HIGHER ENGINEERING MATHEMATICS - II

Dr. Shailesh S. Patel

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GANDHI ROAD, AHMEDABAD.

FOR THE SECOND YEAR B.E. STUDENTS OF
VEER NARMAD SOUTH GUJARAT UNIVERSITY,
OTHER INDIAN UNIVERSITIES
AND FOR THE GATE EXAM

HIGHER ENGINEERING MATHEMATCS - II

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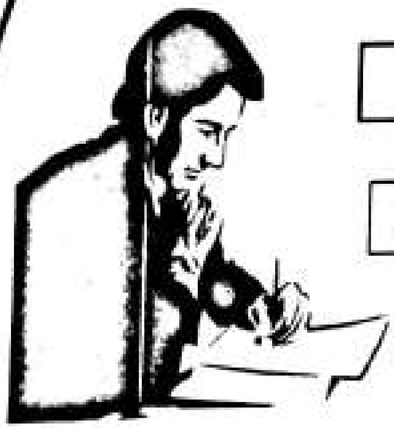

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This Books is
Dedicated to my Teachers
Who have been a
Treasure of
Knowledge and inspiration

FOREWORD

It gives me immense pleasure to write this foreward for the book **"Higher Engineering Mathematics-II"** written for the students of B.E. I year (Second Semester) of Veer Narmad South Gujarat University, Surat. Shri (Dr.) S.S. Patel - the author of this book has really made a sincere and streneous effort to present the subject in a lucid, elegant and analytical manner and in as simple a language as possibe. The sequence of chapters too has been so arranged as to maintain an integrated study of the subject. The book icludes a large number of solved examples which would certainly be of great help to the students in the proper understanding of the subject. Review exercises at the end of each section would also enable the students to prepare for their examinations in an objective and systematic manner. Needless for me to mention that (Dr.) S.S. Patel had a brilliant academic year having secured First Class with distinction at both B. Sc. and M.Sc. examinations of the South Gujarat University in the years 1992 and 1994. He obtained his Doctoral degree from the said University in June 1998. Shri Patel has been with Shri S'ad Vidya Mandal Institute of Technology since February 1998. With such an outstanding academic career and a decade of teaching experience in the field, I am sure that the book prepared by him will certainly prove to be very useful to the concerned students and shall meet the long felt need for a suitable text book covering the syllabus of Veer Narmad South Gujarat University, Surat.

With the publication of this book being third in series, Dr. Patel makes a hat trick and I congratulate him for the same and hope that like his earlier two books, this title too would receive the same response both from the Faculty and the students.

Prin. R. C. Joshi (Retd.)
Secretary,
Shri S'ad Vidya Mandal
Bharuch

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No significant work can be successfully completed without any support and blessings. I am very much grateful to Pri. (Shri) R.C. Joshi (Retd.), Secretary, Shri S'ad Vidya Mandal, Bharuch for his ispiration and positive spirit. I would like to thank my friends of SVMIT, Bharuch. I deeply appreciate my wife Chintal and loving sons Digish and Dhureen for their unwavering support, encouragement, and for helping me to appreciate better everyday while writing this book. Special thanks to my brother-in-law Dr. Hemal for discussing and guiding me about the disease of diabetes (Mathematical approach discussed in sixth chapter). I am also very much thankful to my parents-in-law Smt. Nirmalaben and Shri. Vasantbhai Patel for their constant encouragement and increasing my enthusiasm in all my creativities. I am extermely grateful to the heartily blessings of my parents. Smt. Vimalaben and Shri. Sumantrai Patel without this it would have been impossible to reach at this stage.

Last but not least I am thankful to Shri Hamendrabhai Shai, Shri Bipinbhai Shah and Shri Hirenbhai Shah of ATUI PRAKASHAN for full co-operation and utmost efficiency with which they have brought out the book in the present nice form and also to RAM COMPUTER for efficient typing.

Constructive suggestions for the improvement of this book and intimation of errors and mispirnts will be gratefully acknowledged. Suggestions and feedback can be directly sent to the author through e-mail :eng_maths@yahoo.co.in.

– **Dr. Shailesh S. Patel**

PREFACE

It gives me a great pleasure to present this book entitled, **"HIGHER ENGINEERING MATHEMATICS-II"** to the students pursuing various engineering courses at undergraduate level.

The main Objectives of this book are :

- Discussion of the fundamental concepts of the related topics in elaborative form.
- Every attention has been given to make the explanation quite adequate, simple and systematic.
- Complete coverage of the syllabus of Engineering Mathematics-II at B.E. First year (All Branches, Second Semester) of Veer Narmad South Guajrat University, Surat and also full or some part of the syllabus of other Indian universities.
- Large number of worked out examples have been given to help the students grasping the subject matter quickly which will enhance their self confidence.
- Exercises at the end of each section have been given with answers to self test the solving ability of the students.
- No claim of the originality is made but the treatment of the subject is in my own style.

I am sure that this book will be very much useful to the students of degree engineering students.

– Dr. Shailesh S. Patel

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TO THE READERS

Author and publisher would welcome suggestions towards future edition of this book or the pointing out of any misprint or obscurity. Please write to The Technical Editor. Atul Prakashan. Under Farnandis Bridge, Gandhi Road, Ahmedabad-380 001 or to the author at E-mail : chintshail@yahoo.com

1 PARTIAL DIFFERENTIATION

1.1 INTRODUCTION :

In many engineering and scientific problems, there arise functions depending on two or more independent variables. In this chapter we shall see the concept of differentiation of the functions of these kind, which is known as partial differentiation.

1.2 FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES :

We say that u is a function of the two independent variables x and y whenever some law f assigns a unique value of u (dependent variable) to each pair of values (x, y) belonging to a certain specified set (domain of the function) and is denoted by $u = f(x, y)$. Similarly, we say that u is a function of the n variables $x_1, x_2, \ldots, x_n$ if for each set of the values $(x_1, \ldots, x_n)$ belonging to a certain specified set, there is assigned a corresponding unique value of u. For example, the volume $u = xyz$ of a rectangular parallelopiped is a function of the lengths of the three sides x, y, z.

Geometrical Representation of Functions :

We know that a function of one independent variable is geometrically represented by means of curves. But the functions of two independent variables may be represented by means of surfaces.

For that we consider a rectangular xyu – coordinate system in space and find a point P with the coordinate $u = f(x, y)$, above each point (x, y) of the domain R of the function in the xy–plane. For example, the function

$$u = \sqrt{1 - x^2 - y^2}$$

corresponds the hemisphere above the xy–plane, with unit radius and center at the origin. The linear function $u = ax + by + c$ represents a plane in space.

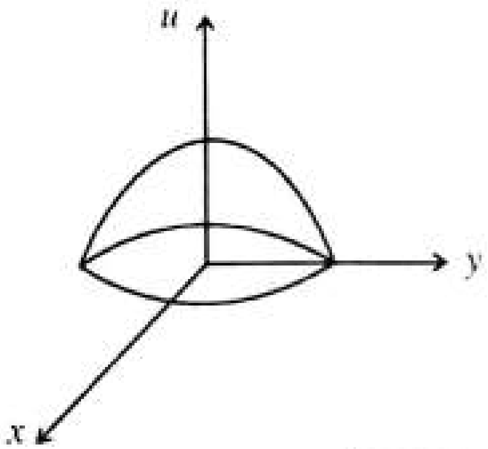


Fig. 1 : Graph of $u = \sqrt{1 - x^2 - y^2}$

If in the function $u = f(x, y)$ one of the independent variables, say y does not occur, then u depends on x only, say $u = g(x)$. Such a function is represented in xyu-space by a cylindrical surface generated by extending the curve $u = g(x)$ on ux-plane towards y-axis (y is arbitrary).

1.3 CONTINUITY :

The function $u = f(x, y)$ is said to be **continuous** at the point (a, b), if for all points (x, y) near (a, b) the value of $f(x, y)$ differs, but little, from the value $f(a, b)$. In other words, if f has the domain R and $Q = (a, b)$ is a point of R, then f is continuous at Q if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$| f(P) - f(Q) | = | f(x, y) - f(a, b) | < \varepsilon \qquad \text{...... (1)}$$

For all $P = (x, y)$ in R for which

$$\overline{PQ} = \sqrt{(x-a)^2 + (y-b)^2} < \delta \qquad \text{...... (2)}$$

If a function is continuous at every point of a domain R, we say that it is continuous in R.

Note : 1. Equation (2) confines a point (x, y) lies in a small disk with center (a, b). Instead of this we could use a small square defined by

$$| x - a | < \delta \text{ and } | y - b | < \delta.$$

2. The sum, difference and product of continuous functions are also continuous. The quotient of continuous functions defines a continuous function at points where the denominator does not vanish.

3. All polynomials are continuous, and all rational functions are continuous at the points where the denominator does not vanish.

4. Continuous functions of continuous functions are themselves continuous.

5. The discontinuity of a function is more than one variable is much more complicated type than a function of a single variable. For example, the function

$$u = \frac{y}{x} \text{ for } x \neq 0 \text{ and } u = 0 \text{ for } x = 0,$$

which is discontinuous along the whole line $x = 0$ instead of at a single point.

A function $u = f(x, y)$ may also be continuous in x for each fixed value of y and continuous in y for each fixed value of x, and however be discontinuous as a function of x and y. For example, the function

$$f(x, y) = \frac{2xy}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0$$

is obviously continuous as a function of x for any fixed $y \neq 0$, as the denominator cannot vanish. For $y = 0$ we have $f(x, a) = 0$, which is also continuous as a function of x. Similarly $f(x, y)$ is also continuous as a function of y for any fixed x. But at every point of the line $y = x$ except at the point $x = y = 0$ we have $f(x, y) = 1$, and there are points of line arbitrary close to the origin. Hence $f(x, y)$ is discontinuous at the point (0, 0).

1.4 THE CONCEPT OF LIMIT :

The notion of limit of a function is closely related to the notion of continuity. Suppose the function $u = f(x, y)$ is defined in a domain R. Let Q = (a, b) be the point in R. We say the function has limit L for (x, y) tending to (a, b) if for every $\varepsilon > 0$ there exist a neighbourhood

$$\overline{PQ} = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

of (a, b) such that

$$| f(P) - L | = | f(x, y) - L | < \varepsilon$$

For all P = (x, y) belonging to R in that neighbourhood.

We can also write the above limit as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Note : If the function $f(x, y)$ is not defined at the point (a, b) of its domain but has a limit L for $(x, y) \to (a, b)$ then we can extend the definition of f at the point (a, b) by putting $f(a, b) = L$, so as the extended function will be continuous at (a, b).

1.5 PARTIAL DERIVATIVES OF A FUNCTION :

1.5.1 Definition and geometrical interpretation :

Let $u = f(x, y)$ be a function of two variables x and y. If we assign to y a definite fixed value $y = y_0$, then the resulting function $u = f(x, y_0)$ will be a function of single variable x. Geometrically this can be represented by cutting the surface $u = f(x, y)$ by the plane $y = y_0$ (Fig. 2) which is a curve.

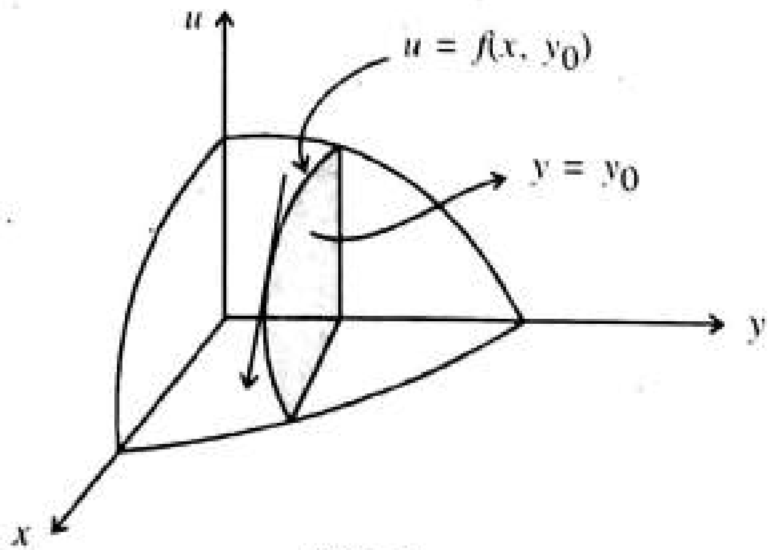


Fig. 2

Assuming that the function $u = f(x, y)$ is defined at the point (x_0, y_0). If we differentiate the function $u = f(x, y_0)$ at the point $x = x_0$ we obtain the partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) :

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \qquad \text{.......... (1)}$$

provided the limit exists.

Geometrically, this partial derivative denotes the tangent of the angle between a line parallel to x – axis and the tangent line to the curve $u = f(x, y_0)$. Thus, it is the slope of the surface $u = f(x, y)$ in the direction of the x – axis.

We can represent the partial derivative (1) by using different notations as follows :

$$\begin{aligned}\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} &= f_x(x_0, y_0) = u_x(x_0, y_0) \\ &= \frac{\partial f}{\partial x} \text{ at } (x_0, y_0) \\ &= D_x f \text{ at } (x_0, y_0)\end{aligned}$$

Note : Here we shall use the special round letter ∂ (read as 'del') instead of the ordinary d which we use in the differentiation of functions of one variable in order to show that we are dealing with function of several variables and the differentiation is with respect to one of the variables.

Similarly as discussed above we can define the partial derivative of the function $u = f(x, y)$ with respect to y at the point (x_0, y_0) by the relation

$$\begin{aligned}\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} &= f_y(x_0, y_0) = u_y(x_0, y_0) \\ &= \frac{\partial f}{\partial y} \text{ at } (x_0, y_0) \\ &= D_y f \text{ at } (x_0, y_0)\end{aligned}$$

Geometrically this derivative with respect to y represents the slope of the curve of intersection of the surface $u = f(x, y)$ with the plane $x = x_0$ perpendicular to the x – axis (Fig. 3).

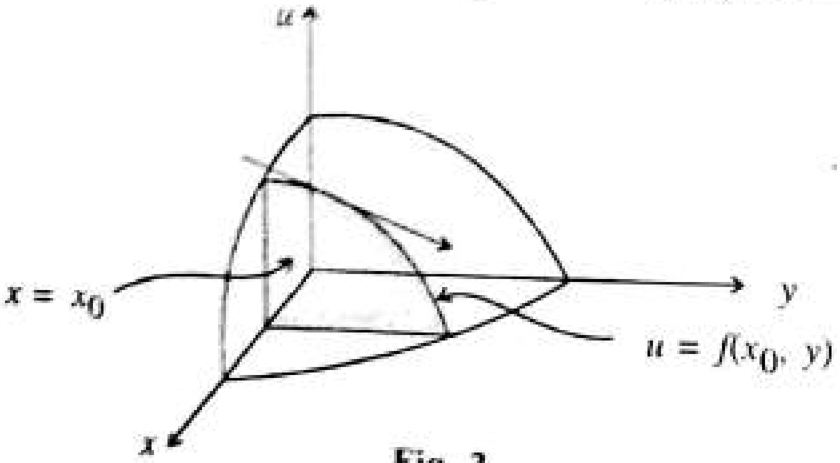


Fig. 3

If the function $u = f(x, y)$ possesses partial derivatives with respect to x and y at every point (x, y) of the region of the function, then they themselves are function of x and y.

$$u_x(x, y) = f_x(x, y) = \frac{\partial f(x, y)}{\partial x} \text{ and } u_y(x, y) = f_y(x, y) = \frac{\partial f(x, y)}{\partial y}.$$

If we have the function of n independent variables, say, $x_1, x_2, \ldots\ldots, x_n$, we define partial derivatives by

$$\frac{\partial f}{\partial x_1} f(x_1, x_2, \ldots\ldots, x_n) = \lim_{h \to 0} \frac{f(x_1 + h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots x_n)}{h}$$

$$= f_{x_1}(x_1, x_2, \ldots\ldots, x_n) = D_{x_1} f(x_1, x_2, \ldots\ldots, x_n)$$

provided the limit exists.

1.5.2 Higher ordered partial derivatives :

As we have that the "first order" partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of the function $u = f(x, y)$ are themselves a functions of x and y, again differentiating such derivatives with respect to one of the variables, we can get "second order" partial derivatives. Continuing the procedure of partial differentiation, we can get higher order partial derivatives.

Note : Here we will indicate the order in which the derivatives are carried out by the order of the subscripts or by the order of the symbols ∂x and ∂y in the denominator from right to left as mentioned below :

- Second order partial derivatives :

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = (D_x)^2 f$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = D_x D_y f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = D_y D_x f$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = (D_y)^2 f$$

• Third order partial derivatives :

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y \partial x^2} = f_{yxx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

and so on.

Note : It is advisable to write the partial derivative at the point (x_0, y_0) as

$$\left[\frac{\partial f(x, y)}{\partial x}\right]_{\substack{x = x_0 \\ y = y_0}} \text{ or } \left[\frac{\partial f(x, y)}{\partial x}\right]_{(x_0, y_0)}$$

instead of writing $\frac{\partial f(x_0, y_0)}{\partial x}$.

e. g. For $f(x, y) = x^2 + 2xy + 4y^2$ then at the point (1, 2), we have

$$\left[\frac{\partial f(x, y)}{\partial x}\right]_{(1, 2)} = f_x(1, 2) = [2x + 2y]_{(1, 2)} = 6,$$

but $\frac{\partial f(1, 2)}{\partial x} = 0$ as $f(1, 2) = 21$ which is a wrong procedure.

Remark : 1. Here we can apply the rules of addition, subtraction, multiplication, quotient, same as in the case of function of a single variable.

2. For the functions of a singe variable, we know that every differentiable functions are continuous. But in the case of functions of two or more variables the existence of partial derivatives does not imply the continuity of the function. e.g. For the function

$$u(x, y) = \frac{2xy}{x^2 + y^2} \text{ with } u(0, 0) = 0$$

We have

$$u_x(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

and $u_y(0, 0) = \lim\limits_{k \to 0} \dfrac{u(0, k) - u(0, 0)}{k} = 0$

Also at the points $(x, y) \neq (0, 0)$ the partial derivatives with respect to x and y exist.

Thus partial derivatives exists everywhere. But, as we have already seen in section 1.3 that it is discontinuous at the origin. Geometrically, we can say that the existence of partial derivatives restricts the behaviour of the function in the directions of the x and y axes only and not in other directions. However, if the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are bounded then the function is continuous.

3. If the "mixed" partial derivatives f_{xy} and f_{yx} of a function $f(x, y)$ are continuous in an open set R, then the equation

$$f_{xy} = f_{yx}$$

holds throughout R. That is the order of differentiation with respect to x and y is immaterial.

4. If $u = f(v)$ and $v = \phi(x, y)$ then u is known as a function of a function of x and y. In this case

$$\frac{\partial u}{\partial x} = \frac{df}{dv} \cdot \frac{\partial v}{\partial x} = f'(v) \frac{\partial v}{\partial x}$$

Similarly $\dfrac{\partial u}{\partial y} = f'(v) \dfrac{\partial v}{\partial y}$

SOLVED EXAMPLES

1. Prove that for the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \quad \textbf{with } f(0, 0) = 0$$

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

Solution : For $h \neq 0$ and $k \neq 0$, we have

$$f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \qquad \text{.......... (1)}$$

$$\text{where } f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0 \qquad \ldots\ldots\ldots\ (2)$$

$$\text{and } f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h}$$

$$= \lim_{h\to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k \qquad \ldots\ldots\ldots\ (3)$$

Thus from (1), (2) and (3)

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1 \qquad \ldots\ldots\ldots\ (4)$$

$$\text{Again } f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \qquad \ldots\ldots\ldots\ (5)$$

$$\text{where } f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k\to 0} \frac{0}{k} = 0 \qquad \ldots\ldots\ldots\ (6)$$

$$\text{and } f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k}$$

$$= \lim_{k\to 0} \frac{hk(h^2 - k^2)}{k(h^2 + k^2)} = h \qquad \ldots\ldots\ldots\ (7)$$

Thus from (5), (6) and (7)

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{h - 0}{h} = 1 \qquad \ldots\ldots\ldots\ (8)$$

$\therefore$ From (4) and (8), we have $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Note : In the above example, f_{xy} and f_{yx} are different, which can only be caused by discontinuity of f_{xy} at origin.

2. If $u = \log(x^2 + y^2)$ prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : We have $u = \log(x^2 + y^2)$

Differentiating u, partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}(2x), \quad \frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}(2y)$$

Again differentiating partially w.r.t y and x respectively, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= 2x \cdot \left(-\frac{1}{x^2 + y^2}\right)(2y) = -\frac{4xy}{x^2 + y^2} \quad \text{.......... (1)}$$

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= 2y \cdot \left(-\frac{1}{x^2 + y^2}\right)(2x) = -\frac{4xy}{x^2 + y^2} \quad \text{.......... (2)}$$

Thus from (1) and (2) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

3. If $z(x + y) = x^2 + y^2$ then prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Solution : We have

$$z = \frac{x^2 + y^2}{x + y}$$

Differentiating partially w.r.t x and y, we get

$$\frac{\partial z}{\partial x} = \frac{(x + y)\,2x - (x^2 + y^2)}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

and $$\frac{\partial z}{\partial y} = \frac{(x + y)\,2y - (x^2 + y^2)}{(x + y)^2} = \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

Now

$$\text{L.H.S.} = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$= \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2 - y^2)}{(x+y)^2}\right]^2 = \left[\frac{2(x-y)}{x+y}\right]^2 = \frac{4(x-y)^2}{(x+y)^2} \qquad \ldots\ldots\ldots (1)$$

$$\text{R.H.S.} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left[1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right]$$

$$= 4\,\frac{x^2 - 2xy + y^2}{(x+y)^2} = 4\,\frac{(x-y)^2}{(x+y)^2} \qquad \ldots\ldots\ldots (2)$$

Thus from (1) and (2), we have

L.H.S. = R.H.S.

4. Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ for $u = \sin^{-1}\left(\dfrac{x}{y}\right)$

Solution : We have $u = \sin^{-1}\left(\dfrac{x}{y}\right)$

Differentiating partially w.r.t. x and y we get,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{x^2}{y^2}}\left(\frac{1}{y}\right) = \frac{y}{x^2 + y^2} \qquad \ldots\ldots\ldots (1)$$

and $\frac{\partial u}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}$ .... (2)

Thus from (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{y}{x^2+y^2} + y\left(-\frac{x}{x^2+y^2}\right)$$

$$= \frac{xy - xy}{x^2+y^2} = 0$$

5. If $u = \log(\tan x + \tan y + \tan z)$, show that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2$$

Solution : We have

$u = \log(\tan x + \tan y + \tan z)$

Differentiating partially w.r.t. x, y and z we get

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x\frac{\partial u}{\partial y} = \frac{2\sin x\cos x\sec^2 x}{\tan x + \tan y + \tan z} = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

[**Note :** The function $u = f(x, y, z)$ is said to symmetrical in x, y, z if it remains same after interchanging x, y, z.]

Since u is symmetrical in x, y, z, we get

$$\sin 2y\frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$$

$$\sin 2z\frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2$$

6. If $\theta = t^n e^{-r^2/4t}$, obtain the value of n for which the relation $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$ satisfied.

Solution : We have

$$\theta = t^n e^{-r^2/4t}$$

Differentiating w.r.t. t, we get

$$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t}\right)$$

$$= t^{n-1} e^{-r^2/4t}\left(n + \frac{r^2}{4t}\right) \qquad \text{.......... (1)}$$

Differentiating θ w.r.t. r, we get

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t}\left(-\frac{2r}{4t}\right) = -\frac{1}{2}\, r\, t^{n-1} e^{-r^2/4t}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 t^{n-1} e^{-r^2/4t}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left\{3r^2 e^{-r^2/4t} + r^3 e^{-r^2/4t}\left(-\frac{2r}{4t}\right)\right\}$$

$$= -\frac{1}{2} t^{n-1} e^{-r^2/4t}\, r^2 \left\{3 - \frac{r^2}{2t}\right\} \qquad \text{.......... (2)}$$

Putting the values of (1) and (2) in the relation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

We get

$$-\frac{1}{2} t^{n-1} e^{-r^2/4t}\left\{3 - \frac{r^2}{2t}\right\} = t^{n-1} e^{-r^2/4t}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \quad -\frac{3}{2} + \frac{r^2}{4t} = n + \frac{r^2}{4t}$$

$$\Rightarrow \quad n = -\frac{3}{2}$$

7. If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution : We have $u = 2(ax + by)^2 - (x^2 + y^2)$

$$\therefore \quad \frac{\partial u}{\partial x} = 4(ax + by)\,a - 2x$$

and $$\frac{\partial^2 u}{\partial x^2} = 4a^2 - 2$$

Again $$\frac{\partial u}{\partial y} = 4(ax + by)\,b - 2y$$

and $$\frac{\partial^2 u}{\partial y^2} = 4b^2 - 2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4(a^2 + b^2) - 4 = 0 \qquad (\because a^2 + b^2 = 1)$$

8. If $u = \dfrac{1}{\sqrt{1 - 2xy + y^2}}$, show that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3$$

Solution : We have $u = (1 - 2xy + y^2)^{-1/2}$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = y(1 - 2xy + y^2)^{-3/2}$$

and $$\frac{\partial u}{\partial y} = \frac{-1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)(1 - 2xy + y^2)^{-3/2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy(1 - 2xy + y^2)^{-3/2} - y(x - y)(1 - 2xy + y^2)^{-3/2}$$

$$= (1 - 2xy + y^2)^{-3/2}(xy - xy + y^2)$$

$$= y^2(1 - 2xy + y^2)^{-3/2} = y^2 u^3$$

9. If $u = (1 - 2xy + y^2)^{-1/2}$ then find the value of

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2 \frac{\partial u}{\partial y}\right]$$

Solution : We have $u = (1 - 2xy + y^2)^{-1/2}$

$\therefore$ As in the above example

$$\frac{\partial u}{\partial x} = y\,(1 - 2xy + y^2)^{-3/2} = y\,u^3$$

and $\dfrac{\partial u}{\partial y} = (x - y)\,(1 - 2xy + y^2)^{-3/2} = (x - y)u^3$

$$\therefore \quad \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\,((1 - x^2)y\,u^3)$$

$$= \frac{\partial}{\partial x}\,\{(y - x^2y)\,u^3\}$$

$$= -\,2xy\,u^3 + (y - x^2y)\,3u^2\,\frac{\partial u}{\partial x}$$

$$= -\,2xy\,u^3 + 3\,(y - x^2y)\,y\,u^5$$

Again $\dfrac{\partial}{\partial y}\left[y^2\dfrac{\partial u}{\partial y}\right] = \dfrac{\partial}{\partial y}\,[y^2\,(x - y)\,u^3]$

$$= \frac{\partial}{\partial y}\,[(xy^2 - y^3)\,u^3]$$

$$= (2xy - 3y^2)\,u^3 + (xy^2 - y^3)\,3u^2\,\frac{\partial u}{\partial y}$$

$$= (2xy - 3y^2)\,u^3 + 3\,(xy^2 - y^3)\,(x - y)\,u^5$$

$$\therefore \quad \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right]$$

$= -\,2xyu^3 + 3(y^2 - x^2y^2)u^5 + (2xy - 3y^2)u^3 + 3\,(xy^2 - y^3)\,(x - y)\,u^5$

$= -\,2xyu^3 + 3y^2u^5 - 3x^2y^2u^5 + 2xyu^3 - 3y^2u^3 + 3x^2y^2u^5 - 3xy^3u^5 - 3xy^3u^5 + 3y^4u^5$

$= 3y^2u^5\,[1 - u^{-2} - 2\,xy + y^2]$

$= 3y^2u^5\,[1 - 1 + 2xy - y^2 - 2xy + y^2]$

$= 0$

10. If $u = (x^2 + y^2 + z^2)^{-1/2}$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$

Solution : We have $u = (x^2 + y^2 + z^2)^{-1/2}$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x) = -xu^3$$

Similarly, we have

$$\frac{\partial u}{\partial y} = -yu^3 \text{ and } \frac{\partial u}{\partial x} = -zu^3$$

$$\therefore \quad \text{L.H.S.} = -x^2u^3 - y^2u^3 - z^2u^3$$
$$= -u^3(x^2 + y^2 + z^2)$$
$$= -u^3 \cdot u^{-2} = -u = \text{R.H.S.}$$

11. If $z = e^{ax+by} f(ax - by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution : We have

$$z = e^{ax+by} f(ax - by)$$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = a\,e^{ax+by} f(ax - by) + a\,e^{ax+by} f'(ax - by)$$
$$= a\,e^{ax+by}[f(ax - by) + f'(ax - by)]$$

and $$\frac{\partial z}{\partial y} = be^{ax+by} f(ax - by) + e^{ax+by} f'(ax - by)\cdot(-b)$$
$$= be^{ax+by}[f(ax - by) - f'(ax - by)]$$

$$\therefore \quad \text{L.H.S.} = b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$$
$$= ab\,e^{ax+by}[f(ax - by) + f'(ax - by) + f(ax - by) - f'(ax - by)]$$
$$= 2ab\,e^{ax+by} f(ax - by)$$
$$= 2abz = R.H.S.$$

12. If $u = e^{r\cos\theta}\cos(r\sin\theta)$ and $v = e^{r\cos\theta}\sin(r\sin\theta)$, prove that

$$\frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution : Differentiating u and v partially w.r.t. r and θ, we get

$$\frac{\partial u}{\partial r} = \cos\theta\, e^{r\cos\theta} \cos(r\sin\theta) - e^{r\cos\theta} \sin(r\sin\theta) \cdot \sin\theta$$

$$= e^{r\cos\theta} [\cos\theta \cdot \cos(r\sin\theta) - \sin\theta \cdot \sin(r\sin\theta)] \qquad \text{..........} \quad (1)$$

$$\frac{\partial u}{\partial \theta} = -\, r\sin\theta\, e^{r\cos\theta} \cos(r\sin\theta) - e^{r\cos\theta} \sin(r\sin\theta) \cdot r\cos\theta$$

$$= re^{r\cos\theta} [-\sin\theta \cdot \cos(r\sin\theta) - \cos\theta \sin(r\sin\theta)] \qquad \text{..........} \quad (2)$$

$$\frac{\partial v}{\partial r} = \cos\theta e^{r\cos\theta} \sin(r\sin\theta) + e^{r\cos\theta} \cos(r\sin\theta) \sin\theta$$

$$= e^{r\cos\theta} [\cos\theta \cdot \sin(r\sin\theta) + \sin\theta \cdot \cos(r\sin\theta)] \qquad \text{..........} \quad (3)$$

$$\frac{\partial v}{\partial \theta} = -\, r\sin\theta\, e^{r\cos\theta} \sin(r\sin\theta) + e^{r\cos\theta} \cos(r\sin\theta) \cdot r\cos\theta$$

$$= re^{r\cos\theta} [-\sin\theta \cdot \sin(r\sin\theta) + \cos\theta \cdot \cos(r\sin\theta)] \qquad \text{..........} \quad (4)$$

$\therefore$ From (1) and (4)

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

and from (2) and (3)

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

13. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{9}{(x+y+z)^2}$$

Solution : We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\,\frac{x^2 - yz + y^2 - xz + z^2 - xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= 3\,\frac{x^2 + y^2 + z^2 - xy - yz - zx}{(x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}$$

$$= \frac{3}{x + y + z}$$

$$\therefore \quad \text{L.H.S.} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= 3\left[-\frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2}\right]$$

$$= -\frac{9}{(x + y + z)^2}$$

Note : In above example, we could also write

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\,\frac{\partial^2 u}{\partial x \partial z} + 2\,\frac{\partial^2 u}{\partial x \partial y} + 2\,\frac{\partial^2 u}{\partial y \partial z}$$

14. If $u = f(x + at) + g(x - at)$, prove that

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Solution : We have $u = f(x + at) + g(x - at)$

$$\therefore \quad \frac{\partial u}{\partial t} = af'(x + at) - ag'(x - at)$$

Again $\dfrac{\partial^2 u}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at)$

Now $\dfrac{\partial u}{\partial x} = f'(x + at) + g'(x - at)$

Again $\dfrac{\partial^2 u}{\partial x^2} = f''(x + at) + g''(x - at)$

$$\therefore \quad \frac{\partial^2 u}{\partial t^2} = a^2 [f''(x + at) + g''(x - at)]$$

$$= a^2 \frac{\partial^2 u}{\partial x^2}$$

15. If $u = (x^2 + y^2 + z^2)^{\frac{m}{2}}$ then find the value of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Solution : Let $r^2 = x^2 + y^2 + z^2 \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$$\therefore \quad u = r^m$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = mr^{m-1} \frac{\partial r}{\partial x} = mr^{m-1} \frac{x}{r} = mxr^{m-2}$$

Again differentiating partially w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(mxr^{m-2})$$

$$= m\left[r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right]$$

$$= m(r^{m-2} + x(m-2)\, r^{m-4}\, x]$$

$$= mr^{m-2}[1 + (m-2)\, x^2\, r^{-2}]$$

Similarly, we get

$$\frac{\partial^2 u}{\partial y^2} = mr^{m-2}[1 + (m-2)y^2\, r^{-2}]$$

and $\dfrac{\partial^2 u}{\partial z^2} = mr^{m-2}[1 + (m-2)z^2\, r^{-2}]$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = mr^{m-2}[3 + (m-2)r^{-2}(x^2 + y^2 + z^2)]$$

$$= mr^{m-2}[3 + (m-2)r^{-2} \cdot r^2]$$

$$= m(m+1)\, r^{m-2}$$

$$= m(m+1)(x^2 + y^2 + z^2)^{\frac{m-2}{2}}$$

16. if $u = f(r)$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

Solution : We have $u = f(r)$ and $r^2 = x^2 + y^2$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} \text{ and } \frac{\partial r}{\partial y} = \frac{y}{r}$$

Now differentiating u partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{xf'(r)}{r}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x}$$

Again differentiating partially w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{f'(r)}{r} + \frac{xf''(r)}{r}\frac{\partial r}{\partial x} + xf'(r)\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r} - \frac{x^2 f'(r)}{r^3}$$

Similarly $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{f'(r)}{r} + \dfrac{y^2 f''(r)}{r^2} - \dfrac{y^2 f'(r)}{r^3}$

$\therefore$ L.H.S. $= \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

$$= \frac{2f'(r)}{r} + \frac{f''(r)}{r^2}(x^2 + y^2) - \frac{f'(r)}{r^3}(x^2 + y^2)$$

$$= \frac{2f'(r)}{r} + \frac{f''(r)}{r^2}r^2 - \frac{f'(r)}{r^3}r^2$$

$$= \frac{2f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= \frac{f'(r)}{r} + f''(r) = \text{R.H.S.}$$

17. If $x^x y^y z^z = c$ show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -[x \log_e x]^{-1}$$

Solution : Taking logarithm on both sides, we get

$x \log x + y \log y + z \log z = \log c$

considering z as a function of x and y differentiating w.r.t. x, we get

$$\log x + x\frac{1}{x} + \log z\frac{\partial z}{\partial x} + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} = 0$$

$$\therefore \quad \log x + 1 + (\log z + 1)\frac{\partial z}{\partial x} = 0$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{1 + \log x}{1 + \log z} \qquad \text{.......... (1)}$$

Similarly $\dfrac{\partial z}{\partial y} = -\dfrac{1+\log y}{1+\log z}$ .... (2)

Differentiating (1) w.r.t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = -(1+\log x)\left[-\frac{1}{(1+\log z)^2}\right]\frac{1}{z}\frac{\partial z}{\partial y}$$

$$= \frac{1+\log x}{(1+\log z)^2}\cdot\frac{1}{z}\cdot\left(-\frac{1+\log y}{1+\log z}\right)$$

$\therefore$ At $x = y = z$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1+\log x}{(1+\log x)^2}\cdot\frac{1}{x}\cdot\left(-\frac{1+\log x}{1+\log x}\right)$$

$$= -\frac{1}{x(\boxed{1}+\log x)}$$

$$= -\frac{1}{x[\log e+\log x]} = -\frac{1}{x\log ex} = -[x\log ex]^{-1}$$

18. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u} = 1$, **prove that**

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$

Solution : We have

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1$$

Differentiating partially w.r.t. x, we get

$$\frac{2x}{a^2+u}-\frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x}-\frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x}-\frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{2x}{V(a^2+u)}$$

where $V = \dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{2y}{V(b^2+u)} \text{ and } \frac{\partial u}{\partial z} = \frac{2z}{V(c^2+u)}$$

$$\therefore \quad \text{L.H.S.} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= \frac{4x^2}{V^2(a^2+u)^2} + \frac{4y^2}{V^2(b^2+u)^2} + \frac{4z^2}{V^2(c^2+u)^2}$$

$$= \frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{V^2} V = \frac{4}{V} \qquad \text{.......... (1)}$$

$$\text{R.H.S.} = 2\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right]$$

$$= 2\left[x\frac{2x}{V(a^2+u)} + y\frac{2y}{V(b^2+u)} + z\frac{2z}{V(c^2+u)}\right]$$

$$= \frac{4}{V}\left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right]$$

$$= \frac{4}{V}(1) = \frac{4}{V} \qquad \text{.......... (2)}$$

Thus from (1) and (2)

L.H.S. = R.H.S.

Note : The derivative of the form $\left(\dfrac{\partial u}{\partial x}\right)_y$ shows that it is the derivative w.r.t. x keeping y as constant.

19. If $x = r\cos\theta$, $y = r\sin\theta$ then obtain

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta$ (ii) $\left(\frac{\partial r}{\partial x}\right)_y$ (iii) $\left(\frac{\partial y}{\partial \theta}\right)_r$ (iv) $\left(\frac{\partial \theta}{\partial y}\right)_x$

Solution : We have $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$

(i) $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$ (θ is kept constant)

(ii) $r^2 = x^2 + y^2$

Differentiating w.r.t. x, keeping y as a constant, we get

$$2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r}$$

(iii) $y = r\sin\theta$

Differentiating w.r.t. θ, keeping r as a constant, we get $\left(\frac{\partial y}{\partial \theta}\right)_r = \sin\theta$

(iv) $\theta = \tan^{-1}\frac{y}{x}$

Differentiating w.r.t. y, keeping x as a constant, we get

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

20. If $u = lx + my$, $v = mx - ly$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2} \text{ and } \left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$$

Solution : We have

$u = lx + my$ and $v = mx - ly$

Solving these two equations for x and y, we get

$$x = \frac{lu+mv}{l^2+m^2},\ y = \frac{mu-lv}{l^2+m^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = l \qquad \text{.......... (1)}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2} \qquad \text{.......... (2)}$$

$\therefore$ From (1) and (2),

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = l \cdot \frac{l}{l^2 + m^2} = \frac{l^2}{l^2 + m^2}$$

To obtain $\left(\frac{\partial y}{\partial v}\right)_x$ we require y as a function of x and v.

$$\therefore \quad v = mx - ly \Rightarrow y = \frac{mx - v}{l}$$

$$\therefore \quad \left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{l} \qquad \text{.......... (3)}$$

To obtain $\left(\frac{\partial v}{\partial y}\right)_u$, we require v as a function of u and y.

$$\therefore \quad v = m\left(\frac{u - my}{l}\right) - ly = \frac{mu - m^2y - l^2y}{l} = \frac{mu - (l^2 + m^2)y}{l}$$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l} \qquad \text{.......... (4)}$$

$\therefore$ From (3) and (4),

$$\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}$$

21. If $u = \log(x^2 + y^2 + z^2)$, show that $x\dfrac{\partial^2 u}{\partial y \partial z} = y\dfrac{\partial^2 u}{\partial z \partial x}$

Solution : We have $u = \log(x^2 + y^2 + z^2)$

Let $r^2 = x^2 + y^2 + z^2 \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\therefore \quad u = \log r^2 = 2\log r$

Differentiating, partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = \frac{2}{r}\frac{\partial r}{\partial x} = \frac{2}{r}\cdot\frac{x}{r} = \frac{2x}{r^2}$$

Again differentiating w.r.t. z, we get

$$\frac{\partial^2 u}{\partial z \partial x} = -\frac{4x}{r^3}\frac{\partial r}{\partial z} = -\frac{4x}{r^3}\frac{z}{r} = -\frac{4xz}{r^4}$$

$$\therefore \quad y\frac{\partial^2 u}{\partial z \partial x} = -\frac{4xyz}{r^4} \qquad \text{.......... (1)}$$

Now differentiating u w.r.t. z, we get

$$\frac{\partial u}{\partial z} = \frac{2}{r}\frac{\partial r}{\partial z} = \frac{2}{r}\cdot\frac{z}{r} = \frac{2z}{r^2}$$

Again differentiating w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y \partial z} = -\frac{4z}{r^3}\cdot\frac{\partial r}{\partial y} = -\frac{4z}{r^3}\cdot\frac{y}{r} = -\frac{4yz}{r^4}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial y \partial z} = -\frac{4xyz}{r^4} \qquad \text{.......... (2)}$$

Thus from (1) and (2),

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x}$$

22. If $u = x^2y + y^2z + z^2x$, prove that

(i) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = 6(x + y + z)$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 2(x + y + z)$$

Solution : We have $u = x^2y + y^2z + z^2x$

Differentiating, partially w.r.t. x, y and z we get

$$\frac{\partial u}{\partial x} = 2xy + z^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz \text{ and } \frac{\partial u}{\partial z} = y^2 + 2zx$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

(i) L.H.S. $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)[x^2 + y^2 + z^2 + 2(xy + yz + zx)]$$

$$= 2x + 2(y + z) + 2y + 2(x + z) + 2z + 2(y + x)$$

$$= 6(x + y + z) = \text{R.H.S.}$$

(ii) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$$= \frac{\partial}{\partial x}(2xy + z^2) = 2y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2yz) = 2z$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}(y^2 + 2zx) = 2x$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 2(x + y + z)$$

23. If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are constants satisfying the equ

$$\frac{\partial u}{\partial t} = A^2 \frac{\partial^2 u}{\partial x^2}, \text{ prove that } Ag = \sqrt{\frac{n}{2}}$$

Solution : We have

$$u = Ae^{-gx}\sin(nt - gx)$$

Differentiating partially w.r.t. t, we get

$$\frac{\partial u}{\partial t} = nAe^{-gx}\cos(nt - gx).$$

Now differentiating w.r.t. x, we get

$$\frac{\partial u}{\partial x} = -gAe^{-gx}\sin(nt-gx) - gAe^{-gx}\cos(nt-gx)$$

$$= -gAe^{-gx}[\sin(nt-gx)+\cos(nt-gx)]$$

Again differentiating w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = g^2Ae^{-gx}[\sin(nt-gx)+\cos(nt-gx)] - gAe^{-gx}[-g\cos(nt-gx) + g\sin(nt-gx)]$$

$$= g^2Ae^{-gx}[\sin(nt-gx)+\cos(nt-gx)+\cos(nt-gx)-\sin(nt-gx)]$$

$$= 2g^2Ae^{-gx}\cos(nt-gx)$$

$\therefore$ Now, we have

$$\frac{\partial u}{\partial t} = A^2\frac{\partial^2 u}{\partial x^2}$$

$\therefore$ $nAe^{-gx}\cos(nt-gx) = 2g^2A^3e^{-gx}\cos(nt-gx)$

$\Rightarrow$ $n = 2g^2A^2$

$\Rightarrow$ $g^2A^2 = \dfrac{n}{2}$

$\Rightarrow$ $Ag = \sqrt{\dfrac{n}{2}}$

24. If $x^xy^yz^z = c$, find the value of $\dfrac{\partial^2 z}{\partial x^2} - 2xy\dfrac{\partial^2 z}{\partial x\partial y} + \dfrac{\partial^2 z}{\partial y^2}$ when $x = y = z = 1$.

