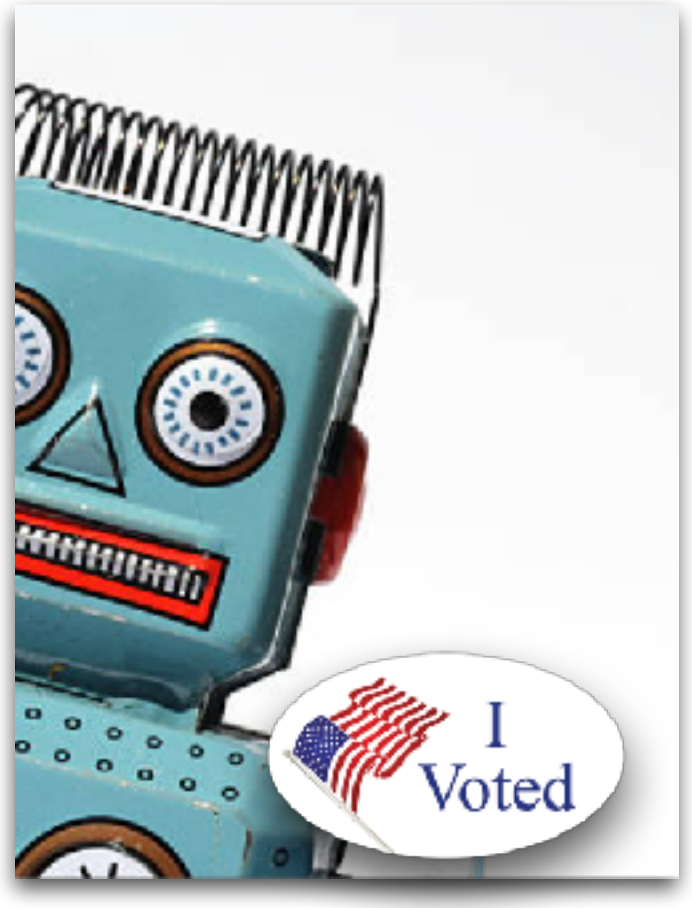
Modeling Democracy

Lecture 11 - **Ranking models**



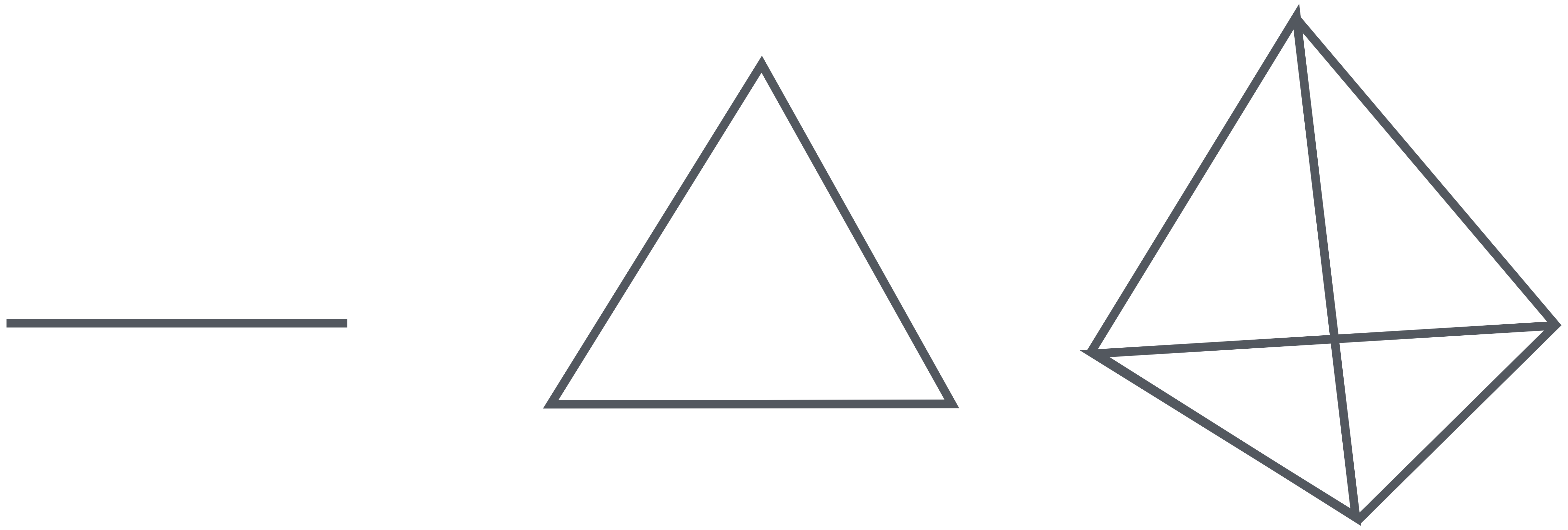
Moon Duchin, Winter 2026 — Wed Feb 11

Simplices, coordinates, and measures

Simplex: polytope formed as convex hull of n points

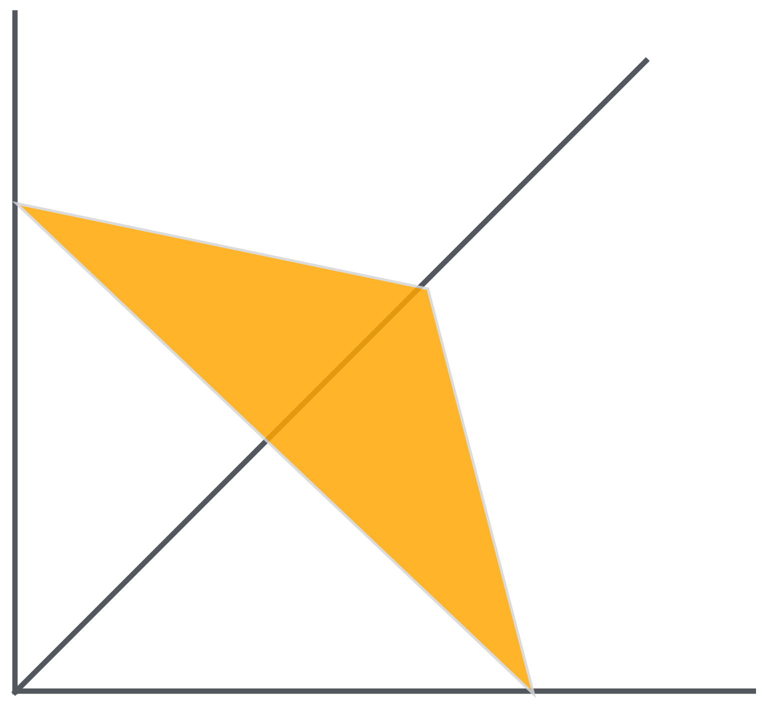
Can form this in Euclidean space as the collection of convex combinations:

