Concepts Review

1. The general first-order linear differential equation has the form $dy/dx + P(x)y = Q(x)$. An integrating factor for this equation is _____.

2. Multiplying both sides of the first-order linear differential equation in Question 1 by its integrating factor makes the left side $\frac{d}{dx}(\_\_\_\_\_)$.

3. The integrating factor for $dy/dx - (1/x)y = x$, where $x > 0$, is _____. When we multiply both sides by this factor, the equation takes the form _____. The general solution to this equation is $y =$ _____.

4. The solution to the differential equation in Question 1 satisfying $y(a) = b$ is called a _____ solution.

Problem Set 6.6

In Problems 1–14, solve each differential equation.

1. $\frac{dy}{dx} + y = e^{-x}$

2. $(x+1)\frac{dy}{dx} + y = x^2 - 1$

3. $(1 - x^2)\frac{dy}{dx} + xy = ax, |x| < 1$

4. $y' + y \tan x = \sec x$

5. $\frac{dy}{dx} - \frac{y}{x} = xe^x$

6. $y' - ay = f(x)$

7. $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$

8. $y' + \frac{2y}{x+1} = (x+1)^3$

9. $y' + yf(x) = f(x)$

10. $\frac{dy}{dx} + 2y = x$ *Hint:* $\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

11. $\frac{dy}{dx} - \frac{y}{x} = 3x^3$; $y = 3$ when $x = 1$.

12. $y' = e^{2x} - 3y$; $y = 1$ when $x = 0$.

13. $xy' + (1 + x)y = e^{-x}$; $y = 0$ when $x = 1$.

14. $\sin x \frac{dy}{dx} + 2y \cos x = \sin 2x$; $y = 2$ when $x = \frac{\pi}{6}$.

15. A tank contains 20 gallons of a solution, with 10 pounds of chemical A in the solution. At a certain instant, we begin pouring in a solution containing the same chemical in a concentration of 2 pounds per gallon. We pour at a rate of 3 gallons per minute while simultaneously draining off the resulting (well-stirred) solution at the same rate. Find the amount of chemical A in the tank after 20 minutes.

16. A tank initially contains 200 gallons of brine, with 50 pounds of salt in solution. Brine containing 2 pounds of salt per gallon is entering the tank at the rate of 4 gallons per minute and is flowing out at the same rate. If the mixture in the tank is kept uniform by constant stirring, find the amount of salt in the tank at the end of 40 minutes.

17. A tank initially contains 120 gallons of pure water. Brine with 1 pound of salt per gallon flows into the tank at 4 gallons per minute, and the well-stirred solution runs out at 6 gallons per minute. How much salt is in the tank after t minutes, $0 \le t \le 60$?

18. A tank initially contains 50 gallons of brine, with 30 pounds of salt in solution. Water runs into the tank at 3 gallons per minute and the well-stirred solution runs out at 2 gallons per minute. How long will it be until there are 25 pounds of salt in the tank?

19. Find the current I as a function of time for the circuit of Figure 3 if the switch S is closed and $I = 0$ at $t = 0$.

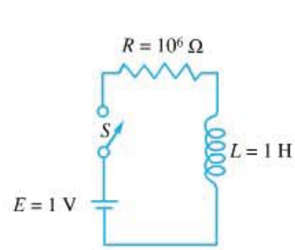

Figure 3

20. Find I as a function of time for the circuit of Figure 4, assuming that the switch is closed and $I = 0$ at $t = 0$.

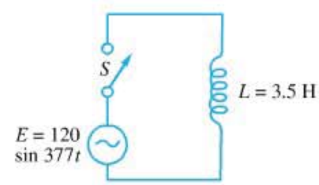

Figure 4

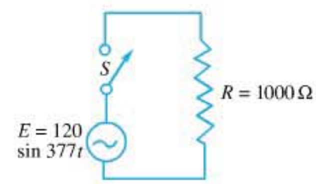

Figure 5

21. Find I as a function of time for the circuit of Figure 5, assuming that the switch is closed and $I = 0$ at $t = 0$.

22. Suppose that tank 1 initially contains 100 gallons of solution, with 50 pounds of dissolved salt, and tank 2 contains 200 gallons, with 150 pounds of dissolved salt. Pure water flows into tank 1 at 2 gallons per minute, the well-mixed solution flows out and into tank 2 at the same rate, and finally, the solution in tank 2 drains away also at the same rate. Let $x(t)$ and $y(t)$ denote the amounts of salt in tanks 1 and 2, respectively, at time t. Find $y(t)$. *Hint:* First find $x(t)$ and use it in setting up the differential equation for tank 2.

23. A tank of capacity 100 gallons is initially full of pure alcohol. The flow rate of the drain pipe is 5 gallons per minute; the flow rate of the filler pipe can be adjusted to c gallons per minute. An unlimited amount of 25% alcohol solution can be brought in through the filler pipe. Our goal is to reduce the amount of alcohol in the tank so that it will contain 100 gallons of 50% solution. Let T be the number of minutes required to accomplish the desired change.

