

NUMERICAL NOTE

You may have noticed that if $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 0, 1, \dots$, then

$$\mathbf{x}_2 = P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0,$$

and, in general,

$$\mathbf{x}_k = P^k\mathbf{x}_0 \quad \text{for } k = 0, 1, \dots$$

To compute a specific vector such as \mathbf{x}_3 , fewer arithmetic operations are needed to compute \mathbf{x}_1 , \mathbf{x}_2 , and \mathbf{x}_3 , rather than P^3 and $P^3\mathbf{x}_0$. However, if P is small—say, 30×30 —the machine computation time is insignificant for both methods, and a command to compute $P^3\mathbf{x}_0$ might be preferred because it requires fewer human keystrokes.

PRACTICE PROBLEMS

1. Suppose the residents of a metropolitan region move according to the probabilities in the migration matrix M in Example 1 and a resident is chosen “at random.” Then a state vector for a certain year may be interpreted as giving the probabilities that the person is a city resident or a suburban resident at that time.
 - a. Suppose the person chosen is a city resident now, so that $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is the likelihood that the person will live in the suburbs next year?
 - b. What is the likelihood that the person will be living in the suburbs in two years?
2. Let $P = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$. Is \mathbf{q} a steady-state vector for P ?
3. What percentage of the population in Example 1 will live in the suburbs after many years?

4.9 EXERCISES

1. A small remote village receives radio broadcasts from two radio stations, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 A.M.
 - a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break. Label the rows and columns.
 - b. Give the initial state vector.
 - c. What percentage of the listeners will be listening to the music station at 9:25 A.M. (after the station breaks at 8:30 and 9:00 A.M.)?
2. A laboratory animal may eat any one of three foods each day. Laboratory records show that if the animal chooses one food on one trial, it will choose the same food on the next trial

with a probability of 60%, and it will choose the other foods on the next trial with equal probabilities of 20%.

- a. What is the stochastic matrix for this situation?
- b. If the animal chooses food #1 on an initial trial, what is the probability that it will choose food #2 on the second trial after the initial trial?



3. On any given day, a student is either healthy or ill. Of the students who are healthy today, 95% will be healthy

tomorrow. Of the students who are ill today, 55% will still be ill tomorrow.

- What is the stochastic matrix for this situation?
 - Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?
 - If a student is healthy today, what is the probability that he or she will be healthy two days from now?
4. The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 40% chance it will be good tomorrow, a 30% chance it will be indifferent, and a 30% chance it will be bad. If the weather is indifferent today, there is a 50% chance it will be good tomorrow, and a 20% chance it will be indifferent. Finally, if the weather is bad today, there is a 30% chance it will be good tomorrow and a 40% chance it will be indifferent.
- What is the stochastic matrix for this situation?
 - Suppose there is a 50% chance of good weather today and a 50% chance of indifferent weather. What are the chances of bad weather tomorrow?
 - Suppose the predicted weather for Monday is 60% indifferent weather and 40% bad weather. What are the chances for good weather on Wednesday?

In Exercises 5–8, find the steady-state vector.

5. $\begin{bmatrix} .1 & .5 \\ .9 & .5 \end{bmatrix}$

6. $\begin{bmatrix} .4 & .8 \\ .6 & .2 \end{bmatrix}$

7. $\begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$

8. $\begin{bmatrix} .4 & .5 & .8 \\ 0 & .5 & .1 \\ .6 & 0 & .1 \end{bmatrix}$

9. Determine if $P = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ is a regular stochastic matrix.

10. Determine if $P = \begin{bmatrix} 1 & .3 \\ 0 & .7 \end{bmatrix}$ is a regular stochastic matrix.

11. a. Find the steady-state vector for the Markov chain in Exercise 1.
b. At some time late in the day, what fraction of the listeners will be listening to the news?

12. Refer to Exercise 2. Which food will the animal prefer after many trials?

13. a. Find the steady-state vector for the Markov chain in Exercise 3.
b. What is the probability that after many days a specific student is ill? Does it matter if that person is ill today?

14. Refer to Exercise 4. In the long run, how likely is it for the weather in Columbus to be good on a given day?

15. [M] The Demographic Research Unit of the California State Department of Finance supplied data for the following migration matrix, which describes the movement of the United States population during 1989. In 1989, about 11.7% of the

total population lived in California. What percentage of the total population would eventually live in California if the listed migration probabilities were to remain constant over many years?