Solution : We have $x^xy^yz^z = c$

Taking logarithm on both sides, we get

$x\log x + y\log y + z\log z = \log c$

Considering z as a function of x and y, differentiating we get

$$x\frac{1}{x} + \log x + z\frac{1}{z}\frac{\partial z}{\partial x} + \log z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z} \qquad \text{.......... (1)}$$

Similarly $\dfrac{\partial z}{\partial y} = -\dfrac{1+\log y}{1+\log z}$ .... (2)

Again, finding the second order partial derivatives, we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\left[\frac{(1+\log z)\dfrac{1}{x} - (1+\log x)\dfrac{1}{z}\dfrac{\partial z}{\partial x}}{(1+\log z)^2}\right]$$

$$= -\left[\frac{\dfrac{1+\log z}{x} - \dfrac{1+\log x}{z}\left(-\dfrac{1+\log x}{1+\log z}\right)}{(1+\log z)^2}\right]$$

$$= -\left[\frac{z\,(1+\log z)^2 + x\,(1+\log x)^2}{xz\,(1+\log z)^3}\right]$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1+\log x}{(1+\log z)^2}\,\frac{1}{z}\,\frac{\partial z}{\partial y}$$

$$= \frac{1}{z}\,\frac{1+\log x}{(1+\log z)^2}\left(-\frac{1+\log y}{1+\log z}\right)$$

$$= -\frac{1}{z}\,\frac{(1+\log x)\,(1+\log y)}{(1+\log z)^2}$$

Similarly

$$\frac{\partial^2 z}{\partial y^2} = -\left[\frac{z\,(1+\log z)^2 + y\,(1+\log y)^2}{yz\,(1+\log z)^3}\right]$$

Now at $x = y = z = 1$, we have

$$\frac{\partial^2 z}{\partial x^2} = -\left[\frac{1+1}{1}\right] = -2,\ \frac{\partial^2 z}{\partial y \partial x} = -1,\ \frac{\partial^2 z}{\partial y^2} = -2$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2} - 2xy\,\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial^2 z}{\partial y^2} = -2 - 2\,(-1) - 2 = -2$$

25. If $z = x\,f(x+y) + y\,g(x+y)$, prove that $\dfrac{\partial^2 z}{\partial x^2} - 2\,\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

Solution : We have

$$z = x\,f(x+y) + y\,g(x+y)$$

We obtain all the second order partial derivatives.

$$\therefore \quad \frac{\partial z}{\partial x} = f(x+y) + x f'(x+y) + y\, g'(x+y)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f'(x+y) + f'(x+y) + x f''(x+y) + y\, g''(x+y)$$

$$= 2f'(x+y) + x f''(x+y) + y\, g''(x+y) \text{..........} \quad (1)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f'(x+y) + x f''(x+y) + g'(x+y) + y\, g''(x+y) \text{.......} \quad (2)$$

$$\frac{\partial z}{\partial y} = x f'(x+y) + g(x+y) + y\, g'(x+y)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = x f''(x+y) + g'(x+y) + g'(x+y) + y\, g''(x+y)$$

$$= xf''(x+y) + 2g'(x+y) + y\, g''(x+y) \quad \text{..........} \quad (3)$$

$\therefore$ From (1), (2) and (3), we get

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 2f'(x+y) + x f''(x+y) + y\, g''(x+y) - 2f'(x+y)$$

$$- 2x f''(x+y) - 2g'(x+y) - 2y g''(x+y) + xf''(x+y) + 2g'(x+y)$$

$$+ y\, g''(x+y) = 0.$$

26. If $u = f(ax^2 + 2hxy + by^2)$ and $v = \phi(ax^2 + 2hxy + by^2)$, prove that

$$\frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right)$$

Solution : We have $v = \phi(ax^2 + 2hxy + by^2)$

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial v}{\partial x} = \phi'(ax^2 + 2hxy + by^2) \cdot (2ax + 2hy) = 2(ax + hy)\phi'$$

$$\frac{\partial v}{\partial y} = \phi'(ax^2 + 2hxy + by^2) \cdot (2hx + 2by) = 2(hx + by)\phi'$$

$$\therefore \quad \frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\,[2(ax + hy) f \cdot \phi']$$

$$= 2\,[h f \cdot \phi' + (ax + hy)\phi' \cdot f' \cdot (2hx + 2by) + (ax + hy) f \cdot \phi'' \cdot (2hx + 2by)] \quad \text{..........} \quad (1)$$

Again,

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\,[2\,(hx + by)\cdot f\cdot \phi']$$

$$= 2\,[h\,f\cdot\phi' + (hx + by)\,\phi'\cdot f'\cdot(2ax + 2hy) + (hx + by)\cdot f\cdot \phi''\cdot (2ax + 2hy)] \qquad \text{.......... (2)}$$

$\therefore$ From (1) and (2)

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$$

27. If $u = r^n\,(3\cos^2\theta - 1)$, find the value of n for which the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0 \text{ is satisfied.}$$

Solution : We have

$$u = r^n\,(3\cos^2\theta - 1)$$

$$\therefore \quad \frac{\partial u}{\partial r} = nr^{n-1}\,(3\cos^2\theta - 1)$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\,[nr^{n+1}\,(3\cos^2\theta - 1)]$$

$$= n\,(n + 1)r^n\,(3\cos^2\theta - 1)$$

Again $\dfrac{\partial u}{\partial\theta} = r^n\,6\cos\theta\,(-\sin\theta) = -\,3r^n\sin 2\theta$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\,(-\,3r^n\sin 2\theta\cdot\sin\theta) = -3r^n\,(2\cos 2\theta\,\sin\theta + \sin 2\theta\,\cos\theta)$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0$$

$$\Rightarrow \quad n\,(n + 1)r^n\,(3\cos^2\theta - 1) + \frac{1}{\sin\theta}\,[-\,3r^n\,(2\cos 2\theta\,\sin\theta + \sin 2\theta\,\cos\theta)] = 0$$

$$\Rightarrow \quad n\,(n + 1)r^n\,(3\cos^2\theta - 1) - 3r^n\,(2\cos 2\theta + 2\cos^2\theta) = 0$$

$$\Rightarrow \quad n\,(n + 1)r^n\,(3\cos^2\theta - 1) - 3r^n\,(4\cos^2\theta - 2 + 2\cos^2\theta) = 0$$

$$\Rightarrow \quad 3n\,(n + 1)\cos^2\theta - n\,(n + 1) - 18\cos^2\theta + 6 = 0$$

$$\Rightarrow \quad (3n\,(n + 1) - 18)\cos^2\theta = n\,(n + 1) - 6$$

$$\Rightarrow \quad 3\,[n\,(n + 1) - 6]\cos^2\theta - [n\,(n + 1) - 6] = 0$$

$$\Rightarrow \quad n\,(n + 1) - 6 = 0$$

$$\Rightarrow \quad n^2 + n - 6 = 0 \Rightarrow (n - 2)\,(n + 3) = 0$$

$$\Rightarrow \quad n = 2,\,-\,3$$

28. If $u = \log (x^3 + y^3 - x^2y - xy^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

Solution : We have $u = \log (x^3 + y^3 - x^2y - xy^2)$

$$= \log [x^2 (x - y) - y^2 (x - y)] = \log [(x^2 - y^2)(x - y)]$$

$$= \log [(x + y)(x - y)^2] = \log (x + y) + 2 \log (x - y)$$

We obtain second order partial derivatives.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x+y} + \frac{2}{x-y}\right)$$

$$= -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x+y} + \frac{2}{x-y}\right)$$

$$= -\frac{1}{(x+y)^2} + \frac{2}{(x-y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x+y} - \frac{2}{x-y}\right)$$

$$= -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{2}{(x+y)^2} + \frac{4}{(x-y)^2} - \frac{1}{(x+y)^2} - \frac{2}{(x-y)^2}$$

$$= -\frac{4}{(x+y)^2}$$

29. If $x = \frac{r}{2}(e^{\theta} + e^{-\theta})$, $y = \frac{r}{2}(e^{\theta} - e^{-\theta})$, prove that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$

Solution : We have $x = \frac{r}{2}(e^{\theta} + e^{-\theta})$, $y = \frac{r}{2}(e^{\theta} - e^{-\theta})$

$\therefore \quad x^2 - y^2 = \frac{r^2}{4}\left(e^{2\theta} + 2 + e^{-2\theta} - e^{2\theta} + 2 - e^{-2\theta}\right) = r^2$

$$\therefore \quad \frac{\partial x}{\partial r} = \frac{e^{\theta} + e^{-\theta}}{2} = \frac{x}{r} \qquad \text{.......... (1)}$$

and $r^2 = x^2 - y^2$

$$\Rightarrow \quad 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{.......... (2)}$$

$\therefore$ From (1) and (2)

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}.$$

30. If $u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$ and if $l^2 + m^2 + n^2 = 1$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Solution : We have $u = 3(lx + my + nz)^2 - (x^2 + y^2 + z^2)$

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = 6(lx + my + nz)\, l - 2x$$

Again differentiating w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = 6l^2 - 2$$

Similarly $\frac{\partial^2 u}{\partial y^2} = 6m^2 - 2, \quad \frac{\partial^2 u}{\partial z^2} = 6n^2 - 2$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(l^2 + m^2 + n^2) - 6$$

$$= 6 - 6 = 0 \qquad (\because l^2 + m^2 + n^2 = 1)$$

31. If $a^2x^2 + b^2y^2 = c^2z^2$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2b^2}{c^4z^3}(x^2 + y^2).$$

Solution : We have $c^2z^2 = a^2x^2 + b^2y^2$

$\therefore$ Differentiating partially w.r.t. x, we get

$$2c^2z\frac{\partial z}{\partial x} = 2a^2x \Rightarrow c^2z\frac{\partial z}{\partial x} = a^2x$$

Again differentiating w.r.t. x, we get

$$c^2 z \frac{\partial^2 z}{\partial x^2} + c^2 \left(\frac{\partial z}{\partial x}\right)^2 = a^2$$

$$\Rightarrow \quad c^2 z \frac{\partial^2 z}{\partial x^2} = a^2 - c^2 \left(\frac{a^2 x}{c^2 z}\right)^2$$

$$= a^2 - \frac{a^4 x^2}{c^2 z^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^2 z^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{a^2 c^2 z^2 - a^4 y^2}{c^4 z^3}$$

Similarly $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{b^2 c^2 z^2 - b^4 y^2}{c^4 z^3}$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{a^2 c^2 z^2 - a^4 x^2}{c^2 z^3} + \frac{b^2 c^2 z^2 - b^4 y^2}{c^4 z^3}$$

$$= \frac{a^2(a^2 x^2 + b^2 y^2) - a^4 x^2 + b^2 (a^2 x^2 + b^2 y^2) - b^4 y^2}{c^4 x^3}$$

$$= \frac{a^4 x^2 + a^2 b^2 y^2 - a^4 x^2 + a^2 b^2 x^2 + b^4 y^2 - b^4 y^2}{c^4 z^3}$$

$$= \frac{a^2 b^2 (x^2 + y^2)}{c^4 z^3}$$

EXERCISE – 1

1. If $f = x^3 + y^3 + z^2 + 3xyz$ find (i) f_x (ii) f_{xx} (iii) f_{xyz}

   Ans. : (i) $3x^2 + 3yz$ (ii) $6x$ (iii) 3

2. If $u = x^3 + y^3 - 3axy$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$

3. If $u = \tan^{-1} \left\{\dfrac{xy}{\sqrt{1 + x^2 + y^2}}\right\}$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{(1 + x^2 + y^2)^{3/2}}$

4. If $u = c\, t^{-1/2} e^{-x^2/4a^2 t}$, prove that $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{a^2} \dfrac{\partial u}{\partial t}$

5. If $x^x y^y z^z = 108$ find $\frac{\partial^2 z}{\partial x \partial y}$ at $x = 1,\ y = 2,\ z = 3.$

$$\textbf{Ans. :}\ -\frac{1}{3}\left[\frac{1 + \log 2}{(1 + \log 3)^3}\right]$$

6. If $u = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

7. If $u = \frac{1}{x^2 + y^2 + z^2}$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$\textbf{Ans. :}\ \frac{2}{\left(x^2 + y^2 + z^2\right)^2}$$

8. If $u = \log\sqrt{x^2 + y^2 + z^2}$, prove that

$$(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1.$$

9. If $u = \sqrt{x^2 + y^2 + z^2}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$

10. If $u = \frac{xy}{x + y}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$

11. If $x = r\cos\theta,\ y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$

(iii) $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$ (iv) $\frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}$

12. If $x = \cos\theta - r\sin\theta,\ y = \sin\theta + r\cos\theta$, prove that

(i) $\frac{\partial \theta}{\partial x} = -\frac{\cos\theta}{r}$ (ii) $\frac{\partial r}{\partial x} = \frac{x}{r}$ (iii) $\frac{\partial^2 \theta}{\partial x^2} = \frac{\cos\theta}{r^3}(\cos\theta - 2r\sin\theta)$

[Hint : $x^2 + y^2 = 1 + r^2,\ x\cos\theta + y\sin\theta = 1$**]**

13. If $u = e^x(x\cos y - y\sin y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

14. If $u = e^{xyz} f\left(\frac{xz}{y}\right)$, prove that

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2xyzu$ (ii) $y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2xyzu$

15. If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$, prove that $\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2 + y^2}{y^2 - x^2}$

16. If $u = (x^2 - y^2) f(xy)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^4 - y^4) f''(xy)$.

17. If $u = x \log (x + r) - r$, where $r^2 = x^2 + y^2$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{x + r}$

18. If $u = \frac{1}{x} [x - y + \psi (x + y)]$, prove that $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = x^2 \frac{\partial^2 u}{\partial y^2}$

19. If $u = ax + 6y + 8z$ satisfies $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$, then find a.

20. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0$.

21. If $u = e^{xy}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

22. If $u = e^{b\theta} \sin (b \log r)$, prove that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

1.6 HOMOGENEOUS FUNCTIONS :

In this section we shall see the concept of homogeneous functions which simplifies the solving procedure by means of special theorems.

Definition : A function $u = f(x, y)$ of two independent variables x and y is said to be homogeneous of degree n if it satisfies.

$$f(\lambda x, \lambda y) = \lambda^n f(x, y), \text{ where } \lambda \text{ is a constant.}$$

If $u = f(x, y)$ is homogeneous of degree n then it can also be expressed as

$$u = f(x, y) = x^n \phi\left(\frac{y}{x}\right)$$

$$= y^n \phi\left(\frac{x}{y}\right)$$

Note : We can extend this definition for the function of three independent variables

$$\text{i.e. if } f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

$$= y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right)$$

$$= z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right)$$

EXAMPLES

1. $u = f(x, y) = x^3 + y^3 + 2x^2y$

$$\therefore \; f(\lambda x, \lambda y) = \lambda^3x^3 + \lambda^3y^3 + 2\lambda^2x^2\lambda y$$

$$= \lambda^3 (x^3 + y^3 + 2x^2y) = \lambda^3 f(x, y)$$

$\therefore$ $u = f(x, y)$ is homogeneous of degree 3.

2. $u = f(x, y) = (y^2 + 4xy)^{3/2}$

$$\therefore \; f(\lambda x, \lambda y) = [\lambda^2y^2 + 4\lambda x \, \lambda y]^{3/2}$$

$$= \lambda^3 [y^2 + 4xy]^{3/2} = \lambda^3 f(x, y)$$

$\therefore$ $f(x, y)$ is homogeneous of degree 3.

3. $u = f(x, y) = x^2 \tan\left(\frac{y}{x}\right)$

$$\therefore \; f(\lambda x, \lambda y) = \lambda^2x^2 \tan\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \lambda^2x^2 \tan\left(\frac{y}{x}\right) = \lambda^2 f(x, y)$$

$\therefore$ $f(x, y)$ is homogeneous of degree 2.

4. $u = f(x, y) = \dfrac{x^{1/2} + y^{1/2}}{x^{1/4} - y^{1/4}}$

$$\therefore \; f(\lambda x, \lambda y) = \frac{\lambda^{1/2} x^{1/2} + \lambda^{1/2} y^{1/2}}{\lambda^{1/4} x^{1/4} - \lambda^{1/4} y^{1/4}}$$

$$= \frac{\lambda^{1/2} (x^{1/2} + y^{1/2})}{\lambda^{1/4} (x^{1/4} - y^{1/4})} = \lambda^{1/4} f(x, y)$$

$\therefore$ $f(x, y)$ is homogeneous of degree $\frac{1}{4}$.

5. $u = f(x, y, z) = x^3 \sin\left(\frac{y}{x}\right) + y^2 z \log\left(\frac{x}{y}\right) + xyz \sin^{-1}\left(\frac{z}{y}\right)$

$$f(\lambda x, \lambda y, \lambda z) = \lambda^3 x^3 \sin\left(\frac{\lambda z}{\lambda x}\right) + \lambda^2 y^2 \lambda z \log\left(\frac{\lambda x}{\lambda y}\right) + \lambda x \lambda y \lambda z \sin^{-1}\left(\frac{\lambda z}{\lambda y}\right)$$

$$= \lambda^3 \left[x^3 \sin\left(\frac{y}{x}\right) + y^2 z \log\left(\frac{x}{y}\right) + xyz \sin^{-1}\left(\frac{z}{y}\right)\right]$$

$$= \lambda^3 f(x, y, z).$$

$\therefore$ $f(x, y, z)$ is homogenous of degree 3.

1.6.1 Euler's Theorem :

Statement : If $u = f(x, y)$ is a homogeneous function of two independent variables x and y of degree n then

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$, for all x, y belong to the domain of the function.

Proof : Given that $u = f(x, y)$ is a homogeneous function of degree n.

$\therefore$ By definition,

$$u = x^n \phi\left(\frac{y}{x}\right) \qquad \text{.......... (1)}$$

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = nx^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= nx^{n-1} \phi\left(\frac{y}{x}\right) - yx^{n-2} \phi'\left(\frac{y}{x}\right) \qquad \text{.......... (2)}$$

Now $\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} \phi'\left(\frac{y}{x}\right)$ .... (3)

Thus from (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) - yx^{n-1} \phi'\left(\frac{y}{x}\right) + yx^{n-1} \phi'\left(\frac{y}{x}\right)$$

$$= nx^n \phi\left(\frac{y}{x}\right)$$

$$= nu \qquad (\because \text{ from (1)})$$

Note : More generally if $u = f(x_1, x_2, ..., x_n)$ is a homogeneous function of $x_1, x_2, ..., x_n$ of degree n then

$$x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + ... + x_n \frac{\partial u}{\partial x_n} = nu.$$

Corollary : If $u = f(x, y)$ is a homogeneous function of x, y of degree n then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)\, u$$

Proof : Given that u is homogeneous function of degree n.

$\therefore$ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \qquad .... (1)$$

Differentiating (1) partially w.r.t. x, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}$$

$$\therefore \quad x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1) \frac{\partial u}{\partial x} \qquad .... (2)$$

Again differentiating (1), partially w.r.t. y, we get

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n \frac{\partial u}{\partial y}$$

$$\therefore \quad x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y} \qquad .... (3)$$

Multiplying (2) by x, (3) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial r} + xy \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\, x \frac{\partial u}{\partial x} + (n-1)\, y \frac{\partial u}{\partial y}$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right]$$

$$= n(n-1)\, u \qquad \text{(From (1))}$$

1.6.2 Modified Euler's Theorem :

Statement : If $u = \phi(v)$, where v is a homogeneous function of x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'}$$

Proof : Given the v is a homogeneous function of degree n.

$\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \qquad \text{.......... (1)}$$

Now $u = \phi(v) \Rightarrow v = f(u)$

where f is a inverse function of ϕ.

Differentiating v w.r.t. x and y, we get

$$\frac{\partial v}{\partial x} = f'(u)\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Putting these values in (1), we get

$$xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nv$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nv}{f'(u)} = n\frac{v}{v'} = n\frac{f(u)}{f'(u)}$$

Corollary : If $u = \phi(v)$, where v is a homogeneous function of x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = h(u)\,[h'(u) - 1]$$

where $h(u) = n\frac{v}{v'}$

Proof : By modified Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'}$$

Since $h(u) = n\frac{v}{v'}$, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = h(u) \qquad \text{.......... (1)}$$

Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=h'(u)\frac{\partial u}{\partial x}$$

$$\therefore\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(h'(u)-1)\frac{\partial u}{\partial x} \qquad \text{.......... (2)}$$

Again differentiating (1), partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=h'\frac{\partial u}{\partial y}$$

$$\therefore\quad x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=(h'(u)-1)\frac{\partial u}{\partial y} \qquad \text{.......... (3)}$$

Multiplying (2) by x, (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[h'(u)-1]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=h(u)\,[h'(u)-1] \quad \text{(From (1))}$$

SOLVED EXAMPLES

1. Verify Euler's theorem for the functions :

(i) $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ **(ii)** $u=\left(\sqrt{x}+\sqrt{y}\right)(x^n+y^n)$

Solution : (i) We have

$$u=f(x,y)=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}.$$

$$\therefore\quad f(\lambda x,\lambda y)=\sin^{-1}\left(\frac{\lambda x}{\lambda y}\right)+\tan^{-1}\left(\frac{\lambda y}{\lambda x}\right)=\lambda^0\left[\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}\right]=\lambda^0 f(x,y)$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0 \qquad \text{.......... (1)}$$

Now differentiating u, partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \qquad \text{..........} \quad (2)$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2} \qquad \text{..........} \quad (3)$$

$\therefore$ From (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= 0 \qquad \text{..........} \quad (4)$$

$\therefore$ Thus from (1) and (4), Euler's theorem is verified.

(ii) We have $u = \left(\sqrt{x}+\sqrt{y}\right)(x^n + y^n)$

$$\therefore \quad f(\lambda x, \lambda y) = \left(\sqrt{\lambda x}+\sqrt{\lambda y}\right)(\lambda^n x^n + \lambda^n y^n)$$

$$= \lambda^{n+1/2}\left(\sqrt{x}+\sqrt{y}\right)(x^n + y^n)$$

$$= \lambda^{n+1/2} f(x, y)$$

$\therefore$ u is a homogeneous function of degree $n+\frac{1}{2}$.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)u \qquad \text{..........} \quad (1)$$

Now differentiating u partially w.r.t. x and y, we get

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}(x^n + y^n) + \left(\sqrt{x} + \sqrt{y}\right) nx^{n-1}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{1}{2}\sqrt{x}\,(x^n + y^n) + nx^n\left(\sqrt{x} + \sqrt{y}\right) \qquad \text{.......... (2)}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}(x^n + y^n) + \left(\sqrt{x} + \sqrt{y}\right) ny^{n-1}$$

$$\therefore \quad y\frac{\partial u}{\partial y} = \frac{1}{2}\sqrt{y}\,(x^n + y^n) + ny^n\left(\sqrt{x} + \sqrt{y}\right) \qquad \text{.......... (3)}$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)(x^n + y^n) + n\left(\sqrt{x} + \sqrt{y}\right)(x^n + y^n)$$

$$= \left(n + \frac{1}{2}\right)\left(\sqrt{x} + \sqrt{y}\right)(x^n + y^n)$$

$$= \left(n + \frac{1}{2}\right)u \qquad \text{..... (4)}$$

Thus from (1) and (2) Euler's theorem is verified.

2. If u is a homogeneous function of n^{th} degree in any number of variables then

that $$\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \ldots..\right]^m (u) = n^m u. \qquad (m \text{ is a postive integer})$$

Solution : We prove this result by using the method of induction on the positive integ

For $m = 1$

Since u is a homogeneous function of degree n, by Euler's theorem we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \ldots.. = nu$$

$\therefore$ The result is true for $m = 1$.

We suppose that the result is true for $m = k$.

i.e. $$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \ldots..\right)^k (u) = n^k u$$

Now, we prove the result for $m = k + 1$

$$\therefore \quad \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots\ldots\right)^{k+1}(u)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots\right)\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots\right)^{k}(u)$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots\right)(n^k u) \quad \text{(By assumption)}$$

$$=n^k\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots\right)(u)$$

$$=n^k\cdot nu \quad \text{(By Euler's theorem)}$$

$$=n^{k+1}u$$

$\therefore$ The result is true for $m = k + 1$.

$\therefore$ By the method of induction the result is true for all positive integer m.

3. If $u=\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{\log x-\log y}{x^2+y^2}$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u$$

Solution : We have $u=f(x,y)=\dfrac{1}{x^2}+\dfrac{1}{xy}+\dfrac{\log\frac{x}{y}}{x^2+y^2}$

$$\therefore \quad f(\lambda x,\lambda y)=\frac{1}{\lambda^2x^2}+\frac{1}{\lambda x\,\lambda y}+\frac{\log\frac{\lambda x}{\lambda y}}{\lambda^2x^2+\lambda^2y^2}$$

$$=\frac{1}{\lambda^2}\left[\frac{1}{x^2}+\frac{1}{xy}+\frac{\log\frac{x}{y}}{x^2+y^2}\right]=\lambda^{-2}f(x,y)$$

$\therefore$ u is a homogeneous function of degree -2.

$\therefore$ By Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u.$$

4. If $u = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

Solution : We have $u = f(x, y, z) = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$

$$\therefore \quad f(\lambda x, \lambda y, \lambda z) = \cos\left(\frac{\lambda x\lambda y + \lambda y\lambda z + \lambda z\lambda x}{\lambda^2 x^2 + \lambda^2 y^2 + \lambda^2 z^2}\right)$$

$$= \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right) = \lambda^0 f(x, y, z)$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

5. If $u = \dfrac{x^2 + y^2}{\sqrt{x + y}}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{3}{2} u$

Solution : We have $u = f(x, y) = \dfrac{x^2 + y^2}{\sqrt{x + y}}$

$$\therefore \quad f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\sqrt{\lambda x + \lambda y}} = \frac{\lambda^2}{\sqrt{\lambda}}\left[\frac{x^2 + y^2}{\sqrt{x + y}}\right] = \lambda^{3/2} f(x, y)$$

$\therefore$ u is a homogeneous function of degree $\dfrac{3}{2}$.

$\therefore$ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$$

6. If $u = x^4 y \sin^{-1}\left(\frac{x}{y}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution : We have $u = x^4 y \sin^{-1}\left(\frac{x}{y}\right) = f(x, y)$.

$$\therefore \quad f(\lambda x, \lambda y) = \lambda^4 x^4 \lambda y \sin^{-1}\left(\frac{\lambda x}{\lambda y}\right) = \lambda^5 x^4 y \sin^{-1}\left(\frac{x}{y}\right) = \lambda^5 f(x, y)$$

$\therefore$ u is a homogeneous function of degree 5.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u$$

7. If $u = f\left(\frac{x}{y}, \frac{y}{x}, \frac{z}{x}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution : We have $u = f\left(\frac{x}{y}, \frac{y}{x}, \frac{z}{x}\right)$

$$\therefore \quad f\left(\frac{\lambda x}{\lambda y}, \frac{\lambda y}{\lambda x}, \frac{\lambda z}{\lambda x}\right) = f\left(\frac{x}{y}, \frac{y}{x}, \frac{z}{x}\right)$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

8. If $u = xyf\left(\frac{y}{x}\right) + yz\phi\left(\frac{y}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$

Solution : We have $u = f(x, y) = xyf\left(\frac{y}{x}\right) + yz\phi\left(\frac{y}{z}\right)$

$$\therefore \quad f(\lambda x, \lambda y) = \lambda x \lambda y f\left(\frac{\lambda y}{\lambda x}\right) + \lambda y \lambda z\, \phi\left(\frac{\lambda y}{\lambda z}\right)$$

$$= \lambda^2\left[xyf\left(\frac{y}{x}\right) + yz\phi\left(\frac{y}{z}\right)\right] = \lambda^2 f(x, y).$$

$\therefore$ u is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$$

9. If $u = x + y + (x+ y) f\left(\frac{y}{x}\right)$, prove that

$$x\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)=y\left(\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x\partial y}\right)$$

Solution : We have $u = x + y + (x + y) f\left(\frac{y}{x}\right)$

which is a homogeneous function of degree 1.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u \qquad \text{.......... (1)}$$

Differentiating (1) partially w.r.t. x and y, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=0 \qquad \text{.......... (2)}$$

and $x\dfrac{\partial^2 u}{\partial x\partial y}+y\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial y}$

$$\Rightarrow x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{.......... (3)}$$

Subtracting (3) from (2), we get

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}-x\frac{\partial^2 u}{\partial y\partial x}-y\frac{\partial^2 u}{\partial y^2}=0$$

$$\Rightarrow x\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)=y\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial x\partial y}\right)$$

10. If $u = x\, f\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right)$**, prove that** $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$

Solution : We have $u = x\, f\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right) = v + w$

where $v = x\, f\left(\dfrac{y}{x}\right)$ which a homogeneous function of degree 1 and $w = g\left(\dfrac{y}{x}\right)$ which is a homogeneous function of degree 0.

$$\therefore \quad x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)\, u = 0 \qquad \text{.......... (2)}$$

$$\text{and} \quad x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = n(n-1)\, u = 0 \qquad \text{.......... (3)}$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= x^2 \frac{\partial^2 (v+w)}{\partial x^2} + 2xy \frac{\partial^2 (v+w)}{\partial x \partial y} + y^2 \frac{\partial^2 (v+w)}{\partial y^2}$$

$$= \left[x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}\right] + \left[x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2}\right]$$

$$= 0 + 0 = 0.$$

11. If $u = x^2 \tan^{-1} \dfrac{y}{x} + y^2 \sin^{-1} \dfrac{x}{y}$**, prove that**

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$$

Solution : We have $u = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) + y^2 \sin^{-1}\left(\dfrac{x}{y}\right) = f(x, y)$

$$\therefore \quad f(\lambda x, \lambda y) = \lambda^2 x^2 \tan^{-1}\left(\frac{\lambda y}{\lambda x}\right) + \lambda^2 y^2 \sin^{-1}\left(\frac{\lambda x}{\lambda y}\right)$$

$$= \lambda^2 \left[x^2 \tan^{-1}\left(\frac{y}{x}\right) + y^2 \sin^{-1}\left(\frac{x}{y}\right)\right] = \lambda^2 f(x, y)$$

$\therefore$ u is a homogeneous function of degree 2.

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)\, u$$

$$= 2u \qquad (\because n = 2)$$

12. If $u = (x^2 + y^2)^{1/3}$, prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$

Solution : We have $u = (x^2 + y^2)^{1/3} = f(x, y)$

$\therefore$ $f(\lambda x, \lambda y) = (\lambda^2 x^2 + \lambda^2 y^2)^{1/3} = \lambda^{2/3}(x^2 + y^2)^{1/3} = \lambda^{2/3} f(x, y)$

$\therefore$ u *is* a homogeneous function of degree $\frac{2}{3}$.

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)\, u$$

$$= \frac{2}{3}\left(\frac{2}{3} - 1\right) u$$

$$= -\frac{2}{9}\, u$$

13. If $u = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right)$, then show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n^2 u$$

Solution : We have

$$u = x^n f\left(\frac{y}{x}\right) + y^{-n} \phi\left(\frac{x}{y}\right) = v + w$$

$\therefore$ where $v = x^n f\left(\frac{y}{x}\right)$ and $w = y^{-n} \phi\left(\frac{x}{y}\right)$ which are homogeneous functio[ns of]

degree n and $-n$ respectively.

$\therefore$ By Euler's theorem,

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv \qquad \text{.........} (1)$$

$$\text{and } x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v \qquad \text{..........} (2)$$

$$\text{Again } x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = -nw \qquad \text{..........} (3)$$

$$\text{and } x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = -n(-n-1)w = n(n+1)w \qquad \text{..........} (4)$$

Now adding (1) and (3), we get

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = nv - nw$$

$$\Rightarrow \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nv - nw \qquad \text{..........} (5)$$

Now adding (2) and (4), we get

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) = n(n-1)v + n(n+1)w.$$

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)v + n(n+1)w \qquad \text{..........} (6)$$

Now adding (6) and (7), we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n(n-1)v + n(n+1)w + nv - nw$$

$$= n^2 v + n^2 w$$

$$= n^2 (v + w) = n^2 u.$$

14. If $x = e^u \tan v$, $y = e^u \sec v$, find the value of

$$\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)\left(x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y}\right)$$

Solution : We have $x = e^u \tan v$ and $y = e^u \sec v$

$$\therefore \quad y^2 - x^2 = e^{2u} \Rightarrow u = \frac{1}{2} \log (y^2 - x^2)$$

$$\text{Again } \frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} \Rightarrow \frac{x}{y} = \sin v \Rightarrow v = \sin^{-1}\left(\frac{x}{y}\right)$$

Now $u = \frac{1}{2}\log w$, where $w = y^2 - x^2$ is a homogeneous function of degree 2. Then by modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{w}{w'} = 2 \cdot \frac{e^{2u}}{2e^{2u}} = 1 \qquad (w = e^{2u})$$

Again v is a homogeneous function of degree 0. Thus by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$$

$$\therefore \quad \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1.0 = 0.$$

15. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution : We have $u = \tan^{-1} v$.

Where $v = \frac{x^3+y^3}{x-y}$ is a homogeneous function of degree 2.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = 2\frac{\tan u}{\sec^2 u}$$
$$= 2\sin u \cos u = \sin 2u$$

16. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Solution : We have $u = \sin^{-1}v \Rightarrow v = \sin u$

Where $v = \frac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = 1\frac{\sin u}{\cos u} = \tan u.$$

17. If $u = \log\left(\dfrac{x^4 + y^4}{x + y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Solution : We have $u = \log v \Rightarrow v = e^u$

Where $v = \dfrac{x^4 + y^4}{x + y}$ is a homogeneous function of degree 3.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = 3\frac{e^u}{e^u} = 3.$$

18. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}\tan u$.

Solution : We have $u = \sin^{-1}v \Rightarrow v = \sin u$

$$\text{Where } v = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = \frac{x^{1/4}\left(1 + \dfrac{y^{1/4}}{x^{1/4}}\right)}{x^{1/5}\left(1 + \dfrac{y^{1/5}}{x^{1/5}}\right)} = x^{1/20}\left[\frac{1 + \dfrac{y^{1/4}}{x^{1/4}}}{1 + \dfrac{y^{1/5}}{x^{1/5}}}\right]$$

$\therefore$ v is a homogeneous function of degree $\dfrac{1}{20}$.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = \frac{1}{20}\frac{\sin u}{\cos u} = \frac{1}{20}\tan u$$

19. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x + y}{x^2 + y^2}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

Solution : We have $u = \operatorname{cosec}^{-1}v \Rightarrow v = \operatorname{cosec} u$

Where $v = \dfrac{x + y}{x^2 + y^2}$ is a homogeneous function of degree -1.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = -\frac{\operatorname{cosec} u}{-\operatorname{cosec} u \cot u} = \tan u$$

20. If $u = \log x + \log y$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

Solution : We have $u = \log x + \log y = \log xy = \log v \Rightarrow v = e^u$

Where $v = xy$ is a homogeneous function of degree 2.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = 2\frac{e^u}{e^u} = 2.$$

21. If $u = e^{x^2 f\left(\frac{x}{y}\right)}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u \log u$

Solution : We have $u = e^v \Rightarrow v = \log u$

Where $v = x^2 f\left(\dfrac{x}{y}\right)$ is a homogeneous function of degree 2.

$\therefore$ By modified Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{v}{v'} = 2\frac{\log u}{1/u} = 2u\log u$$

22. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{144}\tan u\,[\tan^2 u - 11]$$

Solution : We have $u = \sin^{-1} v \Rightarrow v = \sin u$

$$\text{Where } v = \frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}} = \frac{x^{1/4}\left(1 + \dfrac{y^{1/4}}{x^{1/4}}\right)}{x^{1/6}\left(1 + \dfrac{y^{1/6}}{x^{1/6}}\right)} = x^{1/12}\,\phi\left(\frac{y}{x}\right)$$

$\therefore$ v is a homogeneous function of degree $\dfrac{1}{12}$.

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

Where $g(u) = n\dfrac{v}{v'} = \dfrac{1}{12} \cdot \dfrac{\sin u}{\cos u} = \dfrac{1}{12}\tan u$

$\therefore \quad g'(u) = \dfrac{1}{12}\sec^2 u$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left[\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{1}{144}\tan u\,[1 + \tan^2 u - 12]$$

$$= \frac{1}{144}\tan u\,[\tan^2 u - 11]$$

23. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u.$$

Solution : We have $u = \tan^{-1} v \Rightarrow v = \tan u.$

Where $v = \dfrac{x^3+y^3}{x-y}$ is a homogeneous function of degree 2.

$\therefore \quad g(u) = n\dfrac{v}{v'} = 2\dfrac{\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \sin 2u\,[2\cos 2u - 1]$$

$$= 2\sin 2u\cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u \sin u.$$

24. If $u = \sin^{-1}(x^3 + y^3)^{2/5}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{6}{5}\tan u\left[\frac{6}{5}\sec^2 u - 1\right]$$

Solution : We have $u = \sin^{-1} v \Rightarrow v = \sin u$

Where $v = (x^3 + y^3)^{2/5} = \left[x^3\left(1 + \frac{y^3}{x^3}\right)\right]^{2/5} = x^{6/5}\left(1 + \frac{y^3}{x^3}\right)^{2/5} = x^{6/5}\,\phi\left(\frac{y}{x}\right)$

$\therefore$ v is a homogeneous function of degree $\frac{6}{5}$.

$\therefore$ $g(u) = n\frac{v}{v'} = \frac{6}{5}\frac{\sin u}{\cos u} = \frac{6}{5}\tan u$

$\therefore$ By Corollary on modified Euler's theorem, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \frac{6}{5}\tan u\left[\frac{6}{5}\sec^2 u - 1\right]$$

25. If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$ prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$$

Solution : We have $u = \operatorname{cosec}^{-1} v \Rightarrow v = \operatorname{cosec} u$.

$$\text{Where } v = \left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2} = \frac{x^{1/4}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right)^{1/2}}{x^{1/6}\left(1 + \frac{y^{1/3}}{x^{1/3}}\right)^{1/2}} = x^{1/12}\,\phi\left(\frac{y}{x}\right)$$

$\therefore$ v is a homogeneous function of degree $\frac{1}{12}$

$\therefore$ $g(u) = n\frac{v}{v'} = -\frac{1}{12}\frac{\operatorname{cosec} u}{\operatorname{cosec} u \cot u} = -\frac{1}{12}\tan u$

$\therefore$ By corollary of modified Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= -\frac{1}{12} \tan u \left[-\frac{1}{12} \sec^2 u - 1\right]$$

$$= \frac{1}{144} \tan u\,[1 + \tan^2 u - 13]$$

26. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -3$$

Solution : We have $u = \log v \Rightarrow v = e^u$

Where $v = x^3 + y^3 - x^2y - xy^2$ is a homogeneous function of degree 3.

$$\therefore \quad g(u) = n\frac{v}{v'} = 3\frac{e^u}{e^u} = 3$$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= 3\,[0 - 1] = -3$$

27. If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

Solution : We have $u = \sin^{-1} V \Rightarrow V = \sin u$.

Where $v = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree $\frac{1}{2}$.

$$\therefore \quad g(u) = n\frac{v}{v'} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \frac{1}{2} \tan u \left[\frac{1}{2} \sec^2 u - 1 \right]$$

$$= \frac{1}{4} \frac{\sin u}{\cos u} \left[\frac{1}{\cos^2 u} - 2 \right]$$

$$= \frac{1}{4} \frac{\sin u\,(1 - 2\cos^2 u)}{\cos^3 u}$$

$$= -\frac{1}{4} \frac{\sin u \cos 2u}{\cos^3 u}$$

28. If $u = \sec^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ find the value of $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2}$

Solution : We have $u = \sec^{-1} v \Rightarrow v = \sec u$

Where $v = \dfrac{x^2 + y^2}{x - y}$ is a homogeneous function of degree 1.

$$\therefore \quad g(u) = n \frac{v}{v'} = 1 \cdot \frac{\sec u}{\sec u \tan u} = \cot u$$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \cot u\,[-\operatorname{cosec}^2 u - 1]$$

$$= -\cot u\,[1 + \cot^2 u + 1]$$

$$= -\cot u\,[2 + \cot^2 u]$$

29. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right)$, **prove that**

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{400} \tan^2 u\,(\tan^2 u - 19).$$

Solution : We have $u = \sin^{-1} v \Rightarrow v = \sin u$.

$\therefore \quad v = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ is a homogeneous function of degree $\dfrac{1}{20}$

$\therefore \quad g(u) = n\dfrac{v}{v'} = \dfrac{1}{20}\dfrac{\sin u}{\cos u} = \dfrac{1}{20}\tan u$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \frac{1}{20}\tan u\left[\frac{1}{20}\sec^2 u - 1\right]$$

$$= \frac{1}{400}\tan u\left[1 + \tan^2 u - 20\right]$$

$$= \frac{1}{400}\tan u\left[\tan^2 u - 19\right]$$

30. If $u = \log r$, where $r^2 = x^2 + y^2$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1$$

Solution : We have $u = \log r \Rightarrow r = e^u$.

Where $r = \sqrt{x^2 + y^2}$ is a homogeneous function of degree 1.

$\therefore \quad g(u) = n\dfrac{r}{r'} = 1 \cdot \dfrac{e^u}{e^u} = 1$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= 1\,[0 - 1] = -1$$

31. If $f(u) = v\,(x, y, z)$ is a homogeneous function of x, y, z of degree n, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$$

Solution : Since $v\,(x, y, z)$ is a homogeneous function of degree, by Euler's theorem, we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv \qquad \text{..........} \ (1)$$

Now $v = f(u)$

$$\therefore \quad \frac{\partial v}{\partial x} = f'(u)\,\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = f'(u)\,\frac{\partial u}{\partial y},\ \frac{\partial v}{\partial z} = f'(u)\,\frac{\partial u}{\partial z}$$

$\therefore$ From (1), we get

$$x f'(u)\,\frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} + z f'(u)\,\frac{\partial u}{\partial z} = n\, f(u)$$

$$\therefore \quad x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} + z\,\frac{\partial u}{\partial z} = n\,\frac{f(u)}{f'(u)}$$

32. If $u = \tan^{-1}\dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}$, find the value of $x^2\,\dfrac{\partial^2 u}{\partial x^2} + 2xy\,\dfrac{\partial^2 u}{\partial x \partial y} + y^2\,\dfrac{\partial^2 u}{\partial y^2}$

Solution : We have $u = \tan^{-1} v \Rightarrow v = \tan\ u.$

Where $u = \dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1.

$$\therefore \quad g(u) = n\,\frac{v}{v'} = 1\cdot\frac{\tan u}{\sec^2 u} = \sin u \cos u = \frac{1}{2}\sin 2u$$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2xy\,\frac{\partial^2 u}{\partial x \partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= \frac{1}{2}\sin 2u\left[\frac{1}{2}2\cos 2u - 1\right]$$

$$= \frac{1}{2}\sin 2u\,(\cos 2u - 1)$$

$$= \sin u \cos u\,(-\,2\sin^2 u)$$

$$= -\,2\sin^3 u \cos u$$

33. If $y = x\cos u$, find the value of $x^2\,\dfrac{\partial^2 u}{\partial x^2} + 2xy\,\dfrac{\partial^2 u}{\partial x \partial y} + y^2\,\dfrac{\partial^2 u}{\partial y^2}$.

Solution : We have $y = x\cos u \Rightarrow u = \cos^{-1}\dfrac{y}{x} = \cos^{-1} v \Rightarrow v = \cos u$

Where $v = \dfrac{y}{x}$ is a homogeneous function of degree 0.

$\therefore \quad g(u) = n\dfrac{v}{v'} = 0$

$\therefore$ By corollary on modified Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

$$= 0.$$

EXERCISE – 2

1. Verify Euler's theorem for the following :

(a) $u = x^3 - 2x^2y + y^3$ (b) $u = 4x^2 + 4xy + 4y^2$

(c) $u = y^n \log\left(\dfrac{x}{y}\right)$ (d) $u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

2. If $u = \left(\dfrac{x}{y} + \dfrac{y}{x} + \dfrac{z}{x}\right)^n$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

3. If $u = x\, f\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u.$

4. If $u = xy + yz + zx$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2u.$

5. If $u = \cos\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

6. If $u = \sec^{-1}\dfrac{y}{z} + \operatorname{cosec}^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{z}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

7. If $u = x^2y\sin^{-1}\left[\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u.$

8. If $u = \left(\dfrac{x}{y}\right)^{y/x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$.

9. If $u = \sqrt{x^2 - y^2}\,\sin^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} - u = 0.$

10. If $u = \log\left(\dfrac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 3u.$

11. If $u = \sin^{-1}\left(\dfrac{x^2+y^2+z^2}{ax+by+cz}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial x} = \tan u.$

12. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$ prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{2xy^5}{(x^2+y^4)^2}$

13. If $u = x^2\sin^{-1}\dfrac{y}{x} - y^2\cos^{-1}\dfrac{x}{y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2u.$

14. If $u = \tan^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{4}\sin 2u$

15. If $u = \sin^{-1}\sqrt{x^2+y^2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \tan^3 u$

16. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \sin 2u\,(1-4\sin^2$

1.7 THE DIFFERENTIAL OF A FUNCTION :

Let $u = f(x, y)$ be a differentiable function of two independent variables x and y. Suppo that x and y both changes by an increments Δx and Δy respectively, then the correspond change Δu occurs in u. That is,

$$\begin{aligned}\Delta u &= f(x+\Delta x, y+\Delta y) - f(x, y)\\ &= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y)\\ &= \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\Delta x + \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\Delta y\end{aligned}$$

Now taking the limit on both sides as $\Delta x \to 0$ and $\Delta y \to 0$ and using

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta u = du,\quad \lim_{\Delta x\to 0} \Delta x = dx,\quad \lim_{\Delta y\to 0} \Delta y = 0$$

and $$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x, y+\Delta y - f(x, y)}{\Delta y} = \frac{\partial u}{\partial y}$$

We get

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

This differential is called the total differential.

Note : We can generalise this result to the function three or more independent variable, e.g. for $u = f(x, y, z)$

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz.$$

Remark : We shall refer a function with continuous first partial derivatives as a continuously differentiable function and the second – order continuous partial derivatives as twice continuously differentiable functions.

Now if the function $u = f(x, y)$ possesses the continuous partial derivatives of higher order then we can form the differential of the differential du. For that we replace dx by h, and dy by k and regard them as constants, corresponding to the fact that the differential $du = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ is a function of the four independent variables x, y, dx and dy. Thus the second differential of the function is

$$d^2u = d(du) = \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k\right\}h + \frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k\right\}k$$

$$= \frac{\partial^2 u}{\partial x^2}h^2 + 2\,\frac{\partial^2 u}{\partial x \partial y}hk + \frac{\partial^2 u}{\partial y^2}k^2$$

$$= \frac{\partial^2 u}{\partial x^2}dx^2 + 2\,\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2$$

Similarly, we may form the higher differentials

$$d^3u = d\,(d^2u) = \frac{\partial^2 u}{\partial x^3}dx^3 + 3\,\frac{\partial^2 f}{\partial x^2 \partial y}dx^2dy + 3\,\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}\,dy^3$$

and in general

$$d^nu = d(d^{n-1}u)$$

$$= \frac{\partial^n u}{\partial x^n}dx^n + {}^nC_1\,\frac{\partial^n u}{\partial x^{n-1}\partial y}dx^{n-1}dy + \ldots + {}^nC_k\,\frac{\partial^n u}{\partial x^{n-k}\partial y^k}dx^{n-k}dy^k$$

$$+ \ldots + \frac{\partial^n u}{\partial y^n}dy^n$$

We can also express

$$d^n u = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n u$$

where in its binomial expansion we mean that

$$\left(\frac{\partial}{\partial x}dx\right)^n u = \frac{\partial^n u}{\partial x^n}dx^n, \left(\frac{\partial}{\partial x}dx\right)^{n-k}\left(\frac{\partial}{\partial y}dy\right)^k u = \frac{\partial^n u}{\partial x^{n-k}\partial y^k}dx^{n-k}dy^k, \text{ and so on.}$$

Note : For the two functions u and v of x and y, we have $d(uv) = udv + vdu$.

1.8 COMPOSITE FUNCTIONS :

Definition : If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$ then u is called a composite function of the single variable t.

Definition : If $u = f(x, y)$ where $x = \phi\ (r, s)$, $y = \psi\ (r, s)$ then u is called a composite function of the two variables r and s.

1.8.1 Differentiation of composite function :

When $u = f(x, y)$ is a composite function of a single variable t then the total differential coefficient $\frac{du}{dt}$ of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Similarly, when $u = f(x, y)$ is a composite function of r and s then instead of total derivatives, we find the partial derivatives as follows :

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and $$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

Note : To make the concept of composite function meaningful we assume that the functions $x = \phi\ (r, s)$ and $y = \psi\ (r, s)$ have the common domain R and map any point (r, s) of R into points (x, y) for which the function $u = f(x, y)$ is defined, that is, into points of the domain S of f. The composite function

$$u = f\ (x, y) = f\ (\phi\ (r, s), \psi\ (r, s)) = F\ (r, s)$$

is then defined in the region R.

1.9 IMPLICIT FUNCTIONS :

The equation of the form

$$f(x, y) = C$$

is known as an implicit function.

Now f is a function of two variables x, y and y is again a function of x. thus we may regard f as a composite function of x. Thus the derivative of f w.r.t. x is,

$$\frac{df}{dx} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}, \text{ if } f_y \neq 0.$$

To find $\frac{d^2y}{dx^2}$, we find the derivative of $\frac{dy}{dx} = -\frac{f_x}{f_y}$ w.r.t. x.

$$\therefore \quad \frac{d^2y}{dx^2} = -\left[\frac{f_y \frac{df_x}{dx} - f_x \frac{df_y}{dx}}{f_y^2}\right]$$

$$= -\left[\frac{f_y\left\{\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y}\frac{dy}{dx}\right\} - f_x\left\{\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y}\frac{dy}{dx}\right\}}{f_y^2}\right]$$

$$= -\left[\frac{f_y\left\{f_{xx} - f_{xy}\frac{f_x}{f_y}\right\} - f_x\left\{f_{yx} - f_{yy}\frac{f_x}{f_y}\right\}}{f_y^2}\right]$$

$$= -\frac{f_{xx} f_y - f_{xy} f_x - \dfrac{f_x}{f_y}\left(f_y f_{yx} - f_x f_{yy}\right)}{f_y^2}$$

$$= -\frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}$$

SOLVED EXAMPLES

1. If $u = x^2 + y^2$, $x = at^2$, $y = 2at$, find $\dfrac{dz}{dt}$

Solution : Here u is a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 2x \cdot 2at + 2y \cdot 2a$$

$$= 4atx + 4ay$$

$$= 4a\left[t \cdot at^2 + 2at\right]$$

$$= 4a^2 t\left(t^2 + 2\right)$$

2. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

Solution : Here u is a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\,3 + \frac{1}{\sqrt{1-(x-y)^2}}\,(-1)\,12t^2$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}\,3\left(1 - 4t^2\right)$$

$$= \frac{3\left(1-4t^2\right)}{\sqrt{1-\left(3t-4t^3\right)^2}}$$

$$= \frac{3\left(1-4t^2\right)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$= \frac{3\left(1-4t^2\right)}{\sqrt{(1-t^2)\,(1-4t^2)^2}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

3. If $u = x^2 + y^2$, $x = a \cos t$, $y = b \sin t$, find $\frac{du}{dt}$

Solution : Here u is a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 2x\,(-a \sin t) + 2y\,(b \cos t)$$

$$= -2a^2 \cos t \sin t + 2b\, b \sin t \cos t$$

$$= -a^2 \sin 2t + b^2 \sin 2t$$

$$= (b^2 - a^2) \sin 2t$$

4. If $u = \tan^{-1}\left(\frac{x}{y}\right)$, $x = 2t$, $y = 1 - t^2$, prove that $\frac{du}{dt} = \frac{2}{1+t^2}$.

Solution : Here u is a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)2 + \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)(-2t)$$

$$= \frac{2y}{x^2+y^2} + \frac{2t\,x}{x^2+y^2}$$

$$= \frac{2\,(y+t\,x)}{x^2+y^2}$$

$$= \frac{2(1 - t^2 + 2t^2)}{4t^2 + (1-t^2)^2}$$

$$= \frac{2(1+t^2)}{4t^2 + 1 - 2t^2 + t^4} = \frac{2(1+t^2)}{1 + 2t^2 + t^4}$$

$$= \frac{2(1+t^2)}{(1+t^2)^2} = \frac{2}{1+t^2}$$

5. If $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t} \cos 3t$, $z = e^{2t} \sin 3t$, find $\dfrac{du}{dt}$.

Solution : Here u is a composite function of t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x \cdot 2e^{2t} + 2y\,(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z\,(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$

$$= 2e^{2t}\,[2e^{2t} + e^{2t}\cos 3t\,(2\cos 3t - 3\sin 3t) + e^{2t}\sin 3t\,(2\sin 3t + 3\cos 3t)]$$

$$= 2e^{4t}\,[2 + 2\cos^2 3t - 3\sin 3t\cos 3t + 2\sin^2 3t + 3\sin 3t\cos 3t]$$

$$= 2e^{4t}\,[2 + 2] = 8e^{4t}$$

6. If $u = f(v, w)$, where $v = x + y$ and $w = x - y$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial v}.$$

Solution : Here u is a composite function of x and y.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}$$

Adding (1) and (2), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial v}.$$

7. If $z = f(x, y)$, where $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution : Here z is a composite function of u and v.

