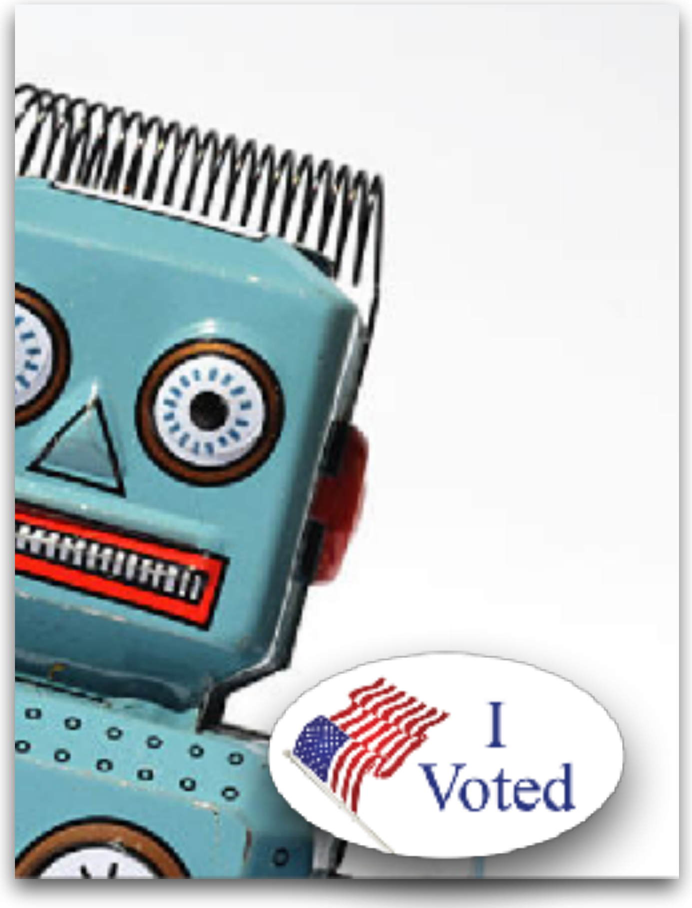
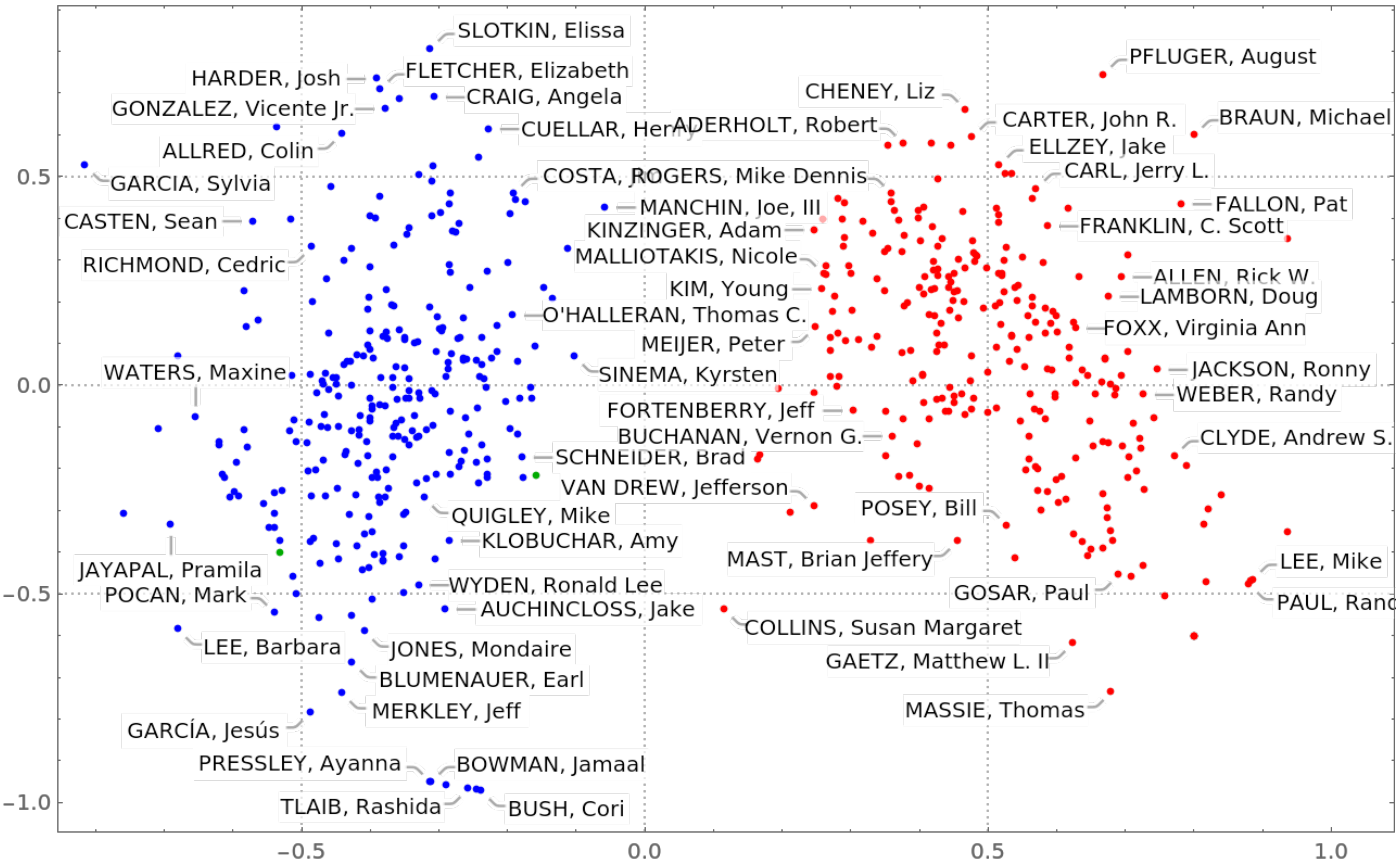
Modeling Democracy

