The exercises that follow give practice with matrix algebra and illustrate typical calculations found in applications.

PRACTICE PROBLEMS

1. Show that $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible and find its inverse.
2. Compute X^TX, where X is partitioned as $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$.

2.4 EXERCISES

In Exercises 1–9, assume that the matrices are partitioned conformably for block multiplication. Compute the products shown in Exercises 1–4.

1. $\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

2. $\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$

3. $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

4. $\begin{bmatrix} I & 0 \\ -E & I \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$

In Exercises 5–8, find formulas for X, Y, and Z in terms of A, B, and C, and justify your calculations. In some cases, you may need to make assumptions about the size of a matrix in order to produce a formula. [*Hint:* Compute the product on the left, and set it equal to the right side.]

5. $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ X & Y \end{bmatrix} = \begin{bmatrix} 0 & I \\ Z & 0 \end{bmatrix}$

6. $\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

7. $\begin{bmatrix} X & 0 & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A & Z \\ 0 & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

8. $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}\begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$

9. Suppose B_{11} is an invertible matrix. Find matrices A_{21} and A_{31} (in terms of the blocks of B) such that the product below has the form indicated. Also, compute C_{22} (in terms of the blocks of B). [*Hint:* Compute the product on the left, and set it equal to the right side.]

$$\begin{bmatrix} I & 0 & 0 \\ A_{21} & I & 0 \\ A_{31} & 0 & I \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \\ 0 & C_{32} \end{bmatrix}$$

10. The inverse of

$$\begin{bmatrix} I & 0 & 0 \\ A & I & 0 \\ B & D & I \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} I & 0 & 0 \\ P & I & 0 \\ Q & R & I \end{bmatrix}.$$

Find P, Q, and R.

In Exercises 11 and 12, mark each statement True or False. Justify each answer.

11. a. If $A = [\,A_1 \quad A_2\,]$ and $B = [\,B_1 \quad B_2\,]$, with A_1 and A_2 the same sizes as B_1 and B_2, respectively, then $A + B = [\,A_1 + B_1 \quad A_2 + B_2\,]$.

b. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then the partitions of A and B are conformable for block multiplication.

12. a. If A_1, A_2, B_1, and B_2 are $n \times n$ matrices, $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, and $B = [\,B_1 \quad B_2\,]$, then the product BA is defined, but AB is not.

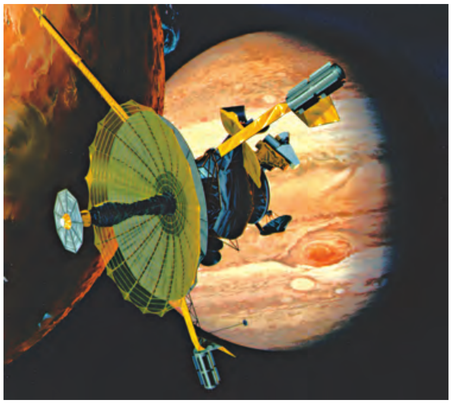
b. If $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then the transpose of A is

$$A^T = \begin{bmatrix} P^T & Q^T \\ R^T & S^T \end{bmatrix}.$$

13. Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$, where B and C are square. Show that A is invertible if and only if both B and C are invertible.

14. Show that the block upper triangular matrix A in Example 5 is invertible if and only if both A_{11} and A_{22} are invertible. [*Hint:* If A_{11} and A_{22} are invertible, the formula for A^{-1} given in Example 5 actually works as the inverse of A.] This fact about A is an important part of several computer algorithms that estimate eigenvalues of matrices. Eigenvalues are discussed in Chapter 5.

15. When a deep space probe is launched, corrections may be necessary to place the probe on a precisely calculated trajectory. Radio telemetry provides a stream of vectors, $\mathbf{x}_1, \ldots, \mathbf{x}_k$, giving information at different times about how the probe's position compares with its planned trajectory. Let X_k be the matrix $[\mathbf{x}_1 \;\cdots\; \mathbf{x}_k]$. The matrix $G_k = X_k X_k^T$ is computed as the radar data are analyzed. When $\mathbf{x}_{k+1}$ arrives, a new G_{k+1} must be computed. Since the data vectors arrive at high speed, the computational burden could be severe. But partitioned matrix multiplication helps tremendously. Compute the column–row expansions of G_k and G_{k+1}, and describe what must be computed in order to *update* G_k to form G_{k+1}.



