Data 102 Spring 2021

Midterm 1 Solutions: All versions

- Please write your solutions using either pen/pencil and paper, or a tablet. Each question should start on a new page. At the end of the exam period (or earlier), please upload your exam to the "Midterm 1" assignment on Gradescope. **It is your responsibility to make sure your work will be legible!**
- We will not answer any questions during the exam. If you think a question is unclear, state your assumptions and answer accordingly.
- You have 80 minutes to work on the exam: you must stop working at 11:00AM PT.
- This exam has 6 questions, for a total of 40 points. **You must complete all 6 questions to receive full credit.** There are multiple versions of this exam.
- Unless otherwise stated, you must show your work to receive full credit.
- You may, without proof, use theorems and facts that were given in the lectures, homework, lab, or discussions.
- **You must complete this honor pledge in order to receive credit on the exam:** We ask that you act in accordance with the honor code. Please copy the following statement by hand and sign your name, and include this in your submission.

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. These answers are my own work.

0. **Make sure you complete the honor pledge on the previous page.**

1. (5 points) For each of the following, answer **True** or **False**. You don't need to justify your answer.

   (a) (1 point) The value that minimizes the posterior risk with respect to zero-one loss is the posterior mean (the mean of the distribution $p(\theta|X)$).

   Solution: False

   (b) (1 point) If, given n p-values, we reject only those p-values at or below α/n, then the false discovery rate will be less than or equal to α.

   Solution: True

   (c) (1 point) Rejecting hypothesis t (as opposed to accepting it) in the LORD algorithm makes it more likely that hypothesis $t+1$ will be accepted.

   Solution: True

   (d) (1 point) When using Gibbs sampling, to update a particular hidden variable, we compute the most likely value of that variable conditioned on all the others.

   Solution: False

   (e) (1 point) When using Gibbs sampling, successive samples (for example, $\theta^{(t)}$ and $\theta^{(t+1)}$) are independent.

   Solution: False

2. (4 points) For each question, **select all that apply**. If none of the answers are correct, write "None". You don't need to justify your answer.

   (a) (1 point) Which sampling algorithm(s) involve an accept/reject step?

   (A) Metropolis-Hastings

   (B) Rejection sampling

   Solution: A, B

   (b) (1 point) Which sampling algorithms require us to know the distribution we're sampling from (including any normalization constants)?

   (A) Metropolis-Hastings

   (B) Rejection sampling

Solution: None

(c) (2 points) Which of the following statements about generalized linear models (GLMs) are true?

(A) The inverse link function in GLMs must always produce nonnegative outputs.

(B) Logistic regression is a kind of GLM.

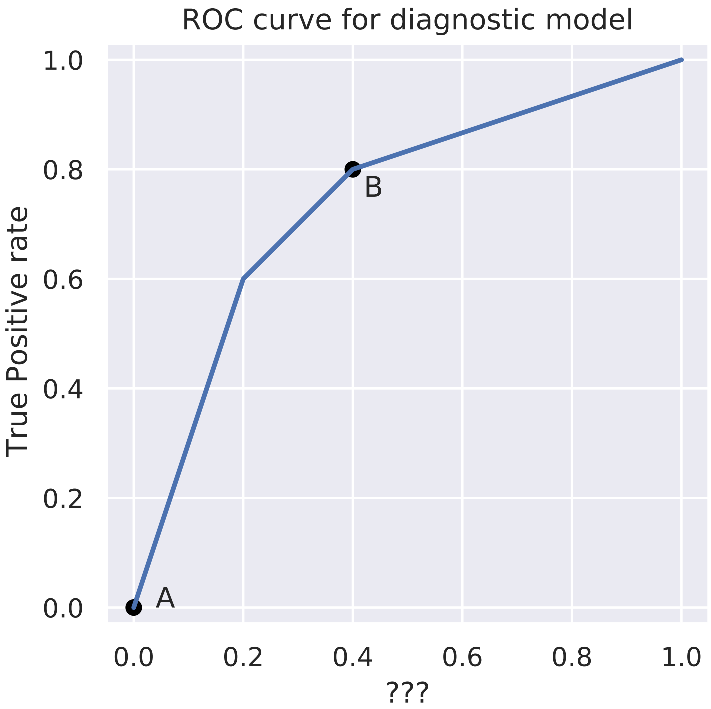
(C) When estimating the parameters of a GLM, one must use Bayesian inference.

