# Data 102, Fall 2023
# Midterm 2

- You have **110 minutes** to complete this exam. There are **5 questions**, totaling **40 points**.
- You may use **two** $8.5 \times 11$ sheet of handwritten notes (front and back), and the provided reference sheet. No other notes or resources are allowed.
- You should write your solutions inside this exam sheet.
- You should write your Student ID on every sheet (in the provided blanks).
- Make sure to write clearly. We can't give you credit if we can't read your solutions.
- Even if you are unsure about your answer, it is better to write down something so we can give you partial credit.
- We have provided a blank pages of scratch paper at the **end** of the exam. No work on this page will be graded.
- You may, without proof, use theorems and facts given in the discussions or lectures, **but please cite them**.
- We don't answer questions individually. If you believe something is unclear, bring your question to us and if we find your question valid we will make a note to the whole class.
- Unless otherwise stated, no work or explanations will be graded for multiple-choice questions.
- Unless otherwise stated, you must show your work for free-response questions in order to receive credit.

| | |
|---|---|
| Last name | |
| First name | |
| Student ID (SID) number | |
| Berkeley email | |
| Name of person to your left | |
| Name of person to your right | |

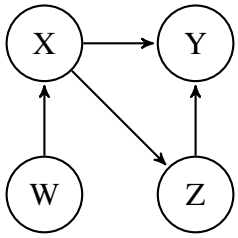
**Honor Code [1 pt]:**
As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others. I am the person whose name is on the exam, and I completed this exam in accordance with the Honor Code.

Signature: _____

# 1 True or False [7 Pts]

For each of the following, determine whether the statement is true or false. For this question, no work will be graded and no partial credit will be assigned.

For parts (a) and (b), consider the following causal DAG:




(a) [1 Pt] When estimating the causal effect of $Z$ on $Y$, the set of nodes $\{W, X\}$ satisfies the backdoor criterion and should therefore be treated as confounders.

● **True** ○ False

(b) [1 Pt] When estimating the causal effect of $X$ on $Z$, $W$ satisfies the exclusion criterion and could therefore be used as an instrumental variable.

● **True** ○ False

(c) [1 Pt] Value iteration is a dynamic programming algorithm for estimating the reward function given the best sequence of state/action pairs.

○ True ● **False**

(d) [1 Pt] In a neural network, using backpropagation and stochastic gradient descent will always find the best set of parameters to minimize the loss on the training set.

○ True ● **False**

(e) [1 Pt] In datasets with outliers, random forests will usually achieve worse test set accuracy than decision trees because bootstrap is sensitive to the presence of outliers.

○ True ● **False**

(f) [1 Pt] In a Markov Decision Process (MDP), the reward for any action is conditionally independent of previous rewards obtained, given the current state.

● **True** ○ False

(g) [1 Pt] Posterior normal approximations can be used for GLM coefficients regardless of whether the prior for those coefficients is normal or not.

● **True** ○ False

# 2 Clickbait for Fun and Profit [6 Pts]

Jason has been experimenting with three different headline styles for a digital media site to maximize user engagement, measured by click-through rates. He is using multi-armed bandits to pick one of three different headline styles to show to website visitors, and has obtained the following data on how many clicks each style received so far:

| Style | Times shown | Clicks received |
|---|---|---|
| A | 30 | 6 |
| B | 10 | 2 |
| C | 20 | 5 |

The company knows that Jason has been using one of the algorithms Explore-then-Commit (ETC), Uniform Confidence Bound (UCB) or Thompson Sampling (TS). For each of the following, determine whether the statement is true or false. **For parts (a) - (d), no work will be graded.**

(a) [1 Pt] Based on the number of times each style has been shown, Jason is definitely not using ETC (as defined in class).

● **True** ○ False

(b) [1 Pt] If using UCB with any confidence level $\delta$, then he won't present A to the next visitor.

● **True** ○ False

(c) [1 Pt] If Jason has been using UCB with a very small nonzero confidence level $\delta$, then he will present Style C to the next visitor.

○ True ● **False**

(d) [1 Pt] If Jason has been using UCB with a confidence level $\delta$ very close to 1, then he will present Style B to the next visitor.

