MATH 2270: Midterm 3 Practice Problems

The following are practice problems for the third exam.

1. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

   (a) Compute the characteristic polynomial of A. Then factor it to find the eigenvalues of A. $p(\lambda) = (\lambda - 2)(\lambda - 3)^2$

   (b) For each eigenvalue, λ, find a basis for the corresponding eigenspace, $V^{(\lambda)}$. Answer: $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

   (c) Use the computations from parts (a) and (b) to write $A = PDP^{-1}$, where D is a diagonal matrix, and P is an invertible matrix. You do not need to compute P^{-1}. Answer:

$$P = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Let H be the subspace of $C(\mathbb{R})$ (continuous functions $\mathbb{R} \to \mathbb{R}$) spanned by $\{\sinh x, \cosh x\}$. Recall (if you don't already know) that $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$ (note the lack of a minus sign). Consider the linear transformation $D \colon H \to H$ defined by $D(f) = f'$, where f' denotes the derivative of f.

   (a) Compute the matrix of D with respect to the basis, $\mathcal{B} = \{\sinh x, \cosh x\}$. *You do not need to show that $\mathcal{B}$ is a basis for H.* Answer: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (b) Use the techniques of Chapter 5 to find a basis for H in which the matrix for D is diagonal. Answer: $\{\sinh x + \cosh x, \sinh x - \cosh x\}$

3. Define $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$ where A is a 3×3 matrix whose eigenvalues are $1, \sqrt{3}$, and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the matrix for T with respect to $\mathcal{B}$ is diagonal? Why or why not? Answer: The matrix A is diagonalizable because it has 3 distinct eigenvalues. Thus, there is a basis in which the matrix for T is diagonal.

4. Give an example of a 2×2 matrix that is not diagonalizable. Answer: There are many correct answers. For example $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

5. Define $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$, where $A = \begin{bmatrix} -6 & -2 \\ 4 & 0 \end{bmatrix}$. Find the matrix for T with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ Answer: $\begin{bmatrix} -4 & 0 \\ 2 & -2 \end{bmatrix}$

6. §4.9 Exercises 4 & 14.

7. §5.6 Exercises 1 & 10.

8. §5.7 Exercises 4 & 10.

9. Find two vectors u, v in $\mathbb{R}^3$ such that $u \cdot v = 7$. Answer: $\begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

10. Determine if $\left\{ \begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} \right\}$ is an orthogonal set. Answer: No.

11. Find a vector $\vec{v}$ pointing in the same direction as $\begin{bmatrix} 5 & -4 & 0 & 3 \end{bmatrix}^T$ with a length of 3. Answer: $\begin{bmatrix} 15/\sqrt{50} \\ -12/\sqrt{50} \\ 0 \\ 9/\sqrt{50} \end{bmatrix}$

12. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$.

    (a) Give a description of $W^{\perp}$ in parametric vector form. Answer: $W^{\perp} = \operatorname{span}\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \}$

    (b) Compute $\operatorname{proj}_W \vec{v}$ where $v = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$. Answer: $\frac{21}{14} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

    (c) Compute $\operatorname{proj}_{W^{\perp}} \vec{v}$. Answer: Take $\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$ and subtract your answer to (b) from it.