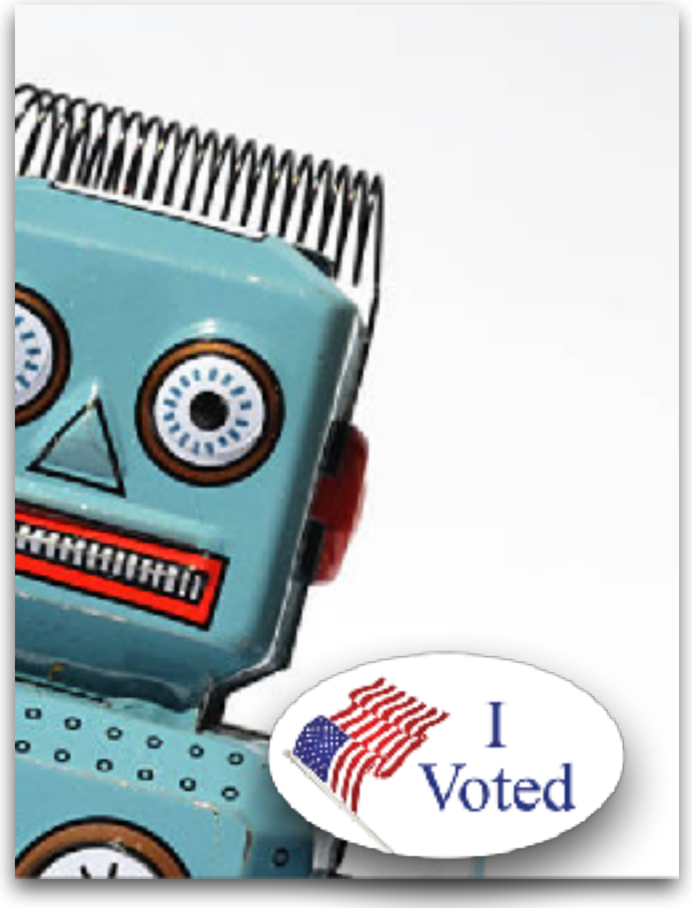
Modeling Democracy

Lecture 4 - **The impossibility theorems**



Moon Duchin, Winter 2026 — Wed Jan 14

IMPOSSIBILITY

SW - single winner
UF - unanimity-fair
PE - Pareto efficient
SM - strongly monotone
SP - strategy proof

IIA - indep of irrelevant alternatives

Arrow (1951)

In a ranking election for $n \geq 3$ candidates,

UF and IIA implies...

Müller-Satterthwaite (1977)

In a SW election for $n \geq 3$ candidates,

PE and SM implies...

Gibbard-Satterthwaite (1973, 1975)

In a SW election for $n \geq 3$ candidates,

PE and SP implies...

Dictatorship!

Let's understand what Dictatorship means and why it has all these good properties: UF, SM, SP.

Dictatorship of the kth voter: their preferences are adopted, irrespective of others
(this works whether single-winner or ranking)

Why UF? because if everyone prefers $X > Y$, so does Dictator, and $X \succ Y$ in output.

Why SM? because moves neutral to X leave them in place on Dictator's ballot; moves favorable to X move them up on Dictator's ballot, and nothing else matters.

Why SP? because Dictator's true preferences are already adopted; nobody else has a strategic move because they're fully irrelevant, and Dictator can't improve on true ballot.

let's play… **Find That Dictator**!

- We're going to use a strategy to probe an unknown voting rule f where we present it with a profile P, then depending on the outcome we modify that to P', P'', etc and keep track of the changes in outcome

- Consider the profile below. One by one, we will alter the columns by moving R to the top and keeping all others in same relative order. Suppose f is Pareto efficient.

#1	#2	#3	#4	#5	•••	#N
R	R	R	V	V		V
V	V	V	O	O		O
O	O	O	T	T		T
T	T	T	E	E		E
E	E	E	R	R		R

- Initially, V must win (Pareto). At the end, R must win. So there's a first column that makes R win. **Aha!** Maybe-dictator.

Examples: plurality, Borda, beatpath (all with tiebreaker). None of these is in fact a dictatorial rule, so our maybe-dictator is not an actual dictator, but we can still run the method. Suppose there are N voters.