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \qquad \text{.......... (1)}$$

Now $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y} \qquad \text{.......... (2)}$$

$\therefore$ Subtracting (2) from (1), we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \left(e^{u} + e^{-v}\right)\frac{\partial z}{\partial x} - \left(e^{-u} - e^{v}\right)\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

8. If $z = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Solution : Here z is a composite function of r and θ.

$$\therefore \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 = \left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right)^2$$

$$= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 \qquad \text{.......... (1)}$$

Now $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta}$

$$= - r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}$$

$$\therefore \frac{1}{r}\frac{\partial z}{\partial \theta} = -\sin\theta \frac{\partial z}{\partial x} + \cos\theta \frac{\partial z}{\partial y}$$

$$\therefore \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(-\sin\theta \frac{\partial z}{\partial x} + \cos\theta \frac{\partial z}{\partial y}\right)^2$$

$$= \sin^2\theta \left(\frac{\partial z}{\partial x}\right)^2 - 2\sin\theta\cos\theta \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 \quad \text{.........} (2)$$

$\therefore$ Adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = (\sin^2\theta + \cos^2\theta)\left(\frac{\partial z}{\partial x}\right)^2 + (\sin^2\theta + \cos^2\theta)\left(\frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

9. If $u = xe^{y}z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^3 x$, find $\dfrac{du}{dx}$.

Solution : Here u is a composite function of x.

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^{y}z + xe^{y}z\frac{(-2x)}{2\sqrt{a^2 - x^2}} + xe^{y}.\,3\sin^2 x\cos x$$

$$= e^{y}z\left[1 - \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{3x\sin^2 x\cos x}{z}\right]$$

$$= e^{y}z\left[1 - \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{3x\sin^2 x\cos x}{\sin^3 x}\right]$$

$$= e^{y}z\left[1 - \frac{x^2}{y} + 3x\cot x\right]$$

10. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

Solution : Let $v = x^2 + 2yz$, $w = y^2 + 2zx$

$\therefore$ $u = f(v, w)$ is a composite function of x, y, z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$$

$$= 2x\frac{\partial u}{\partial v} + 2z\frac{\partial u}{\partial w} \qquad \text{.......... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}$$

$$= 2z\frac{\partial u}{\partial v} + 2y\frac{\partial u}{\partial w} \qquad \text{.......... (2)}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial z}$$

$$= 2y\frac{\partial u}{\partial v} + 2x\frac{\partial u}{\partial w} \qquad \text{.......... (3)}$$

Thus from (1), (2) and (3), we have.

$$\text{L.H.S.} = (y^2 - zx)\left[2x\frac{\partial u}{\partial v} + 2z\frac{\partial u}{\partial w}\right] + (x^2 - zy)\left[2z\frac{\partial u}{\partial v} + 2y\frac{\partial u}{\partial w}\right]$$

$$+ (z^2 - xy)\left[2y\frac{\partial u}{\partial v} + 2x\frac{\partial u}{\partial w}\right]$$

$$= \frac{\partial u}{\partial v}[2xy^2 - 2x^2z + 2x^2z - 2yz^2 + 2y^2 - 2xy^2] + \frac{\partial u}{\partial w}[2y^2z - 2xz^2 + 2x^2z$$

$$- 2yz^2 + 2xz^2 - 2x^2y]$$

$$= 0$$

$$= \text{R.H.S.}$$

11. If $u = f(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution : Let $r = x - y$, $s = y - z$, $t = z - x$.

$\therefore$ $u = f(r, s, t)$ is a composite function of x, y, z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \qquad \text{.......... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad \ldots\ldots\ldots (2$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \qquad \ldots\ldots\ldots (3$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

12. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution : Let $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$

$\therefore$ $u = f(r, s, t)$ is a composite function of x, y, z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}$$

$$\therefore \text{ L.H.S.} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$= 0 = \text{R.H.S.}$$

13. If $u = f(2x - 3y,\ 3y - 4z,\ 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

Solution : Let $r = 2x - 3y$, $s = 3y - 4z$, $t = 4z - 2x$

$\therefore\ u = f(r, s, t)$ is a composite function of x, y, z

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t}$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= -3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= -4\frac{\partial u}{\partial s} + 4\frac{\partial u}{\partial t}$$

$$\therefore \text{L.H.S.} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial r}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$= 0 = \text{R.H.S.}$$

14. If $u = x^2 + y^2 + z^2 - 2xyz = 1$, show that $\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$.

Solution : We have

$$u = x^2 + y^2 + z^2 - 2xyz = 1$$

$$\therefore du = 0 \Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$\Rightarrow (2x - 2yz)\,dx + (2y - 2xz)\,dy + (2z - 2xy)\,dz = 0$$

$$\Rightarrow (x - yz)\,dx + (y - xz)\,dy + (z - xy)\,dz = 0 \qquad \text{.......... (1)}$$

Now $x^2 + y^2 + z^2 - 2xyz = 1$

$$\Rightarrow x^2 - 2xyz = 1 - y^2 - z^2$$

$$\Rightarrow x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2$$

$$\Rightarrow (x - yz)^2 = (1 - y^2)(1 - z^2)$$

$$\Rightarrow x - yz = \sqrt{(1 - y^2)(1 - z^2)}$$

Similarly $y - zx = \sqrt{\left(1-z^2\right)\left(1-x^2\right)}$

and $z - xy = \sqrt{\left(1-x^2\right)\left(1-y^2\right)}$

$\therefore$ From (1), we get

$$\sqrt{\left(1-y^2\right)\left(1-z^2\right)}\,dx + \sqrt{\left(1-z^2\right)\left(1-x^2\right)}\,dy + \sqrt{\left(1-x^2\right)\left(1-y^2\right)}\,dz = 0$$

Dividing by $\sqrt{\left(1-x^2\right)\left(1-y^2\right)\left(1-z^2\right)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

15. If $z = f(x, y)$, where $x = e^u \cos v$, $y = e^u \sin v$, prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$$

Solution : Here z is a composite function of u and v.

$$\therefore \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$= e^u \cos v \frac{\partial f}{\partial x} + e^u \sin v \frac{\partial f}{\partial y}$$

and $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$= - e^u \sin v \frac{\partial f}{\partial x} + e^u \cos v \frac{\partial f}{\partial y}$$

$$\therefore \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = e^{2u}\cos^2 v\left(\frac{\partial f}{\partial x}\right)^2 + 2e^{2u}\cos v \sin v\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + e^{2u}\sin^2 v\left(\frac{\partial f}{\partial y}\right)^2$$

$$+ e^{2u}\sin^2 v\left(\frac{\partial f}{\partial x}\right)^2 - 2e^{2u}\cos v \sin v\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} + e^{2u}\cos^2 v\left(\frac{\partial f}{\partial y}\right)^2$$

$$= e^{2u}\left(\frac{\partial f}{\partial x}\right)^2 + e^{2u}\left(\frac{\partial f}{\partial y}\right)^2$$

$$= e^{2u}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$$

$$\therefore \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial \mathrm{V}}\right)^2\right]$$

16. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$, prove that

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$$

Where ϕ is a function of x, y, z.

Solution : Since ϕ is a function of x, y, z and x, y, z are functions of u, v, w. ϕ is a composite function of u, v, w.

$$\therefore \frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{\sqrt{w}}{2\sqrt{u}}\frac{\partial \phi}{\partial y} + \frac{\sqrt{v}}{2\sqrt{u}}\frac{\partial \phi}{\partial z}$$

$$\Rightarrow u\frac{\partial \phi}{\partial u} = \frac{\sqrt{uw}}{2}\frac{\partial \phi}{\partial y} + \frac{\sqrt{uv}}{2}\frac{\partial \phi}{\partial z} = \frac{1}{2}\left(y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}\right) \quad \text{-------- (1)}$$

Similarly $v\dfrac{\partial \phi}{\partial v} = \dfrac{1}{2}\left(x\dfrac{\partial \phi}{\partial x} + z\dfrac{\partial \phi}{\partial z}\right)$ -------- (2)

and $w\dfrac{\partial \phi}{\partial w} = \dfrac{1}{2}\left(x\dfrac{\partial \phi}{\partial x} + y\dfrac{\partial \phi}{\partial y}\right)$ -------- (3)

Adding (1), (2) and (3), we get

$$u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w} = x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z}$$

17. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, prove that

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \theta}\right)^2$$

where v is a function of x, y and z.

Solution : Since v is a function of x, y, z and x, y, z are functions of r, θ, ϕ; v is a compo function of r, θ, ϕ.

$$\therefore \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= \sin\theta\cos\phi\frac{\partial v}{\partial x} + \sin\theta\sin\phi\frac{\partial v}{\partial y} + \cos\theta\frac{\partial v}{\partial z}$$

$$\therefore \left(\frac{\partial v}{\partial x}\right)^2 = \sin^2\theta\cos^2\phi\left(\frac{\partial v}{\partial x}\right)^2 + \sin^2\theta\sin^2\phi\left(\frac{\partial v}{\partial y}\right)^2 + \cos^2\theta\left(\frac{\partial v}{\partial z}\right)^2 +$$

$$2\sin^2\theta\cos\phi\sin\phi\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + 2\sin\theta\cos\theta\cos\phi\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z} + 2\sin\theta\cos\theta\sin\phi\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} \quad \ldots\ldots (1$$

$$\text{Now } \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial \theta}$$

$$= r\cos\theta\cos\phi\frac{\partial v}{\partial x} + r\cos\theta\sin\phi\frac{\partial v}{\partial y} - r\sin\theta\frac{\partial v}{\partial z}$$

$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = \cos\theta\cos\phi\frac{\partial v}{\partial x} + \cos\theta\sin\phi\frac{\partial v}{\partial y} - \sin\theta\frac{\partial v}{\partial z}$$

$$\therefore \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)^2 = \cos^2\theta\cos^2\phi\left(\frac{\partial v}{\partial x}\right)^2 + \cos^2\theta\sin^2\phi\left(\frac{\partial v}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial v}{\partial z}\right)^2$$

$$+ 2\cos^2\theta\cos\phi\sin\phi\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} - 2\cos\theta\sin\theta\cos\phi\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z}$$

$$- 2\cos\theta\sin\theta\sin\phi\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} \quad \ldots\ldots\ldots (2)$$

$$\text{and } \frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= -r\sin\theta\sin\phi\frac{\partial v}{\partial x} + r\sin\theta\cos\phi\frac{\partial v}{\partial y}$$

$$\therefore \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} = -\sin\phi\frac{\partial v}{\partial x} + \cos\phi\frac{\partial v}{\partial y}$$

$$\therefore \left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\right)^2 = \sin^2\phi\left(\frac{\partial v}{\partial x}\right)^2 + \cos^2\phi\left(\frac{\partial v}{\partial y}\right)^2 - 2\sin\phi\cos\phi\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} \quad \text{.......... (3)}$$

Adding (1), (2) and (3), we get

$$\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\right)^2$$

$$= [\cos^2\phi + \sin^2\phi]\left(\frac{\partial v}{\partial x}\right)^2 + [\sin^2\phi + \cos^2\phi]\left(\frac{\partial v}{\partial y}\right)^2 + [\cos^2\theta + \sin^2\theta]\left(\frac{\partial v}{\partial z}\right)^2$$

$$+ [2\cos\phi\sin\phi - 2\sin\phi\cos\phi]\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}$$

$$= \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2$$

18. If $z = f(x, y)$, $x = e^u \sec v$, $y = e^u \tan v$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

Solution : Here z is a composite function of u and v.

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= e^u\sec v\frac{\partial z}{\partial x} + e^u\tan v\frac{\partial z}{\partial y}$$

$$\therefore \left(\frac{\partial z}{\partial u}\right)^2 = e^{2u}\left[\sec^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \tan^2 v\left(\frac{\partial z}{\partial y}\right)^2 + 2\sec v\tan v\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right] \quad \text{.......... (1)}$$

Now $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

$$= e^u\sec v\tan v\frac{\partial z}{\partial x} + e^u\sec^2 v\frac{\partial z}{\partial y}$$

$$\therefore \cos v \frac{\partial z}{\partial v} = e^u \left[\tan v \frac{\partial z}{\partial x} + \sec v \frac{\partial z}{\partial y} \right]$$

$$\therefore \left[\cos v \frac{\partial z}{\partial v}\right]^2 = e^{2u}\left[\tan^2 v \left(\frac{\partial z}{\partial x}\right)^2 + \sec^2 v \left(\frac{\partial z}{\partial y}\right)^2 + 2 \sec v \tan v \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}\right] \quad ...(2)$$

Subtracting (2) from (1), we get

$$\left(\frac{\partial z}{\partial u}\right)^2 - \left(\cos v \frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\sec^2 v - \tan^2 v\right)\left(\frac{\partial z}{\partial x}\right)^2 + \left(\tan^2 v - \sec^2 v\right)\left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$\therefore e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \left(\cos v \frac{\partial z}{\partial v}\right)^2\right] = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

19. If $w = (u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$, prove that $\frac{\partial w}{\partial v} = 0$ is equivalent to

$$(x + y) \frac{\partial w}{\partial x} + (x - y) \frac{\partial w}{\partial y} = 0.$$

Solution : Here w is a composite function of x and y.

$$\therefore \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$= (2x - 2y) \frac{\partial w}{\partial u}$$

and $$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$= (-2y - 2x) \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$

$$\therefore (x + y) \frac{\partial w}{\partial x} + (x - y) \frac{\partial w}{\partial y} = 2(x^2 - y^2) \frac{\partial w}{\partial u} - 2(x^2 - y^2) \frac{\partial w}{\partial u} + (x - y) \frac{\partial w}{\partial v}$$

$$= (x - y) \frac{\partial w}{\partial v}$$

$$\therefore \frac{\partial w}{\partial v} = 0 \Leftrightarrow (x + y) \frac{\partial w}{\partial x} + (x - y) \frac{\partial w}{\partial y} = 0.$$

20. If $u = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right)$, prove that

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$

Solution : Let $v = \dfrac{y-x}{xy}$, $w = \dfrac{z-x}{xz}$

$\therefore$ $u = f(v, w)$ is a composite function of x, y, z.

Now $v = \dfrac{1}{x} - \dfrac{1}{y}$ and $w = \dfrac{1}{x} - \dfrac{1}{z}$

$$\therefore \frac{\partial v}{\partial x} = -\frac{1}{x^2}, \frac{\partial v}{\partial y} = \frac{1}{y^2}, \frac{\partial v}{\partial z} = 0$$

and $\dfrac{\partial w}{\partial x} = -\dfrac{1}{x^2}, \dfrac{\partial w}{\partial y} = 0, \dfrac{\partial w}{\partial z} = \dfrac{1}{z^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = -\frac{1}{x^2}\frac{\partial u}{\partial v} - \frac{1}{x^2}\frac{\partial u}{\partial w}$$

$$\therefore x^2 \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \qquad \text{.......... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = \frac{1}{y^2}\frac{\partial u}{\partial v}$$

$$\therefore y^2 \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \qquad \text{.......... (2)}$$

and $\dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial v}\dfrac{\partial v}{\partial z} + \dfrac{\partial u}{\partial w}\dfrac{\partial w}{\partial z} = \dfrac{1}{z^2}\dfrac{\partial u}{\partial w}$

$$\therefore z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \qquad \text{.......... (3)}$$

Adding (1), (2) and (3), we get

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial w} = 0.$$

21. If $z = f(u, v)$, $u = lx + my$, $v = ly - mx$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(l^2 + m^2\right)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Solution : Here z is a composite function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = l\,\frac{\partial z}{\partial u} - m\,\frac{\partial z}{\partial v}$$

$$\therefore \quad \frac{\partial}{\partial x} = l\,\frac{\partial}{\partial u} - m\,\frac{\partial}{\partial v}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(l\,\frac{\partial}{\partial u} - m\,\frac{\partial}{\partial v}\right)\left(l\,\frac{\partial z}{\partial u} - m\,\frac{\partial z}{\partial v}\right)$$

$$= l\,\frac{\partial}{\partial u}\left(l\,\frac{\partial z}{\partial u} - m\,\frac{\partial z}{\partial v}\right) - m\,\frac{\partial}{\partial v}\left(l\,\frac{\partial z}{\partial u} - m\,\frac{\partial z}{\partial v}\right)$$

$$= l^2\,\frac{\partial^2 z}{\partial u^2} - 2lm\,\frac{\partial^2 z}{\partial u \partial v} + m^2\,\frac{\partial^2 z}{\partial v^2} \qquad \text{.......... (1)}$$

Now $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = m\,\dfrac{\partial z}{\partial u} + l\,\dfrac{\partial z}{\partial v}$

$$\therefore \quad \frac{\partial}{\partial y} = m\,\frac{\partial}{\partial u} + l\,\frac{\partial}{\partial v}$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(m\,\frac{\partial}{\partial u} + l\,\frac{\partial}{\partial v}\right)\left(m\,\frac{\partial z}{\partial u} + l\,\frac{\partial z}{\partial v}\right)$$

$$= m\,\frac{\partial}{\partial u}\left(m\,\frac{\partial z}{\partial u} + l\,\frac{\partial z}{\partial v}\right) + l\,\frac{\partial}{\partial v}\left(m\,\frac{\partial z}{\partial u} + l\,\frac{\partial z}{\partial v}\right)$$

$$= m^2\,\frac{\partial^2 z}{\partial u^2} + 2lm\,\frac{\partial^2 z}{\partial u \partial v} + l^2\,\frac{\partial^2 z}{\partial v^2} \qquad \text{.......... (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(l^2 + m^2\right)\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$$

22. If $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, prove that

(i) $\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2$

(ii) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}.$

Solution : Here z is a composite function of u and v.

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \cos\alpha \frac{\partial z}{\partial x} + \sin\alpha \frac{\partial z}{\partial y} \qquad \text{..........} (1)$$

$$\therefore \frac{\partial}{\partial u} = \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}$$

$$\therefore \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y}\right)\left(\cos\alpha \frac{\partial z}{\partial x} + \sin\alpha \frac{\partial z}{\partial y}\right)$$

$$= \cos^2\alpha \frac{\partial^2 z}{\partial x^2} + 2\cos\alpha \sin\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 z}{\partial y^2} \qquad \text{..........} (2)$$

$$\text{Now } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y} \qquad \text{..........} (3)$$

$$\therefore \frac{\partial}{\partial v} = -\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}$$

$$\therefore \frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\sin\alpha \frac{\partial z}{\partial x} + \cos\alpha \frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha \frac{\partial^2 z}{\partial x^2} - 2\sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 z}{\partial y^2} \qquad \text{..........} (4)$$

Now from (1),

$$\left(\frac{\partial z}{\partial u}\right)^2 = \cos^2\alpha \left(\frac{\partial z}{\partial x}\right)^2 + \sin^2\alpha \left(\frac{\partial z}{\partial y}\right)^2 + 2\sin\alpha \cos\alpha \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \qquad \text{..........} (5)$$

From (3)

$$\left(\frac{\partial z}{\partial v}\right)^2 = \sin^2\alpha \left(\frac{\partial z}{\partial x}\right)^2 + \cos^2\alpha \left(\frac{\partial z}{\partial y}\right)^2 - 2\sin\alpha \cos\alpha \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \qquad \text{..........} (6)$$

Adding (5) and (6), we get

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \qquad \text{(Result (i))}$$

Now adding (2) and (4), we get

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

23. If $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Solution : We have $u = f(r)$,

and $x = r\cos\theta$, $y = r\sin\theta \Rightarrow r = \sqrt{x^2 + y^2}$

$\therefore$ u is a composite function of x and y.

Again $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$

Now $\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) - \frac{x}{r^2} f'(r) \frac{\partial r}{\partial x} + \frac{x}{r} f''(r) \frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \qquad \text{.......... (1)}$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$ .... (2)

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r} f'(r) - \frac{\left(x^2 + y^2\right)}{r^3} f'(r) + \frac{\left(x^2 + y^2\right)}{r^2} f''(r)$$

$$= \frac{2}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) \qquad (\because x^2 + y^2 = r^2)$$

$$= \frac{1}{r} f'(r) + f''(r).$$

24. Transform two dimensional Laplace equation into polar form.

Solution : The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Take $x = r\cos\theta$, $y = r\sin\theta \Rightarrow r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$$

and $$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Now consider u as a function of r, θ and r, θ as a function of x and y. Thus u is a composite function of x, y.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\therefore \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos\theta\left\{\cos\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right)\right\}$$

$$-\frac{\sin\theta}{r}\left\{\cos\theta\frac{\partial^2 u}{\partial r\partial\theta} - \sin\theta\frac{\partial u}{\partial r} - \frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial\theta} + \sin\theta\frac{\partial^2 u}{\partial\theta^2}\right)\right\}$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial r}{\partial\theta}.\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$$

$$\therefore \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta \left\{\sin\theta \frac{\partial^2 u}{\partial r^2} + \cos\theta\left(-\frac{1}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial \theta}\right)\right\}$$

$$+ \frac{\cos\theta}{r}\left\{\cos\theta \frac{\partial u}{\partial r} + \sin\theta \frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r}\left(-\sin\theta \frac{\partial u}{\partial \theta} + \cos\theta \frac{\partial^2 u}{\partial \theta^2}\right)\right\}$$

$$= \sin^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}$$

25. If $z = f(x, y)$, $u = e^x$, $v = e^y$, prove that $\dfrac{\partial^2 z}{\partial x \partial y} = uv \dfrac{\partial^2 z}{\partial u \partial v}$

Solution : Let $z = f(x, y) = F(u, v)$

$\therefore$ z is a composite function of x and y.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = e^x \frac{\partial z}{\partial u} = u\frac{\partial z}{\partial u}$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\frac{\partial v}{\partial y}$$

$$= e^y \frac{\partial}{\partial v}\left(u\frac{\partial z}{\partial u}\right)$$

$$= v\left(u\frac{\partial^2 z}{\partial v \partial u}\right) = uv\frac{\partial^2 z}{\partial v \partial u}$$

or : We have $\dfrac{\partial z}{\partial x} = u\dfrac{\partial z}{\partial u} \Rightarrow \dfrac{\partial}{\partial x} = u\dfrac{\partial}{\partial v}$

Also $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = e^y\dfrac{\partial z}{\partial v} = v\dfrac{\partial z}{\partial v}$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = u\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial v}\right)$$

$$= u\left[0 + v\frac{\partial^2 z}{\partial u \partial v}\right] = uv\frac{\partial^2 z}{\partial u \partial v}$$

26. If $u = e^x \cos y$, $v = e^x \sin y$, prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(u^2 + v^2\right)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

where f is a function of u and v.

Solution : Here f is a composite function of x and y.

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = e^x \cos y \frac{\partial f}{\partial u} + e^x \sin y \frac{\partial f}{\partial v}$$

$$= u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$$

$$\therefore \frac{\partial}{\partial x} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right)$$

$$= u\frac{\partial}{\partial u}\left(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right) + v\frac{\partial}{\partial v}\left(u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right)$$

$$= u\left\{\frac{\partial f}{\partial u} + u\frac{\partial^2 f}{\partial u^2} + v\frac{\partial^2 f}{\partial u \partial v}\right\} + v\left\{u\frac{\partial^2 f}{\partial v \partial u} + \frac{\partial f}{\partial v} + v\frac{\partial^2 f}{\partial v^2}\right\}$$

$$= u\frac{\partial f}{\partial u} + u^2\frac{\partial^2 f}{\partial u^2} + 2uv\frac{\partial^2 f}{\partial u \partial v} + v\frac{\partial f}{\partial v} + v^2\frac{\partial^2 f}{\partial v^2} \quad \text{..........} \quad (1)$$

Now $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = -e^x \sin y \frac{\partial f}{\partial u} + e^x \cos y \frac{\partial f}{\partial v}$

$$= -v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}$$

$$\therefore \frac{\partial}{\partial y} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)\left(-v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}\right)$$

$$= -v\frac{\partial}{\partial u}\left(-v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}\right) + u\frac{\partial}{\partial v}\left(-v\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}\right)$$

$$= -v\left\{-v\frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial v} + u\frac{\partial^2 f}{\partial u \partial v}\right\} + u\left\{-\frac{\partial f}{\partial u} - v\frac{\partial^2 f}{\partial v \partial u} + u\frac{\partial^2 f}{\partial v^2}\right\}$$

$$= v^2\frac{\partial^2 f}{\partial u^2} - v\frac{\partial f}{\partial u} - 2uv\frac{\partial^2 f}{\partial u \partial v} - u\frac{\partial f}{\partial u} + u^2\frac{\partial^2 f}{\partial v^2} \quad \text{..........} \quad (2)$$

$\therefore$ Adding (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2 + v^2) \frac{\partial^2 f}{\partial u^2} + (u^2 + v^2) \frac{\partial^2 f}{\partial v^2}$$

$$= (u^2 + v^2) \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right)$$

Note : Here $u^2 + v^2 = e^{2x} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{2x} \left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right)$

27. If $u = x^2 - y^2$, $v = 2xy$ and $f(x, y) = \phi(u, v)$, prove that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4 \left(x^2 + y^2 \right) \left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right)$$

Solution : We have $u = x^2 - y^2$, $v = 2xy$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 2x \frac{\partial f}{\partial u} + 2y \frac{\partial f}{\partial v}$$

$u = x^2 - y^2$ and $v = 2xy \Rightarrow$ Differentiating w.r.t. u, we get

$$\left. \begin{aligned} 1 &= 2x \frac{\partial x}{\partial u} - 2y \frac{\partial y}{\partial u} \\ 0 &= 2x \frac{\partial x}{\partial v} - 2y \frac{\partial y}{\partial v} \end{aligned} \right\} \Rightarrow \frac{\partial x}{\partial u} = \frac{x}{2x^2 + 2y^2}, \frac{\partial y}{\partial u} = -\frac{y}{2\left(x^2 + y^2\right)}$$

Again differentiating w.r.t. v, we get

$$\left. \begin{aligned} 0 &= 2x \frac{\partial x}{\partial v} - 2y \frac{\partial y}{\partial v} \\ 1 &= 2x \frac{\partial x}{\partial v} + 2y \frac{\partial y}{\partial v} \end{aligned} \right\} \Rightarrow \frac{\partial x}{\partial v} = \frac{y}{2x^2 + 2y^2}, \frac{\partial y}{\partial v} = \frac{y}{2x^2 + 2y^2}$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2 \left(x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v} \right) \cdot 2 \left(x \frac{\partial f}{\partial u} + y \frac{\partial f}{\partial v} \right)$$

$$= 4 \left[x \left\{ \frac{\partial x}{\partial u} \frac{\partial f}{\partial u} + x \frac{\partial^2 f}{\partial u^2} + \frac{\partial y}{\partial u} \frac{\partial f}{\partial v} + y \frac{\partial^2 f}{\partial u \partial v} \right\} \right.$$

$$\left. + y \left\{ \frac{\partial x}{\partial v} \frac{\partial f}{\partial u} + x \frac{\partial^2 f}{\partial v \partial u} + \frac{\partial y}{\partial v} \frac{\partial f}{\partial v} + y \frac{\partial^2 f}{\partial v^2} \right\} \right]$$

$$= 4\left[\left(x\frac{\partial x}{\partial u} + y\frac{\partial x}{\partial v}\right)\frac{\partial f}{\partial u} + 2xy\frac{\partial^2 f}{\partial u \partial v}\right.$$

$$\left. + x^2\frac{\partial^2 f}{\partial u^2} + \left(x\frac{\partial y}{\partial u} + y\frac{\partial y}{\partial v}\right)\frac{\partial f}{\partial v} + y^2\frac{\partial^2 f}{\partial v^2}\right]$$

$$= 4\left[\frac{1}{2}\frac{\partial f}{\partial u} + 2xy\frac{\partial^2 f}{\partial u \partial v} + x^2\frac{\partial^2 f}{\partial u^2} + y^2\frac{\partial^2 f}{\partial v^2}\right] \quad \text{.......... (1)}$$

Now $\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y} = -2y\dfrac{\partial f}{\partial u} + 2x\dfrac{\partial f}{\partial v}$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2\left(-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right)2\left(-y\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)$$

$$= 4\left[-y\left\{-\frac{\partial y}{\partial u}\frac{\partial f}{\partial u} - y\frac{\partial^2 f}{\partial u^2} + \frac{\partial x}{\partial u}\frac{\partial f}{\partial v} + x\frac{\partial^2 f}{\partial u \partial v}\right\}\right.$$

$$\left. + x\left\{-\frac{\partial y}{\partial v}\frac{\partial f}{\partial u} - y\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial x}{\partial v}\frac{\partial f}{\partial v} + x\frac{\partial^2 f}{\partial v^2}\right\}\right]$$

$$= 4\left[\left(y\frac{\partial y}{\partial u} - x\frac{\partial y}{\partial v}\right)\frac{\partial f}{\partial u} + y^2\frac{\partial^2 f}{\partial u^2} + \left(-y\frac{\partial x}{\partial u} + x\frac{\partial x}{\partial v}\right)\frac{\partial f}{\partial v}\right.$$

$$\left. - 2xy\frac{\partial^2 f}{\partial u \partial v} + x^2\frac{\partial^2 f}{\partial v^2}\right]$$

$$= 4\left[-\frac{1}{2}\frac{\partial f}{\partial u} + y^2\frac{\partial^2 f}{\partial u^2} - 2xy\frac{\partial^2 f}{\partial u \partial v} + x^2\frac{\partial^2 f}{\partial v^2}\right] \quad \text{.......... (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\left[\left(x^2 + y^2\right)\frac{\partial^2 f}{\partial u^2} + \left(x^2 + y^2\right)\frac{\partial^2 f}{\partial v^2}\right]$$

$$= 4\left(x^2 + y^2\right)\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right] = 4\left(x^2 + y^2\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

28. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$.

Solution : We have

$$f(x, y) = x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}$$

$$= -\frac{x^2 + 2xy + y^2}{x^2 + 4xy + y^2}.$$

29. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution : We have

$$f(x, y) = x^3 + y^3 - 3axy = 0$$

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax$$

$$f_{xx} = 6x,\ f_{xy} = -3a,\ f_{yy} = 6y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = -\frac{x^2 - ay}{y^2 - ax}$$

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}\, f_y^2 - 2\, f_x f_y f_{xy} + f_{yy}\, f_x^2}{f_y^3}$$

$$= -\frac{6x\,(3y^2 - 3ax)^2 - 2\,(3x^2 - 3ay)\,(3y^2 - 3ax)\,(-3a) + 6y\,(3x^2 - 3ay)^2}{(3y^2 - 3ax)^3}$$

$$\text{Numerator} = 54x\,(y^4 - 2axy^2 + a^2x^2) + 54a\,(x^2y^2 - ax^3 - ay^3 + a^2xy) + 54y\,(x^4 - 2ax^2y + a^2y^2)$$

$$= 54\,[xy^4 - 2ax^2y^2 + a^2x^3 + ax^2y^2 - a^2x^3 - a^2y^3 + a^3xy + x^4y - 2ax^2y^2 + a^2y^3]$$

$$= 54\,[xy^4 - 3ax^2y^2 + x^4y + a^3xy]$$

$$= 54\,[xy\,(y^3 - 3axy + x^3) + a^3xy]$$

$$= 54\,a^3xy \qquad (\because x^3 + y^3 - 3axy = 0)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{54a^3xy}{27\,(y^2 - ax)^3} = -\frac{2a^3xy}{(y^2 - ax)^3}$$

30. If $(\cos x)^y = (\sin y)^x$, find $\dfrac{dy}{dx}$

Solution : We have $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$\therefore f_x = y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log \sin y$

and $f_y = (\cos x)^y \log \cos x - x(\sin y)^{x-1}(\cos y)$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{y \sin x(\cos x)^{y-1} + (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x - x\cos y(\sin y)^{x-1}}$$

$$= \frac{y\dfrac{\sin x}{\cos x}(\sin y)^x + (\sin y)^x \log \sin y}{(\sin y)^x \log \cos x - x\dfrac{\cos y}{\sin y}(\sin y)^x} \qquad (\because (\cos x)^y = (\sin y)^x)$$

$$= \frac{y\tan x + \log \sin y}{\log \cos x - x\cot y}$$

31. If $z = \sqrt{x^2 + y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, find $\dfrac{dz}{dx}$, when $x = y = a$.

Solution : We have $f(x, y) = x^3 + y^3 + 3axy - 5a^2 = 0$.

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 + 3ay}{3y^2 + 3ax} = -\frac{x^2 + ay}{y^2 + ax}$$

Now $z = \sqrt{x^2 + y^2}$ and $f(x, y) = c$

$\Rightarrow z$ is a composite function of x.

$$\therefore \frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$= \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$$

Now at $x = y = a$, we have

$$\frac{dz}{dx} = \frac{a}{\sqrt{a^2 + a^2}} - \frac{a}{\sqrt{a^2 + a^2}} \cdot \frac{a^2 + a^2}{a^2 + a^2}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

32. If $\phi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z = -1$

Solution : Keeping x as a constant, we have

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\phi_z}{\phi_y}$$

Similarly, we get

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\phi_x}{\phi_z} \text{ and } \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\phi_y}{\phi_x}$$

$$\therefore \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = \left(-\frac{\phi_z}{\phi_y}\right)\left(-\frac{\phi_x}{\phi_z}\right)\left(-\frac{\phi_y}{\phi_x}\right) = -1$$

33. If $f(x, y) = 0$, $\phi(y, z) = 0$, prove that

$$\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial y}$$

Solution : We have $f(x, y) = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

and $\phi(y, z) = 0$

$$\Rightarrow \frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$$

Now $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \left(-\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}\right)\left(-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)$

$$\Rightarrow \frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial x}$$

34. If $u = x\log(xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Solution : We have $f(x, y) = x^3 + y^3 + 3xy - 1 = 0$.

$$\therefore \frac{dy}{dx} = -\frac{fx}{fy} = -\frac{3x^2 + 3y}{3y^2 + 3x} = -\frac{x^2 + y}{y^2 + x}$$

Now $u = x\log(xy)$ and $f(x, y) = 0$

$\Rightarrow u$ is a composite function of x.

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \left(\log(xy) + x\frac{1}{xy}\cdot y\right) + x\frac{1}{xy}\,x\left(-\frac{x^2 + y}{y^2 + x}\right)$$

$$= \log(xy) + 1 - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)$$

35. If $x^y = y^x$, find $\frac{dy}{dx}$.

Solution : We have $f(x, y) = x^y - y^x = 0$

$$\therefore \; f_x = yx^{y-1} - y^x\log y = \frac{y}{x}x^y - x^y\log y \qquad (\because x^y = y^x)$$

and $f_y = x^y\log x - xy^{x-1} = x^y\log x - \frac{x}{y}x^y$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\frac{y}{x}x^y - x^y\log y}{x^y\log x - \frac{x}{y}x^y} = -\frac{y - x\log y}{y\log x - x}$$

$$= \frac{x\log y - y}{x\log x - x}$$

EXERCISE – 3

1. If $z = f(x, y)$, $x = u^2 - v^2$, $y = v^2 - u^2$, prove that $u \dfrac{\partial z}{\partial v} + v \dfrac{\partial z}{\partial u} = 0$.

2. If $z = u^2 + v^2$, $x = u^2 - v^2$, $y = uv$, prove that $\dfrac{\partial z}{\partial x} = \dfrac{x}{z}$.

3. If $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$, prove that

$$\left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 - \sin^2 v \left(\frac{\partial \phi}{\partial v}\right)^2\right]$$

where ϕ is a function of x and y.

4. If $z = f(x, y)$, $x = u \cosh v$, $y = u \sinh v$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2.$$

5. If $z = f(x, y)$, $x = e^r \cos\theta$, $y = e^r \sin\theta$, prove that $y \dfrac{\partial z}{\partial r} + x \dfrac{\partial z}{\partial \theta} = e^{2r} \dfrac{\partial z}{\partial y}$.

6. If $u = x - at$, $v = x + at$, prove that $a^2 \dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial t^2} = 4a^2 \dfrac{\partial^2 z}{\partial u \partial t}$

where z is a function of u and v.

7. If $x = \dfrac{\cos\theta}{r}$, $y = \dfrac{\sin\theta}{r}$ and $z = f(x, y)$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = r^4 \frac{\partial^2 z}{\partial r^2} + r^3 \frac{\partial z}{\partial r} + r^4 \frac{\partial^2 z}{\partial \theta^2}.$$

8. If $x = u + v$, $y = uv$ and $z = f(x, y)$, prove that $u \dfrac{\partial z}{\partial u} + v \dfrac{\partial z}{\partial v} = x \dfrac{\partial z}{\partial x} + 2y \dfrac{\partial z}{\partial y}$.

9. If $u = x \cos\theta - y \sin\theta$, $v = x \sin\theta + y \cos\theta$ and $z = f(u, v)$, and θ is a constant, the prove that $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = u \dfrac{\partial z}{\partial u} + v \dfrac{\partial z}{\partial v}$.

10. If $z = f(x, y)$, $x = uv$, $y = \dfrac{u+v}{u-v}$, prove that $u \dfrac{\partial z}{\partial u} + v \dfrac{\partial z}{\partial v} = 2x \dfrac{\partial z}{\partial x}$.

11. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\dfrac{1}{x}\dfrac{\partial u}{\partial x} + \dfrac{1}{y}\dfrac{\partial u}{\partial y} + \dfrac{1}{z}\dfrac{\partial u}{\partial z} = 0$.

12. If $u = f(r)$, where $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.

13. If $x^x + y^y = a^b$, find $\frac{dy}{dx}$.

$$\textbf{Ans. :} \; -\frac{yx^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}$$

14. If $x^3 - 3ax^2 + y^3 = 0$, prove that $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

15. If $z = x^2y$ and $x^2 + xy + y^2 = 1$, prove that $\frac{dz}{dx} = 2xy - \frac{x^2(2x+y)}{x+2y}$

16. If $x^4 + y^4 = 5a^2xy$, find $\frac{dy}{dx}$.

$$\textbf{Ans. :} \; \frac{5a^2y - 4x^3}{4y^3 - 5a^2x}$$

17. If $x^3 + y^3 = 3ax^2$, find $\frac{d^2y}{dx^2}$.

$$\textbf{Ans. :} \; -\frac{2a^2x^2}{y^5}$$

18. If $f(x, y) = 0$, $\phi(x, z) = 0$, prove that $\frac{\partial \phi}{\partial x} \cdot \frac{\partial f}{\partial y} \cdot \frac{dy}{dz} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial z}$

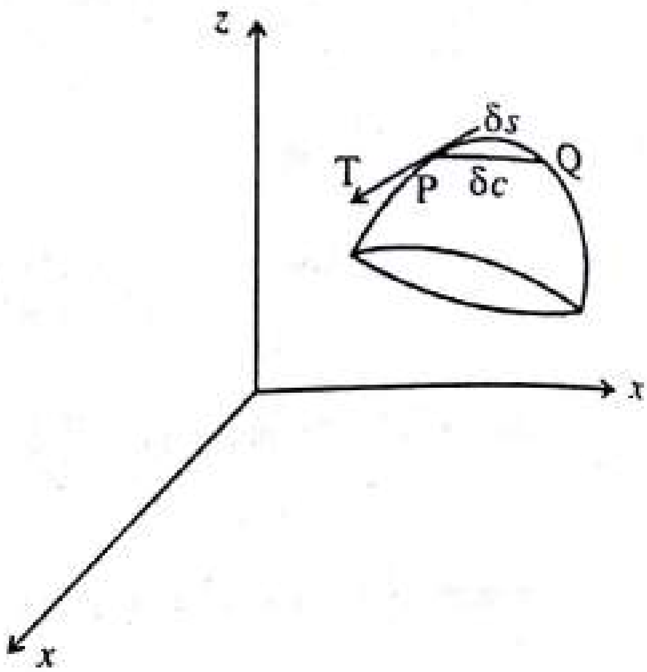
19. If $x^m + y^m = a^m$, find $\frac{d^2y}{dx^2}$.

$$\textbf{Ans. :} \; -(m-1)\, a^m \frac{x^{m-2}}{y^{2m-1}}$$

20. If $u = \phi(x, y)$ and $\psi(x, y) = 0$, show that $\frac{du}{dx} = \frac{\phi_x \psi_y - \phi_y \psi_x}{\psi_y}$.

1.10 TANGENT PLANE AND NORMAL LINE :

Let $F(x, y, z) = c$ be the equation of a surface. Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface. Let acr PQ = δs and chord PQ = δc.



[**Note :** The direction ratios of a line joining two point $A(x_1, y_1)$ and $B(x_2, y_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$, and direction cosines are $\frac{x_2 - x_1}{AB}$, $\frac{y_2 - y_1}{AB}$, $\frac{z_2 - z_1}{AB}$, where AB is the distance between A and B. If the line AB makes an angles α, β, γ with x, y and z axes respectively then

$$\cos\alpha = \frac{x_2 - x_1}{AB}, \cos\beta = \frac{y_2 - y_1}{AB}, \cos\gamma = \frac{z_2 - z_1}{AB},]$$

Now the direction cosines of the chord PQ are

$$\frac{\delta x}{\delta c}, \frac{\delta y}{\delta c}, \frac{\delta z}{\delta c}$$

$$\text{or } \frac{\delta x}{\delta s} \cdot \frac{\delta s}{\delta c}, \frac{\delta y}{\delta s} \cdot \frac{\delta s}{\delta c}, \frac{\delta z}{\delta s} \cdot \frac{\delta s}{\delta c}.$$

When Q → P, we have $\frac{\delta s}{\delta c} = 1$, and the chord PQ tends to a tangent line PT, whose direction cosines are now,

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \quad \text{..........} \quad (1)$$

Again, we have $F(x, y, z) = c$. Differentiating w.r.t. s, we get

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0 \quad \text{..........} \quad (2)$$

[**Note :** If the direction cosines of two lines are l_1, m_1, n_1; and l_2, m_2, n_2 then the two lines are perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$]

From (1) and (2) it is clear that the tangent line PT whose direction cosines are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ is perpendicular to the line having direction ratios $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$, which is known as a normal line to the surface at the point P(x, y).

But the point Q can be joined with P by number of curves in number of direction. All these curves gives number of tangent lines at a point P to the surface which lies in a plane called the tangent plane. Thus the equation of the tangent plane to the surface at P is given by

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0$$

Where (X, Y, Z) is any point on the tangent plane.

Again the equation of the normal line to the surface at the point P is given by

$$\frac{X - x}{\frac{\partial F}{\partial x}} = \frac{Y - y}{\frac{\partial F}{\partial y}} = \frac{Z - z}{\frac{\partial F}{\partial z}}$$

Directional Derivatives :

We know that if $f(x, y)$ is a differentiable function then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the partial derivatives in the directions of x and y axes respectively. But the function can have derivatives in any direction and can be expressed in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, which is known as directional derivatives. The derivative in the direction of the line which makes an angle α with positive x – axis is denoted by $D_{(\alpha)} f(x, y)$ and is defined as

$$D_{(\alpha)} f(x, y) = \cos\alpha\, f_x + \sin\alpha\, f_y.$$

Similarly for the function $f(x, y, z)$ of three variable x, y and z, the directional derivative is given by

$$D_{(\alpha)} f(x, y, z) = \cos\alpha\, f_x + \cos\beta f_y + \cos\gamma\, f_z$$

where α, β, γ are angles made with the x, y and z axes by a directed line.

SOLVED EXAMPLES

1. Find the equations of the tangent plane and normal line to the surface $x^2 + 2y^2 + 3z^2 = 12$ at $(1, 2, -1)$.

Solution : We have

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 - 12 = 0$$

$$\therefore \frac{\partial F}{\partial x} = 2x, \frac{\partial F}{\partial y} = 4y, \frac{\partial F}{\partial z} = 6z$$

$\therefore$ At the point $(1, 2, -1)$,

$$\frac{\partial F}{\partial x} = 2, \frac{\partial F}{\partial y} = 8, \frac{\partial F}{\partial z} = -6$$

$\therefore$ The equation of the tangent plane at $(1, 2, -1)$ is

$$2(x-1) + 8(y-2) + (z+1)(-6) = 0$$

$$\Rightarrow 2x + 8y - 6z = 24$$

$$\Rightarrow x + 4y - 3z = 12$$

and equation of a normal line at $(1, 2, -1)$ is

$$\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$$

2. Find the equations of the tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

Solution : We have $F(x, y, z) = 2xz^2 - 3xy - 4x - 7 = 0$

$$\therefore \frac{\partial F}{\partial x} = 2z^2 - 3y - 4, \frac{\partial F}{\partial y} = -3x, \frac{\partial F}{\partial z} = 4xz$$

$\therefore$ At the point $(1, -1, 2)$,

$$\frac{\partial F}{\partial x} = 7, \frac{\partial F}{\partial y} = -3, \frac{\partial F}{\partial z} = 8$$

$\therefore$ The equation of the tangent plane at $(1, -1, 2)$ is

$$7(x-1) - 3(y+1) + 8(z-2) = 0$$

$$\Rightarrow 7x - 3y + 8z = 26$$

and the equation of the normal line at $(1, -1, 2)$ is

$$\frac{x-1}{7} = \frac{y+1}{-3} = \frac{z-2}{8}$$

3. Prove that the surface $x^2 + 2yz + y^3 = 4$ is perpendicular to the surface $x^2 + 1 = (2 - 4a)y^2 + az^2$ at the point of intersection.

Solution : We have the surfaces are,

$$f(x, y, z) = x^2 - 2yz + y^3 - 4 = 0$$

and $F(x, y, z) = x^2 + 1 - (2 - 4a)y^2 - az^2 = 0$

$$\therefore \frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = -2z + 3y^2,\ \frac{\partial f}{\partial z} = -2y$$

and $\dfrac{\partial f}{\partial x} = 2x,\ \dfrac{\partial f}{\partial y} = -2(2 - 4a)y,\ \dfrac{\partial f}{\partial z} = -2az$

$\therefore$ At the point $(1, -1, 2)$,

$$\frac{\partial f}{\partial x} = 2,\ \frac{\partial f}{\partial y} = -1,\ \frac{\partial f}{\partial z} = 2$$

and $\dfrac{\partial F}{\partial x} = 2,\ \dfrac{\partial F}{\partial y} = 2(2 - 4a),\ \dfrac{\partial F}{\partial z} = -4a$

$\therefore$ The direction ratios of the surface $f(x, y, z) = 0$ are $2, -1, 2$ and that of the surface $F(x, y, z) = 0$ are $2, 2(2 - 4a), -4a$.

Now $(2)(2) + (-1)(4 - 8a) + (2)(-4a)$

$$= 4 - 4 + 8a - 8a = 0 \qquad \text{[See the note in Sec. 1.9]}$$

$\therefore$ The two surfaces are perpendicular.

4. If the plane $3x + 12y - 6z = 17$ touches the conicoid $3x^2 - 6y^2 + 9z^2 = -17$, find the point of contact.

Solution : Let $F(x, y, z) = 3x^2 - 6y^2 + 9z^2 + 17 = 0$

$$\therefore \frac{\partial F}{\partial x} = 6x,\ \frac{\partial F}{\partial y} = -12y,\ \frac{\partial F}{\partial z} = 18z$$

Suppose that (x_0, y_0, z_0) is the point of contact.

$\therefore$ At (x_0, y_0, z_0),

$$\frac{\partial F}{\partial x} = 6x_0,\ \frac{\partial F}{\partial y} = -12y_0,\ \frac{\partial F}{\partial z} = 18z_0$$

The equation of the tangent plane at (x_0, y_0, z_0) is

$$6x_0(x - x_0) - 12y_0(y - y_0) + 18z_0(z - z_0) = 0$$

$$\Rightarrow 6x_0x - 12y_0y + 18z_0z = 6x_0^2 - 12y_0^2 + 18z_0^2$$

$$= 2(3x_0^2 - 6y_0^2 + 9z_0^2)$$

$$= -34$$

$\therefore \quad 3x_0x - 6y_0y + 9z_0z = -17$

$\Rightarrow \quad -3x_0x + 6y_0y - 9z_0z = 17$

$\therefore$ The direction ratios of the tangent plane at (x_0, y_0, z_0) are $-3x_0, 6y_0, -9z_0$ ...(1)

Again the plane $3x + 12y - 6z = 17$ touch the surface. .... (2)

$\therefore$ At this point direction ratio are $3, 12, -6$

$\therefore$ From (1) and (2),

$-3x_0 = 3, 6y_0 = 12, -9z_0 = -6$

$\Rightarrow x_0 = -1, \ y_0 = 2, \ z_0 = \frac{2}{3}$

$\therefore$ The required point is $\left(-1, 2, \frac{2}{3}\right)$.

