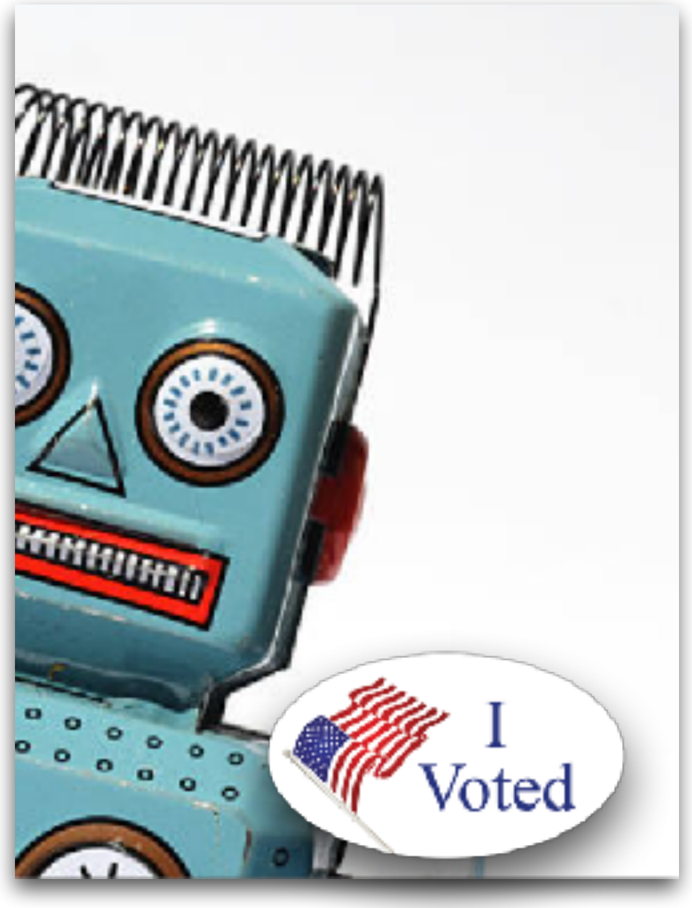
Modeling Democracy

Lecture 6 - **Weighted voting; Hello computer**



Moon Duchin, Winter 2026 — Mon Jan 26

A shot over the bow

WEIGHTED VOTING DOESN'T WORK: A MATHEMATICAL ANALYSIS

John F. Banzhaf III *

In response to recent decisions requiring that both houses of state legislatures be apportioned so that equal numbers of citizens have substantially equal representation, weighted voting has been widely suggested as an alternative to actual reapportionment. Weighted voting has already been adopted in at least three state legislative bodies, including the New Jersey Senate; ordered by one federal district court; considered in several other states; and used by one county governing body for a number of years. In New Jersey and New Mexico, state courts rejected, on state grounds, the weighted systems which had been adopted.[1] Under such systems, in lieu of actual redistricting or reap-

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The author is indebted to Martin Jacobs, Fairleigh Dickinson University, and Donald Wardle, Columbia Law School, for their help in preparing this paper.

1. N.M. Stat. Ann. §§ 2-7-1 to 2-7-13 (Supp. 1964) established a weighted voting plan for the New Mexico state legislature; this plan was later held to conflict with the state

Representative	Number of Votes
A	5
B	1
C	1
D	1
E	1

Representative	Number of Votes
R	8
S	8
T	8
U	8
V	1

What is Banzhaf's concept of power, mathematically?

- It has been assimilated into game theory as follows.

- A **simple cooperative game** is a pair (N, σ) where $\sigma : 2^N \to \{0,1\}$ indicates which subsets of players are "winning coalitions."

- A **weighted voting game** is a triple (N, w, T) where $N = |\mathscr{V}|$, $\mathrm{w} = (w_1, \ldots, w_N)$ is a weight vector, and T is a threshold for success in a vote.

- A player is **decisive** or **pivotal** in a coalition if it wins with them and loses without them.

- A **dummy player** is never decisive. A **veto player** is in every winning coalition.

What is Banzhaf's concept of power, mathematically?

- Player i has the following **Banzhaf power**:

$$\beta_i := c \sum_{S \subset \mathcal{V} \setminus i} \sigma(S \cup i) - \sigma(S)$$

- Here, if you set $c = 1/2^{N-1}$, you can interpret this as the **probability** that player i is decisive in a uniformly random coalition they are in.

- Another option is to choose c so the $\sum \beta_i = 1$, in which case it's some sort of **share** of decisiveness.

belongs to class of **semivalues**: place coefficient inside sum depending only on |S|

in partic, **Shapley value**: all coalitions of size k equally likely

Real examples

- UN Security Council voting can be phrased this way. A measure passes if 9/15 vote in favor, unless vetoed by permanent member.
    - ✳ This amounts to (e.g.) T=49, with weights of 9 for permanent members and 1 for others.

UN Security Council

- European Common Market, formed in 1958 as a federation of six European countries, was governed by a council of ministers with the following vote weights and threshold T=12.
    - ✳ Luxembourg is a dummy!