$$\left\{a_1v_1 + \ldots + a_nv_n : \sum a_i = 1\right\}$$

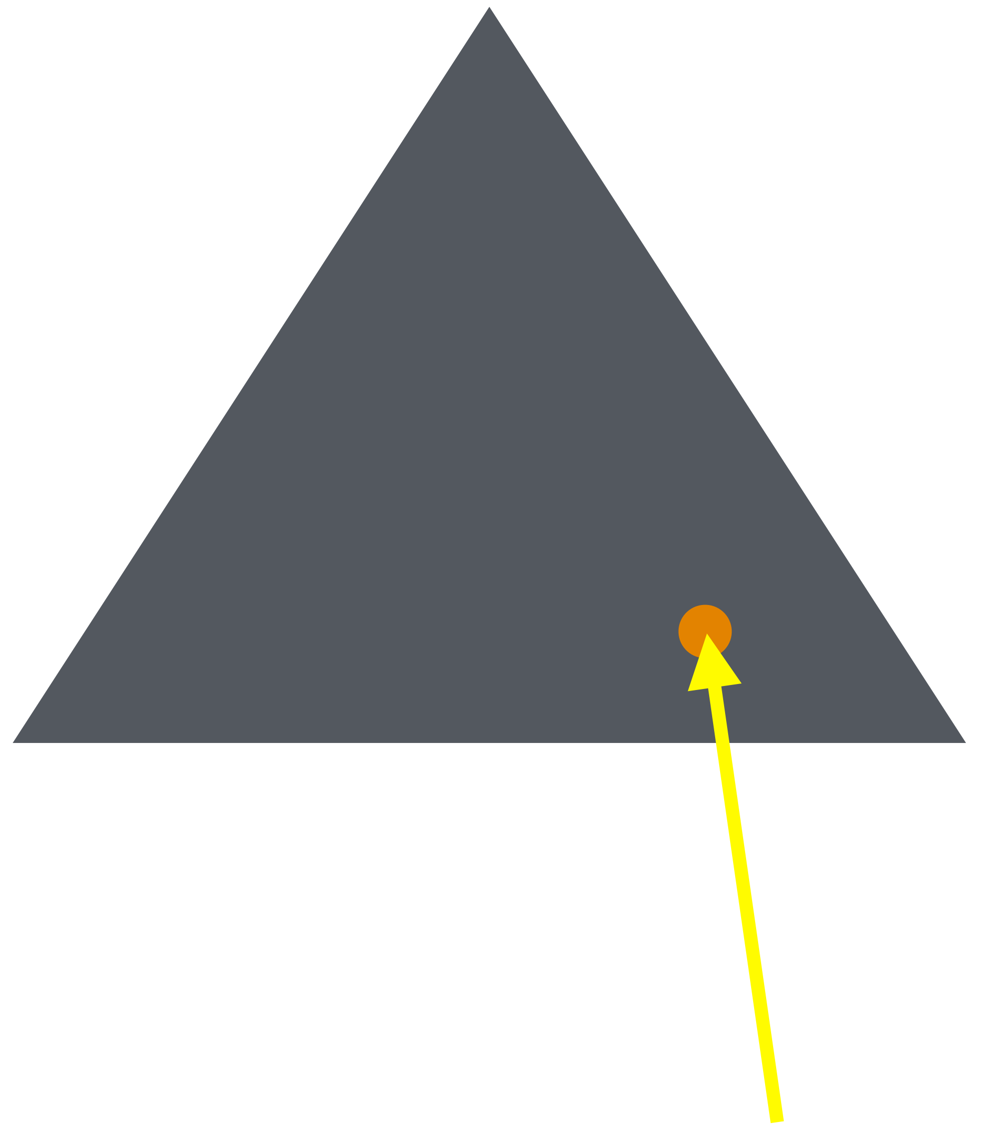


You can think of this as weighted averages of the points.

Standard (regular) n-dim simplex sits in $\mathbb{R}^{n+1}$ as slice of the first orthant, taking v_i as the standard (0,...,0,1,0,...,0)

