Challenge Problem 2: Combinatorial Cupcakes

MATH2603: Discrete Mathematics

Overview: In this Challenge Problem, you'll derive a counting formula that is found in the book without sufficient justification. You'll get to practice several counting techniques; most importantly, this problem is made much easier by using the bijection rule (several times).

Instructions: Complete the exercises below. Your submission should be self-contained; so you should explain what the questions asked, and give context to your work before answering any questions. Your solutions should be complete, clear, and correct. Your solutions should also be accompanied by explanations written in complete sentences that justify your work.

Instructions for submitting your work, and information on how the EMRN rubric will be applied to evaluate this challenge problem, are at the end of the assignment.

Background: The introduction to Chapter 9 describes an innocuous counting problem. A bakery sells 6 different varieties of cupcakes (chocolate, vanilla, red velvet, etc.). How many ways are there to fill a box with 24 cupcakes from the 6 varieties? The order in which the cupcakes are selected is unimportant; all that matters is the number of each variety in the box after they are chosen.

The techniques needed to solve this problem aren't discussed until §9.10, when the book claims (without explanation) that "the number of ways to place n indistinguishable balls into k bins is $\binom{n+k-1}{k-1}$." The goal of this challenge problem is to prove this statement.

Let's call the original problem that was discussed in the textbook "Problem A." Formally, "Problem A" will be the following:

[**Problem A**] If there are k varieties of cupcakes at a cupcake shop, and cupcakes of the same variety are indistinguishable, how many ways are there to place n cupcakes into a box?

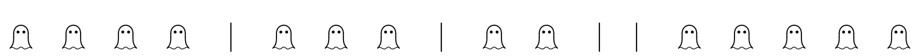
1. Show that this problem [**Problem A**] is equivalent to the problem here:

   [**Problem B**] Given positive integers n and k, how many ways are there to write n as a sum of k integers $x_1, x_2, \ldots, x_k$ with $0 \le x_j \le n$ for all $1 \le j \le k$. That is, determine the number of ways to choose integers $x_1, \ldots, x_k$ such that each x_j is between 0 and k, and so that $\sum_{j=1}^{k} x_j = n$.

   Be explicit in your use of the bijection rule. Provide a thorough explanation justifying the fact that your map is a bijection.

2. Show that [**Problem B**] is equivalent to the following:

   [**Problem C**] *(Ghosts and Walls)* Take n ghosts and place them in a line, then separate the ghosts into rooms using $k-1$ walls. Below is a schematic arrangement of ghosts and walls

   How many ways are there to arrange n ghosts and $k-1$ walls all in a line.

Be explicit in your use of the bijection rule. Provide a thorough explanation justifying the fact that your map is a bijection.

3. Solve the *Ghosts and Walls* problem using counting techniques we discussed in class. Thoroughly explain your counting methodology, and how you arrive at your formula.

Submitting your work: Your work must be neatly typed up using a system that supports mathematical notation. For example, you can use MS Word and its equation editor; or you can write your work in a Jupyter notebook using Markdown and LaTeX. Once it is written up, the work must be saved as a PDF file and then uploaded as a PDF to the area on Blackboard where the original assignment is located. Remember that the work is not actually submitted until you upload the file and click the "Submit" button. Grading and feedback will take place entirely on Blackboard. The following are not allowed: Submissions outside Blackboard (for example through email); files that are not in PDF form; and work that contains any handwriting, though you may *draw a diagram* neatly by hand, scan it, and include it in your submission.

Evaluation: Like all Challenge Problems, your work will be evaluated using the EMRN rubric. Please see the statement of this rubric in the syllabus for an explanation of how it is used. When applied to this Challenge Problem, the following criteria help to assign the grade:

- **E:** The submission consists of a clear, correct, and complete solution. The solution contains no major errors (computation, logic, syntax, or semantic); it is also exceptionally clear and the writeup is professional in its look and style. The solution would be at home in a professional lecture or publication.

- **M:** The submission consists of a clear, correct, and complete solution. The solution contains no major errors (computation, logic, syntax, or semantic) and is neatly and professionally written up.

- **R:** The solution contains at least one, but not several, major errors (computation, logic, syntax, and/or semantic) that require revision. An "R" may also be given for writeups that do not expend sufficient effort to produce a good-looking writeup, or do not provide enough context for the submission to be self-contained.

- **N:** The solution has several significant errors; or the submission is missing large portions of the solution; or the solution is for a significantly altered version of the problem; or the submission is excessively cluttered, messy, difficult to read, or handwritten.