From:

CA	Rest of U.S.	To:
.9821	.0029	California
.0179	.9971	Rest of U.S.

16. [M] In Detroit, Hertz Rent A Car has a fleet of about 2000 cars. The pattern of rental and return locations is given by the fractions in the table below. On a typical day, about how many cars will be rented or ready to rent from the downtown location?

Cars Rented from:

City Airport	Down-town	Metro Airport	Returned to:
.90	.01	.09	City Airport
.01	.90	.01	Downtown
.09	.09	.90	Metro Airport

17. Let P be an $n \times n$ stochastic matrix. The following argument shows that the equation $P\mathbf{x} = \mathbf{x}$ has a nontrivial solution. (In fact, a steady-state solution exists with nonnegative entries. A proof is given in some advanced texts.) Justify each assertion below. (Mention a theorem when appropriate.)
- If all the other rows of $P - I$ are added to the bottom row, the result is a row of zeros.
 - The rows of $P - I$ are linearly dependent.
 - The dimension of the row space of $P - I$ is less than n .
 - $P - I$ has a nontrivial null space.

18. Show that every 2×2 stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form $P = \begin{bmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{bmatrix}$, where α and β are constants between 0 and 1. (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise, there is only one.)

19. Let S be the $1 \times n$ row matrix with a 1 in each column,

$$S = [1 \ 1 \ \cdots \ 1]$$

- Explain why a vector \mathbf{x} in \mathbb{R}^n is a probability vector if and only if its entries are nonnegative and $S\mathbf{x} = 1$. (A 1×1 matrix such as the product $S\mathbf{x}$ is usually written without the matrix bracket symbols.)
- Let P be an $n \times n$ stochastic matrix. Explain why $SP = S$.
- Let P be an $n \times n$ stochastic matrix, and let \mathbf{x} be a probability vector. Show that $P\mathbf{x}$ is also a probability vector.

20. Use Exercise 19 to show that if P is an $n \times n$ stochastic matrix, then so is P^2 .

21. [M] Examine powers of a regular stochastic matrix.

a. Compute P^k for $k = 2, 3, 4, 5$, when

$$P = \begin{bmatrix} .3355 & .3682 & .3067 & .0389 \\ .2663 & .2723 & .3277 & .5451 \\ .1935 & .1502 & .1589 & .2395 \\ .2047 & .2093 & .2067 & .1765 \end{bmatrix}$$

Display calculations to four decimal places. What happens to the columns of P^k as k increases? Compute the steady-state vector for P .

b. Compute Q^k for $k = 10, 20, \dots, 80$, when

$$Q = \begin{bmatrix} .97 & .05 & .10 \\ 0 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix}$$

(Stability for Q^k to four decimal places may require $k = 116$ or more.) Compute the steady-state vector for

Q . Conjecture what might be true for any regular stochastic matrix.

- c. Use Theorem 18 to explain what you found in parts (a) and (b).

22. [M] Compare two methods for finding the steady-state vector \mathbf{q} of a regular stochastic matrix P : (1) computing \mathbf{q} as in Example 5, or (2) computing P^k for some large value of k and using one of the columns of P^k as an approximation for \mathbf{q} . [The *Study Guide* describes a program *nubasis* that almost automates method (1).]

Experiment with the largest random stochastic matrices your matrix program will allow, and use $k = 100$ or some other large value. For each method, describe the time you need to enter the keystrokes and run your program. (Some versions of MATLAB have commands `flops` and `tic ... toc` that record the number of floating point operations and the total elapsed time MATLAB uses.) Contrast the advantages of each method, and state which you prefer.

SOLUTIONS TO PRACTICE PROBLEMS

1. a. Since 5% of the city residents will move to the suburbs within one year, there is a 5% chance of choosing such a person. Without further knowledge about the person, we say that there is a 5% chance the person will move to the suburbs. This fact is contained in the second entry of the state vector \mathbf{x}_1 , where

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

- b. The likelihood that the person will be living in the suburbs after two years is 9.6%, because

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .904 \\ .096 \end{bmatrix}$$

2. The steady-state vector satisfies $P\mathbf{x} = \mathbf{x}$. Since

$$P\mathbf{q} = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} .32 \\ .68 \end{bmatrix} \neq \mathbf{q}$$

we conclude that \mathbf{q} is *not* the steady-state vector for P .

