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A P.D.E. for Pricing Stocks Options when Interest Rates are Stochastic

by

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Abstract

It is long known that the Black-Scholes model assumes that interest rates are constant over the duration of the option. The assumption is incorrect, as in the financial market, interest rates are determined by several factors and frequently change over time. This thesis will derive a partial differential equation for the price of an option on a stock when interest rates are stochastic. An overview of partial differential equations as well as a short description of stochastic processes for finance in continuous time is listed. A short description of interest rate theory is also added for completeness. The derived PDE will finally be transformed into a heat type equation.

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Chapter 1

Introduction

The fundamental assumption about the Black-Scholes model (1973) [6], is that risk free interest rates are known and constant. However, this is not the case as interest rates are based upon prevailing market conditions. Researchers and financial practitioners have been aware of this and much work has been done in modeling bond yield curves (also known as the term structure of interest rates). This paper will derive the Black-Scholes PDE when interest rates are stochastic and examine the pricing of stock options in a stochastic interest rate environment.

1.1 Literature Review

1.1.1 The early beginnings of derivatives pricing

Louis Bachelier pioneered the subject of mathematical finance. He was the first to apply the mathematical rigor to finance in an effort to evaluate options under a Brownian motion. The paper provided the framework for the operations of a stock exchange, describing the various financial instruments that are still in use today. It described two kinds of forwarded dated contracts, namely the stock option and the forward contract.

It must be stated however, that the concept of derivatives were in use long before Bachelier attempted to use applied mathematics to derive prices. Indeed, the Japanese are credited for the first commodities exchange (a forward contracts exchange) with the creation of the Dojima Rice Exchange in 1697. The "Tulip Bulb Mania" in the Netherlands had several owners of options on tulips losing immense amount of money.

The price to pay to enter into the commitments between the parties was simply bartered, and an agreed price reached between them. There was no explicit pricing formula or model. Bachelier (1900) [2] in his PhD thesis was the first to propose the price of an option. The model was not sound as it allowed for negative asset prices as pointed out by Merton (1973) [21] and Smith (1976) [29]. They also pointed out that his formula had ignored any discounting (i.e. the time value concept of money).

Sprenkle (1961) [30] resolved this by using an approach that assumed lognormal returns on assets. His model made allowances for an average rate of growth and a degree of risk averison parameter. While the formula is very similar to the modern option pricing formula present by Black and Scholes (1973), Sprenkle's formula was not used much by the industry. It was thought to be overly complicated due to the amount of parameters that would need to be estimated and not much information was provided by him in his paper on how to compute them.

A few years later, Boness (1964) improved on Sprenkle's work and added discounting to the terminal stock price to account for the time value of money while using the expected rate of return of the asset. Samuelson (1965) [26] suggested further additions to the model, adding a parameter known as an average rate of growth of the option. After noticing how important interest rate are in economic theory, he with the help of his student started investigating further applications. Samuelson and Merton (1969) [27] created an option price model as a function of the asset price and a discount rate. They conceptualized hedging and in the paper, the option value is determined by a hedging strategy where an investor has an option on some amount of stocks.

The Samuelson and Merton (1969) [27] paper paved they way for the modern Black and Scholes (1973) formula. The Black-Scholes formula for the option price was explicitly connected with a hedging strategy. They devised that the expected return on the price of an option must be the risk-free interest rate. The option position in the strategy be dynamically completely hedged by holding a certain amount of the stock.

Eric Benhamou (2015) [3] listed three advantage of using the Black and Scholes (1973) formula is:

1. A hedging strategy that is explicit for the replication of the option that only depends on observable quantities (e.g. volatility, risk-free interest rates, strike price, etc).

2. The risk free rate is considered universal, therefore there should exist a universal price for the option. This allows the user of the formula to ignore risk-aversion as the price must be the same for all buyers. The previous models had various parameters to attempt to account for the different types of investors.

3. The only parameter to estimate is the volatility, this it is much simpler than all the others proposed before.

1.1.2 Background to Interest Rates

There are two main families of models for the term structure of interest rates: equilibrium models and no-arbitrage models. Equilibrium models specifies that the risk-free interest rate is a solution of a stochastic differential equation. These models make use of the local expectation hypothesis and Girsanov theorem to achieve risk-neutrality and be arbitrage free. Merton's model (1973) was one of the first short-rate models. It was modeled by a stochastic process with constant drift and diffusion coefficients. With the development of interest rate derivatives, researchers and practitioners started investigating several other types of models for the short-rate and the term structure of interest rates. Other early models (which are all one-factor short rate models) include Vasicek (1977), Rendleman-Bartter (1978), Dothan (1978) and Brennan and Schwartz (1979).

While equilibrium models can easily be used in practice, no-arbitrage models are more computationally complex and so are harder to implement. These models rely on the use of the current state of the term structure of interest rates while equilibrium models will produce the term structure of interest rates as an output. The drift is in general dependent on time as opposed to equilibrium models. Early models of this type include Ho-Lee (1986), Hull-White (1990), Black-Derman-Toy (1990) and Black-Karasinki (1991). The no-arbitrage model has been generalized with the Heath-Jarrow-Merton (1992) (HJM model) framework [15].

Another no-arbitrage model of interest rates is the Libor Market Model (LMM model) by Brace, Gatarek, and Musiela, Jamshidianm and Miltersen, Sandmann, and Sondermann. The LMM model is expressed in terms of forward rates that are observable on the market while the HJM model is expressed in terms of instantaneous forward rates which is unnatural as this is not observed on the market.

1.1.3 Black-Scholes Model with Stochastic Interest Rates

Several papers have been written that describe how to price options with stochastic interest rates. One of the first was by Rabinovitch (1989) [24]. This paper describes using a Ornstein-Uhlenbeck process (such as the Vasicek or CIR models) and derive an explicit closed-form result for options on stocks and bonds with stochastic interest rates.

The first paper to use the HJM framework for pricing options with stochastic interest was done by Amin and Jarrow (1992) [1]. This was derived by generalizing the framework and embedding it into an economy that contains an arbitrary amount of risky assets. They also accommodated the pricing of American options with stochastic interest rates in the paper.

