A superpopulation model...

Z_i = Treatment indicator
$Y_i(0), Y_i(1)$ = Potential outcomes.
X_i = Covariate vector [e.g. age, gender, ...].

We usually assume that we observe i.i.d. samples
$(Z_i, Y_i(0), Y_i(1), X_i)$ drawn from a superpopulation.
[i.e. a density over $(z, y(1), y(0), x)$.].

The average treatment effect (ATE) is now an expectation:

$$\tau = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)].$$

What goes wrong?

We observe only $\mathbb{E}[Y(1) \mid Z=1]$ and $\mathbb{E}[Y(0) \mid Z=0]$.
In general, these are different from $\mathbb{E}[Y(1)]$ and $\mathbb{E}[Y(0)]$ respectively, i.e. the prima facie treatment effect defined by

$$\tau_{PF} = \mathbb{E}[Y(1) \mid Z=1] - \mathbb{E}[Y(0) \mid Z=0]$$

is not the same as τ.

Can write τ as

$$\begin{aligned}\tau &= \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] \\ &= \mathbb{E}[Y(1) \mid Z=1] \cdot \mathbb{P}(Z=1) + \mathbb{E}[Y(1) \mid Z=0] \cdot \mathbb{P}(Z=0) \\ &\quad - \left(\mathbb{E}[Y(0) \mid Z=1] \cdot \mathbb{P}(Z=1) + \mathbb{E}[Y(0) \mid Z=0] \cdot \mathbb{P}(Z=0).\right)\end{aligned}$$

We do not observe the highlighted terms, so τ

is not identifiable unless we make further assumptions....

What does randomization give us?

$$Z \perp\!\!\!\perp \{Y(1), Y(0)\}.$$

Rem: This is not saying that $Z \perp\!\!\!\perp Y_{obs}$.

This implies $\mathbb{E}[Y(1) | Z=1] = \mathbb{E}[Y(1) | Z=0]$.

$$\Rightarrow \mathbb{E}[Y(1)] = \mathbb{E}[Y(1) | Z=1].$$

Analogously, $\mathbb{E}[Y(0)] = \mathbb{E}[Y(0) | Z=0]$.

$$\Rightarrow \tau = \tau_{PF}$$

How to identify ATE in observational studies.

Unconfoundedness / ignorability / exchangeability:

$$\{Y(1), Y(0)\} \perp\!\!\!\perp Z \mid X$$

Define $\tau(x) = \mathbb{E}[Y(1) | X=x] - \mathbb{E}[Y(0) | X=x]$.

$$\tau_{PF}(x) = \mathbb{E}[Y(1) | X=x, Z=1] - \mathbb{E}[Y(0) | X=x, Z=0].$$

If unconfoundedness holds, then

$$\mathbb{E}[Y(1) | Z=1, X=x] = \mathbb{E}[Y(1) | X=x].$$

$$\mathbb{E}[Y(0) | Z=0, X=x] = \mathbb{E}[Y(0) | X=x]$$

$$\Rightarrow \tau(x) = \tau_{PF}(x).$$

If X is discrete, then

$$\tau = \mathbb{E}[\mathbb{E}[Y(1) - Y(0) \mid X]].$$
$$= \mathbb{E}[\tau(X)].$$
$$= \sum_x \tau(x) \cdot \mathbb{P}(X=x).$$
$$= \sum_x \tau_{PF}(x) \cdot \mathbb{P}(X=x).$$

We know $\tau_{PF}(x)$ and $\mathbb{P}(X=x)$.

$\Rightarrow$ use this to identify τ.

When we have finitely many samples, use plug-in estimators for everything.

$$\hat{\tau}_{PF}(x) = \frac{1}{n_{1,x}} \sum_{\substack{X_i=x \\ Z_i=1}} Y_{i,obs} - \frac{1}{n_{0,x}} \sum_{\substack{X_i=x \\ Z_i=0}} Y_{i,obs},$$

where $n_{1,x} = \#\{i : X_i = x, Z_i = 1\}$.
$n_{0,x} = \#\{i : X_i = x, Z_i = 0\}$.

$$\hat{\mathbb{P}}(X=x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(X_i = x).$$

$$\Rightarrow \hat{\tau} = \sum_x \hat{\tau}_{PF}(x) \cdot \widehat{\mathbb{P}(X=x)}.$$

This relies on assumption that X is discrete.
When X is cts, or X has many levels, then need other methods...

3 methods:
① Regression.

② Propensity score weighting
③ Matching.

Back to kidney stones.

$Z = 1(\text{Treatment } B)$.

$Y(1)$ = Recovery under treatment B.

$Y(0)$ = Recovery under treatment A.

$X = 1(\text{small kidney stones})$.

	Treatment A helps.	Treatment B helps.
Large kidney stones.	69%	73%
Small kidney stones.	87%	93%.
All patients	83%	78%

Proportion of patients with small kidney stones $= 51\%$

$$\tau(1) = \mathbb{E}[Y(1) \mid X=1] - \mathbb{E}[Y(0) \mid X=1]$$

$$\overset{\text{unconfoundedness}}{=} \mathbb{E}[Y(1) \mid X=1, Z=1] - \mathbb{E}[Y(0) \mid X=1, Z=0].$$

$$= 0.93 - 0.87$$

$$= 0.06.$$

$$\tau(0) = 0.73 - 0.69$$

$= 0.04.$

$$\tau = \tau(1)\cdot \mathbb{P}(X=1) + \tau(0)\cdot \mathbb{P}(X=0)$$
$$= 0.06 \cdot 0.51 + 0.04 \cdot 0.49$$
$$\approx 0.05.$$

Methods for estimating ATE under unconfoundedness.

Method 1: Outcome regression.

