This explicit formula for $\mathbf{x}_k$ gives the solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. As $k \to \infty$, $(.92)^k$ tends to zero and $\mathbf{x}_k$ tends to $\begin{bmatrix} .375 \\ .625 \end{bmatrix} = .125\mathbf{v}_1$. ■

The calculations in Example 5 have an interesting application to a Markov chain discussed in Section 4.9. Those who read that section may recognize that matrix A in Example 5 above is the same as the migration matrix M in Section 4.9, $\mathbf{x}_0$ is the initial population distribution between city and suburbs, and $\mathbf{x}_k$ represents the population distribution after k years.

Theorem 18 in Section 4.9 stated that for a matrix such as A, the sequence $\mathbf{x}_k$ tends to a steady-state vector. Now we know *why* the $\mathbf{x}_k$ behave this way, at least for the migration matrix. The steady-state vector is $.125\mathbf{v}_1$, a multiple of the eigenvector $\mathbf{v}_1$, and formula (5) for $\mathbf{x}_k$ shows precisely why $\mathbf{x}_k \to .125\mathbf{v}_1$.

NUMERICAL NOTES

1. Computer software such as Mathematica and Maple can use symbolic calculations to find the characteristic polynomial of a moderate-sized matrix. But there is no formula or finite algorithm to solve the characteristic equation of a general $n \times n$ matrix for $n \geq 5$.
2. The best numerical methods for finding eigenvalues avoid the characteristic polynomial entirely. In fact, MATLAB finds the characteristic polynomial of a matrix A by first computing the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A and then expanding the product $(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$.
3. Several common algorithms for estimating the eigenvalues of a matrix A are based on Theorem 4. The powerful *QR algorithm* is discussed in the exercises. Another technique, called *Jacobi's method*, works when $A = A^T$ and computes a sequence of matrices of the form

$$A_1 = A \quad \text{and} \quad A_{k+1} = P_k^{-1} A_k P_k \quad (k = 1, 2, \ldots)$$

   Each matrix in the sequence is similar to A and so has the same eigenvalues as A. The nondiagonal entries of A_{k+1} tend to zero as k increases, and the diagonal entries tend to approach the eigenvalues of A.
4. Other methods of estimating eigenvalues are discussed in Section 5.8.

PRACTICE PROBLEM

Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

5.2 EXERCISES

Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

1. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 2 \\ 6 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

6. $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$

8. $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for 3×3 determinants described prior to Exercises 15–18 in Section 3.1. [*Note*: Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

9. $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ **10.** $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$ **14.** $\begin{bmatrix} 4 & 0 & -1 \\ -1 & 0 & 4 \\ 0 & 2 & 3 \end{bmatrix}$

For the matrices in Exercises 15–17, list the real eigenvalues, repeated according to their multiplicities.

15. $\begin{bmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ **16.** $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional:

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

20. Use a property of determinants to show that A and A^T have the same characteristic polynomial.

In Exercises 21 and 22, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

21. a. The determinant of A is the product of the diagonal entries in A.

b. An elementary row operation on A does not change the determinant.

c. $(\det A)(\det B) = \det AB$

d. If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

22. a. If A is 3×3, with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then $\det A$ equals the volume of the parallelepiped determined by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

b. $\det A^T = (-1)\det A$.

c. The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

d. A row replacement operation on A does not change the eigenvalues.

A widely used method for estimating eigenvalues of a general matrix A is the *QR algorithm*. Under suitable conditions, this algorithm produces a sequence of matrices, all similar to A, that become almost upper triangular, with diagonal entries that approach the eigenvalues of A. The main idea is to factor A (or another matrix similar to A) in the form $A = Q_1R_1$, where $Q_1^T = Q_1^{-1}$ and R_1 is upper triangular. The factors are interchanged to form $A_1 = R_1Q_1$, which is again factored as $A_1 = Q_2R_2$; then to form $A_2 = R_2Q_2$, and so on. The similarity of $A, A_1, \ldots$ follows from the more general result in Exercise 23.

23. Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

24. Show that if A and B are similar, then $\det A = \det B$.

25. Let $A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. [*Note:* A is the stochastic matrix studied in Example 5 in Section 4.9.]

a. Find a basis for $\mathbb{R}^2$ consisting of $\mathbf{v}_1$ and another eigenvector $\mathbf{v}_2$ of A.

b. Verify that $\mathbf{x}_0$ may be written in the form $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$.

c. For $k = 1, 2, \ldots$, define $\mathbf{x}_k = A^k\mathbf{x}_0$. Compute $\mathbf{x}_1$ and $\mathbf{x}_2$, and write a formula for $\mathbf{x}_k$. Then show that $\mathbf{x}_k \to \mathbf{v}_1$ as k increases.

26. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Use formula (1) for a determinant (given before Example 2) to show that $\det A = ad - bc$. Consider two cases: $a \neq 0$ and $a = 0$.

27. Let $A = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a. Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors of A. [*Note*: A is the stochastic matrix studied in Example 3 of Section 4.9.]

b. Let $\mathbf{x}_0$ be any vector in $\mathbb{R}^3$ with nonnegative entries whose sum is 1. (In Section 4.9, $\mathbf{x}_0$ was called a probability vector.) Explain why there are constants c_1, c_2, c_3 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Compute $\mathbf{w}^T\mathbf{x}_0$, and deduce that $c_1 = 1$.

c. For $k = 1, 2, \ldots$, define $\mathbf{x}_k = A^k\mathbf{x}_0$, with $\mathbf{x}_0$ as in part (b). Show that $\mathbf{x}_k \to \mathbf{v}_1$ as k increases.

28. [**M**] Construct a random integer-valued 4×4 matrix A, and verify that A and A^T have the same characteristic polynomial (the same eigenvalues with the same multiplicities). Do A and A^T have the same eigenvectors? Make the same analysis of a 5×5 matrix. Report the matrices and your conclusions.

29. [**M**] Construct a random integer-valued 4×4 matrix A.

a. Reduce A to echelon form U with no row scaling, and use U in formula (1) (before Example 2) to compute $\det A$. (If A happens to be singular, start over with a new random matrix.)

b. Compute the eigenvalues of A and the product of these eigenvalues (as accurately as possible).

c. List the matrix A, and, to four decimal places, list the pivots in U and the eigenvalues of A. Compute $\det A$ with your matrix program, and compare it with the products you found in (a) and (b).

30. [**M**] Let $A = \begin{bmatrix} -6 & 28 & 21 \\ 4 & -15 & -12 \\ -8 & a & 25 \end{bmatrix}$. For each value of a in the set $\{32, 31.9, 31.8, 32.1, 32.2\}$, compute the characteristic polynomial of A and the eigenvalues. In each case, create a graph of the characteristic polynomial $p(t) = \det(A - tI)$ for $0 \le t \le 3$. If possible, construct all graphs on one coordinate system. Describe how the graphs reveal the changes in the eigenvalues as a changes.

SOLUTION TO PRACTICE PROBLEM

The characteristic equation is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda) - (-4)(4) = \lambda^2 - 3\lambda + 18 \end{aligned}$$

From the quadratic formula,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(18)}}{2} = \frac{3 \pm \sqrt{-63}}{2}$$

It is clear that the characteristic equation has no real solutions, so A has no real eigenvalues. The matrix A is acting on the real vector space $\mathbb{R}^2$, and there is no nonzero vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

5.3 DIAGONALIZATION

In many cases, the eigenvalue–eigenvector information contained within a matrix A can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix. In this section, the factorization enables us to compute A^k quickly for large values of k, a fundamental idea in several applications of linear algebra. Later, in Sections 5.6 and 5.7, the factorization will be used to analyze (and *decouple*) dynamical systems.

The following example illustrates that powers of a diagonal matrix are easy to compute.

EXAMPLE 1 If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \ge 1$$

■

If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute, as the next example shows.

SOLUTION Since A is a triangular matrix, the eigenvalues are 5 and -3, each with multiplicity 2. Using the method in Section 5.1, we find a basis for each eigenspace.

$$\text{Basis for } \lambda = 5: \quad \mathbf{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -3: \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is linearly independent, by Theorem 7. So the matrix $P = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_4\,]$ is invertible, and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

PRACTICE PROBLEMS

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.
2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.
3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

WEB

5.3 EXERCISES

In Exercises 1 and 2, let $A = PDP^{-1}$ and compute A^4.

