Row Reduction Example

Wednesday, August 23, 2017 9:16 PM

We want to perform row operations to the matrix A to get to reduced row echelon form.

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right] \underset{R_1 \leftrightarrows R_4}{\sim} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right]$$

$$\underset{R_3 \rightsquigarrow R_3 + 2R_1}{\overset{R_2 \rightsquigarrow R_2 + R_1}{\sim}} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \underset{R_3 \rightsquigarrow \frac{1}{5} R_3}{\overset{R_2 \rightsquigarrow \frac{1}{2} R_2}{\sim}} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right]$$

- First, we get a non-zero entry in the upper left corner.
- Then, we use that entry to clear out all non-zero entries below it.
- Next, we repeat this process only working in the submatrix boxed in orange.

$$\underset{R_4 \rightsquigarrow R_4 + 3R_2}{\overset{R_3 \rightsquigarrow R_3 - R_2}{\sim}} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right] \underset{R_3 \leftrightarrows R_4}{\sim} \left[\begin{array}{cccc|c} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot positions

- We've arrived at an echelon form. From this, we can read off the pivot positions of the matrix. These are the positions of leading entries in non-zero rows.
- Let's think about what the system of equations looks like at this point

$$x_1 + 4x_2 + 5x_3 - 9x_4 = -7$$

$$x_2 + 2x_3 - 3x_4 = -3$$

$$
\begin{aligned}
x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \\
x_2 + 2x_3 - 3x_4 &= -3 \\
-5x_4 &= 0
\end{aligned}
$$

- We've solved for x_4, but haven't back-substituted yet.
- Now we'll continue and find the <u>reduced</u> echelon form

$$
\underset{R_3 \rightsquigarrow -\frac{1}{5}R_3}{\sim}
\left[\begin{array}{cccc|c}
1 & 4 & 5 & -9 & -7 \\
0 & 1 & 2 & -3 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\underset{\substack{R_1 \rightsquigarrow R_1 + 9R_3 \\ R_2 \rightsquigarrow R_2 + 3R_3}}{\sim}
\left[\begin{array}{cccc|c}
1 & 4 & 5 & 0 & -7 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\underset{R_1 \rightsquigarrow R_1 - 4R_2}{\sim}
\left[\begin{array}{cccc|c}
1 & 0 & -3 & 0 & 5 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

The strategy to get from echelon form to reduced echelon form is to work right to left, and bottom to top:

- Use the right-most pivot entry to get 0's in the col. above that entry.
- Once this is done, move to the next right-most pivot position and repeat.

If we write out the corresponding system of equs. we get:

$$
\begin{aligned}
x_1 \quad - 3x_3 &= 5 \\
x_2 + 2x_3 &= -3 \\
x_4 &= 0
\end{aligned}
$$