3. M in Example 1 is a regular stochastic matrix because its entries are all strictly positive. So we may use Theorem 18. We already know the steady-state vector from Example 4. Thus the population distribution vectors \mathbf{x}_k converge to

$$\mathbf{q} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}$$

Eventually 62.5% of the population will live in the suburbs.

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CHAPTER 4 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.) In parts (a)–(f), $\mathbf{v}_1, \dots, \mathbf{v}_p$ are

vectors in a nonzero finite-dimensional vector space V , and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- a. The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is a vector space.

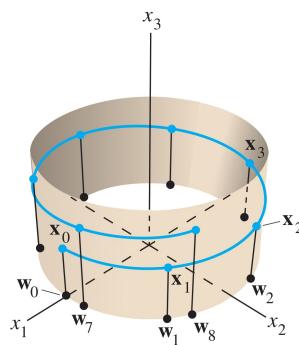


FIGURE 5

Iterates of two points under the action of a 3×3 matrix with a complex eigenvalue.

The phenomenon displayed in Example 7 persists in higher dimensions. For instance, if A is a 3×3 matrix with a complex eigenvalue, then there is a plane in \mathbb{R}^3 on which A acts as a rotation (possibly combined with scaling). Every vector in that plane is rotated into another point on the same plane. We say that the plane is **invariant** under A .

EXAMPLE 8 The matrix $A = \begin{bmatrix} .8 & -.6 & 0 \\ .6 & .8 & 0 \\ 0 & 0 & 1.07 \end{bmatrix}$ has eigenvalues $.8 \pm .6i$ and 1.07 .

Any vector w_0 in the x_1x_2 -plane (with third coordinate 0) is rotated by A into another point in the plane. Any vector x_0 not in the plane has its x_3 -coordinate multiplied by 1.07. The iterates of the points $w_0 = (2, 0, 0)$ and $x_0 = (2, 0, 1)$ under multiplication by A are shown in Fig. 5. ■

PRACTICE PROBLEM

Show that if a and b are real, then the eigenvalues of $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ are $a \pm bi$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

5.5 EXERCISES

Let each matrix in Exercises 1–6 act on \mathbb{C}^2 . Find the eigenvalues and a basis for each eigenspace in \mathbb{C}^2 .

1. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 1 \\ -8 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -5 \\ 1 & 3 \end{bmatrix}$

In Exercises 7–12, use Example 6 to list the eigenvalues of A . In each case, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation, where $-\pi < \varphi \leq \pi$, and give the scale factor r .

7. $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

8. $\begin{bmatrix} 3 & 3\sqrt{3} \\ -3\sqrt{3} & 3 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & .5 \\ -.5 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$

12. $\begin{bmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$

In Exercises 13–20, find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that the given matrix has the form $A = PCP^{-1}$.

13. $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & 5 \\ -2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix}$

17. $\begin{bmatrix} -11 & -4 \\ 20 & 5 \end{bmatrix}$

18. $\begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$

19. $\begin{bmatrix} 1.52 & -.7 \\ .56 & .4 \end{bmatrix}$

20. $\begin{bmatrix} -3 & -8 \\ 4 & 5 \end{bmatrix}$

21. In Example 2, solve the first equation in (2) for x_2 in terms of x_1 , and from that produce the eigenvector $\mathbf{y} = \begin{bmatrix} 2 \\ -1 + 2i \end{bmatrix}$ for the matrix A . Show that this \mathbf{y} is a (complex) multiple of the vector \mathbf{v}_1 used in Example 2.

22. Let A be a complex (or real) $n \times n$ matrix, and let \mathbf{x} in \mathbb{C}^n be an eigenvector corresponding to an eigenvalue λ in \mathbb{C} . Show that for each nonzero complex scalar μ , the vector $\mu\mathbf{x}$ is an eigenvector of A .

Chapter 7 will focus on matrices A with the property that $A^T = A$. Exercises 23 and 24 show that every eigenvalue of such a matrix is necessarily real.

23. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let \mathbf{x} be any vector in \mathbb{C}^n , and let $q = \overline{\mathbf{x}^T A \mathbf{x}}$. The equalities below show that q is a real number by verifying that $\bar{q} = q$. Give a reason for each step.

$$\bar{q} = \overline{\mathbf{x}^T A \mathbf{x}} = \mathbf{x}^T \overline{A \mathbf{x}} = \mathbf{x}^T A \bar{\mathbf{x}} = (\mathbf{x}^T A \bar{\mathbf{x}})^T = \bar{\mathbf{x}}^T A^T \mathbf{x} = q$$

(a) (b) (c) (d) (e)

24. Let A be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector \mathbf{x} in \mathbb{C}^n , then, in fact, λ is real and the real part of \mathbf{x} is an eigenvector of A . [Hint: Compute $\bar{\mathbf{x}}^T A \mathbf{x}$, and use Exercise 23. Also, examine the real and imaginary parts of $A\mathbf{x}$.]
25. Let A be a real $n \times n$ matrix, and let \mathbf{x} be a vector in \mathbb{C}^n . Show that $\operatorname{Re}(A\mathbf{x}) = A(\operatorname{Re} \mathbf{x})$ and $\operatorname{Im}(A\mathbf{x}) = A(\operatorname{Im} \mathbf{x})$.
26. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector \mathbf{v} in \mathbb{C}^2 .
- Show that $A(\operatorname{Re} \mathbf{v}) = a \operatorname{Re} \mathbf{v} + b \operatorname{Im} \mathbf{v}$ and $A(\operatorname{Im} \mathbf{v}) = -b \operatorname{Re} \mathbf{v} + a \operatorname{Im} \mathbf{v}$. [Hint: Write $\mathbf{v} = \operatorname{Re} \mathbf{v} + i \operatorname{Im} \mathbf{v}$, and compute $A\mathbf{v}$.]
 - Verify that if P and C are given as in Theorem 9, then $AP = PC$.

[M] In Exercises 27 and 28, find a factorization of the given matrix A in the form $A = PCP^{-1}$, where C is a block-diagonal matrix with 2×2 blocks of the form shown in Example 6. (For each conjugate pair of eigenvalues, use the real and imaginary parts of one eigenvector in \mathbb{C}^4 to create two columns of P .)

$$27. A = \begin{bmatrix} 26 & 33 & 23 & 20 \\ -6 & -8 & -1 & -13 \\ -14 & -19 & -16 & 3 \\ -20 & -20 & -20 & -14 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 7 & 11 & 20 & 17 \\ -20 & -40 & -86 & -74 \\ 0 & -5 & -10 & -10 \\ 10 & 28 & 60 & 53 \end{bmatrix}$$

SOLUTION TO PRACTICE PROBLEM

Remember that it is easy to test whether a vector is an eigenvector. There is no need to examine the characteristic equation. Compute

$$A\mathbf{x} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} a + bi \\ b - ai \end{bmatrix} = (a + bi) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector corresponding to $\lambda = a + bi$. From the discussion in this section, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ must be an eigenvector corresponding to $\bar{\lambda} = a - bi$.

5.6 DISCRETE DYNAMICAL SYSTEMS

Eigenvalues and eigenvectors provide the key to understanding the long-term behavior, or *evolution*, of a dynamical system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Such an equation was used to model population movement in Section 1.10, various Markov chains in Section 4.9, and the spotted owl population in the introductory example for this chapter. The vectors \mathbf{x}_k give information about the system as time (denoted by k) passes. In the spotted owl example, for instance, \mathbf{x}_k listed the numbers of owls in three age classes at time k .

The applications in this section focus on ecological problems because they are easier to state and explain than, say, problems in physics or engineering. However, dynamical systems arise in many scientific fields. For instance, standard undergraduate courses in control systems discuss several aspects of dynamical systems. The modern *state-space* design method in such courses relies heavily on matrix algebra.¹ The *steady-state response* of a control system is the engineering equivalent of what we call here the “long-term behavior” of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

¹See G. F. Franklin, J. D. Powell, and A. Emami-Naeimi, *Feedback Control of Dynamic Systems*, 5th ed. (Upper Saddle River, NJ: Prentice-Hall, 2006). This undergraduate text has a nice introduction to dynamic models (Chapter 2). State-space design is covered in Chapters 7 and 8.