5. Find $D_{(\alpha)} f(x_0, y_0)$, $\alpha = 0, 30^{\circ}$ for the following functions :

(a) $f(x, y) = ax + by$, a, b are constants, $x_0 = y_0 = 0$.

(b) $f(x, y) = x^2 - y^2$, $x_0 = 1$, $y_0 = 2$.

Solution : (a) We have $f(x, y) = ax + by$

$\therefore \ f_x = a, f_y = b$

(i) For $\alpha = 0$

$$D_{(\alpha)} f(x_0, y_0) = f_x \cos\alpha + f_y \sin\alpha$$
$$= a$$

(ii) For $\alpha = 30^{\circ}$

$$D_{(\alpha)} f(x_0, y_0) = a\frac{\sqrt{3}}{2} + b\frac{1}{2} = \frac{\sqrt{3}\,a + b}{2}$$

(b) We have $f(x, y) = x^2 - y^2$

$\therefore \ f_x = 2x, f_y = -2y$

$\therefore$ At $(1, 2) \Rightarrow f_x = 2, f_y = -4$.

(i) For $\alpha = 0$

$$D_{(\alpha)}(1, 2) = f_x \cos\alpha + f_y \sin\alpha$$
$$= 2$$

(ii) For $\alpha = 30^{\circ}$

$$D_{(\alpha)}(1, 2) = 2\frac{\sqrt{3}}{2} - 4\frac{1}{2} = \sqrt{3} - 2$$

6. **Find the directional derivative of the function $xyz - xy - yz - zx + x + y + z$ at (2, 2, 1) in the direction of (2, 2, 0).**

Solution : We have $f(x, y, z) = xyz - xy - yz - zx + x + y + z$

$\therefore f_x = yz - y - z + 1,\ f_y = xz - x - z + 1,\ f_z = xy - y - x + 1$

$\therefore$ At the point (2, 2, 1), we have

$f_x = 0,\ f_y = 0,\ f_z = 1.$

The direction ratios in the direction of (2, 2, 0) are : 0, 0, – 1.

$\therefore \cos\alpha = 0,\ \cos\beta = 0,\ \cos\gamma = -1$

$$\therefore D_{(\alpha)} f(2, 2, 1) = \cos\alpha\, f_x + \cos\beta\, f_y + \cos\gamma\, f_z$$
$$= (-1)(1) = -1$$

EXERCISE – 4

1. Find the equations of the tangent plane and the normal line to the following surfaces :

(a) $xyz = 6$ at (1, 2, 3).

Ans. : $6x + 3y + 2z = 18;\ \dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-3}{2}$

(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{25} = 1$ at (2, 3, 5)

Ans. : $15x + 10y - 6z = 30;\ \dfrac{x-2}{15} = \dfrac{y-3}{10} = \dfrac{z-5}{-6}$

(c) $2x^2 + y^2 + 2z = 3$ at (2, 1, – 3)

Ans. : $4x + y + z = 6;\ \dfrac{x-2}{4} = \dfrac{y-1}{1} = \dfrac{z+3}{1}$

2. Find $D_{(\alpha)} f(x_0, y_0)$, $\alpha = 60^o, 90^o$ for the following functions :

(a) $f(x, y) = e^x \cos y,\ x_0 = 0,\ y_0 = \pi.$ **Ans. :** $-\dfrac{1}{2}, 0$

(b) $f(x, y) = \cos(x + y),\ x_0 = 0,\ y_0 = 0.$ **Ans. :** 0, 0

3. Find the directional derivative of each of the following functions as indicated :

$xz^2 + y^2 + z^3$ at (1, 0, – 1) in the direction of (2, 1, 0).

Ans. : $-\dfrac{2}{\sqrt{3}}$

2 APPLICATIONS OF PARTIAL DIFFERENTIATION

2.1 INTRODUCTION :

In chapter 1, we have seen about different kinds of partial derivatives for the function of several independent variables. In this chapter, we shall see different applications of partial derivatives, which can be used in estimating the error of occurrence in measurements, Taylor's Series expansion of the functions, Jacobians which is useful in the transformation of variables, in finding extremum values of a function.

2.2 TAYLOR'S THEOREM FOR THE FUNCTION OF TWO VARIABLES :

Statement (without proof) : If $f(x, y)$ is a function of two variables x and y having partial derivatives upto $(n+1)^{\text{th}}$ order then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \ldots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y) + R_n.$$

where R_n denotes the reminder term defined as

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x + \theta h, y + \theta k)$$

where $0 < \theta < 1$.

Note : In Taylor's theorem if $R_n \to 0$ as $n \to \infty$ then it becomes Taylor's series. In the next section we shall see its formal proof.

2.3 TAYLOR'S SERIES :

Statement : If $f(x, y)$ is a function of two variables x and y having partial derivatives of all order, then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \ldots\ldots$$

Proof :

[**Note :** The Taylor's series for the function of a single variable is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots\ldots \Big]$$

We first expand $f(x+h, y+k)$ as a Taylor's series by treating it as a function of a single variable x considering y as a constant.

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x, y+k) + \ldots\ldots \qquad \ldots\ldots\ldots\ (1)$$

Now we expand $f(x, y+k)$ treating it as a function of y only.

$$\therefore\ f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \ldots\ldots \qquad \ldots\ldots\ldots\ (2)$$

Putting (2) in (1), we get

$$f(x+h, y+k) = \left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^2}{\partial y^2} f(x, y) + \ldots\ldots \right]$$

$$+\ h\left[\frac{\partial}{\partial x} f(x, y) + k\frac{\partial^2}{\partial x \partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^3}{\partial x \partial y^2} f(x, y) + \ldots\ldots \right]$$

$$+\ \frac{h^2}{2!}\left[\frac{\partial^2}{\partial x^2} f(x, y) + k\frac{\partial^3}{\partial x^2 \partial y} f(x, y) + \frac{k^2}{2!} \frac{\partial^4}{\partial x^2 \partial y^2} f(x, y) + \ldots\ldots \right]$$

$$+ \ldots\ldots$$

$$= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f(x,y) + \frac{1}{2!}\left(h^2 \frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2} \right) f(x,y)$$

$$+ \ldots\ldots$$

$$= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f(x, y) + \ldots\ldots$$

Remark : We rewrite the above result in the following forms :

(1) Putting $x = a$ and $y = b$, we get

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b} \right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial a} + k\frac{\partial}{\partial b} \right)^2 f(a,b) + \ldots\ldots$$

Here $\frac{\partial}{\partial a} f(a,b) = \left[\frac{\partial}{\partial x} f(x,y)\right]_{(a,b)}$ and $\frac{\partial}{\partial b} f(a,b) = \left[\frac{\partial}{\partial y} f(x,y)\right]_{(a,b)}$

and so on.

(2) Put $a + h = x$ and $b + k = y$ in (1), we get

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial a}+(y-b)\frac{\partial}{\partial b}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial a}+(y-b)\frac{\partial}{\partial b}\right]^2 f(x,y),$$

(3) Putting $a = b = 0$ in (2), we get the series at the point (0, 0) which is known **Maclaurin's Series.** That is

$$f(x, y) = f(0, 0) + \left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)\right] + \frac{1}{2!}\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)\right] + \ldots$$

SOLVED EXAMPLES

1. Expand $e^x \log(1 + y)$ in powers of x and y.

Solution : We have

$f(x, y) = e^x \log(1 + y)$ $\quad\therefore\quad f(0, 0) = 0$

$f_x(x, y) = e^x \log(1 + y)$ $\quad\therefore\quad f_x(0, 0) = 0$

$f_y(x, y) = \frac{e^x}{1+y}$ $\quad f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1 + y)$ $\quad f_{xx}(0, 0) = 0$

$f_{xy}(x, y) = \frac{e^x}{1+y}$ $\quad f_{xy}(0, 0) = 1$

$f_{yy}(x, y) = -\frac{e^x}{(1+y)^2}$ $\quad f_{yy}(0, 0) = -1$

$\therefore$ Using Maclaurin's series, we get

$$f(x, y) = f(0, 0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \ldots$$

$$= y + \frac{1}{2}(2xy - y^2) + \ldots\ldots$$

$$= y + xy - \frac{y^2}{2} + \ldots\ldots$$

2. Expand $e^{ax}\sin by$ in powers of x and y upto third degree terms.

Solution : We have

$$f(x, y) = e^{ax}\sin by$$

$$f_x(x, y) = ae^{ax}\sin by \qquad f(0,0) = 0$$

$$f_y(x, y) = be^{ax}\cos by \qquad f_x(0,0) = 0$$

$$f_{xx}(x, y) = a^2 e^{ax}\sin by \qquad f_y(0,0) = b$$

$$f_{xy}(x, y) = ab\, e^{ax}\cos by \qquad f_{xx}(0,0) = 0$$

$$f_{yy}(x, y) = -b^2 e^{ax}\sin by \qquad f_{xy}(0,0) = ab$$

$$f_{xxx}(x, y) = a^3 e^{ax}\sin by \qquad f_{yy}(0,0) = 0$$

$$f_{xxy}(x, y) = a^2b\, e^{ax}\cos by \qquad f_{xxx}(0,0) = 0$$

$$f_{xyy}(x, y) = -ab^2 e^{ax}\sin by \qquad f_{xxy}(0,0) = a^2b$$

$$f_{yyy}(x, y) = -b^3 e^{ax}\cos by \qquad f_{xyy}(0,0) = 0$$

$$f_{yyy}(0,0) = -b^3$$

$\therefore$ Using Maclaurin's Series, we get

$$f(x, y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) +$$

$$\frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \ldots\ldots$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \ldots\ldots.$$

$$= by + \frac{1}{2!}[2xyab] + \frac{1}{3!}[3x^2ya^2b - 3\ y^3b^3] + \ldots\ldots.$$

$$= by + xyab + \frac{1}{2}(x^2y\ a^2b - y^3b^3) + \ldots\ldots.$$

3. Expand $\tan^{-1}\dfrac{y}{x}$ about the point (1, 1).

Solution : We have

$$f(x, y) = \tan^{-1}\frac{y}{x} \qquad \therefore\ f(1,1) = \frac{\pi}{4}$$

$$f_x(x, y) = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \qquad \therefore\ f_x(1,1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \qquad \therefore \; f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = \frac{2yx}{\left(x^2+y^2\right)^2} \qquad \therefore \; f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = -\frac{\left(x^2+y^2\right)-y\,.\,2y}{x^2+y^2} = -\frac{x^2-y^2}{x^2+y^2} \; \therefore \; f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = -\frac{2xy}{\left(x^2+y^2\right)^2} \qquad \therefore \; f_{yy}(1, 1) = -\frac{1}{2}$$

$\therefore$ Using Taylor's Series

$$f(x, y) = f(a,b)+\left[(x-a)\frac{\partial}{\partial a}+(y-b)\frac{\partial}{\partial b}\right]f(a,b)+\frac{1}{2!}\left[(x-a)\frac{\partial}{\partial a}+(y-b)\frac{\partial}{\partial b}\right]^2 f(a,b)$$

we get

$$f(x, y) = f(1, 1) + (x-1) f_x(1, 1) + (y-1) f_y(1, 1) + \frac{1}{2!}[(x-1)^2 f_{xx}(1,$$
$$2(x-1)(y-1) f_{xy}(1, 1)] + (y-1)^2 f_{yy}(1, 1)] + \ldots\ldots$$

$$= \frac{\pi}{4} - \frac{x-1}{2} + \frac{y-1}{2} + \frac{1}{2!}\left[\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2}\right] + \ldots..$$

$$= \frac{\pi}{4} - \frac{x-1}{2} + \frac{y-1}{2} + \frac{(x-1)^2}{4} - \frac{(y-1)^2}{4} + \ldots..$$

4. Expand $\cos x \cos y$ in powers of x and y upto fourth degree terms.

Solution : We have

$f(x, y) = \cos x \cos y$ $\qquad \therefore \; f(0, 0) = 1$

$f_x(x, y) = -\sin x \cos y$ $\qquad f_x(0, 0) = 0$

$f_y(x, y) = -\cos x \sin y$ $\qquad f_y(0, 0) = 0$

$f_{xx}(x, y) = -\cos x \cos y$ $\qquad f_{xx}(0, 0) = -1$

$f_{xy}(x, y) = \sin x \sin y$ $\qquad f_{xy}(0, 0) = 0$

$f_{yy}(x, y) = -\cos x \cos y$ $\qquad f_{yy}(0, 0) = -1$

$f_{xxx}(x, y) = \sin x \cos y$ $\qquad f_{xxx}(0, 0) = 0$

$f_{xxy}(x, y) = \cos x \sin y$ $\quad f_{xxy}(0, 0) = 0$

$f_{xyy}(x, y) = \sin x \cos y$ $\quad f_{xyy}(0, 0) = 0$

$f_{yyy}(x, y) = \cos x \sin y$ $\quad f_{yyy}(0, 0) = 0$

$f_{x^4}(x, y) = \cos x \cos y$ $\quad f_{x^4}(0, 0) = 1$

$f_{x^3y}(x, y) = -\sin x \sin y$ $\quad f_{x^3y}(0, 0) = 0$

$f_{x^2y^2}(x, y) = \cos x \cos y$ $\quad f_{x^2y^2}(0, 0) = 1$

$f_{xy^3}(x, y) = -\sin x \sin y$ $\quad f_{xy^3}(0, 0) = 0$

$f_{y^4}(x, y) = \cos x \cos y$ $\quad f_{y^4}(0, 0) = 1$

$\therefore$ Using Maclaurin's Series, we get

$$f(x, y) = f(0, 0) + \left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0)\right] + \frac{1}{2!}\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0)\right]$$

$$+ \frac{1}{3!}\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0)\right] + \frac{1}{4!}\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^4 f(0,0)\right] + \ldots\ldots$$

$$= f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}\left[x^3 f_{x^3}(0, 0) + 3x^2 y f_{x^2y}(0, 0) + 3xy^2 f_{xy^2}(0, 0) + y^3 f_{y^3}(0, 0)\right]$$

$$+ \frac{1}{4!}\left[x^4 f_{x^4}(0,0) + 4x^3 y f_{x^3y}(0,0) + 6x^2y^2 f_{x^2y^2}(0,0) + 4xy^3 f_{xy^3}(0,0) + y^4 f_{y^4}(0,0)\right] + \ldots\ldots$$

$$= 1 + \frac{1}{2!}[-x^2 - y^2] + \frac{1}{4!}[x^4 + 6x^2y^2 + y^4] + \ldots\ldots$$

$$= 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \ldots\ldots$$

5. Expand $e^x \cos y$ in the powers of $(x - 1)$ and $\left(y - \frac{\pi}{4}\right)$.

Solution : Here the expansion is at the point $\left(1, \frac{\pi}{4}\right)$.

We have

$$f(x, y) = e^x \cos y \qquad f\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_x(x, y) = e^x \cos y \qquad f\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_y(x, y) = -e^x \sin y \qquad f\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{xx}(x, y) = e^x \cos y \qquad f_{xx}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$f_{xy}(x, y) = -e^x \sin y \qquad f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$$f_{yy}(x, y) = -e^x \cos y \qquad f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$$

$\therefore$ Using Taylor's Series, we get

$$f(x, y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial a} + (y-b)\frac{\partial}{\partial b}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial a} + (y-b)\frac{\partial}{\partial b}\right]^2 f(a,b) + \ldots$$

$$= f\left(1, \frac{\pi}{4}\right) + (x-1) f_x\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(1, \frac{\pi}{4}\right) + \frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1, \frac{\pi}{4}\right)\right.$$

$$\left. + 2(x-1)\left(y - \frac{\pi}{4}\right) f_{xy}\left(1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2 f_{yy}\left(1, \frac{\pi}{4}\right)\right] + \ldots$$

$$= \frac{e}{\sqrt{2}} + (x-1)\frac{e}{\sqrt{2}} - \left(y - \frac{\pi}{4}\right)\frac{e}{\sqrt{2}} + \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}}\right.$$

$$\left. - 2(x-1)\left(y - \frac{\pi}{4}\right)\frac{e}{\sqrt{2}} - \left(y - \frac{\pi}{4}\right)^2 \frac{e}{\sqrt{2}}\right] + \ldots$$

$$= \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2} - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{\left(y - \frac{\pi}{4}\right)^2}{2} + \ldots\right]$$

6. Expand $x^2y + 3y - 2$ in the neighbourhood of the point (1, – 2).

Solution : We have

$f(x, y) = x^2y + 3y - 2$ $\qquad$ $f(1, -2) = -10$

$f_x(x, y) = 2xy$ $\qquad$ $f_x(1, -2) = -4$

$f_y(x, y) = x^2 + 3$ $\qquad$ $f_y(1, -2) = 4$

$f_{xx}(x, y) = 2y$ $\qquad$ $f_{xx}(1, -2) = -4$

$f_{xy}(x, y) = 2x$ $\qquad$ $f_{xy}(1, -2) = 2$

$f_{yy}(x, y) = 0$ $\qquad$ $f_{yy}(1, -2) = 0$

$f_{xxx}(x, y) = 0$ $\qquad$ $f_{xxx}(1, -2) = 0$

$f_{xxy}(x, y) = 2$ $\qquad$ $f_{x^2y}(1, -2) = 2$

$f_{xyy}(x, y) = 0$ $\qquad$ $f_{xy^2}(1, -2) = 0$

$f_{yyy}(x, y) = 0$ $\qquad$ $f_{y^3}(1, -2) = 0$

All other higher derivatives are zero.

∴ Using Taylor's Series, we get

$$f(x, y) = f(1, -2) + (x - 1) f_x(1, -2) + (y + 2) f_y(1, -2)$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{x^2}(1,-2)+2(x-1)(y+2) f_{xy}(1,-2)+(y+2)^2 f_{y^2}(1,-2)\right]$$

$$+\frac{1}{3!}\Big[(x-1)^3 f_{x^3}(1,-2)+3(x-1)^2(y+2) f_{x^2y}(1,-2)+3(x-1)(y+2)^2 f_{xy^2}(1,-2)$$

$$+(y+2)^3 f_{y^3}(1,-2)\Big]$$

$$= -10 - 4(x-1) + 4(y+2) + \frac{1}{2!}[-4(x-1)^2 + 4(x-1)(y+2)] + \frac{1}{3!}[6(x-1)^2(y+2)]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

7. Expand $\sin(x + h)(y + k)$ by Taylor's Series.

Solution : We have $f(x + h, y + k) = \sin(x + h)(y + k)$

$$\Rightarrow f(x, y) = \sin xy$$

∴ $f_x(x, y) = y \cos xy$, $f_y(x, y) = x \cos xy$

$f_{x^2}(x, y) = -y^2 \sin xy$, $f_{xy}(x, y) = \cos xy - xy \sin xy$

$f_{y^2}(x, y) = -x^2 \sin xy$ and so on.

Using Taylor's Series, we get

$$f(x+h, y+k) = f(x, y) + \left[h\, f_x(x, y) + k\, f_y(x, y)\right] + \frac{1}{2!}\left[h^2 f_{x^2}(x, y) + 2hk\, f_{xy}(x, y) + k^2 f_{y^2}(x, y)\right] + \ldots$$

$$= \sin xy + [hy \cos xy + kx \cos xy] + \frac{1}{2!}[-h^2y^2 \sin xy + 2hk(\cos xy - xy \sin xy) - k^2x^2 \sin xy] + \ldots$$

$$= \sin xy + (hy + kx)\cos xy - \frac{h^2y^2}{2}\sin xy + hk(\cos xy - xy \sin xy) - \frac{k^2x^2}{2}\sin xy + \ldots$$

8. Expand $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of h and k.

Solution : We have $f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$

$$\therefore\ f(x, y) = \frac{xy}{x+y}$$

$$f_x = \frac{(x+y)\,y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$f_y = \frac{(x+y)\,x - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}$$

$$f_{xy} = \frac{(x+y)^2\, 2y - y^2\, 2(x+y)}{(x+y)^4} = \frac{(x+y)\left[(x+y)2\,y - 2y^2\right]}{(x+y)^4} = \frac{2xy}{(x+y)^3}$$

$$f_{yy} = -\frac{2x^2}{(x+y)^3}$$

$\therefore$ Using Taylor's Series, we get

$$f(x+h, y+k) = f(x, y) + hf_x + kf_y + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \ldots\ldots$$

$$= \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \ldots$$

9. Expand $(x^2y + \sin y + e^x)$ in powers of $(x-1)$ and $(y-\pi)$.

Solution : The expansion is at the point $(1, \pi)$.

We have

$f(x, y) = x^2y + \sin y + e^x$	$f(1, \pi) = \pi + e$
$f_x(x, y) = 2xy + e^x$	$f_x(1, \pi) = 2\pi + e$
$f_y(x, y) = x^2 + \cos y$	$f_y(1, \pi) = 0$
$f_{xx}(x, y) = 2y + e^x$	$f_{xx}(1, \pi) = 2\pi + e$
$f_{xy}(x, y) = 2x$	$f_{xy}(1, \pi) = 2$
$f_{yy}(x, y) = -\sin y$	$f_{yy}(1, \pi) = 0$

and so on.

$\therefore$ Using Taylor's Series, we get

$$f(x, y) = f(1, \pi) + (x-1) f_x(1, \pi) + (y-\pi) f_y(1, \pi) + \frac{1}{2!}[(x-1)^2 f_{xx}(1, \pi)$$

$$+ 2(x-1)(y-\pi) f_{xy}(1, \pi) + (y-\pi)^2 f_{yy}(1, \pi)] + \ldots$$

$$= \pi + e + (x-1)(2\pi + e) + \frac{1}{2!}[(x-1)^2(2\pi + e) + 4(x-1)(y-\pi)] + \ldots$$

$$= \pi + e + (x-1)(2\pi + e) + \frac{1}{2}(x-1)^2(2\pi + e) + 2(x-1)(y-\pi) + \ldots$$

EXERCISE – 2.1

1. Expand $e^x \sin y$ in powers of x and y upto third degree terms.

   Ans. : $y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots$

2. Expand e^{xy} in the neighbourhood of the point $(1, 1)$.

   Ans. : $e\left[1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + 2(x-1)(y-1) + \frac{1}{2}(y-1)^2 + \ldots\right]$

3. Expand $e^x \cos y$ in powers of x and y upto third degree terms.

$$\text{Ans. : } 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 - \frac{1}{2}xy^2 + \ldots\ldots$$

4. Expand $\sin xy$ near the point $\left(1, \frac{\pi}{2}\right)$ upto second degree terms.

$$\text{Ans. : } 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \ldots\ldots$$

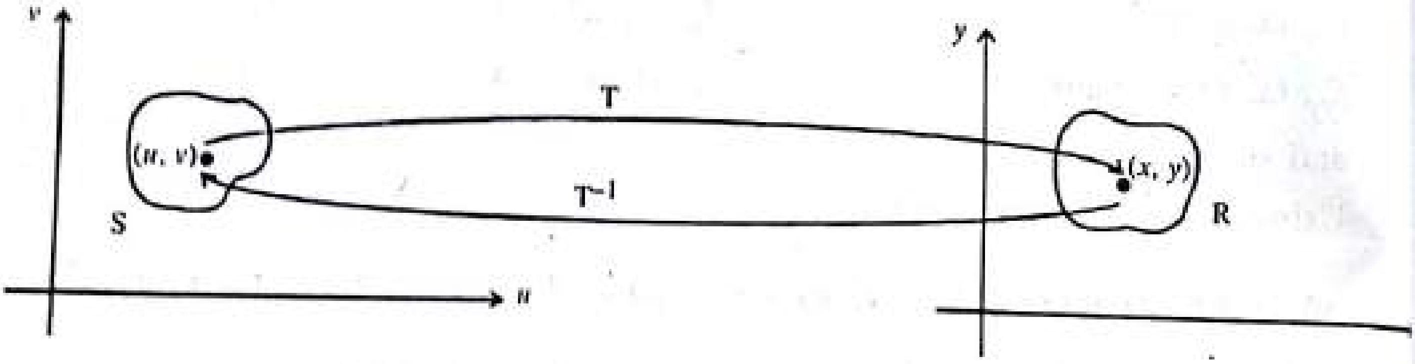
2.4 JACOBIANS :

Definition : Suppose that a point (x, y) is associated with a point (u, v) by mean of relatio

$$x = x(u, v),\ y = y(u, v)$$

then these two equations define a function T which associates every point (u, v) of the regio S in the uv–plane with the points in the region R (called image of T) in the xy–plane.

$$\therefore \quad T(u, v) = (x(u, v),\ y(u, v))$$



The function T is called a transformation from uv–plane to xy–plane. If T is one - o then we can define u, v as a function of x and y, that is

$$u = u(x, y),\ v = v(x, y).$$

The transformation which maps the points (x, y) (image of T) back into (u, v) is calle the inverse of T and denoted by T^{-1}.

Definition : Consider the transformation T from uv–plane to xy–plane defined by the relatio $x = x(u, v)$ and $y = y(u, v)$, then the Jacobian of T is defined and denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

We call this as a Jacobian of x, y with respect to u, v.

Similarly the Jacobian of inverse transformation of T is defined as

$$J(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Again, if x, y, z are functions of u, v, w then the Jacobian of x, y, z with respect to u, v, w is

$$J'(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

2.4.1 PROPERTIES OF JACOBIANS :

Result 1 : If $x = x(u, v)$ and $y = y(u, v)$ and J is the Jacobian of x, y with respect to u, v and if inverse transformation exist then prove that $J \cdot J' = 1$.

Proof : We have $x = x(u, v)$ and $y = y(u, v)$ and as the inverse transformation exists, we can obtain $u = u(x, y)$ and $v = v(x, y)$.

$\therefore$ Differentiating w.r.t. x, y, we get

$$\frac{\partial x}{\partial x} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} \Rightarrow x_u u_x + x_v v_x = 1$$

$$\frac{\partial x}{\partial y} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \Rightarrow x_u u_y + x_v v_y = 0$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} \Rightarrow y_u u_x + y_v v_x = 0$$

$$\frac{\partial y}{\partial y} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \Rightarrow y_u u_y + y_v v_y = 1$$

Now $J \cdot J' = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)}$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} x_u u_x + x_v v_x & x_u u_y + x_v v_y \\ y_u u_x + y_v v_x & y_u u_y + y_v v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Result 2 : If u and v are functions of r and s, and r and s are functions of x, y then

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)}$$

Proof : Here u and v are composite functions of x and y.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y}$$

$$\therefore \quad \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}$$

SOLVED EXAMPLES

1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$ and $\frac{\partial(r, \theta)}{\partial(x, y)}$.

Solution : We have $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad r = \sqrt{x^2 + y^2}, \ \theta = \tan^{-1}\frac{y}{x}$$

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial y}{\partial r} = \sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Again $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}, \ \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$\therefore \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

2. If $u = \frac{x + y}{1 - xy}$, $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u, v)}{\partial(x, y)}$.

Solution : We have $u = \frac{x + y}{1 - xy}$ and $v = \tan^{-1}x + \tan^{-1}y$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{(1 - xy) - (x + y)(-y)}{(1 - xy)^2} = \frac{1 + y^2}{(1 - xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

3. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Solution : We have

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$\therefore \frac{\partial x}{\partial r} = \sin\theta\cos\phi, \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \frac{\partial y}{\partial \phi} = r\sin\theta\sin\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta, \frac{\partial z}{\partial \theta} = -r\sin\theta, \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

($\because$ Taking outside r from 2nd column, $r\sin\theta$ from 3rd column)

$$= r^2\sin\theta\,[\cos\theta\,\{\cos\theta\cos^2\phi + \cos\theta\sin^2\phi\} + \sin\theta\,\{\sin\theta\cos^2\phi + \sin\theta\sin^2\phi\}]$$

[Evaluated over third row]

$$= r^2\sin\theta\,[\cos^2\theta + \sin^2\theta]$$

$$= r^2\sin\theta$$

4. If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u, v)}{\partial(r, \theta)}$.

Solution : We have $u = 2xy$, $v = x^2 - y^2$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2 = -4\,(x^2 + y^2)$$

Again $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\therefore \frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)}$$

$$= -4\,(x^2 + y^2)\,r = -4\,r^2 \cdot r = -4r^3 \qquad (\because r^2 = x^2 + y^2)$$

5. If $x = a \cosh\theta \cos\phi$, $y = a \sinh\theta \sin\phi$, prove that

$$\frac{\partial(x, y)}{\partial(\theta, \phi)} = \frac{a^2}{2}[\cosh 2\theta - \cos 2\phi]$$

Solution : We have $x = a \cosh\theta \cos\phi$ and $y = a \sinh\theta \sin\phi$

$$\therefore \frac{\partial x}{\partial \theta} = a\sinh\theta \cos\phi, \frac{\partial x}{\partial \phi} = -a \cosh\theta \sin\phi$$

$$\frac{\partial y}{\partial \theta} = a\cosh\theta \sin\phi, \frac{\partial y}{\partial \phi} = a\sinh\theta \cos\phi$$

$$\therefore \frac{\partial(x, y)}{\partial(\theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} a \sinh\theta \cos\phi & -a \cosh\theta \sin\phi \\ a \cosh\theta \sin\phi & a \sinh\theta \cos\phi \end{vmatrix}$$

$$= a^2 \sinh^2\theta \cos^2\phi + a^2 \cosh^2\theta \sin^2\phi$$

$$= a^2 [\sinh^2\theta (1 - \sin^2\phi) + (1 + \sinh^2\theta) \sin^2\phi]$$

$$= a^2 [\sinh^2\theta - \sinh^2\theta \sin^2\phi + \sin^2\phi + \sinh^2\theta \sin^2\phi]$$

$$= a^2 [\sinh^2\theta + \sin^2\phi]$$

$$= a^2 \left[\frac{\cosh 2\theta - 1}{2} + \frac{1 - \cos 2\phi}{2}\right] \qquad (\because 2\sin^2 x = 1 - \cos 2x,\ 2\sinh^2 x = \cosh 2x - 1)$$

$$= \frac{a^2}{2}[\cosh 2\theta - \cos 2\phi]$$

6. If $u = x + y + z$, $uv = y + z$, $uvw = z$, find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

Solution : First, we obtain x, y, z as a function of u, v, w.

We have $u = x + y + z$, $uv = y + z$, $uvw = z$

$\therefore u = x + uv \Rightarrow x = u - uv \Rightarrow x = u(1 - v)$

Now $uv = y + uvw \Rightarrow y = uv - uvw \Rightarrow y = uv(1 - w)$ and $z = uvw$

$$\therefore \frac{\partial x}{\partial u} = 1 - v, \frac{\partial x}{\partial v} = -u, \frac{\partial x}{\partial w} = 0$$

$$\frac{\partial y}{\partial u} = v(1-w),\ \frac{\partial y}{\partial v} = u(1-w),\ \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw,\ \frac{\partial z}{\partial v} = uw,\ \frac{\partial z}{\partial w} = uv.$$

$$\therefore\ \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= u^2v \begin{vmatrix} 1-v & -1 & 0 \\ v(1-w) & 1-w & -1 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2v\,[1\cdot\{(1-v)w + vw\} + 1\{(1-v)(1-w) + v(1-w)\}]$$

(evaluated over 3rd column)

$$= u^2v\,[w - vw + vw + (1-w)(1-v+v)]$$

$$= u^2v\,[w + 1 - w] = u^2v$$

7. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Solution : We have $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\therefore\ \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\ \{r\cos^2\theta + r\sin^2\theta\}$$

$$= r.$$

8. If $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$, **find** $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

Solution : We have $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$, $w = \dfrac{xy}{z}$.

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{yx}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ \dfrac{z}{y} & -\dfrac{zx}{y^2} & \dfrac{x}{y} \\ \dfrac{y}{z} & \dfrac{x}{z} & -\dfrac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{yz \cdot xz \cdot xy}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1\,(1 - 1) - 1\,(-1 - 1) + 1\,(1 + 1)$$

$$= 2 + 2 = 4.$$

9. If $u_1 = \dfrac{x_1}{x_n}$, $u_2 = \dfrac{x_2}{x_n}$, ..., $u_{n-1} = \dfrac{x_{n-1}}{x_n}$ and $x_1^2 + x_2^2 + \dots + x_n^2 = 1$,

find $\dfrac{\partial(u_1, u_2, \dots, u_{n-1})}{\partial(x_1, x_2, \dots, x_{n-1})}$.

Solution : We have $u_1 = \dfrac{x_1}{x_n}$, $u_2 = \dfrac{x_2}{x_n}$, ... , $u_{n-1} = \dfrac{x_{n-1}}{x_n}$.

$$\therefore \quad \frac{\partial(u_1, u_2, \dots, u_{n-1})}{\partial(x_1, x_2, \dots, x_{n-1})} = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_{n-1}} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_{n-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_{n-1}}{\partial x_1} & \dfrac{\partial u_{n-1}}{\partial x_2} & \cdots & \dfrac{\partial u_{n-1}}{\partial x_{n-1}} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{x_n} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{x_n} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{x_n} \end{vmatrix}$$

$$= \frac{1}{x_n} \cdot \frac{1}{x_n} \text{ ..... } (n-1) \text{ times}$$

$$= \frac{1}{x_n^{n-1}}$$

10. If $x = e^u \sec v$, $y = e^u \tan v$, prove that $J \cdot J' = 1$.

Solution : We have $x = e^u \sec v$, $y = e^u \tan v$

$$\therefore \quad \frac{\partial x}{\partial u} = e^u \sec v, \quad \frac{\partial x}{\partial v} = e^u \sec v \tan v$$

$$\frac{\partial y}{\partial u} = e^u \tan v, \quad \frac{\partial y}{\partial v} = e^u \sec^2 v$$

Again $x^2 - y^2 = e^{2u} \Rightarrow u = \frac{1}{2} \log (x^2 - y^2)$

and $\frac{y}{x} = \frac{\tan v}{\sec v} = \sin v \Rightarrow v = \sin^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \frac{2x}{x^2 - y^2} = \frac{x}{x^2 - y^2}, \frac{\partial u}{\partial y} = \frac{1}{2} \frac{-2y}{x^2 - y^2} = -\frac{y}{x^2 - y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x\sqrt{x^2 - y^2}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \cdot \left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\therefore J \cdot J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} e^u \sec v & e^u \sec v \tan v \\ e^u \tan v & e^u \sec^2 v \end{vmatrix} \cdot \begin{vmatrix} \frac{x}{x^2 - y^2} & -\frac{y}{x^2 - y^2} \\ -\frac{y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \end{vmatrix}$$

$$= [e^{2u} \sec^3 v - e^{2u} \sec v \tan^2 v] \cdot \left[\frac{x}{(x^2 - y^2)^{3/2}} - \frac{y^2}{x(x^2 - y^2)^{3/2}}\right]$$

$$= e^{2u} \sec v [\sec^2 v - \tan^2 v] \cdot \frac{1}{(x^2 - y^2)^{3/2}} \left[x - \frac{y^2}{x}\right]$$

$$= e^{2u} \sec v \frac{1}{(x^2 - y^2)^{3/2}} \frac{(x^2 - y^2)}{x}$$

$$= \frac{e^{2u} \sec v}{x\sqrt{x^2 - y^2}}$$

$$= \frac{e^{2u} \sec v}{e^u \sec v \cdot e^u} \qquad (\because e^{2u} = x^2 - y^2)$$

$= 1.$

11. If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$, prove that $J \cdot J' = 1$.

Solution : We have $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$

$$\therefore J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$$= -2v(-4uw) + 2w(4uv) = 16\,uvw$$

Now $y + z - x = w^2 + u^2 + u^2 + v^2 - v^2 - w^2 = 2u^2$

$\therefore 2u^2 = y + z - x$

Similarly $2v^2 = x + z - y$, $2w^2 = x + y - z$

Now $4u \frac{\partial u}{\partial x} = -1$, $4u \frac{\partial u}{\partial y} = 1$, $4u \frac{\partial u}{\partial z} = 1$

Similarly $4v \frac{\partial v}{\partial x} = 1$, $4v \frac{\partial v}{\partial y} = -1$, $4v \frac{\partial v}{\partial z} = 1$

$4w \frac{\partial w}{\partial x} = 1$, $4w \frac{\partial w}{\partial y} = 1$, $4w \frac{\partial w}{\partial z} = -1$

$$\therefore J' = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{4u} & \frac{1}{4u} & \frac{1}{4u} \\ \frac{1}{4v} & -\frac{1}{4v} & \frac{1}{4v} \\ \frac{1}{4w} & \frac{1}{4w} & -\frac{1}{4w} \end{vmatrix}$$

$$= \frac{1}{64\,uvw}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{1}{64\,uvw}\left[-1\,(1-1) - 1\,(-2) + 1\,(2)\right]$$

$$= \frac{4}{64\,uvw} = \frac{1}{16\,uvw}$$

$$\therefore\ J \cdot J' = 16\,uvw \cdot \frac{1}{16\,uvw} = 1.$$

(12) **If $z = f(x, y)$, $\phi\,(x, y) = 0$, show that $\frac{\partial \phi}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial(f, \phi)}{\partial(x, y)}$**

Solution : We have $z = f(x, y)$, $\phi(x, y) = 0$.

$\therefore$ z is composite function of x.

$$\therefore\ \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Now $\phi\,(x, y) = 0 \Rightarrow \frac{dy}{dx} = -\dfrac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$

$$\therefore\ \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\left(-\frac{\partial \phi / \partial x}{\partial \phi / \partial y}\right)$$

$$= \left[\frac{\partial z}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial \phi}{\partial x}\right] \div \frac{\partial \phi}{\partial y}$$

$$\therefore\ \frac{\partial \phi}{\partial y}\frac{dz}{dx} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix}$$

$$\therefore\ \frac{\partial \phi}{\partial y}\frac{dz}{dx} = \frac{\partial(f, \phi)}{\partial(x, y)}$$

13. If $x = uv$, $y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

Solution : We first find $\dfrac{\partial(x, y)}{\partial(u, v)}$

$$\frac{\partial x}{\partial u} = v, \ \frac{\partial x}{\partial v} = u$$

and $\dfrac{\partial y}{\partial u} = \dfrac{(u-v)-(u+v)}{(u-v)^2} = \dfrac{-2v}{(u-v)^2}$, $\dfrac{\partial y}{\partial v} = \dfrac{(u-v)+(u+v)}{(u-v)^2} = \dfrac{2u}{(u-v)^2}$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ -\dfrac{2v}{(u-v)^2} & \dfrac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2} = \frac{4uv}{(u-v)^2}$$

We know that $\dfrac{\partial(x, y)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(x, y)} = 1$

$$\therefore \frac{4uv}{(u-v)^2} \frac{\partial(u, v)}{\partial(x, y)} = 1 \Rightarrow \frac{\partial(u, v)}{\partial(x, y)} = \frac{(u-v)^2}{4uv}$$

14. If $u = f(x)$, $v = g\ (x, y)$, $w = h\ (x, y, z)$, prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial w}{\partial z}$$

Hence evaluate $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

Where $x = \cos u$, $y = \cos v \sin u$, $z = \cos w \sin v \sin u$.

Solution : We have

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial u} & 0 \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial w}{\partial x}$$

Now we have

$$x = \cos u,\ y = \cos v \sin u,\ z = \cos w \sin v \sin u$$

$$\therefore\ \frac{\partial x}{\partial u} = -\sin u,\ \frac{\partial y}{\partial v} = -\sin v \sin u,\ \frac{\partial z}{\partial w} = -\sin w \sin v \sin y$$

$\therefore$ By above result, we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w}$$

$$= (-\sin u)(-\sin v \sin u)(-\sin w \sin v \sin u)$$

$$= -\sin^3 u \cdot \sin^2 v \cdot \sin w$$

15. If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Solution : We have $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$

$$\therefore\ \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} yz & xz & xy \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} z(y-x) & x(z-y) & xy \\ y-x & z-y & x+y \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (y - x)(z - y)\begin{vmatrix} z & x & xy \\ 1 & 1 & x+y \\ 0 & 0 & 1 \end{vmatrix} \qquad (\because c_1 \to c_1 - c_2,\ c_2 \to c_2 - c_3)$$

$$= (y - x)(z - y)(z - x) \qquad (\because \text{Evaluated over } 3^{rd} \text{ row})$$

$$= (x - y)(y - z)(z - x)$$

16. If $x = a(u + v)$, $y = b(u - v)$, and $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$.

Solution : We have $x = a(u + v)$, $y = b(u - v)$

$$\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -ab - ab = -2ab$$

Again $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$

$$\therefore \frac{\partial(u, v)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix}$$

$$= 4r^3 \cos^2 2\theta + 4r^3 \sin^2 2\theta = 4r^3$$

$$\therefore \frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(r, \theta)}$$

$$= (-2ab)(4r^3) = -8abr^3$$

17. If $u = \dfrac{x}{\sqrt{1 - r^2}}$, $v = \dfrac{y}{\sqrt{1 - r^2}}$, $w = \dfrac{z}{\sqrt{1 - r^2}}$, where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \left(1 - r^2\right)^{-5/2}$$

Solution : Here $r^2 = x^2 + y^2 + z^2$

$$\therefore\ 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$

Now $u = \frac{x}{\sqrt{1-r^2}}$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\sqrt{1-r^2} - x\frac{(-2r)}{2\sqrt{1-r^2}}\cdot\frac{\partial r}{\partial x}}{1-r^2} = \frac{(1-r^2) + xr\frac{x}{r}}{(1-r^2)^{3/2}} = \frac{(1-r^2)+x^2}{(1-r^2)^{3/2}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{1-r^2}\,\frac{(-2r)}{2\sqrt{1-r^2}}\cdot\frac{y}{r} = \frac{xy}{(1-r^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = -\frac{x}{1-r^2}\cdot\frac{(-2r)}{2\sqrt{1-r^2}}\cdot\frac{z}{r} = \frac{xz}{(1-r^2)^{3/2}}$$

Similarly $\frac{\partial v}{\partial x} = \frac{xy}{(1-r^2)^{3/2}},\ \frac{\partial v}{\partial y} = \frac{1-r^2+y^2}{(1-r^2)^{3/2}},\ \frac{\partial v}{\partial z} = \frac{yz}{(1-r^2)^{3/2}}$

and $\frac{\partial w}{\partial x} = \frac{xz}{(1-r^2)^{3/2}},\ \frac{\partial w}{\partial y} = \frac{yz}{(1-r^2)^{3/2}},\ \frac{\partial w}{\partial z} = \frac{1-r^2+z^2}{(1-r^2)^{3/2}}$

$$\therefore\ \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1-r^2+x^2}{(1-r^2)^{3/2}} & \frac{xy}{(1-r^2)^{3/2}} & \frac{xz}{(1-r^2)^{3/2}} \\ \frac{xy}{(1-r^2)^{3/2}} & \frac{1-r^2+y^2}{(1-r^2)^{3/2}} & \frac{yz}{(1-r^2)^{3/2}} \\ \frac{xz}{(1-r^2)^{3/2}} & \frac{yz}{(1-r^2)^{3/2}} & \frac{1-r^2+z^2}{(1-r^2)^{3/2}} \end{vmatrix}$$

$$= \frac{1}{(1-r^2)^{9/2}} \begin{vmatrix} 1-r^2+x^2 & xy & xz \\ xy & 1-r^2+x^2 & yz \\ xz & yz & 1-r^2+x^2 \end{vmatrix}$$

$$= \frac{1}{zy\,(1-r^2)^{9/2}} \begin{vmatrix} (1-r^2)y & 0 & xz \\ -(1-r^2)x & (1-r^2)z & yz \\ 0 & -(1-r^2)y & 1-r^2+z^2 \end{vmatrix} \begin{matrix} c_1 \to yc_1 - xc_2 \\ c_2 \to zc_2 - yc_3 \end{matrix}$$

$$= \frac{(1-r^2)^2}{yz\,(1-r^2)^{9/2}} \begin{vmatrix} y & 0 & xz \\ -x & z & yz \\ 0 & -y & 1-r^2+z^2 \end{vmatrix}$$

$$= \frac{(1-r^2)^{-5/2}}{yz} [y\{z(1-r^2+z^2)+y^2z\}+xz\,(xy)]$$

$$= (1-r^2)^{-5/2}\,[1-r^2+z^2+y^2+x^2]$$

$$= (1-r^2)^{-5/2} \qquad (\because\ r^2 = x^2+y^2+z^2)$$

18. If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$, find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

Solution : We have $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

we first evaluate $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

$$\therefore\ \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} yz & xz & xy \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \begin{matrix} c_1 \to c_1 - c_2 \\ c_2 \to c_2 - c_3 \end{matrix}$$

$$= 2 \begin{vmatrix} z\,(y-x) & x\,(z-y) & xy \\ x-y & y-z & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x-y)(y-z)\begin{vmatrix} -z & -x & xy \\ 1 & 1 & z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2(x-y)(y-z)(-z+x)$$

$$= -2(x-y)(y-z)(z-x)$$

EXERCISE – 2.2

1. If $u = x + y$, $v = \dfrac{x}{x+y}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$ **Ans. :** $\dfrac{x-y}{(x+y)^2}$

2. If $x = \sin\theta \cos\phi$, $y = \sin\theta \sin\phi$, prove that $J \cdot J' = 1$.

3. If $u = x(1-y)$, $v = xy(1-z)$, $w = xyz$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. **Ans. :** x^2y

4. If $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x(x + 2y - z)$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. **Ans. :** 0

5. If $x = uv$, $y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. **Ans. :** $\dfrac{(u-v)^2}{4uv}$

6. If $x = u(1-v)$, $y = uv$, prove that $\dfrac{\partial(x, y)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(x, y)} = 1$.

7. If $u = e^x \cos y$, $v = e^x \sin y$ *and* $x = lr + sm$, $y = mr - sl$, prove that
$\dfrac{\partial(u, v)}{\partial(x, y)} \cdot \dfrac{\partial(x, y)}{\partial(r, s)} = \dfrac{\partial(u, v)}{\partial(r, s)}$.

8. If $u = x + y + z$, $u^2v = y + z$, $u^3w = z$, prove that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = u^{-5}$.

9. If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta \cos\phi$, $v = r\sin\theta \sin\phi$, $w = r\cos\theta$,
find $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ **Ans. :** $\dfrac{r^2}{4}\sin\theta$

10. If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(u, v)}{\partial(r, \theta)}$. **Ans. :** $-4r^3$

11. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. **Ans. :** $10x + 4$

12. If $u = x^2 - 2y$, $v = x + y$, prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2x + 2$.

2.5 ERRORS AND APPROXIMATION :

Let $z = f(x, y)$ be a continuous function of x and y. Let δx and δy be the errors (small enough) occurs in the measurement of the values of x and y. Then the corresponding error δz occurs in the estimation of the value of z. We can write this in the functional form as :

$$z + \delta z = f(x + \delta x, y + \delta y)$$

$$\Rightarrow \delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

Expanding by using Taylor's Series, we get

$$\delta z = \left[f(x, y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + O(\delta x^2, \delta y^2) \right] - f(x, y)$$

$$\therefore \quad \delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + O(\delta x^2, \delta y^2)$$

Where $O(\delta x^2, \delta y^2)$ contains the terms of higher order of δx and δy, which tends to zero as δx and δy are considered small enough. Thus an approximate error in the value of z is given by :

$$\delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}.$$

Note : For the quantity x :

δx is known as an ***absolute*** error in x.

$\frac{\delta x}{x}$ is known as a ***relative*** error in x.

$\frac{\delta x}{x}$ 100 is known as a ***percentage*** error in x.

SOLVED EXAMPLES

1. If measurements of radius of base and height of a right circular cone are incorrect by – 1% and 2%, prove that there is no error in the volume.

Solution : Let x be the height, y be the radius of base and z be the volume of the cone.

$$\therefore \quad z = \frac{\pi}{3} y^2 x$$ $\pi r^2 h$

$$\delta z = \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y$$

$$= \frac{\pi}{3} y^2 \delta x + \frac{2\pi}{3} yx \, \delta y$$

Now, we have $\frac{\delta x}{x} 100 = 2, \frac{\delta y}{y} 100 = -1$

$$\therefore \quad \delta z = \frac{\pi}{3} y^2 \left(\frac{2x}{100}\right) + \frac{2\pi}{3} yx \left(-\frac{y}{100}\right) = 0.$$

$\therefore$ There is no error in the volume.

2. Obtain the percentage error in the area of an ellipse when an error of 1 percentage is made in measuring the semi major and semi minor axes.

Solution : Let x and y be the semi major and semi minor axes of the ellipse, and A be the area of the ellipse.

$$\therefore \quad A = \pi xy$$

Taking logarithm on both sides, we get

$$\log A = \log \pi + \log x + \log y$$

$$\therefore \quad \frac{\delta A}{A} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

$$\therefore \quad \frac{\delta A}{A} 100 = \frac{\delta x}{x} 100 + \frac{\delta y}{y} 100$$

But it is given that $\frac{\delta x}{x} 100 = 1$ and $\frac{\delta y}{y} 100 = 1$

$$\therefore \quad \frac{\delta A}{A} 100 = 1 + 1 = 2$$

$\therefore$ The error in the area of the ellipse = 2%.

