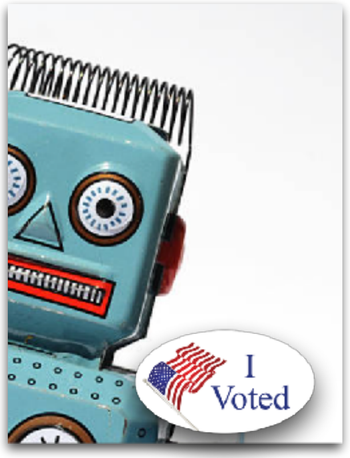
Modeling Democracy

Lecture 3 - **Proving properties**



Moon Duchin, Winter 2026 — Mon Jan 12

More axioms

(Classic) fairness criteria

Majority-fair	Condorcet-fair	Pareto efficient
Unanimity-fair	**Monotonic**	**Strongly monotonic**
Strategy-proof	No (weak) spoilers	Smith-fair
Voter anonymity	Candidate anonymity	Independence of irrelevant alternatives

"favorable move" — X moves up and all others keep relative order

"neutral move" — X stays in place, others may shuffle but don't jump X

- **Unanimity property**— for single-winner, if all rank $X > Y$ then $Y \notin \mathscr{W}$; for ranking output, if all rank $X > Y$ then society ranks $X > Y$
- **Monotonic** — if $X \in \mathscr{W}$ and profile changes only by moves favorable to X, then $X \in \mathscr{W}'$
- **Strongly monotonic** — if $X \in \mathscr{W}$ and profile changes only by moves favorable to or neutral to X, then $X \in \mathscr{W}'$

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- **Strategy-proof** — given a profile P with ballot B, there should be no B', P' so that $f(P')$ is preferable according to B

- **Spoilers** — X is ***not*** a spoiler if, when P, P' differ by removal of candidate X from P, one of these holds. If none of these, then X is a spoiler.
  (a) $\mathscr{W} = \mathscr{W}'$, (b) $\mathscr{W} = \{X\}$, (c) $\mathscr{W}' = \mathscr{W} \setminus \{X\}$.

- **Smith-fair** — only the strong can win. ($\mathscr{W} \subseteq \mathscr{S}$)

removal notation: $P' = P^X$

"weak spoiler" — a spoiler who is outside the Smith set

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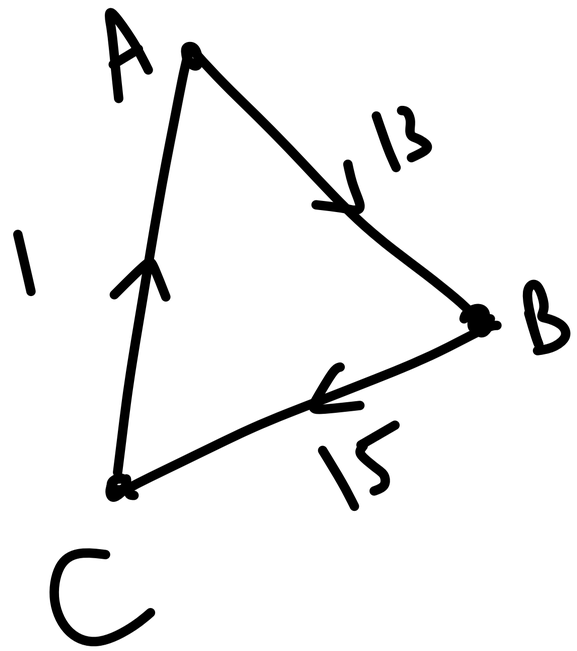
- **Voter anonymity** — independence of voter names or order
- **Candidate anonymity** — independence of candidate names or order
- **IIA** (for rankings) — whether $X > Y$ in final ranking depends only on whether $X > Y$ on ballots

Equivalent formulation of IIA via disqualification:

$$f(P^C) = f(P)^C \quad \forall C \in \mathscr{C}$$

Top-Two example...

x14	x3	x8	x10
A	B	B	C
B	A	C	A
C	C	A	B



As-is, the profile to the left would yield A and B as its top two, then A crushes B in the head-to-head, so A wins.