Common Market vote weights

France	Germany	Italy	Belgium	Netherlands	Luxembourg
4	4	4	2	2	1

Complexity, efficient approximation

- **Theorem**: given a weighted voting game, computing β_i is #P-complete.
- (“recall” that P is **solving** in polynomial time, NP is **checking** in polynomial time, and #P is roughly **counting** solutions for NP problems. If you could compute Banzhaf powers in polynomial time, then you can also count SAT solutions or knapsack solutions in polynomial time.)
- But for large N, never fear: can get good Monte Carlo approximation quickly by uniformly sampling coalitions rather than enumerating over all.
- And for small N, exact calculation is fast.

State	2000Pop	PopShare	2000EV	EVShare	BanzProb	BanzShare
California	33,871,648	0.1204	55	0.1022	23.75%	0.1141
Texas	20,851,820	0.0741	34	0.0632	13.31%	0.0639
New York	18,976,457	0.0674	31	0.0576	12.07%	0.0580
Florida	15,982,378	0.0568	27	0.0502	10.43%	0.0501
Illinois	12,419,293	0.0441	21	0.0390	8.05%	0.0387
Penn	12,281,054	0.0436	21	0.0390	8.05%	0.0387
Ohio	11,353,140	0.0403	20	0.0372	7.66%	0.0368
Michigan	9,938,444	0.0353	17	0.0316	6.49%	0.0312
New Jersey	8,414,350	0.0299	15	0.0279	5.71%	0.0274
Georgia	8,186,453	0.0291	15	0.0279	5.71%	0.0274
N Carolina	8,049,313	0.0286	15	0.0279	5.71%	0.0274
Virginia	7,078,515	0.0252	13	0.0242	4.95%	0.0238
Mass	6,349,097	0.0226	12	0.0223	4.56%	0.0219
Indiana	6,080,485	0.0216	11	0.0204	4.18%	0.0201
Washington	5,894,121	0.0209	11	0.0204	4.18%	0.0201
Tennessee	5,689,283	0.0202	11	0.0204	4.18%	0.0201
Missouri	5,595,211	0.0199	11	0.0204	4.18%	0.0201
Wisconsin	5,363,675	0.0191	10	0.0186	3.79%	0.0182
Maryland	5,296,486	0.0188	10	0.0186	3.79%	0.0182
Arizona	5,130,632	0.0182	10	0.0186	3.79%	0.0182
Minnesota	4,919,479	0.0175	10	0.0186	3.79%	0.0182
Louisiana	4,468,976	0.0159	9	0.0167	3.41%	0.0164
Alabama	4,447,100	0.0158	9	0.0167	3.41%	0.0164
Colorado	4,301,261	0.0153	9	0.0167	3.41%	0.0164
Kentucky	4,041,769	0.0144	8	0.0149	3.03%	0.0146
S Carolina	4,012,012	0.0143	8	0.0149	3.03%	0.0146
Oklahoma	3,450,654	0.0123	7	0.0130	2.65%	0.0127
Oregon	3,421,399	0.0122	7	0.0130	2.65%	0.0127
Connecticut	3,405,565	0.0121	7	0.0130	2.65%	0.0127
Iowa	2,926,324	0.0104	7	0.0130	2.65%	0.0127
Mississippi	2,844,658	0.0101	6	0.0112	2.27%	0.0109
Kansas	2,688,418	0.0096	6	0.0112	2.27%	0.0109
Arkansas	2,673,400	0.0095	6	0.0112	2.27%	0.0109
Utah	2,233,169	0.0079	5	0.0093	1.89%	0.0091
Nevada	1,998,257	0.0071	5	0.0093	1.89%	0.0091
New Mexico	1,819,046	0.0065	5	0.0093	1.89%	0.0091
West Virginia	1,808,344	0.0064	5	0.0093	1.89%	0.0091
Nebraska	1,711,263	0.0061	5	0.0093	1.89%	0.0091
Idaho	1,293,953	0.0046	4	0.0074	1.51%	0.0073
Maine	1,274,923	0.0045	4	0.0074	1.51%	0.0073
NH	1,235,786	0.0044	4	0.0074	1.51%	0.0073
Hawaii	1,211,537	0.0043	4	0.0074	1.51%	0.0073
Rhode Island	1,048,319	0.0037	4	0.0074	1.51%	0.0073
Montana	902,195	0.0032	3	0.0056	1.14%	0.0055
Delaware	783,600	0.0028	3	0.0056	1.14%	0.0055
S Dakota	754,844	0.0027	3	0.0056	1.14%	0.0055
N Dakota	642,200	0.0023	3	0.0056	1.14%	0.0055
Alaska	626,932	0.0022	3	0.0056	1.14%	0.0055
Vermont	608,827	0.0022	3	0.0056	1.14%	0.0055
DC	572,059	0.0020	3	0.0056	1.14%	0.0055
Wyoming	493,782	0.0018	3	0.0056	1.14%	0.0055