○ True ● **False**

(e) [2 Pts] Suppose Jason is using Thompson sampling with uniform priors. Fill in the blanks in the sequence of steps below to describe how he should determine what style to show to the next visitor. **No work outside the blanks will be graded.**

1. Draw one sample each from the following Beta distributions:

   (1) Beta( **7** , 25) (2) Beta(3, **9** ) (3) Beta( **6** , **16** )

2. If the first sample is the largest, show Style **A** .

   If the second sample is the largest, show Style **B** .

   If the third sample is the largest, show Style **C** .

**Solution:** (a) is true because if ETC (as defined in class) were used, the number of times shown would equal 10 for two of the three styles, and 40 for one of the three styles.

(b) is true. At this stage, UCB for A equals:

$$UCB_A(\delta) = \frac{6}{30} + \sqrt{\frac{\log(1/\delta)}{2 \times 30}} = \frac{1}{5} + \sqrt{\frac{\log(1/\delta)}{60}}.$$

On the other hand, UCB for B equals

$$UCB_B(\delta) = \frac{2}{10} + \sqrt{\frac{\log(1/\delta)}{2 \times 10}} = \frac{1}{5} + \sqrt{\frac{\log(1/\delta)}{20}}.$$

It is then clear that $UCB_A(\delta)$ is strictly smaller than $UCB_B(\delta)$ for every $\delta$. So the UCB algorithm will always prefer B to A.

(c) This is false. The UCB for C equals

$$UCB_C(\delta) = \frac{5}{20} + \sqrt{\frac{\log(1/\delta)}{2 \times 20}} = \frac{1}{4} + \sqrt{\frac{\log(1/\delta)}{40}}.$$

If $\delta$ is close to zero, then the dominant term in each UCB is the second term involving $\log(1/\delta)$. As this term is largest for B, the algorithm with very small $\delta$ will choose B (and not C) for the next visitor.

(d) is also false. Using the expressions for the UCB's given above, it is clear that when $\delta$ is close to one, the UCB is simply equal to the observed proportion of click throughs (because $\log(1/\delta) \approx 0$ when $\delta$ is close to 1). In terms of the click through proportions, the best style is C with proportion 0.25. So UCB algorithm with $\delta$ close to 1 will present C (not B) to the next visitor.

(e) The prior is uniform which is same as Beta$(1, 1)$. The posterior for each style is then Beta$(1 + \text{clicks}, 1 + \text{total times} - \text{clicks})$. The Thompson sampling algorithm generates one posterior sample for each style and shows the style for which the obtained posterior sample is the largest.

# 3 Regressions on a Crime Dataset [9 Pts]

Consider the following dataset on arrests from the Introductory Econometrics book by Wooldridge, with information for $n = 2,725$ adult men on the following variables:

- `narr86` ($y$): Number of arrests in the year 1986 (this variable equals zero for 1,970 of the 2,725 men in the dataset)
- `pcnv` ($x_1$): Proportion of previous arrests that led to a conviction
- `tottime` ($x_2$): Total time (in months) in prison since turning 18
- `inc86` ($x_3$): Legal income in 1986 (in hundreds of dollars)
- `qemp86` ($x_4$): Number of quarters employed in 1986
- `black` ($x_5$): Binary variable which equals 1 if the individual is Black and 0 otherwise

(a) [2 Pts] We fit Poisson Regression (MODEL ONE) and Negative Binomial Regression (MODEL TWO) to this data in order to explore the relationship between $y$ and $x_1, x_2, x_3, x_4, x_5$. The summaries of these model fits are given below.

MODEL ONE summary

| | | | |
|---|---|---|---|
| Dep. Variable: | narr86 | No. Observations: | 2725 |
| Model: | GLM | Df Residuals: | 2719 |
| Model Family: | Poisson | Df Model: | 5 |
| Link Function: | Log | Scale: | 1.0000 |
| Method: | IRLS | Log-Likelihood: | -2284.2 |
| Date: | Sun, 05 Nov 2023 | Deviance: | 2893.2 |
| Time: | 23:37:58 | Pearson chi2: | 4.25e+03 |
| No. Iterations: | 6 | Pseudo R-squ. (CS): | 0.1093 |
| Covariance Type: | nonrobust | | |

| | coef | std err | z | P>\|z\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| const | -0.5271 | 0.058 | -9.053 | 0.000 | -0.641 | -0.413 |
| pcnv | -0.4209 | 0.084 | -4.987 | 0.000 | -0.586 | -0.255 |
| tottime | 0.0004 | 0.006 | 0.074 | 0.941 | -0.010 | 0.011 |
| inc86 | -0.0086 | 0.001 | -8.288 | 0.000 | -0.011 | -0.007 |
| qemp86 | -0.0030 | 0.029 | -0.105 | 0.916 | -0.059 | 0.053 |
| black | 0.4945 | 0.069 | 7.189 | 0.000 | 0.360 | 0.629 |