(D) Predictions from GLMs can only ever be accurate when interpolating, not when extrapolating.

(E) A Poisson likelihood model is only appropriate for cases where the variance of the data (y) is much higher than the mean.

Solution: B

3. (9 points) We are trying to construct a spam filter for phone calls (an algorithm that classifies calls into spam vs not spam.) The model that we train has the following ROC curve.



(a) (1 point) What is the x-axis label?

Solution: False Positive Rate.

(b) (2 points) Describe the decision rule that corresponds to the point A. Your answer should be one sentence or less.

Solution: Classify all samples as being negative (not spam).

(c) (2 points) Let S be the binary random variable indicating whether a given call is spam ($S = 1$) or not ($S = 0$), and let D be the binary random variable indicating the algorithm's prediction for the call ($D = 1$ for spam and $D = 0$ otherwise). Write the false positive rate as a conditional probability in terms of S and D.

Solution: $P(D = 1|S = 0)$: you did not need to apply Bayes' rule.

(d) (2 points) We set our score threshold for the model so that the resulting test corresponds to point B. What fraction of calls does this test classify as being spam? Write your answer in terms of the true prevalence of spam calls $\pi_1 = \mathbb{P}(S = 1)$.

Solution: We are interested in the fraction of calls classified as spam, or $P(D = 1)$. Using the law of total probability:

$$\begin{aligned} P(D=1) &= P(D=1|S=0)P(S=0) + P(D=1|S=1)P(S=1) \\ &= \text{FPR} \cdot (1-\pi_1) + \text{TPR} \cdot \pi_1 \\ &= 0.4(1-\pi_1) + 0.8\pi_1 \end{aligned}$$

(e) (2 points) The cost of misclassifying a real (non-spam) call as spam is 5 times more than that of misclassifying a spam call as non-spam. Write a loss function $\ell(d, s)$ that expresses this belief.

Solution:

$$\ell(d,s) = \begin{cases} 0 & \text{if } d = s \\ 1 & \text{if } d = 0, s = 1 \\ 5 & \text{if } d = 1, s = 0 \end{cases}$$

4. (9 points) As part of a medical lab, you that would like to test five hypotheses using what you learned about multiple testing. Four lab tests have already been performed, resulting in the following P-values: $P_1 = \frac{\alpha}{2}, P_2 = \frac{\alpha}{10}, P_3 = \frac{5\alpha}{6}, P_4 = \frac{\alpha}{4}$, for some $0 < \alpha < 1/2$. We are currently waiting on the results for the fifth test, so the value for P_5 is currently unknown. Unless otherwise stated, you should assume that P_5 can take any value in $[0, 1]$.

   Recall the steps of the Benjamini-Hochberg (BH) procedure:

Algorithm 1 The Benjamini-Hochberg Procedure

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input: FDR level $\alpha$, set of $n$ p-values $P_1, \dots, P_n$
Sort the p-values $P_1, \dots, P_n$ in non-decreasing order $P_{(1)} \leq P_{(2)} \leq \cdots \leq P_{(n)}$
Find $K = \max\{i \in \{1, \dots, n\} : P_{(i)} \leq \frac{\alpha}{n} i\}$
Reject the null hypotheses (declare discoveries) corresponding to $P_{(1)}, \dots, P_{(K)}$
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   (a) (2 points) If we use the Bonferroni method to control the family-wise error rate (FWER) at level 2α (**not** α) for all 5 tests, what is the minimum number of rejections made?