Indeed, this was Banzhaf's proposal

and the NY State Court of Appeals bought it

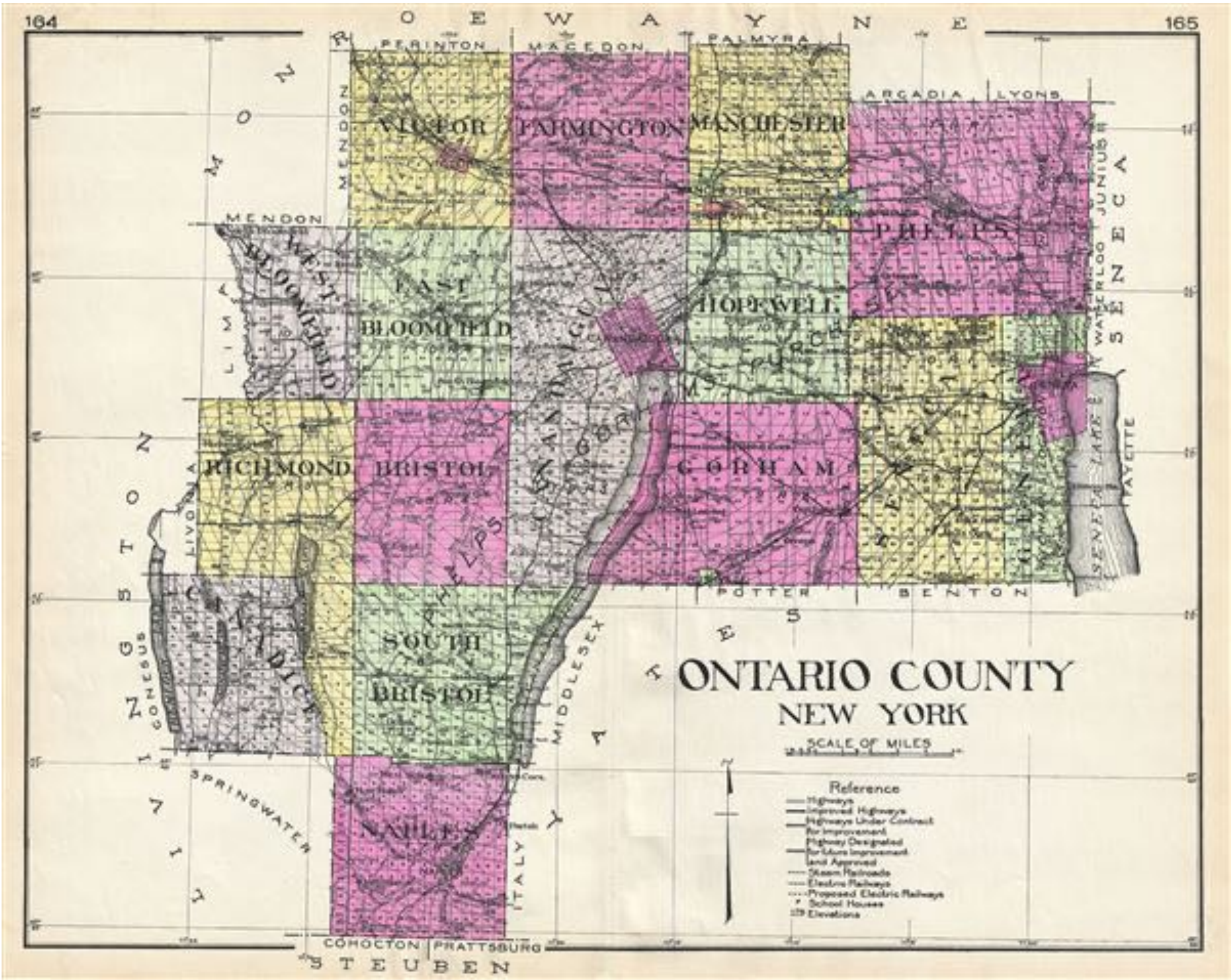
- Banzhaf: seek weights that minimize the discrepancy between Banzhaf share and population share
- This amounts to an **inverse problem** (very hard to solve exactly!)
- The court endorses this approach in *Iannucci v. Board of Supervisors* (1967).