An empirical study of option prices with stochastic interest rate was done by Yong-Jin Kim (2002) [18]. He concluded that incorporating stochastic interest rates into option pricing models has little effect on pricing performances on the data from the Japanese market where long-term dated options are not available. Kung and Lee (2009) [19] has derived a partial differential equation for the Black-Scholes model with stochastic interest rates. The approach used assumed the rate followed the Merton Model, and that the Option price was dependent on the price of a zero coupon bond following this interest rate model. The paper also transformed the partial differential equation into to the standard heat equation. Fang Haowen (2012) [14] then derived the partial differential equation for the Black-Scholes model with stochastic interest rate by assuming that the interest rate is based on the Vasicek model. It assumed that the option price was dependent on the interest rate unlike Kung and Lee before. The equation was then transformed to the a heat type equation. Finally, Shankar Subramaniam (2012) [32] priced an option with Cox-Ingersoll-Ross model with numerical results from the Monte-Carlo method.

1.1.4 Motivation and Problem Formulation

Traditionally, options on stocks have always been priced with a short term maturity date. Short term options expiry cycles are 3 months, 6 months and 9 months. In 1990, a new type of option was introduced by the Chicago Board Options Exchange (CBOE). Long Term Equity Anticipation Security (LEAPS) are options that have maturities which extends for more than 9 months. Another type of option that is more than 9 months are employee stock options. These options are over-the-counter contracts between an employer and employee and can have maturity dates up to 10 years into the future.

The two examples above are enough to conclude that pricing stock options in a stochastic interest rate environment would be beneficial to financial institution that provide these financial contracts. The approach for pricing of stock options with stochastic interest rates in this paper will be to use the local expectation hypothesis in bond pricing followed by a change of numéraire technique. This is different when compared to Merton (1973) [21], Kung and Lee (2009) [19] and Haowen (2012) [14]. The differences will be discussed in the paper. The paper will also derive a finite difference method for finding a solution to the PDE resulting from the change of numéraire technique and will be the contribution to the scientific community.

Chapter 2

Overview of Partial Differential Equations via the Heat Equation

Partial differential equations are used extensively by applied mathematicians modeling physical phenomenon. Differential equations came into being after the invention of calculus by Newton and Leibniz (see Newton and Colson (1736) [23]). PDEs are mathematical equations that will relate a function to it's derivatives, and specifically, a partial differential equations will relate a multivariate functions and their partial derivatives. This chapter will not provide an in depth overview of differential equations (an in depth overview can be found in Evans (1998) [11] and Strauss (1992) [31]). Instead what will be provided is the necessary tools that is required in understanding the partial differential equation used for pricing of financial contracts arising mathematical finance.

2.1 Types of Partial Differential Equations

Partial Differential equations can be classified into three board categories: Parabolic, hyperbolic and elliptic. All three categories are useful in mathematical finance, however this paper is concerned with that of parabolic PDEs as the Black-Scholes equation is an example of such. Specifically, it is a second order linear partial differential equation.

The general linear second order partial differential equation is:

$$L[u] = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + Fu = 0$$

Where $u = u(x,t)$ with t increasing from $t = 0$. Coefficeints A through F may also be functions of x and t. Let $\varepsilon = B^2 - 4AC$, if $\varepsilon < 0$, the linear PDE is elliptic, if $\varepsilon > 0$ the linear PDE is hyperbolic and finally, if $\varepsilon = 0$ the linear PDE will be parabolic.

The Laplace Equation (Elliptic PDE):

$$\frac{\partial^2 u}{\partial x^2} = 0$$

The Heat Equation (Parabolic PDE):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The Wave Equation (Hyperbolic PDE):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2.2 Solutions and Boundary Conditions for the Standard Heat Equation

The Heat Equation models diffusion type phenomena, e.g. the flow of heat from one position to another in an object. The solution to the Heat Equation, therefore, will be a function or procedure that will give the temperature at a given location x at time t. A function is considered a *classical solution* to a PDE if the relevant partial derivatives exists (i.e. it is sufficiently smooth) such that the equation holds at every point in the domain when the function and its derivatives are plugged into the PDE.

The Heat Equation will only be solvable if the correct boundary conditions are met. Without boundary conditions, there are infinitely many solutions. If we consider a rod of length l with insulated sides and given an initial temperature distribution of $f(x)$ degrees C, for $0 < x < l$. We will find the temperature $u(x,t)$ at subsequent times $t > 0$ if the ends of the rod is kept at $0°C$.

The Heat Equation (see derivation in Widder (1976) [34]):

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad \forall x \in [0,1], \quad \forall t > 0,$$

with initial conditions:

$$u(x,0) = f(x), \quad \forall x \in [0,1],$$

and boundary conditions:

$$u(0,t) = u(L,t) = 0, \quad \forall t > 0$$

It is expected that $u \to 0$ as $t \to \infty$.
For the boundary value problem to be solvable, the **well posed** criterion is required which states:

1. There exists a solution (Existence).

2. There is only one solution (Uniqueness).
3. The solution relies only upon the data given in the problem in a continuous stable manner (Stability).

By the Maximum Principle, it can be shown that the Heat Equation with boundary conditions is well posed (see Strauss (1992) [31]). Because of the maximum principle, numerical solutions can be implemented that solves the Heat Equation.

Chapter 3

Stochastic Processes for Finance in Continuous Time

This chapter is based on the book by Shreve (2004) [28] and Björk (2004) [4]. Some important definitions, theorems and results that will be used throughout the rest of the thesis will be mentioned here.

3.1 Basic Stochastic Calculus

Definition 3.1. The Normal distributed random variables

A normal (Gaussian) random variable, $X \sim N(\mu, \sigma^2)$ defined on the real axis, has the probability density function (pdf)

$$\phi_X(x) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and the cumulative density function (cdf) $F_X(x) = \int\limits_{-\infty}^{x} \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dt$

When $\mu = 0$ and $\sigma = 1$, the result is called the *Standard Normal Distribution*.