3. The period of a simple pendulum is given by $T = 2\pi \sqrt{\frac{l}{g}}$. If T is found using $l = 8$ ft., $g = 32$ ft/sec^2, find an approximate error in T if the correct values are $l = 8.05$ ft, $g = 32.01$ ft/sec^2.

Solution : Given that $l = 8$ ft., $g = 32$ ft/sec^2.

Again the errors are :

$$\delta l = 0.05 \text{ ft}, \ \delta g = 0.01 \text{ ft/sec}^2.$$

Now $T = 2\pi \sqrt{\frac{l}{g}}$

Taking logarithm on both sides, we get

$$\log T = \log 2\pi + \frac{1}{2} [\log l - \log g]$$

$$\therefore \quad \frac{\delta T}{T} = \frac{1}{2}\left[\frac{\delta l}{l} - \frac{\delta g}{g}\right]$$

$$\therefore \quad \delta T = \frac{1}{2}\left[\frac{\delta l}{l} - \frac{\delta g}{g}\right] 2\pi \sqrt{\frac{l}{g}}$$

$$= \frac{1}{2}\left[\frac{0.05}{8} - \frac{0.01}{32}\right] 2\pi \sqrt{\frac{8}{32}}$$

$$= 0.00933\pi$$

4. The deflection at the centre of a rod of length l and diameter d supported at its ends and loaded at the centre with a weight w varies as $wl^3 d^{-4}$. If the percentage increase in w, l and d is 3, 2, 1 respectively, find the corresponding percentage increase in the deflection.

Solution : Given that $\frac{\delta w}{w} 100 = 3$, $\frac{\delta l}{l} 100 = 2$, $\frac{\delta d}{d} 100 = 1$.

Now, we have $D \propto wl^3 d^{-4}$

where D is the deflection at the centre of a rod.

$\therefore \quad D = cwl^3 d^{-4}$

where c is the constant of proportionality.

Taking logarithm on both sides, we get

$\log D = \log c + \log w + 3 \log l - 4 \log d$

$$\therefore \quad \frac{\delta D}{D} = \frac{\delta w}{w} + 3\frac{\delta l}{l} - 4\frac{\delta d}{d}$$

$$\therefore \quad \frac{\delta D}{D} 100 = \frac{\delta w}{w} 100 + 3\frac{\delta l}{l} 100 - 4\frac{\delta d}{d} 100$$

$$= 3 + 3(2) - 4(1)$$

$$= 5$$

$\therefore$ The percentage increase in diameter is 5%.

5. Find the possible error in percentage in measuring the parallel resistance r of two resistances, r_1 and r_2 which undergo a positive error of 2% each in their own measurement.

Solution : The resistance r is given by the relation

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

$$\therefore \quad -\frac{\delta r}{r^2} = -\frac{\delta r_1}{r_1^2} - \frac{\delta r_2}{r_2^2}$$

$$\therefore \quad -\frac{1}{r}\left(\frac{\delta r}{r} 100\right) = -\frac{1}{r_1}\left(\frac{\delta r_1}{r_1} 100\right) - \frac{1}{r_2}\left(\frac{\delta r_2}{r_2} 100\right)$$

Given that $\frac{\delta r_1}{r_1} 100 = 2$ and $\frac{\delta r_2}{r_2} 100 = 2$.

$$\therefore \frac{1}{r}\left(\frac{\delta r}{r} 100\right) = \frac{1}{r_1}(2) + \frac{1}{r_2}(2)$$

$$= 2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$= \frac{2}{r}$$

$$\therefore \frac{\delta r}{r} 100 = 2\%$$

6. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm. and 6 cm. respectively. The possible error in each measurement is 0.1 cm. Find approximately, the maximum possible errors in the values computed for volume and lateral surface.

Solution : Let D be the diameter of the base, and h be the height of a can which is a right circular cylinder.

(a) The volume V is given by

$$V = \pi \left(\frac{D}{2}\right)^2 h = \frac{\pi}{4} D^2 h$$

$$\therefore \delta V = \frac{\pi}{4} [2Dh\ \delta D + D^2 \delta h]$$

Given that $D = 4$, $h = 6$ and $\delta D = 0.1$, $\delta h = 0.1$

$$\therefore \delta V = \frac{\pi}{4} [2(4)(6)(0.1) + (16)(0.1)$$

$$= \frac{\pi}{4}(6.4) = 1.6\pi \text{ cm}^3$$

(b) The lateral surface is given by

$$\delta = 2\pi \left(\frac{D}{2}\right) h = \pi Dh$$

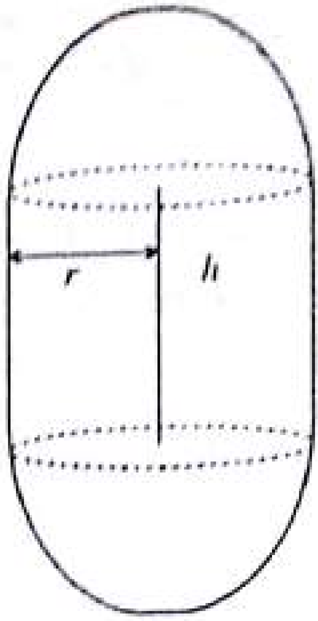
$$\therefore \delta S = \pi [h\delta D + D\ \delta h]$$

$$= \pi [(6)(0.1) + 4(0.1)]$$

$$= \pi (0.6 + 0.4) = \pi$$

7. **A balloon is in the shape of a right circular cylinder of radius 3m and length 6m, and is surmounted by hemispherical ends. If the radius is increased by 1% m and the length by 5% m, find the percentage increase in the volume of a balloon.**

Solution : Let r and h be the radius and height respectively of a right circular cylinder. Thus the volume V of a balloon is given by :



$$V = \pi r^2 h + 2\left(\frac{2}{3}\pi r^3\right)$$

$$\therefore \quad V = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\therefore \quad \delta V = \pi\left[2rh\delta r + r^2\delta h + \frac{4}{3}3r^2\delta r\right]$$

$$\frac{\delta V}{V}100 = \pi\left[2r^2h\frac{\delta r}{r}100 + r^2h\frac{\delta h}{h}100 + 4r^3\frac{\delta r}{r}100\right]\frac{1}{V}$$

Given that $r = 3\ m$, $h = 6\ m$., $\frac{\delta r}{r}100 = 1$, $\frac{\delta h}{h}100 = 5$

$$\therefore \quad V = \pi\left[(9)(6) + \frac{4}{3}27\right] = \pi[54 + 36] = 90\pi$$

$$\therefore \quad \frac{\delta V}{V}100 = \pi[2(9)(6)(1) + (9)(6)(5) + 4(27)(1)]\frac{1}{90\pi}$$

$$= [108 + 270 + 108]\frac{1}{90}$$

$$= \frac{486}{90} = 5.4\%$$

8. **The H. P. required to propel a steamer varies as the cube of the velocity and the square of length. If there is 3% increase in velocity and 4% increase in length, find the corresponding percentage increase in H. P.**

Solution : Let v be the velocity and l be the length of a steamer.

$\therefore$ The required H. P. is given by

$$H \propto v^3 l^2 \Rightarrow H = kv^3 l^2$$

where k is the constant of proportionality.

Taking logarithm on both sides, we get

$$\log H = \log k + 3\log v + 2\log l$$

$$\therefore \quad \frac{\delta H}{H} = 3\frac{\delta v}{v} + 2\frac{\delta l}{l}$$

$$\therefore \quad \frac{\delta H}{H}100 = 3\frac{\delta v}{v}100 + 2\frac{\delta l}{l}100$$

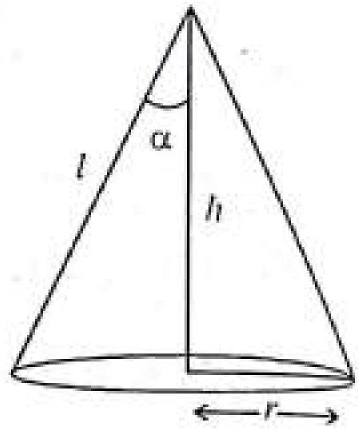
Given that $\frac{\delta V}{V} 100 = 3, \frac{\delta l}{l} 100 = 4$

$\therefore \frac{\delta H}{H} 100 = 3(3) + 2(4) + 17$

$\therefore$ The percentage increase in H. P. is 17%.

9. The height h and semi–vertical angle α of a cone are measured, and from there A, the total area of the cone, including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Also show that, if $\alpha = \frac{\pi}{6}$, an error of 1% in h will be approximately compensated by an error of – 19.8′ in α.

Solution : Let l be the slant height and r be the radius of a cone. Then $l = h\sec\alpha$ and $r = h\tan\alpha$



$\therefore$ The total area

$$\begin{aligned} A &= \pi r^2 + \pi r l \\ &= \pi h^2 \tan^2\alpha + \pi (h\tan\alpha)(h\sec\alpha) \\ &= \pi h^2 [\tan^2\alpha + \tan\alpha \cdot \sec\alpha] \end{aligned}$$

$$\therefore \delta A = 2\pi h [\tan^2\alpha + \tan\alpha \cdot \sec\alpha]\,\delta h + \pi h^2 [2\tan\alpha \sec^2\alpha + \sec^3\alpha + \tan^2\alpha \sec\alpha]\,\delta\alpha$$

Now, put $\alpha = \frac{\pi}{6}, \frac{\delta h}{h} 100 = 1,$

$$\therefore \delta A = 2\pi h\left[\frac{1}{3} + \frac{1}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2 \cdot \frac{1}{\sqrt{3}} \cdot \frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{3} \cdot \frac{2}{\sqrt{3}}\right]\delta\alpha$$

$$= 2\pi h\left(\frac{h}{100}\right) + \frac{\pi h^2}{3\sqrt{3}}(18)\,\delta\alpha$$

$$= \frac{\pi h^2}{50} + \frac{6\pi h^2}{\sqrt{3}}\,\delta\alpha$$

$$= \pi h^2\left[\frac{1}{50} + 2\sqrt{3}\,\delta\alpha\right]$$

Now the error in h is to be compensated by the error in α.

$\therefore \delta A = 0$

$$\therefore \pi h^2\left[\frac{1}{50} + 2\sqrt{3}\,\delta\alpha\right] = 0$$

$\therefore \frac{1}{50} + 2\sqrt{3}\ \delta\alpha = 0$

$\therefore \delta\alpha = -\frac{1}{100\sqrt{3}}$ Rad.

$\therefore \delta\alpha = -\frac{1}{100\sqrt{3}} \cdot \frac{180}{\pi}$ degree

$= -\frac{180}{100\sqrt{3}\pi} \cdot 60$ minutes

$= -\frac{10800}{543.86}$ minutes

$= -19.8'$

10. In estimating the cost of a pile of bricks measured as $2m \times 15m \times 1.2m$, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to $1m^3$ and bricks cost Rs. 1500 per 1000 bricks, find the approximate error in the cost.

Solution : Let x, y, z be the dimensions of a pile of bricks.

$\therefore$ The volume V of a pile is given by,

$V = xyz$

Taking logarithm on both sides, we get

$\log V = \log x + \log y + \log z$

$$\therefore \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$\therefore \frac{\delta V}{V} 100 = \frac{\delta x}{x} 100 + \frac{\delta y}{y} 100 + \frac{\delta z}{z} 100$$

Given that $x = 2\ m$, $y = 15\ m$, $z = 1.2\ m$ and

$$\frac{\delta x}{x} 100 = 1, \frac{\delta y}{y} 100 = 1, \frac{\delta z}{z} 100 = 1.$$

Again $V = xyz = (2)(15)(1.2) = 36\ m^3$

$$\therefore \delta V = [1 + 1 + 1]\ \frac{36}{100} = 1.08\ m^3$$

which is the excess computation in the volume, due to the stretch in the tape.

Now the volume of $1m^3$ contains 450 bricks.

$\therefore$ The excess number of bricks in excess volume counted is

$= 450 \times 1.08 = 451.08 = 452$ bricks approximately.

Now the cost is Rs. 1500 per 1000 bricks.

$\therefore$ The excess cost is $= 452 \times \frac{1500}{1000} = 678$ Rs.

11. A closed rectangular box with unequal sides a, b, c has its edges slightly altered in lengths by amounts $\delta a, \delta b, \delta c$ respectively, so that both its volume and surface area remain uncharged. Prove that

$$\frac{\delta a}{a^2(b-c)} = \frac{\delta b}{b^2(c-a)} = \frac{\delta c}{c^2(a-b)}.$$

Solution : Let V be the volume and S be the surface area of a rectangular box.

$\therefore$ $V = abc$

$\therefore$ $\delta V = bc\,\delta a + ac\,\delta b + ab\,\delta c$

Now $S = 2\,(ab + bc + ca)$

$\therefore$ $\delta S = 2\,[a\delta b + b\delta a + b\delta c + c\delta b + c\delta a + a\delta c]$

$= 2\,[(b + c)\delta a + (a + c)\delta b + (a + b)\delta c]$

As V and S remains unchanged, we have

$\delta V = 0$ and $\delta S = 0$

$\therefore$ $bc\delta a + ac\delta b + ab\delta c = 0$ .... (1)

and $(b + c)\,\delta a + (a + c)\delta b + (a + b)\delta c = 0$ .... (2)

Solving (1) and (2) for δa, δb and δc, we get

$$\frac{\delta a}{\begin{vmatrix} ac & ab \\ a+c & a+b \end{vmatrix}} = \frac{-\delta b}{\begin{vmatrix} bc & ab \\ b+c & a+b \end{vmatrix}} = \frac{\delta c}{\begin{vmatrix} bc & ac \\ b+c & a+c \end{vmatrix}}$$

$$\therefore \quad \frac{\delta a}{ac\,(a+b) - ab\,(a+c)} = \frac{-\delta b}{bc\,(a+b) - ab\,(b+c)} = \frac{\delta c}{bc\,(a+c) - ac\,(b+c)}$$

$$\therefore \quad \frac{\delta a}{a\,(ac-ab)} = \frac{-\delta b}{b\,(bc-ab)} = \frac{\delta c}{c\,(bc-ac)}$$

$$\therefore \quad \frac{\delta a}{a^2\,(b-c)} = \frac{\delta b}{b^2\,(c-a)} = \frac{\delta c}{c\,(a-b)}$$

12. If the sides and angles of a triangle ABC vary in such a way that its circum–radius remains constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

where da, db, dc are small increments in the sides a, b, c respectively.

Solution : The circum–radius R of a triangle ABC is given by

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\therefore\ a = 2R\sin A \Rightarrow da = 2R\cos A\ dA$

$$\therefore\ \frac{da}{\cos A} = 2RdA$$

Similarly, we have

$$\frac{db}{\cos B} = 2RdB,\ \frac{dc}{\cos C} = 2RdC$$

$$\therefore\ \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \qquad \text{.......... (1)}$$

But for any triangle $A + B + C = \pi$

$$\therefore\ dA + dB + dC = 0 \qquad \text{.......... (2)}$$

Thus from (1) and (2), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

13. If $\delta\Delta$ is the error in the area Δ of a triangle ABC with sides a, b, c, occurs due to the errors δb, δc, δA in b, c, A respectively, prove that

$$\frac{\delta\Delta}{\Delta} = \frac{\delta b}{b} + \frac{\delta c}{c} + \cot A\,\delta A.$$

Solution : We know that area of a triangle is given by,

$$\Delta = \frac{1}{2}\,bc\sin A$$

Taking logarithm on both sides, we get

$$\log\Delta = \log b + \log c + \log\sin A + \log\frac{1}{2}$$

$$\therefore\ \frac{\delta\Delta}{\Delta} = \left[\frac{\delta b}{b} + \frac{\delta c}{c} + \frac{1}{\sin A}\cos A \cdot \delta A\right]$$

$$\therefore\ \frac{\delta\Delta}{\Delta} = \frac{\delta b}{b} + \frac{\delta c}{c} + \cot A\cdot\delta A$$

14. With the usual meaning for a, b, c and s if Δ be the area of a triangle, show that the error in Δ due to small error in c, is given by

$$\delta\Delta = \frac{\Delta}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c.$$

Solution : We know that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},\ \text{where } 2s = a + b + c.$$

Taking logarithm on both sides, we get

$$\log\Delta = \frac{1}{2}\,[\log s + \log(s-a) + \log(s-b) + \log(s-c)]$$

$$\therefore \quad \frac{\delta\Delta}{\Delta} = \frac{1}{2}\left[\frac{\delta s}{s} + \frac{\delta s}{s-a} + \frac{\delta s}{s-b} - \frac{\delta s - \delta c}{s-c}\right] \qquad (\because a, b \text{ are constants})$$

But $2\delta s = \delta c$

$$\therefore \quad \frac{\delta\Delta}{\Delta} = \frac{1}{2}\left[\frac{1}{2}\,\frac{\delta c}{s} + \frac{1}{2}\,\frac{\delta c}{s-a} + \frac{1}{2}\,\frac{\delta c}{s-b} - \frac{\frac{\delta c}{2} - \delta c}{s-c}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right]\delta c.$$

15. In a plane triangle ABC, if the sides a, b are constants, prove that the variation of its angles are related by

$$\frac{\delta A}{\sqrt{a^2 - b^2\sin^2 A}} = \frac{\delta B}{\sqrt{b^2 - a^2\sin^2 B}} = \frac{-\delta C}{c}$$

Solution : For the triangle ABC, we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$\therefore \quad a\sin B = b\sin A$

$\therefore \quad a\cos B\;\delta B = b\cos A\;\delta A$

$$\therefore \quad \frac{\delta A}{a\cos B} = \frac{\delta B}{b\cos A} = \frac{\delta A + \delta B}{a\cos B + b\cos A} \qquad \text{.......... (1)}$$

By cosine rule, we have

$$a\cos B + b\cos A = c \qquad \text{.......... (2)}$$

Again for any triangle $A + B + C = \pi$

$$\therefore \quad \delta A + \delta B + \delta C = 0 \qquad \text{.......... (3)}$$

Also $a\cos B = a\sqrt{1 - \sin^2 B}$

$$= a\sqrt{1 - \frac{b^2}{a^2}\sin^2 A} \qquad \left(\because \sin B = \frac{b}{a}\sin A\right)$$

$$= \sqrt{a^2 - b^2\sin^2 A} \qquad \text{.......... (4)}$$

Similarly $b \cos A = \sqrt{b^2 - a^2 \sin^2 B}$ .... (5)

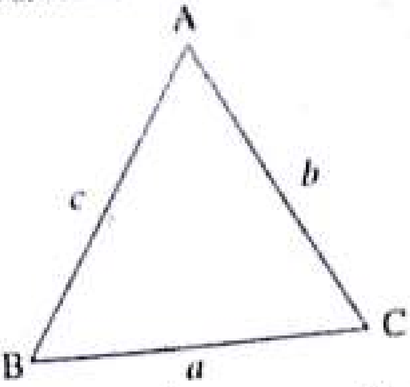
Thus from (1) to (5), we get

$$\frac{\delta A}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{\delta B}{\sqrt{b^2 - a^2 \sin^2 B}} = -\frac{\delta C}{c}$$

16. The angles of a triangle ABC are computed from the sides a, b, c. If small changes δa, δb, δc are made in the sides, prove that

$$\delta A = \frac{a}{2\Delta}\left[\delta a - \delta b \cdot \cos C - \delta c \cdot \cos B\right]$$

where Δ is the area of the triangle.



Solution : For any triangle ABC, we know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \; -\sin A \cdot \delta A = \frac{1}{2}\left[\frac{bc\,\{2b\delta b + 2c\delta c - 2a\delta a\} - (b^2 + c^2 - a^2)\,(b\delta c + c\delta b)}{b^2c^2}\right]$$

$$= \frac{1}{2b^2c^2}\left[\{2b^2c - b^2c - c^3 + a^2c\}\,\delta b + \{2bc^2 - b^3 - bc^2 + a^2b\}\,\delta c - 2abc\,\delta a\right]$$

$$= \frac{1}{2b^2c^2}\left[c\{b^2 + a^2 - c^2\}\,\delta b + b\,\{c^2 + a^2 - b^2\}\,\delta c - 2abc\,\delta a\right]$$

$$= \frac{abc}{b^2c^2}\left[\frac{b^2 + a^2 - c^2}{2ab}\delta b + \frac{c^2 + a^2 - b^2}{2ac}\delta c - \delta a\right]$$

$$= \frac{a}{bc}\left[\cos C \cdot \delta b + \cos B \cdot \delta c - \delta a\right]$$

But $\Delta = \frac{1}{2}\, bc \sin A \Rightarrow bc \sin A = 2\Delta$

$$\therefore \; \delta A = \frac{a}{2\Delta}\left[\delta a - \cos C \cdot \delta b - \cos B \cdot \delta c\right]$$

17. The torsional rigidity N of a length l of a wire is obtained from the formula $N = \frac{8\pi l}{r^4 t^2}$. Find the percentage error in N due to -2% error in l, 2% error in r and 1.5% error in t.

Solution : We have

$$N = \frac{8\pi l}{r^4 t^2}$$

Taking logarithm on both sides, we get

$$\log N = \log 8\pi + \log l - 4 \log r - 2 \log t.$$

$$\therefore \quad \frac{\delta N}{N} = \frac{\delta l}{l} - 4\frac{\delta r}{r} - 2\frac{\delta t}{t}$$

$$\therefore \quad \frac{\delta N}{N}100 = \frac{\delta l}{l}100 - 4\frac{\delta r}{r}100 - 2\frac{\delta t}{t}100.$$

Given that $\frac{\delta l}{l}100 = -2,\ \frac{\delta r}{r}100 = 2,\ \frac{\delta t}{t}100 = 1.5$

$$\therefore \quad \frac{\delta N}{N}100 = -2 - 4(2) - 2(1.5)$$

$$= -2 - 8 - 3 = -13\%$$

18. The quantity Q of water flowing over a triangular notch is given by $Q = CH^{5/2}$, where H is head of water and C is a constant. Find the percentage error in Q if the error in H is that, it is measured 0.198 instead of 0.2.

Solution : We have $Q = CH^{5/2}$

Taking logarithm on both sides, we get

$$\log Q = \log c + \frac{5}{2}\log H$$

$$\therefore \quad \frac{\delta Q}{Q} = \frac{5}{2}\frac{\delta H}{H}$$

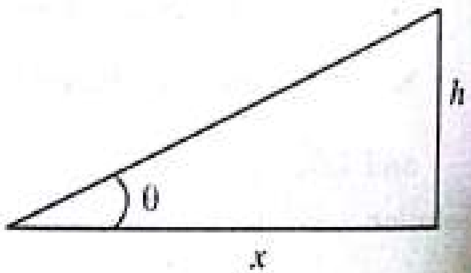
$$\therefore \quad \frac{\delta Q}{Q}100 = \frac{5}{2}\frac{\delta H}{H}100$$

Given that $H = 0.2,\ \delta H = -0.002$

$$\therefore \quad \frac{\delta Q}{Q}100 = \frac{5}{2}\cdot\left(-\frac{0.002}{0.2}\right)100 = -2.5\%$$

19. At a distance of 50 meters from the foot of the, tower the elevation of its top is 30°. If the possible error in measuring the distance and elevation are 2 cm and 0.05 degree respectively, find the approximate error in calculating the height.

Solution : Let h be the height of a tower, x be the horizontal distance and θ is the angle of elevation.



$$\therefore \quad \tan\theta = \frac{h}{x} \Rightarrow h = x\tan\theta$$

$$\therefore \quad \delta h = \delta x \cdot \tan\theta + x\sec^2\theta\cdot\delta\theta$$

Now, given that $x = 50$ meter, $\theta = 30^{\circ}$ and

$$\delta x = 2\text{cm} = \frac{2}{100}\text{ meter} = 0.02\text{ meter},\ \delta\theta = 0.05^{\circ}$$

$\therefore\ \delta h = (0.02)\ (\tan 30^0) + (50)\ (\sec^2 30^0)\ (0.05)$

$$= (0.02)\left(\frac{1}{\sqrt{3}}\right) + (50)\left(\frac{4}{3}\right)(0.05)$$

$$= 0.0115 + 3.3334 = 3.3449 \text{ mt.}$$

20. To sides a, b of a triangle and included angle C are measured. Show that the error δc in the computed length of the side c due to a small error in the angle C is given by $\delta c = a \sin B \cdot \delta C$.

Solution : From the cosine rule of a triangle, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$\therefore\ 2ab \cos C = a^2 + b^2 - c^2$

$\therefore\ -2ab \sin C \cdot \delta C = -2c\,\delta c$ $\qquad$ ($\because$ a, b are unchanged)

$\therefore\ ab \sin C \cdot \delta C = c\delta c$ $\qquad$ .... (1)

Again from the sine rule, we have

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sin C = \frac{c \sin B}{b} \qquad \text{........ (2)}$$

Thus from (1) and (2), we get

$$ab\left(\frac{c\sin B}{b}\right)\delta C = c\delta c.$$

$\therefore\ a\sin B \cdot \delta C = \delta c.$

EXERCISE – 2.3

1. If kinetic energy $K = \frac{wv^2}{2g}$, where g is constant, find approximately the change in kinetic energy as w changes from 49 to 49.5 and v change from 1600 to 1590.

   Ans. : $-\frac{144000}{g}$

2. Find the percentage error in s, where $s = kp^{1/2}\,V^2$ if percentage errors in p and V are respectively 2 and 1.5

   Ans. : 4%

3. The voltage V across a resistor is measured with an error h and the resistance R is measured with an error k. Show that the error in computing the power $w = \frac{V^2}{R}$ generated in the resistor is $\frac{V}{R^2}(2Rh - Vk)$.

4. The sides of a triangle are measured as 12 cm and 15 cm and the angle included between them as 60^0. If the lengths can be measured within 1% accuracy while the angle can be measured within 2% accuracy, find the percentage error in determining (a) area of the triangle (b) length of the opposite side of the triangle.

5. Find the approximate errors in volume and surface area of a rectangular parallelopiped of sides a, b, c due to error h in measuring each side.

Ans. : $(ab + bc + ca)h$, $4(a + b + c)h$

6. The work that must be done to propel a ship of displacement D for a distance S in time t is proportional to $S^2 D^{2/3} t^{-2}$. If the displacement is increased by 1%, the time is diminished by 1% and the distance is increased by 3%, find approximately the percentage increase of work. **Ans. :** $\frac{14}{3}$%

7. The radius of a circle is found to be 100 cm. Find the relative error in the area of the circle due to an error of 1 mm in measuring the radius. **Ans. :** 0.002 m^2

8. The period of oscillation of a pendulum is computed by the formula $T = 2\pi\sqrt{\frac{l}{g}}$

Prove that the percentage error in $T = \frac{1}{2}$ [% error in l – % error in g]. If $l = 6$ cm and relative error in g is equal to $\frac{1}{160}$, find the error in the determination of T ($g = 981$ cm/sec^2). **Ans. :** – 0.0015

9. Find the percentage error in the area of a rectangle if the percentage error in the measurement of its sides are 1%. **Ans. :** 2%

10. Find the percentage error in computing the volume of a right circular cone, if the errors of 2% and 1% are made in measuring the height and base radius respectively.

Ans. : 4%

11. The range of a projectile which starts with a velocity V at an elevation α is given by $R = \frac{V^2 \sin 2\alpha}{g}$. Find the percentage error in R due to an error of 1% in V and an error of $\frac{1}{2}$ % in α.

Ans. : $2 + \alpha \cot$ [illegible]

12. In estimating the cost of a pile of bricks measured as 6 ft × 50 ft × 4 ft, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to ft^3, and bricks cost Rs. 1800 per 1000 bricks, find the approximate error in the cost.

Ans. : Rs. 777.60

2.6 MAXIMA AND MINIMA :

Consider a function $u = f(x, y)$ of two independent variables x, y, whose domain is a certain region R in the xy–plane.

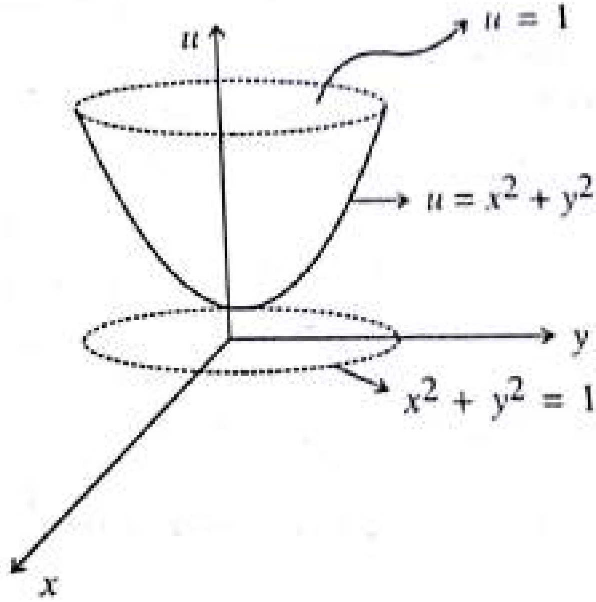
We say that $f(x, y)$ has a (absolute) **maximum** at the point (x_0, y_0) of its domain R if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in R. Such a maximum corresponds to a highest point of the surface S.

We say that $f(x, y)$ has a **strict maximum** at the point (x_0, y_0) of its domain R, if actually, $f(x_0, y_0) > f(x, y)$ for all (x, y) in R that are different from (x_0, y_0). Thus the greatest value is reach only at a single point.

Similarly $f(x, y)$ is said to have a (absolute) **minimum** at the point (x_1, y_1) of R if $f(x_1, y_1) \leq f(x, y)$ for all (x, y) in R. Such a minimum corresponds to a lowest point of the surface S.

Again $f(x, y)$ has a **strict minimum** if $f(x_1, y_1) < f(x, y)$ for all $(x, y) \neq (x_1, y_1)$ in R.

Note : If R is a closed and bounded set and f continuous in R, then there exist points in R where f has its maximum and also points where f has its minimum.



For example, consider the function $u = x^2 + y^2$ defined in the closed disc given by $x^2 + y^2 < 1$. The function represents the surface of paraboloid lying below the plane $u = 1$. Here the maxima of f occur at all the points of the boundary of the circle $x^2 + y^2 = 1$, whereas f has a strict minimum at the origin.

2.6.1. RELATIVE MAXIMA AND MINIMA :

Most of calculus applies directly to the determination of relative maxima or minima, rather than of absolute extrema.

Note : Here we shall use the term **"extremum"** to include either maximum or minimum.

A point (x_0, y_0) of the domain R is a **relative maximum** if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) of R that lie in a sufficiently small neighbourhood of (x_0, y_0). i.e. $f(x_0, y_0) \leq f(x_0 + h, y_0 + k)$, for sufficiently small values of h and k.

Note : The value $f(x_0, y_0)$ at a relative maximum does not have to be the greatest value of f in all of R but is a maximum of f if we restrict ourselves to points sufficiently close to (x_0, y_0).

Similarly a point (x_0, y_0) of the domain R is a **relative minimum** if $f(x_0, y_0) \le f(x, y)$ for all points (x, y) of R that lie in a sufficiently small neighbourhood of (x_0, y_0). i.e. $f(x_0, y_0) \le f(x_0 + h, y_0 + k)$ for sufficiently small values of h and k.

Note : Every absolute extremum also is a relative extremum, but the converse does not hold.

Similarly we can extend the definition of relative extremum for the function $u = f(x, y, z, ...)$ of more than two independent variables.

Now we discuss necessary and sufficient conditions for the existence of relative extremum at the point (x_0, y_0) of the domain R of the function $f(x, y)$.

NECESSARY CONDITIONS : Let the function $f(x, y)$ have partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ at the point (x_0, y_0) of the domain R of the function. For a relative extremum of f to occur at the point (x_0, y_0), it is necessary that

$$f_x(x_0, y_0) = 0,\ f_y(x_0, y_0) = 0.$$

Note : The point (x_0, y_0) is called **stationary** or **critical** point of $f(x, y)$ if the partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ both exist and vanish.

Remark : Every relative extremum is a stationary point, but the converse does not hold.

SUFFICIENT CONDITIONS :

We know that the stationary point (x_0, y_0) of a function $f(x, y)$ is said to be maximum or minimum if and only if the expression

$$f(x_0 + h, y_0 + k) - f(x_0, y_0)$$

is negative or positive respectively. That is, it has the same sign for all sufficiently small values of h and k. If we exand this expression by Taylor's series and use the equations $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, we obtain

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = \frac{1}{2}\ \{h^2 f_{xx}(x_0, y_0) + 2hk\, f_{xy}(x_0, y_0) + k^2 f_{yy}(x_0, y_0)\} + O\ (h^3, k^3).$$

where $O(h^3, k^3)$ contains the remainder part of the series, which tends to zero due to the smallness of h and k.

Thus the sign of $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ is essential, determined by the expression

$$Q\ (h, k) = rh^2 + 2shk + tk^2$$

where $r = f_{xx}(x_0, y_0)$, $s = f_{xy}(x_0, y_0)$, $t = f_{yy}(x_0, y_0)$.

Again $Q\ (h, k)$ is a quadratic expression in h and k.

Note : Here we assume that r, s and t do not all vanish. If it does so, then we must begin with a Taylor's Series extending to terms of higher order.

Now $Q(h, k) = \frac{1}{2r}[h^2r^2 + 2rshk + rtk^2]$

$$= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)]$$

But $(hr + ks)^2$ is always positive. Thus the sigh of $Q(h, k)$ depends on the sign of $rt - s^2$.

1. If $rt - s^2 > 0$ then $Q(h, k)$ has the same sign as that of r for all values of h and k.
   (a) If $r < 0$ then $Q(h, k)$ is negative and hence

   $f(x_0 + h, y_0 + k) - f(x_0, y_0) \leq 0$

   $\therefore$ $f(x, y)$ has a maximum at (x_0, y_0)

   (b) If $r > 0$ then $Q(h, k)$ is positive and hence

   $f(x_0 + h, y_0 + k) - f(x_0, y_0) \geq 0$

   $\therefore$ $f(x, y)$ has a minimum at (x_0, y_0).
2. If $rt - s^2 < 0$ then $Q(h, k)$ changes sign with different values of h and k. Thus there is neither maximum nor minimum at (x_0, y_0). In this case the point (x_0, y_0) is called a saddle point.
3. If $rt - s^2 = 0$ then we cannot conclude about a maximum or minimum point. In this case further investigation is required.

2.6.2 WORKING RULE TO FIND THE EXTREMUM VALUE :

Consider the function $u = f(x, y)$.

Obtain the first and second order derivatives, such as

$p = f_x$, $q = f_y$, $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

1. Take $f_x = 0$, $f_y = 0$ and solve them simultaneously to obtain stationary points.

   Let (x_0, y_0), (x_1, y_1), ... be the stationary points
2. Consider the stationary point (x_0, y_0).

   Obtain the values of r, s and t at (x_0, y_0).

   (a) If $rt - s^2 > 0$ then the extremum values exist.

   $\rightarrow$ If $r < 0$ the value is maximum.

   $\rightarrow$ If $r > 0$ the value is minimum.

   (b) If $rt - s^2 < 0$ the there is no extremum value exist.

   (c) If $rt - s^2 = 0$ then we can't say about extremum value and further investigation is required.
3. Follow the same procedure for other stationary points.

2.6.3 GEOMETRICAL INTERPRETATION :

For the function $u = f(x, y)$ the necessary condition $f_x = 0$ and $f_y = 0$ states that the tangent plane to the surface is horizontal. That is the tangent plane is parallel to the xy – plane. Now if we really have an extreme value, then in the neighbourhood of the point the tangent plane does not intersect the surface. In the case of a saddle point, the plane cuts the surface in a curve that has several branches at the point.

SOLVED EXAMPLES

1. Find the stationary value of $x^3 + y^3 - 3axy$, $a > 0$. Also find the extremum values.

Solution : We have $f(x, y) = x^3 + y^3 - 3axy$

$\therefore$ $f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = t_{yy} = 6y$

Taking $f_x = 0$, $f_y = 0$, we get

$3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$

$\therefore$ $x^2 - ay = 0$ and $y^2 - ax = 0$

Solving these two equations, we get

$$\therefore \frac{y^4}{a^2} - ay = 0 \Rightarrow y^4 - a^3y = 0$$

$\therefore$ $y(y^3 - a^3) = 0 \Rightarrow y = 0, a$

$\therefore$ When $y = 0$ we have $x = 0$

and when $y = a$ we have $x = a$.

$\therefore$ Stationary point are $(0, 0)$, (a, a).

For (0, 0) :

$r = 0$, $s = -3a$, $t = 0 \Rightarrow rt - s^2 = -9a^2 < 0$

$\therefore$ There is no extreme value at $(0, 0)$.

For (a, a) :

$r = 6a$, $s = -3a$, $t = 6a$

$\therefore$ $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

Again $r > 0 \Rightarrow f(x, y)$ is minimum at $(0, 0)$.

$\therefore$ Minimum value $= a^3 + a^3 - 3a^3 = -a^3$.

2. Find the extreme value of $x^2 + y^2 + 6x + 12$.

Solution : We have $f(x, y) = x^2 + y^2 + 6x + 12$

$\therefore$ $f_x = 2x + 6$, $f_y = 2y$

$r = 2$, $s = 0$, $t = 2$

Take $f_x = 0$, $f_y = 0 \Rightarrow 2x + 6 = 0$, $2y = 0 \Rightarrow x = -3$, $y = 0$.

$\therefore$ $(-3, 0)$ is a stationary point.

Again $rt - s^2 = 4 > 0$ and $r > 0$.

$\therefore$ $f(x, y)$ is minimum at $(-3, 0)$.

$\therefore$ minimum value $= f(-3, 0) = 9 - 18 + 12 = 3$.

3. Find the extreme values for $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Solution : We have $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

$\therefore f_x = 3x^2 + 3y^2 - 6x, f_y = 6xy - 6y$

Again $r = 6x - 6$, $s = 6y$, $t = 6x - 6$

Take $f_x = 0$, $f_y = 0$ and solve them simultaneously.

$\therefore$ $3x^2 + 3y^2 - 6x = 0$ and $6xy - 6y = 0$

$\therefore$ $x^2 + y^2 - 2x = 0$ and $y(x - 1) = 0 \Rightarrow y = 0$ or $x = 1$

When $y = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2$

When $x = 1 \Rightarrow 1 + y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$\therefore$ The stationary points are : $(0, 0)$, $(2, 0)$, $(1, 1)$, $(1, -1)$.

	$(0, 0)$	$(2, 0)$	$(1, 1)$	$(1, -1)$
r	-6	6	0	0
s	0	0	6	-6
t	-6	6	0	0
$rt - s^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$

$\therefore$ $f(x, y)$ has maximum at $(0, 0)$ and $f(0, 0) = 4$

and $f(x, y)$ has minimum at $(2, 0)$ and $f(2, 0) = 8 - 12 + 4 = 0$.

4. For $f(x, y) = x^3 + y^3 - 3xy$, examine the extremum values.

Solution : We have $f(x, y) = x^3 + y^3 - 3xy$

$\therefore$ $f_x = 3x^2 - 3y, f_y = 3y^2 - 3x$

$r = 6x$, $s = -3$, $t = 6y$

Take $f_x = 0$, $f_y = 0$ and solve them simultaneously.

$\therefore$ $3x^2 - 3y = 0$ and $3y^2 - 3x = 0$

$\therefore$ $x^2 - y = 0$ and $y^2 - x = 0$

$\therefore$ $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1$

$\therefore$ when $x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = 1$

$\therefore$ The stationary points are $(0, 0)$ and $(1, 1)$.

For (0, 0) :

$r = 0,\ s = -3,\ t = 0$

$\therefore\ rt - s^2 = -9 < 0$

$\therefore$ There is no extremum at (0, 0).

For (1, 1) :

$r = 6,\ s = -3,\ t = 6$

$\therefore\ rt = s^2 = 36 - 9 = 27 > 0$ and $r > 0$

$\therefore$ $f(x, y)$ is minimum at (1, 1).

$\therefore\ f(1, 1) = 1 + 1 - 3 = -1.$

5. Discuss the extrema of $x^3y^2\,(1 - x - y)$.

Solution : We have $f(x, y) = x^3y^2\,(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$

$\therefore\ f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2\,(3 - 4x - 3y)$

$f_y = 2x^3y - 2x^4y - 3x^3y^2 = x^3y\,(2 - 2x - 3y)$

$r = 6xy^2 - 12x^2y^2 - 6xy^3$

$s = 6x^2y - 8x^3y - 9x^2y^2,\ t = 2x^3 - 2x^4 - 6x^3y$

Now take $f_x = 0$, $f_y = 0$ and solve them simultaneously,

$\therefore\ x^2y^2\,(3 - 4x - 3y) = 0$ and $x^3y\,(2 - 2x - 3y) = 0$

$\therefore\ 3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$

$\Rightarrow\ x = \dfrac{1}{2},\ y = \dfrac{1}{3}$

$\therefore$ Stationary point is $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$

$$\therefore\ r = 6 \cdot \frac{1}{2} \cdot \frac{1}{9} - 12 \cdot \frac{1}{4} \cdot \frac{1}{9} - 6 \cdot \frac{1}{2} \cdot \frac{1}{27} = \frac{6}{18} - \frac{12}{36} - \frac{6}{54} = \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$$

$$s = 6 \cdot \frac{1}{4} \cdot \frac{1}{3} - 8\,\frac{1}{8} \cdot \frac{1}{3} - 9 \cdot \frac{1}{4} \cdot \frac{1}{9} = \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$t = 2 \cdot \frac{1}{8} - 2 \cdot \frac{1}{16} - 6 \cdot \frac{1}{8} \cdot \frac{1}{3} = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$\therefore\ r \cdot t - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2 - 1}{144} = \frac{1}{144} > 0$$

Also $r < 0$.

$\therefore$ $f(x, y)$ is minimum at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$

$$\therefore\ f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}.$$

6. Find the extremum values for $\sin x + \sin y + \sin (x + y)$.

Solution : We have $f(x, y) = \sin x + \sin y + \sin(x + y)$.

$\therefore \quad f_x = \cos x + \cos(x + y),\ f_y = \cos y + \cos (x + y).$

$r = -\sin x - \sin (x + y),\ s = -\sin (x + y),\ t = -\sin y - \sin (x + y).$

Now take $f_x = 0$ and $f_y = 0$ and solve them simultaneously.

$\therefore \quad \cos x + \cos(x + y) = 0$ and $\cos y + \cos (x + y) = 0$

$\Rightarrow \cos x - \cos y = 0 \Rightarrow \cos x = \cos y \Rightarrow x = y.$

$\therefore$ From the first equation,

$$\cos x + \cos 2x = 0 \Rightarrow \cos 2x = -\cos x$$
$$\Rightarrow \cos 2x = \cos (\pi - x)$$
$$\Rightarrow 2x = \pi - x \Rightarrow 3x = \pi \Rightarrow x = \frac{\pi}{3}.$$

$\therefore \quad y = \frac{\pi}{3}.$

$\therefore$ Stationary points is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\therefore \quad r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3},\ s = -\frac{\sqrt{3}}{2},\ t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$$

$$\therefore \quad rt - s^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

Also $r < 0$. Thus $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\therefore \quad f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

7. For the plane triangle ABC, find the extreme value of $\cos A \cdot \cos B \cdot \cos C$.

Solution : We have $f(A, B, C) = \cos A \cdot \cos B \cdot \cos C$

Here are three variables A, B and C. Thus we use the relation $A + B + C = \pi$

$$\therefore \quad F(A, B) = f(A, B, C) = \cos A \cdot \cos B \cdot \cos(\pi - A - B)$$
$$= -\cos A \cdot \cos B \cdot \cos(A + B)$$

$$\therefore \quad F_A = [\sin A \cos(A + B) + \cos A \sin (A + B)] \cos B$$
$$= \cos B \cdot \sin (2A + B)$$
$$F_B = \cos A\,[\sin B \cos(A + B) + \cos B \sin (A + B)]$$
$$= \cos A \cdot \sin (A + 2B)$$

$r = F_{AA} = 2\cos B \cos(2A + B)$

$s = F_{AB} = -\sin B \cdot \sin(2A + B) + \cos B \cdot \cos(2A + B) = \cos(2A + 2B)$

$t = F_{BB} = 2\cos A \cdot \cos(A + 2B)$

Now solving the equations $F_A = 0$, $F_B = 0$, we get

$\cos B \sin(2A + B) = 0 \quad \Rightarrow \cos B = 0, \sin(2A + B) = 0$

and $\cos A \cdot \sin(A + 2B) = 0 \quad \Rightarrow \cos A = 0, \sin(A + 2B) = 0$

Thus we have following pairs of equations :

(i) $\cos B = 0, \cos A = 0$

$\Rightarrow B = \frac{\pi}{2}, A = \frac{\pi}{2}$

(ii) $\cos B = 0, \sin(A + 2B) = 0$

$\Rightarrow B = \frac{\pi}{2} \Rightarrow \sin(A + \pi) = 0 \Rightarrow -\sin A = 0 \Rightarrow A = 0$

which is not possible as $A = 0$.

(iii) $\sin(2A + B) = 0, \cos A = 0$

$\therefore A = \frac{\pi}{2} \Rightarrow \sin(\pi + B) = 0 \Rightarrow -\sin B = 0 \Rightarrow B = 0$

which is also not possible as $B = 0$

(iv) $\sin(2A + B) = 0, \sin(A + 2B) = 0$

$\therefore 2A + B = \pi$ and $A + 2B = \pi$

$\therefore A = B = \frac{\pi}{3}$

$\therefore$ The stationary points are $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

Now for $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$r = 0, s = 1, t = 0 \Rightarrow rt - s^2 = -1 < 0$

$\therefore f(x, y)$ does not have any extremum value.

For $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$:

$r = -1, s = -\frac{1}{2}, t = -1$

$\therefore rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$. Also $r < 0$

$\therefore f(x, y)$ has a maximum value of $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\therefore f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{2\pi}{3} = \frac{1}{8}$$

8. Discuss the extremum of $2(x-y)^2 - x^4 - y^4$ leaving aside any doubtful case that may arise.

Solution : We have $f(x, y) = 2(x-y)^2 - x^4 - y^4$.

$\therefore \quad f_x = 4(x-y) - 4x^3,\ f_y = -4(x-y) - 4y^3$

$r = 4 - 12x^2,\ s = -4,\ t = 4 - 12y^2$

Solving the equations $f_x = 0$, $f_y = 0$, we get

$4(x-y) - 4x^3 = 0$ and $-4(x-y) - 4y^3 = 0$

$\Rightarrow x - y - x^3 = 0$ and $-x + y - y^3 = 0$

Adding, we get $-x^3 - y^3 = 0 \Rightarrow x^3 + y^3 = 0 \Rightarrow x = -y$.

$\therefore$ From the 1st equation : $-y - y + y^3 = 0 \Rightarrow y^3 - 2y = 0$

$\Rightarrow y(y^2 - 2) = 0$

$\Rightarrow y = 0, \sqrt{2}, -\sqrt{2}$

From 2nd equation, we get

$y = 0 \Rightarrow x = 0$

$y = \sqrt{2} \Rightarrow -x + \sqrt{2} - 2\sqrt{2} = 0 \Rightarrow -x - \sqrt{2} = 0 \Rightarrow x = -\sqrt{2}$

$y = -\sqrt{2} \Rightarrow -x - \sqrt{2} + 2\sqrt{2} = 0 \Rightarrow -x + \sqrt{2} = 0 \Rightarrow x = \sqrt{2}$

$\therefore$ Stationary points are : $(0, 0)$, $\left(-\sqrt{2}, \sqrt{2}\right)$, $\left(\sqrt{2}, -\sqrt{2}\right)$

	r	s	t	$rt - s^2$
$(0, 0)$	4	-4	4	0
$\left(-\sqrt{2}, \sqrt{2}\right)$	-20	-4	-20	$384 > 0$
$\left(\sqrt{2}, -\sqrt{2}\right)$	-20	-4	-20	$384 > 0$

$\therefore$ The extremum of $f(x, y)$ doubtful at $(0, 0)$.

and $f(x, y)$ has maximum at $\left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$

$\therefore \quad f\left(-\sqrt{2}, \sqrt{2}\right) = 2\left(-\sqrt{2}, -\sqrt{2}\right)^2 - 4 - 4 = 16 - 8 = 8$

$f\left(\sqrt{2}, -\sqrt{2}\right) = 2\left(\sqrt{2} + \sqrt{2}\right)^2 - 4 - 4 = 8$

9. Find the maximum and minimum values of $x^3 + y^3 - 63(x + y) + 12xy$.

Solution : We have $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$.