Until Example 6, we assume that A is diagonalizable, with n linearly independent eigenvectors, $\mathbf{v}_1, \dots, \mathbf{v}_n$, and corresponding eigenvalues, $\lambda_1, \dots, \lambda_n$. For convenience, assume the eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for \mathbb{R}^n , any initial vector \mathbf{x}_0 can be written uniquely as

$$\mathbf{x}_0 = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n \quad (1)$$

This *eigenvector decomposition* of \mathbf{x}_0 determines what happens to the sequence $\{\mathbf{x}_k\}$. The next calculation generalizes the simple case examined in Example 5 of Section 5.2. Since the \mathbf{v}_i are eigenvectors,

$$\begin{aligned}\mathbf{x}_1 &= A\mathbf{x}_0 = c_1 A\mathbf{v}_1 + \dots + c_n A\mathbf{v}_n \\ &= c_1 \lambda_1 \mathbf{v}_1 + \dots + c_n \lambda_n \mathbf{v}_n\end{aligned}$$

In general,

$$\mathbf{x}_k = c_1 (\lambda_1)^k \mathbf{v}_1 + \dots + c_n (\lambda_n)^k \mathbf{v}_n \quad (k = 0, 1, 2, \dots) \quad (2)$$

The examples that follow illustrate what can happen in (2) as $k \rightarrow \infty$.

A Predator–Prey System

Deep in the redwood forests of California, dusky-footed wood rats provide up to 80% of the diet for the spotted owl, the main predator of the wood rat. Example 1 uses a linear dynamical system to model the physical system of the owls and the rats. (Admittedly, the model is unrealistic in several respects, but it can provide a starting point for the study of more complicated nonlinear models used by environmental scientists.)

EXAMPLE 1 Denote the owl and wood rat populations at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$\begin{aligned}O_{k+1} &= (.5)O_k + (.4)R_k \\ R_{k+1} &= -p \cdot O_k + (1.1)R_k\end{aligned} \quad (3)$$

where p is a positive parameter to be specified. The $(.5)O_k$ in the first equation says that with no wood rats for food, only half of the owls will survive each month, while the $(1.1)R_k$ in the second equation says that with no owls as predators, the rat population will grow by 10% per month. If rats are plentiful, the $(.4)R_k$ will tend to make the owl population rise, while the negative term $-p \cdot O_k$ measures the deaths of rats due to predation by owls. (In fact, $1000p$ is the average number of rats eaten by one owl in one month.) Determine the evolution of this system when the predation parameter p is .104.

SOLUTION When $p = .104$, the eigenvalues of the coefficient matrix A for the equations in (3) turn out to be $\lambda_1 = 1.02$ and $\lambda_2 = .58$. Corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$



An initial \mathbf{x}_0 can be written as $\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$. Then, for $k \geq 0$,

$$\begin{aligned}\mathbf{x}_k &= c_1 (1.02)^k \mathbf{v}_1 + c_2 (.58)^k \mathbf{v}_2 \\ &= c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}\end{aligned}$$

(11) becomes

$$\mathbf{x}_k = c_1(1.01)^k \mathbf{v}_1 + c_2(-.03 + .26i)^k \mathbf{v}_2 + c_3(-.03 - .26i)^k \mathbf{v}_3$$

As $k \rightarrow \infty$, the second two vectors tend to zero. So \mathbf{x}_k becomes more and more like the (real) vector $c_1(1.01)^k \mathbf{v}_1$. The approximations in equations (6) and (7), following Example 1, apply here. Also, it can be shown that the constant c_1 in the initial decomposition of \mathbf{x}_0 is positive when the entries in \mathbf{x}_0 are nonnegative. Thus the owl population will grow slowly, with a long-term growth rate of 1.01. The eigenvector \mathbf{v}_1 describes the eventual distribution of the owls by life stages: for every 31 adults, there will be about 10 juveniles and 3 subadults. ■

Further Reading

Franklin, G. F., J. D. Powell, and M. L. Workman. *Digital Control of Dynamic Systems*, 3rd ed. Reading, MA: Addison-Wesley, 1998.

Sandefur, James T. *Discrete Dynamical Systems—Theory and Applications*. Oxford: Oxford University Press, 1990.

