Advanced Exploration 2: An Transitive Closure Algorithm

MATH2603: Discrete Mathematics

Overview: In this Advanced Exploration, you'll have a chance to dive a little deeper into mathematical relations. You'll learn about powers of relations and the transitive closure of a relation, including various ways to represent and compute these things. You'll end by learning an efficient algorithm to compute the transitive closure of a relation; this algorithm is mentioned, but not discussed, in the text.

Instructions: Sections 5.1-5.3 of the text are prerequisite material for this assignment. If you haven't read those sections (which we will discuss during class), then you should do so before continuing. Read Chapter 15 in the textbook (Depictions of Relations); you may have seen some of this material, like matrix multiplication, in other math classes. Next, complete the exercises below. Your solutions should be complete, clear, and correct. Your solutions should also be accompanied by explanations written in complete sentences that justify your work.

Instructions for submitting your work, and information on how the EMRN rubric will be applied to evaluate this exploration, are at the end of the assignment.

Background: Let S be a set and let R be a relation on S. As you saw in §15.1, the transitive closure R^+ of the relation R is the smallest transitive relation that contains R; a relation is a subset of $S \times S$, so containment in this context refers to set containment.

There are several ways to represent a relation: you can write the subset of $S \times S$; you can draw a directed graph whose vertex set is equal to S, with a directed edge from x to y if xRy; or you can construct a binary $|S| \times |S|$ matrix whose rows and columns are labeled by elements of S, with the xy entry being 1 if xRy and 0 otherwise.

The text discusses a method for constructing the directed graph for R^+ by computing the digraphs for $R, R^2, R^3, \ldots, R^{|S|}$, then taking their union. This is a relatively simple process to do by hand when $|S|$ is small, but it's not well suited for input to a computer, and it would be difficult to implement if S is large.

The Problem: Consider the relation on the set $S = \{a, b, c, d, e\}$ given by

$$R = \{(a, b), (a, c), (a, d), (b, a), (c, b), (d, e), (e, c)\}.$$

1. Make a digraph G for the relation R, then construct the adjacency matrix M for G.
2. Write a short paragraph explaining geometrically when two vertices should be connected by a directed edge in the digraph G^+ for the relation R^+.

The algorithm you will implement below is based on the following observation. There is a directed path in G connecting x to y if and only if one of the following holds:

- there is a directed edge from x to y,
- there is a directed path from x to y passing through the vertex a,
- there is a directed path from x to y only passing through vertices from $\{a, b\}$,
- there is a directed path from x to y only passing through vertices from $\{a, b, c\}$,

- there is a directed path from x to y only passing through vertices from $\{a, b, c, d\}$,
- there is a directed path from x to y only passing through vertices from $\{a, b, c, d, e\}$.

3. Construct the matrix $M^{\{a\}}$ whose xy-entry is 1 if either of the first two bullet points above are satisfied. This matrix can be obtained by modifying M, changing a zero to a one when a path from x to y passing through a exists.
4. Next construct the matrix $M^{\{a,b\}}$ whose xy-entry is 1 if any of the first three bullet points are satisfied. Then construct $M^{\{a,b,c\}}$, and so forth. Each time, use the previous matrix to construct the new matrix.
5. The last matrix $M^{\{a,b,c,d,e\}}$ is the adjacency matrix for R^+. Use this to draw the digraph for R^+.
6. Finally, write a few paragraphs describing how this algorithm works in general. Be sure to include: *what* the algorithm does (What is the input? What is the output?); and *how* the algorithm works.

Submitting your work: Your work must be neatly typed up using a system that supports mathematical notation. For example, you can use MS Word and its equation editor; or you can write your work in a Jupyter notebook using Markdown and LaTeX. Once it is written up, the work must be saved as a PDF file and then uploaded as a PDF to the area on Blackboard where the original assignment is located. Remember that the work is not actually submitted until you upload the file and click the Submit button. Grading and feedback will take place entirely on Blackboard. The following are not allowed: Submissions outside Blackboard (for example through email); files that are not in PDF form; and work that contains any handwriting, though you may *draw a diagram* neatly by hand, scan it, and include it in your submission.

Evaluation: Like all Advanced Explorations, your work will be evaluated using the EMRN rubric. Please see the statement of this rubric in the syllabus for an explanation of how it is used. When applied to this Advanced Exploration, the following criteria help to assign the grade:

- **E:** The solution consists of a clear, correct, and complete solution that includes neat figures for questions 1 & 5. The solution contains no major errors (computation, logic, syntax, or semantic); it is also exceptionally clear and the writeup is professional in its look and style. The solution would be at home in a professional lecture or publication.
- **M:** The solution consists of a clear, correct, and complete solution that includes neat figures for questions 1 & 5. The solution contains no major errors (computation, logic, syntax, or semantic) and is neatly and professionally written up.
- **R:** The solution contains at least one, but not several, major errors (computation, logic, syntax, and/or semantic) that require revision. An "R" may also be given for writeups that do not expend sufficient effort to produce a good-looking writeup.
- **N:** The solution has several significant errors; or the submission is missing large portions of the solution; or the solution is for a significantly altered version of the problem; or the submission is excessively cluttered, messy, difficult to read, or handwritten.