Physics 77

Introduction to Computational Techniques in Physics
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Linear Algebra: Solving Systems of Linear Equations

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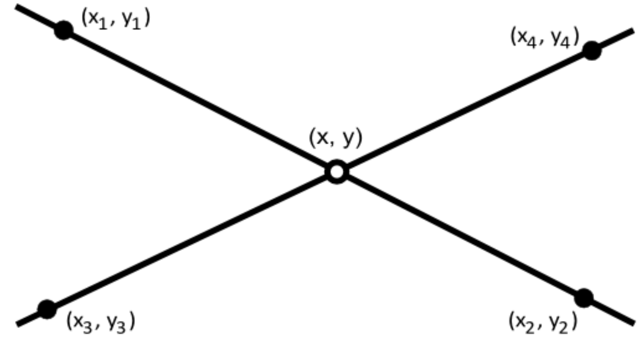
See also Python workbook Lecture09.ipynb

Example: Finding Intersection of Two Lines



- Work out on board
- Example 1:
  - $y = 2-x$
  - $y = x-1$
- Generic example 2:
  - $a_{11}x + a_{12}y = c_1$
  - $a_{21}x + a_{22}y = c_2$

Linear Algebraic Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_1 + a_{23}x_3 + \ldots + a_{2n}x_n &= c_2 \\
a_{31}x_1 + a_{32}x_1 + a_{33}x_3 + \ldots + a_{3n}x_n &= c_3 \\
\vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad\quad & \vdots \qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= c_n
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}
$$

Linear Algebraic Equations

- In Matrix Format

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{c}}$$

- Solution:

$$\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{x}} = \mathbf{A}^{-\mathbf{1}}\vec{\mathbf{c}} \Rightarrow \vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{c}}$$

Linear Algebra Primer: Matrices

- Inverse: $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$
- Transpose: $\mathbf{A}^{T}=\mathbf{A}_{ji}$
- Trace of a Matrix: Tr$\mathbf{A}=\sum_{\mathbf{I}} A_{ii}$
- Symmetric Matrix: $A_{ij}=A_{ji}$
- Conjugate Transpose: $\mathbf{A}^{\dagger}$
- Unitary Matrix: $\mathbf{A}^{-1}=\mathbf{A}^{\dagger}$
- Normal Matrix: $\mathbf{A}\mathbf{A}^{\dagger}=\mathbf{A}^{\dagger}\mathbf{A}$

Determinant of a Matrix

- Determinant is a single parameter that can be used to characterize the matrix.

$$\det \mathrm{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Aside: Levi-Civita Symbol

Define: $$\varepsilon_{ij} = \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i = j \end{cases} \qquad \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

if $\mathbf{A} = [a_{ij}],\ i,j=1..2$: $$det(\mathbf{A}) = \sum_{i,j=1}^{2} \epsilon_{ij} a_{1i} a_{2j}$$

3 dimensions: $$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$$

Famous application:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \mathbf{e}_i a^j b^k$$

$$(\mathbf{a} \times \mathbf{b})^i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a^j b^k \equiv \epsilon_{ijk} a^j b^k$$

(the last expression in Einstein notation, where summation over repeated indices is implied)

Aside: Levi-Civita Symbol

Generalize: $\epsilon_{i_1 i_2 .. i_n} = (-1)^p \epsilon_{12..n} = (-1)^p$ if $i_1 \neq i_2 \neq .. \neq i_n$

zero otherwise

p is *parity* of the permutation from $i_1..i_n$ to 1..*n*, the number of pairwise interchanges to get from order $i_1..i_n$ to 1..*n*

Then in n dimensions: $\mathbf{A}$ = [a_{ij}] and

$$det(\mathbf{A}) = \epsilon_{i_1 i_2 .. i_n} a_{1 i_1} a_{2 i_2} .. a_{n a_n}$$

Cramer's Rule

- M_{ij} is the determinant of the Matrix **A** with the ith row and jth column removed.
- $(-1)^{ij}$ is called the cofactor of element a_{ij}.

$$\text{Det } \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \mathbf{M}_{ij}, \forall j$$

Cramer's Rule

$$\mathbf{x_1} = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ c_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ c_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}, \mathbf{x}_j = \frac{\begin{vmatrix} a_{11} \cdots a_{1\,j-1} & c_1 & a_{1\,j+1} \cdots a_{1n} \\ a_{21} \cdots a_{2\,j-1} & c_2 & a_{2\,j+1} \cdots a_{2n} \\ a_{31} \cdots a_{3\,j-1} & c_3 & a_{3\,j+1} \cdots a_{3n} \\ \vdots & \ddots & \vdots \\ a_{n1} \cdots a_{nj-1} & c_n & a_{n\,j+1} \cdots a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}}$$

Compare with derivations on the board for 2x2 case

Practical Details: Cramer's Rule

- $3n^2$ operations for the determinant.
- $3n^3$ operations for every unknown.
- Unstable for large Matrices.
- Large error propagation.
- Good for small Matrices ($n<20$).