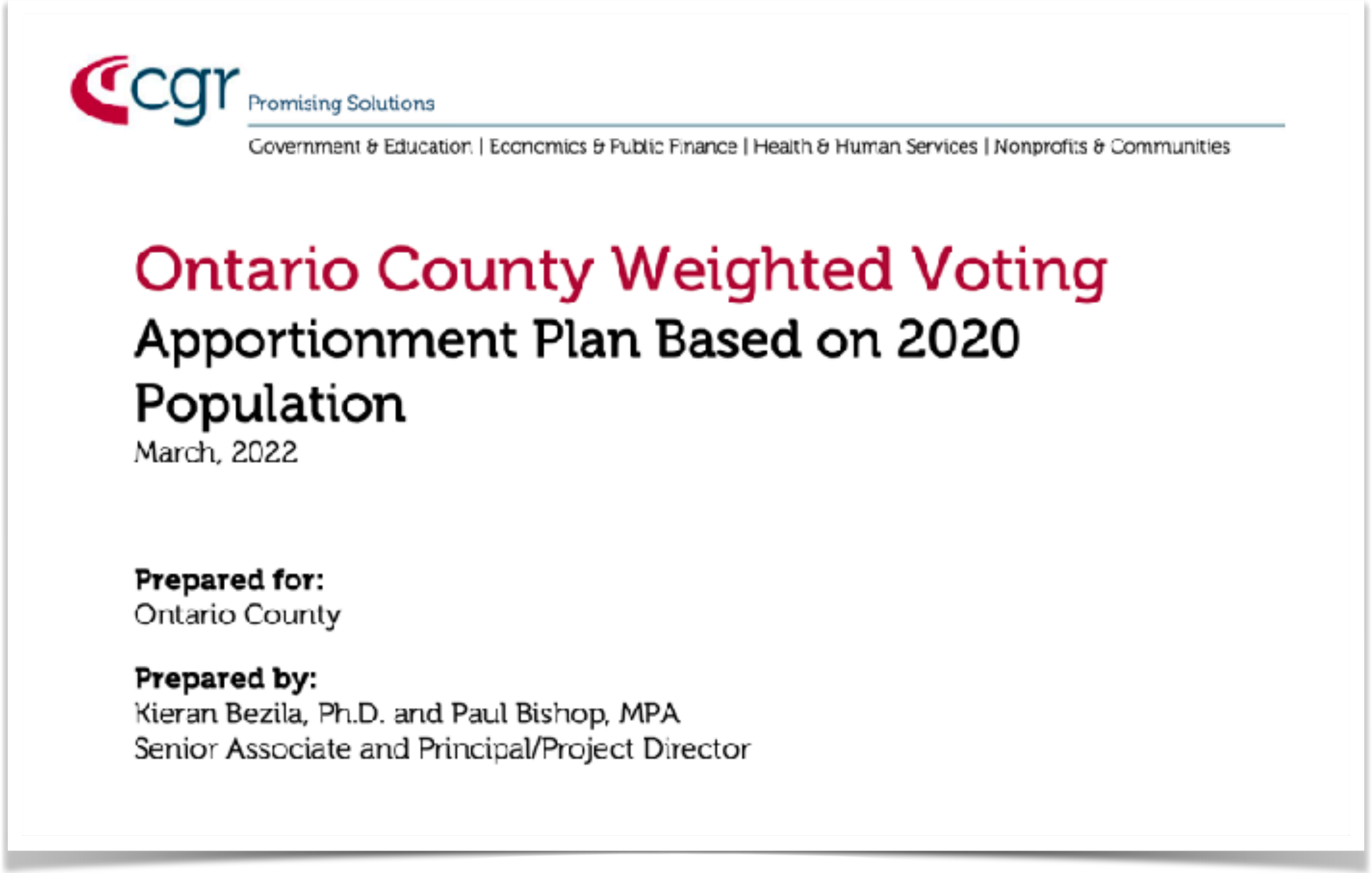
defects". Although the small towns in a county would be separately represented on the board, each might actually be less able to affect the passage of legislation than if the county were divided into districts of equal population with equal representation on the board and several of the smaller towns were joined together in a single district. (See Banzhaf, Weighted Voting Doesn't Work: A Mathematical Analysis, 19 Rutgers L. Rev. 317.) The significant standard for measuring a legislator's voting power, as Mr. Banzhaf points out, is not the number or fraction of votes which he may cast but, rather, his "ability * * *, by his vote, to affect the passage or defeat of a measure"

Unfortunately, it is not readily apparent on its face whether either of the plans before us meets the constitutional standard. Nor will practical experience in the use of such plans furnish relevant data since the sole criterion is the mathematical voting power which each legislator possesses in theory — i.e., the indicia of representation — and not the actual voting power he possesses in fact — i.e., the indicia of influence. In order to measure the mathematical voting power of each member of these county boards of supervisors and compare it with the proportion of the population which he represents, it would be necessary to *253 have the opinions of experts based on computer analyses. The plans,

And today?

- This is still used in many NY counties!
- Their boards of supervisors have to do some business by simple majority (T=1/2) and some by supermajority (T=2/3). Less commonly T=3/5 and T=3/4.
- **Ontario County** has been contracting with think tanks for decades to give them weights that are heuristically optimized for Banzhaf discrepancy.
- For $8000 and a month of time, they computed weights for the 21 “towns” that make up the county.
- Their report gives a truly ad hoc description of the method.

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Ontario County Weighted Voting
Apportionment Plan Based on 2020 Population
March, 2022