Lecture 8 - **Metric distortion and new voting rules**



Moon Duchin, Winter 2026 — Mon Feb 2



SLOTKIN, Elissa
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HARDER, Josh
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FRANKLIN, C. Scott
MALLIOTAKIS, Nicole
RICHMOND, Cedric
ALLEN, Rick W.
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PAUL, Ranc
AUCHINCLOSS, Jake
COLLINS, Susan Margaret
LEE, Barbara
JONES, Mondaire
GAETZ, Matthew L. II
BLUMENAUER, Earl
MERKLEY, Jeff
MASSIE, Thomas
GARCÍA, Jesús
PRESSLEY, Ayanna
BOWMAN, Jamaal
TLAIB, Rashida
BUSH, Cori
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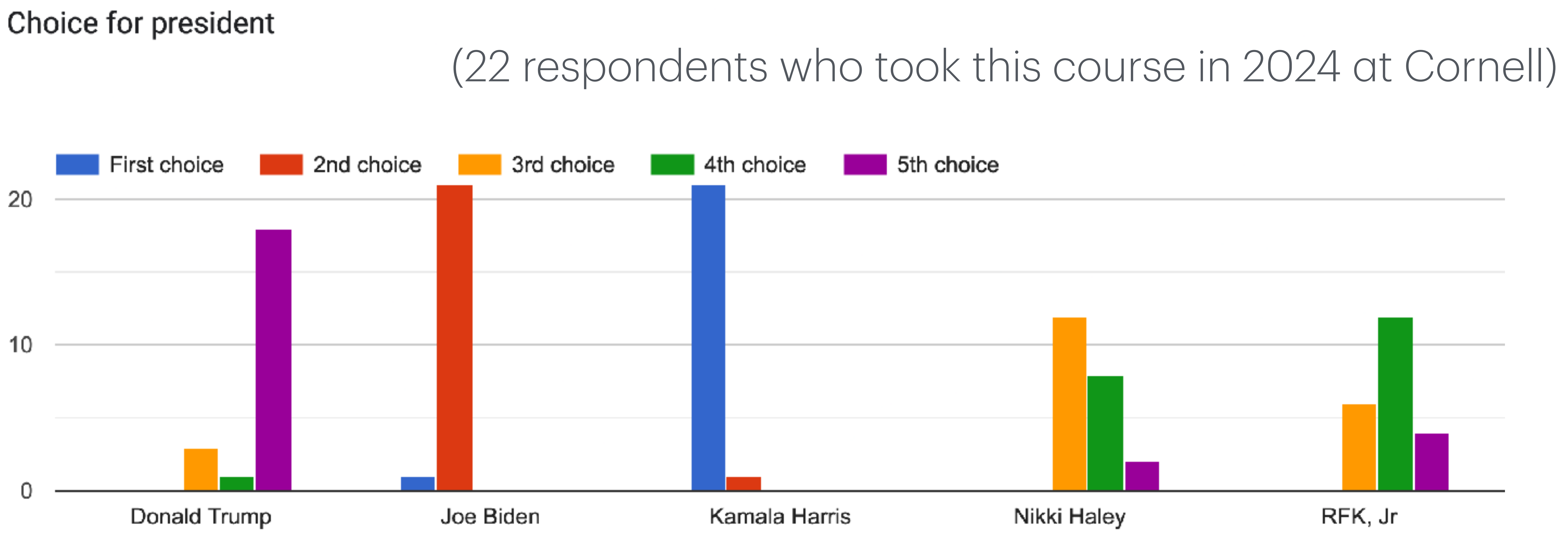

11	4	2	1	1	1	1	1
Harris	Harris	Harris	Harris	Harris	Harris	Biden	Harris
Biden	Biden	Biden	Biden	Biden	Biden	Harris	Biden
Haley	RFK	Trump	Haley	RFK	Trump	RFK	
RFK	Haley	Haley	Trump	Haley	RFK	Haley	
Trump	Trump	RFK	RFK	Trump	Haley	Trump	K,H,T