Tuchinsky, Philip. *Management of a Buffalo Herd*, UMAP Module 207. Lexington, MA: COMAP, 1980.

PRACTICE PROBLEMS

1. The matrix A below has eigenvalues 1 , $\frac{2}{3}$, and $\frac{1}{3}$, with corresponding eigenvectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 :

$$A = \frac{1}{9} \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Find the general solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

2. What happens to the sequence $\{\mathbf{x}_k\}$ in Practice Problem 1 as $k \rightarrow \infty$?

5.6 EXERCISES

1. Let A be a 2×2 matrix with eigenvalues 3 and $1/3$ and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\{\mathbf{x}_k\}$ be a solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.
- Compute $\mathbf{x}_1 = A\mathbf{x}_0$. [Hint: You do not need to know A itself.]
 - Find a formula for \mathbf{x}_k involving k and the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 .
2. Suppose the eigenvalues of a 3×3 matrix A are 3, $4/5$, and $3/5$, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for the specified \mathbf{x}_0 , and describe what happens as $k \rightarrow \infty$.

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In Exercises 3–6, assume that any initial vector \mathbf{x}_0 has an eigenvector decomposition such that the coefficient c_1 in equation (1) of this section is positive.³

3. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .2 in equation (3). (Give a formula for \mathbf{x}_k .) Does the owl population grow or decline? What about the wood rat population?
4. Determine the evolution of the dynamical system in Example 1 when the predation parameter p is .125. (Give a formula for \mathbf{x}_k .) As time passes, what happens to the sizes of the owl and wood rat populations? The system tends toward what is sometimes called an unstable equilibrium. What do you think might happen to the system if some aspect of the model (such as birth rates or the predation rate) were to change slightly?
5. In old-growth forests of Douglas fir, the spotted owl dines mainly on flying squirrels. Suppose the predator-prey matrix for these two populations is $A = \begin{bmatrix} .4 & .3 \\ -p & 1.2 \end{bmatrix}$. Show that if the predation parameter p is .325, both populations grow. Estimate the long-term growth rate and the eventual ratio of owls to flying squirrels.
6. Show that if the predation parameter p in Exercise 5 is .5, both the owls and the squirrels will eventually perish. Find a value of p for which populations of both owls and squirrels tend toward constant levels. What are the relative population sizes in this case?
7. Let A have the properties described in Exercise 1.
 - a. Is the origin an attractor, a repeller, or a saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$?
 - b. Find the directions of greatest attraction and/or repulsion for this dynamical system.
 - c. Make a graphical description of the system, showing the directions of greatest attraction or repulsion. Include a rough sketch of several typical trajectories (without computing specific points).
8. Determine the nature of the origin (attractor, repeller, or saddle point) for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ if A has the properties described in Exercise 2. Find the directions of greatest attraction or repulsion.

In Exercises 9–14, classify the origin as an attractor, repeller, or saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Find the directions of greatest attraction and/or repulsion.

$$9. A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix} \quad 10. A = \begin{bmatrix} .3 & .4 \\ -.3 & 1.1 \end{bmatrix}$$

³ One of the limitations of the model in Example 1 is that there always exist initial population vectors \mathbf{x}_0 with positive entries such that the coefficient c_1 is negative. The approximation (7) is still valid, but the entries in \mathbf{x}_k eventually become negative.