MODEL TWO summary

| | | | |
|---|---|---|---|
| Dep. Variable: | narr86 | No. Observations: | 2725 |
| Model: | NegativeBinomial | Df Residuals: | 2719 |
| Method: | MLE | Df Model: | 5 |
| Date: | Sun, 05 Nov 2023 | Pseudo R-squ.: | 0.04711 |
| Time: | 23:38:02 | Log-Likelihood: | -2182.8 |
| converged: | True | LL-Null: | -2290.7 |
| Covariance Type: | nonrobust | LLR p-value: | 1.174e-44 |

| | coef | std err | z | P>\|z\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| const | -0.5038 | 0.074 | -6.817 | 0.000 | -0.649 | -0.359 |
| pcnv | -0.4806 | 0.103 | -4.651 | 0.000 | -0.683 | -0.278 |
| tottime | 0.0027 | 0.007 | 0.383 | 0.702 | -0.011 | 0.017 |
| inc86 | -0.0083 | 0.001 | -7.174 | 0.000 | -0.011 | -0.006 |
| qemp86 | -0.0135 | 0.035 | -0.387 | 0.698 | -0.082 | 0.055 |
| black | 0.4926 | 0.088 | 5.570 | 0.000 | 0.319 | 0.666 |
| alpha | 1.0191 | 0.115 | 8.826 | 0.000 | 0.793 | 1.245 |

Based only on the information given in the above summaries, which of these two models should be preferred for this dataset? You must justify your answer to receive credit.

◯ Poisson (MODEL ONE)

● **Negative Binomial (MODEL TWO)**

Justification: The Negative Binomial regression should be preferred here. The log-likelihood for Model TWO (-2182.8) is much larger compared to Model ONE (-2284.2). Model TWO has only one additional parameter (the dispersion parameter $\alpha$) compared to Model ONE. Any reasonable model selection criterion (including AIC or BIC) would prefer Model TWO to Model ONE.

(b) [2 Pts] Which of the following are correct interpretations of the coefficients of MODEL TWO? Select all answers that apply.

☐ A. For each additional month in prison since turning 18, the model predicts a 0.0027 increase in the log-odds of being arrested (if all other variables are held fixed).

☐ B. It is possible that a $\underline{90}\%$ confidence interval for the coefficient of `inc86` could include 0.

☐ C. It is possible that a $\underline{99}\%$ confidence interval for the coefficient of `inc86` could include 0.

**Solution:** (a) is false. The log-odds interpretation applies to logistic regression. It is not applicable here for Negative Binomial regression.

(b) is false. From the given summary for Model TWO, the 95% confidence interval for the coefficient of `inc86` is $[-0.011, -0.006]$. The 90% confidence interval should be smaller and contained inside the 95% confidence interval. So the 90% confidence interval cannot include 0.

(c) is also false because the normal quantile corresponding to a 99% confidence interval is $z_{\alpha/2} \approx 2.57$ for $\alpha = 0.01$. This leads to the confidence interval: estimate $\pm$ $z_{\alpha_2}$ $\times$ standard error which, with estimate $= -0.0083$ and standard error $= 0.001$, becomes $[-0.01087, -0.00573]$. So this interval still does not contain 0. This answer requires knowledge of the normal quantile corresponding to the 99% level. If one purely argues from the given 95% interval which is $[-0.011, -0.006]$, one might conclude that the 99% confidence interval might include 0 because the 99% confidence interval should be larger and contain the 95% confidence interval.

(c) [2 Pts] Consider now a third model (MODEL THREE) obtained by Negative Binomial regression for $y$ on $x_1, x_3, x_5$ (in other words, the two variables `tottime` and `qemp86` are

now dropped). The summary for this model is given below. Is MODEL THREE preferable to both MODEL ONE and MODEL TWO? You must provide a **numerical** justification for your answer to receive credit.