Prepared for:
Ontario County

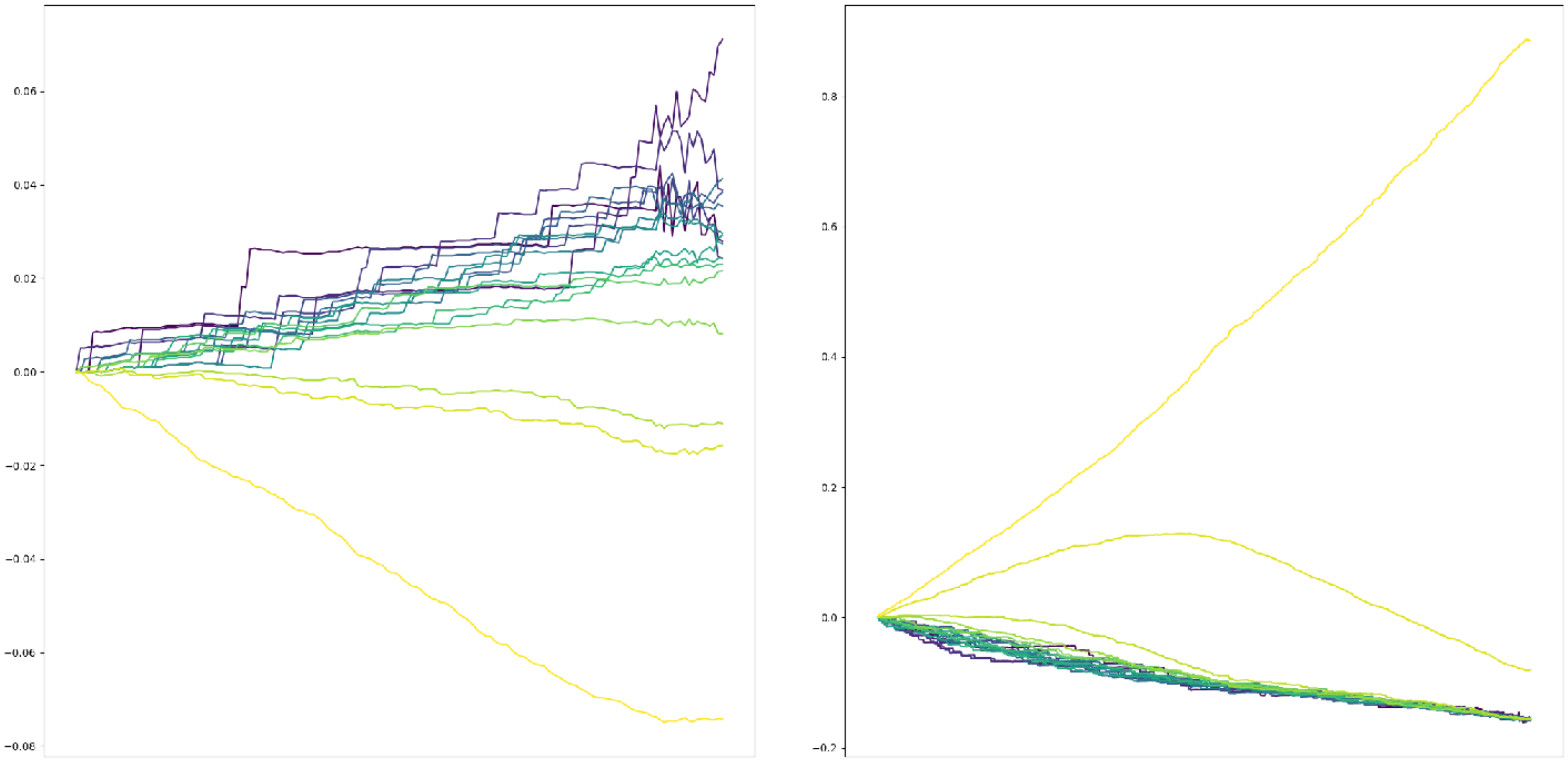
Prepared by:
Kieran Bezila, Ph.D. and Paul Bishop, MPA
Senior Associate and Principal/Project Director

This is where we come in

- We built a Markov chain approach to searching for best weights — guess (starting with population shares) and adjust
- Then we apportioned the weights to add up to 100,000 so the fit with population shares is easy to eyeball
- And there were some surprises!

Ontario County	Town name	Adjusted pop.	Pop. share	$T = 1/2$ weights		$T = 2/3$ weights		$T = 3/4$
				CGR 2022	MGGG	CGR 2022	MGGG	MGGG
Town 1	Town of South Bristol	1641	0.01459	68	1487	68	1438	1229
Town 2	Town of Canadice	1668	0.01483	69	1511	69	1460	1247
Town 3	Town of Bristol	2284	0.02031	96	2105	95	1999	1711
Town 4	Town of Naples	2403	0.02137	100	2210	100	2104	1802
Town 5	Town of Seneca	2644	0.02351	110	2421	110	2313	1987
Town 6	Town of West Bloomfield	2740	0.02437	114	2505	114	2397	2058
Town 7	Town of Richmond	3360	0.02988	142	3123	140	2939	2522
Town 8	Town of Geneva	3473	0.03088	146	3203	144	3042	2607
Town 9	Town of East Bloomfield	3640	0.03237	152	3340	151	3192	2732
Town 10	City of Geneva (5,6)	3679	0.03272	153	3372	153	3223	2760
Town 11	City of Geneva (3,4)	3921	0.03487	163	3582	163	3435	2939
Town 12	Town of Hopewell	3931	0.03496	163	3588	163	3442	2948
Town 13	Town of Gorham	4106	0.03651	170	3741	171	3599	3080
Town 14	City of Canandaigua (2,3)	5140	0.04571	213	4684	214	4505	3855
Town 15	City of Geneva (1,2)	5210	0.04633	216	4741	217	4565	3903
Town 16	City of Canandaigua (1,4)	5436	0.04834	225	4932	226	4767	4076
Town 17	Town of Phelps	6637	0.05902	273	6001	277	5823	4985
Town 18	Town of Manchester	9404	0.08362	381	8381	394	8318	7045
Town 19	Town of Canandaigua	11109	0.09879	444	9750	464	9808	8339
Town 20	Town of Farmington	14170	0.12600	550	12093	601	12710	11576
Town 21	Town of Victor	15860	0.14103	602	13230	711	14921	26599
SUM		112,456		4550	100,000	4745	100,000	100,000
L1 error (discrepancy)				0.00103	0.00013	0.00207	0.00008	0.00012
				7.9× improved		26.3× improved		

Change in weights over course of optimization run



T=1/2

T=3/4

small towns  big towns

Scale is proportional to town size

Weights initialized at pop shares

Ontario County	Town name	Adjusted pop.	Pop. share	$T = 1/2$ weights CGR 2022	$T = 1/2$ weights MGGG	$T = 2/3$ weights CGR 2022	$T = 2/3$ weights MGGG	$T = 3/4$ MGGG
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L1 error (discrepancy)				0.00103	0.00013	0.00207	0.00008	0.00012
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close