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Stress of the MDS embedding: 1001434.755861423
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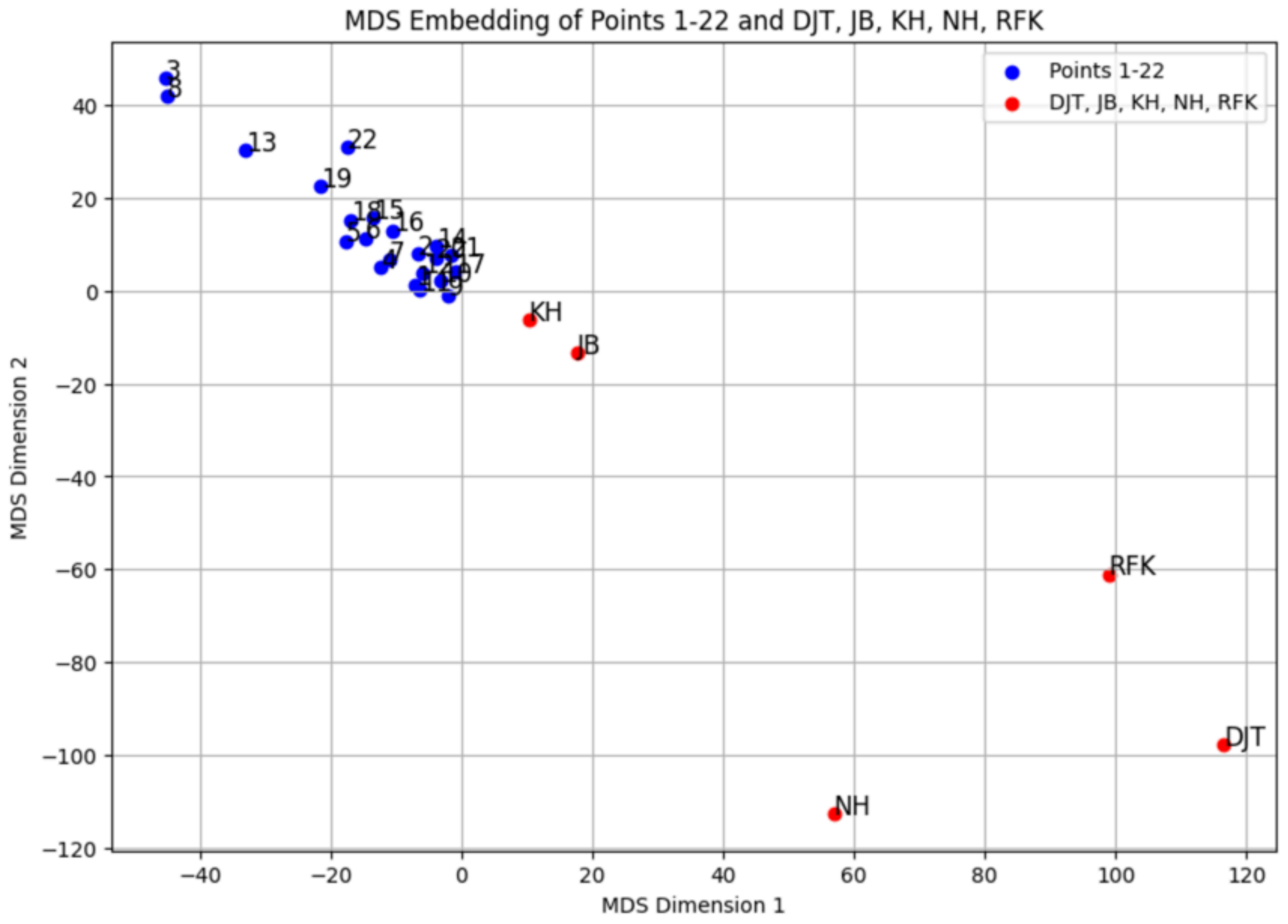
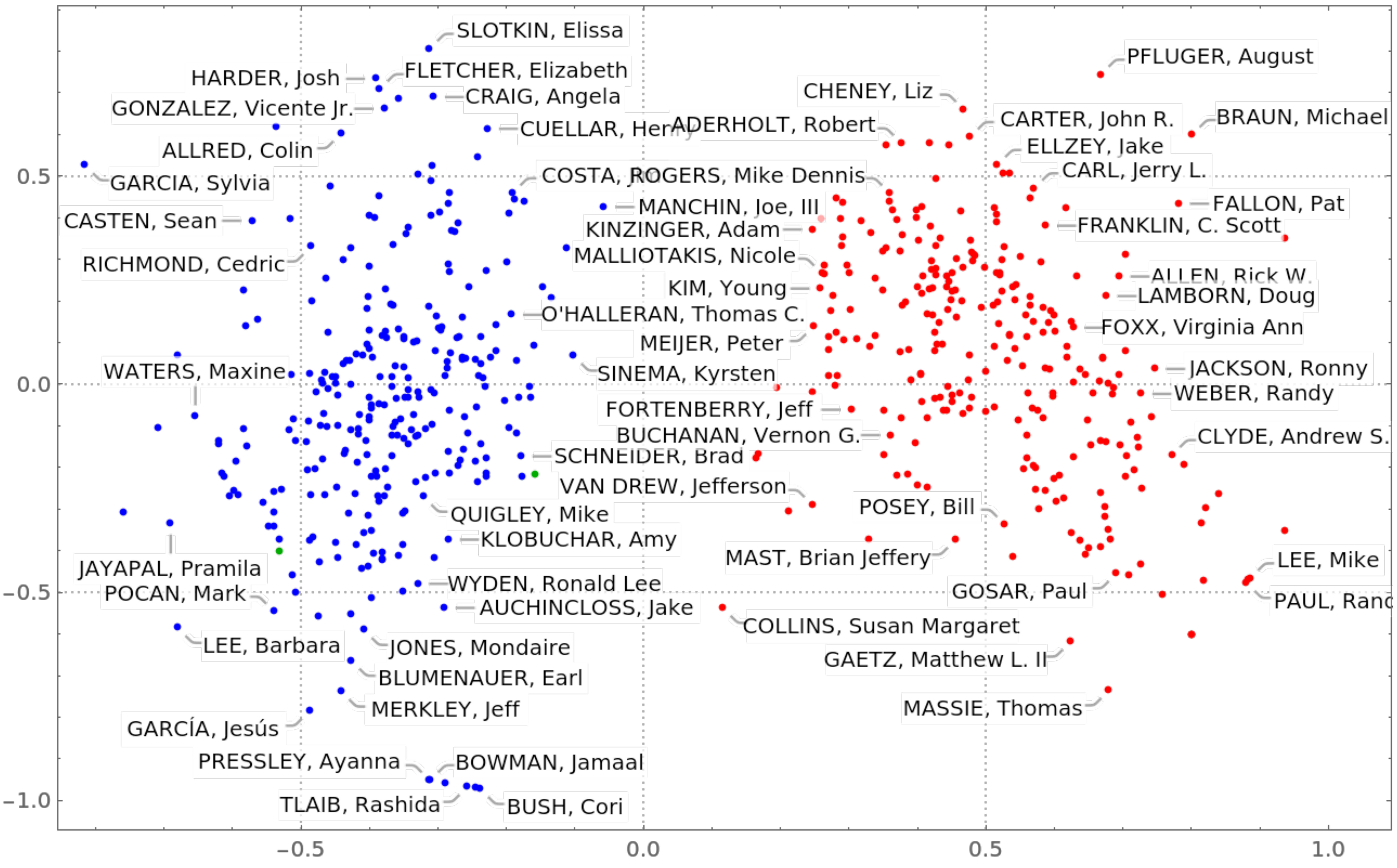


	mean	median
Harris	25.1	20
Biden	37.1	30
Haley	578.1	97
Trump	45,459,241.8	100
Kennedy	45,455,004.3	100