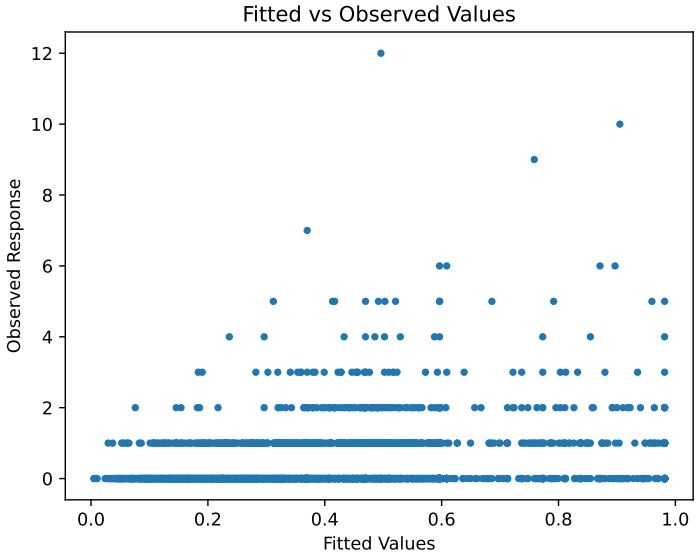
MODEL THREE Summary

| | | | |
|---|---|---|---|
| Dep. Variable: | narr86 | No. Observations: | 2725 |
| Model: | NegativeBinomial | Df Residuals: | 2721 |
| Method: | MLE | Df Model: | 3 |
| Date: | Mon, 06 Nov 2023 | Pseudo R-squ.: | 0.04704 |
| Time: | 00:40:27 | Log-Likelihood: | -2182.9 |
| converged: | True | LL-Null: | -2290.7 |
| Covariance Type: | nonrobust | LLR p-value: | 1.884e-46 |

| | coef | std err | z | P>\|z\| | [0.025 | 0.975] |
|---|---|---|---|---|---|---|
| const | -0.5167 | 0.062 | -8.334 | 0.000 | -0.638 | -0.395 |
| pcnv | -0.4776 | 0.103 | -4.635 | 0.000 | -0.680 | -0.276 |
| inc86 | -0.0086 | 0.001 | -10.860 | 0.000 | -0.010 | -0.007 |
| black | 0.4982 | 0.088 | 5.672 | 0.000 | 0.326 | 0.670 |
| alpha | 1.0174 | 0.115 | 8.821 | 0.000 | 0.791 | 1.243 |

**Solution:** Model THREE has very similar log-likelihood (-2182.9) compared to that of Model TWO (-2182.8). On the other hand, Model THREE has two fewer parameters compared to Model TWO. Thus Model THREE is a smaller (and consequently more interpretable) model having almost the same log-likelihood. So we should prefer Model THREE to Model TWO and also to Model ONE (as we have previously argued that Model ONE is inferior to Model TWO). For a more precise justification, one can calculate the AIC (or BIC) for Model THREE and compare to the corresponding values for Models ONE and TWO.

(d) [3 Pts] Based on the information above and the following plot of the observed response values ($y$) and the fitted values $\hat{y}$ given by MODEL TWO, which of the following statements are true? Select all answers that apply.




■ **A. MODEL TWO is unlikely to be useful for predicting the number of arrests when the number of arrests is large (e.g., 4 or more).**

□ B. The fitted values for MODEL ONE will include many values larger than 1.

□ C. Consider the men for whom $y = 0$ (i.e., they have not been arrested in 1986), but their fitted value from MODEL TWO is more than 0.6. Almost all of them are Black, or have spent many years in prison since turning 18, or both.

**Solution:** The Negative Binomial regression model is given by

$$Y_i \sim \text{Neg-Bin}(\mu_i, \alpha) \quad \text{with } \log \mu_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_m x_{im}.$$

The fitted values in Negative Binomial regression are estimates of $\hat{\mu}_1, \ldots, \hat{\mu}_n$ where

$$\mu_i = \exp\left(\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \cdots + \hat{\beta}_m x_{im}\right)$$

(A) is true because the fitted values for Model TWO are all smaller than 1. This is an indication that the model predictions will all be small (this actually makes sense in this dataset because even though the response consists of counts, more than 90% of the observations have response values equal to 0 or 1).