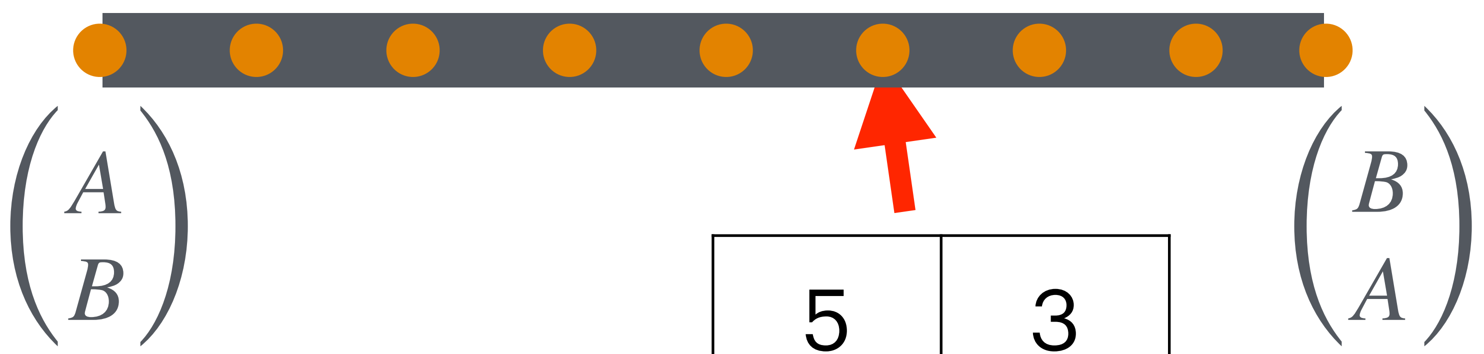


x+y+z=1, x,y,z≥0



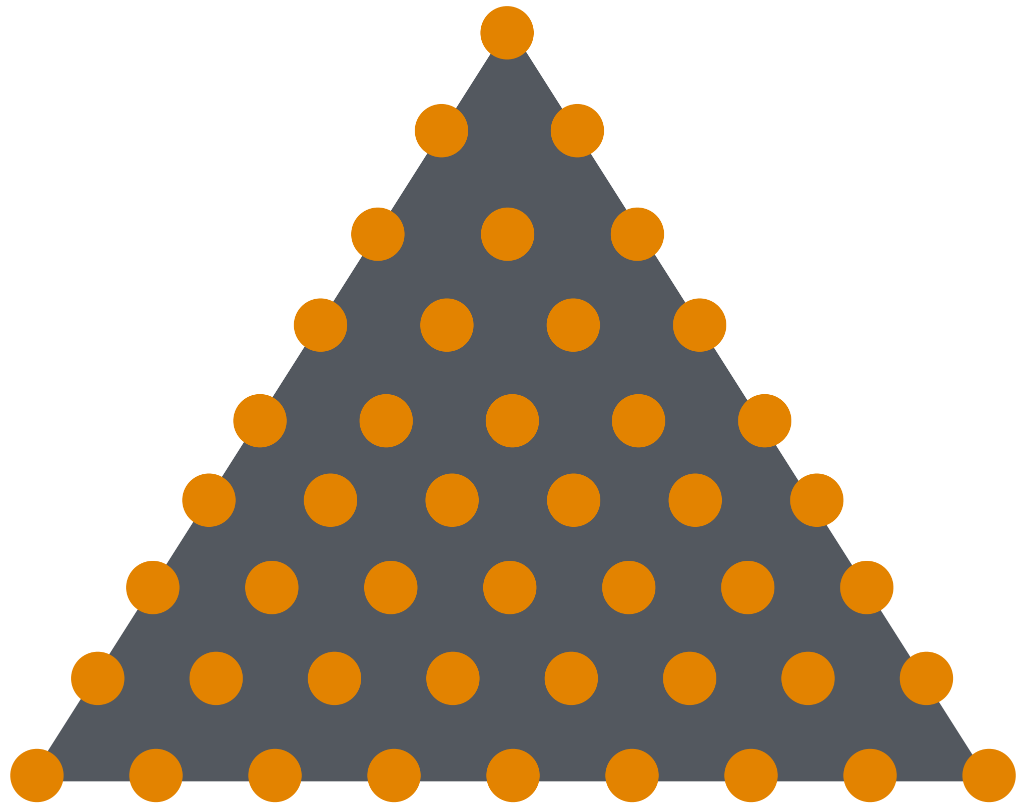
The coordinate construction should make it clear that every point in a simplex is a convex combination of the extreme vertices in a **unique** way, so there's a 1-1 correspondence between points in the simplex and probability vectors on the extreme points.

(vectors with coords summing to 1)

