Problem Set 6.5

In Problems 1–4, solve the given differential equation subject to the given condition. Note that $y(a)$ denotes the value of y at $t = a$.

1. $\frac{dy}{dt} = -6y$, $y(0) = 4$

2. $\frac{dy}{dt} = 6y$, $y(0) = 1$

3. $\frac{dy}{dt} = 0.005y$, $y(10) = 2$

4. $\frac{dy}{dt} = -0.003y$, $y(-2) = 3$

5. A bacterial population grows at a rate proportional to its size. Initially, it is 10,000, and after 10 days it is 20,000. What is the population after 25 days? See Example 2.

6. How long will it take the population of Problem 5 to double? See Example 1.

7. How long will it take the population of Problem 5 to triple? See Example 1.

8. The population of the United States was 3.9 million in 1790 and 178 million in 1960. If the rate of growth is assumed proportional to the number present, what estimate would you give for the population in 2000? (Compare your answer with the actual 2000 population, which was 275 million.)

9. The population of a certain country is growing at 3.2% per year; that is, if it is A at the beginning of a year, it is $1.032A$ at the end of that year. Assuming that it is 4.5 million now, what will it be at the end of 1 year? 2 years? 10 years? 100 years?

10. Determine the proportionality constant k in $dy/dt = ky$ for Problem 9. Then use $y = 4.5e^{kt}$ to find the population after 100 years.

11. A population is growing at a rate proportional to its size. After 5 years, the population size was 164,000. After 12 years, the population size was 235,000. What was the original population size?

12. The mass of a tumor grows at a rate proportional to its size. The first measurement of its size was 4.0 grams. Four months later its mass was 6.76 grams. How large was the tumor six months before the first measurement? If the instrument can detect tumors of mass 1 gram or greater, would the tumor have been detected at that time?

13. A radioactive substance has a half-life of 700 years. If there were 10 grams initially, how much would be left after 300 years?

14. If a radioactive substance loses 15% of its radioactivity in 2 days, what is its half-life?

15. Cesium 137 and strontium 90 are two radioactive chemicals that were released at the Chernobyl nuclear reactor in April 1986. The half-life of cesium 137 is 30.22 years, and that of strontium 90 is 28.8 years. In what year will the amount of cesium 137 be equal to 1% of what was released? Answer this question for strontium 90.

16. An unknown amount of a radioactive substance is being studied. After two days, the mass is 15.231 grams. After eight days, the mass is 9.086 grams. How much was there initially? What is the half-life of this substance?

17. (Carbon Dating) All living things contain carbon 12, which is stable, and carbon 14, which is radioactive. While a plant or animal is alive, the ratio of these two isotopes of carbon remains unchanged, since the carbon 14 is constantly renewed; after death, no more carbon 14 is absorbed. The half-life of carbon 14 is 5730 years. If charred logs of an old fort show only 70% of the carbon 14 expected in living matter, when did the fort burn down? Assume that the fort burned soon after it was built of freshly cut logs.

18. Human hair from a grave in Africa proved to have only 51% of the carbon 14 of living tissue. When was the body buried?

19. An object is taken from an oven at 300°F and left to cool in a room at 75°F. If the temperature fell to 200°F in $\frac{1}{2}$ hour, what will it be after 3 hours?

20. A thermometer registered −20°C outside and then was brought into a house where the temperature was 24°C. After 5 minutes it registered 0°C. When will it register 20°C?

21. An object initially at 26°C is placed in water having temperature 90°C. If the temperature of the object rises to 70°C in 5 minutes, what will be the temperature after 10 minutes?

22. A batch of brownies is taken from a 350°F oven and placed in a refrigerator at 40°F and left to cool. After 15 minutes, the brownies have cooled to 250°F. When will the temperature of the brownies be 110°F?

23. A dead body is found at 10 P.M. to have temperature 82°F. One hour later the temperature was 76°F. The temperature of the room was a constant 70°F. Assuming that the temperature of the body was 98.6°F when it was alive, estimate the time of death.

24. Solve the differential equation for Newton's Law of Cooling for an arbitrary T_0, T_1, and k, assuming that $T_0 > T_1$. Show that $\lim_{t\to\infty} T(t) = T_1$.

25. If \$375 is put in the bank today, what will it be worth at the end of 2 years if interest is 3.5% and is compounded as specified?

(a) Annually
(b) Monthly
(c) Daily
(d) Continuously

26. Do Problem 25 assuming that the interest rate is 4.6%.

27. How long does it take money to double in value for the specified interest rate?

(a) 6% compounded monthly
(b) 6% compounded continuously

28. Inflation between 1999 and 2004 ran at about 2.5% per year. On this basis, what would you expect a car that would have cost \$20,000 in 1999 to cost in 2004?

29. Manhattan Island is said to have been bought by Peter Minuit in 1626 for \$24. Suppose that Minuit had instead put the \$24 in the bank at 6% interest compounded continuously. What would that \$24 have been worth in 2000?

30. If Methuselah's parents had put \$100 in the bank for him at birth and he left it there, what would Methuselah have had at his death (969 years later) if interest was 4% compounded annually?

