Data Sciences Initiative
Physics 105
Physics 111A
Project 9b: Improving the Accuracy of Fourier Results

1 Keywords

- Discrete Fourier Transform
- Data Acquisition
- Signal to Noise Ratio
- Sample Rate
- Doppler Radar

2 Purpose of this Project

The purpose of this project is best explained with a numerical example. Suppose we are doing data acquisition with a sample rate of 44.1 kHz, and we acquire 16384 samples. We do a Discrete Fourier Transform and compute Fourier coefficients. Among the many coefficients we get, we notice some large coefficients belonging to the following frequencies:

1095.50 Hz

1098.19 Hz

1100.89 Hz

If it is know that the source was a pure frequency (for example 1100.00 Hz) how can we determine this frequency using the computed Fourier coefficients?

The point of this project is that we can determine frequencies present in sampled data with higher precision than just calculating a discrete Fourier transform and choosing the largest coefficients.

3 Statement of problem

The file "signal.txt" contains audio data. Column 1 is the time in seconds, and column 2 is the audio signal (16-bit, mono). The sample rate is $44.1\,kHz$, and column 1 is simply the zero-based line number divided by 44100. As is clear from the file, there are 16384 samples.

The root mean square value for column 2 is about 2863.5. If we subtract the following function from column 2, then the root mean square value is 1710.3

$$f_1(t) = 3248.812\ \cos[(548.7288\,Hz)2\pi\,t - 0.732184] \tag{1}$$

The task is to find another function of this form, $f_2(t)$, so that the root mean square value of column 2 minus f_1 minus f_2 is less than 843. [Hint: the frequency of $f_2(t)$ will be about twice that of $f_1(t)$, but the exact value should be determined from your Fourier calculations, not by doubling the value given above.]

Note that this problem is stated without any reference to specific Fourier conventions.

4 Overview of solution

In this section, I'll use specific conventions for a discrete Fourier transform. The reference

`http://en.wikipedia.org/wiki/Discrete_Fourier_transform`

contains the following definition

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \qquad k = 0, \ldots, N-1 \tag{2}$$

Let's consider an input of the following form

$$x_n = A\,e^{ian} \tag{3}$$

where A and a are complex constants and a is not an integer times $2\pi/N$. We can evaluate the sum in closed form, and the formula for the Fourier coefficients is

$$X_k = A\,\frac{1 - e^{i\left(a - \frac{2\pi}{N}k\right)N}}{1 - e^{i\left(a - \frac{2\pi}{N}k\right)}} \tag{4}$$

This simiplifies to

$$X_k = A\,\frac{1-e^{iaN}}{1-e^{i\left(a-\frac{2\pi}{N}k\right)}} \tag{5}$$

Given a value for a, an integer k_0 can be chosen so that the real part of δ, defined below, is greater than 0 and less than 2π

$$a = \frac{2\pi}{N}k_0 + \frac{\delta}{N} \tag{6}$$

The result for A/X_k is

$$\frac{A}{X_k} = \frac{1-e^{i(2\pi k_0 - 2\pi k + \delta)/N}}{1-e^{i\delta}} \tag{7}$$

For the special case of a real a, we see that the reciprocal Fourier coefficients lie uniformly spaced on a circle in the complex plane, and the circle contains the origin. In practice, we will usually work with just a few of the larger Fourier coefficients (k near k_0), and N will be large, so the points in the complex plane will lie on a nearly straight line segment that goes through the origin.

5 A method for computing the constants in Eq. (1)

In this section, we'll consider the case of input data that contains just one pure oscillation at a frequency that is not commensurate with the sampling rate. This is of interest just to see how the algebra works out for computing the amplitude, frequency and phase.