(B) is false. Model ONE is Poisson regression which has smaller dispersion compared to Negative Binomial regression. Therefore the fitted values of Model ONE will be restricted to an even smaller range (compared to the fitted values of model TWO). So it cannot happen that Model ONE will have many fitted values larger than 1.

(C) is false. This question concerns individuals for which the response value is 0 but the model gives a somewhat high prediction. If all the covariate variables are equal to 0, then the fitted value is $\exp(-0.5038)$ which already exceeds 0.6. However, for the fitted value to be even larger, say to exceed 0.7, contributions from variables with positive coefficients would be necessary (and the only two variables with positive coefficients in Model TWO are `black` and `tottime`). Actually, there was a typo in this question and we wanted to have 0.7 as opposed to 0.6 (in which case (c) would be true).

# 4 Gambler vs Casino [9 Pts]

Bugsy goes to the casino and plays $n$ independent games of chance. Each game, he has probability $p$ of winning, and probability $1-p$ of losing, where $0 < p \leq 0.5$. Let $X$ be the number of times he wins. The casino uses concentration inequalities to guarantee that Bugsy won't win too often: in other words, they want to bound the probability that $X/n$ is not much larger than $p$, with probability at least $0.95$.

(a) [2 Pts] Using Chebyshev's inequality, show that

$$\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{20p(1-p)}{n}}\right) \geq 0.95$$

*Hint: you should use the fact that if* $\mathbb{P}(|a-b| < c) \leq q$*, then* $\mathbb{P}(a - b < c) \leq q$*.*

**Solution:** Write

$$\begin{aligned}
&\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{20p(1-p)}{n}}\right) \\
&= 1 - \mathbb{P}\left(\frac{X}{n} \geq p + \sqrt{\frac{20p(1-p)}{n}}\right) \\
&= 1 - \mathbb{P}\left(\frac{X}{n} - p \geq \sqrt{\frac{20p(1-p)}{n}}\right) \\
&\geq 1 - \mathbb{P}\left(\left(\frac{X}{n} - p\right)^2 \geq \frac{20p(1-p)}{n}\right) \\
&\geq 1 - \frac{\mathbb{E}\left(\frac{X}{n} - p\right)^2}{20p(1-p)/n} \\
&= 1 - \frac{\text{var}(X/n)}{20p(1-p)/n} = 1 - \frac{p(1-p)/n}{20p(1-p)/n} = 1 - \frac{1}{20} = 0.95.
\end{aligned}$$

(b) [3 Pts] Using Hoeffding's inequality, show that

$$\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{\ln 20}{2n}}\right) \geq 0.95.$$

**Solution:** The Hoeffding inequality applied to $X \sim \text{Bin}(n, p)$ is given by:

$$\mathbb{P}\{X \geq t\} \leq \exp\left(-2\frac{(t-np)^2}{n}\right) \quad \text{for } t \geq np.$$

Below we use the above inequality for $t = np + \sqrt{n(\log 20)/2}$. We write

$$\begin{aligned}
&\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{\ln 20}{2n}}\right) \\
&= 1 - \mathbb{P}\left(\frac{X}{n} \geq p + \sqrt{\frac{\ln 20}{2n}}\right) \\
&= 1 - \mathbb{P}\left\{X \geq np + \sqrt{n\frac{\log 20}{2}}\right\} \\
&= 1 - \mathbb{P}\left\{X \geq t\right\} \quad \text{with } t = np + \sqrt{n(\log 20)/2} \\
&\geq 1 - \exp\left(-2\frac{(t-np)^2}{n}\right) \\
&= 1 - \exp\left(-2\frac{n(\log 20)/2}{n}\right) = 1 - \frac{1}{20} = 0.95
\end{aligned}$$

(c) [2 Pts] The casino manager argues that Hoeffding's inequality will always produce a "better" guarantee: in other words, that for the same probability $0.95$, that $p + \sqrt{\frac{\ln 20}{2n}} < p + \sqrt{\frac{20p(1-p)}{n}}$. Find a value of $p$ that proves the casino manager wrong. You do not need to find all such values of $p$: providing one is enough.

