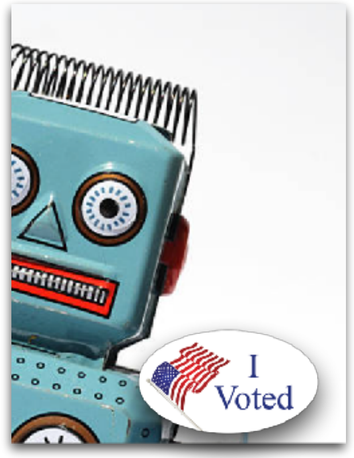
Modeling Democracy

Lecture 12 - **What (else) could proportionality mean?**



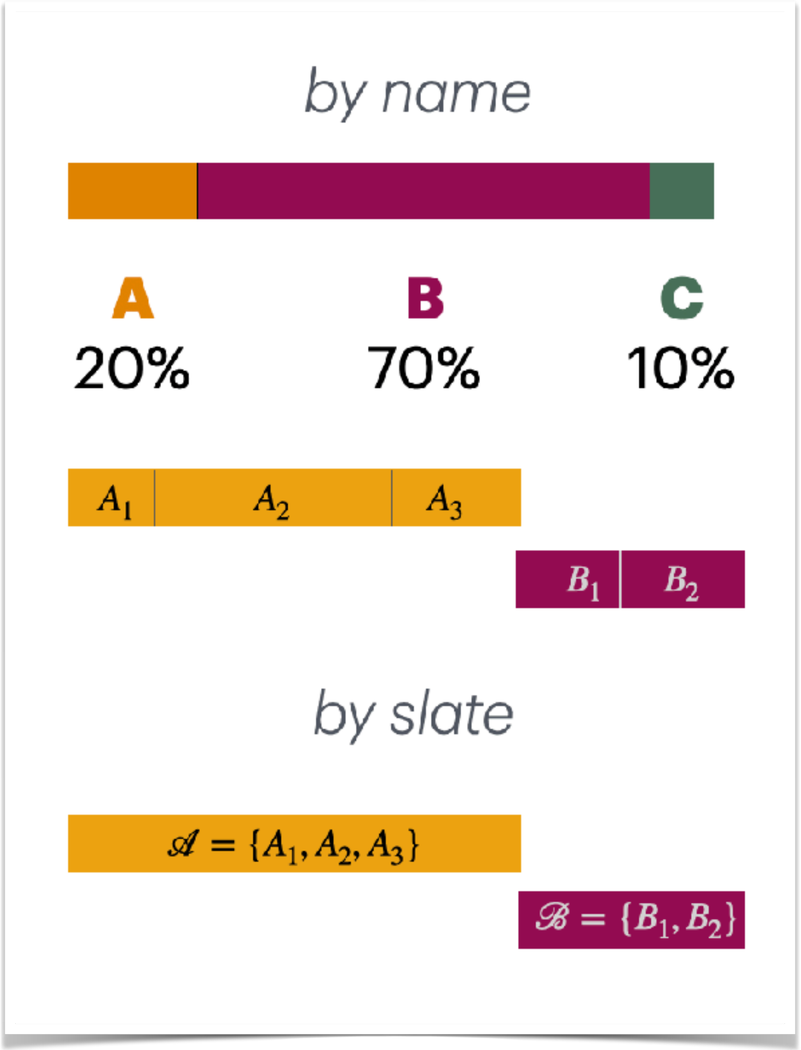
Moon Duchin, Winter 2026 — Mon Feb 16

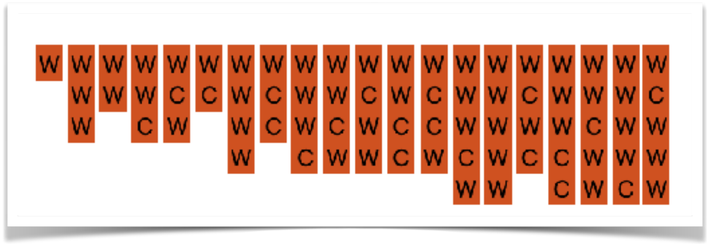
Generative model recap

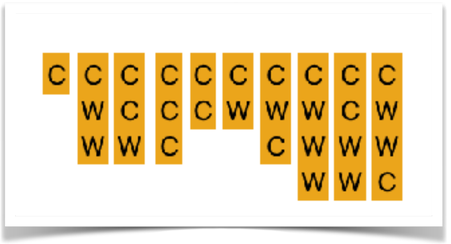
PL / **impulsive**: sample from utilities without replacement

BT / **deliberative**: probability of a ranking is built from probs of each pairwise comparison

Cambridge: alternation pattern and length are drawn from Cambridge, MA historical voting

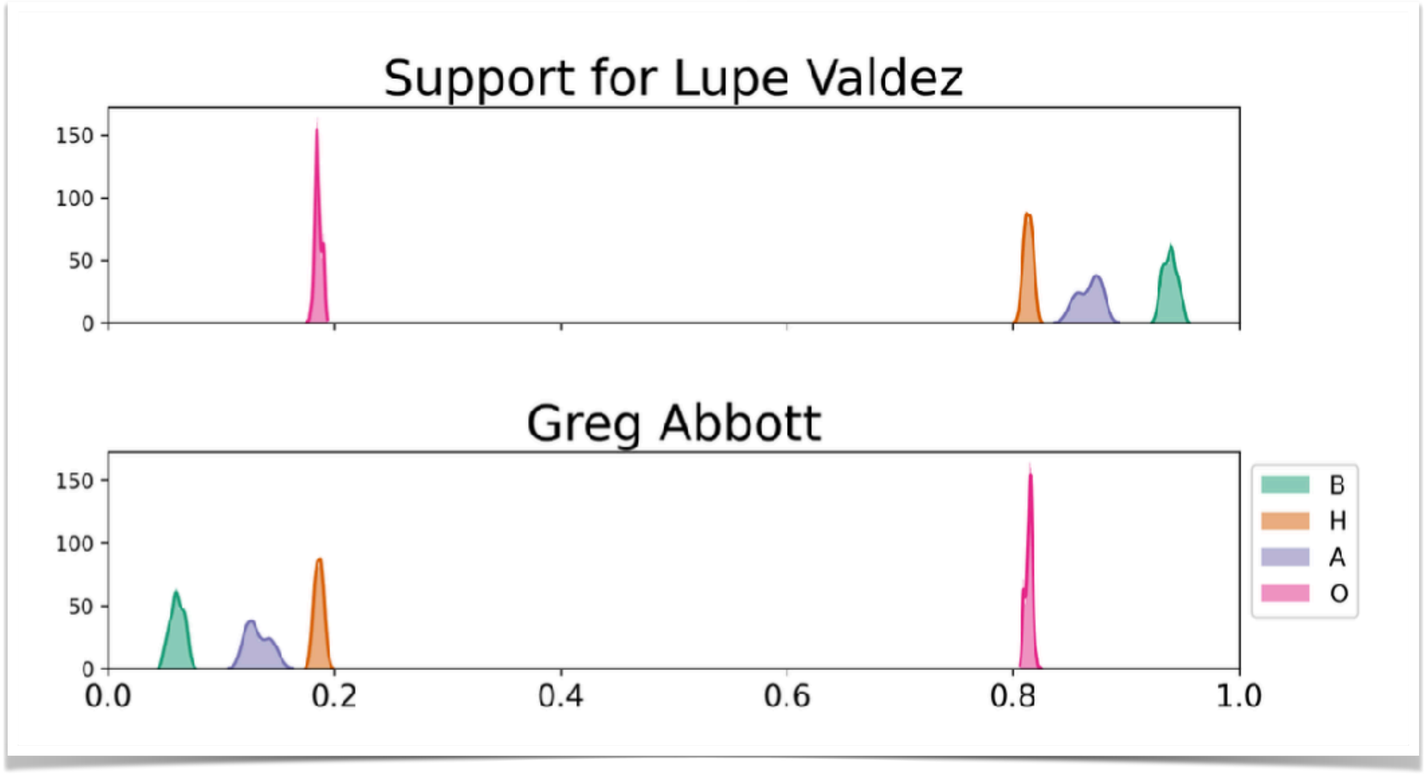






what's missing?