Using Eq. (5), we get

$$X_k/A = \frac{1-e^{iaN}}{1-e^{i\left(a-\frac{2\pi}{N}k\right)}} \tag{8}$$

$$= \frac{1-\alpha^N}{1-\alpha\, f^k} \tag{9}$$

where α and f are defined to be

$$\alpha = e^{ia} \tag{10}$$

$$f = e^{-2\pi\, i/N} \tag{11}$$

For any choice of distinct k and ℓ

$$\frac{X_k}{X_\ell} = \frac{1 - \alpha\, f^\ell}{1 - \alpha\, f^k} \tag{12}$$

from which follows

$$(1 - \alpha\, f^k)\, \frac{X_k}{X_\ell} = 1 - \alpha\, f^\ell \tag{13}$$

and

$$\frac{X_k}{X_\ell} - 1 = \alpha \left(f^k\, \frac{X_k}{X_\ell} - f^\ell \right) \tag{14}$$

so that α is given by

$$\alpha = \frac{X_k - X_\ell}{f^k\, X_k - f^\ell\, X_\ell} \tag{15}$$

Once α is known, Eq. (9) can be used to calculate A.

If one were to try to use this approach for data from a real experiment, one should probably choose k and ℓ differing by one, and have X_k and X_ℓ as large as possible, so they are known with great precision.

6 Frequency calibration

From the formula

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{\frac{2\pi i}{N} kn} \qquad n = 0, \ldots, N-1 \tag{16}$$

we see that X_1 is the coefficient of

$$\frac{1}{N}\, e^{\frac{2\pi i}{N} n} \qquad n = 0, \ldots, N-1 \tag{17}$$

which is a wave that goes through a single cycle as n goes from 0 to $N - 1$. With Δt denoting the time between samples (a second divided by 44100, in our example), X_1 is the coefficient of a wave that has a period of $N\, \Delta t$. This wave has a frequency of $1/(N\, \Delta t)$. This frequency is cycles per second, not radians per second. By the same logic, X_k is associated with frequency $k/(N\, \Delta t)$.

7 Methods for use with actual data

Our starting point is Eq. (7), which we copy here for convenience

$$\frac{A}{X_k} = \frac{1 - e^{i(2\pi k_0 - 2\pi k + \delta)/N}}{1 - e^{i\delta}} \tag{18}$$

For k near k_0

$$\frac{A}{X_k} = \frac{-i\left(2\pi k_0 - 2\pi k + \delta\right)/N}{1 - e^{i\delta}} \tag{19}$$

which can be written as

$$\frac{1}{X_k} = \frac{2\pi i}{NA\left(1 - e^{i\delta}\right)}\left(k - k_0 - \frac{\delta}{2\pi}\right) \tag{20}$$

This is a simple linear function of k. The two points in the complex plane that are closest to the origin are $1/X_{k_0}$ and $1/X_{k_0+1}$. If δ is real, then the points define a line that goes through the origin.

7.1 A simple approach

Let's first consider a simple approach that uses just two Fourier coefficients, rather than a more complicated analysis involving a greater number of coefficients.

As explained above, k_0 has the property that the two largest coefficients for the spectral peak in question are X_{k_0} and X_{k_0+1}. From Eq. (20) we have

$$\frac{X_{k_0}}{X_{k_0+1}} = \frac{1 - \frac{\delta}{2\pi}}{0 - \frac{\delta}{2\pi}} \tag{21}$$

$$= 1 - \frac{2\pi}{\delta} \tag{22}$$

so that

$$\frac{2\pi}{\delta} = 1 - \frac{X_{k_0}}{X_{k_0+1}} \tag{23}$$

$$= \frac{X_{k_0+1} - X_{k_0}}{X_{k_0+1}} \tag{24}$$

and

$$\delta = 2\pi \frac{X_{k_0+1}}{X_{k_0+1} - X_{k_0}} \tag{25}$$

[If for some reason one is convinced that there should be zero attenuation then δ could be rounded to the nearest real number at this point in the calculation.] Now since k_0 and δ are known, Eq. (20) can be used to calculate A. In order to reduce errors, the larger of X_{k_0} and X_{k_0+1} should be used for this.

7.2 Using more than two Fourier coefficients

[In this subsection, we will consider methods that use more than just the two largest Fourier coefficients. This will involve certain weighted averages. This text is just a place holder for future work.]