*Hint:* $\ln(20) \approx 3$

**Solution:** We need to find a value of $p$ such that the Hoeffding bound is worse than the Chebyshev bound. In other words, we want $p$ such that

$$p + \sqrt{\frac{\ln 20}{2n}} \geq p + \sqrt{\frac{20p(1-p)}{n}}$$

The above is equivalent to

$$\frac{\ln 20}{2n} \geq \frac{20p(1-p)}{n} \iff \frac{\ln 20}{2} \geq 20p(1-p).$$

Clearly this will be true if $p$ is very close to 0. So $p = 10^{-5}$ proves the casino manager wrong.

(d) [2 Pts] Which of the following statements are true? Select all answers that apply.

- ☐ A. The bound from Chebyshev's inequality is "tight": in other words, the true probability $\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{20p(1-p)}{n}}\right)$ is always at most 0.96.

□ B. The bound from Hoeffding's inequality is "tight": in other words, the true probability $\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{\ln 20}{2n}}\right)$ is always at most 0.96.

■ **C. If we use Chernoff's bound with the Binomial MGF and the optimal value of $\lambda$, then we will obtain a better bound than the result from parts (a) or (b).**

**Solution:** Generally both Chebyshev and Hoeffding are quite loose. For example, when $n = 100$ and $p = 0.5$, one can check on the computer that $\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{20p(1-p)}{n}}\right)$ for $X \sim \text{Bin}(n, p)$ is more than 0.9999. Also $\mathbb{P}\left(\frac{X}{n} < p + \sqrt{\frac{\ln 20}{2n}}\right)$ is more than 0.99. So both A and B are false. C is true. We have seen in class (Lecture 18) that Chernoff bound is stronger than the Hoeffding bound. It is also stronger than Chebyshev as can be verified directly (this makes sense because the Chernoff bound uses information on the MGF while the Chebyshev bound uses only the variance).

# 5 Can Taking Data 102 Make You Happier? [8 Pts]

The Data 102 staff want to determine whether taking Data 102 causes an increase in student happiness. They collect the following data from a random sample of Data Science majors:

- `happiness`: a number from 0 to 10 indicating how happy the student is
- `ds102`: whether or not the student has taken Data 102 (binary)
- `GPA`: the student's GPA
- `ds140`: whether or not the student has taken Data 140 (binary)

(a) [2 Pts] They assume that GPA is the only confounding variable for the effect of taking Data 102 on happiness. Which of the following statements, if true, would make this assumption incorrect? Select all answers that apply.

- ☐ A. Happiness causes an increase in GPA
- ■ **B. Taking Data 140 increases the chances of a student taking Data 102, and also increases their happiness**
- ☐ C. Students who work together on a Data 102 project are likely to increase each others' happiness

**Solution:** Even if happiness causes increased GPA, GPA Choice B means that taking Data 140 is a confounding variable, since it has a causal effect on their happiness and their decision to take Data 102.

Choice C does not affect whether GPA is a confounding variable (although it is a violation of SUTVA).

**For the remainder of this question, assume that they are correct, and that GPA is the only confounding variable for the effect of taking Data 102 on happiness.**

(b) [3 Pts] Suppose their sample only contains four students, and their data is as follows. Compute the IPW estimate for the average treatment effect. You do not need to simplify arithmetic expressions to receive full credit.

| `happiness` | `ds102` | `GPA` | `ds140` | propensity score |
|---|---|---|---|---|
| 4 | 0 | 3.2 | 1 | 0.8 |
| 9 | 1 | 2.7 | 0 | 0.9 |
| 5 | 1 | 3.1 | 0 | 0.2 |
| 6 | 0 | 3.4 | 1 | 0.4 |

**Solution:** The treatment is `ds102`, and the outcome is `happiness`. The IPW formula is:

$$\begin{aligned}\hat{\tau}_{IPW} &= \frac{1}{n}\left[\sum_{i:Z_i=1}\frac{Y_i}{e(X_i)} - \sum_{i:Z_i=0}\frac{Y_i}{1-e(X_i)}\right] \\ &= \frac{1}{4}\left[\frac{9}{0.9} + \frac{5}{0.2} - \frac{4}{0.2} - \frac{6}{0.6}\right]\end{aligned}$$

(c) [3 Pts] Alan learns that several semesters ago, Data 102 enrollment was cut in half just before the semester started. Half the students were removed at random, and were enrolled in an alternative course. He then discovers that some of the students who were removed got re-enrolled, because they didn't have prerequisites for the alternative course.