ATE $\tau = \mathbb{E}[\tau(X)] = \int \tau(x)\cdot p(x)dx.$

where $p(x)$ = density of X, and

$\tau(x) = \mathbb{E}[Y(1) \mid X=x] - \mathbb{E}[Y(0) \mid X=x].$

is called the Conditional Average Treatment Effect (CATE) function.

Outcome regression comprises 2 steps:

Step 1: Estimate $\tau(x)$ via $\hat{\tau}(x)$.

Step 2: Use plug-in estimator for τ. (i.e. we substitute $p(x)$ with empirical distribution $\hat{p}(x)$ [= uniform distribution on $\{X_1, \dots, X_n\}$]).

Hence, $\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} \hat{\tau}(X_i).$

To estimate τ is tricky.

Many strategies, we will state 2 of these.

Strategy 1: Fit joint model for $\mu(x,z) = \mathbb{E}[Y_{obs} \mid X=x, Z=z]$

observe that $\tau(x) = \mu(x,1) - \mu(x,0).$

Strategy 2: Fit 2 models, one each for

- $\mu_{(1)}(x) = \mathbb{E}[Y_{obs} \mid X = x, Z = 1]$
- $\mu_{(0)}(x) = \mathbb{E}[Y_{obs} \mid X = x, Z = 0]$.

Observe that $\tau(x) = \mu_{(1)}(x) - \mu_{(0)}(x)$.

What can we use to estimate $\mu, \mu_{(1)}, \mu_{(0)}$?

- Linear model.
- GLM.

} Model may be misspecified.

- ML methods. $\rightsquigarrow$ problems with overfitting & extrapolation.

<u>Fact</u>: Use Strategy 1 with a linear model, i.e. regress $Y \sim X + Z$, get is a model.

$$Y = \hat{\alpha} + \hat{\beta}^T X + \hat{\gamma} Z + \varepsilon.$$

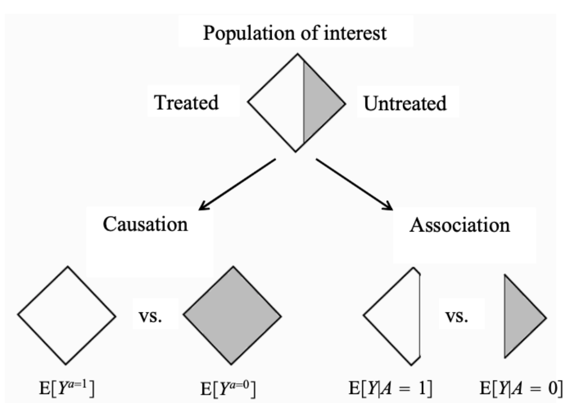
Then $\hat{\tau} = \hat{\gamma}$.

Pf. $\hat{\tau}(x) = \hat{\mu}(x,1) - \hat{\mu}(x,0)$. $[\hat{\mu}(x,z) = \hat{\alpha} + \hat{\beta}^T x + \hat{\gamma} z]$

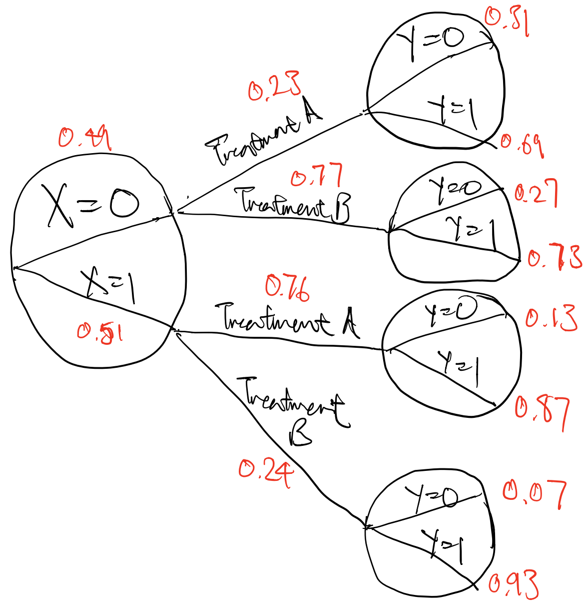
$$= \hat{\gamma}.$$

$$\Rightarrow \hat{\tau} = \hat{\gamma}. \quad \blacksquare$$

<u>Method 2: Inverse propensity score weighting</u>.



Observed:



See slides / lecture video for explanation of above.

<u>Thm</u>. Assume $\{Y(1), Y(0)\} \perp\!\!\!\perp Z \mid X$, then.

① $$\mathbb{E}\left[\frac{Y_{obs} \cdot Z}{e(X)}\right] = \mathbb{E}[Y(1)].$$

② $$\mathbb{E}\left[\frac{Y_{obs}(1-Z)}{1-e(X)}\right] = \mathbb{E}[Y(0)].$$

Hence, $\tau = \mathbb{E}\left[\frac{Y_{obs} \cdot Z}{e(X)}\right] - \mathbb{E}\left[\frac{Y_{obs}(1-Z)}{1-e(X)}\right].$

For finite samples, we have the inverse propensity weighting (IPW) estimator.

$$\hat{\tau}_{IPW} = \frac{1}{n}\sum_{i=1}^{n} \frac{Y_{i,obs} Z_i}{\hat{e}(X_i)} - \frac{1}{n}\sum_{i=1}^{n} \frac{Y_{i,obs}(1-Z_i)}{\hat{e}(X_i)}.$$

where $\hat{e}(x)$ is an estimator of $e(X)$.

<u>Fact</u>. Assume $\{Y(1), Y(0)\} \perp\!\!\!\perp Z \mid X$, then

$\{Y(0), Y(1)\} \perp\!\!\!\perp Z \mid e(X)$

Proofs of Thm, Fact are easy.