The probe Galileo was launched October 18, 1989, and arrived near Jupiter in early December 1995.

16. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. If A_{11} is invertible, then the matrix $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the **Schur complement** of A_{11}. Likewise, if A_{22} is invertible, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is called the Schur complement of A_{22}. Suppose A_{11} is invertible. Find X and Y such that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \tag{7}$$

17. Suppose the block matrix A on the left side of (7) is invertible and A_{11} is invertible. Show that the Schur complement S of A_{11} is invertible. [*Hint:* The outside factors on the right side of (7) are always invertible. Verify this.] When A and A_{11} are both invertible, (7) leads to a formula for A^{-1}, using S^{-1}, A_{11}^{-1}, and the other entries in A.

18. Let X be an $m \times n$ data matrix such that $X^T X$ is invertible, and let $M = I_m - X(X^T X)^{-1} X^T$. Add a column $\mathbf{x}_0$ to the data and form

$$W = \begin{bmatrix} X & \mathbf{x}_0 \end{bmatrix}$$

Compute $W^T W$. The $(1, 1)$-entry is $X^T X$. Show that the Schur complement (Exercise 16) of $X^T X$ can be written in the form $\mathbf{x}_0^T M \mathbf{x}_0$. It can be shown that the quantity $(\mathbf{x}_0^T M \mathbf{x}_0)^{-1}$ is the $(2, 2)$-entry in $(W^T W)^{-1}$. This entry has a useful statistical interpretation, under appropriate hypotheses.

In the study of engineering control of physical systems, a standard set of differential equations is transformed by Laplace transforms into the following system of linear equations:

$$\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix} \tag{8}$$

where A is $n \times n$, B is $n \times m$, C is $m \times n$, and s is a variable. The vector $\mathbf{u}$ in $\mathbb{R}^m$ is the "input" to the system, $\mathbf{y}$ in $\mathbb{R}^m$ is the "output," and $\mathbf{x}$ in $\mathbb{R}^n$ is the "state" vector. (Actually, the vectors $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}$ are functions of s, but this does not affect the algebraic calculations in Exercises 19 and 20.)

19. Assume $A - sI_n$ is invertible and view (8) as a system of two matrix equations. Solve the top equation for $\mathbf{x}$ and substitute into the bottom equation. The result is an equation of the form $W(s)\mathbf{u} = \mathbf{y}$, where $W(s)$ is a matrix that depends on s. $W(s)$ is called the *transfer function* of the system because it transforms the input $\mathbf{u}$ into the output $\mathbf{y}$. Find $W(s)$ and describe how it is related to the partitioned *system matrix* on the left side of (8). See Exercise 16.

20. Suppose the transfer function $W(s)$ in Exercise 19 is invertible for some s. It can be shown that the inverse transfer function $W(s)^{-1}$, which transforms outputs into inputs, is the Schur complement of $A - BC - sI_n$ for the matrix below. Find this Schur complement. See Exercise 16.

$$\begin{bmatrix} A - BC - sI_n & B \\ -C & I_m \end{bmatrix}$$

21. a. Verify that $A^2 = I$ when $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

b. Use partitioned matrices to show that $M^2 = I$ when

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

22. Generalize the idea of Exercise 21 by constructing a 6×6 matrix $M = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ C & 0 & D \end{bmatrix}$ such that $M^2 = I$. Make C a nonzero 2×2 matrix. Show that your construction works.

23. Use partitioned matrices to prove by induction that the product of two lower triangular matrices is also lower triangular. [*Hint*: A $(k + 1) \times (k + 1)$ matrix A_1 can be written in the form below, where a is a scalar, $\mathbf{v}$ is in $\mathbb{R}^k$, and A is a $k \times k$ lower triangular matrix. See the *Study Guide* for help with induction.]

$$A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$$

SG The Principle of Induction 2–19

24. Use partitioned matrices to prove by induction that for $n = 2, 3, \ldots$, the $n \times n$ matrix A shown below is invertible and B is its inverse.