Definition 3.2. The Log normal random variable

Let us assume we have $X \sim N(\mu, \sigma)$, if $Y = \exp[X]$, then Y is called a log normal variable and thus defined by

$$\phi_Y(y) = \phi_X\left(\frac{\ln y - \mu}{\sigma}\right)$$

Definition 3.3. Stochastic Process

A *Stochastic Process* is a collection of random variables on a probability space $(\Omega, \mathscr{F}, P)$.

$$(X_t)_{t \in T}, T \in [0, \infty)$$

and for each $t \in T$ we have that

$$\omega \to X_t(\omega), \omega \in \Omega$$

Definition 3.4. Brownian Motion

The *Stochastic Process* $W = (W_t : t \geq 0)$ on the probability space $(\Omega, \mathscr{F}, P)$ is called $Brownian Motion$ if the following properties hold almost surely (i.e. for an event $E \in \mathscr{F}$ happens almost surely if $P[E] = 1$):

1 $W_0 = 0$.

2 The increments are independent and stationary, i.e. if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables.

3 The increments of $W_{t+h} - W_t$ are normally distributed, $N(0,h)$ for all $h,t \geq 0$

4 W_t has continuous trajectories.

Definition 3.5. 1-dimensional Itō Process

Let W_t be a $Brownian Motion$, then the Itō Process X_t on the probability space $(\Omega, \mathscr{F}, P)$ is defined as an adapted stochastic process that can be expressed as the sum of an integral with respect to both the $Brownian Motion$ and time satisfying:

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dWs$$

Such that the following conditions hold almost surely

$$P[\int_0^t \sigma(X_s, s)^2 ds < \infty, \forall t \geq 0] = 1$$

$$P[\int_0^t \mu(X_s, s)ds < \infty, \forall t \geq 0] = 1$$

The process can be represented as a stochastic differential equation as:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

Theorem 3.6. *Existence and uniqueness*

The the stochastic differential equation form of the Itō Process in Definition 3.5 will have a solution that exists and is unique. Let μ and σ satisfy the following:

$$|\mu(x,t)| + |\sigma(x,t)| \leq C(1 + |x|) \;\; for \;\; x \in \mathbb{R}, \;\; t \in [0,T]$$

for some constant C, which guarantees global existence and Lipshitz condition

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq D|x - y|$$

$$x, y \in \mathbb{R}, \;\; t \in [0,T]$$

for some constant D, which guarantees local uniqueness, where $\mathscr{F}_t$ is the filtration generated by $W = (W_t : t \in \mathbb{R})$, then the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

has a unique t-continuous solution, i.e. the solution is continuous in time.

Theorem 3.7. *Itō's formula*

Let X_t be a stochastic process given by Definition 3.3. Further, let $g(t,x)$ be a twice differentiable function in space on, then $Y_t = g(t,X_t)$ is an Itō process and $Y_t \in \mathbb{R}^2$

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2$$

and from here one may use the following rules:

$$dt \times dt = 0$$

$$dt \times dW_t = 0$$

$$dW_t \times dW_t = dt$$

Definition 3.8. Geometric Brownian Motion

A geometric Brownian motion is defined as solutions to $dS_t = \mu S_t dt + \sigma S_t dW_t$ with Theorem 3.6 holding. We assume that the daily asset returns follows a normal distribution

$$Y_t = \ln(\frac{S_t}{S_0})$$

Using Theorem 3.7, we get

$$dY_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

with solution

$$Y_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

Solving we get

$$S_t = S_0 \exp[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t]$$

The expected value is $E[S_t] = S_0 e^{rt}$, S_t is log normally distribution and the following holds

$$\ln(\frac{S_t}{S_0}) \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$$

Definition 3.9. Correlation of Stochastic Processes

Let us assume we have 2 stochastic processes that follows the Geometric Brownian motion. The realized pairwise correlation of the asset returns is by the estimator

$$\rho_R(S_1,S_2) = \frac{\sum_{i=1}^{N}(\log\frac{S_{1,t_1}}{S_{1,t_{i-1}}} - \widehat{S}_1)(\log\frac{S_{2,t_i}}{S_{2,t_{i-1}}} - \widehat{S}_2)}{\sqrt{\sum_{i=1}^{N}(\log\frac{S_{1,t_1}}{S_{1,t_{i-1}}} - \widehat{S}_1)^2 \sum_{i=1}^{N}(\log\frac{S_{2,t_i}}{S_{2,t_{i-1}}} - \widehat{S}_2)^2}}$$

with

$$\widehat{S}_i = \frac{1}{N}\sum_{i=1}^{N}\log\frac{S_{t_i}}{S_{t_{i-i}}}$$

From this it is easily seen that the covariance $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ and $\rho_{ii} = 1$.

Definition 3.10. Construction of Correlated Brownian Motion

Let us assume we have n normally distributed random variables i.e., $\mathbf{X} = (X_1, ..., X_n)^\top$ where $X_i \sim N(0,1)$ with drift $\boldsymbol{\mu} = (\mu_1 ... \mu_n)^\top$ and variance-covariance matrix $\boldsymbol{\Sigma}$ then:

$$\boldsymbol{\mu} + \mathrm{Chol}(\boldsymbol{\Sigma})\mathbf{X} \sim \mathbf{N}_n(\mu, \boldsymbol{\Sigma})$$

where Chol represents the Cholesky Decomposition of the Variance-Covariance matrix.
The simplest case with $n = 2$ yields:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 [\rho dW_1 + (1-\rho)^{1/2} dW_2]$$

Definition 3.11. The Black–Scholes-Merton Model

The Black–Scholes World

1. The stock price follows a Geometric Browning Motion with constant drift and volatility.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transactions costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no risk-less arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest r is constant and the same for all maturities.

Definition 3.12. Risk Free Asset

This is an asset of which the price follows the following process:

$$dB(t) = rB(t)dt$$

with $r \geqslant 0$. This r is called the risk free interest rate.