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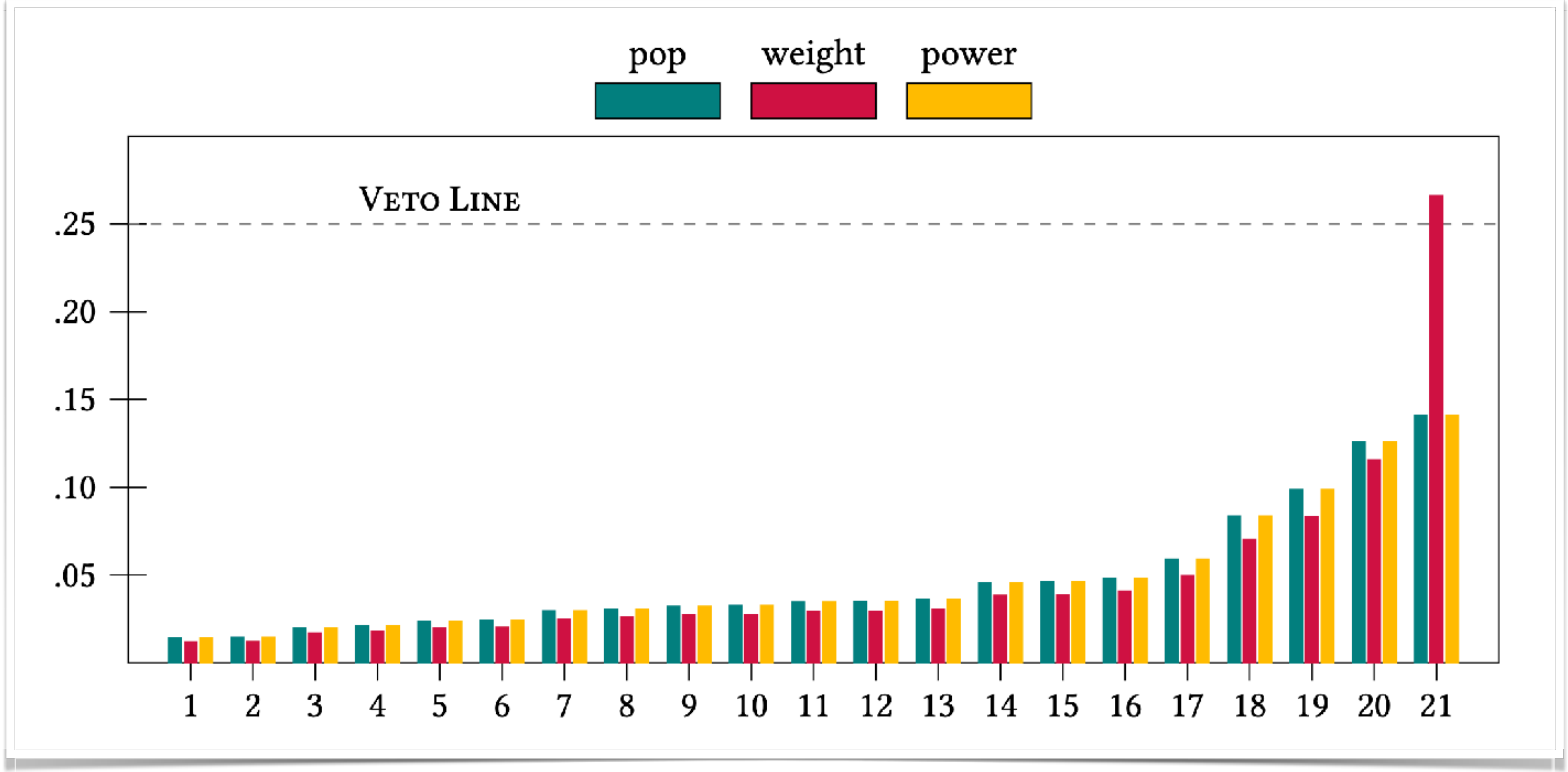
not close!



I'd estimate that there are 200+ papers on Banzhaf and Shapley values, and nobody seems to discuss this "undeserving veto player" problem.

New paper: a practical fix with some nice theoretical guarantees: at threshold T, voters have propensity p=T to vote yes!