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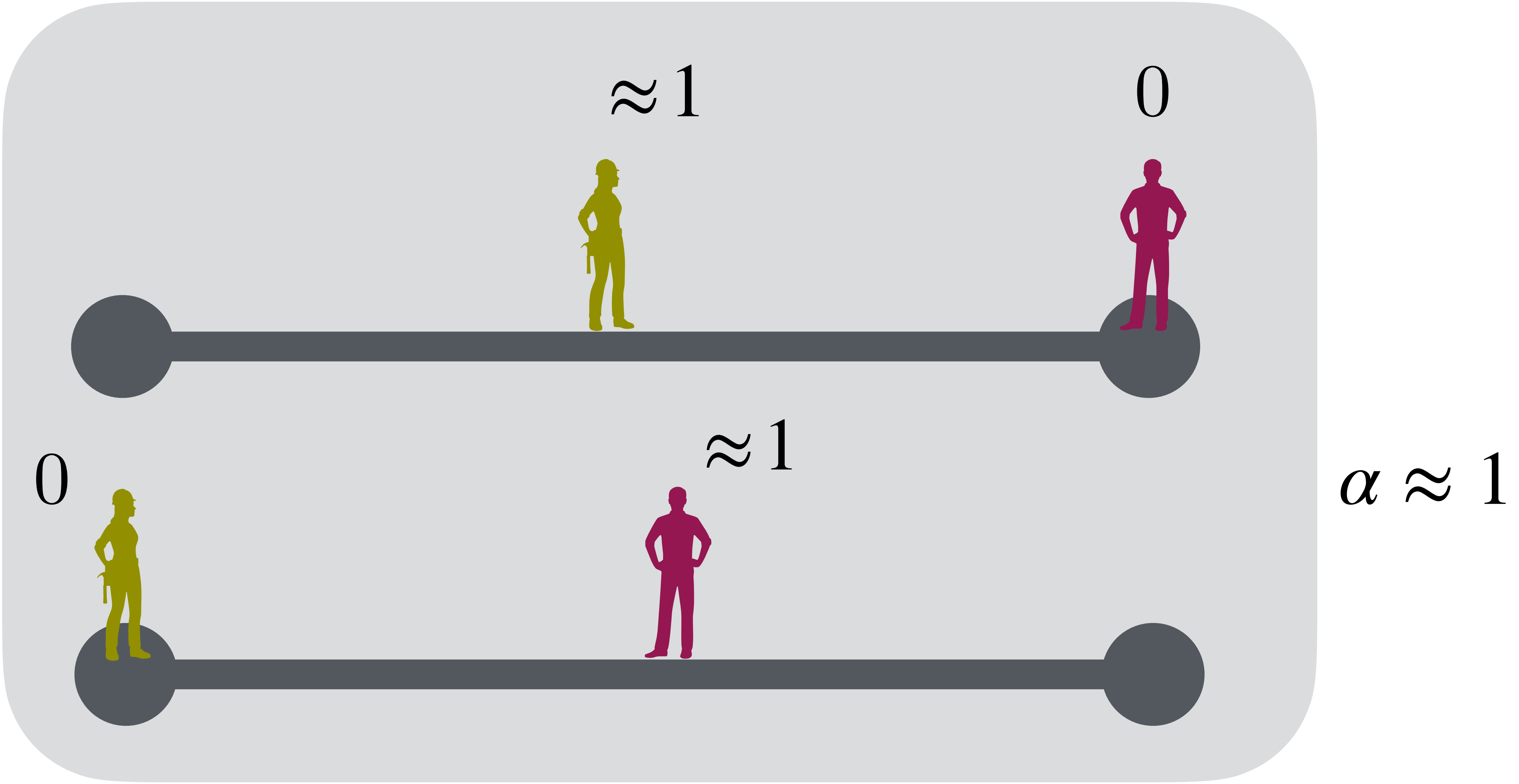
indifference is a hard case.

Ruling out voter indifference

α of an embedding is a measure of voter first-to-worst preference strength: highest ratio for any voter of lowest to highest distance to candidate

note these ratios are always ≤1, so the overall $\alpha \leq 1$. (i.e., $\alpha = 1$ is unrestrictive)

$\alpha < 1$ rules out total indifference



Often can get better bounds when $\alpha < 1$

A simple rule with good metric distortion

- **Random Dictatorship**: choose a voter uniformly at random to be Dictator
- In a single-winner election, this amounts to setting candidate's probability of winning proportional to their first-place vote share

Theorem:

Expected metric distortion of Random Dictator is no worse than $3 - \frac{2}{N}$.

Stronger Theorem (with controlled indifference α):

Expected metric distortion of Random Dictator is no worse than $2 + \alpha - \frac{2}{N}$.