Key attribute: **Polarization**



there are socially relevant groups that vote differently
(to say the least)

We can incorporate this with **mixture models** — create two (or more) **blocs** of voters and specify different parameters

interlude:
you can do all this in VoteKit

And also have big repo of **Scottish STV** election records.

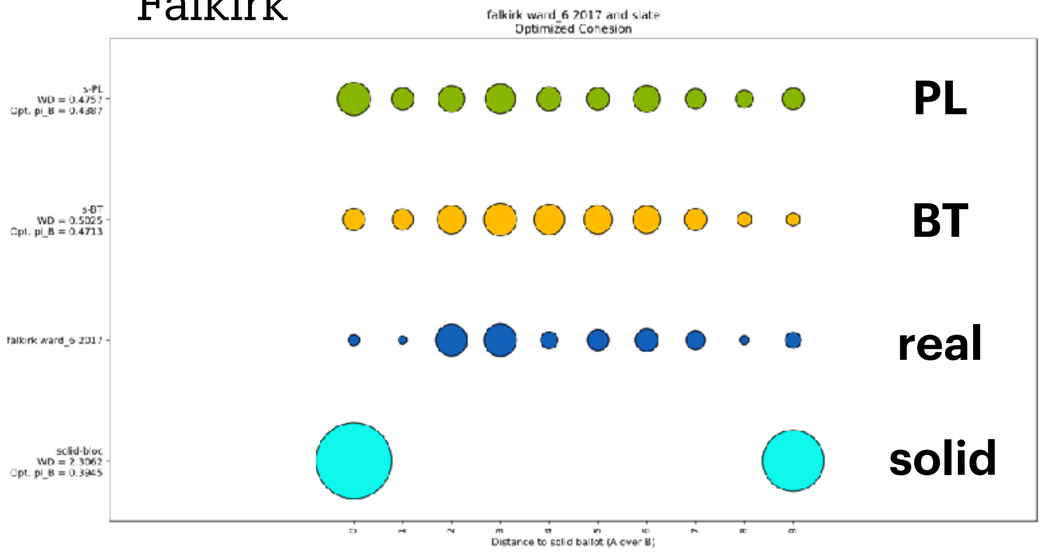
Toward model validation

ballot $\begin{pmatrix} A \\ A \\ B \\ A \\ B \end{pmatrix}$ is **one swap** from $\begin{pmatrix} A \\ A \\ A \\ B \\ B \end{pmatrix}$

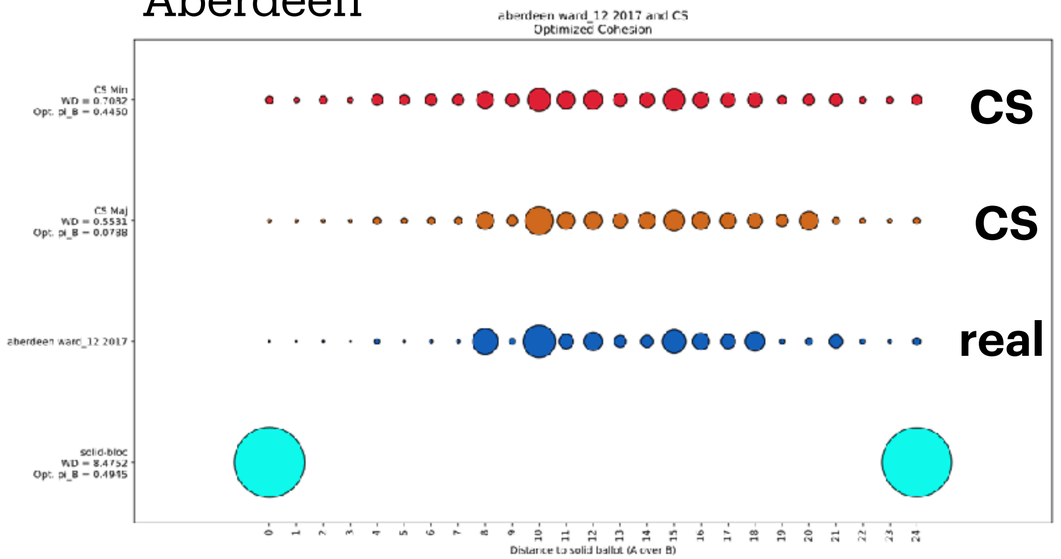
measure ***swap distance to sorted A>B*** on database of Scottish local government elections

with Lib Dem / Labour / Green as **Slate B**

Falkirk



Aberdeen



example application:

Proportionality

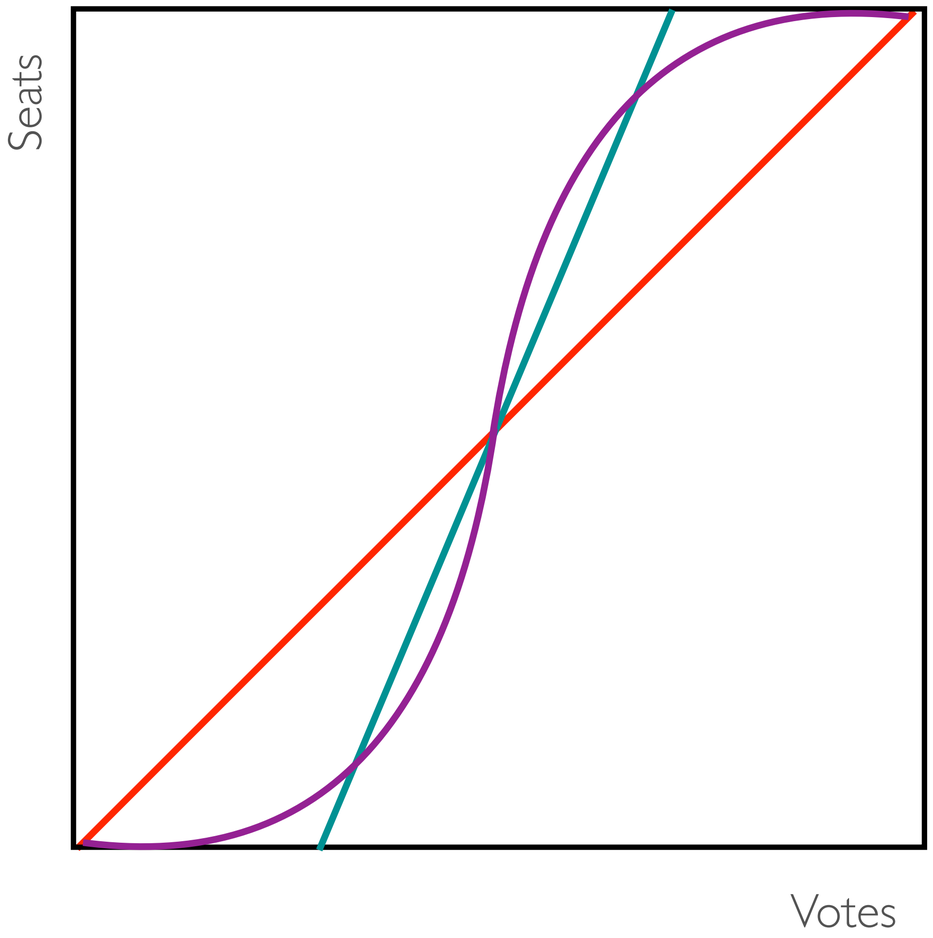
Standard account from political science

Proportionality: seat share for a group is about the same as vote share

Requires a way of identifying **bloc** and **slate**, such as party or race

- **Proportionality**: ideal S=V
- "**Efficiency gap**": ideal 2S-4V=1
- **Partisan symmetry**: ideal S(V) symmetric