$\therefore \quad f_x = 3x^2 - 63 + 12y, f_y = 3y^2 - 63 + 12x$

$r = 6x, s = 12, t = 6y$

$\therefore$ Solving $f_x = 0$ and $f_y = 0$, we get

$3x^2 - 63 + 12y = 0$ and $3y^2 - 63 + 12x = 0$

$\Rightarrow x^2 + 4y = 21$ and $y^2 + 4x = 21$

Subtracting 2nd equation from the 1st one, we get

$x^2 - y^2 + 4y - 4x = 0$

$\Rightarrow (x - y)(x + y) - 4(x - y) = 0$

$\Rightarrow x - y = 0$ and $x + y - 4 = 0$

Now, when $x - y = 0 \Rightarrow x = y$

$\Rightarrow$ From 1st equation : $x^2 + 4x - 21 = 0$

$\Rightarrow (x + 7)(x - 3) = 0 \Rightarrow x = -7, 3$

$\therefore \quad x = -7 = y$ and $x = 3 = y$.

Now when $x + y - 4 = 0 \Rightarrow$ From the 1st equation

$\Rightarrow x^2 + 4(4 - x) - 21 = 0 \Rightarrow x^2 - 4x - 5 = 0$

$\Rightarrow (x - 5)(x + 1) = 0 \Rightarrow x = 5, -1$

$\therefore \quad x = 5 \Rightarrow y = 1$ and $x = -1 \Rightarrow y = 5$

$\therefore$ The stationary points are : $(-7, -7), (3, 3), (5, 1), (-1, 5)$.

	r	s	t	$rt - s^2$
$(-7, -7)$	-46	12	-42	$1620 > 0$
$(3, 3)$	18	12	18	$180 > 0$
$(5, 1)$	-30	12	-6	$-324 < 0$
$(-1, 5)$	-6	12	13	$-324 < 0$

$\therefore \quad f(x, y)$ has maximum at $(-7, -7)$ and minimum at $(3, 3)$.

$\therefore \quad f(-7, -7) = -343 - 343 + 882 + 588 = 784$

and $f(3, 3) = 27 + 27 - 378 + 108 = -216$.

10. Find the extreme values of $xy(a - x - y)$, $a > 0$.

Solution : We have $f(x, y) = xya - x^2y - xy^2$

$\therefore \quad f_x = ay - 2xy - y^2 = y(a - 2x - y)$

$f_y = ax - x^2 - 2xy = x(a - x - 2y)$

$\therefore$ $r = -2y$, $s = a - 2x - 2y$, $t = -2x$.

Solving $f_x = 0$ and $f_y = 0$, we get

$y(a - 2x - y) = 0 \Rightarrow y = 0,\ a - 2x - y = 0$

and $x(a - x - 2y) = 0 \Rightarrow x = 0,\ a - x - 2y = 0$

$\therefore$ we have the following pairs of equations :

(*i*) $y = 0,\ x = 0$.

(*ii*) $y = 0$ and $a - x - 2y = 0 \Rightarrow x = a$

(*iii*) $a - 2x - y = 0$ and $x = 0 \Rightarrow x = 0$ and $y = a$

(*iv*) $a - 2x - y = 0$ and $a - x - 2y = 0 \Rightarrow x = \frac{a}{3},\ y = \frac{a}{3}$

$\therefore$ The stationary points are : $(0, 0)$, $(a, 0)$, $(0, a)$, $\left(\frac{a}{3}, \frac{a}{3}\right)$

	r	s	t	$rt - s^2$
$(0, 0)$	0	a	0	$-a^2 < 0$
$(a, 0)$	0	$-a$	$-2a$	$-a^2 < 0$
$(0, a)$	$-2a$	$-a$	0	$-a^2 < 0$
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$-\frac{2a}{3}$	$-\frac{a}{3}$	$-\frac{2a}{3}$	$\frac{a^2}{3} > 0$

$\therefore$ $f(x, y)$ has maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$\therefore\quad f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^2}{9}\left(\frac{a}{3}\right) = \frac{a^3}{27}$$

11. Find the extreme values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Solution : We have $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

$\therefore$ $f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

$r = 6x - 30$, $s = 6y$, $t = 6x - 30$

Solving $f_x = 0$ and $f_y = 0$, we get

$3x^2 + 3y^2 - 30x + 72 = 0$ and $6xy - 30y = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 = 0$ and $y(x - 5) = 0$

$\therefore$ Second equation $\Rightarrow y = 0,\ x = 5$

From the first equation :

when $y = 0 \Rightarrow x^2 - 10x + 24 = 0 \Rightarrow (x-6)(x-4) = 0 \Rightarrow x = 4, 6$

when $x = 5 \Rightarrow 25 + y^2 - 50 + 24 = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1$.

$\therefore$ Stationary points are : (4, 0), (6, 0), (5, 1), (5, – 1).

	r	s	t	$rt - s^2$
(4, 0)	– 6	0	– 6	$36 > 0$
(6, 0)	6	0	6	$36 > 0$
(5, 1)	0	6	0	$-36 < 0$
(5, – 1)	0	– 6	0	$-36 < 0$

$\therefore$ $f(x, y)$ has maximum at (4, 0) and minimum at (6, 0).

$\therefore$ $f(4, 0) = 64 - 240 + 288 = 112$.

$f(6, 0) = 216 - 540 + 432 = 108$

12. Find the extreme values of $x^2 + y^2 + xy + x - 4y + 5$.

Solution : We have $f(x, y) = x^2 + y^2 + xy + x - 4y + 5$

$\therefore$ $f_x = 2x + y + 1,\ f_y = 2y + x - 4$.

$r = 2,\ s = 1,\ t = 2$

Solving $f_x = 0$ and $f_y = 0$, we get

$2x + y + 1 = 0$ and $2y + x - 4 = 0$

$\Rightarrow 2x + y + 1 - 4y - 2x + 8 = 0 \Rightarrow -3y + 9 = 0 \Rightarrow y = 3$

$\therefore$ $x = 4 - 2y = -2$

$\therefore$ Stationary point is (– 2, 3).

Now $rt - s^2 = 4 - 1 = 3 > 0$. Also $r = 2 > 0$

$\therefore$ $f(x, y)$ has minimum at (–2, 3)

$\therefore$ $f(-2, 3) = 4 + 9 - 6 - 2 - 12 + 5 = -2$.

13. Find the extreme values of $x^2y^2 - 5x^2 - 8xy - 5y^2$.

Solution : We have $f(x, y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

$\therefore$ $f_x = 2xy^2 - 10x - 8y,\ f_y = 2x^2y - 8x - 10y$.

$r = 2y^2 - 10,\ s = 4xy - 8,\ t = 2x^2 - 10$.

Solving $f_x = 0$ and $f_y = 0$, we get

$2xy^2 - 10x - 8y = 0$ and $2x^2y - 8x - 10y = 0$

$\Rightarrow xy^2 - 5x - 4y = 0$ and $x^2y - 4x - 5y = 0$

First equation $\Rightarrow x = \dfrac{4y}{y^2 - 5}$

$\therefore$ Second equation $\Rightarrow \dfrac{16y^2}{(y^2-5)^2}\, y - \dfrac{16y}{y^2-5} - 5y = 0$

$\Rightarrow y\,[16y^2 - 16(y^2 - 5) - 5(y^2 - 5)^2] = 0$

$\Rightarrow y = 0,\ 16y^2 - 16y^2 + 80 - 5(y^2 - 5)^2 = 0$

Now $y = 0 \Rightarrow x = 0$

and $80 - 5(y^2 - 5)^2 \Rightarrow (y^2 - 5)^2 = 16$

$\Rightarrow y^2 - 5 = \pm 4$

$\Rightarrow y^2 = 5 \pm 4 \Rightarrow y^2 = 9,\ y^2 = 1$

$\Rightarrow y = \pm 3,\ y = \pm 1$

when $y = 3 \Rightarrow x = \dfrac{12}{4} = 3$

$y = -3 \Rightarrow x - \dfrac{12}{4} = -3$

$y = 1 \Rightarrow x = \dfrac{4}{-4} = -1$

$y = -1 \Rightarrow x = -\dfrac{4}{-4} = 1$

$\therefore$ Stationary point are : (0, 0), (3, 3), (– 3, – 3), (– 1, 1), (1, – 1)

(x, y)	r	s	t	$rt - s^2$
(0, 0)	– 10	– 8	– 10	36 > 0
(3, 3)	8	28	8	– 720 < 0
(– 3, – 3)	8	28	8	– 720 < 0
(– 1, 1)	– 8	– 12	– 8	– 80 < 0
(1, – 1)	– 8	– 12	– 8	– 80 < 0

$\therefore$ $f(x, y)$ has maximum at (0, 0).

$\therefore$ $f(0, 0) = 0$

14. Divide a given number *a* into three positive parts, such that their sum is *a* and product is maximum.

Solution : Let x, y, z be the positive parts of the number a.

$\therefore$ $x + y + z = a$ and $f(x, y, z) = xyz$ which is to be maximized.

$\therefore$ $f(x, y, z) = F(x, y) = xy(a - x - y)$.

Finding extreme values of $F(x, y)$ as in example – 10, we get the maximum value of $F(x, y)$ at $x = \frac{a}{3}$, $y = \frac{a}{3}$

$$\therefore \quad z = a - \frac{a}{3} - \frac{a}{3} = \frac{a}{3}$$

and $f\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right) = F\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^2}{27}$

15. Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Solution : Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices and (x, y) be the interior point of a triangle. Thus the required sum of the distances is given by

$$f(x, y) = (x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + (x - x_3)^2 + (y - y_3)^2$$

$$\therefore \quad f_x = 2(x - x_1) + 2(x - x_2) + 2(x - x_3) = 6x - 2(x_1 + x_2 + x_3)$$

$$f_y = 2(y - y_1) + 2(y - y_2) + 2(y - y_3) = 6y - 2(y_1 + y_2 + y_3)$$

and $r = 2 + 2 + 2 = 6$, $s = 0$, $t = 2 + 2 + 2 = 6$.

Solving $f_x = 0$ and $f_y = 0$, we get

$$x = \frac{x_1 + x_2 + x_3}{3} \text{ and } y = \frac{y_1 + y_2 + y_3}{3}$$

Again $rt - s^2 = 36 > 0$ and $r > 0$

$$\therefore \quad f(x, y) \text{ is minimum at } \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

16. In a plane triangle ABC, if the perimeter is constant, prove that its area is maximum when the triangle is equilateral.

Solution : Let a, b, c be the sides of a triangle ABC.

$\therefore$ Perimeter $2s = a + b + c$.

Now Area of a triangle is :

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(a+b-s)}$$

$$\therefore \quad f(a, b) = A^2 = s(s-a)(s-b)(a+b-s).$$

$$\therefore \quad f_a = s(s-b)[(a+b-s)(-1) + (s-a)(1)] = s(s-b)(2s - 2a - b)$$

$$f_b = s(s-a)[(a+b-s)(-1) + (s-b)(1)] = s(s-a)(2s - a - 2b)$$

$r = f_{aa} = -2s(s-b)$, $s = f_{ab} = s[(2s-2a-b)(-1)+(s-b)(-1)] = s(2a+2b-3s)$

$t = f_{bb} = -2s(s-a)$

Solving $f_a = 0$ and $f_b = 0$, we get.

$s(s-b)(2s-2a-b) = 0$ and $s(s-a)(2s-a-2b) = 0$

$\Rightarrow (s-b)(2s-2a-b) = 0$ and $(s-a)(2s-a-2b) = 0$ $\quad (\because s \neq 0)$

Now, first equation $\Rightarrow s-b = 0,\ 2s-2a-b = 0$

$\therefore$ From second equation :

When $s = b \Rightarrow (b-a)(2b-a-2b) = 0 \Rightarrow (a-b)a = 0 \Rightarrow a = b$ $\quad (\because a \neq 0)$

$\therefore\ a = s$ and $b = s$

When $2s-2a-b = 0 \Rightarrow b = 2s-2a$

$\Rightarrow (s-a)(2s-a-4s+4a) = 0$

$\Rightarrow (s-a)(-2s+3a) = 0 \Rightarrow a = s$ or $a = \frac{2}{3}s$

Now $a = s \Rightarrow b = 2s-2a = 0$

and $a = \frac{2s}{3} \Rightarrow b = 2s - \frac{4s}{3} = \frac{2s}{3}$

$\therefore$ The stationary points are : (s, s), $(s, 0)$, $\left(\frac{2s}{3}, \frac{2s}{3}\right)$

(a, b)	r	s	t	$rt - s^2$
(s, s)	0	s^2	0	$-s^4 < 0$
$(s, 0)$	$-2s^2$	$-s^2$	0	$-s^4 < 0$
$\left(\frac{2s}{3}, \frac{2s}{3}\right)$	$-\frac{2s^2}{3}$	$-\frac{s^2}{3}$	$-\frac{2s^2}{3}$	$\frac{s^4}{3} > 0$

$\therefore\ f(a, b)$ has maximum at $\left(\frac{2s}{3}, \frac{2s}{3}\right)$

$\therefore\ c = 2s - a - b = 2s - \frac{2s}{3} - \frac{2s}{3} = \frac{2s}{3}$

$\therefore\ a = b = c.$

$\therefore$ The area is maximum when the triangle is equilateral.

17. A rectangular box, open at the top, is to have a given capacity of 64 cm^3. Find the dimensions of the box requiring least material for its construction.

Solution : Let x, y, z be the length, breadth and height of the rectangular box respectively and S be its surface area.

Since the box has given capacity, its volume is constant. That is $xyz = 64$.

Now $S = 2(xz + yz) + xy = 2(x + y)\dfrac{64}{xy} + xy$

$\therefore\ f(x, y) = S = 128\left(\dfrac{1}{y} + \dfrac{1}{x}\right) + xy.$

$\therefore\ f_x = -\dfrac{128}{x^2} + y,\ f_y = -\dfrac{128}{y^2} + x$

$r = \dfrac{256}{x^3},\ s = 1,\ t = \dfrac{256}{y^3}$

Solving $f_x = 0$ and $f_y = 0$, we get

$-\dfrac{128}{x^2} + y = 0$ and $-\dfrac{128}{y^2} + x = 0$

First equation $\Rightarrow y = \dfrac{128}{x^2}$

$\therefore$ Second equation $\Rightarrow -128\left(\dfrac{x^4}{128 \times 128}\right) + x = 0$

$$\Rightarrow x\left(\frac{x^3}{128} - 1\right) = 0$$

$$\Rightarrow x = 0,\ x^3 = 128$$

$$\Rightarrow x = 0,\ x = 4\sqrt[3]{2}$$

Now $x = 0$ is not possible.

$\therefore\ x = 4\sqrt[3]{2} \Rightarrow y = \dfrac{128}{16.2^{2/3}} = \dfrac{8}{2^{2/3}} = 4\sqrt[3]{2}$

Now $r = \dfrac{256}{128} = 2,\ t = 1,\ s = \dfrac{256}{128} = 2$

$\therefore\ rt - s^2 = 3 > 0$ and $r > 0$.

$\therefore\ f(x, y)$ (i.e.S) is minimum at $\left(4\sqrt[3]{2}, 4\sqrt[3]{2}\right)$

$x = 4\sqrt[3]{2},\ y = 4\sqrt[3]{2}$

and $z = \dfrac{64}{xy} = \dfrac{64}{16.2^{2/3}} = \dfrac{4}{2^{2/3}} = 2\sqrt[3]{2}$

EXERCISE – 2.4

1. Find the stationary points and hence extremum values of the following function :

(a) $x^2 + y^2 + 6x + 12$ **Ans. :** Minimum at $(-3, 0)$; value = 3

(b) $xy + \frac{a^3}{x} + \frac{a^3}{y}$ **Ans. :** Minimum at (a, a); value = $3a^2$

(c) $2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$ **Ans. :** Maximum at $\left(\frac{3a}{2}, -\frac{a}{2}\right)$, and $\left(\frac{a}{2}, \frac{a}{2}\right)$;

Minimum at $\left(\frac{a}{2}, -\frac{a}{2}\right)$

(d) $x^4 + y^4 - x^2 - y^2 + 1$ **Ans. :** Maximum value 1 at $(0, 0)$;

Minimum value $\frac{1}{2}$ at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$.

(e) $x^4 + x^2y + y^2$ **Ans. :** Minimum value 0 at $(0, 0)$

(f) $x^3 + 4xy + 3x^2 + y^2$ **Ans. :** Minimum value $-\frac{4}{27}$ at $\left(\frac{2}{3}, -\frac{4}{3}\right)$

(g) $\sin x \sin y \sin(x + y)$ **Ans. :** Maximum value $\frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$;

Minimum value $-\frac{3\sqrt{3}}{8}$ at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$.

2. Prove that the function $2x^2y + x^2 - y^2 + 2y$ has no extremum value.

3. Prove that the function $x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has no extremum value at $(0, 0)$.

2.7 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS :

We have seen in section 2.6 that for the function $u = f(x, y, z)$ of three variables x, y, z which are not independent, but subjected to some condition, say, $\phi(x, y, z) = c$ then we have converted $u = f(x, y, z)$ as a function of two variables by eliminating one of the three variables, by using the relation between them. Sometimes this method becomes tedious. Thus to find the relative maximum or minimum we use the Lagrange's method of undetermined multipliers.

Let $u = f(x, y, z)$ be the function of three variables x, y, z. where x, y, z are related by the constraint $\phi(x, y, z) = c$, c is constant.

We know that the necessary conditions for the stationary points are :

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

$$\therefore \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \qquad \text{.......... (1)}$$

Again, we have $\phi(x, y, z) = c$

$\therefore \quad d\phi = 0$

$$\therefore \quad \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \qquad \text{.......... (2)}$$

Multiplying (2) by λ and adding it to (1), we get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0 \qquad \text{.......... (3)}$$

where λ is a parameter independent of x, y and z; also known as Lagrange's multiplier.

Now form the function,

$$\therefore \quad F(x, y, z) = f(x, y, z) + \lambda \, \phi(x, y, z) \qquad \text{.......... (4)}$$

which is known as Lagrange's function.

The differential of $F(x, y, z)$ is

$$dF = \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz \qquad \text{.......... (5)}$$

From (3) and (4), we have

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0$$

$\therefore$ The necessary conditions for the relative extremum of $F(x, y, z)$ are :

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$$

That is

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$$

This rule is known as Lagrange's method of undetermined multipliers.

Note : The above conditions are necessary only. The points obtained by this method may be relative extrema or not. If the method determines the point uniquely and the exceptional case does not occur anywhere in the region under discussion, then we can be sure that we have really found the point where the extreme value occurs.

SOLVED EXAMPLES

1. Find the dimensions of the rectangle of perimeter l which has maximum area.

Solution : Let x, y be the length and breadth of the rectangle.

$\therefore$ Perimeter $l = 2x + 2y \Rightarrow \phi(x, y) = 2x + 2y - l = 0$

and Area $f(x, y) = xy$ which is to be maximized.

Consider the Lagrange's function,

$$F(x, y) = f(x, y) + \lambda\, \phi(x, y) = xy + \lambda\,(2x + 2y - l)$$

$\therefore$ For stationary value, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$.

$$\therefore \quad y + 2\lambda = 0 \quad \text{.......... (1)}$$

$$x + 2\lambda = 0 \quad \text{.......... (2)}$$

$$\therefore \quad (1) \Rightarrow \lambda = -\frac{y}{2} \text{ and } (2) \Rightarrow \lambda = -\frac{x}{2}$$

$$\therefore \quad -\frac{y}{2} - \frac{x}{2} \Rightarrow x = y$$

But $2x + 2y = l \Rightarrow 2x + 2x = l \Rightarrow x = \frac{l}{4} = y$

$\therefore$ The rectangle of maximum area is a square.

2. If $u = x^2 + y^2 + z^2$, where $ax + by + cz = p$, find the stationary value of u.

Solution : Consider the Lagrange function

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\,(ax + by + cz - p)$$

$$\therefore \quad \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$$

$$\therefore \quad 2x + \lambda a = 0 \quad \Rightarrow \lambda = -\frac{2x}{a} \quad \text{.......... (1)}$$

$$\therefore \quad 2y + \lambda b = 0 \quad \Rightarrow \lambda = -\frac{2y}{b} \quad \text{.......... (2)}$$

$$\therefore \quad 2z + \lambda c = 0 \quad \Rightarrow \lambda = -\frac{2z}{c} \quad \text{.......... (3)}$$

From (1) and (2),

$$-\frac{2x}{a} = -\frac{2y}{b} \Rightarrow y = \frac{b}{a}x$$

From (1) and (3),

$$-\frac{2x}{a} = -\frac{2z}{c} \Rightarrow z = \frac{c}{a}x$$

But $ax + by + cz = p \Rightarrow ax + b\frac{bx}{a} + c\frac{cx}{a} = p$

$$\Rightarrow x\left(\frac{a^2+b^2+c^2}{a}\right) = p$$

$$\Rightarrow x = \frac{ap}{a^2+b^2+c^2}$$

Now $y = \frac{b}{a}x = \frac{bp}{a^2+b^2+c^2}$ and $z = \frac{c}{a}x = \frac{cp}{a^2+b^2+c^2}$

$$u = x^2 + y^2 + z^2 = \frac{a^2p^2 + b^2p^2 + c^2p^2}{\left(a^2+b^2+c^2\right)^2} = \frac{p^2}{a^2+b^2+c^2}$$

3. Find a point on the plane $2x + 3y - z = 5$ which is nearest to the origin.

Solution : Let (x, y, z) be the point on the plane.

$\therefore$ The distance from origin to (x, y, z) is

$$d = \sqrt{x^2 + y^2 + z^2}$$

Now we minimize the function,

$$f(x, y, z) = d^2 = x^2 + y^2 + z^2$$

under the condition $\phi(x, y, z) = 2x + 3y - z = 5$.

Consider the Lagrange function,

$$F(x, y, z) = f(x, y, z) + \lambda\,\phi(x, y, z)$$
$$= x^2 + y^2 + z^2 + \lambda(2x + 3y - z - 5)$$

$$\therefore \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$$

$\Rightarrow 2x + 2\lambda = 0 \Rightarrow \lambda = -x$ .... (1)

$2y + 3\lambda = 0 \Rightarrow \lambda = -\frac{2y}{3}$ .... (2)

$2z - \lambda = 0 \Rightarrow \lambda = z$ .... (3)

From (1) and (2),

$$-x = -\frac{2y}{3} \Rightarrow y = \frac{3x}{2}$$

From (1) and (3),

$$-x = 2z \Rightarrow z = -\frac{x}{2}$$

But we have $2x + 3y - z = 5$,

$$\therefore\ 2x + \frac{9x}{2} + \frac{x}{2} = 5 \Rightarrow \frac{14x}{2} = 5 \Rightarrow x = \frac{5}{7}$$

Thus $y = \frac{3x}{2} = \frac{3}{2} \cdot \frac{5}{7} = \frac{15}{14}$ and $z = -\frac{x}{2} = -\frac{5}{14}$

$\therefore\ \left(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14}\right)$ is the nearest point from the origin.

4. If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$, find the values of x, y, z such that $x + y + z$ is minimum.

Solution : Let $f(x, y, z) = x + y + z$ which is to be minimized,

and $\phi(x, y, z) = \frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6 = 0$.

Consider the Lagrange function,

$F(x, y, z) = f(x, y, z) + \lambda\, \phi\,(x, y, z)$

$$= x + y + z + \lambda \left(\frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6\right)$$

The necessary conditions are : $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

$$\therefore\ 1 - \frac{3\lambda}{x^2} = 0 \Rightarrow \lambda = \frac{x^2}{3} \qquad \text{.......... (1)}$$

$$1 - \frac{4\lambda}{y^2} = 0 \Rightarrow \lambda = \frac{y^2}{4} \qquad \text{.......... (2)}$$

$$1 - \frac{5\lambda}{z^2} = 0 \Rightarrow \lambda = \frac{z^2}{5} \qquad \text{.......... (3)}$$

$\therefore$ From (1) and (2) $\Rightarrow \frac{x^2}{3} = \frac{y^2}{4} \Rightarrow y\ \frac{2}{\sqrt{3}}\ x$

From (1) and (3) $\Rightarrow \frac{x^2}{3} = \frac{z^2}{5} \Rightarrow z\ \sqrt{\frac{5}{3}}\ x$

But we have $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$

$$\Rightarrow \frac{3}{x}+\frac{4\sqrt{3}}{2x}+\frac{5\sqrt{3}}{\sqrt{5}x}=6$$

$$\Rightarrow \frac{1}{x}\left[3+2\sqrt{3}+\sqrt{15}\right]=6$$

$$\Rightarrow x=\frac{3+2\sqrt{3}+\sqrt{15}}{6}=\frac{\sqrt{3}\left(\sqrt{3}+2+\sqrt{5}\right)}{6}=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{3}$$

Now $y=\frac{2x}{\sqrt{3}}=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{4}$

$$z=\sqrt{\frac{5}{3}}\,x=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{5}$$

5. Find the maximum value of $u=x^py^qz^r$ when the variables x, y, z are subject to the condition $ax+by+cz=p+q+r$.

Solution : We have $u=f(x,y,z)=x^py^qz^r$

and $\phi(x,y,z)=ax+by+cz-p-q-r=0$

From the Lagrange function,

$$F(x,y,z)=f(x,y,z)+\lambda\,\phi(x,y,z)$$

$$=x^py^qz^r+\lambda\,(ax+by+cz-p-q-r).$$

Necessary conditions are : $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0.$

$$\therefore\ px^{p-1}y^qz^r+\lambda\,a=0\Rightarrow\lambda=-\frac{px^{p-1}y^qz^r}{a} \qquad \text{..........} (1)$$

$$qx^py^{q-1}z^r+\lambda\,b=0\Rightarrow\lambda=-\frac{qx^py^{q-1}z^r}{b} \qquad \text{..........} (2)$$

$$rx^py^qz^{r-1}+\lambda\,c=0\Rightarrow\lambda=-\frac{rx^py^qz^{r-1}}{c} \qquad \text{..........} (3)$$

Equations (1) and (2) $\Rightarrow -\frac{px^{p-1}y^qz^r}{a}=-\frac{qx^py^{q-1}z^r}{b}$

$$\Rightarrow\frac{py}{a}=\frac{qx}{b}\Rightarrow y=\frac{aq}{bp}\,x$$

Equations (1) and (3) $\Rightarrow -\dfrac{px^{p-1}y^q z^r}{a} = -\dfrac{rx^p y^q z^{r-1}}{c}$

$$\Rightarrow \frac{pz}{a} = \frac{rx}{c} \Rightarrow z = \frac{ar}{cp}x$$

But we have $ax + by + cz = p + q + r$

$$\Rightarrow ax + \frac{aq}{p}x + \frac{ar}{p}x = p + q + r$$

$$\Rightarrow ax\frac{(p+q+r)}{p} = p + q + r \Rightarrow x = \frac{p}{a}$$

Now $y = \dfrac{aq}{bp}x = \dfrac{aq}{bp}\cdot\dfrac{p}{a} = \dfrac{q}{b}$ and $z = \dfrac{ar}{cp}\cdot\dfrac{p}{a} = \dfrac{r}{c}$

$$\therefore\ u = \left(\frac{p}{a}\right)^p\left(\frac{q}{b}\right)^q\left(\frac{r}{c}\right)^r$$

6. Find the maximum value of $x^m y^n$ where x and y are positive and subject to the constraint $x + y = a$.

Solution : Let $f(x, y) = x^m y^n$, $\phi(x, y) = x + y - a = 0$.

Form the Lagrange function,

$$F(x, y) = f(x, y) + \lambda\,\phi(x, y)$$
$$= x^m y^n + \lambda\,(x + y - a)$$

Necessary conditions are : $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$.

$\therefore\ mx^{m-1}y^n + \lambda = 0 \Rightarrow \lambda = -\,mx^{m-1}y^n$ .... (1)

and $nx^m y^{n-1} + \lambda = 0 \Rightarrow \lambda = -\,nx^m y^{n-1}$ .... (2)

$\therefore$ From (1) and (2),

$$-\,mx^{m-1}y^n = -\,nx^m y^{n-1}$$

$$\Rightarrow my = nx \Rightarrow y = \frac{nx}{m}$$

Now, we have $x + y = a$.

$$\therefore\ x + \frac{nx}{m} = a \Rightarrow x\frac{(m+n)}{m} = a \Rightarrow x = \frac{am}{m+n}$$

$$\therefore\ y = \frac{nx}{m} = \frac{na}{m+n}$$

$$\therefore \quad f(x, y) = x^m y^n = \frac{a^m m^m}{(m+n)^m} \cdot \frac{a^n n^n}{(m+n)^n}$$

$$= \frac{a^{m+n} m^m n^n}{(m+n)^{m+n}}$$

7. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution : Let $2x$, $2y$ and $2z$ be the length, breadth and height of the rectangular solid. Thus volume

$V = (2x)(2y)(2z) = 8\,xyz$ which is to be maximized.

Again, let a be the radius of the sphere. Thus

$x^2 + y^2 + z^2 = a^2.$

Consider the Lagrange function,

$F(x, y, z) = 8xyz + \lambda\,(x^2 + y^2 + z^2 - a^2).$

Necessary conditions are : $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$

$$\therefore \quad 8yz + 2\lambda x = 0 \Rightarrow \lambda = -\frac{4yz}{x} \qquad \text{...........} \ (1)$$

$$8xz + 2\lambda y = 0 \Rightarrow \lambda = -\frac{4xz}{y} \qquad \text{...........} \ (2)$$

$$8xy + 2\lambda z = 0 \Rightarrow \lambda = -\frac{4xy}{z} \qquad \text{...........} \ (3)$$

From (1) and (2),

$$-\frac{4yz}{x} = -\frac{4xz}{y} \Rightarrow x^2 = y^2$$

From (1) and (3),

$$-\frac{4yz}{x} = -\frac{4xy}{z} \Rightarrow x^2 = z^2$$

Now, we have $x^2 + y^2 + z^2 = a^2$

$$\Rightarrow x^2 + x^2 + x^2 = a^2 \Rightarrow 3x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly $y = \frac{a}{\sqrt{3}}$, $z = \frac{a}{\sqrt{3}}$

$\therefore \quad x = y = z \Rightarrow$ The rectangular solid is a cube.

8. Find the dimensions of the rectangular box, open at the top with maximum volume and given surface area.

Solution : Let x, y, z be the dimensions of the box. Thus surface area

$$s = 2\,(xz + yz) + xy = c$$

and the volume

$$V = xyz \text{ which is to be maximized.}$$

Consider the Lagrange function, $F(x, y, z) = xyz + \lambda\,(2xz + 2yz + xy - c)$.

Necessary conditions are : $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0.$

$$\therefore\quad yz + \lambda\,(2z + y) = 0 \quad \Rightarrow \lambda = -\frac{yz}{2z + y} \qquad \text{.......... (1)}$$

$$xz + \lambda\,(2z + x) = 0 \quad \Rightarrow \lambda = -\frac{xz}{2z + x} \qquad \text{.......... (2)}$$

$$xy + \lambda\,(2x + 2y) = 0 \quad \Rightarrow \lambda = -\frac{xy}{2x + 2y} \qquad \text{.......... (3)}$$

From (1) and (2),

$$-\frac{yz}{2z + y} = -\frac{xz}{2z + x} \Rightarrow 2yz + xy = 2xz + xy$$

$$\Rightarrow 2yz = 2xz \Rightarrow y = x.$$

From (1) and (3)

$$-\frac{yz}{2z + y} = -\frac{xy}{2x + 2y} \Rightarrow 2xz + 2yz = 2xz + xy$$

$$\Rightarrow 2yz = xy \Rightarrow 2z = x$$

Now we have $2\,(xz + yz) + xy = s$

$$\therefore\quad 2x \cdot \frac{x}{2} + 2x \cdot \frac{x}{2} + x \cdot x = s$$

$$\therefore\quad x^2\,(1 + 1 + 1) = s$$

$$\Rightarrow x^2 = \frac{s}{3} \Rightarrow x = \sqrt{\frac{s}{3}}$$

$$\therefore\quad y = \sqrt{\frac{s}{3}} \text{ and } z = \frac{1}{2}\sqrt{\frac{s}{3}}$$

$$v = xyz = \sqrt{\frac{s}{3}} \cdot \sqrt{\frac{s}{3}} \cdot \frac{1}{2}\sqrt{\frac{s}{3}} = \frac{1}{2}\left(\frac{s}{3}\right)^{3/2}$$

9. If $u = a^3x^2 + b^3y^2 + c^3z^2$, where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, prove that the stationary values of u are given by $x = \frac{\Sigma a}{a}$, $y = \frac{\Sigma a}{b}$, $z = \frac{\Sigma a}{c}$, where $\Sigma a = a + b + c$.

Solution : We have

$$u = a^3x^2 + b^3y^2 + c^3z^2 \text{ and } \phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$$

Consider the Lagrange function,

$$F(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

Necessary conditions are : $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

$$\therefore\ 2a^3x - \frac{\lambda}{x^2} = 0 \Rightarrow \lambda = 2a^3x^3 \qquad \text{.......... (1)}$$

$$2b^3y - \frac{\lambda}{y^2} = 0 \Rightarrow \lambda = 2b^3y^3 \qquad \text{.......... (2)}$$

$$2c^3z - \frac{\lambda}{z^2} = 0 \Rightarrow \lambda = 2c^3z^3 \qquad \text{.......... (3)}$$

$\therefore$ From (1) and (2),

$$a^3x^3 = b^3y^3 \Rightarrow y = \frac{ax}{b}$$

From (1) and (3),

$$a^2x^3 = c^2z^3 \Rightarrow z = \frac{ax}{c}$$

Now, we have $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\therefore\ \frac{1}{x} + \frac{b}{ax} + \frac{c}{ax} = 1 \Rightarrow \frac{1}{x}\left(\frac{a+b+c}{a}\right) = 1$$

$$\therefore\ x = \frac{a+b+c}{a} = \frac{\Sigma a}{a}$$

$$y = \frac{ax}{b} = \frac{\Sigma a}{b},\ z = \frac{ax}{c} = \frac{\Sigma a}{c}$$

10. If $xyz = 8$, find the values of x, y, z for which $u = \dfrac{5xyz}{x + 2y + 4z}$ is maximum.

Solution : We have $u = \dfrac{5xyz}{x + 2y + 4z}$ and $\phi(x, y, z) = xyz - 8 = 0$

Form the Lagrange function,

$$F(x, y, z) = \frac{40}{x + 2y + 4z} + \lambda(xyz - 8) \qquad (\because xyz = 8)$$

Necessary conditions are : $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0.$

$$\therefore \quad -\frac{40}{(x + 2y + 4z)^2} + \lambda yz = 0 \Rightarrow \lambda = \frac{40}{yz(x + 2y + 4z)^2} \quad \text{.........} \ (1)$$

$$-\frac{80}{(x + 2y + 4z)^2} + \lambda xz = 0 \Rightarrow \lambda = \frac{80}{xz(x + 2y + 4z)^2} \quad \text{.........} \ (2)$$

$$-\frac{160}{(x + 2y + 4z)^2} + \lambda xy = 0 \Rightarrow \lambda = \frac{160}{xy(x + 2y + 4z)^2} \quad \text{.........} \ (3)$$

From (1) and (2),

$$\frac{40}{yz(x + 2y + 4z)^2} = \frac{80}{xz(x + 2y + 4z)^2} \Rightarrow \frac{1}{y} = \frac{2}{x} \Rightarrow y = \frac{x}{2}$$

From (1) and (3)

$$\frac{40}{yz(x + 2y + 4z)^2} = \frac{160}{xy(x + 2y + 4z)^2} \Rightarrow \frac{1}{z} = \frac{4}{x} \Rightarrow z = \frac{x}{4}$$

We have $xyz = 8 \Rightarrow x \cdot \dfrac{x}{2} \cdot \dfrac{x}{4} = 8 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$$\therefore \quad y = \frac{x}{2} = 2, \ z = \frac{x}{4} = 1$$

$\therefore \ x = 4, \ y = 2, \ z = 1.$

11. Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225.$$

Solution : We have $\phi(x, y) = x^2 + 8xy + 7y^2 - 225 = 0.$

Let (x, y) be any point on the hyperbola. Then the distance from origin to (x, y) is given by

$$d = \sqrt{x^2 + y^2}$$

Let $f(x, y) = d^2 = x^2 + y^2$.

From the Lagrange function,

$F(x, y) = x^2 + y^2 + \lambda\,(x^2 + 8xy + 7y^2 - 225)$

Necessary conditions are : $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$

$\therefore \quad 2x + \lambda\,(2x + 8y) = 0 \qquad \text{.........} \quad (1)$

$\quad 2y + \lambda\,(8x + 14y) = 0 \qquad \text{.........} \quad (2)$

Equation (1) $\Rightarrow (1 + \lambda)x + 4\lambda y = 0 \qquad \text{.........} \quad (3)$

Equation (2) $\Rightarrow 4\lambda x + (1 + 7\,\lambda)y = 0 \qquad \text{.........} \quad (4)$

For the nontrivial solution, we have

$$\begin{vmatrix} 1+\lambda & 4\lambda \\ 4\lambda & 1+7\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 + \lambda)\,(1 + 7\lambda) - 16\lambda^2 = 0$

$\Rightarrow 9\lambda^2 - 8\lambda - 1 = 0 \Rightarrow (\lambda - 1)\,(9\lambda + 1) = 0$

$\Rightarrow \lambda = 1$ or $-\dfrac{1}{9}$

When $\lambda = 1$:

From (3) $\Rightarrow 2x + 4y = 0 \Rightarrow x = -4y$

$\therefore \quad x^2 + 8xy + 7y^2 = 225 \Rightarrow 16y^2 - 32y^2 + 7y^2 = 225$

$\therefore \quad 9y^2 = -225$

which has no real solution.

When $\lambda = -\dfrac{1}{9}$:

From (3) $\Rightarrow \dfrac{8}{9}x - \dfrac{4}{9}y = 0 \Rightarrow 2x - y = 0 \Rightarrow y = 2x$

$\therefore \quad x^2 + 8xy + 7y^2 = 225 \Rightarrow x^2 + 16x^2 + 28x^2 = 225$

$\qquad \Rightarrow 45x^2 = 225 \Rightarrow x^2 = 5$

and $y^2 = 4x^2 = 20$

$\Rightarrow$ shortest distance $d = \sqrt{x^2 + y^2}$

$\qquad = \sqrt{5 + 20} = 5.$

12. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

Solution : Let (x, y, z) be any point on the sphere. Then the distance from (3, 4, 12) to (x, y, z) is given by

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

Again $\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Let $f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$.

Consider the Lagrange function,

$$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda\,(x^2 + y^2 + z^2 - 1).$$

Necessary conditions are : $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

$$\therefore\ 2(x-3) + 2\lambda x = 0 \quad \Rightarrow \lambda = -\frac{x-3}{x} \qquad \text{.......... (1)}$$

$$2(y-4) + 2\lambda y = 0 \quad \Rightarrow \lambda = -\frac{y-4}{y} \qquad \text{.......... (2)}$$

$$2(z-12) + 2\lambda z = 0 \quad \Rightarrow \lambda = -\frac{z-12}{z} \qquad \text{.......... (3)}$$

$\therefore$ From (1) and (2),

$$-\frac{x-3}{x} = -\frac{y-4}{y} \Rightarrow \frac{x-3}{x} = \frac{y-4}{y}$$
$$\Rightarrow xy - 3y = xy - 4x$$
$$\Rightarrow -3y = -4x \Rightarrow y = \frac{4}{3}x$$

From (1) and (3),

$$-\frac{x-3}{x} = -\frac{z-12}{z} \Rightarrow xz - 3z = xz - 12x$$
$$\Rightarrow -3z = -12x \Rightarrow z = 4x$$

Now we have $x^2 + y^2 + z^2 = 1$

$$\Rightarrow x^2 + \frac{16x^2}{9} + 16x^2 = 1 \Rightarrow \frac{169x^2}{9} = 1 \Rightarrow x^2 = \frac{9}{169}$$

$$\therefore\ y^2 = \frac{16x^2}{9} = \frac{16}{9}\cdot\frac{9}{169} = \frac{16}{169}$$

$$z^2 = 16x^2 = 16\cdot\frac{9}{169} = \frac{144}{169}$$

$$x = \pm\frac{3}{13}, y = \pm\frac{4}{13}, z = \pm\frac{12}{13}$$

$\therefore$ We have the point $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$

At $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) \Rightarrow d = \sqrt{\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2} = 12$

At $\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right) \Rightarrow d = \sqrt{\left(-\frac{3}{13}-3\right)^2 + \left(-\frac{4}{13}-4\right)^2 + \left(-\frac{12}{13}-12\right)^2} = 14$

$\therefore$ The maximum distance = 14 and

minimum distance = 12.

13. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 16$.

Solution : Let (x, y, z) be any point on the sphere. Thus the distance from (1, 2, 2) to (x, y, z) is given by

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

let $f(x, y, z) = d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$ and $\phi(x, y, z) = x^2 + y^2 + z^2 - 16 = 0$.

From the Lagrange function.

$F(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2 + \lambda(x^2 + y^2 + z^2 - 16)$

Necessary conditions are : $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0.$

$\therefore\ 2(x-1) + 2\lambda x = 0 \Rightarrow \lambda = -\frac{x-1}{x}$ .... (1)

$2(y-2) + 2\lambda y = 0 \Rightarrow \lambda = -\frac{y-2}{y}$ .... (2)

$2(z-2) + 2\lambda z = 0 \Rightarrow \lambda = -\frac{z-2}{z}$ .... (3)

$\therefore$ **From (1) and (2),**

$$-\frac{x-1}{x} = -\frac{y-2}{y} \Rightarrow xy - y = xy - 2x \Rightarrow y = 2x$$

From (1) and (3),

$$-\frac{x-1}{x} = -\frac{z-2}{z} \Rightarrow xz - z = xz - 2x \Rightarrow z = 2x$$

Now, we have $x^2 + y^2 + z^2 = 16$

$\therefore\ x^2 + 4x^2 + 4x^2 = 16 \Rightarrow x^2 = \frac{16}{9} \Rightarrow x = \pm \frac{4}{3}$

$\therefore\ y = \pm \frac{8}{3},\ z = \pm \frac{8}{3}$

For $\left(\frac{4}{3}, \frac{8}{3}, \frac{8}{3}\right)$,

$$d = \sqrt{\left(\frac{4}{3} - 1\right)^2 + \left(\frac{8}{3} - 2\right)^2 + \left(\frac{8}{3} - 2\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$

which is the shortest distance.

14. Find the maximum volume of a rectangular parallelopiped which can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution : Let (x, y, z) be any point on the rectangular parallelopiped. Thus the dimensions are $2x$, $2y$, $2z$.

$\therefore$ Volume $V = 8xyz$

Let $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

Consider the Lagrange function,

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Necessary conditions are : $\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$

$$\therefore\ 8yz + \frac{2\lambda x}{a} = 0 \Rightarrow \lambda = -\frac{4yza}{x} \qquad \text{.......... (1)}$$

$$8xz + \frac{2\lambda y}{b} = 0 \Rightarrow \lambda = -\frac{4xzb}{y} \qquad \text{.......... (2)}$$

$$8xy + \frac{2\lambda z}{c} = 0 \Rightarrow \lambda = -\frac{4xyc}{z} \qquad \text{.......... (3)}$$

From (1) and (2),

$$-\frac{4yza}{x} = -\frac{4xzb}{y} \Rightarrow ay^2 = bx^2 \Rightarrow y^2 = \frac{b}{a}x^2$$

From (1) and (3),

$$-\frac{4yza}{x} = -\frac{4xyc}{z} \Rightarrow az^2 = cx^2 \Rightarrow z^2 = \frac{c}{a}x^2$$

Now, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\therefore \frac{x^2}{a^2} + \frac{1}{b^2}\cdot\frac{b^2}{a^2}x^2 + \frac{1}{c^2}\cdot\frac{c^2}{a^2}x^2 = 1$$

$$\Rightarrow 3x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{3}}$$

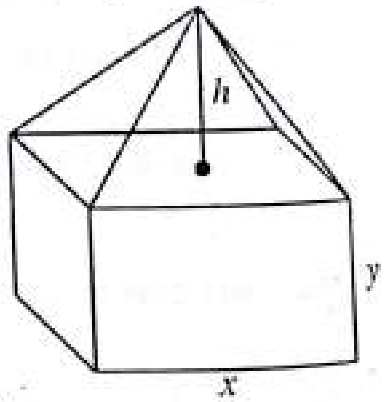
$$\therefore y = \frac{b}{a}x = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{a}x = \frac{c}{\sqrt{3}}$$

$$\therefore v = 8xyz = 8\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}\cdot\frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$$

15. A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h if the canvas required for its construction is to be minimum for the tent to have a given capacity.

Solution : The volume of the tent V is given by

$$V = x^2y + \frac{1}{3}x^2h = C.$$



The surface area of a tent is :

$$S = 4xy + x\sqrt{4h^2 + x^2}$$

Consider the Lagrange function,

$$F(x, y, h) = 4xy + x\sqrt{x^2 + 4h^2} + \lambda\left(x^2y + \frac{1}{3}x^2h - V\right)$$

(Area of a regular pyramid $= \frac{1}{2} \times$ slant height $\times$ perimeter of base)

Necessary conditions are : $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial h} = 0.$

$$\therefore 4y + \sqrt{x^2 + 4h^2} + x\frac{2x}{2\sqrt{x^2 + 4h^2}} + \lambda\left(2xy + \frac{2}{3}xh\right) = 0 \qquad \text{.......... (1)}$$

and $4x + \lambda x^2 = 0$

Equation (2) $\Rightarrow 4 + \lambda x = 0 \Rightarrow \lambda = -\dfrac{4}{x}$ .... (2)

Now $\dfrac{\partial F}{\partial h} = 0 \Rightarrow \dfrac{4hx}{\sqrt{x^2 + 4h^2}} + \dfrac{\lambda x^2}{3} = 0$ .... (3)

$\therefore$ From (2) and (3),

$$\frac{4hx}{\sqrt{x^2 + 4h^2}} = -\frac{x^2}{3}\left(-\frac{4}{x}\right) \Rightarrow 3h = \sqrt{x^2 + 4h^2}$$

$$\Rightarrow 9h^2 = x^2 + 4h^2 \Rightarrow x^2 = 5h^2 \Rightarrow x = \sqrt{5}h$$

From (1),

$$4y + \sqrt{5h^2 + 4h^2} + \frac{5h^2}{\sqrt{5h^2 + 4h^2}} - 8y - \frac{8}{3}h = 0$$

$$\Rightarrow 4y = 3h + \frac{5h^2}{3h} - \frac{8}{3}h = 3h + \frac{5}{3}h - \frac{8}{3}h = 2h$$

$$\Rightarrow y = \frac{1}{2}h$$

16. Find the extreme values of $x^2 + y^2 + z^2$ subject to the conditions $x^2 + \dfrac{y^2}{2} + \dfrac{z^2}{3} = 2$ and $3x + 2y + z = 0$.