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

2. $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

In Exercises 3 and 4, use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer.

3. $\begin{bmatrix} a & 0 \\ 2(a-b) & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

In Exercises 5 and 6, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

5. $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}$
$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

6. $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$
$$= \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{bmatrix}$$

Diagonalize the matrices in Exercises 7–20, if possible. The real eigenvalues for Exercises 11–16 and 18 are included below the matrix.

7. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$ $\lambda = -1, 5$

12. $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ $\lambda = 2, 5$

13. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ $\lambda = 1, 5$

14. $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $\lambda = 2, 3$

15. $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\lambda = 0, 1$

16. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$ $\lambda = 0$

17. $\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$ $\lambda = -2, -1, 0$

19. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

In Exercises 21 and 22, A, B, P, and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer. (Study Theorems 5 and 6 and the examples in this section carefully before you try these exercises.)

21. a. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

b. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

c. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d. If A is diagonalizable, then A is invertible.

22. a. A is diagonalizable if A has n eigenvectors.

b. If A is diagonalizable, then A has n distinct eigenvalues.

c. If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.

d. If A is invertible, then A is diagonalizable.

23. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

27. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

28. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

29. A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix A in Example 2. With $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, use the information in Example 2 to find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$.

30. With A and D as in Example 2, find an invertible P_2 unequal to the P in Example 2, such that $A = P_2 D P_2^{-1}$.

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.

[**M**] Diagonalize the matrices in Exercises 33–36. Use your matrix program's eigenvalue command to find the eigenvalues, and then compute bases for the eigenspaces as in Section 5.1.

33. $\begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$

34. $\begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$

35. $\begin{bmatrix} 13 & -12 & 9 & -15 & 9 \\ 6 & -5 & 9 & -15 & 9 \\ 6 & -12 & -5 & 6 & 9 \\ 6 & -12 & 9 & -8 & 9 \\ -6 & 12 & 12 & -6 & -2 \end{bmatrix}$

36. $\begin{bmatrix} 24 & -6 & 2 & 6 & 2 \\ 72 & 51 & 9 & -99 & 9 \\ 0 & -63 & 15 & 63 & 63 \\ 72 & 15 & 9 & -63 & 9 \\ 0 & 63 & 21 & -63 & -27 \end{bmatrix}$

NUMERICAL NOTE

An efficient way to compute a $\mathcal{B}$-matrix $P^{-1}AP$ is to compute AP and then to row reduce the augmented matrix $[\,P \quad AP\,]$ to $[\,I \quad P^{-1}AP\,]$. A separate computation of P^{-1} is unnecessary. See Exercise 15 in Section 2.2.

PRACTICE PROBLEMS

1. Find $T(a_0 + a_1t + a_2t^2)$, if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[\,T\,]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

2. Let A, B, and C be $n \times n$ matrices. The text has shown that if A is similar to B, then B is similar to A. This property, together with the statements below, shows that "similar to" is an *equivalence relation*. (Row equivalence is another example of an equivalence relation.) Verify parts (a) and (b).
   a. A is similar to A.
   b. If A is similar to B and B is similar to C, then A is similar to C.

5.4 EXERCISES

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{b}_1) = 3\mathbf{d}_1 - 5\mathbf{d}_2, \quad T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2, \quad T(\mathbf{b}_3) = 4\mathbf{d}_2$$

Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{D}$.

2. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces V and W, respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(\mathbf{d}_1) = 3\mathbf{b}_1 - 3\mathbf{b}_2, \quad T(\mathbf{d}_2) = -2\mathbf{b}_1 + 5\mathbf{b}_2$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

3. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V, and let $T : \mathbb{R}^3 \to V$ be a linear transformation with the property that

$$T(x_1, x_2, x_3) = (2x_3 - x_2)\mathbf{b}_1 - (2x_2)\mathbf{b}_2 + (x_1 + 3x_3)\mathbf{b}_3$$

   a. Compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$.
   b. Compute $[T(\mathbf{e}_1)]_{\mathcal{B}}$, $[T(\mathbf{e}_2)]_{\mathcal{B}}$, and $[T(\mathbf{e}_3)]_{\mathcal{B}}$.
   c. Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$.