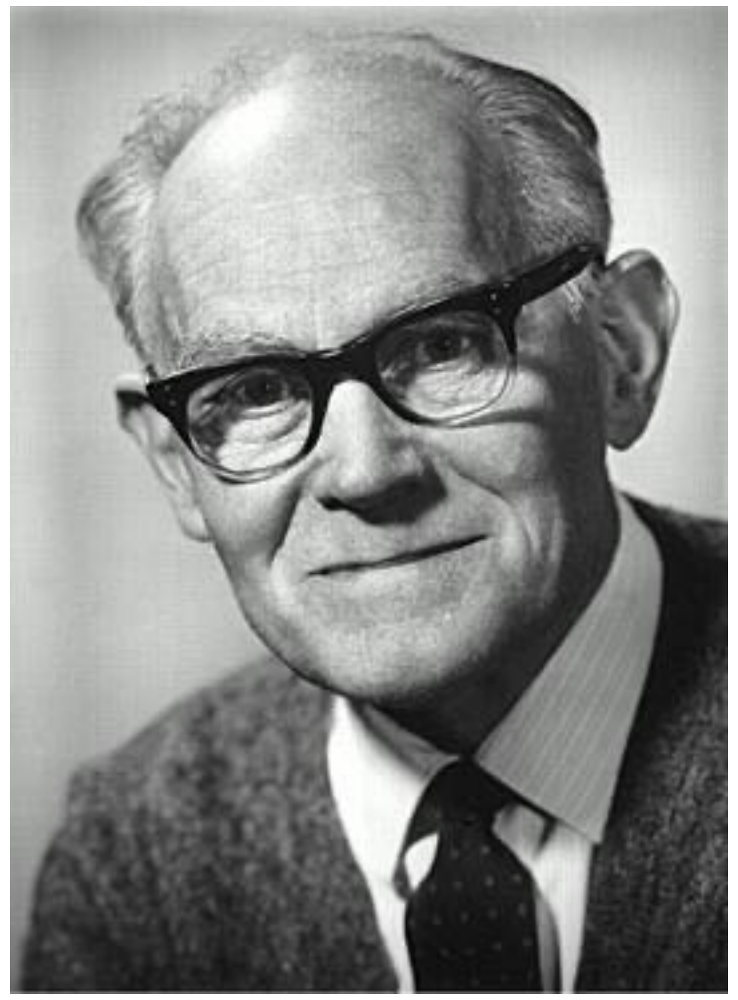
Theorem (dR-D-P-TF): Suppose $\mathcal{G}_n$ is any sequence of weighted voting games where populations are bounded, new players are added one at a time, and threshold is fixed. Suppose that players vote yes with propensity p. Then

$$\frac{\beta_i^p(\mathcal{G}_n)}{\beta_j^p(\mathcal{G}_n)} \to \frac{w_i}{w_j} \iff p = T \text{ or } w_i = w_j.$$

Furthermore, smaller players have a power boost (relative to weights) when $p < T$, while larger players have a power boost when $p > T$.

Let's shift to individual voter perspective

- What is the probability that a voter in a first-past-the-post election is decisive?
- Say voter i is decisive if their candidate wins with their vote but would have lost without their vote.
- Standard model: suppose each of n other voters flips a fair coin. You need them to split evenly.
- $p = \frac{\binom{n}{n/2}}{2^n} \sim \frac{2}{\sqrt{2\pi n}}$. So it's on the order of $1/\sqrt{n}$.
- "**Penrose square root law**"



Lionel Penrose

Galton Professor of Eugenics, 1945-63

Professor of Human Genetics 1963-65

Bernoulli model

- But in fact if we want the model to have some given expected outcome, say set to match the polling average, then we're flipping a **weighted** coin.

> [I]magine that a particular candidate's victory is worth $33 billion to the common good (suppose she is a civic-minded, financial wizard), that there are 122,293,322 voters (as in the 2004 U.S. presidential election), and that the probability of any given voter supporting our financial wizard is 50.5%. With the stakes artificially raised, one might expect that individual votes are impactful. But the expected value to the common good of one's vote for the financial wizard is a mere $\$4.77 \times 10^{-2650}$.

- So voting is clearly irrational.

Alternative model

from Barnett, "Why You Should Vote to Change the Outcome" (2020)

- **First**, Barnett points out that Bernoulli model is obviously wrong
  - Bernoulli $B(n, p)$ has mean $\mu = np$ and variance $\sigma^2 = np(1 - p)$, so the 95% confidence interval with $p = .505$ and 100M voters is roughly $[0.5049,\ 0.5051]$.
  - Ridiculous! Far too narrow.
- **Second**, he uses economics "cost of voting" framework — voting has some cost, such as missed work, transportation, inconvenience. Maybe \$100.
- "Rationally," one should only vote if benefit exceeds cost.

Sufficient conditions

- **Stakes condition**: average social benefit of your candidate, b, is more than twice the cost of voting: $b > 2c$.
- **Chances condition**: decisive probability at least one over number of voters: $d \geq 1/N$.
- Why this suffices for rationality: $\mathbb{E} = \dfrac{bNd}{2} \geq \dfrac{b}{2} > c$
- Why believe **stakes** is reasonable? Example: Iraq/Afghanistan military expenses in the wake of Sept 11 estimated at \$1.4-\$3.5 trillion, more than \$4000 per U.S. citizen. Total tax revenue roughly \$10K per citizen.
- For chances, more detailed arguments.

More Barnett

- Here are three modeling assumptions Barnett wants to consider.
    - (a) **reasonably close** – both candidates have at least 10% chance of winning,
    - (b) "**partial unimodularity**" – the likely leader is more likely to get 50% than any particular value less than that,
    - (c) "**narrow upsets**" – given that the likely leader loses, the chance of getting 45-50% of the vote is at least half.
- How are they argued for? The first is just something that will sometimes seem reasonable based on polling. The others are "true of every election model with which I am familiar."
- Barnett very attached to the point of view that given our knowledge going into election day, there is a **fact of the matter** regarding the probability of each outcome, and polling average is close to the "true degree of support."

