Lecture 10: Bayesian regression

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Announcements

- Jacob's OH moved to Wednesday this week (1:30-2:30)
- Midterm next Wednesday

Recap

- Bayesian models
- Inference via sampling (MCMC)

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This time: Bayesian perspective on regression

Linear Regression: Review

Observe data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$, where $x^{(i)} \in \mathbb{R}^d$ and $y^{(i)} \in \mathbb{R}$

Minimize loss function $L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}; \beta)$

Example:

- $\ell(x, y; \beta) = (y - \beta^\top x)^2$ (least squares regression)
- Other examples?

Linear Classification: Review

Observe data $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$ as before, but this time $y^{(i)} \in \{0, 1\}$ **(classification)**

Still minimize loss function $L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}; \beta)$

$$\ell(x, y; \beta) = -y \log \sigma(\beta^\top x) - (1 - y) \log(1 - \sigma(\beta^\top x))$$

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(Recall $\sigma(z) = \frac{1}{1+\exp(-z)}$)

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- Where does logistic loss come from?
- How to generalize (e.g. to counting; $y \in \{0, 1, 2, \ldots\}$)

Linear Regression, Bayesian Interpretation

Consider linear Gaussian model: $y^{(i)} \mid x^{(i)}, \beta \sim N(\beta^\top x^{(i)}, 1)$

Likelihood function: $p(y \mid x, \beta) = \exp(-(y - \beta^\top x)^2/2)/\sqrt{2\pi}$

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Least squares regression $\leftrightarrow$ MLE under Gaussian likelihood!

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Ridge regression $\leftrightarrow$ MAP under Gaussian likelihood + prior!

Sampling from the posterior

Suppose we want full posterior over β. Proportional to:

$$p(\beta \mid x^{(1:n)}, y^{(1:n)}) \propto \exp(-\tfrac{1}{2}\|\beta\|_2^2/\lambda^2) \cdot \prod_{i=1}^{n} \exp(-\tfrac{1}{2}(y^{(i)} - \beta^\top x^{(i)})^2).$$

In this case, can show posterior over β is Gaussian, compute closed form. But could also do Gibbs sampling:

$$p(\beta_j \mid x^{(1:n)}, y^{(1:n)}, \beta_{-j}) \propto \exp(-\tfrac{1}{2}{\beta_j}^2/\lambda^2) \cdot \prod_{i=1}^{n} \exp(-\tfrac{1}{2}(y^{(i)} - \beta_{-j}^\top x_{-j}^{(i)} - \beta_j x_j^{(i)})^2)$$

In practice, use an off-the-shelf sampling library such as PyMC3

Linear regression on wind turbine data

[Jupyter demo]

Regression on count data

Number of turbines isn't an arbitrary real number, but integer count in $\{0, 1, 2 \ldots\}$

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Power of Bayesian thinking: just swap in new likelihood!

Poisson regression on turbine data

[Jupyter demo]

Pitfalls of Bayes

Peril of Bayesian thinking: at the mercy of your model

Poisson distribution too narrow, leads to overconfident posterior

Common issue (esp. with count data): **overdispersion**

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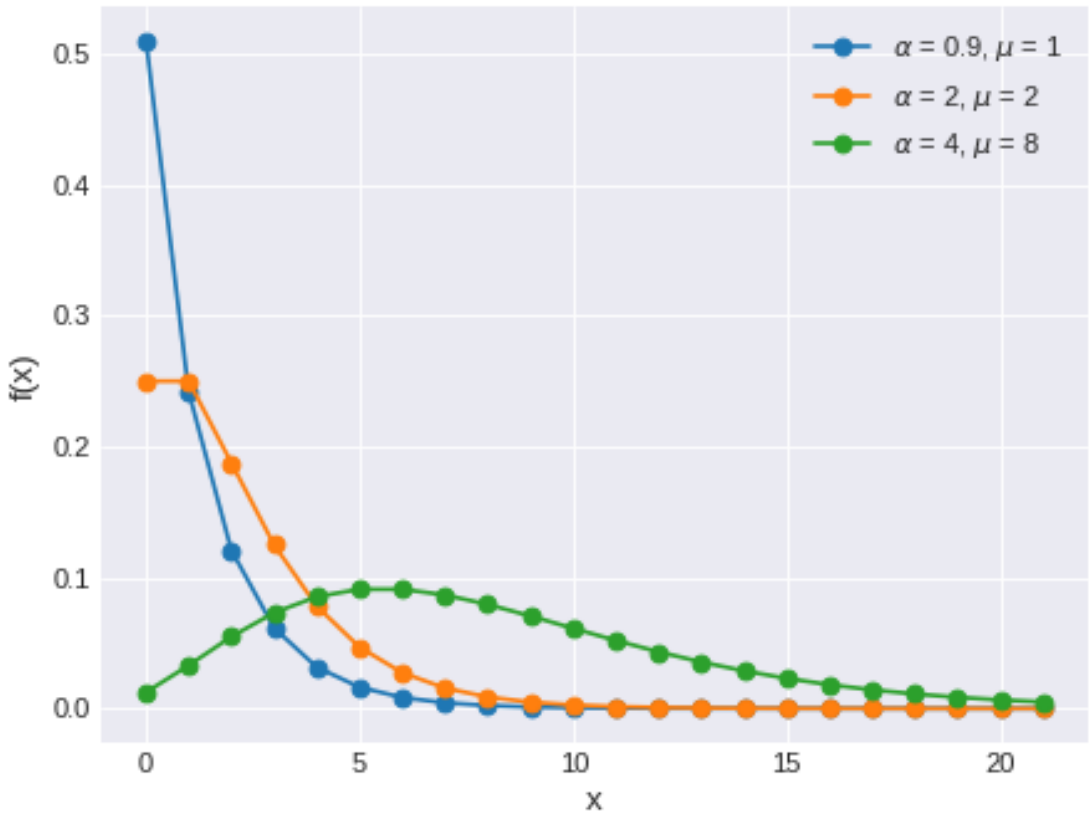
Typical fix: negative binomial distribution

$$p_{\mu,\alpha}(k) \propto \binom{k+\alpha-1}{k} \left(\frac{\mu}{\mu+\alpha}\right)^k$$

Mean μ, overdispersion α (variance $\mu \cdot (1+\mu/\alpha)$)

Negative binomial plots



[Credit: PyMC3 docs]

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[Jupyter demo]

Logistic regression revisited

Recall loss function for logistic regression: $L(\theta) = \frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}; \beta)$, where

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- Exponentiate to make positive, normalize to add up to 1
- Generalization: softmax $\exp(z_j)/\sum_{j'} \exp(z_{j'})$

Generalized Linear Models

(Inverse) Link function + likelihood. Many libraries handle them!

Regression	Inverse link function	Link function	Likelihood
Linear	identity	identity	Gaussian
Logistic	sigmoid	logit	Bernoulli
Poisson	exponential	log	Poisson
Negative binomial	exponential	log	Negative binomial

Discussion: modeling assumptions

What other modeling assumptions might be violated for the turbine data?