# Food for Thought 7

Due Tuesday, October 24

*Spend the rest of today's class period working through these problems. I encourage you to work with your classmates and discuss the problems. If you are finished with the assignment at the end of class today, then you can turn it in today. If you would like to work on the assignment more, take it home over the weekend and turn it in on Tuesday. This assignment will be graded for* ***effort*** *(which means you have written down thoughtful, complete solutions to each problem), not correctness. Solutions to these problems will be posted on Canvas on Monday for future reference.*

1. Give examples of the following:

   (a) A square matrix $A$ such that $\text{rank}A = 2$ and $\det A = 0$.

   (b) A vector space $V$ whose objects are matrices such that $\dim V = 100$.

2. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for a vector space $V$. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

3. This problem will introduce the concept of *change of basis.*

   (a) Invent a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ of $\mathbb{R}^2$ (*No zero entries allowed in any of the vectors. To make the numbers work out well, you may want to choose the vectors to have integer entries and to arrange that the determinant of the matrix* $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ *is equal to 1.*

   (b) Find the $\mathcal{B}$-coordinates of the standard basis vectors, $\vec{e}_1, \vec{e}_2$. Do you see any relationship between these vectors $[\vec{e}_1]_\mathcal{B}, [\vec{e}_2]_\mathcal{B}$ and the matrix mentioned above?

(c) Now repeat. Invent a second basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ of $\mathbb{R}^2$ *(preferably with the same properties as above.)*

(d) Find the $\mathcal{C}$-coordinates of the standard basis vectors.

(e) Now find the $\mathcal{C}$-coordinates of the vectors $\vec{b}_1$ and $\vec{b}_2$.

(f) *Finale:* Given a random vector, written in $\mathcal{B}$-coordinates, find a matrix $A$ such that multiplication by $A$ gives the $\mathcal{C}$-coordinates of that vector. As a starting point, you can start with vectors whose $\mathcal{B}$-coordinates are $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\2\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$.

(g) The above matrix is called the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, and is typically denoted by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Explain in your own words what is happening in this problem.

4. Is the set $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ a basis for $M_{2\times 2}$, the vector space of $2 \times 2$ matrices? Explain how you know.

5. If $A$ is a $6 \times 4$ matrix, what is the smallest possible dimension of Nul$A$? Explain.