SOLUTION Since $\tan x = \dfrac{\sin x}{\cos x}$ we can make the substitution $u = \cos x$, $du = -\sin x\, dx$, to obtain

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{-1}{\cos x}(-\sin x\, dx) = -\ln|\cos x| + C \quad \blacksquare$$

Similarly, $\int \cot x\, dx = \ln|\sin x|$.

EXAMPLE 10 Evaluate $\int \sec x \csc x\, dx$.

SOLUTION For this one we use the trig identity $\sec x \csc x = \tan x + \cot x$. Then

$$\int \sec x \csc x\, dx = \int (\tan x + \cot x)\, dx = -\ln|\cos x| + \ln|\sin x| + C \quad \blacksquare$$

Concepts Review

1. The function ln is defined by $\ln x =$ _____. The domain of this function is _____ and its range is _____.

2. From the preceding definition, it follows that $D_x \ln x =$ _____ for $x > 0$.

3. More generally, for $x \neq 0$, $D_x \ln|x| =$ _____ and so $\int (1/x)\, dx =$ _____.

4. Some common properties of ln are $\ln(xy) =$ _____, $\ln(x/y) =$ _____, and $\ln(x^r) =$ _____.

Problem Set 6.1

1. Use the approximations $\ln 2 \approx 0.693$ and $\ln 3 \approx 1.099$ together with the properties stated in Theorem A to calculate approximations to each of the following. For example, $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3 \approx 0.693 + 1.099 = 1.792$.

(a) $\ln 6$ (b) $\ln 1.5$ (c) $\ln 81$

(d) $\ln \sqrt{2}$ (e) $\ln\left(\frac{1}{36}\right)$ (f) $\ln 48$

2. Use your calculator to make the computations in Problem 1 directly.

In Problems 3–14, find the indicated derivative (see Examples 1 and 2). Assume in each case that x is restricted so that ln is defined.

3. $D_x \ln(x^2 + 3x + \pi)$

4. $D_x \ln(3x^3 + 2x)$

5. $D_x \ln(x - 4)^3$

6. $D_x \ln \sqrt{3x - 2}$

7. $\dfrac{dy}{dx}$ if $y = 3 \ln x$

8. $\dfrac{dy}{dx}$ if $y = x^2 \ln x$

9. $\dfrac{dz}{dx}$ if $z = x^2 \ln x^2 + (\ln x)^3$

10. $\dfrac{dr}{dx}$ if $r = \dfrac{\ln x}{x^2 \ln x^2} + \left(\ln \dfrac{1}{x}\right)^3$

11. $g'(x)$ if $g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$

12. $h'(x)$ if $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $f'(81)$ if $f(x) = \ln \sqrt[4]{x}$

14. $f'\left(\dfrac{\pi}{4}\right)$ if $f(x) = \ln(\cos x)$

In Problems 15–26, find the integrals (see Examples 4, 5, and 6).

15. $\displaystyle\int \frac{1}{2x + 1}\, dx$

16. $\displaystyle\int \frac{1}{1 - 2x}\, dx$

17. $\displaystyle\int \frac{6v + 9}{3v^2 + 9v}\, dv$

18. $\displaystyle\int \frac{z}{2z^2 + 8}\, dz$

19. $\displaystyle\int \frac{2 \ln x}{x}\, dx$

20. $\displaystyle\int \frac{-1}{x(\ln x)^2}\, dx$

21. $\displaystyle\int_0^3 \frac{x^4}{2x^5 + \pi}\, dx$

22. $\displaystyle\int_0^1 \frac{t + 1}{2t^2 + 4t + 3}\, dt$

23. $\displaystyle\int \frac{x^2}{x - 1}\, dx$

24. $\displaystyle\int \frac{x^2 + x}{2x - 1}\, dx$

25. $\displaystyle\int \frac{x^4}{x + 4}\, dx$

26. $\displaystyle\int \frac{x^3 + x^2}{x + 2}\, dx$

In Problems 27–30, use Theorem A to write the expressions as the logarithm of a single quantity.

27. $2 \ln(x + 1) - \ln x$

28. $\frac{1}{2}\ln(x - 9) + \frac{1}{2}\ln x$

29. $\ln(x - 2) - \ln(x + 2) + 2 \ln x$

30. $\ln(x^2 - 9) - 2 \ln(x - 3) - \ln(x + 3)$

In Problems 31–34, find dy/dx by logarithmic differentiation (see Example 8).

31. $y = \dfrac{x + 11}{\sqrt{x^3 - 4}}$

32. $y = (x^2 + 3x)(x - 2)(x^2 + 1)$

33. $y = \dfrac{\sqrt{x + 13}}{(x - 4)\sqrt[3]{2x + 1}}$

34. $y = \dfrac{(x^2 + 3)^{2/3}(3x + 2)^2}{\sqrt{x + 1}}$

In Problems 35–38, make use of the known graph of $y = \ln x$ to sketch the graphs of the equations.