   Solution: We compare all P-values with the threshold $\frac{2\alpha}{5}$. Hence, we reject at least two P-values: P_2 and P_5.

   (b) (3 points) Suppose $P_5 \geq P_3$. If we run the BH procedure to control the false discovery rate at level α for all 5 tests, what are the possible sets of P-values that could be rejected? For example, a possible answer could be "$\{P_1, P_2\}$ and $\{P_1\}$".).

   Solution: We consider 2 cases:

   Case 1: $\frac{5\alpha}{6} < P_5 \leq \alpha$. P_5 is the largest P-value, and is compared with α. Since it lies under the threshold, all P-values are rejected.

   Case 2: $P_5 > \alpha$. P_5 is the largest P-value, and is compared with α, while P_3 is compared with $\frac{4\alpha}{5}$. Both lie above their corresponding thresholds and are accepted. All other P-values lie beneath their thresholds are rejected.

   Hence, the possible rejection sets are $\{P_1, P_2, P_3, P_4, P_5\}$ and $\{P_1, P_2, P_4\}$.

   (c) (2 points) What is the largest value of P_5 so that BH makes the maximum number of rejections? You may write your answer in terms of α.

   Solution: Based on the previous solution, $P_5 = \alpha$.

   (d) (2 points) Suppose that for all five P-values, the null hypothesis is true. In this case, is it true that FWER = FDR? Explain why or why not. *Hint: We use the convention that 0/0 = 0.*

Solution: True, because then TP $= 0$, and

$$\begin{aligned}
\text{FDR} &= E[\text{FDP}] \\
&= E\left[\frac{\text{FP}}{\text{TP} + \text{FP}}\right] \\
&= E\left[\mathbb{1}(\text{FP} > 0)\right] \\
&= \text{FWER}
\end{aligned}$$

5. (8 points) **Grab Bag:** The parts of this question are all completely unrelated to each other.

   (a) (2 points) Suppose we use a generalized linear model (GLM) to predict the number of Piazza followups for a Data 102 lecture thread. We'll let y be the number of followup posts, and we'll use the number of lecture videos, the length of the lecture videos, and the length of the discussion worksheet to construct x. Specify a **link function** and a **likelihood model** so that the predictions are always valid followup post counts.

   Solution:

   Likelihood model: Negative Binomial (or Poisson)

   Link function: log (we also accepted exp, since that's the inverse link function)

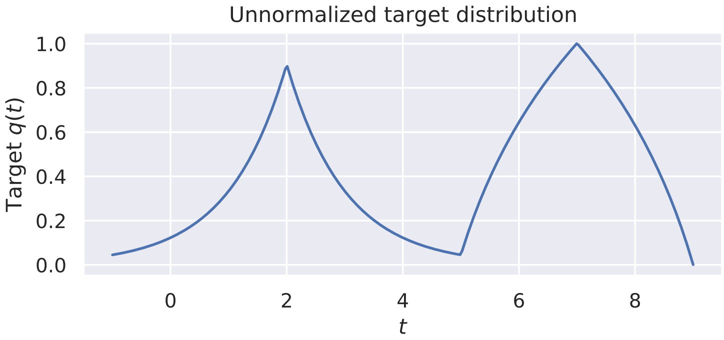
   (b) (4 points) Suppose that the prior over random variable θ has the following form: $p(\theta) \propto \theta^{a-1}e^{-b\theta}$. This is a **Gamma distribution** with parameters a and b (called *shape* and *rate* respectively).

   We observe two variables x and y that are conditionally independent given θ. We know that $x|\theta \sim$ Exponential(θ), and $y|\theta \sim$ Poisson(θ). Show that the posterior distribution $p(\theta|x, y)$ is also a Gamma distribution, and express its two parameters in terms of the prior parameters a and b and the observed values x and y.