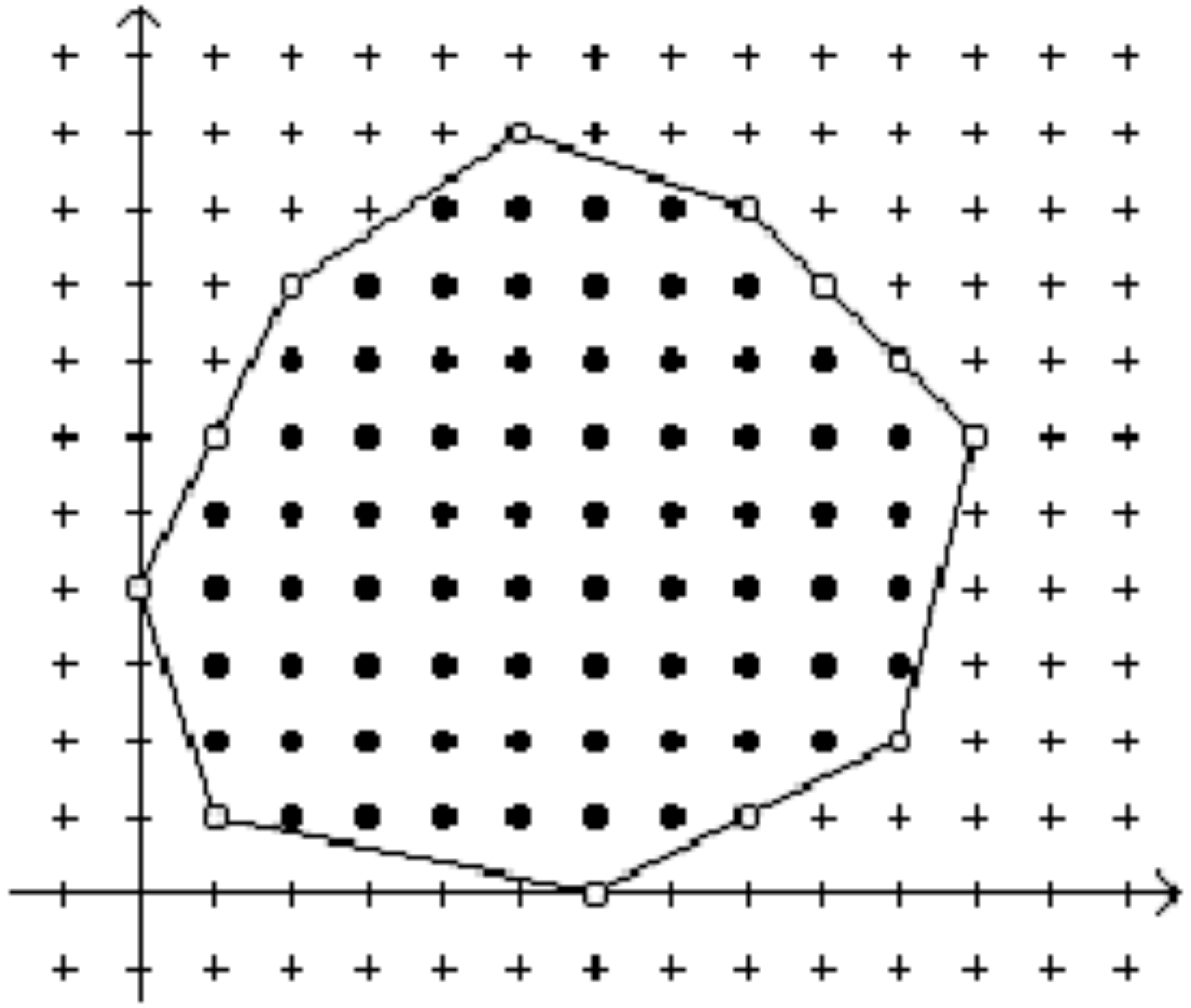


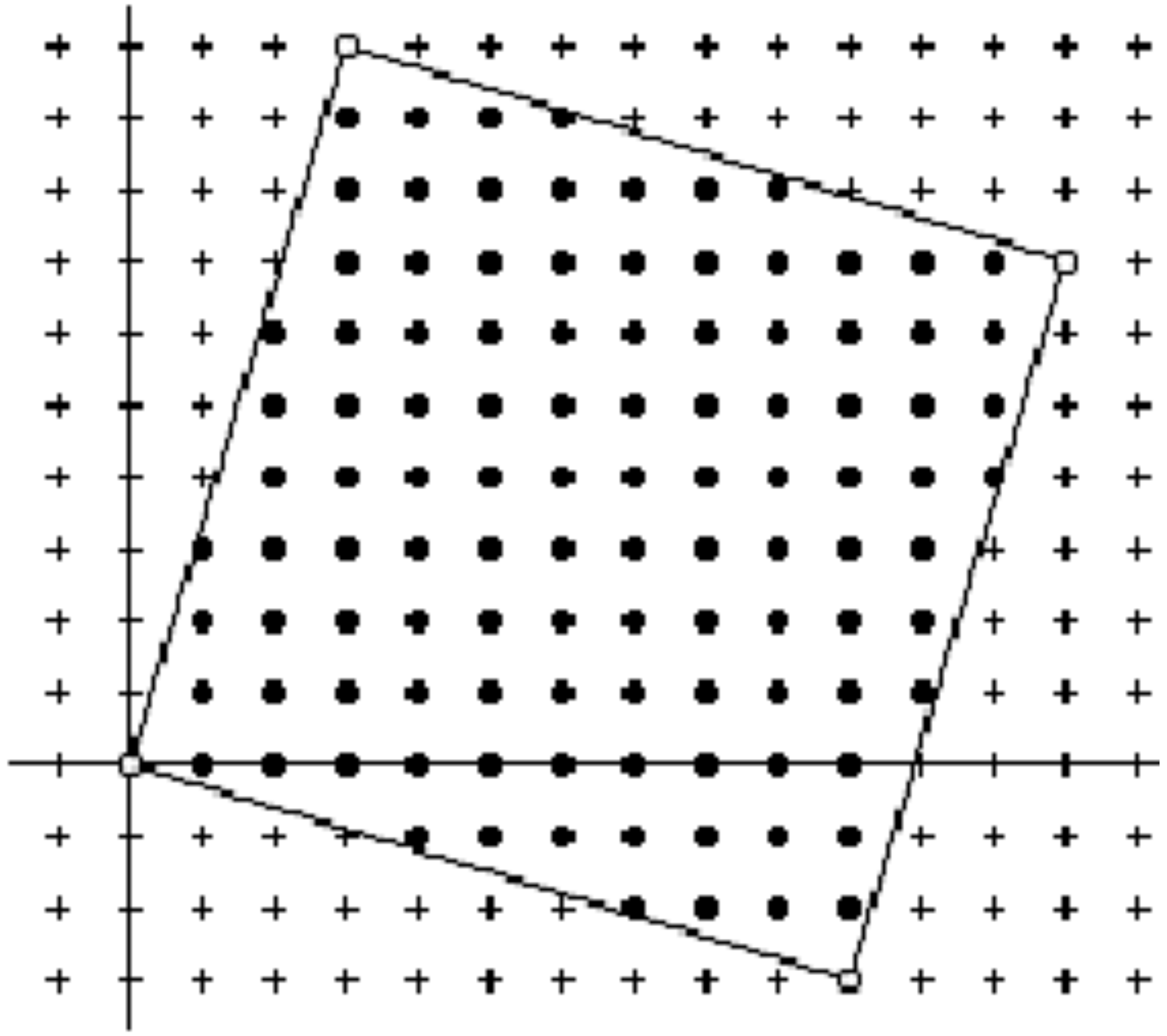
5	3
B	A
A	B

Next, observe that for a fixed number of voters, the set of possible profiles sits as a lattice in the simplex on the possible ballots. For example, this image shows 8 voters and two ballot types.



8 voters and
3 ballot types





Classical fact (well studied by Gauss, Riemann, Sierpinski, ...)

When a lattice gets finer and finer, the count of lattice points in a region is proportional to its area/volume.

generative models

"Impartial Culture"