4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V and let $T : V \to \mathbb{R}^2$ be a linear transformation with the property that

$$T(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3) = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ -2x_1 + 5x_3 \end{bmatrix}$$

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

5. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $(t + 3)\mathbf{p}(t)$.
   a. Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.
   b. Show that T is a linear transformation.
   c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

6. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) + 2t^2\mathbf{p}(t)$.
   a. Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.
   b. Show that T is a linear transformation.
   c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

7. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1t + a_2t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$

is linear. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

8. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Find $T(4\mathbf{b}_1 - 3\mathbf{b}_2)$ when T is a linear transformation from V to V whose matrix relative to $\mathcal{B}$ is

$$[\,T\,]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

9. Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

a. Find the image under T of $\mathbf{p}(t) = 5 + 3t$.

b. Show that T is a linear transformation.

c. Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

10. Define $T : \mathbb{P}_3 \to \mathbb{R}^4$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-2) \\ \mathbf{p}(3) \\ \mathbf{p}(1) \\ \mathbf{p}(0) \end{bmatrix}$.

a. Show that T is a linear transformation.

b. Find the matrix for T relative to the basis $\{1, t, t^2, t^3\}$ for $\mathbb{P}_3$ and the standard basis for $\mathbb{R}^4$.

In Exercises 11 and 12, find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$, where $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

11. $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

12. $A = \begin{bmatrix} -6 & -2 \\ 4 & 0 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

In Exercises 13–16, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

13. $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

16. $A = \begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}$

17. Let $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, for $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Verify that $\mathbf{b}_1$ is an eigenvector of A but that A is not diagonalizable.

b. Find the $\mathcal{B}$-matrix for T.

18. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 3×3 matrix with eigenvalues 5, 5, and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix for T is a diagonal matrix? Discuss.

Verify the statements in Exercises 19–24. The matrices are square.

19. If A is invertible and similar to B, then B is invertible and A^{-1} is similar to B^{-1}. [*Hint:* $P^{-1}AP = B$ for some invertible P. Explain why B is invertible. Then find an invertible Q such that $Q^{-1}A^{-1}Q = B^{-1}$.]

20. If A is similar to B, then A^2 is similar to B^2.

21. If B is similar to A and C is similar to A, then B is similar to C.

22. If A is diagonalizable and B is similar to A, then B is also diagonalizable.

23. If $B = P^{-1}AP$ and $\mathbf{x}$ is an eigenvector of A corresponding to an eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding also to λ.

24. If A and B are similar, then they have the same rank. [*Hint:* Refer to Supplementary Exercises 13 and 14 in Chapter 4.]

25. The *trace* of a square matrix A is the sum of the diagonal entries in A and is denoted by tr A. It can be verified that $\text{tr}(FG) = \text{tr}(GF)$ for any two $n \times n$ matrices F and G. Show that if A and B are similar, then tr A = tr B.

26. It can be shown that the trace of a matrix A equals the sum of the eigenvalues of A. Verify this statement for the case when A is diagonalizable.

27. Let V be $\mathbb{R}^n$ with a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$; let W be $\mathbb{R}^n$ with the standard basis, denoted here by $\mathcal{E}$; and consider the identity transformation $I : \mathbb{R}^n \to \mathbb{R}^n$, where $I(\mathbf{x}) = \mathbf{x}$. Find the matrix for I relative to $\mathcal{B}$ and $\mathcal{E}$. What was this matrix called in Section 4.4?

28. Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, let W be the same space V with a basis $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, and let I be the identity transformation $I : V \to W$. Find the matrix for I relative to $\mathcal{B}$ and $\mathcal{C}$. What was this matrix called in Section 4.7?

29. Let V be a vector space with a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Find the $\mathcal{B}$-matrix for the identity transformation $I : V \to V$.