Plot for 2-party case



let’s study

Proportionality as a *tendency*

Proportionality for rankings

Let's take inspiration from generative models and from voting rights law, where cohesion is more probabilistic than set-theoretic

- can set blocs A, B, and their sizes N_A, N_B
- π_A, π_B — cohesion of group A and group B voters; their tendency to vote for members of their own slates
- candidate strength — preference intervals (such as 10:1:1 support among A_1, A_2, A_3) or Dirichlet parameters ($\alpha = .5$ among the A candidates), etc

We seek a definition that

(a) uses this ***kind*** of data,

(b) applies across all of these conditions, and

(c) is a matter of degree rather than yes/no

Proportionality defined by **combined support**

Proportionality attained when *seat share for B equals their share of combined support*

$$S_B = N_B \cdot \pi_B + N_A\,(1 - \pi_A)$$

how many B · cohesion for B · how many A · crossover for B

note special case S=V!

(when $N_B = 1$ and voters *choose one*)

election	(r, s, k)	first place pref. π_B	first place pref. proportionality	top-k Borda share π_B	top-k Borda share proportionality	STV outcome
North Ayrshire 2022 Ward 1	(8, 4, 5)	0.17	0.87 seats	0.24	1.19 seats	0 seats
Angus 2012 Ward 8	(4, 2, 4)	0.24	0.96 seats	0.26	1.02 seats	1 seat
Clackmannanshire 2012 Ward 2	(5, 3, 4)	0.32	1.27 seats	0.31	1.25 seats	1 seat
Aberdeen 2022 Ward 12	(7, 3, 4)	0.31	1.26 seats	0.36	1.42 seats	1 seat
Aberdeen 2017 Ward 12	(6, 4, 4)	0.33	1.33 seats	0.41	1.63 seats	1 seat
Falkirk 2017 Ward 6	(3, 3, 4)	0.34	1.35 seats	0.43	1.71 seats	2 seats
Renfrewshire 2017 Ward 1	(4, 4, 4)	0.37	1.49 seats	0.46	1.84 seats	1 seat
Fife 2022 Ward 21	(4, 4, 4)	0.46	1.86 seats	0.51	2.02 seats	2 seats
Glasgow 2012 Ward 16	(7, 5, 4)	0.60	2.40 seats	0.58	2.32 seats	3 seats

Scottish examples ($N_B = 1$)