More Barnett

- Offers empirical evidence for "narrow upsets."
- Now we close it out by a simple argument. Consider all outcomes (shown as rows in the table here) with 45-50% for the leader. There are N/20 such outcomes.
- Trailing candidate has at least 10% chance of winning, but conditioned on that, half the probability is in 45-50% range for the leader, so that range has at least 5% probability mass = 1/20.
- The "top bucket" of this range—the outcome with 50%, where you are decisive—has the highest probability of the range. If all were equal, each would be 1/N, so this means the decisive outcome has probability > 1/N.

[25] Senate races from 2010, 2012, 2014, 2016, and 2018 were analyzed. The RealClearPolitics.com polling average was used to identify leading candidates. Upsets occurred in Colorado 2010, Nevada 2010, Montana 2012, North Dakota 2012, Kansas 2014, North Carolina 2014, New Hampshire 2016, Pennsylvania 2016, Wisconsin 2016, Arizona 2018, Florida 2018, and Indiana 2018. Of these, only in the Kansas 2014 race was the margin of victory greater than ten percentage points.

1,000,001 possibilities in total

Votes for Donald	Votes for Daisy	Outcome
1,000,000	0	Donald wins regardless of what you do
999,999	1	Donald wins regardless of what you do
999,998	2	Donald wins regardless of what you do
...	...	...
500,001	499,999	Donald wins regardless of what you do
500,000	*500,000*	*Your vote is decisive*
499,999	500,001	Daisy wins regardless of what you do
...	...	...
2	999,998	Daisy wins regardless of what you do
1	999,999	Daisy wins regardless of what you do
0	1,000,000	Daisy wins regardless of what you do

Naive application to weighted voting

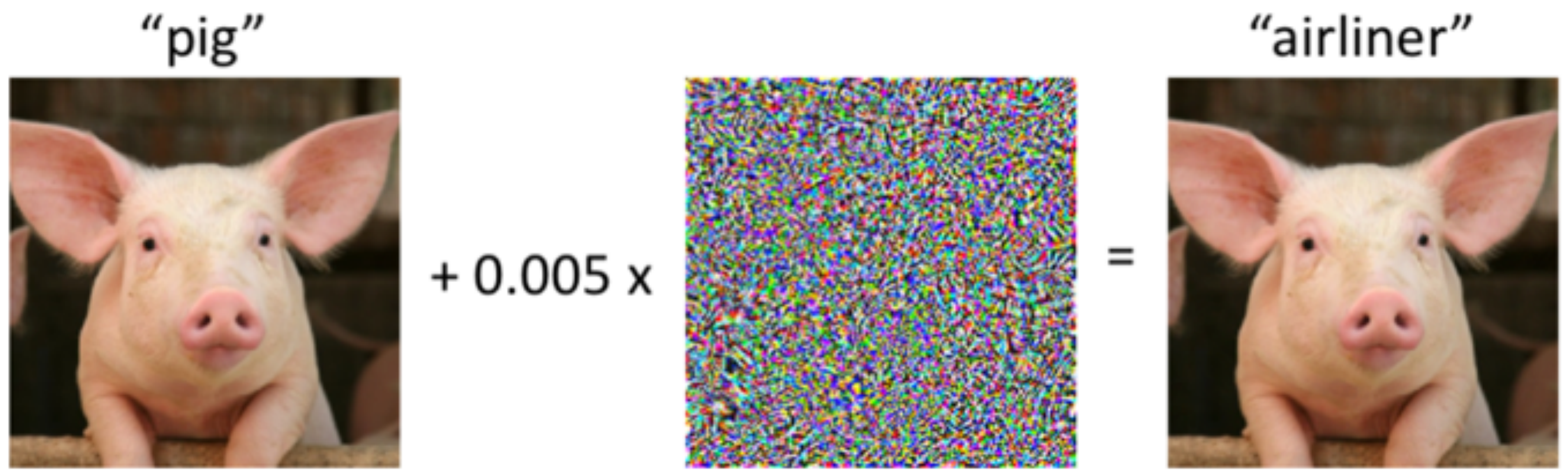
- Town i has population n_i and weight w_i and its rep has Banzhaf power β_i (all written as shares).
- We've been solving for $\beta_i \approx n_i$.
- But now suppose the probability of a voter casting a decisive vote in town i is p_i.
- Then don't we want $p_i\beta_i$ to be constant?
    - If $p_i \propto 1/\sqrt{n_i}$ then our objective would be $\beta_i \propto \sqrt{n_i}$.
    - OTOH, if $p_i \propto 1/n_i$, then our objective would be $\beta_i \propto n_i$ (as we had before!)
    - If decisive probability is higher in competitive districts, this counsels you to massively boost the voting weight of a representative of landslide districts! Not reasonable.
- A more reasonable interpretation is: try to make all districts competitive.
  (But you can't do that in the fixed-district world.)

How much do you count?

- There are other applications of measuring power or contribution drawn from collective game theory

- Shapley-Shubik value is another such alternative, applied in finance to help business partners split profits

- Both of these, plus more ideas like Integrated Gradients, now used in ML for **feature attribution**



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ML algorithms often require explanation

- Consider an algorithm to help predict probability of loan default
- Customer data $x = (5, 3, 2, 5, 5, 20, 0.8, 0, 40, 0.7)$— labeled high probability of default, loan denied
- U.S. law requires company to provide an explanation of the **reasons** for denial
- Input data is the pixels, DENY is the classification — how responsible was each datum?
- Banzhaf, Shapley, etc now actually used for this purpose