HW 1

Solutions

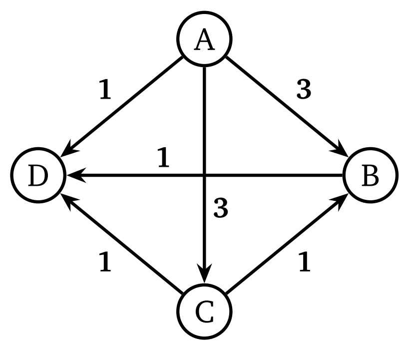
Modeling Democracy

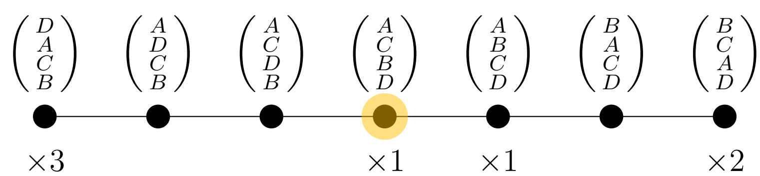
Duchin, Winter 2026

General instructions: Please TeX your solutions. Ask for help with math or TeX or anything else if needed!

Problem 1. Practice all of the voting systems we discussed on this simple reduced preference schedule. (That is: Plurality, PWC, Borda, Condo-Borda, Top-Two, IRV, Sequential, STV, Coombs, Secondality, Smith, Beatpath, Dodgson, Kemeny, Dictatorship, Ranked Pairs, and any others of your choice.) For some of these you will have to specify ancillary information, like a sequential order or a tiebreaking protocol.

1	2	3	1
A	B	D	A
B	C	A	C
C	A	C	B
D	D	B	D





Solution 1. *For single-winner methods, let's assume alphabetical tiebreakers. D wins by plurality. A wins by many other methods (PWC, Condo-Borda, Dodgson, Smith, Beatpath) by virtue of being Condorcet. A also wins by Borda, Top-Two, IRV, Coombs, and even Secondality.*

To see this for Borda (with scoring 3-2-1-0), note that A has $2(3) + 3(2) + 2(1) = 14$ *Borda points, while C has 10 and B,D have 9. For Top-Two, D has the most FPV, while A,B tie; since we're using alphabetical tiebreakers, A and D advance to the instant runoff and A wins. In IRV we have a threshold of 4 votes needed to be elected. We first eliminate C with no FPV, then A or B is up for elimination. Since alphabetical tiebreakers give priority to A, we eliminate B. With just A and D left, A wins 4 to 3. On the other hand, Coombs eliminates D first, then B, leaving A vs. C head-to-head, so A wins. A and C tie in Secondality, but the tiebreaker gives it to A.*

If we move to STV to elect 2 winners, the Droop quota is just over 1/3 of the ballots, which is 3 votes needed to win. D is elected in the first round with no surplus, so those votes don't transfer. Next, A *and B have 2 first place votes each, not yet at threshold. We eliminate C and that leaves A and B, each with two votes. So once again we rely on the tiebreaker, and the winners are* $\{D, A\}$*. Interestingly, if we took this exact election and scaled it up so the vote totals per column were 10, 20, 30, 10, we'd get a different dynamic. Now the quota is* $\lfloor 70/3 \rfloor + 1 = 24$*. This time the D ballots transfer with weight* $6/30 = 1/5$*, which sends 6 more votes to A, and that gives A enough support to clear threshold without another elimination or tiebreak.*

Any of A, B, or D might win by Dictatorship, depending on which voter is Dictator.

One method that requires different reasoning is Kemeny, where I try to find the complete permutation that minimizes the sum of distances to the cast ballots (counted with multiplicity).

We can study a portion of the ballot graph on four candidates, shown above, that contains all the ballots cast in this profile. I've put the number of each ballot into the figure. The path is a geodesic, *meaning that there are no shortcuts between any pair of its points. That means to calculate the distance between two ballots in this profile, it suffices to consider the distance along this path. An easy calculation shows that the "cost" of* $\begin{pmatrix} A \\ C \\ B \\ D \end{pmatrix}$ *(the sum of its distances to the ballots) is* $3(3)+1(0)+1(1)+2(3) = 16$*, while moving to either side increases the cost. So the winning ranking in a Kemeny election is* $A \succ C \succ B \succ D$*.*

Ranked pairs finds the same answer a different way. Recall that it calls for "turning on" the arrows from highest margin to lowest, rejecting only those that form a directed cycle. However, there are no cycles in this graph, and there's a Condorcet order $ACBD$*. In graph theory, this is called a* topological sort*—it's a sequence where all arrows in the graph point from earlier to later nodes in the list.*

Problem 2. Show that domsets are nested. (That is, if X and Y are both dominating sets for a given preference profile, then $X \subseteq Y$ or $Y \subseteq X$.)