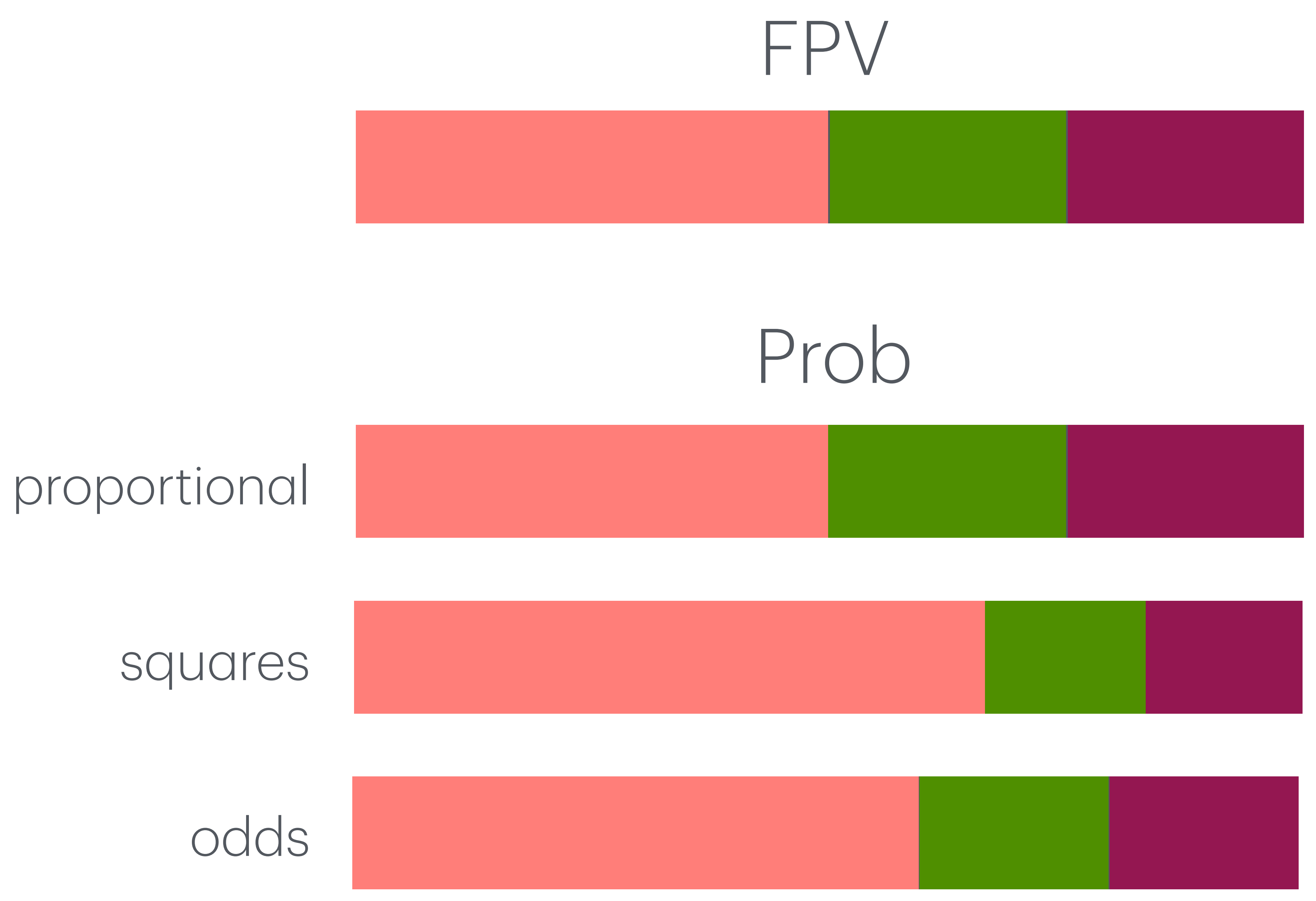
Some different weighting schemes

Suppose FPV distribution is 1/2, 1/4, 1/4

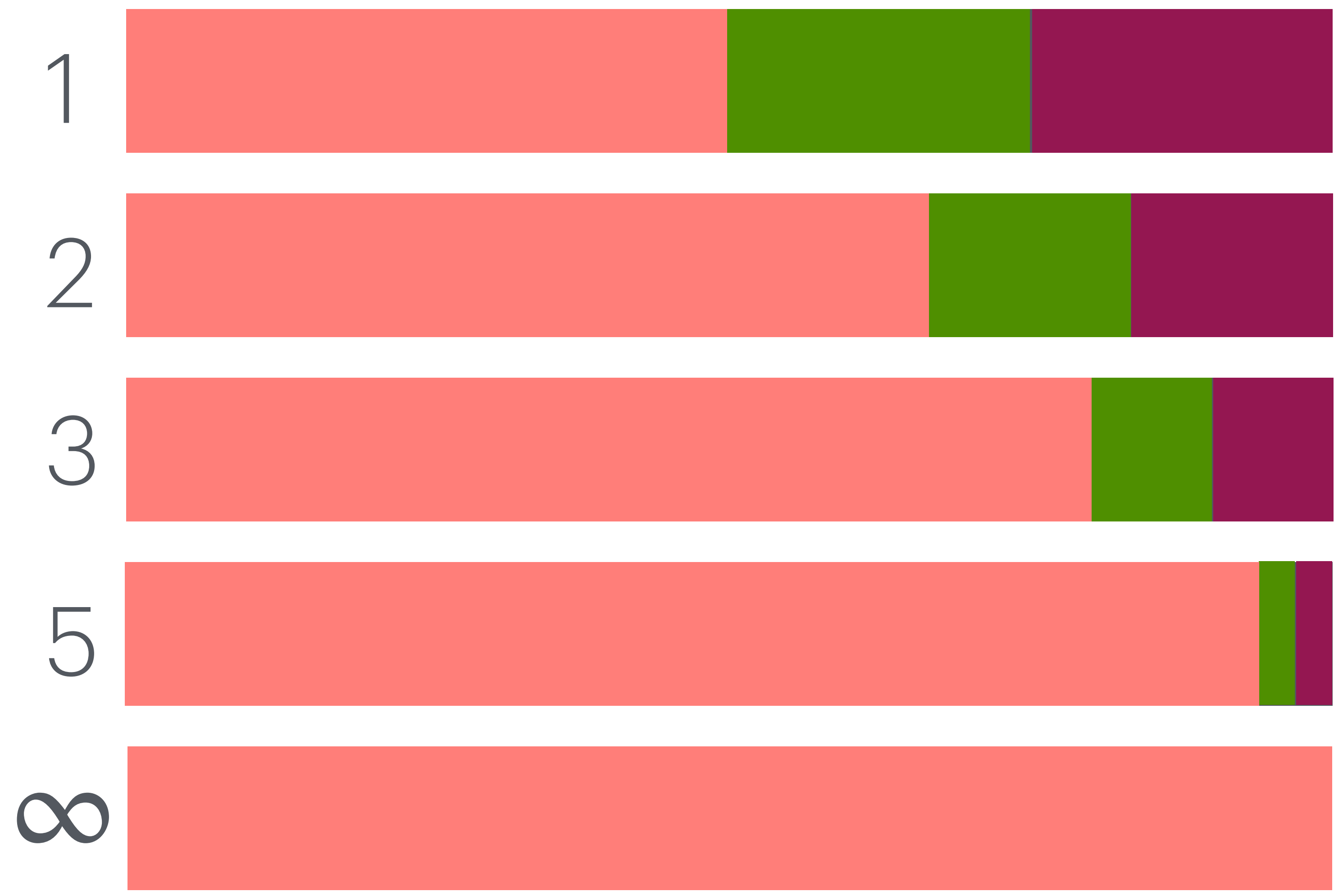
proportional to FPV:
weights 1/2, 1/4, 1/4
(X is 2 times as much as Y,Z)
X selected with prob 1/2 (50%)

