MATH2270: Midterm 1 Practice Problem Answers

The following are practice problems for the first exam.

1. Inventions:

   (a) Give an example of a linearly independent set of vectors in $\mathbb{R}^3$ that contains as many vectors as possible. Possible Answer: $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

   (b) Give an example of a 3×3 matrix A in reduced row echelon form such that the first and third columns of A are pivot columns, and $A\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

   $$\begin{bmatrix}1 & -2 & 0\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$$

   (c) Give an example of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ such that Span$\{\vec{v}, \vec{w}\}$ is a line. Possible answer: $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \vec{w} = \begin{bmatrix}2\\4\\6\end{bmatrix}$

2. True/False: Determine if each statement is true or false.

   (a) A free variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix. Answer: False

   (b) A consistent linear system always has infinitely many solutions. Answer: False

   (c) If A is a 2×3 matrix, then the linear transformation $x \mapsto Ax$ has domain $\mathbb{R}^3$. Answer: True

   (d) If A is a 2×3 matrix, then the linear transformation $x \mapsto Ax$ cannot be onto. Answer: False

   (e) If the columns of a matrix A are linearly dependent, then the equation $Ax = 0$ has only one solution. Answer: False

   (f) If the columns of a matrix A are linearly dependent, the linear transformation $x \mapsto Ax$ is one-to-one. Answer: False

3. For what values of h and k is the following system consistent?

   $$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

   Answer: $k = -3h$, $h \in \mathbb{R}$

4. Give a parametric description of the solutions to the equation $A\vec{x} = \vec{0}$ where A is the matrix shown below:

   $$\begin{bmatrix}1 & -2 & 3 & -6 & 5 & 0\\ 0 & 0 & 0 & 1 & 4 & -6\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

   Answer: $\left\{x_2\begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix} + x_5\begin{bmatrix}-29\\0\\0\\-4\\1\\0\end{bmatrix} \mid x_2, x_3, x_5 \in \mathbb{R}\right\}$

5. Determine if the vector $\vec{b} = \left[\begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}\right]$ is in $\text{Span}\{v_1, v_2\}$ where

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \vec{v_2} = \begin{bmatrix} 5 \\ 9 \\ 7 \end{bmatrix}$$

If the answer is yes, then write $\vec{b}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

Answer: Yes. $\vec{b} = -3\vec{v_1} + \vec{v_2}$

6. If $\vec{b}$ is in the span of the vectors $\vec{v_1}, \dots, \vec{v_k}$, what can you say about solutions to the matrix equation $A\vec{x} = \vec{b}$ where A is the matrix whose columns are $\vec{v_1}, \dots, \vec{v_k}$ (i.e., $A = [\vec{v_1}\ \vec{v_2}\ \dots\ \vec{v_k}]$)?

Answer: The equation $A\vec{x} = \vec{b}$ is consistent.

7. Is the set of vectors $\left\{ \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right] \right\}$ linearly independent? Why or why not? Answer: No

8. Is the set of vectors $\{\text{Span}\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}\}$ linearly independent? Why or why not? Answer: No

9. Determine if the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$, whose standard matrix is A, is 1-1. Is it onto?

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Answer: Yes

10. Suppose $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $S\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 3 \\ -2 \\ -1 \end{smallmatrix}\right]$ and $S\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} -1 \\ 2 \\ 2 \end{smallmatrix}\right]$.

   (a) Find $S\left(\begin{bmatrix} -3 \\ 3 \end{bmatrix}\right)$. Answer: $\left[\begin{smallmatrix} -12 \\ 12 \\ 9 \end{smallmatrix}\right]$

   (b) Find the standard matrix for S. Answer: $\left[\begin{smallmatrix} 3 & -1 \\ -2 & 2 \\ -1 & 2 \end{smallmatrix}\right]$

11. If the columns of a 6×4 matrix A are linearly dependent, then what is the maximum number of pivot positions of A? Answer: 3