Can the staff use the random enrollment decision as an instrumental variable? For each of the assumptions necessary for instrumental variables, specify whether the random enrollment decision satisfies the criterion, and explain why or why not.

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# 6 Congratulations [0 Pts]

Congratulations! You have completed Midterm 2.

- **Make sure that you have written your student ID number on *every other page* of the exam.** You may lose points on pages where you have not done so.
- Also ensure that you have **signed the Honor Code** on the cover page of the exam for 1 point.
- If more than 10 minutes remain in the exam period, you may hand in your paper and leave. If $\leq$ 10 minutes remain, please **sit quietly** until the exam concludes.

[Optional, 0 pts] What's on your mind?

# Midterm 2 Reference Sheet

**Useful Distributions:**

| Distribution | Support | PDF/PMF | Mean | Variance | Mode |
|---|---|---|---|---|---|
| $X \sim \text{Poisson}(\lambda)$ | $x = 0, 1, 2, \ldots$ | $\frac{\lambda^x e^{-\lambda}}{x!}$ | $\lambda$ | $\lambda$ | $\lfloor \lambda \rfloor$ |
| $X \sim \text{Binomial}(n, p)$ | $x \in \{0, 1, \ldots, n\}$ | $\binom{n}{x} p^x (1-p)^{1-x}$ | $np$ | $np(1-p)$ | $\lfloor (n+1)p \rfloor$ |
| $X \sim \text{Beta}(\alpha, \beta)$ | $0 \leq x \leq 1$ | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha}{\alpha+\beta}\frac{\beta}{\alpha+\beta}\frac{1}{\alpha+\beta+1}$ | $\frac{\alpha-1}{\alpha+\beta-2}$ |
| $X \sim \text{Gamma}(\alpha, \beta)$ | $x \geq 0$ | $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ | $\frac{\alpha-1}{\beta}$ |
| $X \sim \mathcal{N}(\mu, \sigma^2)$ | $x \in \mathbb{R}$ | $\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$ | $\mu$ | $\sigma^2$ | $\mu$ |
| $X \sim \text{Exponential}(\lambda)$ | $x \geq 0$ | $\lambda \exp(-\lambda x)$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | 0 |

**Conjugate Priors:** For observations $x_i$, $i = 1, \ldots, n$:

| Likelihood | Prior | Posterior |
|---|---|---|
| $x_i \vert \theta \sim \text{Bernoulli}(\theta)$ | $\theta \sim \text{Beta}(\alpha, \beta)$ | $\theta \vert x_{1:n} \sim \text{Beta}\left(\alpha + \sum_i x_i, \beta + \sum_i (1 - x_i)\right)$ |
| $x_i \vert \mu \sim \mathcal{N}(\mu, \sigma^2)$ | $\mu \sim \mathcal{N}(\mu_0, 1)$ | $\mu \vert x_{1:n} \sim \mathcal{N}\left(\frac{\sigma^2}{\sigma^2+n}\left(\mu_0 + \frac{1}{\sigma^2}\sum_i x_i\right), \frac{\sigma^2}{\sigma^2+n}\right)$ |
| $x_i \vert \lambda \sim \text{Exponential}(\lambda)$ | $\lambda \sim \text{Gamma}(\alpha, \beta)$ | $\lambda \vert x_{1:n} \sim \text{Gamma}\left(\alpha + n, \beta + \sum_i x_i\right)$ |

**Generalized Linear Models**

| Regression | Inverse link function | Likelihood |
|---|---|---|
| Linear | identity | Gaussian |
| Logistic | sigmoid | Bernoulli |
| Poisson | exponential | Poisson |
| Negative binomial | exponential | Negative binomial |

Some powers of $e$:

| $x$ | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = e^x$ | 1.05 | 1.11 | 1.22 | 1.35 | 1.49 | 1.65 | 1.82 | 2.01 | 2.23 | 2.46 | 2.72 |