Definition 3.13. Martingale

A stochastic process $\{M_t\}_{t \geqslant 0}$ on $(\Omega, \mathscr{F}, P)$ is a martingale with respect to the filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$ on $(\Omega, \mathscr{F}, P)$ if the following properties holds:

1 M_t is $\mathscr{F}_t$ - measurable or M_t is $\mathscr{F}_t$ adapted for all t

2 $E[|M_t|] < \infty$ for all t

3 $E[|M_t|\mathscr{F}_s^M] = M_s$ for all $0 \leqslant s < t < \infty$

Theorem 3.14. *Girsanov Theorem*

We assume that we have an objective probability measure P (i.e. there is a specified P-Dynamics of all asset prices in the primary market or for this thesis, the interest rate). Let W^P be a Brownian Motion following the P-Dynamics on $(\Omega, \mathscr{F}, P, \underline{\mathscr{F}})$ and let φ be an adapted coloumn vector process. A fixed T is chosen and a process L is defined on $[0,T]$ by:

$$dL_t = \varphi_t^* L_t dW_t^P$$
$$L_0 = 1$$

i.e.

$$L_t = \exp^{\int_0^t \varphi_s^* dW_s^P - \frac{1}{2}\int_0^t ||\varphi_s||^2 ds}$$

Assume that:

$$E^P[L_T] = 1$$

and define the new probability measure Q on $\mathscr{F}_T$ by:

$$L_T = \frac{dQ}{dP}$$

Then:

$$dW_t^P = \varphi_t dt + dW_t^Q$$

where W^Q is a Q-Dynamic Brownian Motion. This Q is known as the risk-neutral probability measure defined such that the asset price and the discounted asset price are equal under this measure.

Definition 3.15. Contingent Claim

This is a stochastic variable $X = \Phi(Z)$ where Z is a stochastic process driving the asset price and Φ is a desired pay off function.

Theorem 3.16. *Risk Neutral Valuation Formula*

Given the contingent claim $\Phi(S_t)$ for an European option, the arbitrage free price $\Pi(t,\Phi)$ of this claim is given $\Pi(t,\Phi) = e^{-rT}E^Q[\Phi(S_t)|\mathscr{F}_t^S]$ where Q denotes the risk-neutral martingale measure using a risk free asset as a numerate and $\mathscr{F}_t^S$ the filtration which contains all the information about S up until time t. This involves making use of the Girsanov theorem to move from real probability measure P on the contingent claim.

Theorem 3.17. *Feynman-Kac Theorem*

Assuming we have an asset with geometric Brownian motion $dX = \mu X dt + \sigma X dW$ as in Definition 3.5. Let function $f(t,X(t))$ be the price at time t of the derived security that matures at end time T with final price being a contingent claim given at $f(T,X(T)) = \Phi(X(T))$ and satisfies the following partial differential equation:

$$\frac{\partial f(t,x)}{\partial t} + \mu \frac{\partial f(t,x)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(t,x)}{\partial x^2} = rf(t,x)$$

Then there is a unique solution that is a conditional expectation:

$$C(t,x) = E^Q[\Phi(X(T))|X(t) = x]$$

This theorem is very important as it links a partial differential equation and a stochastic process.

Theorem 3.18. *Black–Scholes Partial Differential Equation*

The Black–Scholes equation, is a partial differential equation that describes the price of the option over time and is derived as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where σ is the volatility of the asset returns, r is the risk free interest rate, S is the asset price modeled by a geometric Brownian motion process with the drift r in the risk-neutral world and V is the price for the option.

If we set a contingent claim such that the pay off $\Phi(S(T)) = \max[S(T) - K, 0]$ and using Itō formula (Theorem 3.7), Risk Neutral Valuation Formula (Theorem 3.16) and the Feynman-Kac theorem (Theorem 3.17), we will get conditional expectation of:

$$V(t,S) = e^{-r(T-t)}E^Q[\Phi(S(T))|\mathscr{F}_t^S], S(t) = S]$$

There exists a closed form solution, the Black–Scholes Formula:

$$V(t,S) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln[S/K] + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Chapter 4

Interest Rate Theory

4.1 Bond Pricing

A zero-coupon bond is a financial asset that promises a single payment upon maturity in the future at an agreed interest rate. By modeling the evolution of the short rate r_t, the price of a zero-coupon bond $P(t,T)$ at any time t maturing at time T can be found as:

$$P(t,T) = E\left[\exp\left(-\int_{t=s}^{T} r_s ds\right) |\mathscr{F}_s\right] \tag{4.1}$$

The instantaneous forward rate as now formulated as:

$$f(t,T) = -\frac{\partial}{\partial T}\ln(P(t,T)) \tag{4.2}$$

A coupon bearing bond can be decomposed into several zero coupon bond and the price calculated after (see Jamshidian (1989) [17]).

4.2 Interest Rates

The interest rate at which an entity can borrow money for a short amount of time from time t, is known as the spot rate (denoted by r_t or $r(t)$). The purpose of a short model is to describe the future evolution of the interest rate by the future evolution of the spot rate. Historically, short rate models were the earliest successful models of the term structure of interest rates. It takes the stochastic state variable to be the instantaneous forward rate. The stochastic differential equation describing the dynamics of the spot rate is in the form given by Definition 3.3:

$$dr_t = A(r_t,t)dt + B(r_t,t)dW_t \tag{4.3}$$

There are several types of short rate models. They are classified in terms of how many random factors that determines the evolution of the spot rate over time. They are classified as: one-factor, two-factor, and multi-factor models.

4.2.1 One-Factor Short Rate Models

One-Factor continuous time model of the short-term interest rate, is a stochastic process of the form:

$$dr_t = (\alpha - \beta r_t)dt + \sigma r_t^{\gamma} dW_t \tag{4.4}$$

where r is the interest rate, dW is a standard Brownian motion, σ is a constant term and α, β, and γ are parameters. There are several one-factor models, and a description of some of the most commonly used models will now be presented.