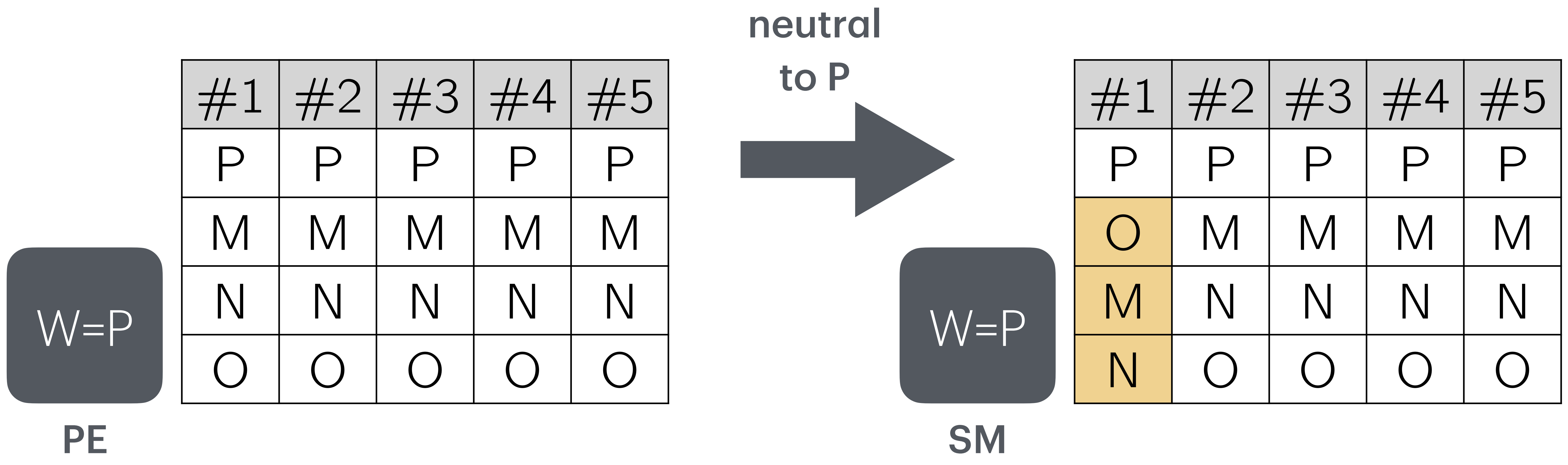
Plurality: if we started with a profile with candidate V on top of all ballots, we'd need to flip candidate R to the top on $\left\lfloor \frac{N}{2} + 1 \right\rfloor$ ballots to change the winner. So the maybe-dictator is about halfway through.

Borda: let's suppose we're using scoring where first place is $m - 1$ points, down to zero points for last place. We need to minimize k s.t. $k(m-1) > k(m-2) + (N-k)(m-1)$. (Candidate V needs to surpass the Borda points of candidate R.) The inequality simplifies to $k > N \cdot \dfrac{m-1}{m}$. This is well more than halfway—with 7 candidates, it's at the 6/7 mark.

Beatpath: R starts out anti-Condorcet, so has no beatpaths to anyone. This persists until they beat someone head-to-head, at which point they beat everyone head-to-head, become Condorcet, and win by beatpath. So the maybe-dictator is the same as in plurality.

Constructive-ish proof of Müller-Satterthwaite

(SW+PE+SM $\Rightarrow$ )



W=P
PE
#1 #2 #3 #4 #5
P P P P P
M M M M M
N N N N N
O O O O O
neutral to P
W=P
SM
#1 #2 #3 #4 #5
P P P P P
O M M M M
M N N N N
N O O O O