Solution : Let $f(x, y, z) = x^2 + y^2 + z^2$,

$$\phi(x, y, z) = x^2 + \frac{y^2}{2} + \frac{z^2}{3} - 2 = 0, \ \psi(x, y, z) = 3x + 2y + z = 0$$

Construct the Lagrange function,

$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z) + \mu\, (3x + 2y + z)$

$$= x^2 + y^2 + z^2 + \lambda\left(x^2 + \frac{y^2}{2} + \frac{z^2}{3} - 2\right) + \mu\, (3x + 2y + z)$$

Necessary conditions are : $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0.$

.... (1)

$\therefore$ $2x + 2\lambda x + 3\mu = 0$ .... (2)

$2y + \lambda y + 2\mu = 0$ .... (3)

$2z + \dfrac{2\lambda z}{3} + \mu = 0$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get

$$2(x^2 + y^2 + z^2) + 2\lambda\left(x^2 + \frac{y^2}{2} + \frac{z^2}{3}\right) + \mu\,(3x + 2y + z) = 0.$$

$$\Rightarrow 2f + 2\lambda\,(2) + \mu\,(0) = 0 \Rightarrow 2\lambda + f = 0 \Rightarrow \lambda = -\frac{f}{2}$$

$\therefore$ From (1),

$$2x\left(1 - \frac{f}{2}\right) + 3\mu = 0 \Rightarrow x(2 - f) = -3\mu \Rightarrow x = -\frac{3\mu}{2 - f}$$

From (2),

$$2y\left(1 - \frac{f}{4}\right) + 2\mu = 0 \Rightarrow \frac{y\,(4 - f)}{2} = -2\mu \Rightarrow y = -\frac{4\mu}{4 - f}$$

From (3),

$$2z\left(1 - \frac{f}{6}\right) + \mu = 0 \Rightarrow \frac{z\,(6 - f)}{3} = -\mu \Rightarrow z = -\frac{3\mu}{6 - f}$$

But $3x + 2y + z = 0$

$$\Rightarrow -\frac{9\mu}{2 - f} - \frac{8\mu}{4 - f} - \frac{3\mu}{6 - f} = 0$$

$$\Rightarrow -\mu\left[\frac{9}{2 - f} + \frac{8}{4 - f} + \frac{3}{6 - f}\right] = 0$$

$$\Rightarrow 9\,(4 - f)\,(6 - f) + 8\,(2 - f)\,(6 - f) + 3\,(2 - f)\,(4 - f) = 0$$

$$\Rightarrow 9\,(24 - 10f + f^2) + 8\,(12 - 8f + f^2) + 3\,(8 - 6f + f^2) = 0$$

$$\Rightarrow 20f^2 - 172f + 336 = 0$$

$$\Rightarrow 5f^2 - 43f + 84 = 0$$

$$\Rightarrow (5f - 28)\,(f - 3) = 0$$

$$\therefore\ f = \frac{28}{5} \text{ or } 3.$$

17. Prove that the extreme values of $u = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$, where $lx + my + nz = 0$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ are the roots of the equation $\dfrac{l^2a^4}{1 - a^2u} + \dfrac{m^2b^4}{1 - b^2u} + \dfrac{n^2c^4}{1 - c^2u} = 0.$

Solution : We have $u = \dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}$.

Let $\phi(x, y, z) = lx + my + nz = 0$, $\psi(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$.

Construct the Lagrange function,

$$F(x, y, z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \lambda(lx + my + nz) + \mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$\therefore$ The necessary conditions are : $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial z} = 0$

$$\therefore \frac{2x}{a^4} + \lambda l + \frac{2\mu x}{a^2} = 0 \qquad \text{..........} (1)$$

$$\frac{2y}{b^4} + \lambda m + \frac{2\mu y}{b^2} = 0 \qquad \text{..........} (2)$$

$$\frac{2z}{c^4} + \lambda n + \frac{2\mu z}{c^2} = 0 \qquad \text{..........} (3)$$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get

$$2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda(lx + my + nz) + 2\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$\therefore$ $2u + \lambda(0) + 2\mu(1) = 0 \Rightarrow \mu = -u$

From (1),

$$\frac{2x}{a^4} + \lambda l - \frac{2\mu x}{a^2} = 0 \Rightarrow \frac{2x}{a^2}\left(\frac{1}{a^2} - u\right) = -\lambda l \Rightarrow x = -\frac{\lambda l a^4}{2(1 - a^2 u)}$$

Similarly from (2) and (3), we get

$$y = -\frac{\lambda m b^4}{2(1 - b^2 u)}, \quad z = -\frac{\lambda n c^4}{2(1 - c^2 u)}$$

But $lx + my + nz = 0$

$$\therefore -\frac{\lambda l^2 a^4}{2(1 - a^2 u)} - \frac{\lambda m^2 a^4}{2(1 - b^2 u)} - \frac{\lambda n^2 c^4}{2(1 - c^2 u)} = 0$$

$$\Rightarrow \frac{l^2 a^4}{1 - a^2 u} + \frac{m^2 a^4}{1 - b^2 u} + \frac{n^2 c^4}{1 - c^2 u} = 0$$

$\therefore$ The extreme values are given by the roots of this equation.

EXERCISE – 2.5

1. Find the minimum distance from the point $(2, 1, -3)$ to the plane $2x + y - 2z = 4$.

   Ans. : $7/3$

2. If $u = x^2 + y^2 + z^2$, where $x + y + z = 1$ show that u is stationary when $x = y = z = \frac{1}{3}$.

3. Find the stationary values of the function $x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

   Ans. : $(0, 0, 1), (0, 0, -1)$.

4. Find the minimum values of $x^2 + y^2 + z^2$ when $xyz = a^3$.

   Ans. : $3a^2$

5. Find the dimensions of the rectangular box, open at the top, with given capacity and minimum surface area.

6. Find the maxima of $f(x, y, z) = xy^2z^3$ subject to the conditions $x + y + z = 6$, $x > 0$, $y > 0$, $z > 0$.

   Ans. : $(1, 2, 3)$, 108

7. If $u = \dfrac{x^2}{a^3} + \dfrac{y^2}{b^3} + \dfrac{z^2}{c^3}$ with $x + y + z = 1$, show that the stationary values of u are given by, $x = \dfrac{a^3}{\Sigma a^3}$, $y = \dfrac{b^3}{\Sigma a^3}$, $z = \dfrac{c^3}{\Sigma a^3}$, where $\Sigma a^3 = a^3 + b^3 + c^3$

8. Find the shortest and the longest distances from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

   Ans. : $\sqrt{6}, 3\sqrt{6}$

9. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 cm^2.

   Ans. : $(12, 12, 6)$

10. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

    Ans. : $\dfrac{4096}{25\sqrt{5}}$

11. Find the minimum value of the function $x^2 + y^2 + z^2$ given that $xy + yz + zx = 3a^2$.

    Ans. : $3a^2$, (a, a, a), $(-a, -a, -a)$

12. Find the shortest distance from the point $(1, 0)$ to the parabola $y^2 = 4x$.

    Ans. : 1 from $(0, 0)$

3 DIFFERENTIAL EQUATIONS

3.1 INTRODUCTION :

In "Higher Engineering Mathematics–I" we have seen first order differential equations with their solution techniques. But in normal practice we can have differential equations of higher order of different kinds. In this chapter we shall discuss the higher ordered linear differential equations. These equations have a wide range of applications in physical sciences and engineering.

3.2 DEFINITIONS :

1. A differential equation is an equation containing derivatives or differentials of one or more dependent variables with respect to one or more independent variables :

   e. g. We list the following differential equations :

$$dy = (x + \sin x)\, dx \qquad \text{.......... (1)}$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \qquad \text{.......... (2)}$$

$$y = \sqrt{x}\, \frac{dy}{dx} + \frac{k}{dy/dx} \qquad \text{.......... (3)}$$

$$k\, \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad \text{.......... (4)}$$

$$\frac{\partial^2 u}{\partial t^2} = k\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \qquad \text{.......... (5)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\, \frac{\partial u}{\partial t} \qquad \text{.......... (6)}$$

A differential equation can be classified into two categories depending upon the nature of derivatives involved in the equation.

2. A differential equation containing a single independent variable and the derivatives with respect to it is called **an ordinary differential equation.**

   Thus equations (1) to (4) are all ordinary differential equations.

3. A differential equation containing more than one independent variables and the partial derivatives with respect to them is called a **partial differential equation.**

Thus equations (5) and (6) are partial differential equations.

4. The **order** of the differential equation is the order of the highest derivative involved in the differential equation.

Equations (1) and (3) are of first order, equations (4) and (6) are of second order, equation (5) is of third order and equation (2) is of fourth order.

5. The **degree** of a differential equation is the **degree** of the highest derivative, which occur in it when the differential coefficients have been made from radicals and fractions if any.

Note : This definition of degree does not require the variables x, t, u, etc. to be free from radicals and fractions.

Equations (1), (2), (6) are of first degree.

Rewrite equation (3) making free from fractions, we get $y\left(\frac{dy}{dx}\right) = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$ which is of second degree.

Making equation (4) free from radicals, we get $k^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$ which is of second degree.

Equation (5) is of second degree.

6. A differential equation is said to be **linear** if the dependent variable and every derivatives in the equation occurs in the first degree only and they should not be multiplied together. A differential equation which is not linear is called a **nonlinear** differential equation.

Equations (1) and (6) are linear, equations (2), (3), (4) and (5) are all nonlinear.

7. A **solution** (or the primitive) of the differential equation is a relation between the dependent and independent variables which satisfies the differential equation. The process of finding all the solutions is called integrating (or solving) the differential equation.

e.g. 1. $y = e^{3x}$ is the solution of $\frac{dy}{dx} = 3y$ (verify)

2. $y = e^{2x}$ and $y = e^{5x}$ are the solutions of $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$. (verify)

Note : Here $y = c\,e^{3x}$ is also a solution of the given differential equation for any real value of the constant c, which is known as an arbitrary constant.

5. If a solution of the differential equation contains the number of arbitrary constants which is equal to the order of the differential equation is called the **general solution** or the **complete primitive** of the differential equation.

e.g. $y = c_1 e^{2x} + c_2 e^{5x}$ is the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0.$$

6. A solution obtained from a general solution by giving particular values to one or more arbitrary constants is called a **particular solution** of the differential equation.

e.g. $y = e^{2x} + 6\ e^{5y}$, $y = 2e^{2x} - e^{5x}$ etc. are the particular solutions of

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0.$$

Note : 1. Because of arbitrary constants, the differential equations can have infinitely many solutions. However the general solution is always unique if it exists.

2. The solution of differential equation if it exists is always a continuous function.

3.3 HIGHER ORDERED LINEAR DIFFERENTIAL EQUATIONS :

A more general form of the ordinary differential equation is :

$$F(x, y, y_1, y_2, \ldots\ldots, y_n) = 0$$

which is of n^{th} order. where $y_1, y_2, \ldots\ldots, y_n$ are the derivatives with their usual meaning.

Now, we write the general form of n^{th} order ordinary linear differential equation as :

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots\ldots + A_2(x)\frac{d^2y}{dx^2} + A_1(x)\frac{dy}{dx} + A_0(x)\,y = F(x) \ \ldots\ldots\ldots (1)$$

where $A_0(x), A_1(x), \ldots\ldots, A_n(x)$ are the coefficients of the equation, which may be functions of x or constants. We assume that the coefficients and the input function $F(x)$ are continuous functions on some interval **I**, and also we assume that $A_n(x) \neq 0$ on **I**.

When the input function $F(x)$ is zero then equation (1) reduces to the associated **homogeneous equation.**

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots\ldots + A_n(x)\frac{dy}{dx} + A_0(x)y = 0 \qquad \ldots\ldots\ldots\ldots (2)$$

The differential equation obtained after dividing equation (1) by $A_n(x)$ is called the **normal form** of the differential equation (1). That is

$$y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \ldots\ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad \ldots\ldots\ldots\ldots (3)$$

Note : The general solution of equation (3) will contain n arbitrary constants. Thus, for the application point of view we require n auxiliary conditions to determine the values of n arbitrary constants. Generally these conditions are in the form of initial conditions such as $y(x_0), y'(x_0), \ldots\ldots, y^{(n-1)}(x_0)$ all are specified at the same initial point x_0 of the interval I

3.4 IMPORTANT DEFINITIONS :

1. **Wronskian :** The Wronskian of n functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ is defined and denoted by the determinant.

$$W(y_1, y_2, \ldots\ldots, y_n)(x) = \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ y_1' & y_2' & \ldots\ldots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots\ldots & y_n^{(n-1)} \end{vmatrix}$$

2. **Linear Dependent of Functions :** The n functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ are said to be **linearly dependent** if there exist constant $c_1, c_2, \ldots\ldots, c_n$ (not all zero) such that the linear combination

$$c_1 y_1 + c_2 y_2 + \ldots.. + c_n y_n = 0. \qquad \ldots\ldots\ldots. (1)$$

If, however, the identity (1) implies that

$$c_1 = c_2 = \ldots.. = c_n = 0$$

then $y_1, y_2, \ldots\ldots, y_n$ are said to be linearly independent.

Now from the identity (1), we have (Assuming all are differentiable upto $(n-1)^{th}$ order)

$$c_1 y_1 + c_2 y_2 + \ldots.. + c_n y_n = 0$$

$$\therefore\ c_1 y_1' + c_2 y_2' + \ldots.. + c_n y_n' = 0$$

$$c_1 y_1'' + c_2 y_2'' + \ldots.. + c_n y_n'' = 0$$

$$\vdots$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots.. + c_n y_n^{(n-1)} = 0$$

The above equations constitute a homogeneous system of algebraic linear equations in terms of $c_1, c_2, \ldots\ldots, c_n$. Thus for the nontrivial solution, that is atleast one of the $c_1, c_2, \ldots\ldots, c_n$ is non zero, we must have

$$\begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ y_1' & y_2' & \ldots\ldots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots\ldots & y_n^{(n-1)} \end{vmatrix} = 0$$

That is $W(y_1, y_2, \ldots.., y_n)(x) = 0$

Thus, if the Wronskian of the functions $y_1, y_2, \ldots\ldots, y_n$ is zero at every point x of **I** then the functions $y_1, y_2, \ldots\ldots, y_n$ are linearly dependent.

Note : If Wronskian does not vanish identically, then $y_1, y_2, \ldots\ldots, y_n$ are **linearly independent.**

3.5 EXISTENCE AND UNIQUENESS THEOREM :

If $a_0(x), a_1(x), \ldots\ldots, a_{n-1}(x)$, and $f(x)$ are continuous functions on an open interval **I** containing the point x_0, then the **initial valued problem (IVP).**

$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + a_1(x) y'(x) + a_0(x)\, y = f(x)$ with the initial conditions

$y(x_0) = \alpha_1,\ y'(x_0) = \alpha_2, \ldots\ldots, y^{(n-1)}(x_0) = \alpha_n$

has a unique solution on **I** for every $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.

3.6 SUPER POSITION PRINCIPLE :

Consider a linear homogeneous ordinary differential equation

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + a_1(x) y' + a_0(x)\, y = 0 \qquad \ldots\ldots\ldots (1)$$

If $y_1, y_2, \ldots\ldots, y_n$ are the solutions of equation (1) then their linear combination

$$y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n \qquad \ldots\ldots\ldots (2)$$

where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants, is also a solution of equation (1).

If the n solutions $y_1, y_2, \ldots\ldots, y_n$ are linearly independent then the solution (2) is called the general solution of an equation (1).

3.7 DIFFERENTIAL OPERATOR :

Let $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ... The symbols D, D^2, ... are called differential operators.

The power of D indicates the number of times the operation of differentiation must be carried out.

e.g. $D^5 \cos x$ shows that we must differentiate $\cos x$, five times.

We note the following results, which are valid :

1. $D^m + D^n = D^n + D^m$
2. $D^m D^n = D^n D^m = D^{n+m}$
3. $D(y_1 + y_2) = Dy_1 + Dy_2$
4. $(D - a_1)(D - a_2) = (D - a_2)(D - a_1)$, where a_1 and a_2 are constants.

$\therefore$ We can write the differential equation

$$y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots\ldots + a_1(x)\, y' + a_0(x)\, y = f(x)$$

or $$\frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_1(x) \frac{dy}{dx} + a_0(x)\, y = f(x)$$

as

$$(D^n + a_{n-1}(x)\, D^{n-1} + \ldots + a_1(x)\, D + a_0(x))\; y = f(x)$$

$$\Rightarrow P(D)y = f(x)$$

Where P(D) is an operator now. If $P_1(D)$ and $P_2(D)$ are two operators, then $P_1(D)$ $P_2(D)$ is also an opertor, such that

$$P_1(D)\, P_2(D) = P_2(D)\, P_1(D).$$

Again if α is some constant then

$$P(D)\,(\alpha y) = \alpha\, P(D)y$$

Note : 1. Here P(D) is an algebraic equation in terms of D. Thus it obviously satisfies the fundamental laws of algebra.

2. Negative power of D, that is D^{-1} is equivalent to an integration.

$$e.g.\; D^{-1}x = \int x\, dx = \frac{x^2}{2}$$

Here arbitrary constant is not added because the main object of D^{-1} is to find an integral only but not the complete integral.

Here $(D^{-1})^5 = D^{-5} \Rightarrow$ The operation of integration is to perform five times.

Again we can write,

$$D^{-1} = \frac{1}{D} \text{ and } DD^{-1} = 1.$$

3.8 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

A differential equation of the form

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0 y = R(x) \qquad \ldots\ldots\ldots (1)$$

Where R(x) is a function of x only, and $a_0, a_1, \ldots, a_{n-1}$ are constants is called a linear differential equation **with constant coefficients** of order n.

Using the symbol D, equation (1) can be rewritten as

$$D^n y + a_{n-1}D^{n-1}y + \ldots + a_1 Dy + a_0 y = R(x)$$

$$\Rightarrow (D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)\; y = R(x)$$

$$\Rightarrow f(D)y = R(x) \qquad \ldots\ldots\ldots (2)$$

Where $f(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0$ is a polynomial operator in terms of D of degree n.

Consider the homogeneous part of the differential equation (1).

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0 y = 0$$

or $f(D)y = 0$ .... (3)

By the superposition principle, if $y_1, y_2, ..., y_n$ are n linearly independent solutions of equation (3) then, the linear combination

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n$$

where $c_1, c_2, ..., c_n$ are arbitrary constants, is also a solution of the equation (3), and hence it is a general solution of equation (3).

Let $u = c_1 y_1 + c_2 y_2 + ... + c_n y_n$.

$\therefore$ $f(D)\, u = 0$ .... (4)

Now, let v be any particular solution of equation (2) not involving any arbitrary constants. Thus we have $f(D)\, v = R(x)$ .... (5)

Let $y = u + v$.

$$\begin{aligned} \therefore\ f(D)\,(u+v) &= f(D)u + f(D)v \\ &= 0 + R(x) \qquad (\because \text{ from (4) and (5)}) \\ &= R(x). \end{aligned}$$

$\therefore$ $y = u + v$, i.e. $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n + v$ is the general solution of equation (1).

The part $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is known as the **Complementary Function** (which we will denote by C. F. or y_c or y_H) and the part v is known as the particular integral (which we will denote by P. I. or y_P).

Thus the general solution of equation (1) is :

$$y = \text{C. F.} + \text{P. I.}$$

Where C. F. involves n arbitrary constants and P. I. does not involve any arbitrary constant.

Note : Here the P. I. exists due to the presence of $R(x)$ only in (1). If the equation is given with $R(x) = 0$, then its general solution will not involve P. I. That is the general solution of such an equation with $R(x) = 0$ is given by y = C. F. only.

.9 METHOD OF FINDING COMPLEMENTARY FUNCTION :

Consider the homogeneous part of the differential equation (1) in section 3.8.

$$(D_n + a_{n-1} D^{n-1} + ... + a_1 D + a_0)\, y = 0 \qquad .... (1)$$

Assume that $y = e^{mx}$ is a solution of this equation. Since

$$Dy = me^{mx},\ D^2 y = m^2 e^{mx},\ .....,\ D^n y = m^n e^{mx}$$

Equation (1) becomes

$$(m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0)\, e^{mx} = 0$$

But $e^{mx} \neq 0$.

$\therefore \quad m^n + a_{n-1} m^{n-1} + ... + a_1 m + a_0 = 0$ .... (2)

which is a polynomial of m of degree n.

Thus $y = e^{mx}$ will be a solution of equation (1) only for those values of m obtained from equation (2).

Equation (2) is called the **Auxiliary Equation (A. E.)** and is obtained directly by putting D = m in f(D) = 0.

The auxiliary equation (2) has n roots, say, $m_1, m_2, ..., m_n$. Therefore three cases arise depending on the nature of the n roots.

Case – I : When all the roots are real and distinct.

If the roots $m_1, m_2, ..., m_n$ are all real and distinct, then $e^{m_1 x}, e^{m_2 x}, ..., e^{m_n x}$ are n distinct and linearly independent solutions of equation (1). Thus the C. F. in this case is

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x} \quad .... (3)$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

Case – II : When the roots are equal.

Suppose that m_1 and m_2 are real and equal, and $m_3, ..., m_n$ are real and distinct. Then the solution (3) becomes

$$y = c_1 e^{m_1 x} + c_2 e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$= (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

Now $c_1 + c_2$ can be replaced by a single constant say c. Therefore, now, there are only $(n - 1)$ arbitrary constants and hence it is not the general solution.

To obtain the general solution, consider the differential equation $(D - m_1)^2 y = 0$

$(\because f(D) = (D - m_1)(D - m_2)(D - m_3) ..... (D - m_n))$

in which the two roots are equal.

$\therefore \quad (D - m_1)(D - m_1) y = 0$

Let $(D - m_1) y = v \Rightarrow (D - m_1) v = 0$

$$\Rightarrow \frac{dv}{dx} = m_1 v \Rightarrow \frac{dv}{v} = m_1 \, dx$$

$$\Rightarrow \log v = m_1 x + \log c_1$$

$$\Rightarrow v = e^{m_1 x + \log c_1} \Rightarrow v = c_1 e^{m_1 x}$$

$\therefore$ We have

$$(D - m_1) y = v \Rightarrow \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$$

$\therefore$ I. F. $= e^{-\int m_1 dx} = e^{-m_1 x}$

$\therefore$ The solution of this equation is,

$$y \cdot e^{-m_1 x} = \int c_1 e^{m_1 x} \cdot e^{-m_1 x}\, dx + c_2$$

$$= c_1 x + c_2$$

$\therefore$ $y = (c_1 x + c_2)\, e^{m_1 x}$

$\therefore$ The C. F. for the equation (1) becomes

$$y_c = (c_1 x + c_2)\, e^{m_1 x} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

Similarly if all the roots are equal, i.e. $m_1 = m_2 = \ldots\ldots = m_n$ then C. F. of equation (1) is given by

$$y_c = (c_1\, x^{n-1} + c_2\, x^{n-2} + \ldots\ldots + c_n)\, e^{m_1 x}$$

Case – III : When the roots are complex.

The complex roots always occurs in a pair of conjugate complex numbers. Let m_1 and m_2 be conjugate complex numbers, say

$$m_1 = a + ib,\ m_2 = a - ib \text{ (where } i^2 = 1)$$

$\therefore$ The C. F. for equation (1) is,

$$y_c = c_1\, e^{(a+ib)x} + c_2\, e^{(a-ib)x} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{ax}\left\{c_1\, e^{ibx} + c_2\, e^{-ibx}\right\} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{ax}\left\{c_1\, (\cos bx + i \sin bx) + c_2\, (\cos bx - i \sin bx)\right\} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{ax}\left\{A_1 \cos bx + A_2 \sin bx)\right\} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

where $A_1 = c_1 + c_2$, $A_2 = i\,(c_1 - c_2)$

Rewriting this solution as,

$$y_c = e^{ax}\left\{c_1 \cos bx + c_2 \sin bx)\right\} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

Similarly if the complex roots are repeated, say m_1, m_2, m_3, m_4 occurs in the form $a \pm ib$ twice, then

$$y_c = e^{ax}\left\{(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx\right\} + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}.$$

Note : 1. We can also write the expression $e^{ax}\,\{c_1 \cos bx + c_2 \sin bx\}$ as $c_1 e^{ax} \sin (bx + c_2)$ or $c_1 e^{ax} \cos (bx + c_2)$.

2. If the auxiliary equation has a pair of irrational rots i.e. they are $a \pm \sqrt{b}$, where β is positive then the corresponding term in C. F. will be

$$e^{\alpha x}\left\{c_1 \cosh\sqrt{\beta}\, x + c_2 \sinh\sqrt{\beta} x\right\}$$

or $c_1 e^{\alpha x} \sinh\left(\sqrt{\beta}\, x + c_2\right)$ or $c_1 e^{\alpha x} \cosh\left(\sqrt{\beta}\, x + c_2\right)$.

SOLVED EXAMPLES

1. Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$.

Solution : We have $(D^2 - 3D - 4)y = 0$

$\therefore$ The auxiliary equation is $m^2 - 3m - 4 = 0$

$\Rightarrow (m - 4)(m + 1) = 0$

$\Rightarrow m = 4, -1$

$\therefore$ The general solution is

$y = c_1 e^{4x} + c_2 e^{-x}$

2. Solve $y'' - 5y' + 6y = 0$.

Solution : We have $(D^2 - 5D + 6)\, y = 0$

$\therefore$ Auxiliary equation is : $m^2 - 5m + 6 = 0$

$\Rightarrow (m - 3)(m - 2) = 0 \Rightarrow m = 2, 3$

$\therefore$ The general solution is

$y = c_1 e^{2x} + c_2 e^{3x}$

3. Solve $\dfrac{d^2y}{dx^2} + (a + b)\dfrac{dy}{dx} + aby = 0$.

Solution : We have $(D^2 + (a + b)\, D + ab)\, y = 0$

$\therefore$ Auxiliary equation is : $m_2 + (a + b)\, m + ab = 0$

$\Rightarrow (m + a)(m + b) = 0 \Rightarrow m = -a, -b$

$\therefore$ The general solution is

$y = c_1 e^{-ax} + c_2 e^{-bx}$

4. Solve $y''' + 6y'' + 3y' - 10y = 0$.

Solution : The auxiliary equation is : $m^3 + 6m^2 + 3m - 10 = 0$

$\Rightarrow (m - 1)(m^2 + 7m + 10) = 0$

$\Rightarrow (m-1)(m+2)(m+5)=0$

$\Rightarrow m = 1, -2, -5$

$\therefore$ The general solution is

$y = c_1e^x + c_2e^{-2x} + c_3e^{-5x}$

5. Solve $(D^3 - 6D^2 + 11D - 6)\ y = 0$.

Solution : The auxiliary equation is :

$m^3 - 6m^2 + 11m - 6 = 0$

$\Rightarrow (m-1)(m^2 - 5m + 6) = 0$

$\Rightarrow (m-1)(m-2)(m-3) = 0$

$\Rightarrow m = 1, 2, 3$

$\therefore$ The general solution is

$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

6. Solve $y'' - 3y' + 2y = 0$, given that when $x = 0$, $y = 0$ and $y' = 0$.

Solution : The auxiliary equation is :

$m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

$\therefore$ The general solution is given by

$y = c_1e^x + c_2e^{2x}$ .... (1)

Now given that $y(0) = 0$ and $y'(0) = 0$

$\therefore$ $y' = c_1e^x + 2c_2e^{2x}$ .... (2)

$\therefore$ From (1) and (2)

$0 = c_1 + c_2$ and $0 = c_1 + 2c_2$

Solving, we get

$c_1 = 0$ and $c_2 = 0$

$\therefore$ The required solution is :

$y = 0$

7. Solve $(D^3 - 3D + 2)\ y = 0$.

Solution : The auxiliary equation is :

$m^3 - 3m + 2 = 0$

$\Rightarrow (m-1)(m^2 + m - 2) = 0$

$\Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

$\therefore$ The general solution is :

$$y = (c_1 + c_2x)\, e^x + c_3 e^{-2x}$$

8. Solve $y'' - 4y' + 4y = 0$.

Solution : The auxiliary equation is :

$$m^2 - 4m + 4 = 0 \Rightarrow (m - 2)^2 = 0$$

$$\Rightarrow m = 2, 2.$$

$\therefore$ The general solution is,

$$y = (c_1 + c_2x)\, e^{2x}$$

9. Solve $y''' - 4y'' + 5y' - 2y = 0$.

Solution : The auxiliary equation is :

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$\Rightarrow (m - 1)(m^2 - 3m + 2) = 0$$

$$\Rightarrow (m - 1)(m - 1)(m - 2) = 0$$

$$\Rightarrow m = 1, 1, 2.$$

$\therefore$ The general solution is

$$y = (c_1 + c_2x)\, e^x + c_3\, e^{2x}$$

10. Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution : The auxiliary equation is :

$$m^3 - 5m^2 + 8m - 4 = 0$$

$$\Rightarrow (m - 1)(m^2 - 4m + 4) = 0$$

$$\Rightarrow (m - 1)(m - 2)^2 = 0$$

$$\Rightarrow m = 1, 2, 2.$$

$\therefore$ The general solution is given by

$$y = c_1e^x + (c_2 + c_3x)\, e^{2x}$$

11. Solve $(D^4 + 2D^3 - 3D^2 - 4D + 4)\, y = 0$.

Solution : The auxiliary equation is :

$$m^4 + 2m^3 - 3m^2 - 4m + 4 = 0$$

$$\Rightarrow (m - 1)(m^3 + 3m^2 - 4) = 0$$

$$\Rightarrow (m - 1)(m - 1)(m^2 + 4m + 4) = 0$$

$$\Rightarrow (m - 1)^2 (m + 2)^2 = 0$$

$\Rightarrow m = 1, 1, 2, 2$

$\therefore$ The general solution is :

$y = (c_1 + c_2x)\, e^x + (c_3 + c_4x)\, e^{2x}$

12. Solve $(D^4 - D^3 - 9D^2 - 11\,D - 4)\, y = 0$

Solution : The auxiliary equation is :

$m^4 - m^3 - 9m^2 - 11m - 4 = 0$

$\Rightarrow (m + 1)\,(m^3 - 2m^2 - 7m - 4) = 0$

$\Rightarrow (m + 1)\,(m + 1)\,(m^2 - 3m - 4) = 0$

$\Rightarrow (m + 1)^2\,(m + 1)\,(m - 4) = 0$

$\Rightarrow (m + 1)^3\,(m - 4) = 0$

$\Rightarrow m = -1, -1, -1, 4.$

$\therefore$ The general solution is

$y = (c_1 + c_2x + c_3x^2)\, e^{-x} + c_4e^{4x}$

13. Solve $y''' - 8y = 0$.

Solution : The auxiliary equation is :

$m^3 - 8 = 0 \Rightarrow (m - 2)\,(m^2 + 2m + 4) = 0$

$$\Rightarrow m = 2 \text{ and } m = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$\Rightarrow m = 2 \text{ and } m = -1 \pm \sqrt{3}i$$

$\therefore$ The general solution is

$$y = c_1e^{2x} + e^{-x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$$

14. Solve $(D^4 + K^4)\, y = 0$.

Solution : The auxiliary equation is :

$$m^4 + k^4 = 0 \Rightarrow m^4 + 2k^2m^2 + k^4 - 2k^2m^2 = 0$$

$$\Rightarrow (m^2 + k^2)^2 - 2k^2m^2 = 0$$

$$\Rightarrow (m^2 - \sqrt{2}\,km + k^2)\,(m^2 + \sqrt{2}\,km + k^2) = 0$$

$$\Rightarrow m^2 - \sqrt{2}\,km + k^2 = 0 \text{ and } m^2 + \sqrt{2}\,km + k^2 = 0$$

$$\Rightarrow m = \frac{\sqrt{2}k \pm \sqrt{2k^2 - 4k^2}}{2} \text{ and } m = \frac{-\sqrt{2}k \pm \sqrt{2k^2 - 4k^2}}{2}$$

$$\Rightarrow m = \frac{\sqrt{2}k \pm \sqrt{2}ki}{2} \text{ and } m = \frac{-\sqrt{2}k \pm \sqrt{2}ki}{2}$$

$$\Rightarrow m = \frac{k}{\sqrt{2}} \pm \frac{k}{\sqrt{2}} i \text{ and } m = -\frac{k}{\sqrt{2}} \pm \frac{k}{\sqrt{2}} i$$

∴ The general solution is

$$y = e^{\frac{k}{\sqrt{2}}x}\left\{c_1 \cos\frac{k}{\sqrt{2}}x + c_2 \sin\frac{k}{\sqrt{2}}x\right\} + e^{-\frac{k}{\sqrt{2}}x}\left\{c_3 \cos\frac{k}{\sqrt{2}}x + c_4 \sin\frac{k}{\sqrt{2}}x\right\}$$

15. Solve $(D^4 - K^4)\, y = 0$.

Solution : The auxiliary equation is

$$m^4 - k^4 = 0 \Rightarrow (m^2 - k^2)(m^2 + k^2) = 0$$

$$\Rightarrow m = \pm k,\ m = \pm ki$$

∴ The general solutin is :

$$y = c_1 e^{kx} + c_2 e^{-kx} + c_3 \cos kx,\ c_4 \sin kx$$

16. Solve $(D^2 + 6D + 4)\, y = 0$.

Solution : The auxiliary equation is :

$$m^2 + 6m + 4 = 0 \Rightarrow m = \frac{-6 \pm \sqrt{36-16}}{2} = -3 \pm \sqrt{5}$$

∴ The general solution is

$$y = e^{-3x}(c_1 \cosh\sqrt{5}x + c_2 \sinh\sqrt{5}x)$$

17. Solve $(D^4 - 2D^3 + 3D^2 - 2D + 1)\, y = 0$

Solution : The auxiliary equation is :

$$m^4 - 2m^3 + 3m^2 - 2m + 1 = 0$$

$$\Rightarrow (m^2 - m + 1)^2 = 0$$

$$\Rightarrow m^2 - m + 1 = 0 \text{ twice}$$

$$\Rightarrow m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \text{ twice}$$

∴ The general solution is

$$y = e^{\frac{1}{2}x}\left\{(c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right\}$$

18. Solve $(D^4 - 2D^3 - 2D - 1)\ y = 0.$

Solution : The auxiliary equation is :

$$m^4 - 2m^3 - 2m - 1 = 0$$

$$\Rightarrow m^4 - 1 - 2m\ (m^2 + 1) = 0$$

$$\Rightarrow (m^2 + 1)\ (m^2 - 1 - 2m) = 0$$

$$\Rightarrow m^2 + 1 = 0,\ m^2 - 2m - 1 = 0$$

$$\Rightarrow m = \pm\, i,\ m = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$\therefore$ The general solution is

$$y = c_1 \cos x + c_2 \sin x + e^x \left(c_3 \cosh \sqrt{2}\,x + c_4 \sinh \sqrt{2}\,x\right)$$

19. Solve $(D^2 + 1)^3\ (D^2 + D + 1)^2\ y = 0$

Solution : The auxiliary equation is :

$$(m^2 + 1)^3\ (m^2 + m + 1)^2\ y = 0$$

$\Rightarrow m^2 + 1 = 0$ three times, and $m^2 + m + 1 = 0$ two times

$\Rightarrow m = \pm$ i three times, and $m = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$ two times

$\therefore$ The general solution is

$$y = (c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x +$$

$$e^{-\frac{1}{2}x} \left((c_7 + c_8 x) \cos \frac{\sqrt{3}}{2} x + (c_9 + c_{10} x) \sin \frac{\sqrt{3}}{2} x \right)$$

20. Solve $(D^2 \pm K^2)\ y = 0.$

Solution : The auxiliary equation is :

$$m^2 + k^2 = 0 \text{ or } m^2 - k^2 = 0$$

when $m^2 + k^2 = 0 \Rightarrow m = \pm\, ki$

$\therefore$ The solution of $(D^2 + K^2)\ y = 0$ is,

$$y = c_1 \cos kx + c_2 \sin kx$$

Now $m^2 - k^2 = 0 \Rightarrow m = \pm\, k$

$\therefore$ The solution of $(D^2 - K^2)\ y = 0$ is,

$$y = c_3 e^{kx} + c_4 e^{-kx}.$$

21. Solve $(D^2 + 1)\, y = 0$, given $y = 2$ for $x = 0$, and $y = -2$ for $x = \frac{\pi}{2}$.

Solution : The auxiliary equation is :

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$\therefore$ The general solution is

$$y = c_1 \cos x + c_2 \sin x \qquad \text{..........} (1)$$

Now, given that $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = -2$.

$\therefore$ From (1), we have

$$c_1 = 2 \text{ and } c_2 = -2$$

$\therefore$ The required solution is

$$y = 2\cos x - 2 \sin x = 2 (\cos x - \sin x)$$

22. Solve $(D^6 + 6D^4 + 9D^2)\, y = 0$.

Solution : The auxiliary eqatuion is :

$$m^6 + 6m^4 + 9m^2 = 0$$

$$\Rightarrow m^2 (m^4 + 6m^2 + 9) = 0$$

$$\Rightarrow m^2 (m^2 + 3)^2 = 0 \Rightarrow m = 0, 0 \text{ and } m = \pm \sqrt{3}i \text{ twice.}$$

$\therefore$ The general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x) \cos\sqrt{3}x + (c_5 + c_6 x) \sin\sqrt{3}x.$$

EXERCISE – 3.1

❑ **Solve the following differential equations :**

1. $(D^4 + 2D^2 + 1)\, y = 0$. **Ans. :** $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$
2. $(D^3 - 4D^2 + 5D - 2)\, y = 0$. **Ans. :** $y = (c_1 + c_2 x)e^x + c_3 e^{2x}$
3. $(D^4 - 81)\, y = 0$. **Ans. :** $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 3x + c_4 \sin 3x$
4. $(D^4 + D^2 + 1)\, y = 0$.

   Ans. : $y = e^{x/2}\left\{c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right\} + e^{-x/2}\left\{c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x\right\}$
5. $(D^3 - 7D + 6)\, y = 0$. **Ans. :** $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$
6. $(D^4 + 13D^2 + 36)\, y = 0$. **Ans. :** $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$
7. $(D^4 + 4D^3 - 5D^2 - 36D - 36)\, y = 0$. **Ans. :** $y = c_1 e^{-3x} + c_2 e^{3x} + (c_3 + c_4 x) e^{-2x}$

8. $(D^4 - 7D^3 + 18D^2 - 20D + 8)\ y = 0.$ **Ans. :** $y = c_1e^x + (c_2 + c_3x + c_4x^2)\ e^{2x}$

9. $(D^2 + D - 30)\ y = 0.$ **Ans. :** $y = c_1e^{5x} + c_2e^{-6x}$

10. $(D^4 - 2D^3 + 5D^2 - 8D + 4)\ y = 0.$ **Ans. :** $y = (c_1 + c_2x)\ e^x + c_3 \cos 2x + c_4 \sin 2x$

3.10 METHODS OF FINDING PARTICULAR INTEGRAL :

We have already seen in section 3.8 that the general solution of

$$(D^n + a_{n-1}\ D^{n-1} + ... + a_1\ D + a_0)\ y = R(x) \quad .... (1)$$

is given by

$$y = C.\ F. + P.\ I.$$

In section 3.9, we already discussed the method of finding C. F. Now let us see the methods of finding particular integral (P. I.), which does not contain any arbitrary constants.

Rewriting equation (1) as,

$$f(D)\ y = R(x) \quad .... (2)$$

If we let $y = \frac{1}{f(D)}$ R(x) some function of x which satisfies eqatuion (2). That is

$$f(D)\left\{\frac{1}{f(D)}\,R(x)\right\} = R(x)$$

$$\Rightarrow R(x) = R(x)$$

Here $\frac{1}{f(D)}$ is called the inverse operator of f(D).

$\therefore$ The particular integral of (2) is,

$$\frac{1}{f(D)}\,R(x)$$

3.10.1 GENERAL METHOD OF FINDING P. I. :

THEROREM : If X is a function of x, then

$$\frac{1}{D - a}\,X = e^{ax}\int X e^{-ax} dx$$

Proof : Let $y = \frac{1}{D - a}\,X$

On operating by D – a, we get

$$(D - a)\ y = X$$

$$\therefore \frac{dy}{dx} - ay = X$$

which is a linear differential equation.

$\therefore$ I. F. $= e^{-a\int dx} = e^{-ax}$

$\therefore$ The solution is given by

$$y \cdot e^{-ax} = \int e^{-ax} X \, dx$$

$$\Rightarrow y = e^{ax} \int X e^{-ax} \, dx$$

$$\therefore \quad \frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$$

Note : Here the constant of integration is not added, as the P. I. does not contain any arbitrary constant.

Working rule : To find P. I. for the equation

$$f(D)\, y = R(x)$$

factorize $f(D)$. That is

$$f(D) = (D-\alpha_1)(D-\alpha_2) \ldots\ldots (D-\alpha_n)$$

$$\therefore \quad \text{P. I.} = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots\ldots(D-\alpha_n)} R(x)$$

$$= \frac{1}{D-\alpha_1} \cdot \frac{1}{D-\alpha_2} \ldots\ldots \frac{1}{D-\alpha_{n-1}} \cdot \frac{1}{D-\alpha_n} R(x)$$

$$= \frac{1}{D-\alpha_1} \cdot \frac{1}{D-\alpha_2} \ldots\ldots \frac{1}{D-\alpha_{n-1}} \left\{ e^{\alpha_n x} \int e^{-\alpha_n x} R(x)\, dx \right\}$$

One by one operation of the inverse operators upon the resulting functions, we get the required P. I.

Or, we can find P. I. by using partial fractions. That is

$$\text{P. I.} = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots\ldots(D-\alpha_n)} R(x)$$

$$= \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots\ldots + \frac{A_n}{D-\alpha_n} \right] R(x)$$

$$= A_1 \frac{1}{D-\alpha_1} R(x) + \ldots\ldots + A_n \frac{1}{D-\alpha_n} + R(x)$$

$$= A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} R(x)\, dx + \ldots\ldots + A_n e^{\alpha_n x} \int e^{-\alpha_n x} R(x)\, dx$$

Note : The above method is more general and can be used to evaluate P. I. in any problem. However it is advisable to use this method only for those R(x) which are not covered in the special cases (will be discussed later).

SOLVED EXAMPLES

1. Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

Solution : The auxiliary equation is :

$$m^2 + a^2 = 0 \Rightarrow m = \pm\, ai$$

$\therefore$ C. F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{Now P. I.} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D + ai)(D - ai)} \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D - ai} \sec ax - \frac{1}{D + ai} \sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int e^{-aix} \sec ax\, dx - e^{-aix}\int e^{aix} \sec ax\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int \frac{\cos ax - i\sin ax}{\cos ax}\, dx - e^{-aix}\int \frac{\cos ax + i\sin ax}{\cos ax}\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int (1 - i\tan ax)\, dx - e^{-aix}\int (1 + i\tan ax)\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\left\{x - \frac{i}{a}\log\sec ax\right\} - e^{-aix}\left\{x + \frac{i}{a}\log\sec ax\right\}\right]$$

$$= \frac{1}{2ai}\left[x\left\{e^{aix} - e^{-aix}\right\} - \frac{i}{a}\log\sec ax\left\{e^{aix} + e^{-aix}\right\}\right]$$

$$= \frac{x}{a}\left\{\frac{e^{aix} - e^{-aix}}{2i}\right\} - \frac{1}{a^2}\log\sec ax\left\{\frac{e^{aix} + e^{-aix}}{2}\right\}$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}(\log\sec ax)\cos ax.$$

$\therefore$ The general solution is y = C. F. + P. I.

2. Solve $\dfrac{d^2y}{dx^2} + a^2y = \tan ax.$

Solution : The auxiliary equation is :

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$$\therefore \quad y_c = c_1\cos ax + c_2\sin ax$$

Now $y_p = \dfrac{1}{D^2 + a^2} \tan ax$

$$= \frac{1}{(D+ai)(D-ai)} \tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \tan ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} \tan ax - \frac{1}{D+ai} \tan ax\right]$$

Now $\dfrac{1}{D-ai} \tan ax = e^{axi}\displaystyle\int e^{-axi} \tan ax\, dx$

$$= e^{axi}\int(\cos ax - i\sin ax) \tan ax\, dx$$

$$= e^{axi}\int\left(\sin ax - i\,\frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{axi}\int\left(\sin ax - i\,\frac{1-\cos^2 ax}{\cos ax}\right) dx$$

$$= e^{axi}\int[\sin ax - i\,(\sec ax - \cos ax)]\, dx$$

$$= e^{axi}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + i\,\frac{\sin ax}{a}\right]$$

$$= e^{axi}\left[-\frac{1}{a}\{\cos ax - i\sin ax\} - \frac{i}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$= -\frac{e^{axi}}{a}\left[e^{-axi} + i\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right]$$

$$= -\frac{1}{a}\left[1 + i\,e^{axi}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right].$$

Similarly, we get (replacing i by $-i$)

$$\frac{1}{D+ai}\tan ax = -\frac{1}{a}\left[1 - i\,e^{-axi}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$\therefore\ y_p = \frac{1}{2ai}\left[-\frac{i}{a}\left(e^{axi}+e^{-axi}\right)\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{1}{a^2}\cos ax\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

The general solution is $y = y_c + y_p$.

3. Solve $\frac{d^2y}{dx^2} + a^2y = \operatorname{cosec} ax$.

Solution : The auxiliary equation is :

$$m^2 + a^2 = 0 \Rightarrow m = \pm\, ai$$

$$\therefore\ y_c = c_1\cos ax + c_2\sin ax$$

Now $y_p = \frac{1}{D^2+a^2}\operatorname{cosec} ax$

$$= \frac{1}{(D+ai)(D-ai)}\operatorname{cosec} ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\operatorname{cosec} ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}\operatorname{cosec} ax - \frac{1}{D+ai}\operatorname{cosec} ax\right] \qquad \text{.......... (1)}$$

Now $\frac{1}{D-ai}\operatorname{cosec} ax = e^{aix}\int e^{-aix}\operatorname{cosec} ax\, dx$

$$= e^{aix}\int\frac{\cos ax - i\sin ax}{\sin ax}\,dx$$

$$= e^{aix}\int(\cot ax - i)\,dx$$

$$= e^{aix}\left[\frac{1}{a}\log\sin ax - i\,x\right] \qquad \text{.......... (2)}$$

Again replacing i by $-i$, we get

$$\frac{1}{D+ai}\operatorname{cosec} ax = e^{-aix}\left[\frac{1}{a}\log\sin ax + i\,x\right] \qquad \text{.......... (3)}$$

$\therefore$ From (1), (2) and (3),

$$y_p = \frac{1}{2ai}\left[\frac{1}{a}\log\sin ax\left(e^{iax} - e^{-iax}\right) - ix\left(e^{iax} + e^{-iax}\right)\right]$$

$$= \frac{1}{a}(\log\sin ax)(\sin ax) - \frac{x}{a}\cos ax$$

$\therefore$ The general solution is

$y = y_c + y_p$.

EXERCISE – 3.2

☐ **Solve the following differential equations :**

1. $(D^2 + 9)\,y = \sec 3x$

   Ans. : $y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \cdot \log$ |cos3x|

2. $(D^2 + 1)\,y = \operatorname{cosec} x$. **Ans. :** $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log$ |sinx|

3. $(D^2 + 4)\,y = \tan 2x$. **Ans. :** $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \cdot \log\tan\left(\frac{\pi}{4} + \right.$

3.10.2 SOME SPECIAL CASES :

Case – I : To find P. I. when R(x) is of the form e^{ax}, where a is a constant.

Result : Prove that $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$, **if** $f(a) \neq 0$

$$= x\frac{1}{f'(a)}e^{ax}, \text{ if } f(a) = 0.$$

Proof : By simple differentiation, we know that

$$De^{ax} = ae^{ax},\ D^2e^{ax} = a^2e^{ax},\ \ldots\ldots,\ D^ne^{ax} = a^ne^{ax}$$

$$\therefore f(D)e^{ax} = \{D^n + a_{n-1}D^{n-1} + \ldots\ldots + a_1 D + a_0\}\,e^{ax}$$

$$= \{a^n + a_{n-1}a^{n-1} + \ldots\ldots a_1 . a + a_0\}\,e^{ax}$$

$$= f(a)\,e^{ax}$$

Operating on both sides with $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}f(D)\,e^{ax} = \frac{1}{f(D)}f(a)\,e^{ax}$$

$$\Rightarrow e^{ax} = f(a)\frac{1}{f(D)}e^{ax}$$

$$\Rightarrow \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \qquad (\because f(a) \neq 0)$$

Now if $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $D - a$ is the factor of $f(D)$.

Let $f(D) = \phi(D)(D - a)$.