proportional to squares:
weights 1/4, 1/16, 1/16
(X is 4 times as much as Y,Z)
X selected w/ prob 1/4 over 3/8 = 2/3 (66.7%)

proportional to odds:
weights 1, 1/3, 1/3
(X is 3 times as much as Y,Z)
X selected w/ prob 1 over 5/3 = 3/5 (60%)



Effect of proportional to power?



- Boosts FPV leaders, interpolates from Random Dictator ($q = 1$) to Deterministic Plurality ($q \to \infty$)
- Interpretation: tracks likelihood of randomly sampling q voters and having them agree on their first choice

Effect of odds?

- In general, odds is keyed to a measurement of how much information is needed to change your mind. Bayes Rule in odds form: *posterior odds equals prior odds times likelihood ratio.*
- Example: prior is that E is as likely as not-E. Suppose you get successive independent updates favoring E by 2 to 1.
  - Probabilities: 1/2 → 2/3 → 4/5 → 8/9 → 16/17
  - Odds: (1:1) → (2:1) → (4:1) → (8:1) → (16:1)
- and *log odds* increases in constant increments!

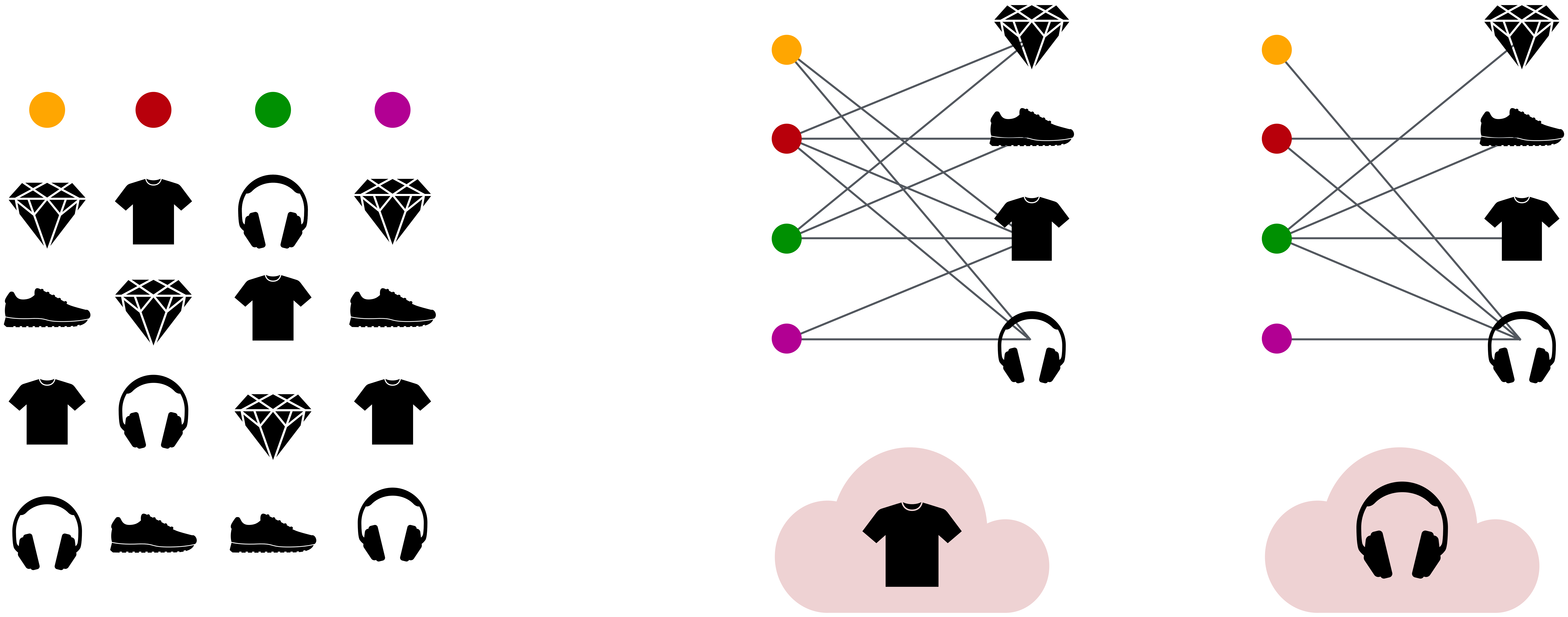
More interesting rules

$N = |V|$ $k = \#\text{seats}$
$m = |C|$

- we saw Random Dictatorship: choose a voter uniformly at random to be Dictator
- **Plurality Veto**: give points by FPV, take them away by querying voters sequentially and subtracting a point from their least favorite still standing.
- **Smart Dictatorship** / **Proportional to Odds**: let F be first-place share and fix a parameter $\beta \geq 0$. If any candidates have $F(X) \geq \frac{1+\beta}{2}$, they win. Else, $\mathbb{P}(X) \propto \dfrac{F(X)}{1 - \left(\frac{2}{1+\beta}\right) F(X)}$.
- **Power-Proportional**, (p,q)-rules: do Random Dictator with probability p and Proportional to q^th power with probability 1-p. For example, $\left(\frac{m-2}{m-1}, \quad 2\right)$.
- **Plurality Matching**: solve a perfect matching problem in a "domination graph."