Vasicek model

Oldrich Vasicek introduced this model in 1977 [33]. We will get this model when we set γ to 0 in Equation 4.4.

$$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t \tag{4.5}$$

This can be rewritten as:

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{4.6}$$

where:

- a is the *Speed of Mean Reversion*, $0 \le a \le 1$.
- b is the *Mean Reversion Level*.
- $\alpha = ab$.
- r_t is the *short rate* at time t.
- σ is the volatility.
- W_t is the Brownian Motion Process at time t.

Vasicek in his paper [33] demonstrated that the price of a zero-coupon bond that pays $1 at maturity is given by:

$$P(t,T) = A(t,T)\exp(-B(t,T)r_t) \tag{4.7}$$

where:

$$B(t,T) = \frac{1 - \exp(-a(T-t))}{a} \tag{4.8}$$

and:

$$A(t,T) = \exp\left[\frac{(B(t,T) - T + t)(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right] \tag{4.9}$$

Cox-Ingersoll-Ross model

Cox et al. (1985) [8] extended the model presented by Vasicek, and occurs when $\gamma = 0.5$. The CIR model is therefore:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{4.10}$$

The addition of the square root term ensures that the spot rate is always positive unlike the Vasicek model. The short rate will be positive for as long as a and b are positive values. As the interest rate increase, the standard deviation will also increase. The price of the zero coupon bond can be found using Equation (4.7), where:

$$B(t,T) = \frac{2\exp((h(T-t)-1))}{(h+a)(\exp((h(T-t)-1)) + 2h} \tag{4.11}$$

and:

$$A(t,T) = \left(\frac{2h\exp((a+h)(T-t)/2)}{(h+a)(\exp((h(T-t)-1)) + 2h}\right)^{2ab/\sigma^2} \tag{4.12}$$

with:

$$h = \sqrt{a^2 + 2\sigma^2} \tag{4.13}$$

Ho-Lee model

The Ho-Lee (1986) model is the first no-arbitrage model for interests proposed to the financial community [16]. The model occurs when β and γ of Equation 4.4 is set 0 and α is is now a function of time. This results in:

$$dr_t = \alpha(t)dt + \sigma dW_t \tag{4.14}$$

where $\alpha(t)$ is time dependent drift and the standard deviation σ is constant. The choice of α is based on the instantaneous short rate standard deviation as well as it being a fit of the initial structure. We will have:

$$\alpha(t) = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t \tag{4.15}$$

Equation (4.15) above shows that the average direction that the short rate will be moving in the future is approximately equal to the slope of the instantaneous forward curve. The slope of the forward curve defines the average direction that the short rate is moving at any given time. The price at time t of a zero-coupon bond with a single payment at time T is given by:

$$P(t,T) = A(t,T)\exp(-r_t(T-t)) \tag{4.16}$$

and:

$$\ln(A(t,T)) = \ln\left(\frac{P(0,T)}{P(0,t)}\right) + (T-t)\frac{\partial P(0,t)}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2 \tag{4.17}$$

Hull-White model

When mean reversion is added to the Ho-Lee model, the Hull-White model will result. It is an extension of the Vasicek model with time dependence added to the α term in Equation 4.5:

$$dr_t = (\alpha_t - \beta r_t)dt + \sigma dW_t \tag{4.18}$$

The α_t term can be found by:

$$\alpha(t) = \frac{\partial f(0,t)}{\partial t} + \beta f(0,t) + \frac{\sigma^2}{2a}(1 - e^{2at}) \tag{4.19}$$

The bond price at time T is given by:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{4.20}$$

where,

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{4.21}$$

$$\ln(A(t,T)) = \ln\left[\frac{P(0,T)}{P(0,t)}\right] + B(t,T)f(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})(e^{2at} - 1) \tag{4.22}$$

Black-Derman-Toy model

The Black-Derman-Toy (1990) model [5] is a very popular model due to the fact that is log-normal. This attribute ensures that interest rates are non-negative. Several authors have shown that the implied continuous time limit for the BDT model is given by the following stochastic differential equation Nazir(2009) [22]:

$$d(\ln(r_t)) = \left(a(t) + \frac{\sigma^{'}(t)}{\sigma(t)}\right)dt + \sigma(t)dW_t \tag{4.23}$$

where $\sigma^{'}$ is the first derivative of the volatility function. If $\sigma(t)$ is a descreasing fucntion, then the derivative will be smaller than 0 and the model will satisfy the mean reversion property. If $\sigma(t)$ is an increasing function, then the derivative will be greater than 0 and the model will not have a mean reversion property. Finally, if the derivative is of σ is 0, the model becomes specific:

$$d(\ln(r_t)) = a(t)dt + \sigma dW_t \tag{4.24}$$

The solution to this diffrential equation will yield:

$$\frac{r_t}{r_0} = \exp\left(\int_0^t a(s)ds + \sigma(W_t - W_0)\right) \tag{4.25}$$

Chapter 5

Black-Scholes Model with Stochastic Interest Rates

5.1 Model Derivation

In deriving the model all requirements of the Black-Scholes-Merton model in Definition 3.11 be the same expect for the requirement about the interest rate. Assume we have 2 stochastic process representing an asset price S and interest rate r:

$$dS_t = \mu S_t dt + \sigma_S S_t dW_t^S \tag{5.1}$$

$$dr_t = \alpha(r_t, t)dt + \sigma_r(r_t, t)dW_t^r \tag{5.2}$$

where W_t^S and W_t^r for all $t \geq 0$ are standard Browning motion with covariance $cov(dW_t^S, dW_t^r) = \rho dt$. We also assume that the model is in the risk-neutral world Q using Girsanov Theorem given by Theorem 3.14.

