Food for Thought 10

Due Wednesday, December 6

*Spend the rest of today's class period working through these problems. I encourage you to work with your classmates and discuss the problems. If you are finished with the assignment at the end of class today, then you can turn it in today. If you would like to work on the assignment more, take it home by Wednesday. This assignment will be graded for **effort** (which means you have written down thoughtful, complete solutions to each problem), not correctness. Solutions to these problems will be posted on Canvas on Wednesday for future reference.*

1. Give examples of the following:

   (a) A symmetric 2×2 matrix A with $\text{rank} A = 1$.

   (b) A matrix A such that the function $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is a positive definite quadratic form.

   (c) A matrix that is diagonalizable, but not orthogonally diagonalizable.

2. In class I stated the following: If A is a symmetric $n \times n$ matrix and $\mathbf{v}_1$ and $\mathbf{v}_2$ are an eigenvectors with distinct eigenvalues λ_1 and λ_2, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The equation below is a "proof" of this fact, with no explanation given. Your job is to provide an explanation for each equality in the equation. Then write one or two sentences explaining why this proves the fact above

$$\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) \stackrel{1}{=} A\mathbf{v}_1 \cdot \mathbf{v}_2 \stackrel{2}{=} \mathbf{v}_1^T A^T \mathbf{v}_2 \stackrel{3}{=} \mathbf{v_1}^T A \mathbf{v}_2 \stackrel{4}{=} \mathbf{v}_1 \cdot A\mathbf{v}_2 \stackrel{5}{=} \lambda_2(\mathbf{v_1} \cdot \mathbf{v}_2)$$

3. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

4. Given a singular value decomposition $A = U\Sigma V^T$, find a singular value decomposition of A^T. How are the singular values of A and A^T related? In what situation would it be easier to compute an SVD for A^T than A? (Think about possible sized for A and A^T, and how that affects SVD calculations.)

5. Let A and B be symmetric $n \times n$ matrices whose eigenvalues are all positive. Show that the eigenvalues of $A + B$ are all positive. (Hint: Consider quadratic forms.)

6. Show that if A is an $n \times n$ symmetric matrix, then $(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.