MATH 2270: Homework 7 Worksheet

due October 18, 2017

Instructions: Do the following problems on a separate sheet of paper. Show all of your work.

1. **Vector spaces and differential equations:** Consider the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that have the form
$$f(t) = c_1 \cos(t) + c_2 \sin(t)$$
As a set, we write
$$V = \{f\colon \mathbb{R} \to \mathbb{R} \mid f(t) = c_1 \cos(t) + c_2 \sin(t)\}$$
   (a) Show that V is a vector space.
   (b) Give two examples of vectors in V.
   (c) Consider the differential equation $y'' = -y$. Show that all of the functions/vectors $f(t) \in V$ are solutions to this differential equation.
   (d) What do you think is the dimension of this vector space? Can you write V as the span of some vectors? Can you find a basis for V?

   It turns out that for certain differential equations (such as the one above), the set of solutions to equation forms a vector space!!

2. **Vector space of continuous functions:** The set of all continuous real valued functions defined on a closed interval is denoted by $C([a,b])$. Specifically
$$C([a,b]) = \{f\colon [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$$
   (a) What facts about continuous functions would you need to know in order to prove that $C([a,b])$ is a vector space.
   (b) Give three interesting examples of vectors in $C([0,1])$.
   (c) Show that $H = \{f \in C([a,b]) \mid f(a) = f(b)\}$ is a subspace of $C([a,b])$.
   (d) Give two examples of vectors in H.

3. **Intersection of subspaces:** Let H and K be subspaces of a vector space V. The *intersection* of H and K, written as $H \cap K$, is the set of $\vec{v} \in V$ that belong to both H and K. Show that $H \cap K$ is a subspace of V. Draw a picture depicting this where H and K are each two-dimensional subspaces of $\mathbb{R}^3$. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not necessarily a subspace.

   As a remark, $H \cap K$ is "the biggest subspace" contained in both H and K. Can you guess how we might make the term "biggest" precise?

4. **Sum of subspaces:** Given subspaces H and K of a vector space V, the *sum* of H and K, written as $H + K$ is the set of all vectors in V that can be written as the sum of two vectors, one in H and one in K. As a set, we write
$$H + K = \{\vec{u} + \vec{v} \mid \vec{u} \in H \text{ and } \vec{v} \in K\}$$
   (a) Show that $H + K$ is a subspace of V.

(b) Show that H is a subspace of $H + K$. (The same argument shows that K is a subspace of $H + K$.

As a remark, $H + K$ is "the smallest subspace" containing both H and K. Can you guess how we might make the term "smallest" precise?

5. **Polynomials and $\mathbb{R}^n$:** Define $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

(a) Give three examples of vectors in $\mathbb{P}_2$.

(b) Let $p \in \mathbb{P}_2$ be the polynomial $p(t) = 3 + 5t + 7t^2$. Find $T(p(t))$.

(c) Show that T is a linear transformation.

(d) Find a polynomial $q \in \mathbb{P}_2$ that is in the kernel of T. Describe the range of T.