NAME: MATH2270-004

November 10, 2017

Midterm 3

(100 points)

Instructions: Do all the problems on **both sides** of each page. Show all your work and box your answers. If you get stuck on a problem, skip it and come back to it at the end.

1. Let T be a linear transformation $T\colon \mathbb{R}^8 \to \mathbb{R}^5$.

   (a) [**4 points**] What is the largest possible dimension for $\ker T$? In this case, what is the dimension of the image of T? Explain your reasoning in one or two *complete sentences.*

   (b) [**4 points**] What is the smallest possible dimension for $\ker T$? In this case, what is the dimension of the image of T? Explain your reasoning in one or two *complete sentences.*

2. In a certain pond, the population of zombie frogs and zombie flies is described by the following dynamical system:

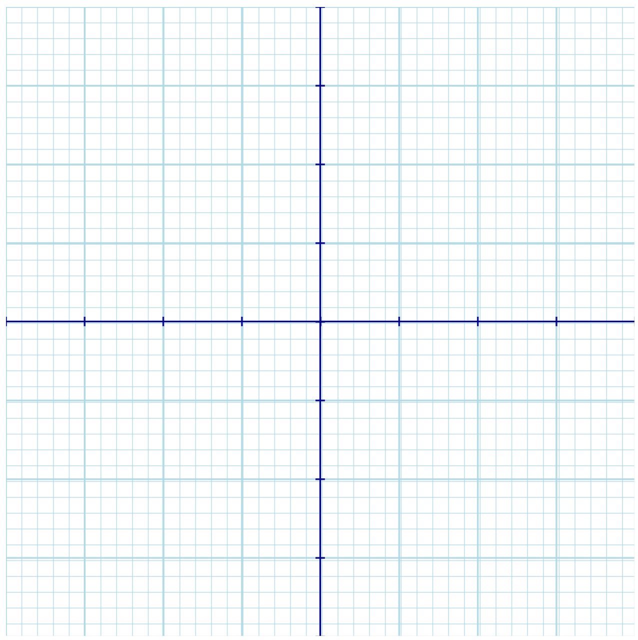
$$g_{k+1} = .38g_k + .24y_k$$
$$y_{k+1} = -.36g_k + 1.22y_k$$

where g_k and y_k denote the number of zombie frogs and zombie flies respectively at time k. The fly populations are measured in thousands of flies. The eigenvalues for this system are $\lambda_1 = 1.1$ and $\lambda_2 = 0.5$ with corresponding eigenvectors $v_1 = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$ and $v_2 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ respectively.

(a) **[7 points]** Use the information provided to write the matrix $A = \left[\begin{smallmatrix} .38 & .24 \\ -.36 & 1.22 \end{smallmatrix}\right]$ as $A = PDP^{-1}$, where P is an invertible matrix P and D is a diagonal matrix. You do not need to compute P^{-1}.

(b) **[7 points]** Given an initial population of 3 frogs and 4,000 flies (so $g_0 = 3$ and $f_0 = 4$), write a general formula for the number of frogs and flies at time k.

(c) **[8 points]** Classify the origin as an attractor, repeller, or saddle point. Then give a graphical description of the solutions to this dynamical system that includes several trajectories.



3. **Inventions:**

(a) **[5 points]** Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$. Invent a *different* basis $\mathcal{C}$ of $\mathbb{P}_2$ so that each column of the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ has exactly one non-zero entry, and that entry is 1.

(b) **[5 points]** Invent a 2×2 matrix A such that $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ is an eigenvector of A with associated eigenvalue $\lambda = 3$.

4. Consider the linear transformation $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ defined by $T(p(t)) = \left[\begin{smallmatrix} p(-1) \\ p(1) \end{smallmatrix}\right]$.

   (a) **[3 points]** Compute $T(1 + t + t^2)$.

   (b) **[3 points]** Compute $T(2 + 6t + t^2)$.

   (c) **[3 points]** Compute $T(-1 + t^2)$.

(d) **[6 points]** Is the set of polynomials $\mathcal{B} = \{1 + t + t^2, 2 + 6t + t^2, -1 + t^2\}$ a basis for $\mathbb{P}_2$? Use a coordinate mapping to reduce the problem to computing the determinant of a 3×3 matrix.

(e) **[5 points]** Write down the matrix for T relative to the basis $\mathcal{B}$. *(No additional calculations are necessary.)*

(f) **[5 points]** Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Write down the change of basis matrix $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}$.

5. Consider the following matrix:

$$A = \begin{bmatrix} 4 & 0 & 6 \\ -2 & 1 & -5 \\ -3 & 0 & -5 \end{bmatrix}$$

The characteristic polynomial of A is $p(\lambda) = (\lambda - 1)^2(\lambda + 2)$.

(a) [**7 points**] Find a basis for the eigenspace $V^{(-2)}$.

(b) [**7 points**] Find a basis for the eigenspace $V^{(1)}$.

(c) **[5 points]** Use your calculations in parts (a) and (b) to determine whether or not the matrix A diagonalizable. Give an explanation of your reasoning.

6. **[8 points]** Consider the set of functions

$$S = \{f\colon [0,1] \to \mathbb{R} \mid f \text{ is infinitely differentiable and } f'(0) = 0\}$$

S is a subset of $C^\infty([0,1])$ (the vector space of infinitely differentiable functions $f\colon [0,1] \to \mathbb{R}$). Is S a subspace of $C^\infty([0,1])$? If it is, show that S satisfies the definition of a subspace; if it's not, explain which part of the definition fails?

7. **[8 points]** Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates counter-clockwise about the origin by π (or 180°). Determine whether or S any non-trivial eigenspaces. If it does, describe them (for example as the span of a vector) and give the associated eigenvalue. Explain your reasoning in one or two *complete sentences*. You may use a picture if you like.

"Why, Mr. Anderson? Why, why? Why do you do it? Why, why get up? Why keep fighting? Do you believe you're fighting…for something? For more than your survival? Can you tell me what it is? Do you even know? Is it freedom? Or truth? Perhaps peace? Could it be for love? Illusions, Mr. Anderson. Vagaries of perception. Temporary constructs of a feeble human intellect trying desperately to justify an existence that is without meaning or purpose." –Agent Smith