_j$ occuring in the complete splitting of σ may be negative, so that $\mathrm{tw}(f_\#^t(\sigma))$ may be less than $t|\omega(\psi)|$. In this case, just replace σ by a sufficiently high iterate so that the exponent is positive.

Now write ψ in terms of the generators $\psi = s_1 s_2 \cdots s_p$ so that for any conjugacy class w, by repeatedly applying Lemma 6.10 we obtain

$$\mathrm{tw}(s_1(s_2 \cdots s_p(w))) \leq \mathrm{tw}(s_2(s_3 \cdots s_p(w))) + D_2 \leq \mathrm{tw}(s_3 \cdots s_p(w)) + 2D_2 \leq \ldots \leq \mathrm{tw}(w) + pD_2$$

so that, $D_2|\psi|_{\mathrm{Out}(F_n)} \geq \mathrm{tw}(\psi(w)) - \mathrm{tw}(w)$. Applying this inequality to the circuit $f_\#^{t_0-1}(\sigma)$ just constructed, and letting w be the conjugacy class $\rho^{-1}(f_\#^{t_0-1}(\sigma))$, we have

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_2}\left[\mathrm{tw}(\psi(w)) - \mathrm{tw}(w)\right] \geq \frac{1}{D_2}\left[\mathrm{tw}(f_\#^{t_0}(\sigma)) - \mathrm{tw}(f_\#^{t_0-1}(\sigma))\right] - \frac{2K_2}{D_2} \geq \frac{1}{D_2}|\omega(\psi)| - \frac{2K_2}{D_2}$$

The second inequality is justified by Lemma 6.11 and the third uses the property of σ established in (1) above. Since ω was chosen to be largest coordinate of $\Omega_{comp}(\psi)$ and Ω_{comp} is injective, the proof is complete. □

7. The Mixed Case

There are no additional difficulties with the mixed case since both the distance function on CV_n and Alibegović's twisting function are well suited for dealing with outer automorphisms whose growth is neither purely exponential nor purely polynomial. Consequently, for an element ψ of an abelian subgroup H, if the image of ψ is large under PF_H then we can use CV_n to show that $|\psi|_{\mathrm{Out}(F_n)}$ is large, and if the image is large under Ω_{comp} then we can use the methods from §6 to show $|\psi|_{\mathrm{Out}(F_n)}$ is large. The injectivity of Ω [FH09, Lemma 4.6] exactly says that if $|\psi|_H$ is large, then at least one of the aforementioned quantities must be large as well.

Theorem 7.1. *Abelian subgroups of* $\mathrm{Out}(F_n)$ *are undistorted.*

Proof. Assume, by passing to a finite index subgroup, that H is rotationless. By [FH09, Lemma 4.6], the map $\Omega\colon H \to \mathbb{Z}^{N+K}$ is injective. Choose a minimal generating set for H and write $H = \langle \phi_1, \ldots, \phi_k \rangle$. The restriction of Ω to the first N coordinates is precisely the map PF_H from section 4. Choose k coordinates of Ω so that the restriction Ω_π to those coordinates is injective. Let $PF_{\Lambda_1}, \ldots, PF_{\Lambda_l}$ be the subset of the chosen coordinates corresponding to expansion factor homomorphisms. Pass to a finite index subgroup of H and choose generators so that $\Omega_\pi(\phi_i) = (0, \ldots, PF_{\Lambda_i}(\phi_i), \ldots, 0)$ for $1 \leq i \leq l$. Now we proceed as in the proofs of Theorems 4.2 and 6.12.

Fix a basepoint $* \in \mathrm{CV}_n$ and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k}$ in H. We may assume without loss that ψ is generic in H (again, it suffices to prove that generic elements are uniformly undistorted). Replace the ϕ_i's by their inverses if necessary to ensure that all p_i's are non-negative. Then, for each of the first l coordinates of Ω_π, replace Λ_i by its paired lamination if necessary (Lemma 4.1) to ensure that $PF_{\Lambda_i}(\psi) > 0$. Look at the coordinates of $\Omega_\pi(\psi)$ and pick out the one with the largest absolute value. We first consider the case where the largest coordinate corresponds to an expansion factor homomorphism PF_{Λ_j}. We have already arranged that $PF_{\Lambda_j}(\psi) > 0$.

By Theorem 3.3, the translation length of ψ is the maximum of the Perron-Frobenius eigenvalues associated to the EG strata of a relative train track representative f of ψ. Since ψ is generic and the first l coordinates of Ω_π are non-negative, $\{\Lambda_1, \ldots, \Lambda_l\} \subset \mathcal{L}(\psi)$. Each Λ_i is associated to an EG stratum of f. For such a stratum, the logarithm of the PF eigenvalue is $PF_{\Lambda_i}(\psi)$. Just as in the proof of Theorem 4.2, for each $1 \leq i \leq l$, we have that $PF_{\Lambda_i}(\psi) = p_i PF_{\Lambda_i}(\phi_i)$. So the translation length of ψ acting on Outer Space is

$$\tau(\psi) \geq \max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq l\}$$

The inequality is because there may be other laminations in $\mathcal{L}(\psi)$. Just as in Theorem 4.2, we have

$$d(*, * \cdot \psi) \leq D_1 |\psi|_{\mathrm{Out}(F_n)}$$

where $D_1 = \max_{s \in S} d(*, * \cdot s)$. Let $K_1 = \min\{PF_{\Lambda_i^\pm}(\phi_j^\pm) \mid 1 \leq i \leq l, 1 \leq j \leq k\}$. Then we have

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_1} d(*, * \cdot \psi) \geq \frac{1}{D_1}\tau(\psi) \geq \frac{1}{D_1}\max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\} \geq \frac{K_1}{D_1}\max\{p_i\}$$

We now handle the case where the largest coordinate of $\Omega_\pi(\psi)$ corresponds to a comparison homomorphism ω. Let $\mathcal{G}$ be the finite set of marked graphs provided by Proposition 5.2 and let $f\colon G \to G$ be a CT for ψ where $G \in \mathcal{G}$. Define K_2 exactly as in the proof of Theorem 6.12 so that $\mathrm{tw}(\rho(w)) \geq \mathrm{tw}(w) - K_2$ for all conjugacy classes w and any marking or inverse marking of the finitely many marked graphs in $\mathcal{G}$. The construction of the completely split circuit σ satisfying equation (1) given in the polynomial case works without modification in our current setting, where the comparison homomorphism ω in equation (1) is the coordinate of Ω_π which is largest in absolute value.