W=P

#1	#2	#3	#4	#5
P	P	P	P	P
M	M	M	M	M
N	N	N	N	N
O	O	O	O	O

W=P

#1	#2	#3	#4	#5
P	P	P	P	P
O	M	M	M	M
M	N	N	N	N
N	O	O	O	O

W=P or O

#1	#2	#3	#4	#5
O	O	P	P	P
P	P	M	M	M
M	M	N	N	N
N	N	O	O	O

W=P

#1	#2	#3	#4	#5
O	O	O	P	P
P	P	P	O	M
M	M	M	M	N
N	N	N	N	O

W=O

#1	#2	#3	#4	#5
O	O	O	O	P
P	P	P	P	M
M	M	M	M	N
N	N	N	N	O

W=O

#1	#2	#3	#4	#5
O	O	O	O	M
M	M	M	P	N
N	N	N	M	P
P	P	P	N	O

$W \neq M$
$W \neq N$
$W \neq O$

#1	#2	#3	#4	#5
O	O	O	P	M
M	M	M	O	N
N	N	N	M	P
P	P	P	N	O

W=P

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	P
P	P	P	O	O

$W \neq M$
$W \neq N$
$W \neq O$

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

W=P

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

Our constructed pathological profile

neutral to P

#1	#2	#3	#4	#5
O	O	N	P	O
M	M	O	M	M
N	N	M	N	N
P	P	P	O	P

favorable to P

#1	#2	#3	#4	#5
P	O	N	P	O
O	P	O	M	P
M	M	M	N	M
N	N	P	O	N

W=P

Completely arbitrary profile where #4 ranks P first

We've shown that ∃ voter k who is **decisive** for P: if they rank P first, P wins.

Observe that if k decisive for P and a different voter j is decisive for a different candidate Q, there's a contradiction (if j top-ranks P and k top-ranks Q, things break.)

And the argument can be run for every candidate, so k must be decisive for all.

So we have found a globally decisive voter, i.e., *dictator*!

Now show Gib-Sat follows from Mul-Sat

- I must show SW+SP+PE⇒SM.

- Given f(P)=X, consider changes to ballot B so that all below X in B are still below X on B′ — i.e,, changes neutral or favorable to X. If X still wins, then f is SM.

- Suppose f(P′)=Y≠X. Since f assumed strategy-proof, Y must be below X on B. (Else the B ↦ B′ shift is strategic for the true-B voter, obtaining the preferred winner Y.) By assumption that the B ↦B′ shift was favorable to X, we know Y is still below them on modified ballot B′.

- But then the reverse change, B′ ↦ B, is a successful strategic vote. (Honest voter B′ prefers X to Y, but can change to B and make X win!)

f(P′)=X, so f is SM.

Sen's proof of Arrow's Theorem

Assume voting rule f satisfies UF and IIA. "Decisive": group unanimity suffices for outcome.

Decisiveness Lemma: if a group G is decisive for any pair X,Y, then G is globally decisive.

Proof: Suppose G decisive for X,Y, and take two other candidates A,B. Suppose G voters prefer A>X>Y>B. Suppose all others like A>X and Y>B but we don't know about other comparisons. By hypothesis, $X \succ Y$*. But A>X and Y>B are unanimous, so* $A \succ X$ *and* $Y \succ B$*. So* $A \succ B$*. This was decided with arbitrary A,B preferences outside G, so G is decisive for A,B.* $\square$

Contraction Lemma: if a group G is decisive and |G|>1, then a proper subset is also decisive.

Proof: Suppose $G = G_1 \sqcup G_2$ *with* $X > Y, Z$ *in* G_1 *and* $X, Z > Y$ *in* G_2*. Since G decisive,* $X \succ Y$*. Now consider the set of X,Z preferences* P_{XZ}*. By IIA, that's enough to get a social ranking of X,Z. Suppose* $Z \succeq X$*. Then* $Z \succ Y$ *— but we only know* G_2 *ranking of* Z, Y*, so this means they are decisive. Exactly similarly, if* $P_{XZ} \mapsto X \succ Z$*, then* G_1 *is decisive.* $\square$

Sen's proof

Decisiveness Lemma: if a group G is decisive for any pair X,Y, then G is globally decisive.

Contraction Lemma: if a group G is decisive and |G|>1, then a proper subset is also decisive.

Proof of Arrow's Theorem from these lemmas:

by Pareto, the full set of voters is decisive. But then apply contraction to find a singleton — this is a dictator.

interlude:
you can do things in VoteKit

TRAININGS
Tue 10-12, DSI 353
Thu 1-3, DSI 353
Fri 1:30-3:30 Ry 177

Rounding out Unit I — intro to classical (pre-1990) social choice

- **Apportionment**: how to allocate indivisible goods fairly — in particular, legislative seats!
- **Weighted voting**: a scheme for measuring the power of voters (or representatives)

Running list of non-HW exercises

- A Condorcet 3-cycle exists on 3 candidates when some three cyclically permuted columns have been selected n_1, n_2, n_3 times and those numbers satisfy the triangle inequality

- If a preference profile consists of complete rankings, then all margins in the PWCG have the same *parity* (all even or all odd)

- Sen's proof of Arrow via decisiveness and contraction constructs pathological profiles in both lemmas that are suggestive of a group's decisiveness. Finish the proof in both cases by reducing from arbitrary profiles (of the kind needed to witness the property) to the ones in the proof.