Gaussian Elimination

- Divide each row by the leading element.
- Subtract row 1 from all other rows.
- Move to the second row an continue the process.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} \frac{a_{11}}{a_{11}} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{11}}{a_{11}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{a_{31}}{a_{31}} & \frac{a_{32}}{a_{31}} & \frac{a_{33}}{a_{31}} & \cdots & \frac{a_{3n}}{a_{31}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{n1}} & \frac{a_{n2}}{a_{n1}} & \frac{a_{n3}}{a_{n1}} & \cdots & \frac{a_{nn}}{a_{n1}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_2 \\ \vdots \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} \\ \vdots \\ \frac{c_n}{a_{n1}} \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{a_{32}}{a_{31}} - \frac{a_{12}}{a_{11}} & \frac{a_{33}}{a_{31}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{3n}}{a_{31}} - \frac{a_{1n}}{a_{11}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{a_{n2}}{a_{n1}} - \frac{a_{12}}{a_{11}} & \frac{a_{n3}}{a_{n1}} - \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{nn}}{a_{n1}} - \frac{a_{1n}}{a_{11}} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_2 \\ \vdots \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_3}{a_{31}} - \frac{c_1}{a_{11}} \\ \vdots \\ \frac{c_n}{a_{n1}} - \frac{c_1}{a_{11}} \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & 1 & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & 1 & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ \vdots \\ c'_n \end{bmatrix}$$

- Back substitution

$$\mathbf{x_n} = c'_n$$

$$\mathbf{x_i} = c'_i - \sum_{j=i+1}^{n} a'_{ij} \mathbf{x_j}$$

Gaussian Elimination: Practical Details

- Division by zero: May occur in the forward elimination steps.

- Round-off error: Prone to round-off errors.

Gaussian Elimination: Example

Consider the system of equations:

Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 1 & -.7 & 0 \\ 0 & 1 & -588.23524 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ -588.33323 \\ 0.9992235 \end{bmatrix}$$

Gaussian Elimination: Example

Back Substitution

$$\begin{bmatrix} 1 & -.7 & 0 \\ 0 & 1 & -588.23524 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -588.3323 \\ 0.9992235 \end{bmatrix}$$

$$x_3 = 0.99993$$

$$x_2 = -1.5$$

$$x_1 = -0.3500$$

Gaussian Elimination: Example

Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Gaussian Elimination: Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

Partial Pivoting

Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the kth step of forward elimination, find the maximum of

$$\left|a_{kk}\right|,\left|a_{k+1,k}\right|,\ldots\ldots\ldots\ldots\ldots,\left|a_{nk}\right|$$

If the maximum of the values is $\left|a_{pk}\right|$ In the pth row, $k \le p \le n$, then switch rows p and k.

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 3x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -.7 & 0 \\ 0 & -0.0034 & 2 \\ 0 & -0.9 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 2.000333 \\ -.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.0034| and |-0.9| or 0.0034 and 0.9

The largest absolute value is 0.9, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 1 & -.7 & 0 \\ 0 & -0.0034 & 2 \\ 0 & -0.9 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ 2.000333 \\ -.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -.7 & 0 \\ 0 & -0.9 & -1 \\ 0 & -0.0034 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 \\ -0.5 \\ 2.000333 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & 1 & 1.1111 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.55555 \\ 0.999223 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 1 & -0.7 & 0 \\ 0 & 1 & 1.1111 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.55555 \\ 0.999223 \end{bmatrix}$$

$$x_3 = 0.999234$$

$$x_2 = -1$$

$$x_1 = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0.99923 \end{bmatrix} \quad [X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Matrix Factorization

- Assume **A** can be written as **A=VU** where **V** and **U** are triangular Matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_{11} & 0 & 0 & \cdots & 0 \\ v_{21} & v_{22} & 0 & \cdots & 0 \\ v_{31} & v_{32} & v_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Matrix Factorization

How can this be used?

Given $[A][X]=[C]$

Decompose $[\mathrm{A}]$ into $[V]$ and $[U]$, so $[V][U][X]=[C]$

Then solve $[\mathrm{V}][\mathrm{Z}]=[C]$ for $[Z]$

And then solve $[U][\mathrm{X}]=[Z]$ for $[X]$

Matrix Factorization

Method: Decompose **[A]** to [V] and [U]

$$[A]=[V]\cdot[U]=\begin{bmatrix}1 & 0 & 0\\ v_{21} & 1 & 0\\ v_{31} & v_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[V] is obtained using the *multipliers* that were used in the forward elimination process

LU Decomposition: Example

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\text{Row2} - \left[\frac{\text{Row1}}{25}\right] \times (64) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\text{Row3} - \left[\frac{\text{Row1}}{25}\right] \times (144) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

LU Decomposition: Example

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \qquad \text{Row3} - \left[\frac{\text{Row2}}{-4.8}\right] \times (-16.8) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

LU Decomposition: Example

Finding the [*V*] matrix

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ v_{21} & 1 & 0 \\ v_{31} & v_{32} & 1 \end{bmatrix}$$

$$v_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$v_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$v_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

LU Decomposition: Example

$$[V] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does $[V][U] = [A]$?

$$[V][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} =$$

Exercise for the reader

LU Decomposition: Example

Example: Solving simultaneous linear equations using VU factorization

Solve the following set of linear equations using VU Factorization

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*V*] and [*U*] matrices

$$[A] = [V][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

LU Decomposition: Example

Complete the forward substitution to solve for $[Z]$: $[L][Z] = [X]$

$$
\begin{aligned}
z_1 &= 106.8 \\
z_2 &= 177.2 - 2.56 z_1 \\
&= 177.2 - 2.56(106.8) \\
&= -96.2 \\
z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\
&= 279.2 - 5.76(106.8) - 3.5(-96.21) \\
&= 0.735
\end{aligned}
$$

$$
[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}
$$

LU Decomposition: Example

Set $[U][X]=[Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}=\begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$0.7a_3 = 0.735$$

LU Decomposition: Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$= 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$= \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$= 19.70$$

LU Decomposition: Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5\mathrm{a}_2 - \mathrm{a}_3}{25}$$

$$= \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$= 0.2900$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$