Using this circuit and defining $w = \rho^{-1}(f_\#^{t_0-1}\sigma)$, the inequalities and their justifications in the proof of Theorem 6.12 now apply verbatim to the present setting to conclude

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_2}\max\{|\omega(\psi)| \mid \omega \in \Omega_\pi\}| - \frac{2K_2}{D_2}$$

We have thus shown that the image of H under Ω_π undistorted. Since Ω_π is injective, it is a quasi-isometric embedding of H into $\mathbb{Z}^k$, so the theorem is proved. □

We conclude by proving the rank conjecture for $\mathrm{Out}(F_n)$. The maximal rank of an abelian subgroup of $\mathrm{Out}(F_n)$ is $2n-3$, so Theorem 7.1 gives a lower bound for the geometric rank of $\mathrm{Out}(F_n)$: $\mathrm{rank}\,\mathrm{Out}(F_n) \leq 2n-3$. The other inequality follows directly from the following result, whose proof we sketch below.

Theorem 7.2. *If G has virtual cohomological dimension $k \geq 3$, then $\mathrm{rank}(G) \leq k$.*

The virtual cohomological dimension of $\mathrm{Out}(F_n)$ is $2n-3$ [CV86]. Thus, for $n \geq 3$, we have:

Corollary 7.3. *The geometric rank of $\mathrm{Out}(F_n)$ is $2n-3$, which is the maximal rank of an abelian subgroup of $\mathrm{Out}(F_n)$.*

Proof of 7.2. Let $G' \leq G$ be a finite index subgroup whose cohomological dimension is k. Since G is quasi-isometric to its finite index subgroups, we have $\mathrm{rank}(G') = \mathrm{rank}(G)$. A well known theorem of Eilenberg-Ganea [EG57] provides the existence of a $k-$dimensional CW complex X which is a $K(G', 1)$. By Švarc-Milnor, it suffices to show that there can be no quasi-isometric embedding of $\mathbb{R}^{k+1}$ into the universal cover $\tilde{X}$. Suppose for a contradiction that $f\colon \mathbb{R}^{k+1} \to \tilde{X}$ is such a map. The first step is to replace f by a continuous quasi-isometry f' which is a bounded distance from f. This is done using the "connect-the-dots argument" whose proof is sketched in [SW02]. The key point is that $\tilde{X}$ is *uniformly contractible*. That is, for every r, there is an $s = s(r)$, such that any continuous map of a finite simplicial complex into X whose image is contained in an r-ball is contractible in an $s(r)$-ball.

It is a standard fact [Hat02, Theorem 2C.5] that X may be replaced with a simplicial complex of the same dimension so that $\tilde{X}$ may be assumed to be simplicial. We now construct a cover $\mathcal{U}$ of the simplicial complex $\tilde{X}$ whose nerve is equal to the barycentric subdivision of $\tilde{X}$. The cover $\mathcal{U}$ has one element for each cell of $\tilde{X}$. For each vertex v, the set $U_v \in \mathcal{U}$ is a small neighborhood of v. For each i-cell, σ, Define U_σ by taking a sufficiently small neighborhood of $\sigma \setminus \bigcup_{\sigma' \in \tilde{X}^{(i-1)}} U_{\sigma'}$ to ensure that $U_\sigma \cap \tilde{X}^{(i-1)} = \emptyset$. The key property of $\mathcal{U}$ is that all $(k+2)-$fold intersections are necessarily empty because the dimension of the barycentric subdivision of $\tilde{X}$ is equal to $\dim(\tilde{X})$.

Since we have arranged f to be continuous, we can pull back the cover just constructed to obtain a cover $\mathcal{V} = \{f^{-1}(U)\}_{U\in\mathcal{U}}$ of $\mathbb{R}^{k+1}$. Since the elements of $\mathcal{U}$ are bounded, and f is a quasi-isometric embedding, the elements of $\mathcal{V}$ are bounded as well. The intersection pattern of the elements of $\mathcal{V}$ is exactly the same as the intersection pattern of elements of $\mathcal{U}$. But the cover $\mathcal{U}$ was constructed so that any intersection of $(k+2)$ elements is necessarily empty. Thus, we have constructed a cover of $\mathbb{R}^{k+1}$ by bounded sets with no $(k+2)-$fold intersections. We will contradict the fact that the Lebesgue covering dimension of any compact subset of $\mathbb{R}^{k+1}$ is $k+1$. Let K be compact in $\mathbb{R}^{k+1}$ and let $\mathcal{V}'$ be an arbitrary cover of K. Let δ be the constant provided by the Lebesgue covering Lemma applied to $\mathcal{V}'$. Since the elements of $\mathcal{V}$ are uniformly bounded, we can scale them by a single constant to obtain a cover of K whose sets have diameter $< \delta/3$. Such a cover is necessarily a refinement of $\mathcal{V}'$, but has multiplicity $k+1$. This contradicts the fact that K has covering dimension $k+1$ so the theorem is proved. □

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