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & & 0 \\ 1 & 1 & 1 & & 0 \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & \cdots & -1 & 1 \end{bmatrix}$$

For the induction step, assume A and B are $(k+1) \times (k+1)$ matrices, and partition A and B in a form similar to that displayed in Exercise 23.

25. Without using row reduction, find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

26. [**M**] For block operations, it may be necessary to access or enter submatrices of a large matrix. Describe the functions or commands of a matrix program that accomplish the following tasks. Suppose A is a 20×30 matrix.

a. Display the submatrix of A from rows 5 to 10 and columns 15 to 20.

b. Insert a 5×10 matrix B into A, beginning at row 5 and column 10.

c. Create a 50×50 matrix of the form $C = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$.

[*Note:* It may not be necessary to specify the zero blocks in C.]

27. [**M**] Suppose memory or size restrictions prevent a matrix program from working with matrices having more than 32 rows and 32 columns, and suppose some project involves 50×50 matrices A and B. Describe the commands or operations of the matrix program that accomplish the following tasks.

a. Compute $A + B$.

b. Compute AB.

c. Solve $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b}$ in $\mathbb{R}^{50}$, assuming that A can be partitioned into a 2×2 block matrix $[A_{ij}]$, with A_{11} an invertible 20×20 matrix, A_{22} an invertible 30×30 matrix, and A_{12} a zero matrix. [*Hint:* Describe appropriate smaller systems to solve, without using any matrix inverses.]

SOLUTIONS TO PRACTICE PROBLEMS

1. If $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible, its inverse has the form $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$. Verify that

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} W & X \\ AW + Y & AX + Z \end{bmatrix}$$

So W, X, Y, Z must satisfy $W = I$, $X = 0$, $AW + Y = 0$, and $AX + Z = I$. It follows that $Y = -A$ and $Z = I$. Hence

$$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The product in the reverse order is also the identity, so the block matrix is invertible, and its inverse is $\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}$. (You could also appeal to the Invertible Matrix Theorem.)

2. $X^T X = \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{bmatrix}$. The partitions of X^T and X are automatically conformable for block multiplication because the columns of X^T are the rows of X. This partition of $X^T X$ is used in several computer algorithms for matrix computations.

2.5 MATRIX FACTORIZATIONS

A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices. Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data. In the language of computer science, the expression of A as a product amounts to a *preprocessing* of the data in A, organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.

b. To factor the matrix $\begin{bmatrix} 1 & -8 \\ -.5 & 5 \end{bmatrix}$ into the product of transfer matrices, as in equation (6), look for R_1 and R_2 in Fig. 4 to satisfy

$$\begin{bmatrix} 1 & -R_1 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ -.5 & 5 \end{bmatrix}$$

From the $(1, 2)$-entries, $R_1 = 8$ ohms, and from the $(2, 1)$-entries, $1/R_2 = .5$ ohm and $R_2 = 1/.5 = 2$ ohms. With these values, the network in Fig. 4 has the desired transfer matrix. ■

A network transfer matrix summarizes the input–output behavior (the design specifications) of the network without reference to the interior circuits. To physically build a network with specified properties, an engineer first determines if such a network can be constructed (or *realized*). Then the engineer tries to factor the transfer matrix into matrices corresponding to smaller circuits that perhaps are already manufactured and ready for assembly. In the common case of alternating current, the entries in the transfer matrix are usually rational complex-valued functions. (See Exercises 19 and 20 in Section 2.4 and Example 2 in Section 3.3.) A standard problem is to find a *minimal realization* that uses the smallest number of electrical components.

PRACTICE PROBLEM

Find an LU factorization of $A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$. [*Note*: It will turn out that A has only three pivot columns, so the method of Example 2 will produce only the first three columns of L. The remaining two columns of L come from I_5.]

2.5 EXERCISES

In Exercises 1–6, solve the equation $A\mathbf{x} = \mathbf{b}$ by using the LU factorization given for A. In Exercises 1 and 2, also solve $A\mathbf{x} = \mathbf{b}$ by ordinary row reduction.