35. $y = \ln|x|$

36. $y = \ln\sqrt{x}$

37. $y = \ln\left(\dfrac{1}{x}\right)$

38. $y = \ln(x - 2)$

39. Sketch the graph of $y = \ln\cos x + \ln\sec x$ on $(-\pi/2, \pi/2)$, but think before you begin.

40. Explain why $\lim_{x\to 0} \ln\dfrac{\sin x}{x} = 0$.

41. Find all local extreme values of $f(x) = 2x^2 \ln x - x^2$ on its domain.

42. The rate of transmission in a telegraph cable is observed to be proportional to $x^2 \ln(1/x)$, where x is the ratio of the radius of the core to the thickness of the insulation $(0 < x < 1)$. What value of x gives the maximum rate of transmission?

43. Use the fact that $\ln 4 > 1$ to show that $\ln 4^m > m$ for $m > 1$. Conclude that $\ln x$ can be made as large as desired by choosing x sufficiently large. What does this imply about $\lim_{x\to\infty} \ln x$?

44. Use the fact that $\ln x = -\ln(1/x)$ and Problem 43 to show that $\lim_{x\to 0^+} \ln x = -\infty$.

45. Solve for x: $\displaystyle\int_{1/3}^{x} \frac{1}{t}\,dt = 2\int_{1}^{x} \frac{1}{t}\,dt$.

46. Prove the following statements.

(a) Since $1/t < 1/\sqrt{t}$ for $t > 1$, $\ln x < 2\left(\sqrt{x} - 1\right)$ for $x > 1$.

(b) $\lim_{x\to\infty} (\ln x)/x = 0$.

47. Calculate

$$\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right]$$

by writing the expression in brackets as

$$\left[\frac{1}{1 + 1/n} + \frac{1}{1 + 2/n} + \cdots + \frac{1}{1 + n/n}\right]\frac{1}{n}$$

and recognizing the latter as a Riemann sum.

C **48.** A famous theorem (the Prime Number Theorem) says that the number of primes less than n for large n is approximately $n/(\ln n)$. About how many primes are there less than 1,000,000?

49. Find and simplify $f'(1)$.

(a) $f(x) = \ln\left(\dfrac{ax - b}{ax + b}\right)^c$, where $c = \dfrac{a^2 - b^2}{2ab}$.

(b) $f(x) = \displaystyle\int_1^u \cos^2 t\,dt$, where $u = \ln(x^2 + x - 1)$.

50. Evaluate $\displaystyle\int_0^{\pi/3} \tan x\,dx$.

51. Evaluate $\displaystyle\int_{\pi/4}^{\pi/3} \sec x \csc x\,dx$.

52. Evaluate $\displaystyle\int \frac{\cos x}{1 + \sin x}\,dx$.

53. The region bounded by $y = (x^2 + 4)^{-1}$, $y = 0$, $x = 1$, and $x = 4$, is revolved about the y-axis, generating a solid. Find its volume.

54. Find the length of the curve $y = x^2/4 - \ln\sqrt{x}$, $1 \le x \le 2$.

55. By appealing to the graph of $y = 1/x$, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

56. Prove **Napier's Inequality,** which says that, for $0 < x < y$,

$$\frac{1}{y} < \frac{\ln y - \ln x}{y - x} < \frac{1}{x}$$

CAS **57.** Let $f(x) = \ln(1.5 + \sin x)$.

(a) Find the absolute extreme points on $[0, 3\pi]$.

(b) Find any inflection points on $[0, 3\pi]$.

(c) Evaluate $\displaystyle\int_0^{3\pi} \ln(1.5 + \sin x)\,dx$.

CAS **58.** Let $f(x) = \cos(\ln x)$.

(a) Find the absolute extreme points on $[0.1, 20]$.

(b) Find the absolute extreme points on $[0.01, 20]$.

(c) Evaluate $\displaystyle\int_{0.1}^{20} \cos(\ln x)\,dx$.

CAS **59.** Draw the graphs of $f(x) = x\ln(1/x)$ and $g(x) = x^2\ln(1/x)$ on $(0, 1]$.

(a) Find the area of the region between these curves on $(0, 1]$.

(b) Find the absolute maximum value of $|f(x) - g(x)|$ on $(0, 1]$.

CAS **60.** Follow the directions of Problem 59 for $f(x) = x\ln x$ and $g(x) = \sqrt{x}\ln x$.

Answers to Concepts Review: **1.** $\int_1^x (1/t)\,dt$; $(0, \infty)$; $(-\infty, \infty)$ **2.** $1/x$ **3.** $1/x$; $\ln|x| + C$ **4.** $\ln x + \ln y$; $\ln x - \ln y$; $r\ln x$

6.2 Inverse Functions and Their Derivatives

Our stated aim for this chapter is to expand the number of functions in our repertoire. One way to manufacture new functions is to take old ones and "reverse" them. When we do this for the natural logarithm function, we will be led to the natural exponential function, the subject of Section 6.3. In this section, we study the general problem of reversing (or inverting) a function. Here is the idea.

A function f takes a number x from its domain D and assigns to it a single value y from its range R. If we are lucky, as in the case of the two functions