(a) Evaluate T if $c = 5$ and both pipes are opened.

(b) Evaluate T if $c = 5$ and we first drain away a sufficient amount of the pure alcohol and then close the drain and open the filler pipe.

(c) For what values of c (if any) would strategy (b) give a faster time than (a)?

(d) Suppose that $c = 4$. Determine the equation for T if we initially open both pipes and then close the drain.

EXPL **24.** The differential equation for a falling body near the earth's surface with air resistance proportional to the velocity v is $dv/dt = -g - av$, where $g = 32$ feet per second per second is the acceleration of gravity and $a > 0$ is the *drag coefficient*. Show each of the following:

(a) $v(t) = (v_0 - v_\infty)e^{-at} + v_\infty$, where $v_0 = v(0)$, and

$$v_\infty = -g/a = \lim_{t\to\infty} v(t)$$

the so-called terminal velocity.

(b) If $y(t)$ denotes the altitude, then

$$y(t) = y_0 + tv_\infty + (1/a)(v_0 - v_\infty)(1 - e^{-at})$$

25. A ball is thrown straight up from ground level with an initial velocity $v_0 = 120$ feet per second. Assuming a drag coefficient of $a = 0.05$, determine each of the following:

(a) the maximum altitude

(b) an equation for T, the time when the ball hits the ground

26. Mary bailed out of her plane at an altitude of 8000 feet, fell freely for 15 seconds, and then opened her parachute. Assume that the drag coefficients are $a = 0.10$ for free fall and $a = 1.6$ with the parachute. When did she land?

27. For the differential equation $\dfrac{dy}{dx} - \dfrac{y}{x} = x^2, x > 0$, the integrating factor is $e^{\int(-1/x)\,dx}$. The general antiderivative $\int\left(-\frac{1}{x}\right)dx$ is equal to $-\ln x + C$.

(a) Multiply both sides of the differential equation by $\exp\left(\int\left(-\frac{1}{x}\right)dx\right) = \exp(-\ln x + C)$, and show that $\exp(-\ln x + C)$ is an integrating factor for every value of C.

(b) Solve the resulting equation for y, and show that the solution agrees with the solution obtained when we assumed that $C = 0$ in the integrating factor.

28. Multiply both sides of the equation $\dfrac{dy}{dx} + P(x)y = Q(x)$ by the factor $e^{\int P(x)\,dx + C}$.

(a) Show that $e^{\int P(x)\,dx+C}$ is an integrating factor for every value of C.

(b) Solve the resulting equation for y, and show that it agrees with the general solution given before Example 1.

Answers to Concepts Review: **1.** $\exp\left(\int P(x)\,dx\right)$ **2.** $y\exp\left(\int P(x)\,dx\right)$ **3.** $1/x$; $\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = 1$; $x^2 + Cx$ **4.** particular

6.7 Approximations for Differential Equations

In the previous section we studied a number of differential equations that arise from physical applications. For each equation, we were always able to find an **analytic solution**; that is, we found an explicit function that satisfies the equation. Many differential equations do not have such analytic solutions, so for these equations we must settle for approximations. In this section, we will study two ways to approximate a solution to a differential equation; one method is graphical and the other is numerical.

A Function of Two Variables

The function f depends on two variables. Since $y'(x) = f(x, y)$, the slope of a solution depends on *both* the x- and y-coordinates. Functions of two or more variables were introduced in Section 0.5. We will study them further in Chapter 12.

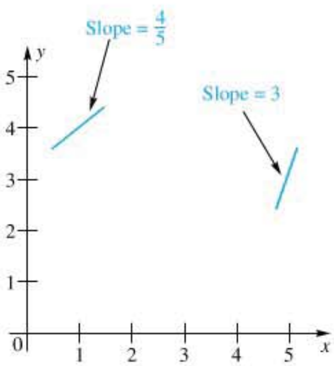

Figure 1

Slope Fields Consider a first-order differential equation of the form

$$y' = f(x, y)$$

This equation says that at the point (x, y) the slope of a solution is given by $f(x, y)$. For example, the differential equation $y' = y$ says that the slope of the curve passing through the point (x, y) is equal to y.

For the differential equation $y' = \frac{1}{5}xy$, at the point $(5, 3)$ the slope of the solution is $y' = \frac{1}{5}\cdot 5\cdot 3 = 3$; at the point $(1, 4)$ the slope is $y' = \frac{1}{5}\cdot 1\cdot 4 = \frac{4}{5}$. We can indicate graphically this latter result by drawing a small line segment through the point $(1, 4)$ having slope $\frac{4}{5}$ (see Figure 1).

If we repeat this process for a number of ordered pairs (x, y), we obtain a **slope field**. Since plotting a slope field is a tedious job if done by hand, the task is best suited for computers; *Mathematica* and *Maple* are capable of plotting slope fields. Figure 2 shows a slope field for the differential equation $y' = \frac{1}{5}xy$. Given an initial condition, we can follow the slopes to get at least a rough approximation to the particular solution. We can often see from the slope field the behavior of all solutions to the differential equation.