MATH 2270: Midterm 3 Practice Problems

The following are practice problems for the third exam.

1. Inventions:
   (a) Invent a 2-dimensional subspace of $\mathbb{P}_3$.
   (b) Invent a linearly independent set consisting of infinitely many vectors in $\mathbb{P}$.
   (c) Invent a linear transformation $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ such that $\ker T$ is one-dimensional.
   (d) Invent a basis for the subspace H of $\mathbb{P}_3$ defined by $H = \{p(t) \in \mathbb{P}_3 \mid p(1) = 0\}$.
   (e) Let $\mathcal{S} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$. Invent a basis $\mathcal{B}$ of $\mathbb{P}_2$ so that the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{S}}{\mathcal{P}}$ is upper triangular.
   (f) Find a vector space V so that for any linear transformation $T\colon V \to \mathbb{R}^3$, the dimension of the kernel of T is at least 2.
   (g) Invent a 3×3 matrix A so that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvectors of A with eigenvalue $\lambda = 1$.
   (h) Invent a 2×2 matrix A whose only eigenvalue is 3.

2. Answer each of the following true/false questions, and then give an explanation of your reasoning.
   (a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
   (b) $\mathbb{P}^2$ is a subspace of $\mathbb{P}^3$.
   (c) $C([0,1])$ is a subspace of $C([0,2])$.
   (d) A change of basis matrix is never invertible.
   (e) Eigenspaces for a linear transformation $T\colon V \to V$ are subspaces that are T-invariant.
   (f) A linear transformation of an n-dimensional vector space has at least n different eigenvalues.
   (g) If $T\colon V \to V$ is a linear transformation, then T is represented by a unique matrix.

3. Determine whether or not each of the following linear transformations has a non-trivial eigenspace. If it does, describe the eigenspace and associated eigenvalue. Explain your reasoning.
   (a) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that rotates counter-clockwise about the origin by $\pi/2$.
   (b) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that reflects across the line $y = x$.
   (c) The linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that takes a vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} x \\ 2y \end{smallmatrix}\right]$.
   (d) The linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ that reflects in the xy-plane.

4. Show that $H = \{f \in C(\mathbb{R}) \mid f(0) = 0\}$ is a subspace of $C(\mathbb{R})$.

5. Consider the linear transformation $D\colon \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p) = p'$. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ and let $\mathcal{C} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$.

(a) Find the matrix for the linear transformation D with respect to the basis $\mathcal{B}$.

(b) Let $p(t) = -2t^3 + 3t^2 - 10t + 1$. Find $[p(t)]_{\mathcal{C}}$.

(c) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

6. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A. Then factor it to find the eigenvalues of A.

(b) For each eigenvalue, λ, find a basis for the corresponding eigenspace, $V^{(\lambda)}$.

(c) Use the computations from parts (a) and (b) to write $A = PDP^{-1}$, where D is a diagonal matrix, and P is an invertible matrix. You do not need to compute P^{-1}.

7. Let H be the subspace of $C(\mathbb{R})$ (continuous functions $\mathbb{R} \to \mathbb{R}$) spanned by $\{\sinh x, \cosh x\}$. Recall (if you don't already know) that $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$ (note the lack of a minus sign). Consider the linear transformation $D\colon H \to H$ defined by $D(f) = f'$, where f' denotes the derivative of f.

(a) Compute the matrix of D with respect to the basis, $\mathcal{B} = \{\sinh x, \cosh x\}$. *You do not need to show that $\mathcal{B}$ is a basis for H.*

(b) Use the techniques of Chapter 5 to find a basis for H in which the matrix for D is diagonal.

8. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x) = Ax$ where A is a 3×3 matrix whose eigenvalues are $1, 3$, and -2. Does there exist a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that the matrix for T with respect to $\mathcal{B}$ is diagonal? Why or why not?

9. Give an example of a 2×2 matrix that is not diagonalizable.

10. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$, where $A = \left[\begin{smallmatrix} -6 & -2 \\ 4 & 0 \end{smallmatrix}\right]$. Find the matrix for T with respect to the basis $\mathcal{B} = \{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]\}$

11. §4.9 Exercises 4 & 14.

12. §5.6 Exercises 1 & 10.