[**M**] In Exercises 30 and 31, find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ where $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

30. $A = \begin{bmatrix} 6 & -2 & -2 \\ 3 & 1 & -2 \\ 2 & -2 & 2 \end{bmatrix}$,

$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

31. $A = \begin{bmatrix} -7 & -48 & -16 \\ 1 & 14 & 6 \\ -3 & -45 & -19 \end{bmatrix}$,

$\mathbf{b}_1 = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

32. [**M**] Let T be the transformation whose standard matrix is given below. Find a basis for $\mathbb{R}^4$ with the property that $[\,T\,]_{\mathcal{B}}$ is diagonal.

$$A = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$$

NUMERICAL NOTE

You may have noticed that if $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, \ldots$, then

$$\mathbf{x}_2 = P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0,$$

and, in general,

$$\mathbf{x}_k = P^k\mathbf{x}_0 \quad \text{for } k = 0, 1, \ldots$$

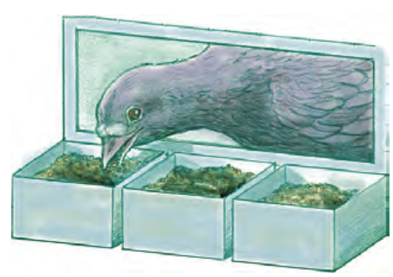
To compute a specific vector such as $\mathbf{x}_3$, fewer arithmetic operations are needed to compute $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, rather than P^3 and $P^3\mathbf{x}_0$. However, if P is small—say, 30×30—the machine computation time is insignificant for both methods, and a command to compute $P^3\mathbf{x}_0$ might be preferred because it requires fewer human keystrokes.

PRACTICE PROBLEMS

1. Suppose the residents of a metropolitan region move according to the probabilities in the migration matrix M in Example 1 and a resident is chosen "at random." Then a state vector for a certain year may be interpreted as giving the probabilities that the person is a city resident or a suburban resident at that time.
   a. Suppose the person chosen is a city resident now, so that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the likelihood that the person will live in the suburbs next year?
   b. What is the likelihood that the person will be living in the suburbs in two years?
2. Let $P = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$. Is $\mathbf{q}$ a steady-state vector for P?
3. What percentage of the population in Example 1 will live in the suburbs after many years?

4.9 EXERCISES

1. A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.
   a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.
   b. Give the initial state vector.
   c. What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?
2. A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food on one trial, it will choose the same food on the next trial with a probability of 60%, and it will choose the other foods on the next trial with equal probabilities of 20%.
   a. What is the stochastic matrix for this situation?
   b. If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?



3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy

tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

a. What is the stochastic matrix for this situation?

b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?

c. If a student is healthy today, what is the probability that he or she will be healthy two days from now?

4. The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 40% chance it will be good tomorrow, a 30% chance it will be indifferent, and a 30% chance it will be bad. If the weather is indifferent today, there is a 50% chance it will be good tomorrow, and a 20% chance it will be indifferent. Finally, if the weather is bad today, there is a 30% chance it will be good tomorrow and a 40% chance it will be indifferent.

a. What is the stochastic matrix for this situation?

b. Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?

c. Suppose the predicted weather for Monday is 60% indifferent weather and 40% bad weather. What are the chances for good weather on Wednesday?

In Exercises 5–8, find the steady-state vector.

5. $\begin{bmatrix} .1 & .5 \\ .9 & .5 \end{bmatrix}$

6. $\begin{bmatrix} .4 & .8 \\ .6 & .2 \end{bmatrix}$

7. $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$

8. $\begin{bmatrix} .4 & .5 & .8 \\ 0 & .5 & .1 \\ .6 & 0 & .1 \end{bmatrix}$

9. Determine if $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ is a regular stochastic matrix.

10. Determine if $P = \begin{bmatrix} 1 & .3 \\ 0 & .7 \end{bmatrix}$ is a regular stochastic matrix.

11. a. Find the steady-state vector for the Markov chain in Exercise 1.

b. At some time late in the day, what fraction of the listeners will be listening to the news?

12. Refer to Exercise 2. Which food will the animal prefer after many trials?

13. a. Find the steady-state vector for the Markov chain in Exercise 3.

b. What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?