Solution 2. *This one's pretty easy: if some two domsets were not nested, then there would be an element* $x \in X \setminus Y$ *and another element* $y \in Y \setminus X$*. Since* X *is a domset, all arrows point out, so* x *is preferred to* y *head-to-head. But by symmetry we have a problem:* Y *is also a domset, so* y *is preferred to* x *head-to-head.* $\lightning$

Problem 3. Show that the winner of a sequential election is always "strong" (i.e., belongs to the Smith set).

Solution 3. *The winner of a sequential election, say X, has the property that for every other candidate, say Y, there is a chain of directed arrows from X to Y. This is because every other candidate was eliminated by someone, who was eliminated by someone, and so on, and finally eliminated by X. But this means it's impossible for Y to be in a domset that does not include X; the forward chain from X to Y would have to enter the domset, which is impossible.*

Problem 4. Show that a Condorcet candidate can be a losing spoiler.

Solution 4. *One simple example is*

$\times 51$	$\times 50$	$\times 20$
A	C	B
B	B	C
C	A	A

Here, A is the plurality winner, but if B is disqualified, then 51 people prefer A to C while 70 people prefer the reverse. This means that without B, the plurality winner switches to C, which means B is a losing spoiler.

However, they're Condorcet because preferred head-to-head to both A (by a margin of 70-51) and C (by a margin of 71-50). So a Condorcet candidate can be a losing spoiler.

Incidentally, it's notable that A is anti-Condorcet, losing to both *of the other two head-to-head, but nonetheless wins the plurality election.*

Problem 5.

(a) Show that beatpath elimination ($\triangleright$) is transitive.

(b) Conclude that the beatpath method is well-defined (same winner(s) no matter what order you consider candidates in) and that $|\mathcal{W}| \geq 1$.

(c) Show that the beatpath method has the unanimity property.

(d) Show that beatpath is Smith-fair.

Solution 5. *(a) I need to show that $A \triangleright B \triangleright C$ implies $A \triangleright C$. Let the strongest (and unmatched) beatpath $A \twoheadrightarrow B$ be called P, respectively Q goes $B \twoheadrightarrow C$. Finally, let R be the strongest beatpath $C \twoheadrightarrow A$, and we'll check if it's unmatched. Suppose their strengths are p, q, r—recalling that the strength of a beatpath is its smallest margin on any single arrow. I claim that $p, q > r$. Consider the concatenated paths $R.P$ from C to B and $Q.R$ from B to A. Because of the assumed eliminations, these must be weaker than Q and P, respectively. That means $\min(r, p) < q$ and $\min(q, r) < p$. This forces r to be strictly smallest of the three, so the beatpath $P.Q$ from A to C is stronger than any from C to A, which is what we needed to show.*

(b) This one is a cool argument. I want to show that it's impossible that every candidate is eliminated. Start with some candidate A_1 and suppose they are eliminated by A_2 and they are eliminated by A_3 and so on (with A_j not necessarily distinct from the previous ones). Either this terminates before everyone has been eliminated, which means some candidate is not eliminated, or else this must at some point repeat a candidate name. But that would be a contradiction, because by transitivity it would lead to some candidate eliminating themselves, which is impossible.

(c) If everyone prefers X to Y, then the arrow $X \to Y$ in the pairwise comparison graph has strength N, the number of voters. This must be an unmatched beatpath because if Y had a beatpath of strength N back to X it would mean Y is unanimously preferred to B_1 who is unanimously preferred to B_2 and so on, with finally some B_k unanimously preferred to X. But this means that X is above Y on every ballot, and also below Y on every ballot. $\lightning$

(d) I must show that the winner(s) of a beatpath election always belong to the Smith set. This is fairly easy: nobody outside the Smith set has any *beatpath to someone inside it, no matter the strength. So all of the weak (non-Smith) candidates are immediately eliminated by any of the strong (Smith) ones.*