But if the three B>A>C voters switch A and B — a move favorable to A and neutral to C — then the top two are A and C, and C will win.

This shows that **Top Two is not monotonic**, since a move favorable to A turned them from winner to loser.

Plurality is monotonic.

Argument:

Moves favorable to A can only increase their FPV, can only decrease others'

PWC is monotonic.

Argument:

Moves favorable to A can only change arrows incident to A in the graph; out-arrows can get stronger, ties can become out-arrows; in-arrows can get weaker or can become ties or out-arrows.

These moves don't affect arrows between other candidates.

Therefore, A's number of H2H wins stays the same or goes up; others stay the same or go down. Thus A can't change from winner to loser.

we must show moves favorable to A can't change them from winner to loser.

Beatpath is UF.

HW!!

Borda is UF.

Argument:

If every voter prefers X>Y then every voter gives more Borda points to X than Y, so it's impossible that Y ends up with more points than X overall.

we must show that if all voters prefer X>Y, then Y can't be a winner.

Spoilers

5	2	1	4	3
F	F	S	S	D
X	S	X	D	X
S	X	D	X	S
D	D	F	F	F

Consider S under **Plurality**

Consider F under **Top Two**

Say a system is “two-way-fair” if it does the only reasonable thing in a case with $n = 2$ candidates: *the one with more support wins.*

Proposition: Any “two-way-fair” system admits spoilers!

1	1	1
A	B	C
B	C	A
C	A	B

Case 1: three winners, $\mathscr{W} = \{A, B, C\}$ — B spoils

Case 2: two winners, WLOG $\mathscr{W} = \{A, B\}$ — B spoils

Case 3: one winner, WLOG $\mathscr{W} = \{A\}$ — B spoils

So: **Are spoilers bad?**

Strategy

simple example

x4	x3	x2
B	A	C
A	B	A
C	C	B

plurality

$W = \{B\}$

More intricate example – Sequential election

(alphabetical tiebreaker)

x8	x12	x10	x10	x4
M	C	O	S	S
O	S	M	M	O
S	O	C	C	C
C	M	S	O	M

one MOSC voter reverses ballot completely.

Try it!

What is missing??

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New norms and axioms

- **Social choice** highlights Condorcet property, monotonicity, strategy-proofness
- **Political science** highlights rotation in office, responsiveness, symmetry
- **Law** highlights non-exclusion, non-dilution
- **Everyday voters** highlight competitiveness and proportionality

My research has tried to operationalize **responsiveness**, **non-dilution**, and **proportionality** to bring them into the social choice framework

Impossibility

SW - single winner
UF - unanimity - fair
PE - Pareto efficient
SM - strongly monotonic
SP - strategy proof

IIA - indep of irrelevant alternatives

Arrow (1951)

In a ranking election for $n \geq 3$ candidates,

UF and IIA implies...

Müller-Satterthwaite (1977)

In a SW election for $n \geq 3$ candidates,

PE and SM implies...

Gibbard-Satterthwaite (1973, 1975)

In a SW election for $n \geq 3$ candidates,

PE and SP implies...

Dictatorship!

Let's understand what Dictatorship means and why it has all these good properties: UF, SM, SP.

Dictatorship of the kth voter: their preferences are adopted, irrespective of others
(this works whether single-winner or ranking)

Why UF? because if everyone prefers $X > Y$, so does Dictator, and $X \succ Y$ in output.

Why SM? because moves neutral to X leave them in place on Dictator's ballot; moves favorable to X move them up on Dictator's ballot, and nothing else matters.

Why SP? because Dictator's true preferences are already adopted; nobody else has a strategic move because they're fully irrelevant, and Dictator can't improve on true ballot.