Plurality matching

- Define a **domination graph** for candidate A as follows: bipartite with voters on the left and candidates on the right. Connect a voter i to a candidate X if $A \succeq_i X$.



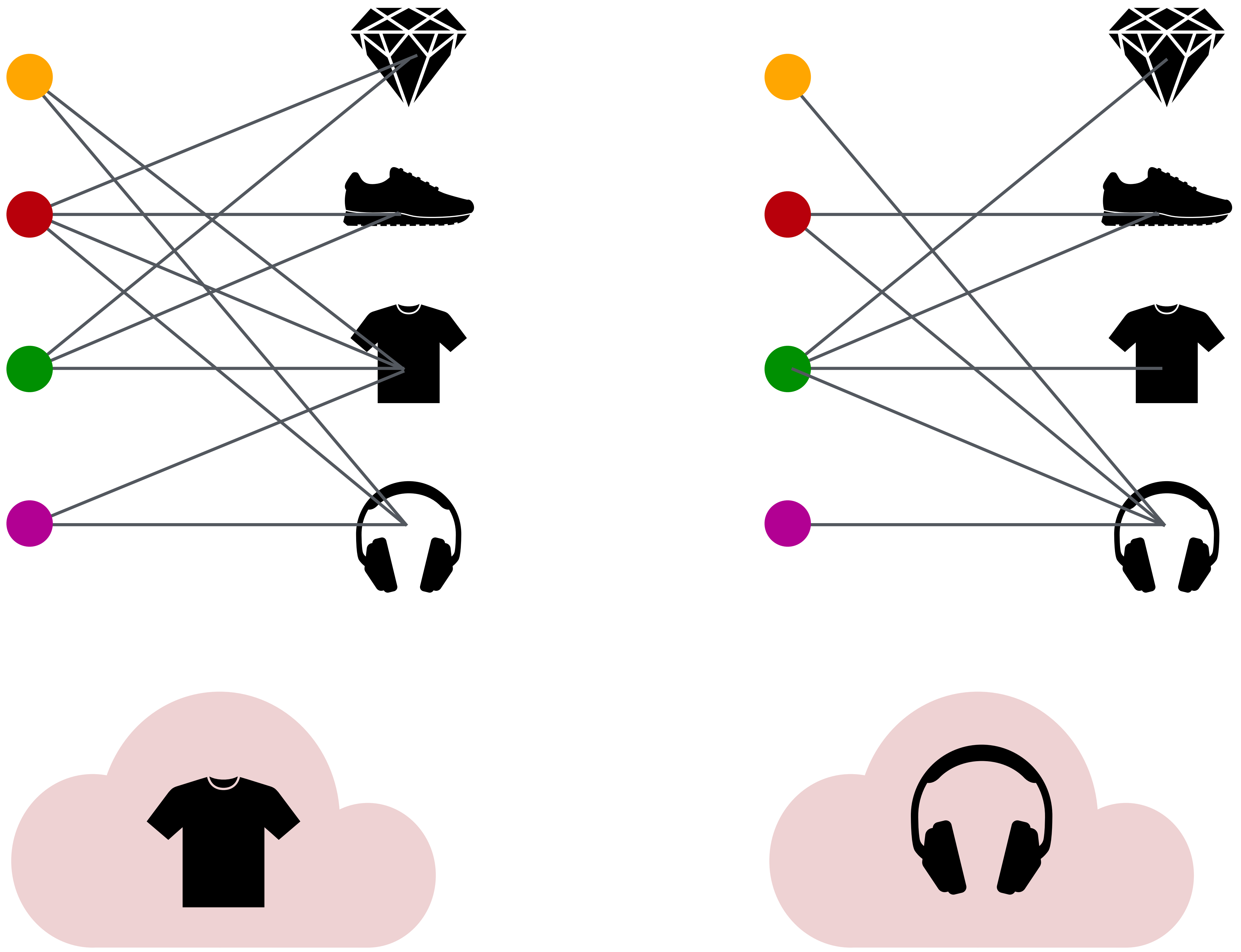
Fractional perfect matching given node weights:

set of edge weights so that edges incident to vertex sum to weight of vertex

(usual perfect matching is special case of binary weights and unit-weighted nodes)

Lemma: For any ranked election and any probability weights, some candidate has a fractional perfect matching. Can be found in polynomial time.

cf. Hall



New voting rule(s): take any weights, such as uniform-uniform, or uniform-plurality. Return as the winner anyone with a FPM.