8 Real-valued functions of time

The DFT defined in Eq. (2) defines a mapping which we will write using the following compact notation

$$x_n \mapsto X_k \tag{26}$$

Using this notation, the result stated in Eq. (5) can be written

$$A\, e^{ian} \mapsto A\, \frac{1 - e^{iaN}}{1 - e^{i\left(a - \frac{2\pi}{N} k\right)}} \tag{27}$$

From this we get the related result

$$A^\star\, e^{-ia^\star n} \mapsto A^\star\, \frac{1 - e^{-ia^\star N}}{1 - e^{-i\left(a^\star + \frac{2\pi}{N} k\right)}} \tag{28}$$

Note that the left-hand side is the complex conjugate of the left-hand side of the previous statement, but the right-hand side is not the complex conjugate of the right-hand side of the previous statement.

Let ϕ be an angle with the property

$$A = |A|\, e^{i\phi} \tag{29}$$

and let a_r and a_i denote the real and imaginary parts of a. Then we have

$$|A|\, e^{i\phi}\, e^{i(a_r + i\, a_i)n} + |A|\, e^{-i\phi}\, e^{-i(a_r - i\, a_i)n} \mapsto A\, \frac{1 - e^{iaN}}{1 - e^{i\left(a - \frac{2\pi}{N} k\right)}} + A^\star\, \frac{1 - e^{-ia^\star N}}{1 - e^{-i\left(a^\star + \frac{2\pi}{N} k\right)}} \tag{30}$$

The left-hand side can be written as

$$2\,|A|\,e^{-a_i\,n}\cos(a_r\,n+\phi) \tag{31}$$

The right-hand side of Eq. (30) is dominated by the first term (the DFT of $A\,e^{ian}$) when we look at coefficients near k_0. So we see that the analysis of the previous section together with the knowledge that the input signal was real allows us to determine the constants in the form shown in Eq. (31). In order to write Eq. (31) as a function of time, one uses $n = t/\Delta t$, where Δt denotes the time between samples (a second divided by 44100, in our example).

8.1 Example 1

Looking at the file “FourierCoefficients.txt” near $550\,Hz$ we have

$$\begin{aligned} k_0 &= 203 && (32)\\ X_{k_0} &= -1835479.0910978063 + i\,3488898.64003292 && (33)\\ X_{k_0+1} &= 10266966.625773327 - i\,23711771.544650428 && (34) \end{aligned}$$

Using Eq. (25) we get

$$\delta = 5.45298 - i\,0.0545988 \tag{35}$$

[In a subsequent discussion we can talk about rounding this to the nearest real number because the input signal was a steady sound from an electronic keyboard.] The value of A is given by Eq. (20)

$$A = 1162.86 - i\,1068.75 \tag{36}$$

Making use of the definition in Eq. (6), the expression in Eq. (31) becomes

$$3158.79\,e^{t/6.80453\,s}\cos[(548.741\,Hz)2\pi\,t - 0.743253] \tag{37}$$

which can be compared with Eq. (1), which was determined by optimizing the three-parameter form. The small exponential growth in Eq. (37) is just a few percent over the duration of the recording. I have quoted the results to six digits for the time being. A further effort would be required to determine the accuracy of each of the numbers.

8.2 Example 2

Looking at the file “FourierCoefficients.txt” near $1100\,Hz$ we have

$$\begin{aligned} k_0 &= 407 && (38) \\ X_{k_0} &= 3195761.805273114 - i\,4693752.920151186 && (39) \\ X_{k_0+1} &= -8849498.51986139 + i\,12337585.350921521 && (40) \end{aligned}$$

Using Eq. (25) we get

$$\delta = 4.57311 + i\,0.0304572 \tag{41}$$

The value of A is given by Eq. (20)

$$A = -1061.00 + i\,104.417 \tag{42}$$

Making use of the definition in Eq. (6), the expression in Eq. (31) becomes

$$2132.25\, e^{-t/12.1981\,s} \cos[(1097.46\,Hz)2\pi\, t + 3.04349] \tag{43}$$

9 A useful task

A useful task for students would be to write some code that creates a test data file that contains a pure single-frequency source being sampled at $44.1\,kHz$, and output 16384 samples. Then we compute a Discrete Fourier Transform of this data, and analyze it with the methods discussed in this document.