11. $A = \begin{bmatrix} .4 & .5 \\ -.4 & 1.3 \end{bmatrix}$
12. $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$
13. $A = \begin{bmatrix} .8 & .3 \\ -.4 & 1.5 \end{bmatrix}$
14. $A = \begin{bmatrix} 1.7 & .6 \\ -.4 & .7 \end{bmatrix}$
15. Let $A = \begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}$. The vector $\mathbf{v}_1 = \begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$ is an eigenvector for A , and two eigenvalues are .5 and .2. Construct the solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ that satisfies $\mathbf{x}_0 = (0, .3, .7)$. What happens to \mathbf{x}_k as $k \rightarrow \infty$?
16. [M] Produce the general solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ when A is the stochastic matrix for the Hertz Rent A Car model in Exercise 16 of Section 4.9.
17. Construct a stage-matrix model for an animal species that has two life stages: juvenile (up to 1 year old) and adult. Suppose the female adults give birth each year to an average of 1.6 female juveniles. Each year, 30% of the juveniles survive to become adults and 80% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (j_k, a_k)$, where the entries in \mathbf{x}_k are the numbers of female juveniles and female adults in year k .
 - a. Construct the stage-matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.
 - b. Show that the population is growing, compute the eventual growth rate of the population, and give the eventual ratio of juveniles to adults.
 - c. [M] Suppose that initially there are 15 juveniles and 10 adults in the population. Produce four graphs that show how the population changes over eight years: (a) the number of juveniles, (b) the number of adults, (c) the total population, and (d) the ratio of juveniles to adults (each year). When does the ratio in (d) seem to stabilize? Include a listing of the program or keystrokes used to produce the graphs for (c) and (d).
18. A herd of American buffalo (bison) can be modeled by a stage matrix similar to that for the spotted owls. The females can be divided into calves (up to 1 year old), yearlings (1 to 2 years), and adults. Suppose an average of 42 female calves are born each year per 100 adult females. (Only adults produce offspring.) Each year, about 60% of the calves survive, 75% of the yearlings survive, and 95% of the adults survive. For $k \geq 0$, let $\mathbf{x}_k = (c_k, y_k, a_k)$, where the entries in \mathbf{x}_k are the numbers of females in each life stage at year k .
 - a. Construct the stage-matrix A for the buffalo herd, such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k \geq 0$.
 - b. [M] Show that the buffalo herd is growing, determine the expected growth rate after many years, and give the expected numbers of calves and yearlings present per 100 adults.

Next, use equation (7) to write

$$\mathbf{x}_1(t) = \begin{bmatrix} i \\ 2 \end{bmatrix} e^{-2t} (\cos 5t + i \sin 5t)$$

The real and imaginary parts of \mathbf{x}_1 provide real solutions:

$$\mathbf{y}_1(t) = \begin{bmatrix} -\sin 5t \\ 2 \cos 5t \end{bmatrix} e^{-2t}, \quad \mathbf{y}_2(t) = \begin{bmatrix} \cos 5t \\ 2 \sin 5t \end{bmatrix} e^{-2t}$$

Since \mathbf{y}_1 and \mathbf{y}_2 are linearly independent functions, they form a basis for the two-dimensional real vector space of solutions of $\mathbf{x}' = A\mathbf{x}$. Thus the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -\sin 5t \\ 2 \cos 5t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \cos 5t \\ 2 \sin 5t \end{bmatrix} e^{-2t}$$

To satisfy $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, we need $c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, which leads to $c_1 = 1.5$ and $c_2 = 3$. Thus

$$\mathbf{x}(t) = 1.5 \begin{bmatrix} -\sin 5t \\ 2 \cos 5t \end{bmatrix} e^{-2t} + 3 \begin{bmatrix} \cos 5t \\ 2 \sin 5t \end{bmatrix} e^{-2t}$$

or

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -1.5 \sin 5t + 3 \cos 5t \\ 3 \cos 5t + 6 \sin 5t \end{bmatrix} e^{-2t}$$

See Fig. 5. ■

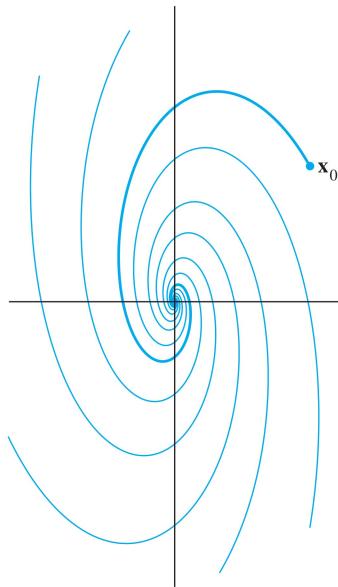


FIGURE 5

The origin as a spiral point.

In Fig. 5, the origin is called a **spiral point** of the dynamical system. The rotation is caused by the sine and cosine functions that arise from a complex eigenvalue. The trajectories spiral inward because the factor e^{-2t} tends to zero. Recall that -2 is the real part of the eigenvalue in Example 3. When A has a complex eigenvalue with positive real part, the trajectories spiral outward. If the real part of the eigenvalue is zero, the trajectories form ellipses around the origin.