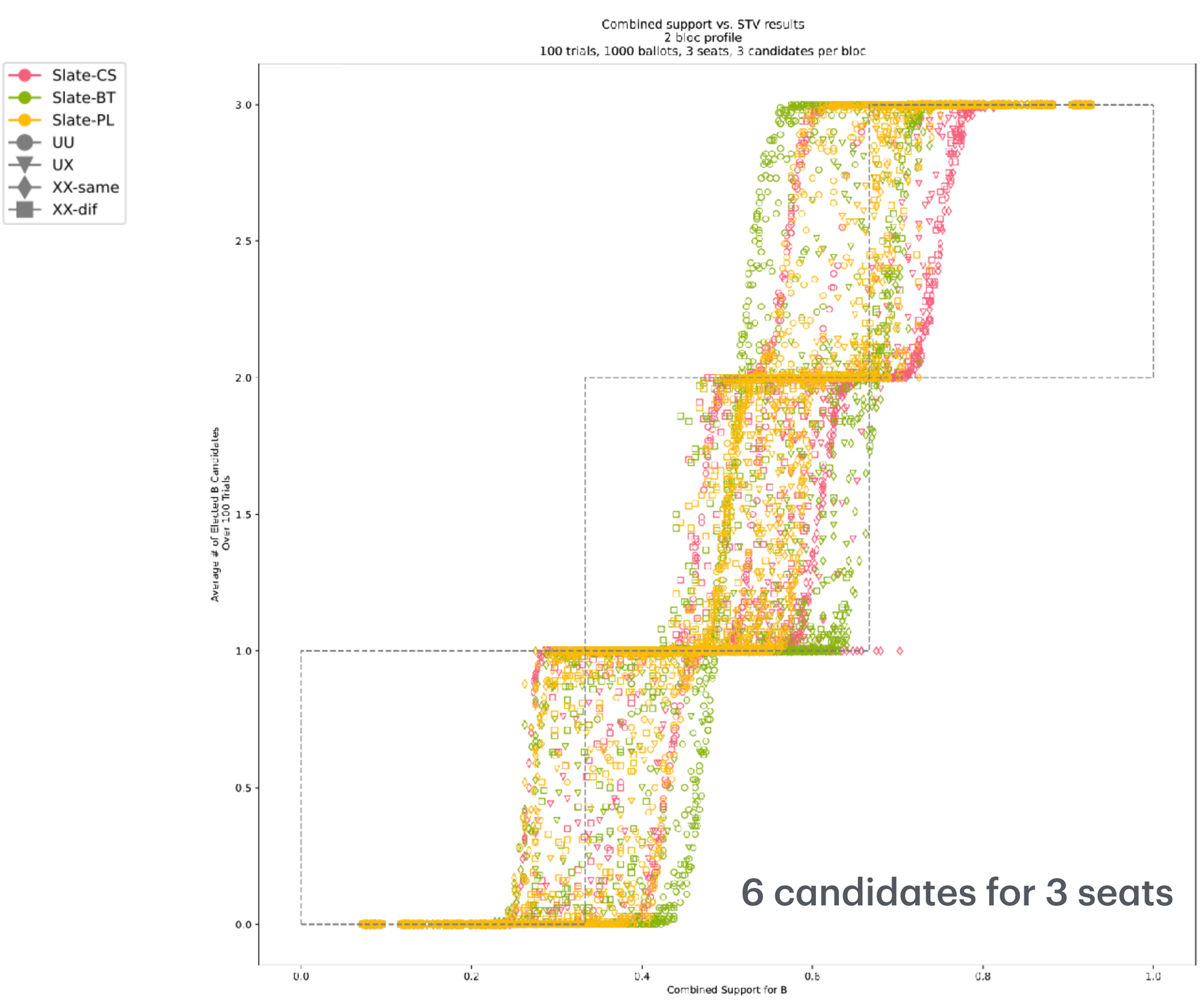
r = # A candidates

s = # B candidates

k = # seats

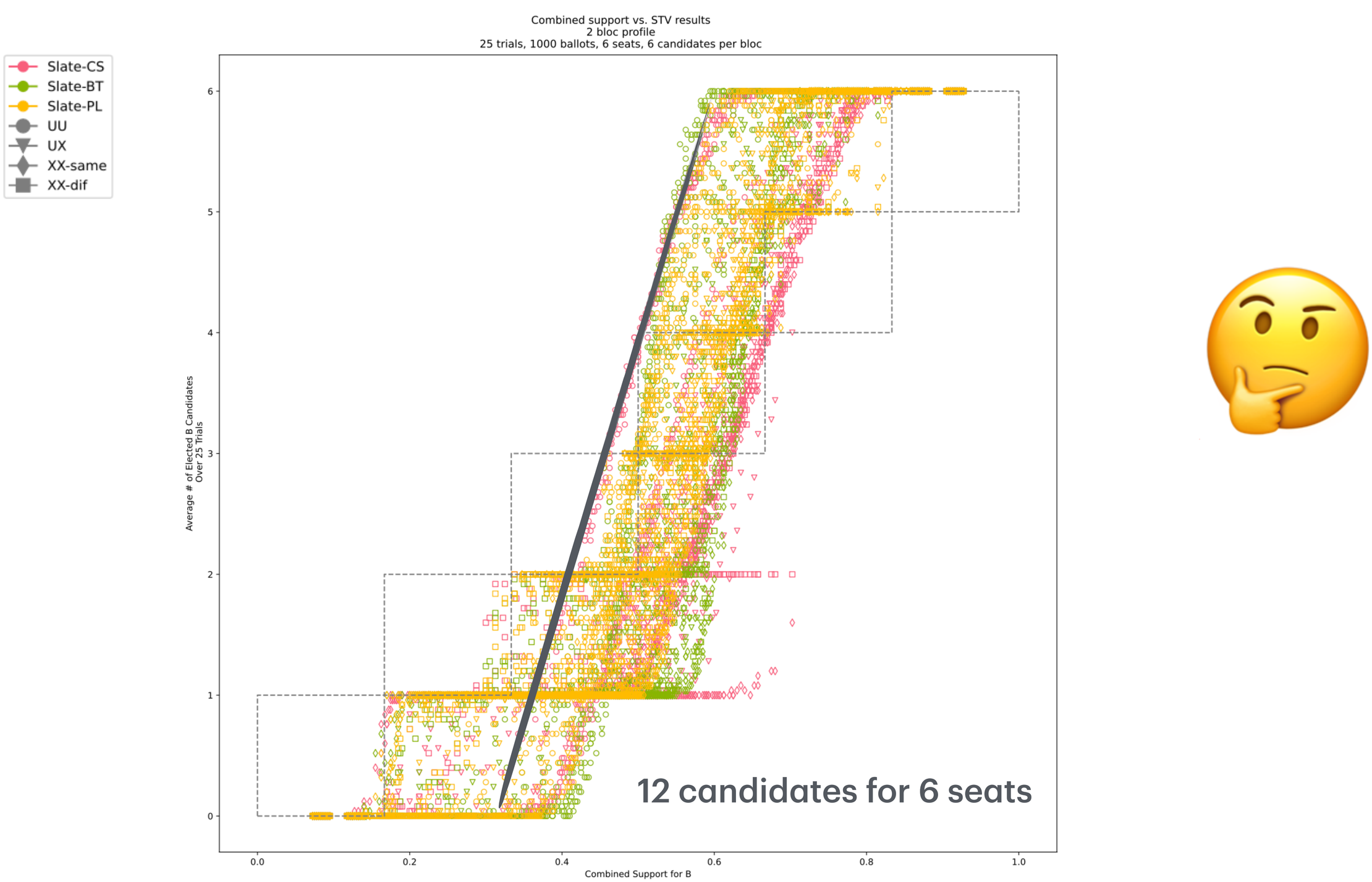
π_B = tendency to vote B, expressed through FPV or Borda

outcomes: mostly proportional!



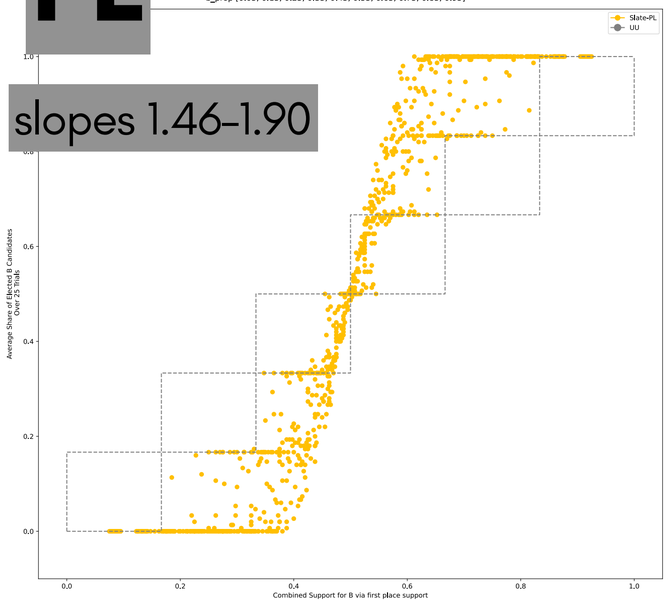
Now we'll do a 2-bloc setup and "turn all the knobs," varying generative model, bloc sizes, cohesion, and candidate strength

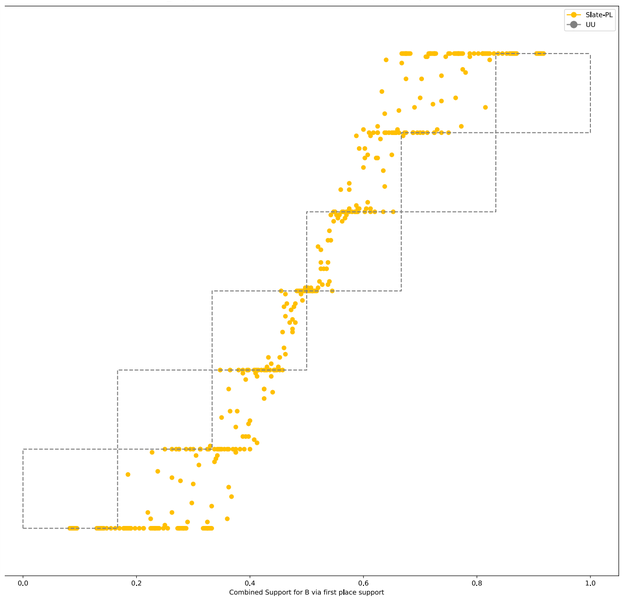
STV outcomes: maybe a little less proportional than you'd hope!

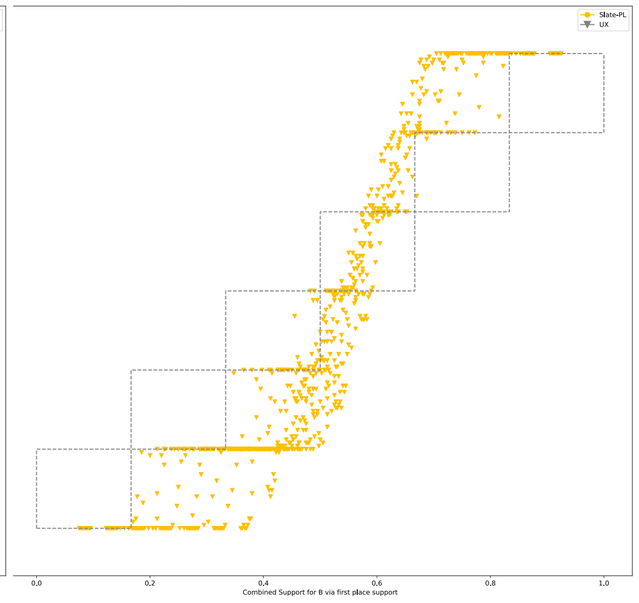

Combined support vs. STV results
2 bloc profile
25 trials, 1000 ballots, 6 seats, 6 candidates per bloc
Slate-CS
Slate-BT
Slate-PL
UU
UX
XX-same
XX-dif
Average # of Elected B Candidates
Over 25 Trials
Combined Support for B
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6
12 candidates for 6 seats

