Data 102 (Fall 2022) **Names:**
Discussion Worksheet 3
09/09/2022

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1. **Bias-variance.** Decompose the squared-error loss between a fixed (non-random) parameter θ and its estimator $\delta(X)$ into bias and variance terms. (Recall that the squared error is $\mathbb{E}[(\delta(X) - \theta)^2]$.)

   Solution:

$$\begin{aligned}
\mathbb{E}(\delta(X) - \theta)^2 &= \mathbb{E}(\delta(X) - \theta + \mathbb{E}\delta(X) - \mathbb{E}\delta(X))^2 \\
&= \mathbb{E}(\delta(X) - \mathbb{E}\delta(X))^2 - 2\mathbb{E}(\delta(X) - \mathbb{E}\hat{\delta}(x))\mathbb{E}(\theta - \mathbb{E}\delta(X)) + \mathbb{E}(\theta - \mathbb{E}\delta(X))^2 \\
&= \underbrace{\mathbb{E}(\delta(X) - \mathbb{E}\delta(X))^2}_{\text{Variance}} + \mathbb{E}(\underbrace{\theta - \mathbb{E}\delta(X)}_{\text{Bias}})^2.
\end{aligned}$$

2. **ROC Curves.** Consider the toy dataset in the table below; Y is the label, X_1, X_2 are features, and we use the prediction function $f(X_1, X_2)$.

Table 1: Example dataset

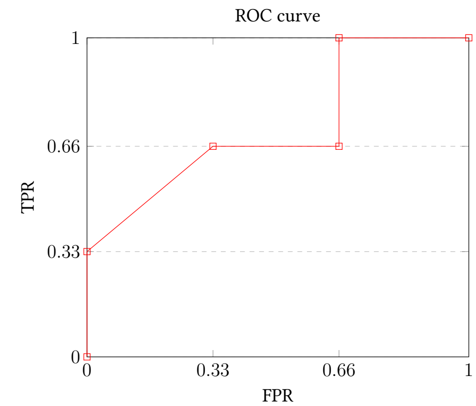
Y	$f(X_1, X_2)$	X_1	X_2
0	-1	-1	0.5
0	-0.5	-1	0.75
1	0	-1	1
1	1	0.2	-0.3
0	0.25	-0.25	0
1	0.25	-0.05	-0.3

   (a) Draw the ROC curve for the prediction function f with respect to the label Y.

   Solution: At a given decision threshold α, if $f(X_1, X_2) > \alpha$, then the decision is a positive classification, and if $f(X_1, X_2) \leq \alpha$, then the decision is a negative classification. Since the model $f(X_1, X_2)$ only takes five different values on this dataset, only five different decision thresholds lead to different true and false positive rates.

   - $\alpha < -1$: TPR $= 1$, FPR $= 1$
   - $-1 \leq \alpha < -0.5$: TPR $= 1$, FPR $= \frac{2}{3}$
   - $-0.5 \leq \alpha < 0$: TPR $= \frac{2}{3}$, FPR $= \frac{2}{3}$
   - $0 \leq \alpha < 0.25$: TPR $= \frac{2}{3}$, FPR $= \frac{1}{3}$
   - $0.25 \leq \alpha < 1$: TPR $= \frac{1}{3}$, FPR $= 0$
   - $1 \leq \alpha$: TPR $= 0$, FPR $= 0$

   Plotting each (FPR, TPR) combination as a point on a plot results in the following ROC curve:



(b) Is it possible to choose a (possibly randomized) decision threshold for f, such that the expected true positive rate is $\frac{1}{3}$, and the expected false positive rate is $\frac{2}{3}$?

Solution: No, this is not possible for this dataset. Every deterministic threshold corresponds to a point on the ROC curve drawn above, and every randomized threshold corresponds to a *convex combination* of points on the ROC curve. By inspecting the picture above, we see that $(\frac{2}{3}, \frac{1}{3})$ is slightly outside of the convex hull.

3. **LORD Procedure**:
You want to control the FDR with LORD at level α and are currently at time step $t = 5$.

Algorithm 1 The LORD Procedure

Input FDR level α, non-increasing sequence $\{\gamma_t\}_{t=1}^{\infty}$ such that $\sum_{t=1}^{\infty} \gamma_t = 1$, $W_0 = \alpha$

Set $\alpha_1 = \gamma_1 W_0$.
for $t = 1, 2, \ldots,$ **do**
    p-value P_t arrives.
    if $P_t \leq \alpha_t$ **then**
        Reject P_t.
    Update $\alpha_{t+1} = \gamma_{t+1} W_0 + \alpha \sum_{j=1}^{\infty} \gamma_{t+1-\tau_j} 1\{\tau_j < t\}$, where τ_j is the time of the j'th rejection.

Set $\gamma_t = 2^{-t}$.

(a) The only discovery you've made so far was at time step $t = 1$. How small must the 5'th p-value be in order for you to make a discovery at $t = 5$?

Solution: At most $(\frac{1}{32} + \frac{1}{16})\alpha = \frac{3}{32}\alpha$. At $t = 4$, you compute $\alpha_5 = (1/32)\alpha + \alpha(\gamma_{5-1} = 1/16)$.

(b) The only discovery you've made so far was at time step $t = 2$. How small must the 5'th p-value be in order for you to make a discovery at $t = 5$?

Solution: At most $(\frac{1}{32} + \frac{1}{8})\alpha = \frac{5}{32}\alpha$. At $t = 4$, you compute $\alpha_5 = (1/32)\alpha + \alpha(\gamma_{5-2} = 1/8)$.