PRACTICE PROBLEMS

A real 3×3 matrix A has eigenvalues $-.5$, $.2 + .3i$, and $.2 - .3i$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 + 2i \\ 4i \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 - 2i \\ -4i \\ 2 \end{bmatrix}$$

1. Is A diagonalizable as $A = PDP^{-1}$, using complex matrices?
2. Write the general solution of $\mathbf{x}' = A\mathbf{x}$ using complex eigenfunctions, and then find the general real solution.
3. Describe the shapes of typical trajectories.

5.7 EXERCISES

1. A particle moving in a planar force field has a position vector \mathbf{x} that satisfies $\mathbf{x}' = A\mathbf{x}$. The 2×2 matrix A has eigenvalues 4 and 2 , with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find the position of the particle at time t , assuming that $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

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2. Let A be a 2×2 matrix with eigenvalues -3 and -1 and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\mathbf{x}(t)$ be the position of a particle at time t . Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

In Exercises 3–6, solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $t \geq 0$, with $\mathbf{x}(0) = (3, 2)$. Classify the nature of the origin as an attractor, repeller, or saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. Find the directions of greatest attraction and/or repulsion. When the origin is a saddle point, sketch typical trajectories.

$$3. A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$$

$$4. A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

In Exercises 7 and 8, make a change of variable that decouples the equation $\mathbf{x}' = A\mathbf{x}$. Write the equation $\mathbf{x}(t) = P\mathbf{y}(t)$ and show the calculation that leads to the uncoupled system $\mathbf{y}' = D\mathbf{y}$, specifying P and D .

7. A as in Exercise 5

8. A as in Exercise 6

In Exercises 9–18, construct the general solution of $\mathbf{x}' = A\mathbf{x}$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories.

$$9. A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

$$11. A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$$

$$12. A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}$$

$$13. A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$$

$$14. A = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}$$

$$15. [M] A = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix}$$

$$16. [M] A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$$

$$17. [M] A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$$

$$18. [M] A = \begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$$

19. [M] Find formulas for the voltages v_1 and v_2 (as functions of time t) for the circuit in Example 1, assuming that $R_1 = 1/5$ ohm, $R_2 = 1/3$ ohm, $C_1 = 4$ farads, $C_2 = 3$ farads, and the initial charge on each capacitor is 4 volts.

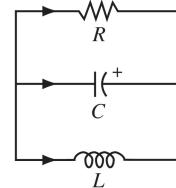
20. [M] Find formulas for the voltages v_1 and v_2 for the circuit in Example 1, assuming that $R_1 = 1/15$ ohm, $R_2 = 1/3$ ohm, $C_1 = 9$ farads, $C_2 = 2$ farads, and the initial charge on each capacitor is 3 volts.

21. [M] Find formulas for the current i_L and the voltage v_C for the circuit in Example 3, assuming that $R_1 = 1$ ohm, $R_2 = .125$ ohm, $C = .2$ farad, $L = .125$ henry, the initial current is 0 amp, and the initial voltage is 15 volts.

22. [M] The circuit in the figure is described by the equation

$$\begin{bmatrix} i'_L \\ v'_C \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/(RC) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

where i_L is the current through the inductor L and v_C is the voltage drop across the capacitor C . Find formulas for i_L and v_C when $R = .5$ ohm, $C = 2.5$ farads, $L = .5$ henry, the initial current is 0 amp, and the initial voltage is 12 volts.



SOLUTIONS TO PRACTICE PROBLEMS

- Yes, the 3×3 matrix is diagonalizable because it has three distinct eigenvalues. Theorem 2 in Section 5.1 and Theorem 5 in Section 5.3 are valid when complex scalars are used. (The proofs are essentially the same as for real scalars.)
- The general solution has the form

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1+2i \\ 4i \\ 2 \end{bmatrix} e^{(2+.3i)t} + c_3 \begin{bmatrix} 1-2i \\ -4i \\ 2 \end{bmatrix} e^{(2-.3i)t}$$

The scalars c_1, c_2, c_3 here can be any complex numbers. The first term in $\mathbf{x}(t)$ is real. Two more real solutions can be produced using the real and imaginary parts of the