1. $A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ -2 & -3 & -4 & 12 \\ 3 & 0 & 4 & -36 \\ -5 & -3 & -8 & 49 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 3 & 0 & 12 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find an LU factorization of the matrices in Exercises 7–16 (with L unit lower triangular). Note that MATLAB will usually produce a permuted LU factorization because it uses partial pivoting for numerical accuracy.

7. $\begin{bmatrix} 2 & 5 \\ -3 & -4 \end{bmatrix}$

8. $\begin{bmatrix} 6 & 4 \\ 12 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix}$

10. $\begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ -3 & -2 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 0 & 5 & 2 \\ -6 & 3 & -13 & -3 \\ 4 & 9 & 16 & 17 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -3 & 4 \\ -4 & 8 & -7 \\ 6 & -5 & 14 \\ -6 & 9 & -12 \\ 8 & -6 & 19 \end{bmatrix}$

17. When A is invertible, MATLAB finds A^{-1} by factoring $A = LU$ (where L may be permuted lower triangular), inverting L and U, and then computing $U^{-1}L^{-1}$. Use this method to compute the inverse of A in Exercise 2. (Apply the algorithm in Section 2.2 to L and to U.)

18. Find A^{-1} as in Exercise 17, using A from Exercise 3.

19. Let A be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that A is invertible and A^{-1} is lower triangular. [*Hint:* Explain why A can be changed into I using only row replacements and scaling. (Where are the pivots?) Also, explain why the row operations that reduce A to I change I into a lower triangular matrix.]

20. Let $A = LU$ be an LU factorization. Explain why A can be row reduced to U using only replacement operations. (This fact is the converse of what was proved in the text.)

21. Suppose $A = BC$, where B is invertible. Show that any sequence of row operations that reduces B to I also reduces A to C. The converse is not true, since the zero matrix may be factored as $0 = B \cdot 0$.

Exercises 22–26 provide a glimpse of some widely used matrix factorizations, some of which are discussed later in the text.

22. (*Reduced LU Factorization*) With A as in the Practice Problem, find a 5×3 matrix B and a 3×4 matrix C such that $A = BC$. Generalize this idea to the case where A is $m \times n$, $A = LU$, and U has only three nonzero rows.

23. (*Rank Factorization*) Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.
    a. Show that A is the sum of four outer products. (See Section 2.4.)
    b. Let $m = 400$ and $n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

24. (*QR Factorization*) Suppose $A = QR$, where Q and R are $n \times n$, R is invertible and upper triangular, and Q has the property that $Q^TQ = I$. Show that for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What computations with Q and R will produce the solution?

WEB

25. (*Singular Value Decomposition*) Suppose $A = UDV^T$, where U and V are $n \times n$ matrices with the property that $U^TU = I$ and $V^TV = I$, and where D is a diagonal matrix with positive numbers $\sigma_1, \dots, \sigma_n$ on the diagonal. Show that A is invertible, and find a formula for A^{-1}.

26. (*Spectral Factorization*) Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix

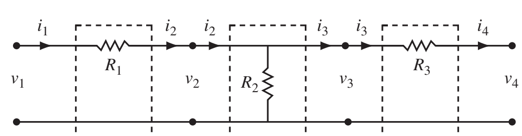
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that this factorization is useful when computing high powers of A. Find fairly simple formulas for A^2, A^3, and A^k (k a positive integer), using P and the entries in D.

27. Design two different ladder networks that each output 9 volts and 4 amps when the input is 12 volts and 6 amps.

28. Show that if three shunt circuits (with resistances R_1, R_2, R_3) are connected in series, the resulting network has the same transfer matrix as a single shunt circuit. Find a formula for the resistance in that circuit.

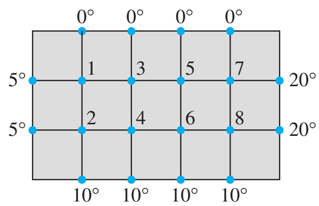
29. a. Compute the transfer matrix of the network in the figure below.



b. Let $A = \begin{bmatrix} 3 & -12 \\ -1/3 & 5/3 \end{bmatrix}$. Design a ladder network whose transfer matrix is A by finding a suitable matrix factorization of A.