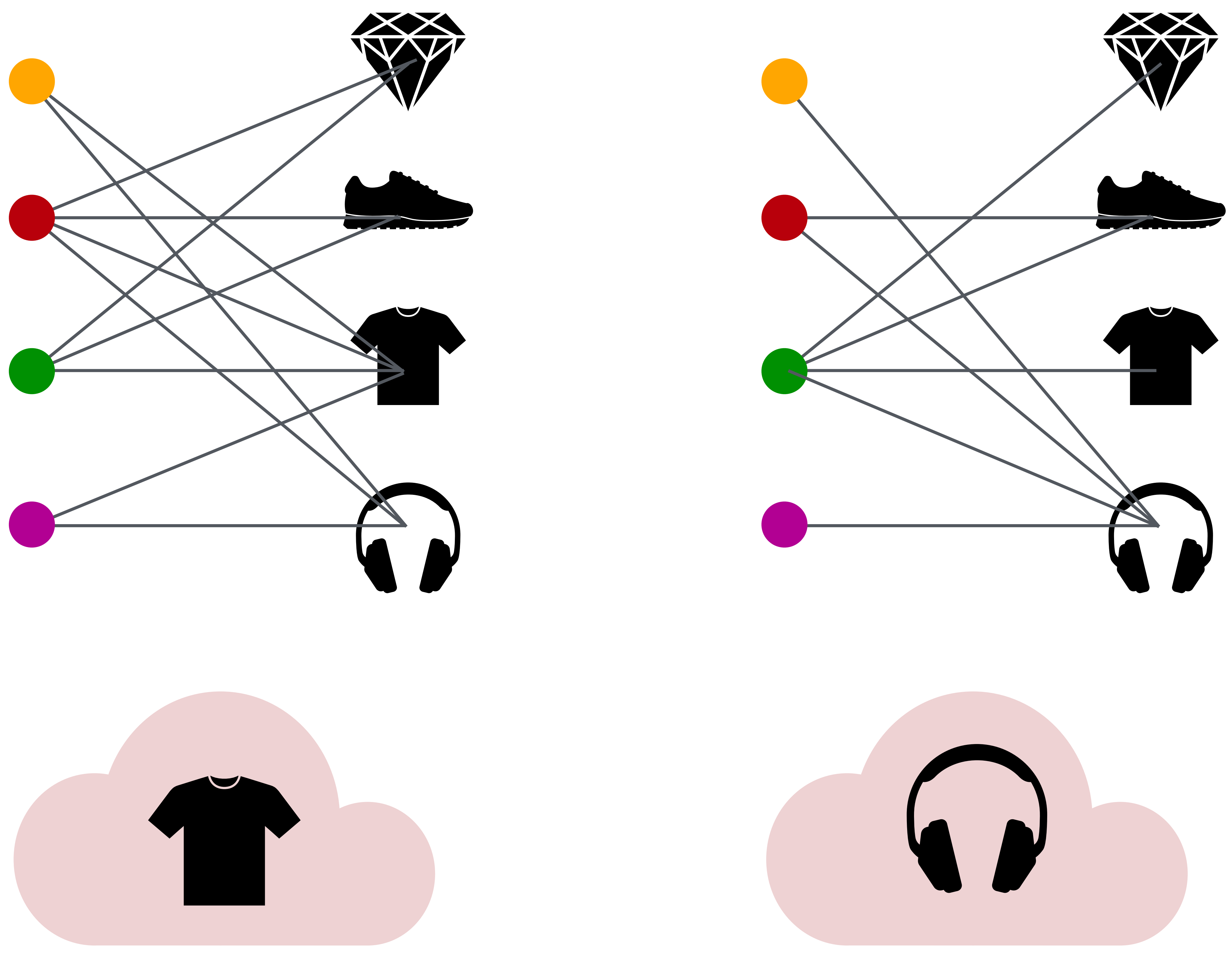
What does it mean?

New voting rule(s): take any weights, such as uniform-uniform, or uniform-plurality. Return as the winner anyone with a FPM.

For a set of voters S and a candidate A, let the **witness set** T be the candidates that some member of S ranks at or below A

Uniform matching: for any set that makes up λ share of voters, the winner's witness set makes up at least λ share of candidates

Plurality matching: for any set that makes up λ share of voters, the winner's witness set accounts for at least λ share of first-place votes



metric distortion theorems

- *Anshelevich—Postl 2016, Kempe 2019*

  Random Dictatorship has expected distortion $3 - \frac{2}{N}$. Distortion $3 - \frac{2}{m}$ is optimal among rules that only use FPV. The $\left(\frac{m-2}{m-1}, \ 2\right)$ rule achieves this.

- *Gkatzelis—Halpern—Shah 2020*

  Plurality Matching has distortion $2 + \alpha$, so under general assumptions its distortion is 3, which is optimal for deterministic rules.

  Smart Dictatorship (proportional to odds) also gets optimal distortion $2 + \alpha - \frac{2}{m}$ for every α.

- Kizilkaya—Kempe 2023

  Plurality Veto gets distortion 3 much more simply than Plurality Matching!

- *Elkind-Skowron 2017*

  Others showed Plurality and Borda have linear distortion. All scoring rules have super-constant distortion. **STV** has distortion $O(\log m)$, making it "reasonable."

proof that distortion 3 is realizable

“Proof from the book” by Pras Ramakrishnan

PV winner

your veto

$\forall i \in \mathscr{C}$

$$
\begin{aligned}
\sum_{v \in V} d(j^*, v) &\le \sum_{v \in V} d(j_v, v) && (j^* \succeq_v j_v) \\
&\le \sum_{v \in V} (d(i, v) + d(i, j_v)) && \text{(triangle inequality)} \\
&= \sum_{v \in V} d(i, v) + \sum_{j \in C} \mathrm{plu}(j) d(i, j) && (j \text{ is vetoed } \mathrm{plu}(j) \text{ times}) \\
&= \sum_{v \in V} d(i, v) + \sum_{j \in C} \sum_{v \in P_j} d(i, j) \\
&\le \sum_{v \in V} d(i, v) + \sum_{j \in C} \sum_{v \in P_j} (d(i, v) + d(j, v)) && \text{(triangle inequality)} \\
&\le \sum_{v \in V} d(i, v) + \sum_{j \in C} \sum_{v \in P_j} 2d(i, v) && (v \in P_j \text{ means } j \succeq_v i) \\
&= 3 \sum_{v \in V} d(i, v)
\end{aligned}
$$

P_j is set of voters ranking j first

Key point: we run the veto process through every voter, so $|P_j| = \mathrm{plu}(j)$ (each candidate eventually loses all their FPVs by veto)

$$
\begin{aligned}
\sum_{v\in V} d(j^*,v) &\leq \sum_{v\in V} d(j_v,v) \\
&\leq \sum_{v\in V} (d(i,v)+d(i,j_v)) \\
&= \sum_{v\in V} d(i,v) + \sum_{j\in C} \operatorname{plu}(j) d(i,j) \\
&= \sum_{v\in V} d(i,v) + \sum_{j\in C} \sum_{v\in P_j} d(i,j) \\
&\leq \sum_{v\in V} d(i,v) + \sum_{j\in C} \sum_{v\in P_j} (d(i,v)+d(j,v)) \\
&\leq \sum_{v\in V} d(i,v) + \sum_{j\in C} \sum_{v\in P_j} 2d(i,v) \\
&= 3\sum_{v\in V} d(i,v)
\end{aligned}
$$