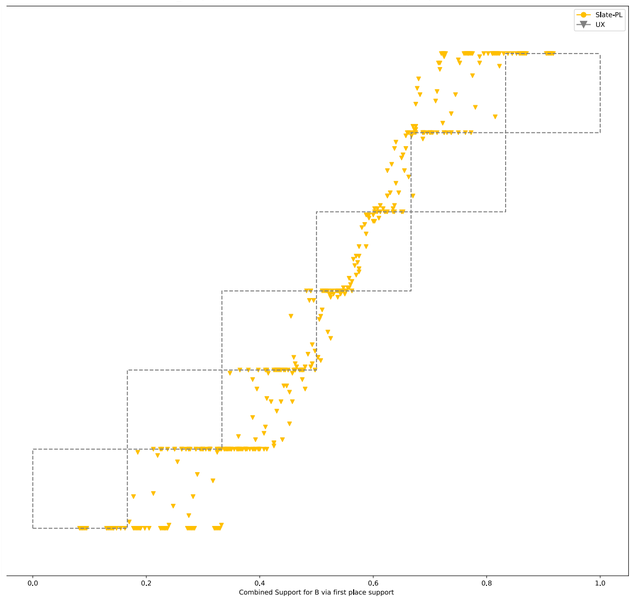
PL

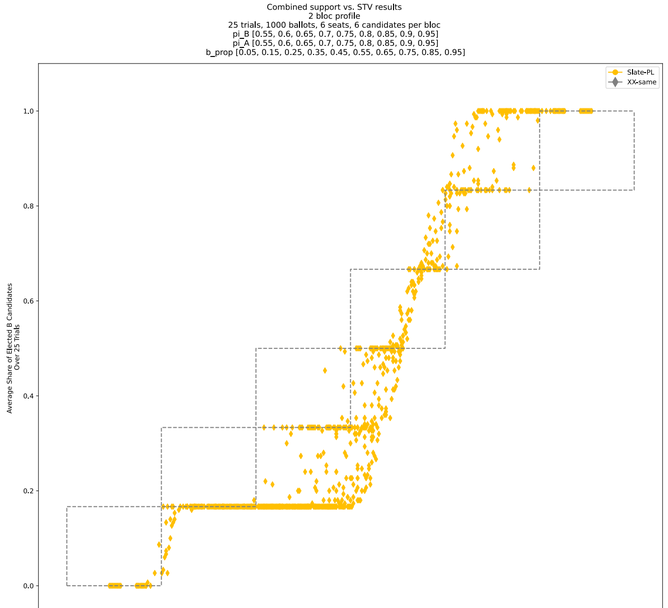
slopes 1.46–1.90

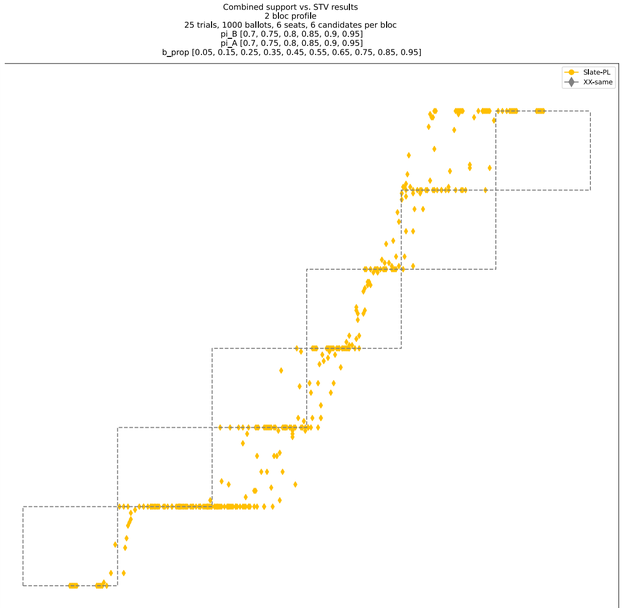


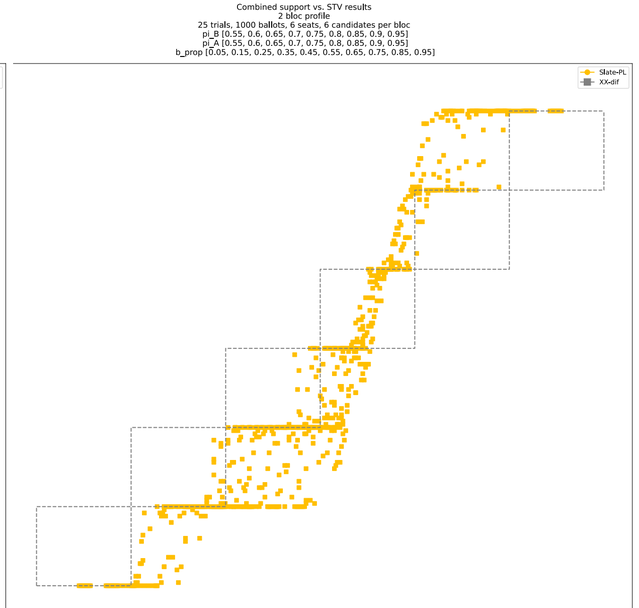


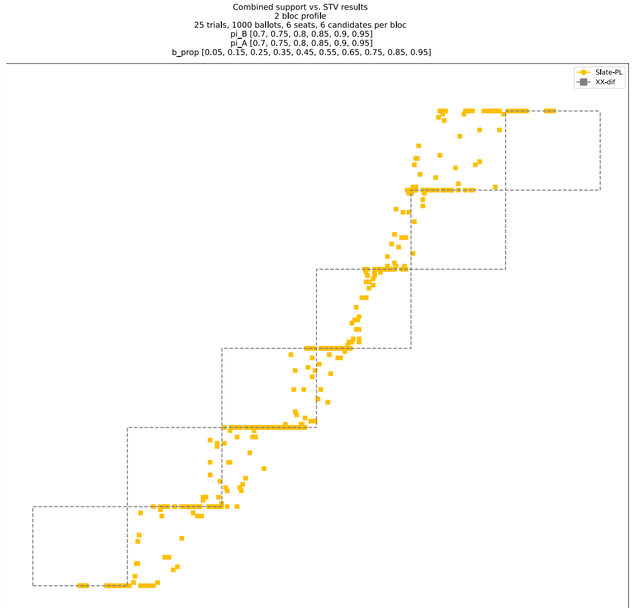






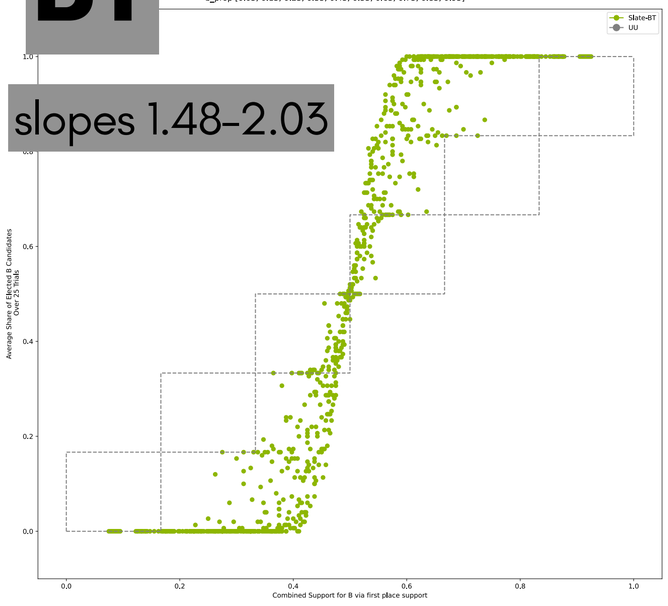


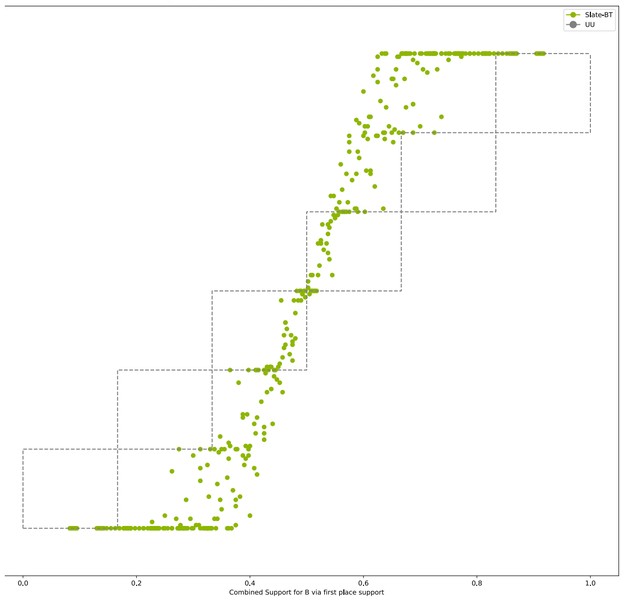


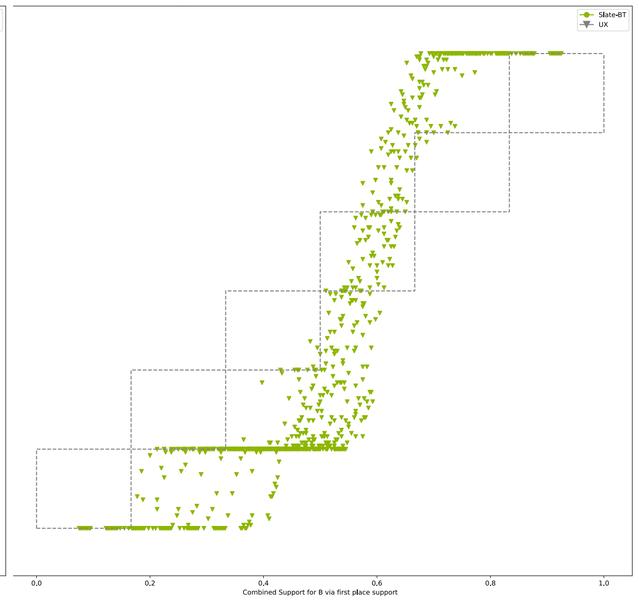


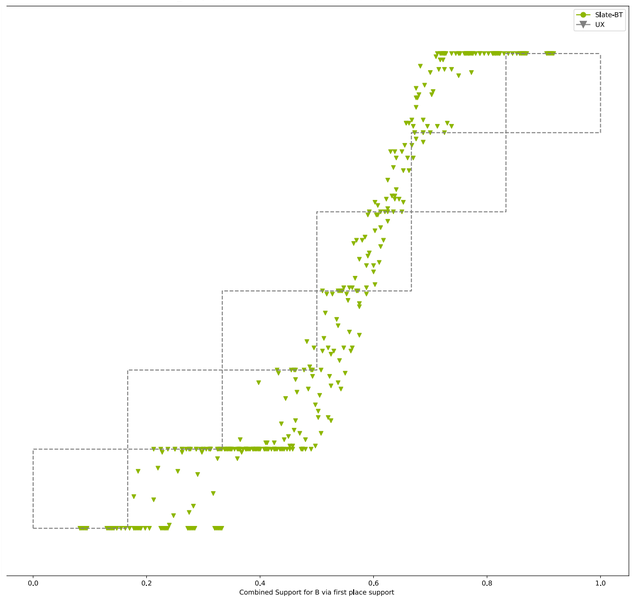
BT

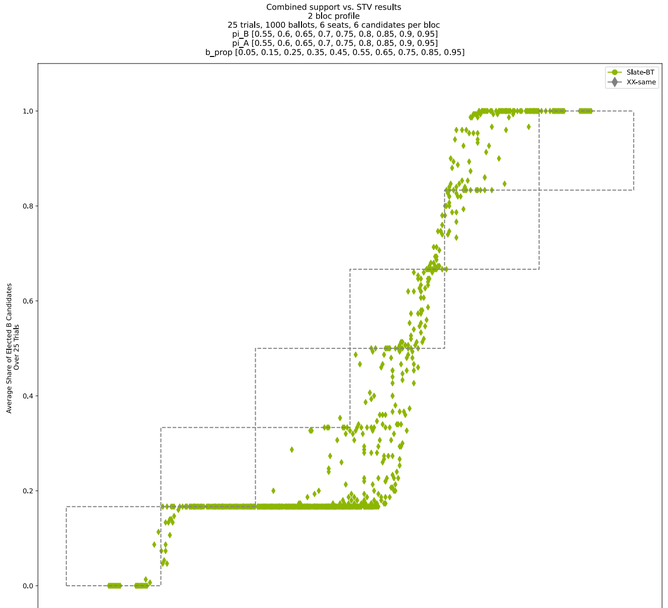
slopes 1.48–2.03