14. Refer to Exercise 4. In the long run, how likely is it for the weather in Columbus to be good on a given day?

15. [**M**] The Demographic Research Unit of the California State Department of Finance supplied data for the following migration matrix, which describes the movement of the United States population during 1989. In 1989, about 11.7% of the total population lived in California. What percentage of the total population would eventually live in California if the listed migration probabilities were to remain constant over many years?

$$\begin{array}{cl} \text{From:} & \\ \begin{array}{cc} \text{CA} & \text{Rest of U.S.} \end{array} & \text{To:} \\ \begin{bmatrix} .9821 & .0029 \\ .0179 & .9971 \end{bmatrix} & \begin{array}{l} \text{California} \\ \text{Rest of U.S.} \end{array} \end{array}$$

16. [**M**] In Detroit, Hertz Rent A Car has a fleet of about 2000 cars. The pattern of rental and return locations is given by the fractions in the table below. On a typical day, about how many cars will be rented or ready to rent from the downtown location?

$$\begin{array}{cl} \text{Cars Rented from:} & \\ \begin{array}{ccc} \text{City Airport} & \text{Downtown} & \text{Metro Airport} \end{array} & \text{Returned to:} \\ \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix} & \begin{array}{l} \text{City Airport} \\ \text{Downtown} \\ \text{Metro Airport} \end{array} \end{array}$$

17. Let P be an $n \times n$ stochastic matrix. The following argument shows that the equation $P\mathbf{x} = \mathbf{x}$ has a nontrivial solution. (In fact, a steady-state solution exists with nonnegative entries. A proof is given in some advanced texts.) Justify each assertion below. (Mention a theorem when appropriate.)

a. If all the other rows of $P - I$ are added to the bottom row, the result is a row of zeros.

b. The rows of $P - I$ are linearly dependent.

c. The dimension of the row space of $P - I$ is less than n.

d. $P - I$ has a nontrivial null space.

18. Show that every 2×2 stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$, where α and β are constants between 0 and 1. (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise, there is only one.)

19. Let S be the $1 \times n$ row matrix with a 1 in each column,

$$S = [\,1 \quad 1 \quad \cdots \quad 1\,]$$

a. Explain why a vector $\mathbf{x}$ in $\mathbb{R}^n$ is a probability vector if and only if its entries are nonnegative and $S\mathbf{x} = 1$. (A 1×1 matrix such as the product $S\mathbf{x}$ is usually written without the matrix bracket symbols.)

b. Let P be an $n \times n$ stochastic matrix. Explain why $SP = S$.

c. Let P be an $n \times n$ stochastic matrix, and let $\mathbf{x}$ be a probability vector. Show that $P\mathbf{x}$ is also a probability vector.

20. Use Exercise 19 to show that if P is an $n \times n$ stochastic matrix, then so is P^2.

21. [**M**] Examine powers of a regular stochastic matrix.

a. Compute P^k for $k = 2, 3, 4, 5$, when

$$P = \begin{bmatrix} .3355 & .3682 & .3067 & .0389 \\ .2663 & .2723 & .3277 & .5451 \\ .1935 & .1502 & .1589 & .2395 \\ .2047 & .2093 & .2067 & .1765 \end{bmatrix}$$

Display calculations to four decimal places. What happens to the columns of P^k as k increases? Compute the steady-state vector for P.

b. Compute Q^k for $k = 10, 20, \ldots, 80$, when

$$Q = \begin{bmatrix} .97 & .05 & .10 \\ 0 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$$

(Stability for Q^k to four decimal places may require $k = 116$ or more.) Compute the steady-state vector for Q. Conjecture what might be true for any regular stochastic matrix.

c. Use Theorem 18 to explain what you found in parts (a) and (b).

22. [**M**] Compare two methods for finding the steady-state vector $\mathbf{q}$ of a regular stochastic matrix P: (1) computing $\mathbf{q}$ as in Example 5, or (2) computing P^k for some large value of k and using one of the columns of P^k as an approximation for $\mathbf{q}$. [The *Study Guide* describes a program *nulbasis* that almost automates method (1).]