Let $V_t = V(S_t, r_t, t)$ represent the price of the call on an European option, with $V_t = (S_t - K)^+$. The the subcript t will be dropped throughout for convenience when needed. Using similar arguments as Haowen (2012) [14], by Δ hedging the derivation of the Black-Scholes with stochastic interest rates begins:

$$\Pi_t = V_t - \Delta_S S_t - \Delta_P P_t \tag{5.3}$$

Where Π_t represents the total worth of the portfolio, with Δ_S of stock priced at S_t and Δ_P of zero-coupon bond priced at P_t. In a risk free economy, the portfolio is self financing and we have:

$$d\Pi_t = r_t \Pi_t dt \tag{5.4}$$

Here, this derivation will differ to that of Haowen (2012) [14]. He assumes that the bond price can be represented as a solution to a stochastic differential equation, in this paper, it will be derived explicitly and the local expectation hypothesis used to link the price of the bond with the risk free interest rate.

Let $P_t = P(r_t, t; T)$, by Itō's Lemma (in depth explanation of this lemma is in Chapter 2 of Hanson (2007) [13]),

$$dP_t = \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(dr)^2 \tag{5.5}$$

According to Equation (5.2):

$$dP_t = \left[\frac{\partial P}{\partial t} + \alpha(r_t,t)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2(r_t,t)\frac{\partial^2 P}{\partial r^2}\right]dt + \sigma_r(r_t,t)\frac{\partial^2 P}{\partial r^2}dW_t^r \tag{5.6}$$

By Local Expectations Hypothesis (see Equation (15) of Cox et al. (1981) [7]) we will have:

$$r_t P_t = \frac{\partial P}{\partial t} + \alpha(r_t,t)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2(r_t,t)\frac{\partial^2 P}{\partial r^2} \tag{5.7}$$

and

$$-\sigma_P(\sigma_r,t)P = \sigma_r(r_t,t)\frac{\partial P}{\partial r} \tag{5.8}$$

Therefore

$$dP = r_t P dt - \sigma_P(\sigma_r,t)P dW_t^r \tag{5.9}$$

Equation (5.8) is negative because of the inverse relationship between bond prices and yield (Theorem 1 of Malkiel (1962) [20]). The Local Expectation Hypothesis is applicable here as we are in the risk-neutral measure Q.

Using Itō's Lemma on V_t:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr)^2 + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{\partial^2 V}{\partial S \partial r}dSdr \tag{5.10}$$

$$\frac{\partial^2 V}{\partial r^2}(dr)^2 = \sigma_r^2\frac{\partial^2 V}{\partial r^2}dt \tag{5.11}$$

$$\frac{\partial^2 V}{\partial S^2}(dS)^2 = \sigma_S^2 S^2\frac{\partial^2 V}{\partial S^2}dt \tag{5.12}$$

$$\frac{\partial^2 V}{\partial S \partial r}dSdr = \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r}dt \tag{5.13}$$

Making these substitutions into Equation (5.10):

$$dV_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2\frac{\partial^2 V}{\partial r^2} + \rho\sigma_S\sigma_r S\frac{\partial^2 V}{\partial S \partial r}\right]dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial S}dS \tag{5.14}$$

The incremental change in portfolio Π is:

$$d\Pi = dV_t - \Delta_S dS_t - \Delta_P dP_t$$

Substituting into $d\Pi$ Equations (5.14) and (5.9):

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \rho\sigma_S\sigma_r S \frac{\partial^2 V}{\partial S \partial r}\right] dt$$

$$+ \left[\frac{\partial V}{\partial S} - \Delta_S\right] dS + \alpha(r_t,t)\frac{\partial V}{\partial r} dt + \sigma_r(r_t,t)\frac{\partial V}{\partial r} dW_t^r - \Delta_P\left[r_t P dt - \sigma_P(\sigma_r,t) P dW_t^r\right] \quad (5.15)$$

For the the portfolio to be a riskless hedge, we see from the equation above that $\Delta_S = \frac{\partial V}{\partial S}$ and $\Delta_P = \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r}$. This assumptions differs from that of Haowen (2012) [14]. This assumption makes better intuitive logic as change in bond price should be sensitive to the change in option price with respect to interest rate. Haowen (2012) [14] used the change in option price with respect to time. Using Equation (5.8), and simplifying, Equation (5.15) now becomes:

$$d\Pi_t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \rho\sigma_S\sigma_r S \frac{\partial^2 V}{\partial S \partial r} + \alpha(r_t,t)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt \quad (5.16)$$

Equation (5.4) now becomes:

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \rho\sigma_S\sigma_r S \frac{\partial^2 V}{\partial S \partial r} + \alpha(r_t,t)\frac{\partial V}{\partial r} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt$$

$$= \left[r_t V - r_t S \frac{\partial V}{\partial S} - \frac{\partial V}{\partial r} / \frac{\partial P}{\partial r} r_t P\right] dt \quad (5.17)$$

Equating the coefficients of dt in Equation (5.17), a Partial Differential Equation for the Black-Scholes with Stochastic Interest Rates is derived as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \rho\sigma_S\sigma_r S \frac{\partial^2 V}{\partial S \partial r} + \alpha(r_t,t)\frac{\partial V}{\partial r} + r_t S \frac{\partial V}{\partial S} - r_t V = 0,$$

$$r_t \in \mathbb{R}, \quad S \in \mathbb{R}^+, \quad t \in [0,T] \quad (5.18)$$

as

$$t = T, \quad V(S,t,T) = (S_T - K)^+, \quad (r \in \mathbb{R}, S \in \mathbb{R}^+) \quad (5.19)$$

5.2 Conversion to Heat Equation

The Risk Neutral Valuation Formula given by Theorem 3.16 (more details in Chapter 10 of Björk (2004) [4]) and the Feynman-Kac theorem given by Theorem 3.16, results in:

$$V_t = E^Q\left[e^{-\int_t^T r_\tau d\tau}(S_T - K)^+ | r(t) = r_t, S(t) = S_t\right] \quad (5.20)$$

This will be a problem when attempts are made to find a solution. Computationally, to find the expected value, it will be required that the joint distribution of the stochastic interest rate and the asset under Q be found and we integrate with respect to this distribution, i.e, computation of a double integral. The next step in the derivation is to use a change of numéraire to assist in reducing computational complexity.