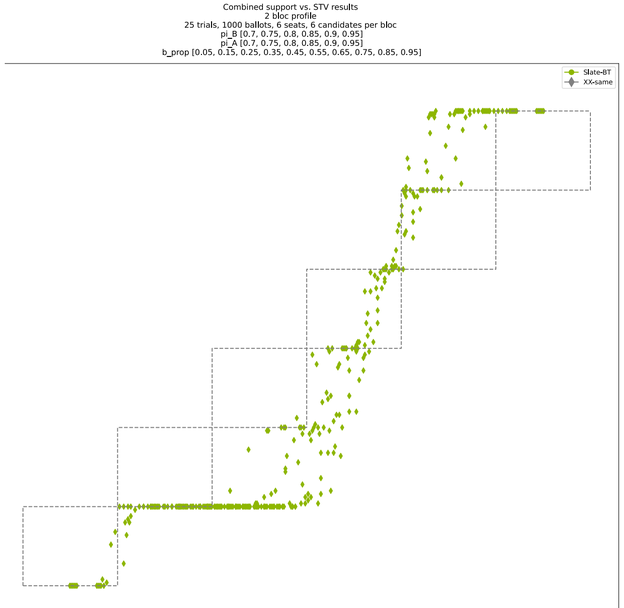


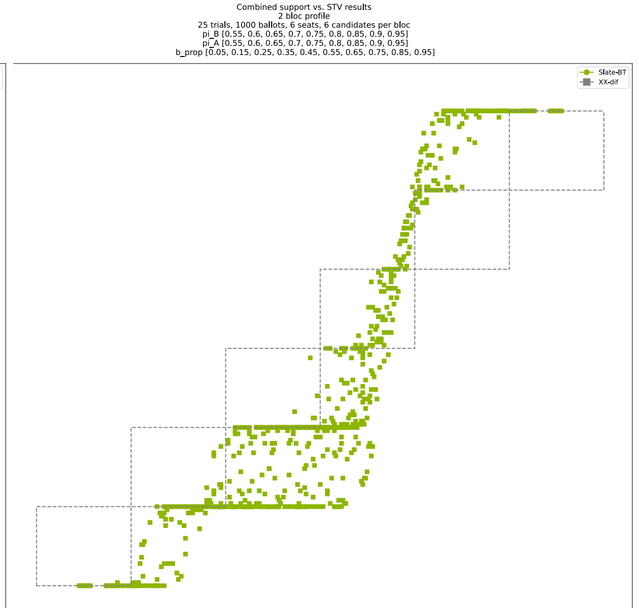


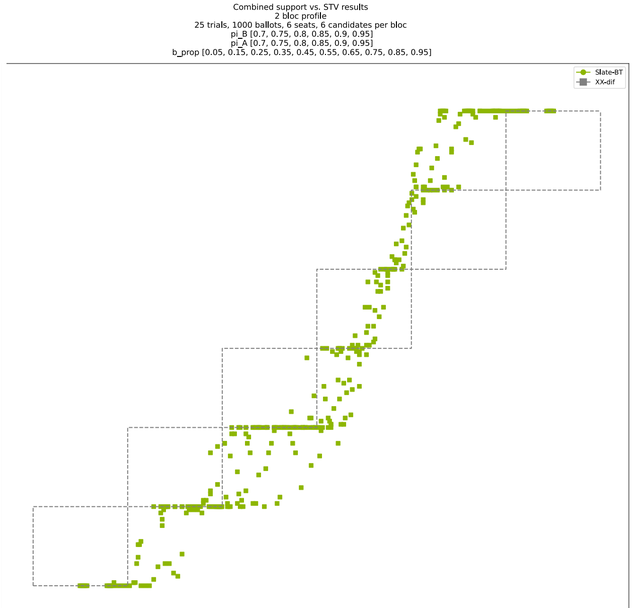






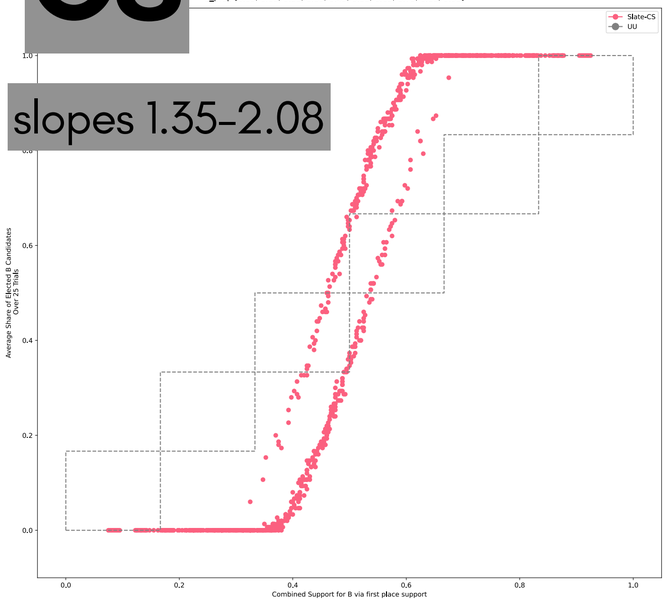


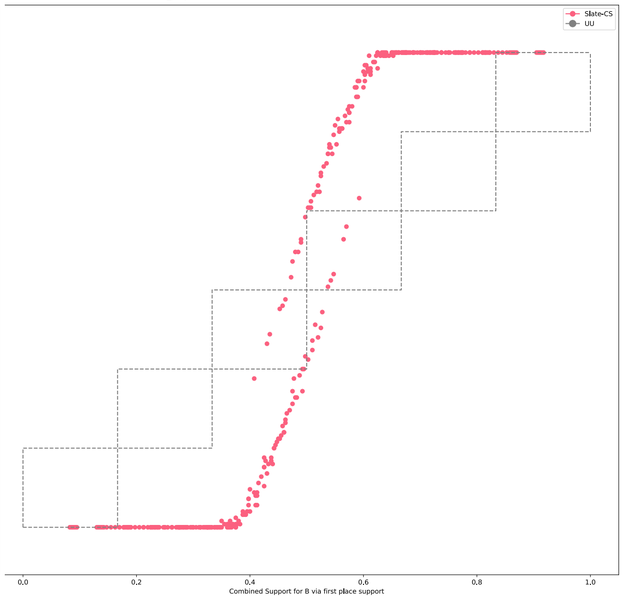


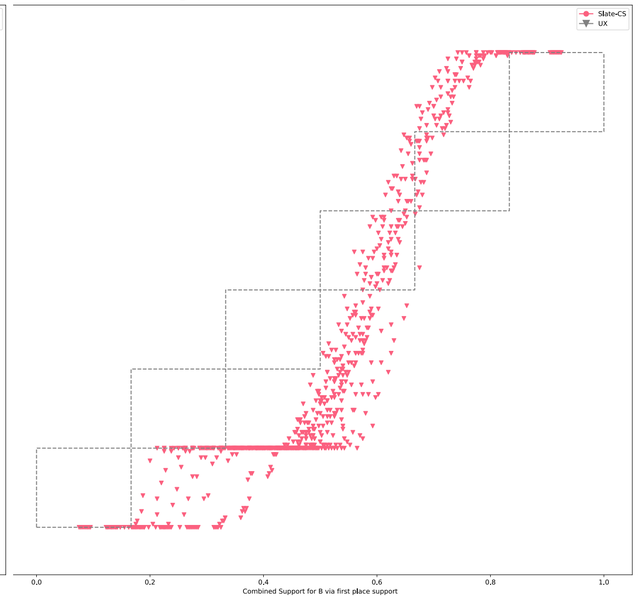


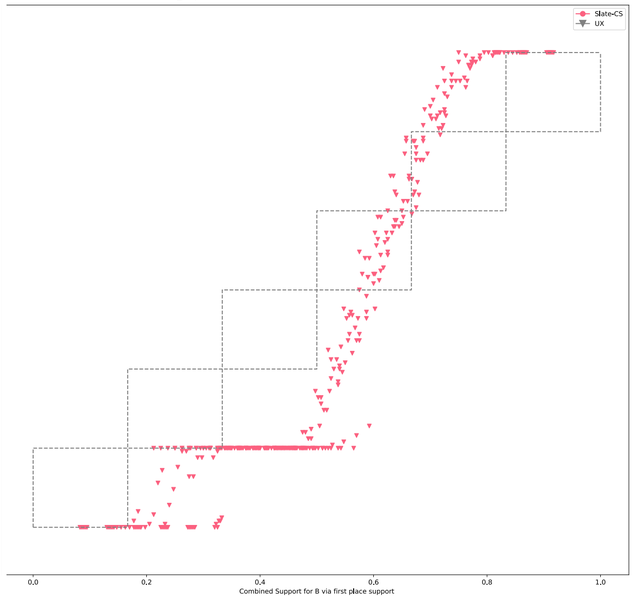
CS

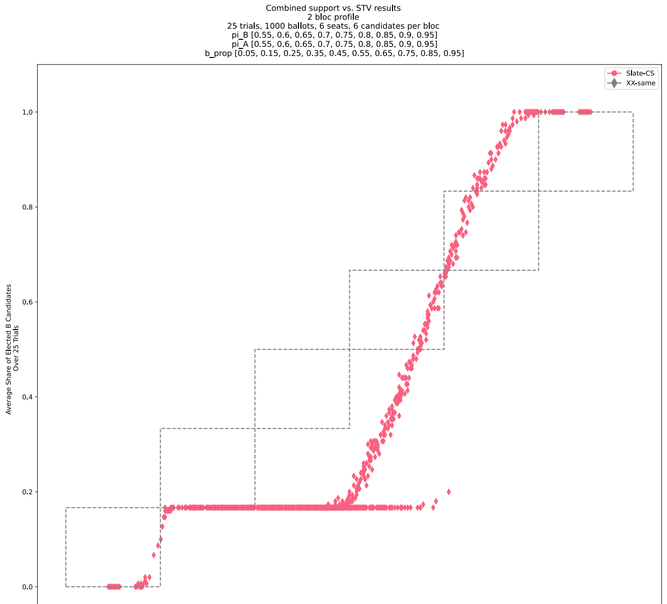
slopes 1.35–2.08

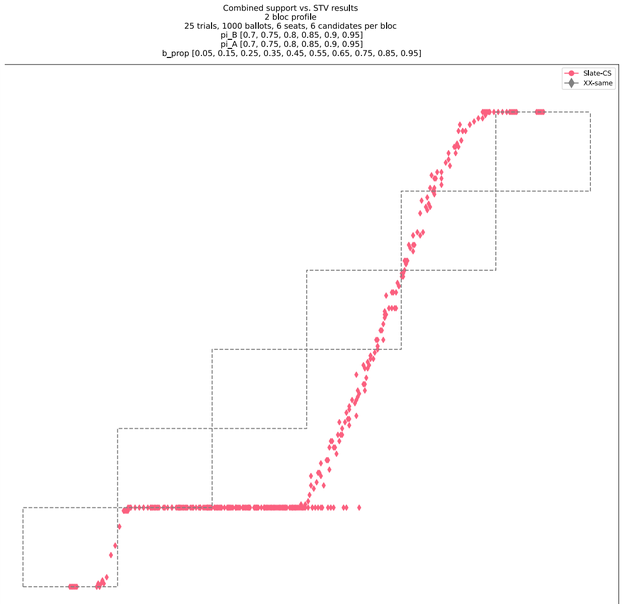


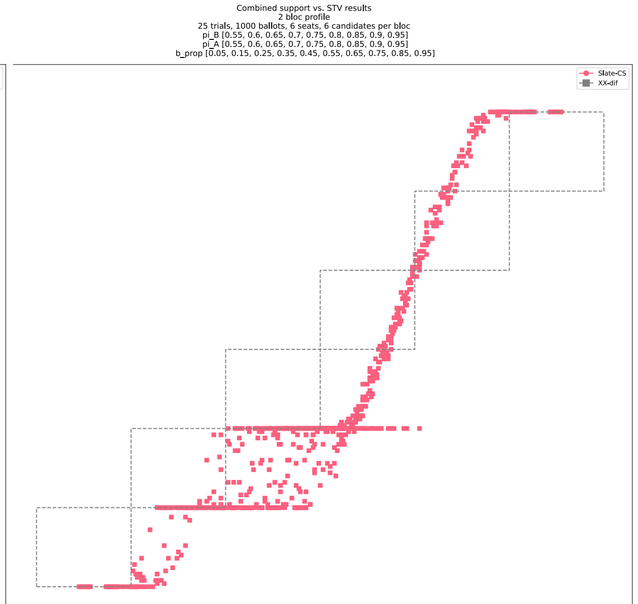