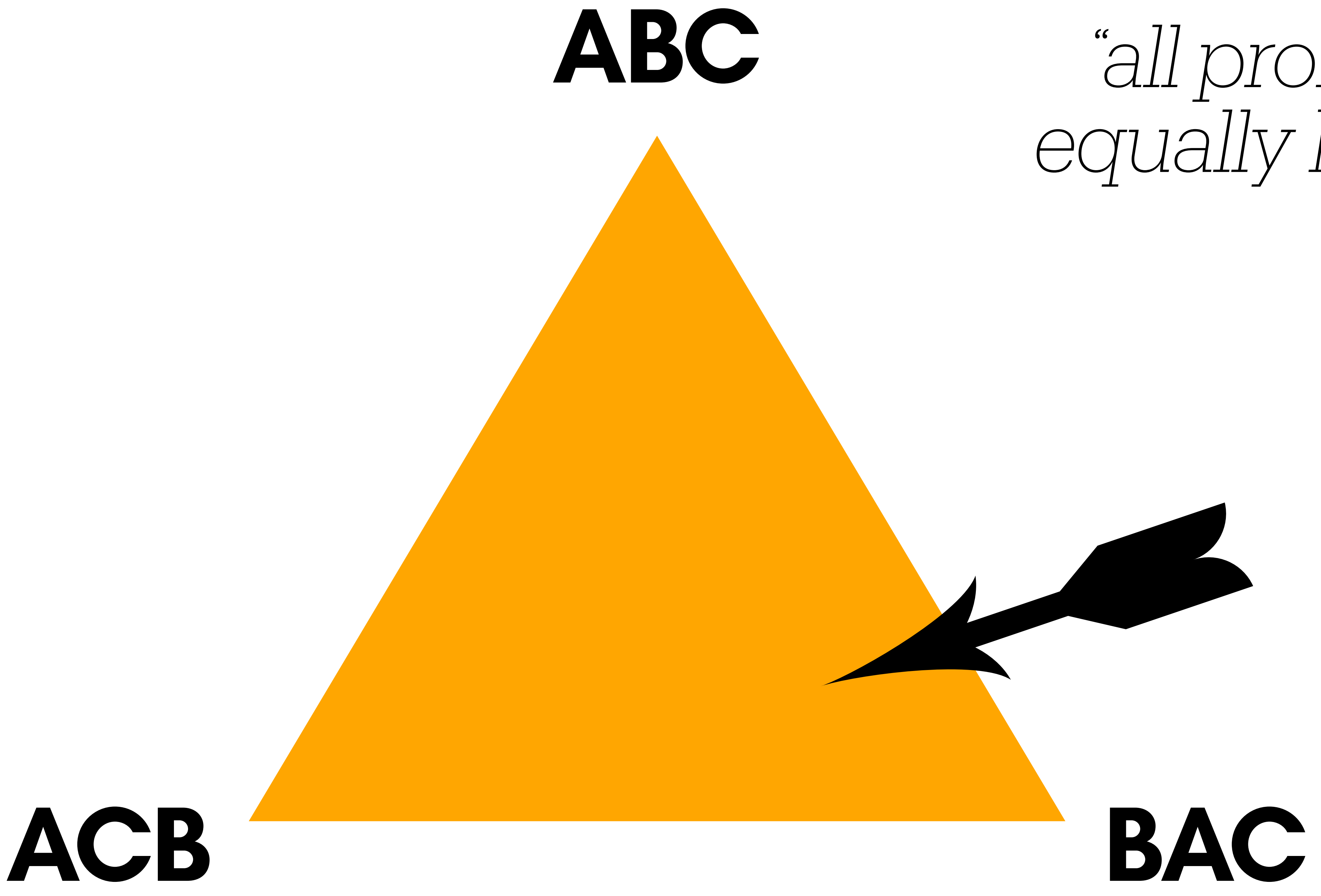
all permutations equally likely

1/6	1/6	1/6	1/6	1/6	1/6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

draw from these probabilities

"Impartial Anonymous Culture"

Lebesgue measure on ballot simplex



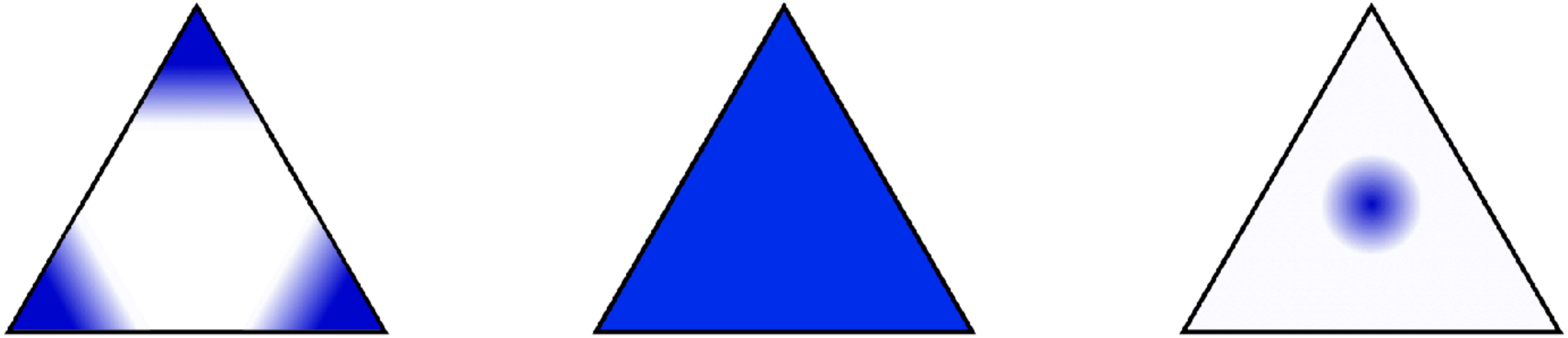
"all profiles equally likely"

20%	70%	10%
A	B	A
B	A	C
C	C	B

use this as your profile
OR
draw from these probabilities

Dirichlet distributions

a family of measures on the simplex



$\alpha \to 0$ $\qquad$ $\alpha = 1$ $\qquad$ $\alpha \to \infty$

We'll use two ways:

- Dirichlet on **ballot** simplex will get us a probability on profiles (IC, IAC are special cases)
- Dirichlet on **candidate** simplex will get us preference weights on candidates

Tideman–Plassmann, *The Structure of the Election-Generating Universe*

None of the 11 models discussed so far are based on the belief that the associated distributions of *P* might actually describe rankings in actual elections. IAC, IC, UUP, DC, and UP assume that various components of *p* are equally likely, for the sake of algebraic tractability. $IAC_b(k_b)$, $IAC_t(k_t)$, $IAC_c(k_c)$ and SSP seek to describe rankings that have meaningful interpretations for the problem of defining probabilities of observing Condorcet's paradox. The Borda and Condorcet models are rationalizations of claims about how one ought to determine the winner in an election.

By contrast, their realistic model is… spatial voting!

are these realistic?
can we do better?