By using a zero coupon bond as a numéraire, a T measure can be defined (Theorem 2 of Geman et. al (1995) [12]) resulting in:

$$\frac{V_t}{P(r_t,t,T)} = E^T\left[\left(\frac{S_T}{P(r_T,T,T)} - K\right)^+\right] \tag{5.21}$$

This can be rearranged as:

$$\widehat{V}_t = E^T\left[\left(\widehat{S}_T - K\right)^+ |\widehat{S}(t) = S_t\right] \tag{5.22}$$

where:

$$\widehat{V}_t = \frac{V_t}{P(r_t,t,T)} \tag{5.23}$$

$$\widehat{S}_t = \frac{S_t}{P(r_t,t,T)} \tag{5.24}$$

Therefore, to solve problem (5.18) with (5.19), a transformation of independent variable y can be used where:

$$y = \frac{S_t}{P(r_t,t,T)} \tag{5.25}$$

and unknown function:

$$\widehat{V}(y,t) = \frac{V(S_t,r_t,t)}{P(r_t,t,T)} \tag{5.26}$$

Using the chain rule for partial derivatives, the derivation of partials required to complete the derivation are now shown below:

$$\frac{\partial V}{\partial t} = \widehat{V}\frac{\partial P}{\partial t} + P\frac{\partial \widehat{V}}{\partial t} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial t} \tag{5.27}$$

$$\frac{\partial V}{\partial r} = \widehat{V}\frac{\partial P}{\partial r} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial r} \tag{5.28}$$

$$\frac{\partial V}{\partial S} = \frac{\partial \widehat{V}}{\partial y} \tag{5.29}$$

$$\frac{\partial^2 V}{\partial r^2} = \widehat{V}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial^2 P}{\partial r^2} + y^2\frac{\partial^2 \widehat{V}}{\partial y^2}\frac{1}{P}\left(\frac{\partial P}{\partial r}\right)^2 \tag{5.30}$$

$$\frac{\partial^2 V}{\partial r \partial S} = -y\frac{\partial^2 \widehat{V}}{\partial y^2}\frac{1}{P}\frac{\partial P}{\partial r} \tag{5.31}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{P}\frac{\partial^2 \widehat{V}}{\partial y^2} \tag{5.32}$$

As an example, the derivation of Equation (5.30) will now be shown. From Equation (5.28):

$$\frac{\partial^2 V}{\partial r^2} = \widehat{V}\frac{\partial^2 P}{\partial r^2} + \frac{\partial \widehat{V}}{\partial r}\frac{\partial P}{\partial r} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial P}{\partial r}\frac{\partial^2 \widehat{V}}{\partial y \partial r} - \frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial r}\frac{\partial y}{\partial r} \tag{5.33}$$

From the chain rule we see that:

$$\frac{\partial \widehat{V}}{\partial r}\frac{\partial P}{\partial r} = \frac{\partial \widehat{V}}{\partial y}\frac{\partial P}{\partial r}\frac{\partial y}{\partial r}$$

So Equation (5.33) can be simplified to:

$$\frac{\partial^2 V}{\partial r^2} = \widehat{V}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial \widehat{V}}{\partial y}\frac{\partial^2 P}{\partial r^2} - y\frac{\partial P}{\partial r}\frac{\partial^2 \widehat{V}}{\partial y \partial r} \tag{5.34}$$

Now define u such that:

$$u = \frac{\partial \widehat{V}}{\partial y}$$

Then:

$$\frac{\partial^2 \widehat{V}}{\partial y \partial r} = \frac{\partial}{\partial r}u = \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 \widehat{V}}{\partial y^2}\frac{\partial y}{\partial r} = -y\frac{\partial^2 \widehat{V}}{\partial y^2}\frac{1}{P}\frac{\partial P}{\partial r}$$

Place this into Equation (5.34) will yield Equation (5.30).
Substituting Equations (5.27) - (5.32) into Equation (5.18) and dividing by P will give:

$$\begin{aligned}
&\frac{\partial \widehat{V}}{\partial t} + \frac{1}{P}\left[\frac{\partial P}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} + \alpha(r_t,t)\frac{\partial P}{\partial r} - r_t P\right]\widehat{V} \\
&+\frac{1}{P}\left[\frac{\partial P}{\partial t} + \frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} + \alpha(r_t,t)\frac{\partial P}{\partial r} - r_t\frac{S}{y}\right] y\frac{\partial \widehat{V}}{\partial y} \\
&+\frac{1}{2}\left[\sigma_S^2\frac{S^2}{P^2} - 2\rho\sigma_S\sigma_r y\frac{S}{P^2}\frac{\partial P}{\partial r} + y^2\left(\sigma_r\frac{1}{P}\frac{\partial P}{\partial r}\right)^2\right]\frac{\partial^2 \widehat{V}}{\partial y^2} = 0
\end{aligned} \tag{5.35}$$

Using the local expectation hypothesis, we get:

$$\frac{\partial \widehat{V}}{\partial t} + \frac{1}{2}\widehat{\sigma}^2(t)y^2\frac{\partial^2 \widehat{V}}{\partial y^2} = 0 \tag{5.36}$$

where:

$$\widehat{\sigma}^2(t) = \sigma_S^2 - 2\rho\sigma_S\sigma_r\frac{1}{P}\frac{\partial P}{\partial r} + \left(\sigma_r\frac{1}{P}\frac{\partial P}{\partial r}\right)^2 \tag{5.37}$$

With terminal condition:

$$\widehat{V}(y,T) = (y-K)^+ \tag{5.38}$$

This differs from Haowen (2012) [14] who uses the bond price PDE to yield the same result. Using Equation (5.8):

$$\widehat{\sigma}^2(t) = \sigma_S^2 + 2\rho\sigma_S\sigma_P + \sigma_P^2 \tag{5.39}$$

Equation (5.36) is a heat type equation discussed in the previous section, and therefore by the maximum principle it is well posed and a numerical solution can be implemented to solve it.