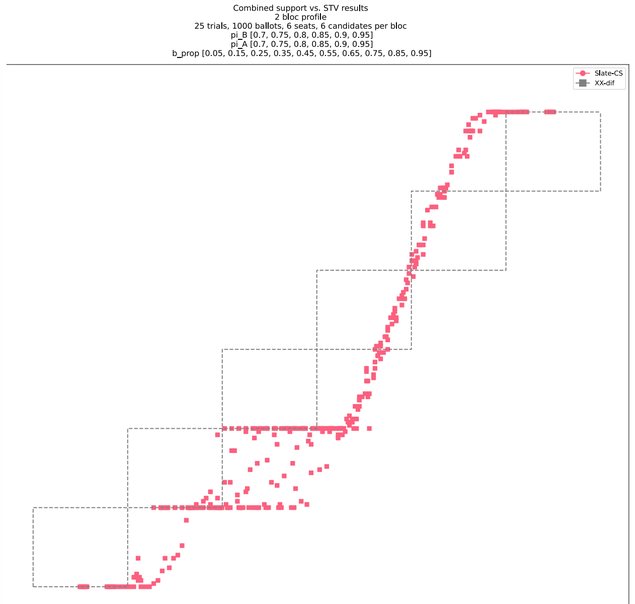




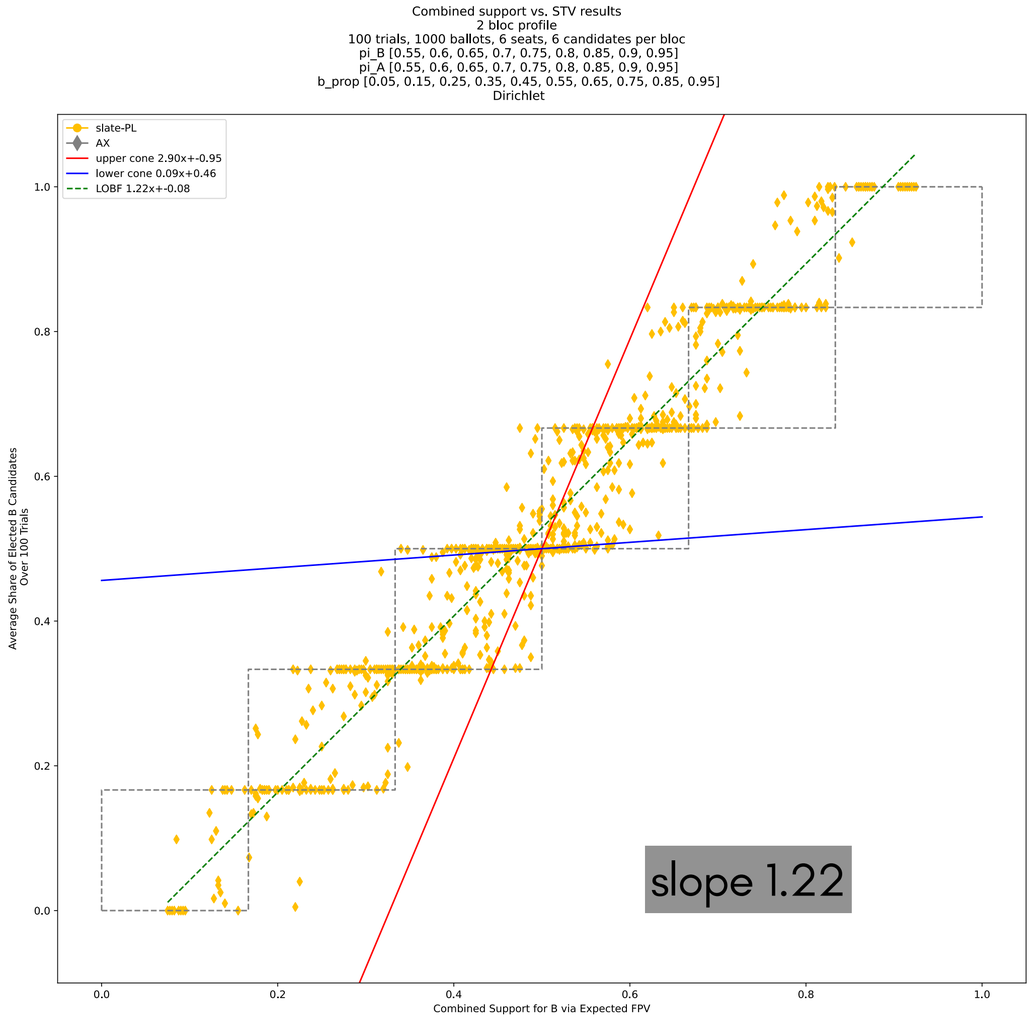








What's going on?



Switch from rigid (1:1:1) / (10:1:1) preference intervals to Dirichlet model of "strong" or "uniform" candidates and the change is substantial — now slopes of fit lines much closer to 1

There is important work to do in understanding the **conditions that produce (dis)proportionality**

And theoretical results?

COROLLARY 4.4 (BOUNDING DISPROPORTIONALITY FOR STV WITH FIXED CANDIDATE ORDERS). *Suppose we consider STV under the same conditions as above (slate-PL, fixed candidate order, sufficiently large k). Then under simultaneous election, disproportionality can get arbitrarily severe as the election gets large. However, under one-at-a-time election, the asymptotic ratio of seats to votes for the minority party satisfies*

$$\frac{2}{3} \leq \frac{S_B}{\beta} \leq 2,$$

where β is the support for B candidates.

Benadè-Donnay-Duchin-Weighill