30. Find a different factorization of the transfer matrix A in Exercise 29, and thereby design a different ladder network whose transfer matrix is A.

31. [**M**] Consider the heat plate in the following figure (refer to Exercise 33 in Section 1.1).



The solution to the steady-state heat flow problem for this plate is approximated by the solution to the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (5, 15, 0, 10, 0, 10, 20, 30)$ and

$$A = \begin{bmatrix} 4 & -1 & -1 & & & & & \\ -1 & 4 & 0 & -1 & & & & \\ -1 & 0 & 4 & -1 & -1 & & & \\ & -1 & -1 & 4 & 0 & -1 & & \\ & & -1 & 0 & 4 & -1 & -1 & \\ & & & -1 & -1 & 4 & 0 & -1 \\ & & & & -1 & 0 & 4 & -1 \\ & & & & & -1 & -1 & 4 \end{bmatrix}$$

WEB

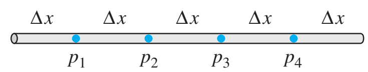
The missing entries in A are zeros. The nonzero entries of A lie within a band along the main diagonal. Such *band matrices* occur in a variety of applications and often are extremely large (with thousands of rows and columns but relatively narrow bands).

a. Use the method in Example 2 to construct an LU factorization of A, and note that both factors are band matrices (with two nonzero diagonals below or above the main diagonal). Compute $LU - A$ to check your work.

b. Use the LU factorization to solve $A\mathbf{x} = \mathbf{b}$.

c. Obtain A^{-1} and note that A^{-1} is a dense matrix with no band structure. When A is large, L and U can be stored in much less space than A^{-1}. This fact is another reason for preferring the LU factorization of A to A^{-1} itself.

32. [**M**] The band matrix A shown below can be used to estimate the unsteady conduction of heat in a rod when the temperatures at points $p_1, \ldots, p_4$ on the rod change with time.[2]



The constant C in the matrix depends on the physical nature of the rod, the distance Δx between the points on the rod, and the length of time Δt between successive temperature measurements. Suppose that for $k = 0, 1, 2, \ldots$, a vector $\mathbf{t}_k$ in $\mathbb{R}^4$ lists the temperatures at time $k\Delta t$. If the two ends of the rod are maintained at 0°, then the temperature vectors satisfy the equation $A\mathbf{t}_{k+1} = \mathbf{t}_k$ $(k = 0, 1, \ldots)$, where

$$A = \begin{bmatrix} (1+2C) & -C & & \\ -C & (1+2C) & -C & \\ & -C & (1+2C) & -C \\ & & -C & (1+2C) \end{bmatrix}$$

a. Find the LU factorization of A when $C = 1$. A matrix such as A with three nonzero diagonals is called a *tridiagonal matrix*. The L and U factors are *bidiagonal matrices*.

b. Suppose $C = 1$ and $\mathbf{t}_0 = (10, 15, 15, 10)^T$. Use the LU factorization of A to find the temperature distributions $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, and $\mathbf{t}_4$.

[2] See Biswa N. Datta, *Numerical Linear Algebra and Applications* (Pacific Grove, CA: Brooks/Cole, 1994), pp. 200–201.

SOLUTION TO PRACTICE PROBLEM

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

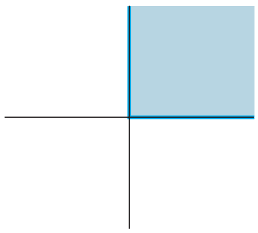
$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Divide the entries in each highlighted column by the pivot at the top. The resulting columns form the first three columns in the lower half of L. This suffices to make row reduction of L to I correspond to reduction of A to U. Use the last two columns of I_5

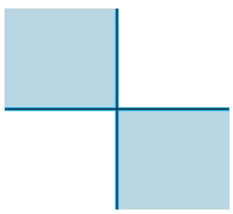
2.8 EXERCISES

Exercises 1–4 display sets in $\mathbb{R}^2$. Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of $\mathbb{R}^2$. (For instance, find two vectors in H whose sum is *not* in H, or find a vector in H with a scalar multiple that is not in H. Draw a picture.)