   Solution:

$$\begin{aligned} p(\theta|x,y) &\propto p(x,y|\theta)p(\theta) \\ &= p(x|\theta)p(y|\theta)p(\theta) \\ &\propto \theta e^{-\theta x}\ \frac{\theta^y e^{-\theta}}{y!}\ \theta^{a-1}e^{-b\theta} \\ &\propto e^{-\theta(x+1+b)}\theta^{y+a} \\ &= \text{Gamma}(y+a+1, x+b+1) \end{aligned}$$

   (c) (2 points) Suppose we use rejection sampling to approximate the following unnormalized target distribution $q(t)$ for random variable t, where $t \in [-1, 9]$:



Provide a sampling distribution $p(t)$ that we can use with rejection sampling, and find a constant M such that $Mq(t) \leq p(t)$.

Solution: Since $t \in [-1, 9]$, we can use a uniform distribution:

$\boxed{p(t) = \text{Uniform}[-1, 9]}$

This sampling distribution $p(t)$ has a height of 0.1. Since the maximum value of $q(t)$ is 1, $M = 0.1$.

6. (5 points) **Bayesian fidget spinners**

Nat's company makes and sells fidget spinners. Suppose that when the factory manufactures them, each fidget spinner is defective with probability q: we'll call this the *defect rate*. She receives n boxes full of fidget spinners. For each box, she randomly pulls out fidget spinners until she finds a defective one, and records how many fidget spinners she pulled out (including the defective one). She calls this number x_i for box i ($i = 1, \ldots, n$).

She defines the following Bayesian probability model:

$$q \sim \text{Beta}(\alpha, \beta)$$
$$x_i | q \sim \text{Geometric}(q)$$

(You may assume that the number of fidget spinners in each box is much larger than the number she pulls out, so her model is valid.)

(a) (2 points) Nat is sure that the defect rate is very close to 0.1, and wants her prior distribution $p(q)$ to reflect that certainty. Which of the following is the best choice for the parameters of her prior distribution (i.e., α and β)? **Select only one answer.**

(A) $\alpha = 1, \beta = 9$
(B) $\alpha = 10, \beta = 90$
(C) $\alpha = 9, \beta = 1$
(D) $\alpha = 90, \beta = 10$

Solution:

We want a strong prior, to reflect Nat's certainty. With Beta distributions, stronger priors correspond to higher values of α and β. So, we must choose B or D.

We want a prior whose mean is 0.1. Since the mean of the beta distribution is $\alpha/(\alpha + \beta)$, only A and B satisfy this.

So, the correct answer is $\boxed{\text{B}}$.

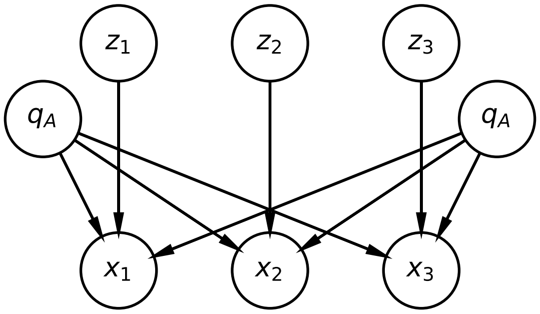
(b) (3 points) Nat discovers that her boxes are actually from two different factories, A and B, which have *different* defect rates q_A and q_B. Unfortunately, the boxes aren't labeled with which factory they came from. Nat defines a new random variable z_i:

$$z_i = \begin{cases} 1 & \text{if box } i \text{ came from factory A} \\ 0 & \text{if box } i \text{ came from factory B} \end{cases}$$

She assumes that within any particular box, all the fidget spinners inside are from the same factory.

Suppose there are exactly three boxes. Draw a graphical model illustrating the relationship between q_A, q_B, z_i, and x_i, for $i = 1, 2, 3$.

Solution: This is very similar to the Gaussian mixture model used in class (for heights and sex in lecture, and dog weights and size in discussion).


z_1
z_2
z_3
q_A
q_A
x_1
x_2
x_3