new models,
using classic ranking ideas

rankings from preference intervals

Plackett-Luce: voter fills i… …ithout replacement according to pref…

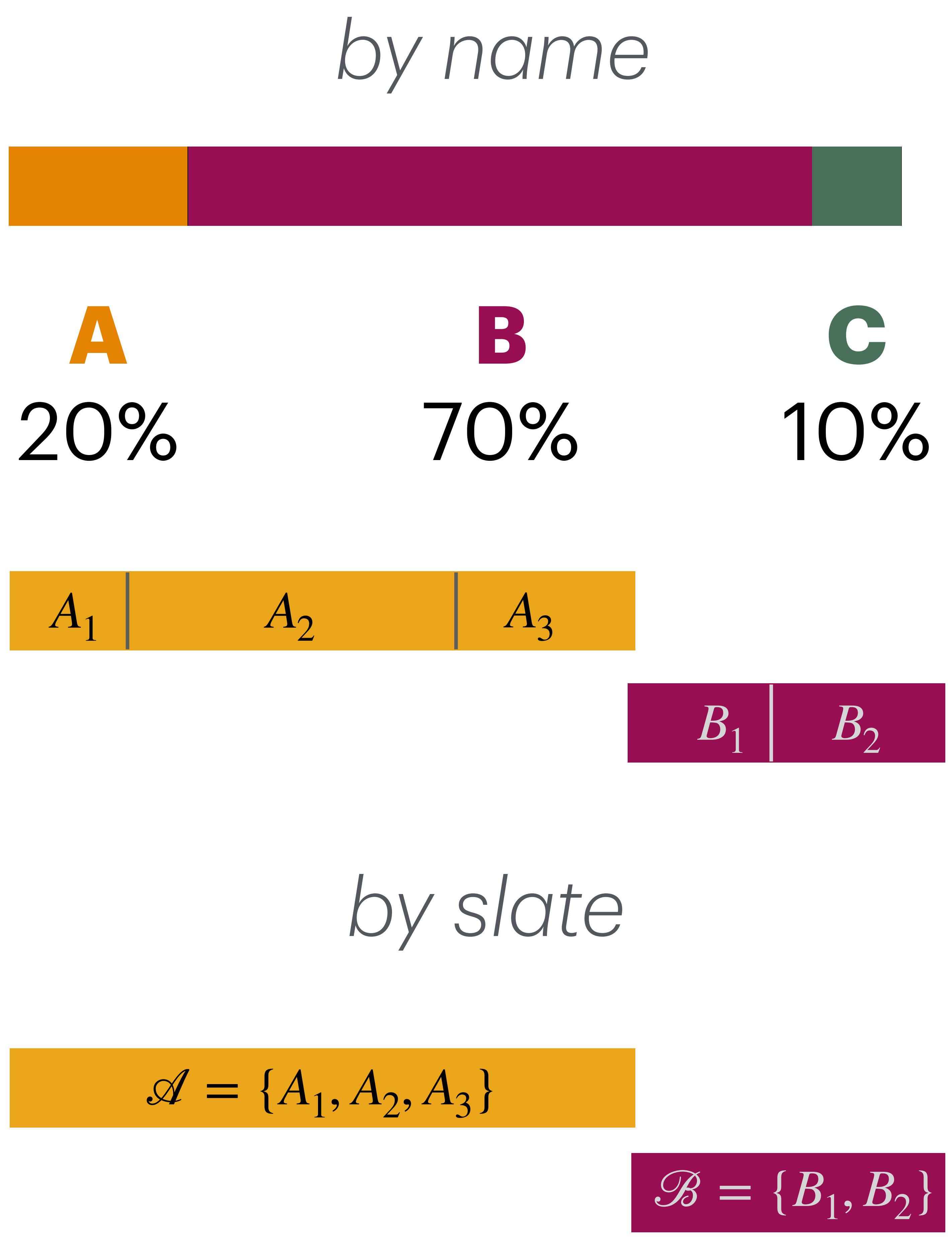
impulsive voter

Bradley-Terry: probability of ABC ballot i… …ional to

$$\left(\frac{\ell(A)}{\ell(A)+\ell(B)}\right)\left(\frac{\ell(A)}{\cdots}\right)\left(\frac{\ell(B)}{\ell(B)+\ell(C)}\right)$$

deliberative voter

(product of prob… pairwise comparisons A>B, A>C, B>C)



Examples

$\begin{pmatrix} A \\ A \\ B \\ B \end{pmatrix}$ has $A>B$ in 4 ways, $B>A$ 0 ways

so prob weight $(.8)^4$

whereas $\begin{pmatrix} A \\ B \\ A \\ B \end{pmatrix}$ has $(.8)^3(.2)^1$

DATA GENERATION: Let's use **Slate-BT**

80% A | 20% B ← 1 must pick my group's preference of slates

A_1, A_2 B_1, B_2 ← 2 who's winning?

3 choose candidate strength settings

A_1 —— A_2

$\alpha \simeq 0$

(strong pref.)

B_1 —— B_2

$\alpha \simeq \infty$

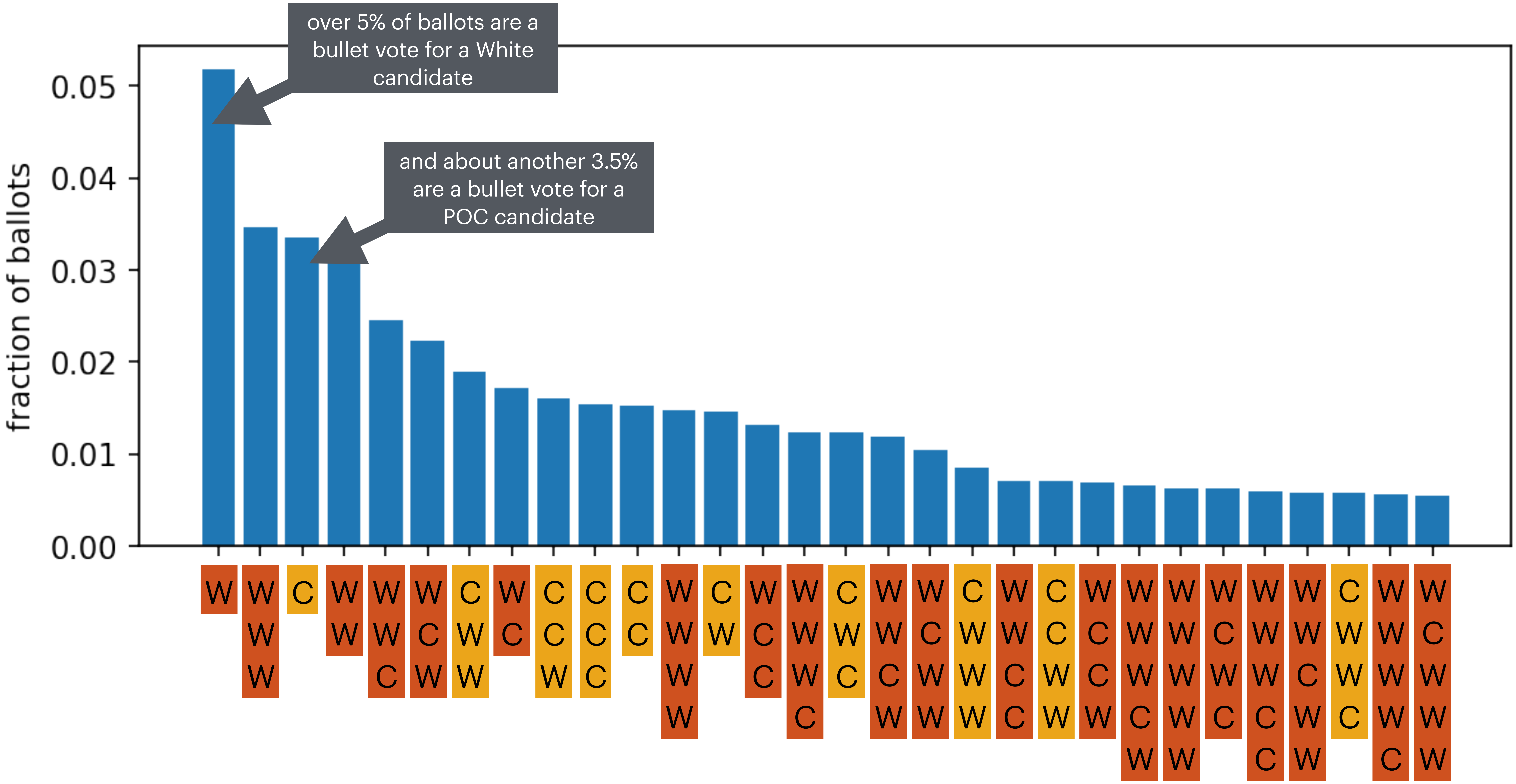
(indifferent)