(c) The only discovery you've made so far was at time step $t = 3$. How small must the 5'th p-value be in order for you to make a discovery at $t = 5$?

Solution: At most $(\frac{1}{32} + \frac{1}{4})\alpha = \frac{9}{32}\alpha$. At $t = 4$, you compute $\alpha_5 = (1/32)\alpha + \alpha(\gamma_{5-3} = 1/4)$.

(d) The only discovery you've made so far was at time step $t = 4$. How small must the 5'th p-value be in order for you to make a discovery at $t = 5$?

Solution: At most $\frac{1}{32}\alpha$. At $t = 4$, you compute $\alpha_5 = (1/32)\alpha + 0$, where 0 is due to the indicator. In this version of LORD, a rejection at the previous step isn't immediately counted, due to the strict inequality in the indicator. However, the fact that this threshold is more stringent than the previous 3 shouldn't distract from the main takeaway of LORD's algorithm, which is that more recent rejections make a current rejection more likely.

(e) You're made discoveries at steps $t = 1, 2, 3, 4$. How small must the 5'th p-value be in order for you to make a discovery at $t = 5$?

Solution: At step $t = 4$, you compute $\alpha_5 = \frac{1}{32}\alpha + \alpha(\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + 0) = \frac{15}{32}\alpha$.

4. Alice has a bag with 3 red marbles, 2 blue marbles, and 1 green marble. Norman has a bag with 1 red marbles, 1 blue marble, and 4 green marbles. You observe two samples with replacement from either Alice or Norman, and want to figure out which is which. You want to have the highest TPR while keeping the FPR at $\frac{1}{9}$. What decision rule do you pick? What is the corresponding TPR? Assume that **Norman is the null** and **Alice is the alternative**.

   To help you get started, the table below writes out the probabilities for all 9 outcomes:

	P_A	P_N
RR	$1/4$	$1/36$
RB	$1/6$	$1/36$
RG	$1/12$	$1/9$
BR	$1/6$	$1/36$
BB	$1/9$	$1/36$
BG	$1/18$	$1/9$
GR	$1/12$	$1/9$
GB	$1/18$	$1/9$
GG	$1/36$	$4/9$

Solution: We want to test the hypotheses: $\begin{cases} H_0 : \text{samples came from Norman} \\ H_1 : \text{samples came from Alice} \end{cases}$

So $\delta = 1$ corresponds to picking Alice, and $\delta = 0$ corresponds to picking Norman. Using the intuition from Neyman-Pearson, we want to set $\delta = 1$ for outcomes that have a large likelihood ratio P_A/P_N. We compute these ratios in the table below:

	P_A	P_N	LR	δ
RR	$1/4$	$1/36$	9	1
RB	$1/6$	$1/36$	6	1
RG	$1/12$	$1/9$	$3/4$	0
BR	$1/6$	$1/36$	6	1
BB	$1/9$	$1/36$	4	1
BG	$1/18$	$1/9$	$1/2$	0
GR	$1/12$	$1/9$	$3/4$	0
GB	$1/18$	$1/9$	$1/2$	0
GG	$1/36$	$4/9$	$9/144$	0

Pick the likelihood-ratio threshold to be $3/4$; i.e. reject the null if LR $> 3/4$. Then,

$$\begin{aligned}\mathbb{P}(\text{reject the null}|\text{null is true}) &= \mathbb{P}(\text{you observed RR, RB, BR, or BB}|\text{Norman sampled})\\ &= \frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{4}{36}=\frac{1}{9}.\end{aligned}$$

The corresponding TPR is

$$\begin{aligned}\mathbb{P}(\text{reject the null}|\text{null is false}) &= \mathbb{P}(\text{you observed RR, RB, BR, or BB}|\text{Alice sampled})\\ &= \frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{9}=\frac{25}{36}.\end{aligned}$$

5. **Benjamini-Yekutieli procedure** (Challenge Question)

Suppose you are testing n hypotheses and want to control the FDR at level α. It turns out that Benjamini-Hochberg is only guaranteed to work when the hypotheses are *independent* or *positively correlated.* Construct an example with negatively correlated hypotheses where Benjamini-Hochberg fails.

Remark: The Benjamini-Yekutieli procedure, a generalization of Benjamini-Hochberg, controls the FDR regardless of independence assumptions, and therefore is guaranteed to work in all cases. It is shown below. The only difference from Benjamini-Hochberg is the $c(n)$ function highlighted in red.

Algorithm 2 The Benjamini-Yekutieli Procedure

Input FDR level α, set of n p-values $P_1, \ldots, P_n$

1: Sort the p-values $P_1, \ldots, P_n$ in non-decreasing order $P_{(1)} \leq P_{(2)} \leq \cdots \leq P_{(n)}$

2: Find $K = \max\{i \in \{1, \ldots, n\} : P_{(i)} \leq \frac{\alpha}{n \cdot c(n)} i\}$, where

$$c(n) = \begin{cases} 1 & \text{tests are independent or positively correlated (this is just B-H)} \\ \sum_{j=1}^{n} \frac{1}{j} & \text{tests are dependent or negatively correlated} \end{cases} \tag{1}$$

3: Reject the null hypotheses (declare discoveries) corresponding to $P_{(1)}, \ldots, P_{(K)}$

Feedback Form

On a scale of 1-5, where 1 = much too slow and 5 = much too fast, how was the pace of the discussion section?

1 2 3 4 5

Which problem(s) did you find most useful?

Which were least useful?

Any other feedback?