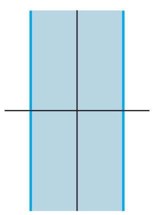
1.



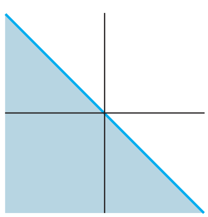
2.



3.



4.



5. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

6. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 5 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 2 \end{bmatrix}$. Determine if $\mathbf{u}$ is in the subspace of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix},$$

$$\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}.$$

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Col A?

c. Is $\mathbf{p}$ in Col A? Why or why not?

8. Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix},$$

and $\mathbf{p} = \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}$. Determine if $\mathbf{p}$ is in Col A, where $A = [\,\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3\,]$.

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

10. With $\mathbf{u} = \begin{bmatrix} -5 \\ 5 \\ 3 \end{bmatrix}$ and A as in Exercise 8, determine if $\mathbf{u}$ is in Nul A.

In Exercises 11 and 12, give integers p and q such that Nul A is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$.

11. $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \\ 3 & 3 & 4 \end{bmatrix}$

13. For A as in Exercise 11, find a nonzero vector in Nul A and a nonzero vector in Col A.

14. For A as in Exercise 12, find a nonzero vector in Nul A and a nonzero vector in Col A.

Determine which sets in Exercises 15–20 are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

15. $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

16. $\begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

17. $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

18. $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$

19. $\begin{bmatrix} 3 \\ -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A subspace of $\mathbb{R}^n$ is any set H such that (i) the zero vector is in H, (ii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are in H, and (iii) c is a scalar and $c\mathbf{u}$ is in H.

b. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the same as the column space of the matrix $\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix}$.

c. The set of all solutions of a system of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^m$.

d. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

e. Row operations do not affect linear dependence relations among the columns of a matrix.

22. a. A subset H of $\mathbb{R}^n$ is a subspace if the zero vector is in H.

b. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

c. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is a subspace of $\mathbb{R}^n$.

d. Let H be a subspace of $\mathbb{R}^n$. If $\mathbf{x}$ is in H, and $\mathbf{y}$ is in $\mathbb{R}^n$, then $\mathbf{x}+\mathbf{y}$ is in H.

e. The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

Exercises 23–26 display a matrix A and an echelon form of A. Find a basis for Col A and a basis for Nul A.

23. $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

24. $A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

27. Construct a 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Col A, but $\mathbf{b}$ is not the same as any one of the columns of A.

28. Construct a 3×3 matrix A and a vector $\mathbf{b}$ such that $\mathbf{b}$ is *not* in Col A.

29. Construct a nonzero 3×3 matrix A and a nonzero vector $\mathbf{b}$ such that $\mathbf{b}$ is in Nul A.

30. Suppose the columns of a matrix $A = [\mathbf{a}_1 \cdots \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is a basis for Col A.

In Exercises 31–36, respond as comprehensively as possible, and justify your answer.

31. Suppose F is a 5×5 matrix whose column space is not equal to $\mathbb{R}^5$. What can be said about Nul F?

32. If B is a 7×7 matrix and Col $B = \mathbb{R}^7$, what can be said about solutions of equations of the form $B\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^7$?

33. If C is a 6×6 matrix and Nul C is the zero subspace, what can be said about solutions of equations of the form $C\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbb{R}^6$?

34. What can be said about the shape of an $m \times n$ matrix A when the columns of A form a basis for $\mathbb{R}^m$?

35. If B is a 5×5 matrix and Nul B is *not* the zero subspace, what can be said about Col B?

36. What can be said about Nul C when C is a 6×4 matrix with linearly independent columns?

[M] In Exercises 37 and 38, construct bases for the column space and the null space of the given matrix A. Justify your work.

37. $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 5 & 3 & 2 & -6 & -8 \\ 4 & 1 & 3 & -8 & -7 \\ 5 & 1 & 4 & 5 & 19 \\ -7 & -5 & -2 & 8 & 5 \end{bmatrix}$

WEB Column Space and Null Space

WEB A Basis for Col A