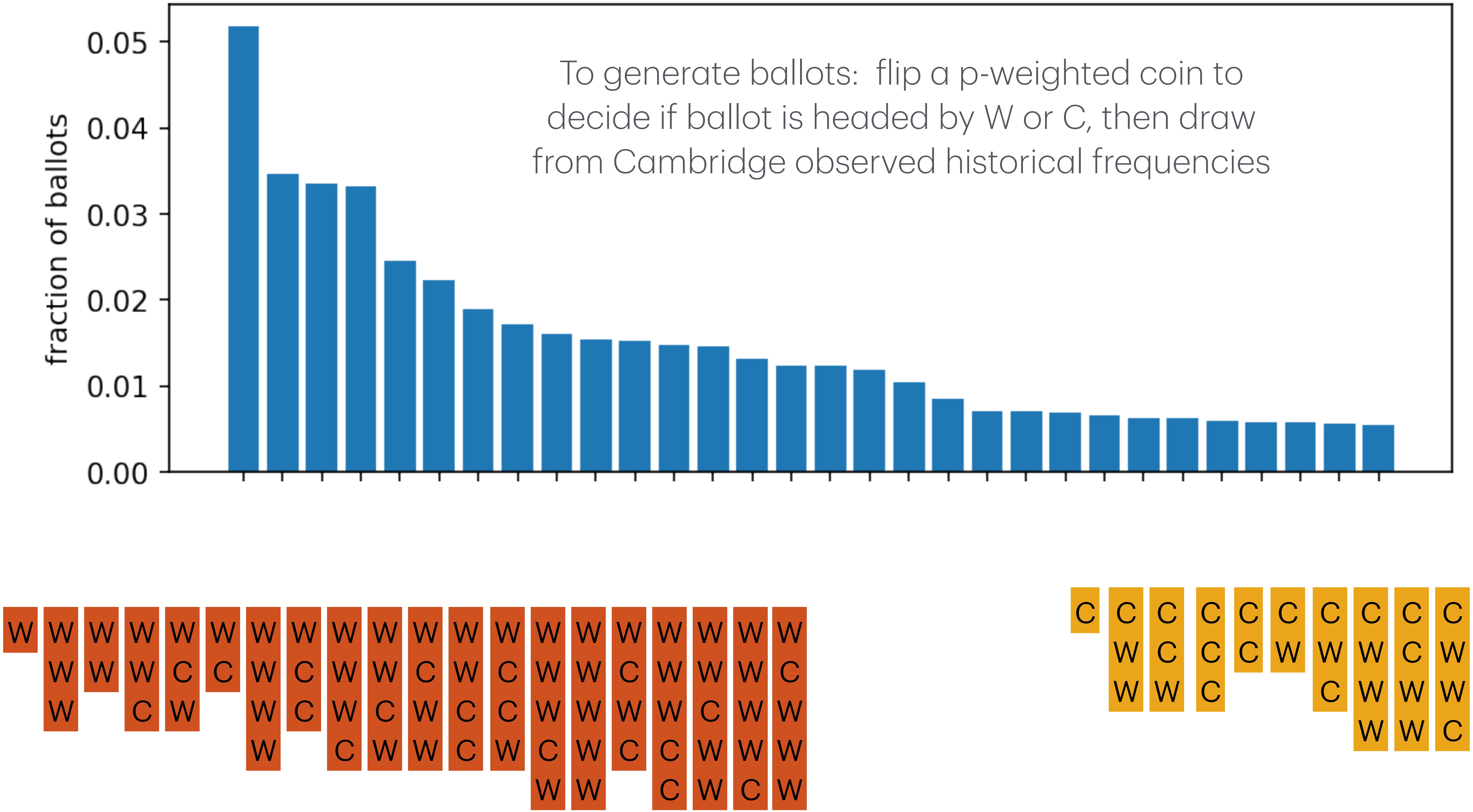
EXECUTE THE STEPS

$\begin{pmatrix} A \\ A \\ B \\ B \end{pmatrix}$, then $\begin{cases} A_2 > A_1, \\ B_1 > B_2 \end{cases}$ to get $\begin{pmatrix} A_2 \\ A_1 \\ B_1 \\ B_2 \end{pmatrix}$.