Experiment with the largest random stochastic matrices your matrix program will allow, and use $k = 100$ or some other large value. For each method, describe the time *you* need to enter the keystrokes and run your program. (Some versions of MATLAB have commands `flops` and `tic ... toc` that record the number of floating point operations and the total elapsed time MATLAB uses.) Contrast the advantages of each method, and state which you prefer.

SOLUTIONS TO PRACTICE PROBLEMS

1. a. Since 5% of the city residents will move to the suburbs within one year, there is a 5% chance of choosing such a person. Without further knowledge about the person, we say that there is a 5% chance the person will move to the suburbs. This fact is contained in the second entry of the state vector $\mathbf{x}_1$, where

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

b. The likelihood that the person will be living in the suburbs after two years is 9.6%, because

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .904 \\ .096 \end{bmatrix}$$

2. The steady-state vector satisfies $P\mathbf{x} = \mathbf{x}$. Since

$$P\mathbf{q} = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}\begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} .32 \\ .68 \end{bmatrix} \neq \mathbf{q}$$

we conclude that $\mathbf{q}$ is *not* the steady-state vector for P.

3. M in Example 1 is a regular stochastic matrix because its entries are all strictly positive. So we may use Theorem 18. We already know the steady-state vector from Example 4. Thus the population distribution vectors $\mathbf{x}_k$ converge to

$$\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$$

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Eventually 62.5% of the population will live in the suburbs.

CHAPTER 4 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.) In parts (a)–(f), $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a nonzero finite-dimensional vector space V, and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a vector space.

(11) becomes

$$\mathbf{x}_k = c_1(1.01)^k\mathbf{v}_1 + c_2(-.03 + .26i)^k\mathbf{v}_2 + c_3(-.03 - .26i)^k\mathbf{v}_3$$

As $k \to \infty$, the second two vectors tend to zero. So $\mathbf{x}_k$ becomes more and more like the (real) vector $c_1(1.01)^k\mathbf{v}_1$. The approximations in equations (6) and (7), following Example 1, apply here. Also, it can be shown that the constant c_1 in the initial decomposition of $\mathbf{x}_0$ is positive when the entries in $\mathbf{x}_0$ are nonnegative. Thus the owl population will grow slowly, with a long-term growth rate of 1.01. The eigenvector $\mathbf{v}_1$ describes the eventual distribution of the owls by life stages: for every 31 adults, there will be about 10 juveniles and 3 subadults. ■

Further Reading

Franklin, G. F., J. D. Powell, and M. L. Workman. *Digital Control of Dynamic Systems*, 3rd ed. Reading, MA: Addison-Wesley, 1998.

Sandefur, James T. *Discrete Dynamical Systems—Theory and Applications*. Oxford: Oxford University Press, 1990.

Tuchinsky, Philip. *Management of a Buffalo Herd*, UMAP Module 207. Lexington, MA: COMAP, 1980.

PRACTICE PROBLEMS

1. The matrix A below has eigenvalues 1, $\frac{2}{3}$, and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Find the general solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

2. What happens to the sequence $\{\mathbf{x}_k\}$ in Practice Problem 1 as $k \to \infty$?

5.6 EXERCISES

1. Let A be a 2×2 matrix with eigenvalues 3 and 1/3 and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

a. Compute $\mathbf{x}_1 = A\mathbf{x}_0$. [*Hint:* You do not need to know A itself.]

b. Find a formula for $\mathbf{x}_k$ involving k and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

2. Suppose the eigenvalues of a 3×3 matrix A are 3, 4/5, and 3/5, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for the specified $\mathbf{x}_0$, and describe what happens as $k \to \infty$.

In Exercises 3–6, assume that any initial vector $\mathbf{x}_0$ has an eigenvector decomposition such that the coefficient c_1 in equation (1) of this section is positive.[3]

3. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .2 in equation (3). (Give a formula for $\mathbf{x}_k$.) Does the owl population grow or decline? What about the wood rat population?

4. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .125. (Give a formula for $\mathbf{x}_k$.) As time passes, what happens to the sizes of the owl and wood rat populations? The system tends toward what is sometimes called an unstable equilibrium. What do you think might happen to the system if some aspect of the model (such as birth rates or the predation rate) were to change slightly?

5. In old-growth forests of Douglas fir, the spotted owl dines mainly on flying squirrels. Suppose the predator–prey matrix for these two populations is $A = \begin{bmatrix} .4 & .3 \\ -p & 1.2 \end{bmatrix}$. Show that if the predation parameter p is .325, both populations grow. Estimate the long-term growth rate and the eventual ratio of owls to flying squirrels.

6. Show that if the predation parameter p in Exercise 5 is .5, both the owls and the squirrels will eventually perish. Find a value of p for which populations of both owls and squirrels tend toward constant levels. What are the relative population sizes in this case?

7. Let A have the properties described in Exercise 1.

a. Is the origin an attractor, a repeller, or a saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$?

b. Find the directions of greatest attraction and/or repulsion for this dynamical system.

c. Make a graphical description of the system, showing the directions of greatest attraction or repulsion. Include a rough sketch of several typical trajectories (without computing specific points).

8. Determine the nature of the origin (attractor, repeller, or saddle point) for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if A has the properties described in Exercise 2. Find the directions of greatest attraction or repulsion.

In Exercises 9–14, classify the origin as an attractor, repeller, or saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Find the directions of greatest attraction and/or repulsion.

9. $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$ **10.** $A = \begin{bmatrix} .3 & .4 \\ -.3 & 1.1 \end{bmatrix}$

11. $A = \begin{bmatrix} .4 & .5 \\ -.4 & 1.3 \end{bmatrix}$ **12.** $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$

13. $A = \begin{bmatrix} .8 & .3 \\ -.4 & 1.5 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1.7 & .6 \\ -.4 & .7 \end{bmatrix}$

15. Let $A = \begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}$. The vector $\mathbf{v}_1 = \begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$ is an eigenvector for A, and two eigenvalues are .5 and .2. Construct the solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ that satisfies $\mathbf{x}_0 = (0, .3, .7)$. What happens to $\mathbf{x}_k$ as $k \to \infty$?

16. **[M]** Produce the general solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ when A is the stochastic matrix for the Hertz Rent A Car model in Exercise 16 of Section 4.9.

17. Construct a stage-matrix model for an animal species that has two life stages: juvenile (up to 1 year old) and adult. Suppose the female adults give birth each year to an average of 1.6 female juveniles. Each year, 30% of the juveniles survive to become adults and 80% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (j_k, a_k)$, where the entries in $\mathbf{x}_k$ are the numbers of female juveniles and female adults in year k.

a. Construct the stage-matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.

b. Show that the population is growing, compute the eventual growth rate of the population, and give the eventual ratio of juveniles to adults.

c. **[M]** Suppose that initially there are 15 juveniles and 10 adults in the population. Produce four graphs that show how the population changes over eight years: (a) the number of juveniles, (b) the number of adults, (c) the total population, and (d) the ratio of juveniles to adults (each year). When does the ratio in (d) seem to stabilize? Include a listing of the program or keystrokes used to produce the graphs for (c) and (d).

18. A herd of American buffalo (bison) can be modeled by a stage matrix similar to that for the spotted owls. The females can be divided into calves (up to 1 year old), yearlings (1 to 2 years), and adults. Suppose an average of 42 female calves are born each year per 100 adult females. (Only adults produce offspring.) Each year, about 60% of the calves survive, 75% of the yearlings survive, and 95% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (c_k, y_k, a_k)$, where the entries in $\mathbf{x}_k$ are the numbers of females in each life stage at year k.

a. Construct the stage-matrix A for the buffalo herd, such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.

b. **[M]** Show that the buffalo herd is growing, determine the expected growth rate after many years, and give the expected numbers of calves and yearlings present per 100 adults.

[3] One of the limitations of the model in Example 1 is that there always exist initial population vectors $\mathbf{x}_0$ with positive entries such that the coefficient c_1 is negative. The approximation (7) is still valid, but the entries in $\mathbf{x}_k$ eventually become negative.