31. Find the value of \$1000 at the end of 1 year when the interest is compounded continuously at 5%. This is called the **future value.**

32. Suppose that after 1 year you have \$1000 in the bank. If the interest was compounded continuously at 5%, how much money did you put in the bank one year ago? This is called the **present value.**

33. It will be shown later for small x that $\ln(1 + x) \approx x$. Use this fact to show that the doubling time for money invested at p percent compounded annually is about $70/p$ years.

34. The equation for logistic growth is

$$\frac{dy}{dt} = ky(L - y)$$

Show that this differential equation has the solution

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-Lkt}}$$

Hint: $\dfrac{1}{y(L - y)} = \dfrac{1}{Ly} + \dfrac{1}{L(L - y)}$.

35. Sketch the graph of the solution in Problem 34 when $y_0 = 6.4$, $L = 16$, and $k = 0.00186$ (a *logistic model* for world population; see the discussion at the beginning of this section). Note that $\lim_{t\to\infty} y = 16$.

36. Find each of the following limits.

(a) $\lim_{x\to 0}(1 + x)^{1000}$ (b) $\lim_{x\to 0}(1)^{1/x}$

(c) $\lim_{x\to 0^+}(1 + \varepsilon)^{1/x}, \varepsilon > 0$ (d) $\lim_{x\to 0^-}(1 + \varepsilon)^{1/x}, \varepsilon > 0$

(e) $\lim_{x\to 0}(1 + x)^{1/x}$

37. Use the fact that $e = \lim_{h\to 0}(1 + h)^{1/h}$ to find each limit.

(a) $\lim_{x\to 0}(1 - x)^{1/x}$ *Hint:* $(1 - x)^{1/x} = [(1 - x)^{1/(-x)}]^{-1}$

(b) $\lim_{x\to 0}(1 + 3x)^{1/x}$ (c) $\lim_{n\to\infty}\left(\dfrac{n + 2}{n}\right)^n$

(d) $\lim_{n\to\infty}\left(\dfrac{n - 1}{n}\right)^{2n}$

38. Show that the differential equation

$$\frac{dy}{dt} = ay + b$$

has solution

$$y = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$$

Assume that $a \neq 0$.

39. Consider a country with a population of 10 million in 1985, a growth rate of 1.2% per year, and immigration from other countries of 60,000 per year. Use the differential equation of Problem 38 to model this situation and predict the population in 2010. Take $a = 0.012$.

40. Important news is said to diffuse through an adult population of fixed size L at a time rate proportional to the number of people who have not heard the news. Five days after a scandal in City Hall was reported, a poll showed that half the people had heard it. How long will it take for 99% of the people to hear it?

EXPL *Besides providing an easy way to differentiate products, logarithmic differentiation also provides a measure of the **relative** or **fractional rate of change**, defined as y'/y. We explore this concept in Problems 41–44.*

41. Show that the relative rate of change of e^{kt} as a function of t is k.

42. Show that the relative rate of change of any polynomial approaches zero as the independent variable approaches infinity.

43. Prove that if the relative rate of change is a positive constant then the function must represent exponential growth.

44. Prove that if the relative rate of change is a negative constant then the function must represent exponential decay.

45. Assume that (1) world population continues to grow exponentially with growth constant $k = 0.0132$, (2) it takes $\frac{1}{2}$ acre of land to supply food for one person, and (3) there are 13,500,000 square miles of arable land in the world. How long will it be before the world reaches the maximum population? *Note:* There were 6.4 billion people in 2004 and 1 square mile is 640 acres.

GC **46.** The Census Bureau estimates that the growth rate k of the world population will decrease by roughly 0.0002 per year for the next few decades. In 2004, k was 0.0132.

(a) Express k as a function of time t, where t is measured in years since 2004.

(b) Find a differential equation that models the population y for this problem.

(c) Solve the differential equation with the additional condition that the population in 2004 ($t = 0$) was 6.4 billion.

(d) Graph the population y for the next 300 years.

(e) With this model, when will the population reach a maximum? When will the population drop below the 2004 level?

GC **47.** Repeat Exercise 46 under the assumption that k will decrease by 0.0001 per year.

EXPL **48.** Let E be a differentiable function satisfying $E(u + v) = E(u)E(v)$ for all u and v. Find a formula for $E(x)$. *Hint:* First find $E'(x)$.

GC **49.** Using the same axes, draw the graphs for $0 \le t \le 100$ of the following two models for the growth of world population (both described in this section).

(a) Exponential growth: $y = 6.4e^{0.0132t}$

(b) Logistic growth: $y = 102.4/(6 + 10e^{-0.030t})$

Compare what the two models predict for world population in 2010, 2040, and 2090. *Note:* Both models assume that world population was 6.4 billion in 2004 ($t = 0$).

GC **50.** Evaluate:

(a) $\lim_{x\to 0}(1 + x)^{1/x}$ (b) $\lim_{x\to 0}(1 - x)^{1/x}$

The limit in part (a) determines e. What special number does the limit in part (b) determine?

Answers to Concepts Review: **1.** ky; $ky(L - y)$ **2.** 8 **3.** half-life **4.** $(1 + h)^{1/h}$