Cambridge Sampler

based on actual Cambridge, MA ballots

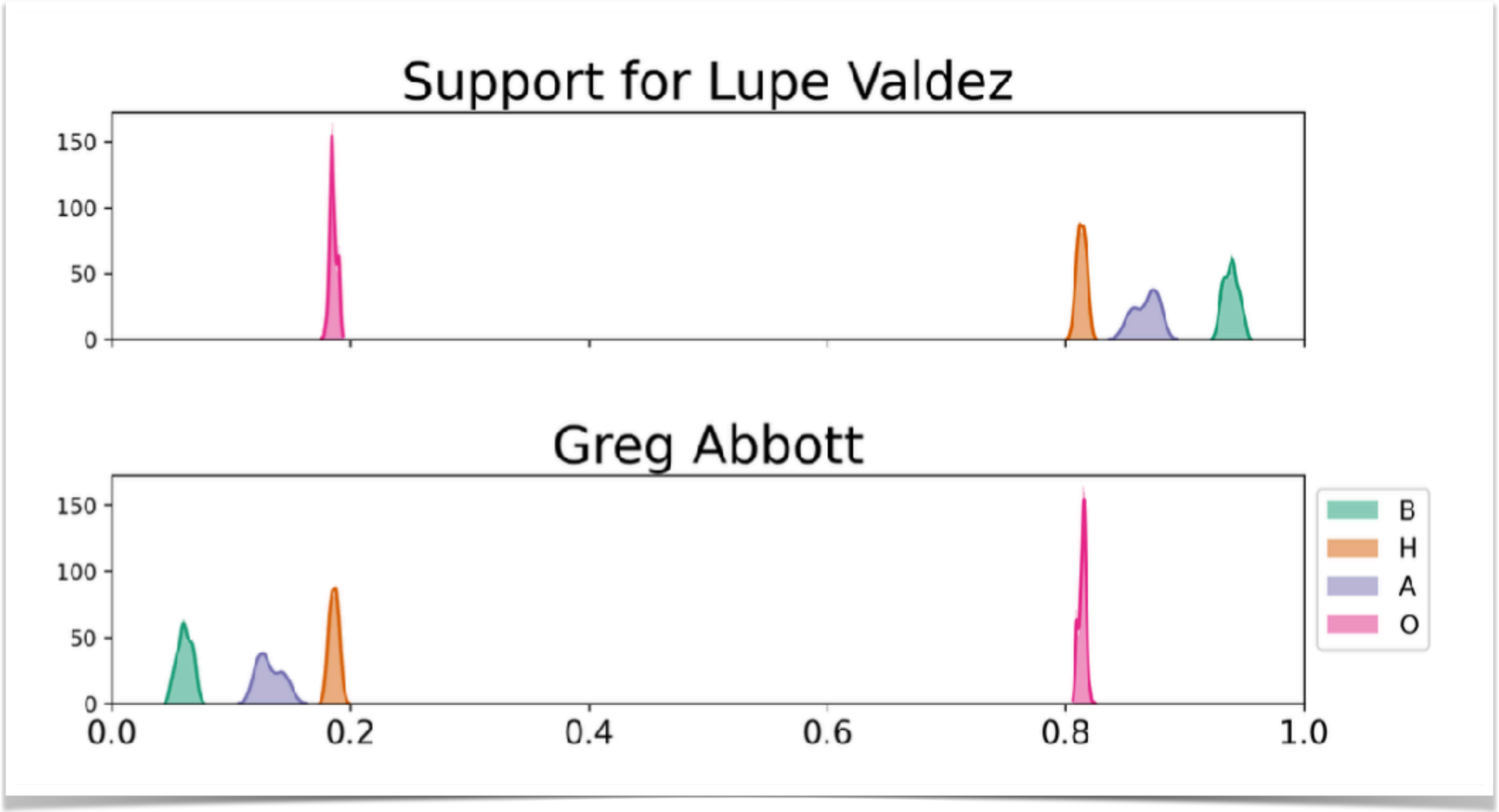




fraction of ballots
0.00
0.01
0.02
0.03
0.04
0.05
To generate ballots: flip a p-weighted coin to decide if ballot is headed by W or C, then draw from Cambridge observed historical frequencies
W
W W W
W W
W W C
W C W
W C
W W W W
W C C
W W W C
W W C W
W C W W
W W C C
W C C W
W W W C W
W W W W W
W C W C
W W W C C
W W C W W
W W W W C
W C W W W
C
C W W
C C W
C C C
C C
C W
C W C
C W W W
C C W W
C W W C

what's missing?

Key attribute: **Polarization**



there are socially relevant groups that vote differently (to say the least)

We can incorporate this with **mixture models** — create two (or more) **blocs** of voters and specify different parameters

interlude:
you can do all this in VoteKit

And also have big repo of **Scottish STV** election records.

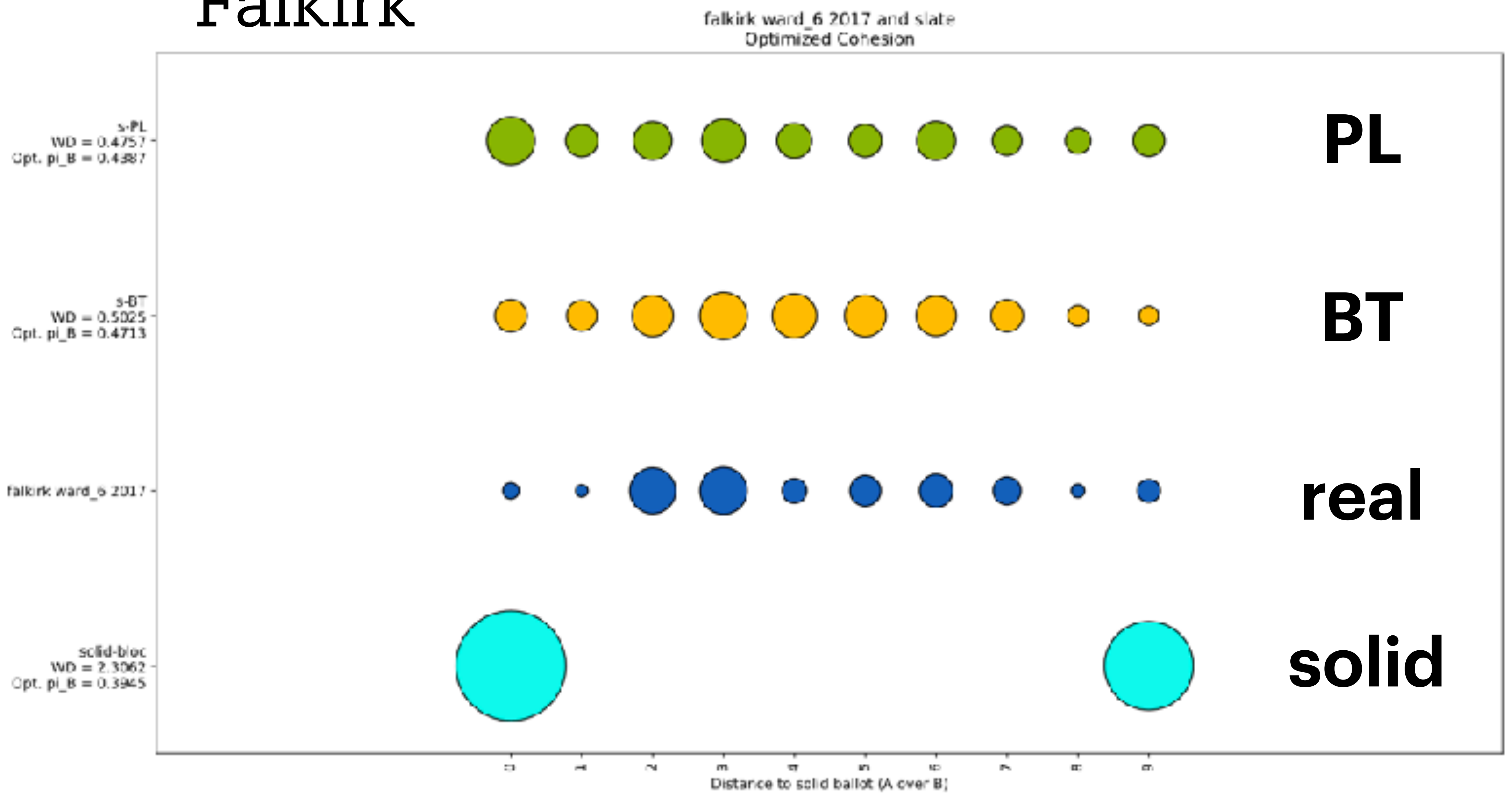
Toward model validation

ballot $\begin{pmatrix} A \\ A \\ B \\ A \\ B \end{pmatrix}$ is **one swap** from $\begin{pmatrix} A \\ A \\ A \\ B \\ B \end{pmatrix}$

measure ***swap distance to sorted A>B*** on database of Scottish local government elections

with Lib Dem / Labour / Green as **Slate B**

Falkirk



Aberdeen