5.3 Alternatives and Closed-Form Solution

Alternatively, Equation (5.36) can be found be using the Zero Bond Price partial differential equation (this was done by Haowen (2012) [14]). The PDE for the bond price can be derived as (see Equation (33) in Cox et al. (1985) [8]):

$$\begin{gathered}\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_P^2(r_t,t)\frac{\partial^2 P}{\partial r^2} + \alpha(r_t,t)\frac{\partial P}{\partial r} - r_t P = 0, \\ r_t \in \mathbb{R}, \quad P \in \mathbb{R}^+, \quad t \in [0,T]\end{gathered} \tag{5.40}$$

With terminal condition:

$$P(r,T) = 1 \tag{5.41}$$

Kung and Lee (2009) [19] arrived at this result by making change of variables as well and using the chain rule for partial derivatives, rather than a change of numéraire technique as presented here. The technique is very similar to that of Merton (1973) [21], however Merton made use of the Local Expectation Hypothesis theorem.

A closed solution formula for pricing of options with stochastic interest rates is:

$$\widehat{V}(y,t) = yN(d_1) - KN(d_2) \tag{5.42}$$

where:

$$d_1 = \frac{\ln(\frac{y}{K}) + \frac{1}{2}\int_t^T \widehat{\sigma}^2(\tau)d\tau}{\sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau}} \tag{5.43}$$

$$d_2 = d_1 - \sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau} \tag{5.44}$$

Merton's model arrived at this result as well by the following:

$$V(S,r,t) = SN(d_1^*) - KP(r,t;T)N(d_2^*) \tag{5.45}$$

Where P is the zero coupon bond price and:

$$d_1^* = \frac{\ln(\frac{S}{K}) - \ln(P(r,t;T)) + \frac{1}{2}\int_t^T \widehat{\sigma}^2(\tau)d\tau}{\sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau}} \tag{5.46}$$

$$d_2^* = d_1 - \sqrt{\int_t^T \widehat{\sigma}^2(\tau)d\tau} \tag{5.47}$$

It is easy to see that Equations (5.45) to Equation (5.47) can have the zero coupon bond price factored out, and essentially converted to the Equation (5.42) to Equation (5.44). The models are therefore equal as is was demonstrated by Haowen (2012) [14].

Chapter 6

Concluding Remarks

6.1 Conclusions

This thesis as illustrated how to derive the Black-Scholes partial differential equation in a stochastic interest rate environment. It has also demonstrated how to transform the model into a heat equation type where a numerical algorithm can be applied to yield a solution. The change of numéraire technique was particularly useful in assisting in reducing the computational complexity. It will help in expanding this for several assets, or other more exotic options and allows for the use of any interest rate model, a few of which were listed in Chapter 4. Crank-Nicolson (1947) [9], Dawson et al. (1991) [10] and Recktenwald (2011) [25] are excellent resources to explore possible algorithm derivation and implementation of Equation (5.36). Of all the finite difference methods, the Crank-Nicolson algorithm should be used as it is known to be numerically stable and converges faster as opposed to other methods. Additionally, as pointed out in Paul Wilmott's book [35], for the explicit method to be stable α needs to be greater than zero but less than 0.5. Another option would have been to implement a Monte Carlo algorithm to find the price of the option. The next step in this paper, would be to actually implement a numerical algorithm and also investigate numerical solutions for other types of contingent claims e.g. the American option and Barrier option. Rigourous tests would also need to be applied to test the numerical stability and convergence.

6.2 Summary of reflection of objectives in the Thesis

6.2.1 Objective 1: Knowledge and understanding

The author presented by the literature review, the history of the Black-Scholes formula and illustrated why from a modeling perspective it was good at inception. There is also a description of different interest rate models in use today, and lastly, there is a section that has described early methods for pricing stock options in a stochastic interest rate environment. The derived model in Chapter 5 is shown to be similar to Merton's version of the Black-Scholes model with interest rate. The author's model in this paper was implemented with the interest rate as a function of the option price while Merton used a bond and also used the local expectation hypothesis like in Merton's paper rather than using a bond price partial differential equation as in Haowen.

6.2.2 Objective 2: Deeper methodological knowledge

Sections before the model derivation described some necessary underlying theory that is required for deriving the Black–Scholes PDE with stochastic interest rate. These are theory of Partial Differential Equations, Stochastic Processes and basic Interest Rate Theory. Throughout the derivation, certain key concepts where referenced, these include the local expectation hypothesis, the change of numéraire technique and the relationship between interest rates and bond prices. Finally, references were listed for the numerical procedures indicating that the author had done sufficient research in this area albeit not the focus of the paper.

6.2.3 Objective 3: Ability to critically and systematically integrate knowledge and to analyse

The author used sources primarily from other papers in the field of financial mathematics and economics found using Google Scholar as well as text books that are sufficient in mathematical rigor and other students thesis. Without these references, the author would not have been able to arrive at a model derivation and to eventually solve it with a numerical method.

6.2.4 Objective 4: Ability to critically, independently determine, formulate problems and carry out advanced tasks

The model derivation is based on the derivation of the standard Black-Scholes model. The author used established principles financial mathematics and economics to discover a way to implement the Black-Scholes model with Stochastic Interest rates. Once the model was derived, and the need for a numerical solution established, the author critically analyzed relevant literature to transform the model in to a heat type equation which would be easier to implement numerical.

6.2.5 Objective 5: Communication in context, both national and international level

The author has written the paper to be as fundamental as possible and making references to advanced papers when necessary. This was done in an effort to make the paper approachable by a wide audience. The model is relevant in the sense that interest rates are not fixed. This may not be a problem for single asset options, but most certainly will be useful for more complex options that will have longer maturities and depend on several assets or have strict barrier price requirements.

6.2.6 Objective 7: Ability to make judgement by taking in account relevant factors: scientific,social, ethical

The author believes that this work will be helpful in providing better pricing mechanisms in the financial market for long term Stock Options. Specifically, many companies will offer employee stock options which may have longer term maturities. Insurance companies will also find this concept useful as they will be able to hedge risks on a long term basis. Typically, insurance companies invest in assets that are stable and provide income over time (such as dividend paying stocks).

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