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{1}{\phi(D)(D-a)} e^{ax}$$

$$= \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{D-a} \cdot \left\{\frac{e^{ax}}{\phi(a)}\right\} \qquad (\because \phi(a) \neq 0)$$

$$= \frac{1}{\phi(a)} \frac{1}{D-a} e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax} \int e^{-ax} \cdot e^{ax}\, dx$$

$$= \frac{1}{\phi(a)} x\, e^{ax}$$

Now $f(D) = (D - a)\phi(D) \Rightarrow f'(D) = \phi(D) + (D - a)\phi'(D)$

$$\Rightarrow f'(a) = \phi(a)$$

$$\therefore \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}, \text{ when } f(a) = 0.$$

Again if, $\phi(a) = 0$, i.e. $f(D)$ has one more factor $D - a$, then following the same procedure as above, we get

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$$

Continuing in this way, assume that the factor $D - a$ occurs r times then

$$\frac{1}{f(D)} e^{ax} = x^r \frac{1}{f^{(r)}(a)} e^{ax}$$

SOLVED EXAMPLES

1. Solve $y'' - 3y' + 2y = e^x$.

Solution : The auxiliary equation is :

$$m^2 - 3m + 2 = 0 \quad \Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore y_c = c_1 e^x + c_2 e^{2x}$$

Now $y_p = \dfrac{1}{D^2 - 3D + 2} e^x$

$= x \dfrac{1}{2D - 3} e^x$

$= -xe^x$

$\therefore$ The general solution is

$y = y_c + y_p$

2. Solve $(D^2 - 2D + 1)\ y = e^x$.

Solution : The auxiliary equation is :

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1 \text{ twice}$$

$\therefore\ y_c = (c_1 + c_2 x)\ e^x$

Now $y_p = \dfrac{1}{D^2 - 2D + 1} e^x$

$= x \dfrac{1}{2D - 2} e^x$

$= x^2 \dfrac{1}{2} e^x = \dfrac{x^2}{2} e^x$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2 x)\ e^x + \frac{x^2}{2} e^x.$$

3. Solve $(D^3 - 5D^2 + 7D - 3)\ y = e^{2x} \cosh x$.

Solution : The auxiliary equation is :

$m^3 - 5m^2 + 7m - 3 = 0$

$\Rightarrow (m-1)(m^2 - 4m + 3) = 0$

$\Rightarrow (m-1)(m-1)(m-3) = 0$

$\Rightarrow m = 1, 1, 3.$

$\therefore\ y_c = (c_1 + c_2 x)e^x + c_3 e^{3x}.$

Now $y_p = \dfrac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \cosh x$

$= \dfrac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \left(\dfrac{e^x + e^{-x}}{2} \right)$

$$= \frac{1}{2}\left[\frac{1}{D^3 - 5D^2 + 7D - 3} e^{3x} + \frac{1}{D^3 - 5D^2 + 7D - 3} e^{x}\right]$$

$$= \frac{1}{2}\left[x \frac{1}{3D^2 - 10D + 7} e^{3x} + x \frac{1}{3D^2 - 10D + 7} e^{x}\right]$$

$$= \frac{1}{2}\left[x \frac{1}{27 - 30 + 7} e^{3x} + x^2 \frac{1}{6D - 10} e^{x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{4} e^{3x} - \frac{x^2}{4} e^{x}\right]$$

$\therefore$ The general solution is

$$y = y_c + y_p$$

$$= (c_1 + c_2 x) e^{x} + c_3 e^{3x} + \frac{x}{8} e^{3x} - \frac{x^2}{8} e^{x}$$

4. Solve $(D^2 + D - 6)\, y = e^{2x}$.

Solution : The auxiliary equation is :

$$m^2 + m - 6 = 0 \Rightarrow (m - 2)(m + 3) = 0 \Rightarrow m = 2, -3.$$

$$\therefore\ y_c = c_1 e^{2x} + c_2 e^{-3x}$$

$$\text{Now } y_p = \frac{1}{D^2 + D - 6} e^{2x}$$

$$= x \frac{1}{2D + 1} e^{2x}$$

$$= \frac{x}{5} e^{2x}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-3x} + \frac{x}{5} e^{2x}$$

5. Solve $(D^2 - 4D + 3)\, y = 2e^{3x}$.

Solution : The auxiliary equation is :

$$m^2 - 4m + 3 = 0 \Rightarrow (m - 3)(m - 1) = 0 \Rightarrow m = 1, 3.$$

$$\therefore\ y_c = c_1 e^{x} + c_2 e^{3x}$$

Now $y_p = \dfrac{1}{D^2 - 4D + 3} 2e^{3x}$

$$= 2 \frac{1}{D^2 - 4D + 3} e^{3x}$$

$$= 2x \frac{1}{2D - 4} e^{3x}$$

$$= xe^{3x}$$

6. Solve $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = e^x$.

Solution : The auxiliary equation is :

$$m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$$

$$\Rightarrow (m - 1)(m^3 - m^2 + 4m - 4) = 0$$

$$\Rightarrow (m - 1)(m - 1)(m^2 + 4) = 0$$

$$\Rightarrow m = 1, 1, \pm 2i$$

$$\therefore\ y_c = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x$$

Now $y_p = \dfrac{1}{D^4 - 2D^3 + 5D^2 - 8D + 4} e^x$

$$= x \frac{1}{4D^3 - 6D^2 + 10D - 8} e^x$$

$$= x^2 \frac{1}{12D^3 - 12D + 10} e^x$$

$$= \frac{x^2}{10} e^x$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2 x)\, e^x + c_3 \cos 2x + c_4 \sin 2x + \frac{x^2}{10} e^x$$

7. Solve $(D^3 - 7D + 6)\, y = e^{2x}$.

Solution : The auxiliary equation is :

$$m^3 - 7m + 6 = 0$$

$$\Rightarrow (m - 1)(m^2 + m - 6) = 0$$

$$\Rightarrow (m - 1)(m - 2)(m + 3) = 0$$

$$\Rightarrow m = 1, 2, -3$$

$$\therefore\ y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}$$

Now $y_p = \frac{1}{D^3 - 7D + 6} e^{2x}$

$$= x \frac{1}{3D^2 - 7} e^{2x}$$

$$= \frac{x}{5} e^{2x}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x} + \frac{x}{5} e^{2x}.$$

8. Solve $y''' - y = (e^x + 1)^2$.

Solution : The auxiliary equation is :

$$m^3 - 1 = 0 \Rightarrow (m - 1)(m^2 + m + 1) = 0$$

$$\Rightarrow m = 1, \frac{-1 \pm \sqrt{1-4}}{2} = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore y_c = c_1 e^x + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

Now $y_p = \frac{1}{D^3 - 1}\left(e^x + 1\right)^2$

$$= \frac{1}{D^3 - 1}\left(e^{2x} + 2e^x + 1\right)$$

$$= \frac{1}{D^3 - 1} e^{2x} + 2\frac{1}{D^3 - 1} e^x + \frac{1}{D^3 - 1} 1$$

$$= \frac{1}{7} e^{2x} + 2x \frac{1}{3D^2} e^x + \frac{1}{D^3 - 1} e^{0x}$$

$$= \frac{e^{2x}}{7} + \frac{2}{3} x e^x - 1$$

$\therefore$ The general solution is

$$y = y_c + y_p$$

$$= c_1 e^x + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{e^{2x}}{7} + \frac{2}{3} x e^x - 1$$

9. Solve $y'' - 6y' + 9y = 6e^{3x} - 5\log 2$

Solution : The auxiliary equation is :

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3, 3$$

$$\therefore \quad y_c = (c_1 + c_2x)\, e^{3x}$$

$$\text{Now } y_p = \frac{1}{D^2 - 6D + 9}\left(6e^{3x} - 5\log 2\right)$$

$$= 6\,\frac{1}{D^2 - 6D + 9}\, e^{3x} - 5\log 2\,\frac{1}{D^2 - 6D + 9}\, e^{0x}$$

$$= 6x\,\frac{1}{2D - 6}\, e^{3x} - 5\log 2\,\frac{1}{9}\, e^{0x}$$

$$= 6x^2\,\frac{1}{2}\, e^{3x} - \frac{5}{9}\log 2$$

$$= 3x^2e^{3x} - \frac{5}{9}\log 2$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2x)\, e^{3x} + 3x^2e^{3x} - \frac{5}{9}\log 2.$$

10. Solve $(D^2 - a^2)\, y = \sinh ax$.

Solution : The auxiliary equation is :

$$m^2 - a^2 = 0 \Rightarrow m = \pm\, ai$$

$$\therefore \quad y_c = c_1 \cos ax + c_2 \sin ax$$

$$\text{Now } y_p = \frac{1}{D^2 - a^2}\sinh ax$$

$$= \frac{1}{D^2 - a^2}\left(\frac{e^{ax} - e^{-ax}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - a^2}\, e^{ax} - \frac{1}{D^2 - a^2}\, e^{-ax}\right]$$

$$= \frac{1}{2}\left[x\,\frac{1}{2D}\, e^{ax} - x\,\frac{1}{2D}\, e^{-ax}\right]$$

$$= \frac{1}{2}\left[\frac{x}{2a}\, e^{ax} + \frac{x}{2a}\, e^{-ax}\right]$$

$$= \frac{x}{4a}\left[e^{ax} + e^{-ax}\right]$$

$$= \frac{x}{2a}\cosh ax$$

$\therefore$ The general solution is, $y = y_c + y_p$.

11. Solve $D^2y - 3Dy + 2y = \cosh x$.

Solution : The auxiliary equation is :

$$m^2 - 3m + 2 = 0 \Rightarrow (m-2)(m-1) = 0 \Rightarrow m = 1, 2.$$

$$\therefore\ y_c = c_1e^x + c_2e^{2x}.$$

$$\text{Now } y_p = \frac{1}{D^2 - 3D + 2}\cosh x$$

$$= \frac{1}{D^2 - 3D + 2}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3D + 2}e^x + \frac{1}{D^2 - 3D + 2}e^{-x}\right]$$

$$= \frac{1}{2}\left[x\frac{1}{2D - 3}e^x + \frac{1}{6}e^{-x}\right]$$

$$= \frac{1}{2}\left[-x\,e^x + \frac{1}{6}e^{-x}\right]$$

$$= -\frac{x}{2}e^x + \frac{1}{12}e^{-x}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} - \frac{1}{2}x\,e^x + \frac{1}{12}e^{-x}$$

12. Solve $y''' + 6y'' + 11y' + 6y = e^{2x}$.

Solution : The auxiliary equation is :

$$m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m+1)(m^2 + 5m + 6) = 0$$

$$\Rightarrow (m+1)(m+2)(m+3) = 0 \Rightarrow m = -1, -2, -3.$$

$$\therefore\ y_c = c_1\,e^{-x} + c_2\,e^{-2x} + c_3\,e^{-3x}$$

$$\text{Now } y_p = \frac{1}{D^3 + 6D^2 + 11D + 6}e^{2x}$$

$$= \frac{1}{8 + 24 + 22 + 6}e^{2x}$$

$$= \frac{1}{60}e^{2x}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1\,e^{-x} + c_2\,e^{-2x} + c_3\,e^{-3x} + \frac{1}{60}e^{2x}.$$

13. Solve $y'' + 31y' + 240y' + 240y = 272\ e^{-x}$.

Solution : The auxiliary equation is :

$$m^2 + 31m + 240 = 0$$

$$\Rightarrow (m + 15)(m + 16) = 0 \Rightarrow m = -15, -16$$

$$\therefore\ y_c = c_1 e^{-15x} + c_2 e^{-16x}$$

$$\text{Now } y_p = \frac{1}{D^2 + 31D + 240} 272\ e^{-x}$$

$$= 272 \frac{1}{D^2 + 31D + 240} e^{-x}$$

$$= 272 \frac{1}{210} e^{-x}$$

$$= \frac{136}{105} e^{-x}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{-15x} + c_2 e^{-16x} + \frac{136}{105} e^{-x}$$

14. Solve $y'' - 2ky' + k^2y = e^x$, $k \neq 1$.

Solution : The auxiliary equation is :

$$m^2 - 2km + k^2 = 0$$

$$\Rightarrow (m - k)^2 = 0 \Rightarrow m = k, k.$$

$$\therefore\ y_c = (c_1 + c_2x) e^{kx}$$

$$\text{Now } y_p = \frac{1}{D^2 - 2kD + k^2} e^x$$

$$= \frac{1}{1 - 2k + k^2} e^x$$

$$= \frac{1}{(k-1)^2} e^x$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2x) e^{kx} + \frac{e^x}{(k-1)^2}.$$

EXERCISE – 3.3

❑ **Solve the following differential equations :**

1. $(D^2 + D - 2)\,y = e^x$. **Ans. :** $y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} x e^x$

2. $(D^2 - 7D + 6)\,y = e^{2x}$. **Ans. :** $y = c_1 e^x + c_2 e^{6x} - \frac{1}{4} e^{2x}$

3. $(D^2 + D + 1)\,y = e^{-x}$. **Ans. :** $y = c_1 e^{-x/2} \cos \left\{\frac{\sqrt{3}}{2} x + c_2\right\} + e^{-x}$

4. $(D^2 - 3D + 2)\,y = e^{5x}$. **Ans. :** $y = c_1 e^x + c_2 e^{2x} + \frac{1}{12} e^{5x}$

5. $(D^2 - 3D + 2)\,y = \cosh x$. **Ans. :** $y = c_1 e^{2x} + c_2 e^x - \frac{1}{2} x e^x + \frac{1}{12} e^{-x}$

6. $(D^2 - 1)\,y = \cosh x$. **Ans. :** $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x \sinh x$

7. $(D^2 + 4D + 4)\,y = e^{2x} - e^{-2x}$. **Ans. :** $y = (c_1 + c_2 x) e^{-2x} + \frac{1}{16} e^{2x} - \frac{1}{2} x^2 e^{-2x}$

8. $(D + 2)(D - 1)^3\,y = e^x$. **Ans. :** $y = (c_1 + c_2 x + c_3 x^2) e^x + c_2 e^{-2x} + \frac{1}{18} x^3 e^x$

9. $(D^3 + 1)\,y = (e^x + 1)^2$

   Ans. : $y = c_1 e^{-x} + e^{x/2} \left\{c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right\} + 1 + e^x + \frac{1}{9} e^{2x}$

10. $(D^2 - 2kD + k^2)\,y = e^{kx}$

    $y = (c_1 + c_2 x) e^{kx} + \frac{x^2}{2} e^{kx}$.

Case – II : To find P. I. when R(x) is of the form $\sin ax$ or $\cos ax$, where a is a constant.

Result : Prove that

$$\frac{1}{\phi(D)^2} \sin ax = \frac{1}{\phi(-a)^2} \sin ax, \text{ when } \phi\,(-a)^2 \neq 0$$

$$= x \frac{1}{\phi'(-a^2)} \sin ax, \text{ when } \phi\,(-a^2) = 0$$

where $f(D) = \phi(D^2)$

Proof : By ordinary differentiation, we have

$$D\sin ax = a\cos ax$$

$$D^2\sin ax = -a^2\sin ax \qquad \text{.......... (1)}$$

$$D^3\sin ax = -a^3\cos ax$$

$$D^4\sin ax = a^4\sin ax = (-a^2)^2\sin ax \qquad \text{.......... (2)}$$

....

....

In general

$$(D^2)^n \sin ax = (-a^2)^n \sin ax \qquad \text{.......... (3)}$$

$\therefore$ From (1), (2) and (3), we have

$$[(D^2)^n + a_{n-1}(D^2)^{n-1} + \ldots + a_1(D^2)^1 + a_0]\sin ax$$

$$= [(-a^2)^n + a_{n-1}(-a^2)^{n-1} + \ldots + a_1(-a^2)^1 + a_0]\sin ax$$

$\Rightarrow \phi(D^2)\sin ax = \phi(-a^2)\sin ax$

Operating on both sides by $\dfrac{1}{\phi(D^2)}$, we get

$$\frac{1}{\phi(D^2)}\phi(D^2)\sin ax = \frac{1}{\phi(D^2)}\phi(-a^2)\sin ax$$

$$\therefore \quad \sin ax = \phi(-a^2)\frac{1}{\phi(D^2)}\sin ax$$

Since $\phi(-a^2) \neq 0$, we have

$$\frac{1}{\phi(-a^2)}\sin ax = \frac{1}{\phi(D^2)}\sin ax$$

That is $\dfrac{1}{\phi(D^2)}\sin ax = \dfrac{1}{\phi(-a^2)}\sin ax$

Now if $\phi(-a^2) = 0$ then $D^2 + a^2$ is the factor of $\phi(D^2)$.

$$\therefore \quad \frac{1}{\phi(D^2)}\sin ax = \text{Imaginary part of } \frac{1}{\phi(D^2)}e^{aix}$$

$$= \text{I. P. of } x\frac{1}{\phi'(-a^2)}e^{aix} \qquad (\because \phi(-a^2) = 0)$$

$$= x\frac{1}{\phi'(-a^2)}\sin ax, \text{ when } \phi'(-a^2) \neq 0$$

Again if $\phi(D^2)$ has r factors $D^2 + a^2$ repeated then $\frac{1}{\phi(D^2)} \sin ax = x^r \frac{1}{\phi^{(r)}(-a^2)} \sin ax$, when $\phi^{(r-1)}(-a^2) = 0$

Similarly, we can prove that

$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax, \text{ when } \phi(-a^2) \neq 0$$

$$= x \frac{1}{\phi'(-a^2)} \cos ax, \text{ when } \phi(-a^2) = 0$$

$$= x^r \frac{1}{\phi^{(r)}(-a^2)} \cos ax, \text{ when } \phi^{(r-1)}(-a^2) = 0$$

SOLVED EXAMPLES

1. Solve $y'' - y' - 2y = \sin 2x$.

Solution : The auxiliary equation is :

$$m^2 - m - 2 = 0 \Rightarrow (m+1)(m-2) = 0 \Rightarrow m = -1, 2$$

$\therefore$ $y_c = c_1 e^{-x} + c_2 e^{2x}$

Now $y_p = \frac{1}{D^2 - D - 2} \sin 2x$

$$= \frac{1}{-4 - D - 2} \sin 2x$$

$$= -\frac{1}{D+6} \sin 2x$$

$$= -\frac{D-6}{(D+6)(D-6)} \sin 2x$$

$$= -\frac{D-6}{D^2 - 36} \sin 2x$$

$$= -\frac{D-6}{-40} \sin 2x$$

$$= \frac{1}{40} (D - 6) \sin 2x$$

$$= \frac{1}{40} \{D \sin 2x - 6 \sin 2x\}$$

$$= \frac{1}{40} \{2\cos 2x - 6 \sin 2x\}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{20} \{\cos 2x - 3 \sin 2x\}$$

2. Solve $(D^2 + 9)y = \cos 4x$.

Solution : The auxiliary equation is :

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$\therefore \quad y_c = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{Now } y_p = \frac{1}{D^2 + 9} \cos 4x$$

$$= \frac{1}{-16 + 9} \cos 4x$$

$$= -\frac{1}{7} \cos 4x$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7} \cos 4x$$

3. Solve $(D^2 - 5D + 6)\, y = \sin 3x$.

Solution : The auxiliary equation is :

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow (m - 2)(m - 3) = 0 \Rightarrow m = 2, 3$$

$$\therefore \quad y_c = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{Now } y_p = \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$= \frac{1}{-9 - 5D + 6} \sin 3x$$

$$= -\frac{1}{5D + 3} \sin 3x$$

$$= -\frac{5D - 3}{25D^2 - 9} \sin 3x$$

$$= -\frac{5D - 3}{-225 - 9} \sin 3x$$

$$= \frac{1}{234} (5D - 3) \sin 3x$$

$$= \frac{1}{234} (15 \cos 3x - 3 \sin 3x)$$

$$= \frac{1}{78} (5 \cos 3x - \sin 3x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78} (5 \cos 3x - \sin 3x)$$

4. **Solve $(D^2 - 2D + 1)\, y = \cos 3x$.**

Solution : The auxiliary equation is :

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$\therefore\ y_c = (c_1 + c_2 x)\, e^x$$

$$\text{Now } y_p = \frac{1}{D^2 - 2D + 1} \cos 3x$$

$$= \frac{1}{-9 - 2D + 1} \cos 3x$$

$$= -\frac{1}{2(D+4)} \cos 3x$$

$$= -\frac{D-4}{2(D^2 - 16)} \cos 3x$$

$$= -\frac{1}{2} \frac{D-4}{-9-16} \cos 3x$$

$$= \frac{1}{50} (D - 4) \cos 3x$$

$$= \frac{1}{50} (-3 \sin 3x - 4\cos 3x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = (c_1 + c_2 x)e^x - \frac{1}{50} (3\sin 3x + 4\cos 3x)$$

5. **Solve $(D^2 - 8D + 9)\, y = 40 \sin 5x$.**

Solution : The auxiliary equation is :

$$m^2 - 8m + 9 = 0 \Rightarrow m = \frac{8 \pm \sqrt{64 - 36}}{2} = 4 \pm \sqrt{7}$$

$$\therefore\ y_c = c_1\, e^{4x} \cosh\left(\sqrt{7}x + c_2\right)$$

$$\text{Now } y_p = \frac{1}{D^2 - 8D + 9}\, 40 \sin 5x$$

$$= 40 \frac{1}{-25 - 8D + 9} \sin 5x$$

$$= -\frac{40}{8} \frac{1}{D+2} \sin 5x$$

$$= -5\,\frac{D-2}{D^2-4}\,\sin 5x$$

$$= -\frac{5}{-29}\,(D-2)\,\sin 5x$$

$$= \frac{5}{29}\,(5\cos 5x - 2\sin 5x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1\,e^{4x}\cosh\left(\sqrt{7}x + c_2\right) + \frac{5}{29}\,(5\cos 5x - 2\sin 5x)$$

6. Solve $y''' - 3y'' + 9y' - 27y = \cos 3x$

Solution : The auxiliary equation is :

$$m^3 - 3m^2 + 9m - 27 = 0$$

$$\Rightarrow (m-3)(m^2+9) = 0 \Rightarrow m = 3, \pm 3i$$

$$\therefore\ y_c = c_1\,e^{3x} + c_2\cos 3x + c_3\sin 3x$$

Now $y_p = \dfrac{1}{D^3 - 3D^2 + 9D - 27}\cos 3x$

$$= x\,\frac{1}{3D^2 - 6D + 9}\cos 3x$$

$$= x\,\frac{1}{-27 - 6D + 9}\cos 3x$$

$$= -\frac{x}{6}\,\frac{1}{D+3}\cos 3x$$

$$= -\frac{x}{6}\,\frac{D-3}{D^2-9}\cos 3x$$

$$= -\frac{x}{6}\cdot\frac{1}{(-18)}\,(D-3)\cos 3x$$

$$= \frac{x}{108}\,(-3\sin 3x - 3\cos 3x)$$

$$= -\frac{x}{36}\,(\sin 3x + \cos 3x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1\,e^{3x} + c_2\cos 3x + c_3\sin 3x - \frac{x}{36}\,(\sin 3x + \cos 3x)$$

7. **Solve $(D^3 + D)y = \cos x$.**

Solution : The auxiliary equation is :

$$m^3 + m = 0 \Rightarrow m(m^2 + 1) = 0 \Rightarrow m = 0, \pm i$$

$$\therefore\ y_c = c_1 + c_2 \cos x + c_3 \sin x$$

$$\text{Now } y_p = \frac{1}{D^3 + D} \cos x$$

$$= x \frac{1}{3D^2 + 1} \cos x$$

$$= -\frac{x}{2} \cos x$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 + c_2 \cos x + c_3 \sin x - \frac{x}{2} \cos x$$

8. **Solve $(D^2 - 4D + 1)\ y = a\sin 2x$.**

Solution : The auxiliary equation is :

$$m^2 - 4m + 1 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore\ y_c = c_1 e^{2x} \cosh\left(\sqrt{3}x + c_2\right)$$

$$\text{Now } y_p = \frac{1}{D^2 - 4D + 1} a \sin 2x$$

$$= a \frac{1}{D^2 - 4D + 1} \sin 2x$$

$$= a \frac{1}{-4 - 4D + 1} \sin 2x$$

$$= -a \frac{1}{4D + 3} \sin 2x$$

$$= -a \frac{4D - 3}{16D^2 - 9} \sin 2x$$

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$$= \frac{a}{73} (4D - 3) \sin 2x$$

$$= \frac{a}{73} (8\cos 2x - 3\sin 2x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{2x} \cosh\left(\sqrt{3}x + c_2\right) + \frac{a}{73} (8\cos 2x - 3\sin 2x).$$

9. Solve $(D^2 + 2D + 10)y + 37 \sin 2x = 0$.

Solution : Rewriting the solution as

$$(D^2 + 2D + 10)\, y = -37 \sin 2x$$

The auxiliary equation is :

$$m^2 + 2m + 10 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$

$\therefore \quad y_c = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$

Now $y_p = \dfrac{1}{D^2 + 2D + 10}(-37 \sin 2x)$

$$= -37 \frac{1}{D^2 + 2D + 10} \sin 2x$$

$$= -37 \frac{1}{-4 + 2D + 10} \sin 2x$$

$$= -\frac{37}{2} \frac{1}{D+3} \sin 2x$$

$$= -\frac{37}{2} \frac{D-3}{D^2-9} \sin 2x$$

$$= -\frac{37}{2} \frac{D-3}{-13} \sin 2x$$

$$= \frac{37}{26}(D-3) \sin 2x$$

$$= \frac{37}{26}(2\cos 2x - 3\sin 2x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{37}{26}(2\cos 2x - 3\sin 2x)$$

10. Solve $(D^2 + 9)y = \cos 2x + \sin 2x$.

Solution : The auxiliary equation is :

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$\therefore \quad y_c = c_1 \cos 3x + c_2 \sin 3x$

Now $y_p = \frac{1}{D^2+9}(\cos 2x + \sin 2x)$

$$= \frac{1}{D^2+9}\cos 2x + \frac{1}{D^2+9}\sin 2x$$

$$= \frac{1}{5}\cos 2x + \frac{1}{5}\sin 2x$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x + \frac{1}{5}\cos 2x + \frac{1}{5}\sin 2x.$$

11. Solve $(D^3 + 1)\ y = \cos 2x$

Solution : The auxiliary equation is :

$$m^3 + 1 = 0 \Rightarrow (m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1,\ \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow m = -1,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$\therefore\ y_c = c_1 e^{-x} + e^{-\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

Now $y_p = \frac{1}{D^3+1}\cos 2x$

$$= \frac{1}{-4D+1}\cos 2x$$

$$= \frac{-4D-1}{(16D^2-1)}\cos 2x$$

$$= \frac{-4D-1}{-64-1}\cos 2x$$

$$= -\frac{1}{65}(-4D-1)\cos 2x$$

$$= -\frac{1}{65}(8\sin 2x - \cos 2x)$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 e^{-x} + e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{65}(8\sin 2x - \cos 2x)$$

12. Solve $(D^2 + 4)y = e^x + \sin 2x$.

Solution : The auxiliary equation is :

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{Now } y_p = \frac{1}{D^2+4}(e^x + \sin 2x)$$

$$= \frac{1}{D^2+4} e^x + \frac{1}{D^2+4} \sin 2x$$

$$= \frac{1}{5} e^x + x \frac{1}{2D} \sin 2x$$

$$= \frac{1}{5} e^x - \frac{x}{4} \cos 2x$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x.$$

13. Solve $(D^2 + 4)\,y = \sin^2 x$.

Solution : The auxiliary equation is :

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{Now } y_p = \frac{1}{D^2+4} \sin^2 x$$

$$= \frac{1}{D^2+4}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{D^2+4} e^{0x} - \frac{1}{D^2+4} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{4} e^{0x} - x \frac{1}{2D} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{4} - \frac{x}{4} \sin 2x\right\}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} - \frac{x}{8} \sin 2x$$

14. Solve $(D^2 + 1)\, y = \sin x \sin 2x$

Solution : The auxiliary equation is :

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \; y_c = c_1 \cos x + c_2 \sin x$$

Now $y_p = \dfrac{1}{D^2+1} \sin x \sin 2x$

$$= \frac{1}{D^2+1}\left(\frac{-\cos 3x + \cos x}{2}\right)$$

$$= \frac{1}{2}\left\{-\frac{1}{D^2+1}\cos 3x + \frac{1}{D^2+1}\cos x\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{-9+1}\cos 3x + x\,\frac{1}{2D}\cos x\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{8}\cos 3x + \frac{x}{2}\sin x\right\}$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{16}\cos 3x + \frac{x}{4}\sin x.$$

15. Solve $(D^2 + 9)\,(D^2 + 1)\, y = \cos 3x$

Solution : The auxiliary equation is :

$$(m^2 + 9)\,(m^2 + 1) = 0 \Rightarrow m = \pm 3i, \pm i$$

$$\therefore \; y_c = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos x + c_4 \sin x$$

Now $y_p = \dfrac{1}{(D^2+9)(D^2+1)} \cos 3x$

$$= \frac{1}{D^2+9}\cdot\frac{1}{D^2+1}\cos 3x$$

$$= -\frac{1}{8}\,\frac{1}{D^2+9}\cos 3x$$

$$= -\frac{1}{8}\cdot x\,\frac{1}{2D}\cos 3x$$

$$= -\frac{x}{48}\sin 3x$$

$\therefore$ The general solution is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos x + c_4 \sin x - \frac{x}{48}\sin 3x.$$

16. Solve $(D^2 + 4)\, y = \sin 2x$, given that $y = 0$ and $\frac{dy}{dx} = 2$ when $x = 0$.

Solution : The auxiliary equation is :

$$m^2 + 4 = 0 \Rightarrow m = \pm\, 2i$$

$$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$$

Now $y_p = \frac{1}{D^2+4} \sin 2x$

$$= x \frac{1}{2D} \sin 2x$$

$$= -\frac{x}{4} \cos 2x$$

$\therefore$ The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{4} \cos 2x \qquad \text{.......... (1)}$$

and $\frac{dy}{dx} = -2c_1 \sin 2x + 2c_2 \cos 2x - \frac{1}{4} \cos 2x + \frac{x}{2} \sin 2x \qquad \text{.......... (2)}$

Now given that $y(0) = 0$ and $y'(0) = 2$.

$\therefore$ From (1) and (2), we have

$$0 = c_1$$

and $2 = 2c_2 - \frac{1}{4} \Rightarrow 2c_2 = \frac{9}{4} \Rightarrow c_2 = \frac{9}{8}$.

$\therefore$ The required solution is

$$y = \frac{9}{8} \sin 2x - \frac{x}{4} \cos 2x.$$

17. Solve $\left(D^2 + w_0^2\right) y = a \cos wx$ and discuss the nature of solution as $w \to w_0$

Solution : The auxiliary equation is :

$$m^2 + w_0^2 = 0 \Rightarrow m = \pm\, w_0 i$$

$$\therefore \quad y_c = c_1 \cos w_0 x + c_2 \sin w_0 x.$$

Now $y_p = \frac{1}{D^2 + w_0^2} a \cos wx$

$$= \frac{a}{w_0^2 - w^2} \cos wx \text{ when } w^2 \neq w_0^2$$

When $w = w_0$ then

$$y_p = \frac{1}{D^2 + w_0^2}\, a\cos wx$$

$$= ax \frac{1}{2D} \cos wx$$

$$= \frac{ax}{2w} \sin wx = \frac{ax}{2w_0} \sin w_0 x$$

$\therefore$ The general solution is

$$y = c_1 \cos w_0 x + c_2 \sin w_0 x + \frac{a}{w_0^2 - w^2} \cos wx, \text{ when } w^2 \neq w_0^2$$

and $y = c_1 \cos w_0 x + c_2 \sin w_0 x + \dfrac{ax}{2w_0} \sin w_0 x$, when $w = w_0$.

The solution written later is the required solution when $w \to w_0$.

18. Solve $(D^4 - m^4)\, y = \sin mx$.

Solution : The auxiliary equation is :

$$M^4 - m^4 = 0 \Rightarrow (M^2 - m^2)(M^2 + m^2) = 0$$

$$\Rightarrow M = \pm m, \pm mi$$

$$\therefore\ y_c = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

Now $y_p = \dfrac{1}{D^4 - m^4} \sin mx$

$$= x \frac{1}{4D^3} \sin mx$$

$$= \frac{x}{4} \frac{1}{(-m^2 D)} \sin mx$$

$$= -\frac{x}{4m^2} \cdot \frac{1}{D} \sin mx$$

$$= \frac{x}{4m^3} \cos mx$$

$\therefore$ The general solution is

$$y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3} \cos mx$$

19. Solve $(D^4 + 2a^2D^2 + a^4)\,y = \cos ax$.

Solution : The auxiliary equation is :

$$m^4 + 2a^2m^2 + a^4 = 0 \Rightarrow (m^2 + a^2)^2 = 0 \Rightarrow m = \pm\, ai \text{ twice.}$$

$\therefore\; y_c = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax.$

$$\text{Now } y_p = \frac{1}{D^4 + 2a^2 D^2 + a^4}\cos ax$$

$$= x\,\frac{1}{4D^3 + 4a^2 D}\cos ax$$

$$= x^2\,\frac{1}{12D^2 + 4a^2}\cos ax$$

$$= x^2\,\frac{1}{-12a^2 + 4a^2}\cos ax$$

$$= -\frac{x^2}{8a^2}\cos ax$$

$\therefore$ The general solution is,

$$y = (c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax - \frac{x^2}{8a^2}\cos ax$$

20. Solve $(D^2 + 9)\,y = 2\sin 3x + \cos 3x$.

Solution : The auxiliary equation is :

$$m^2 + 9 = 0 \Rightarrow m = \pm\, 3i$$

$\therefore\; y_c = c_1\cos 3x + c_2\sin 3x.$

$$\text{Now } y_p = \frac{1}{D^2 + 9}(2\sin 3x + \cos 3x)$$

$$= 2\,\frac{1}{D^2 + 9}\sin 3x + \frac{1}{D^2 + 9}\cos 3x$$

$$= 2x\,\frac{1}{2D}\sin 3x + x\,\frac{1}{2D}\cos 3x$$

$$= -\frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

$\therefore$ The general solution is

$$y = c_1\cos 3x + c_2\sin 3x - \frac{x}{3}\cos 3x + \frac{x}{6}\sin 3x$$

21. **Solve** $(D-1)^2 (D^2+1)^2 y = \sin^2 \frac{x}{2} + e^x$

Solution : The auxiliary equation is :

$$(m-1)^2 (m^2+1)^2 = 0 \Rightarrow m = 1, 1 \text{ and } m = \pm i \text{ twice.}$$

$$\therefore y_c = (c_1 + c_2 x) e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x$$

$$\text{Now } y_p = \frac{1}{(D-1)^2 (D^2+1)^2} \left[\sin^2 \frac{x}{2} + e^x\right]$$

$$= \frac{1}{(D-1)^2 (D^2+1)^2} \left(\frac{1-\cos x}{2}\right) + \frac{1}{(D-1)^2 (D^2+1)^2} e^x$$

$$= \frac{1}{2} \frac{1}{(D-1)^2 (D^2+1)^2} e^{0x} - \frac{1}{2} \frac{1}{(D-1)^2 (D^2+1)^2} \cos x + \frac{1}{4} \cdot \frac{1}{(D-1)^2} e^x$$

$$= \frac{1}{2} - \frac{1}{2} \frac{1}{(D^2+1)^2} \cdot \frac{1}{(-2D)} \cos x + \frac{x}{4} \frac{1}{2(D-1)} e^x \quad (\because (D-1)^2 = D^2 - 2D + 1)$$

$$= \frac{1}{2} + \frac{1}{4} \frac{1}{(D^2+1)^2} \sin x + \frac{x^2}{8} e^x$$

$$= \frac{1}{2} + \frac{x}{4} \frac{1}{2(D^2+1)2D} \sin x + \frac{x^2}{8} e^x$$

$$= \frac{1}{2} + \frac{x}{16} \frac{1}{D^2+1} (-\cos x) + \frac{x^2}{8} e^x$$

$$= \frac{1}{2} - \frac{1}{16} x^2 \cdot \frac{1}{2D} \cos x + \frac{x^2}{8} e^x$$

$$= \frac{1}{2} - \frac{x^2}{32} \sin x + \frac{x^2}{8} e^x$$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x + \frac{1}{2} - \frac{x^2}{32} \sin x + \frac{x^2}{8} e^x.$$

22. **Solve** $D^2 (D^2+9) y = \cos 3x + 5.$

Solution : The auxiliary equation is :

$$m^2 (m^2+9) = 0 \Rightarrow m = 0, 0, \pm 3i$$

$$\therefore y_c = c_1 + c_2 x + c_3 \cos 3x + c_4 \sin 3x$$

$$\text{Now } y_p = \frac{1}{D^2(D^2+9)}(\cos 3x + 5)$$

$$= \frac{1}{D^2(D^2+9)}\cos 3x + 5\,\frac{1}{D^2(D^2+9)}e^{0x}$$

$$= x\,\frac{1}{D^2}\cdot\frac{1}{2D}\cos 3x + 5\,\frac{1}{D^2}\cdot\frac{1}{9}e^{0x}$$

$$= \frac{x}{2}\cdot\frac{1}{D}\cdot\frac{1}{-9}\cos 3x + \frac{5}{9}\,\frac{1}{D^2}\cdot 1$$

$$= -\frac{x}{18}\,\frac{1}{D}\cos 3x + \frac{5}{9}\,\frac{1}{D}\,x$$

$$= -\frac{x}{54}\sin 3x + \frac{5}{18}x^2$$

$\therefore$ The general solution is

$$y = c_1 + c_2x + c_3\cos 3x + c_4\sin 3x - \frac{x}{54}\sin 3x + \frac{5}{18}x^2$$

EXERCISE – 3.4

❑ **Solve the following differential equations :**

1. $(D^2 - 2D + 5)\,y = \sin 3x.$

Ans. : $y = e^{2x}(c_1\cos 2x + c_2\sin 2x) + \frac{1}{26}(3\cos 3x - 2\sin$

2. $(D^2 - 3D + 2)\,y = \sin 3x.$ **Ans. :** $y = c_1e^x + c_2e^{2x} + \frac{1}{130}(9\cos 3x - 7\sin$

3. $(D^2 - 2D + 1)\,y = \cos 3x.$ **Ans. :** $y = (c_1 + c_2x)\,e^x - \frac{1}{50}(3\sin 3x + 4\cos$

4. $(D^3 - 2D^2 + 3)\,y = \cos x.$

Ans. : $y = c_1e^{-x} + e^{3x/2}\left\{c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right\} + \frac{1}{26}(5\cos x - \sin$

5. $(\operatorname{cosec}x \cdot D^4 + \operatorname{cosec}x)\,y = \sin 2x$

Ans. : $y = e^{x/\sqrt{2}}\left\{c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right\} + e^{-x/\sqrt{2}}\left\{c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right.$

$+ \frac{1}{4}\cos x - \frac{1}{164}\cos 3$

6. $(D^4 + 8D^2 + 16)\,y = \cos 2x$.

Ans. : $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x + \frac{1}{32} - \frac{x^2}{64}\cos 2x$

7. $(D^2 + 4)\,y = 8\cos 2x$. **Ans. :** $y = c_1\cos 2x + c_2\sin 2x + 2x\sin 2x$

8. $(D^2 - 4D + 3)\,y = e^{2x} + \cos x$. **Ans. :** $y = c_1 e^x + c_2 e^{3x} - e^{2x} + \frac{1}{10}(-2\sin x + \cos x)$

9. $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$.

Ans. : $y = c_1 e^x + e^x(c_2\cos x + c_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)$

10. $(D^2 - 2D - 3)y = 2e^{2x} + 10\sin 3x$, given that $y(0) = 2$ and $y'(0) = 4$.

Ans. : $y = \frac{29}{12}e^{3x} - \frac{1}{12}e^{-x} - \frac{2}{3}e^{2x} + \frac{1}{3}[\cos 3x - 2\sin 3x]$.

Case – III : To find P. I. when R(x) is of the form x^m or a polynomial of degree m, m being positive integer.

Working rule : To evaluate $\frac{1}{f(D)}x^m$, take out common the lowest degree term in D from $f(D)$, so that remaining factor in the denominator is of the form $[1 + \phi(D)]$ or $[1 - \phi(D)]$ which is then taken in the numerator with a negative index. Then expand $[1 \pm \phi(D)]^{-1}$ in powers of D by the binomial theorem and operate upon x^m with the expansion obtained.

Remark : When $R(x) = x^m$ or a polynomial of degree m, then expand $[1 \pm \phi(D)]^{-1}$ upto the term containing D^m only, because terms after D^m, operated upon x^m becomes zero. That is $D^r x^m = 0$ for $r > m$.

Some expansions to be remembered :

1. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$
2. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$
3. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$
4. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots$
5. $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$
6. $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots$

SOLVED EXAMPLES

1. Solve $(D^2 - 4)y = x^2$.

Solution : The auxiliary equation is :

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

$$\therefore \; y_c = c_1 e^{2x} + c_2 e^{-2x}$$

Now
$$y_p = \frac{1}{D^2 - 4} x^2$$

$$= \frac{1}{-4\left(1 - \frac{D^2}{4}\right)} x^2$$

$$= -\frac{1}{4}\left[1 - \frac{D^2}{4}\right]^{-1} x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \ldots\right] x^2$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4}\cdot 2\right] = -\frac{1}{4}\left(x^2 + \frac{1}{2}\right)$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$$

2. Solve $(D^3 - D^2 - 6D)y = x^2 + 1$.

Solution : The auxiliary equation is :

$$m^3 - m^2 - 6m = 0$$

$$\therefore \; m(m^2 - m - 6) = 0 \Rightarrow m(m - 3)(m + 2) = 0$$

$$\Rightarrow m = 0, 3, -2$$

$$\therefore \; y_c = c_1 + c_2 e^{3x} + c_3 e^{-2x}$$

Now
$$y_p = \frac{1}{D^3 - D^2 - 6D}\left(x^2 + 1\right)$$

$$= \frac{1}{-6D\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)}\left(x^2 + 1\right)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}\left(x^2 + 1\right)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{7D^2}{36}\right](x^2+1)$$

$$= -\frac{1}{6D}\left[x^2+1-\frac{1}{6}\cdot 2x+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

$\therefore$ The general solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

3. **Solve $(D^2 + D - 6)y = x$.**

Solution : The auxiliary equation is :

$$m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0 \Rightarrow m = 2, -3$$

$\therefore$ $y_c = c_1 e^{2x} + c_2 e^{-3x}$

Now $y_p = \dfrac{1}{D^2 + D - 6} x$

$$= -\frac{1}{6}\left[1-\left(\frac{D}{6}+\frac{D^2}{6}\right)\right]^{-1} x$$

$$= -\frac{1}{6}\left[1+\left(\frac{D}{6}+\frac{D^2}{6}\right)\right] x$$

$$= -\frac{1}{6}\left[1+\frac{D}{6}\right] x$$

$$= -\frac{1}{6}\left[x+\frac{1}{6}\right]$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{6}\left[x+\frac{1}{6}\right]$$

4. Solve $(D^3 + 3D^2 + 2D)y = x^2$.

Solution : The auxiliary equation is :

$$m^3 + 3m^2 + 2m = 0$$

$$\Rightarrow m(m^2 + 3m + 2) = 0 \Rightarrow m(m+1)(m+2) = 0 \Rightarrow m = 0, -1, -2$$

$$\therefore \; y_c = c_1 + c_2 e^{-x} + c_3 e^{-2x}$$

Now $y_p = \dfrac{1}{D^3 + 3D^2 + 2D} x^2$

$$= \frac{1}{2D}\left[1 + \left(\frac{3}{2}D + \frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3}{2}D + \frac{D^2}{2}\right) + \left(\frac{3}{2}D + \frac{D^2}{2}\right)^2\right] x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D - \frac{D^2}{2} + \frac{9}{4}D^2\right] x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3}{2}D + \frac{7}{4}D^2\right] x^2$$

$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}\cdot 2x + \frac{7}{4}\cdot 2\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{7}{2}x\right]$$

$\therefore$ The general solution is

$$y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{7}{2}x\right]$$

5. Solve $(D^3 + 8)y = x^4 + 2x + 1$.

Solution : The auxiliary equation is :

$$m^3 + 8 = 0 \Rightarrow (m+2)(m^2 - 2m + 4) = 0$$

$$\Rightarrow m = -2, \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$\Rightarrow m = -2, 1 \pm \sqrt{3}i$$

$$\therefore \; y_c = c_1 e^{-2x} + e^x\{c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\}$$

Now $y_p = \frac{1}{D^3+8}\left(x^4+2x+1\right)$

$$= \frac{1}{8}\left[1+\frac{D^3}{8}\right]^{-1}\left(x^4+2x+1\right)$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}+\frac{D^6}{64}\right]\left(x^4+2x+1\right)$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{1}{8}\cdot 24x\right]$$

$$= \frac{1}{8}\left[x^4-x+1\right]$$

$\therefore$ The general solution is

$$y = c_1\, e^{-2x} + e^x\left\{c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right\} + \frac{1}{8}\left[x^4-x+1\right]$$

6. Solve $(D^2 + 2D + 1)\, y = 2x + x^2$.

Solution : The auxiliary equation is :

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1.$$

$\therefore\ y_c = (c_1 + c_2 x)\, e^{-x}$

Now $y_p = \frac{1}{D^2+2D+1}\left(2x+x^2\right)$

$$= \frac{1}{(1+D)^2}\left(2x+x^2\right)$$

$$= [1+D]^{-2}\,(2x+x^2)$$

$$= [1-2D+3D^2]\,(2x+x^2)$$

$$= 2x + x^2 - 2\,(2+2x) + 3(2)$$

$$= x^2 - 2x + 2$$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)\, e^{-x} + x^2 - 2x + 2.$$

7. Solve $(D^2 + D - 2)\, y = x + \sin x$.

Solution : The auxiliary equation is :

$$m^2 + m - 2 = 0 \Rightarrow (m-1)(m+2) = 0 \Rightarrow m = 1, -2$$

$\therefore\ y_c = c_1\, e^x + c_2\, e^{-2x}$

Now $y_p = \frac{1}{D^2 + D - 2}(x + \sin x)$

$$= \frac{1}{D^2 + D - 2}x + \frac{1}{D^2 + D - 2}\sin x$$

$$= -\frac{1}{2}\left[1 - \left(\frac{D}{2} + \frac{D^2}{2}\right)\right]^{-1} x + \frac{1}{D - 3}\sin x$$

$$= -\frac{1}{2}\left[1 + \frac{D}{2} + \frac{D^2}{2}\right] x + \frac{D + 3}{D^2 - 9}\sin x$$

$$= -\frac{1}{2}\left[x + \frac{1}{2}\right] - \frac{1}{10}(D + 3)\sin x$$

$$= -\frac{x}{2} - \frac{1}{4} - \frac{1}{10}(\cos x + 3\sin x)$$

$\therefore$ The general solution is

$$y = c_1 e^x + c_2 e^{-2x} - \frac{x}{2} - \frac{1}{4} - \frac{1}{10}(\cos x + 3\sin x)$$

8. Solve $(D^2 - 5D + 6)y = x + \sin 3x$

Solution : The auxiliary equation is :

$$m^2 - 5m + 6 = 0 \Rightarrow (m - 2)(m - 3) = 0 \Rightarrow m = 2, 3.$$

$\therefore$ $y_c = c_1 e^{2x} + c_2 e^{3x}$.

Now $y_p = \frac{1}{D^2 - 5D + 6}(x + \sin 3x)$

$$= \frac{1}{D^2 - 5D + 6}x + \frac{1}{D^2 - 5D + 6}\sin 3x$$

$$= \frac{1}{6}\left[1 + \left(-\frac{5D}{6} + \frac{D^2}{6}\right)\right]^{-1} x + \frac{1}{-5D - 3}\sin 3x$$

$$= \frac{1}{6}\left[1 - \left(-\frac{5D}{6} + \frac{D^2}{6}\right)\right] x - \frac{5D - 3}{25D^2 - 9}\sin 3x$$

$$= \frac{1}{6}\left[x + \frac{5}{6}\right] + \frac{1}{234}(5D - 3)\sin 3x$$

$$= \frac{x}{6} + \frac{5}{36} + \frac{1}{234}(15\cos 3x - 3\sin 3x)$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{x}{6} + \frac{5}{36} + \frac{1}{234}(15\cos 3x - 3\sin 3x)$$

9. Solve $(D^2 - 5D + 6)\,y = x + e^{4x}$.

Solution : The auxiliary equation is :

$$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3.$$

$$\therefore\ y_c = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{Now } y_p = \frac{1}{D^2 - 5D + 6}\left(x + e^{4x}\right)$$

$$= \frac{1}{D^2 - 5D + 6}x + \frac{1}{D^2 - 5D + 6}e^{4x}$$

$$= \frac{1}{6}\left[1 + \left(-\frac{5D}{6} + \frac{D^2}{6}\right)\right]^{-1} x + \frac{1}{2}e^{4x}$$

$$= \frac{1}{6}\left[1 - \left(-\frac{5D}{6} + \frac{D^2}{6}\right)\right] x + \frac{1}{2}e^{4x}$$

$$= \frac{1}{6}\left(x + \frac{5}{6}\right) + \frac{1}{2}e^{4x}$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}\left(x + \frac{5}{6}\right) + \frac{1}{2}e^{4x}$$

10. Solve $(D^3 - 3D^2 + 2D)\,y = 4 + 60\,e^{5x}$.

Solution : The auxiliary equation is :

$$m^3 - 3m^2 + 2m = 0$$

$$\Rightarrow m(m^2 - 3m + 2) = 0 \Rightarrow m(m-1)(m-2) = 0 \Rightarrow m = 0, 1, 2.$$

$$\therefore\ y_c = c_1 + c_2 e^{x} + c_3 e^{2x}.$$

$$\text{Now } y_p = \frac{1}{D^3 - 3D^2 + 2D}\left(4 + 60e^{5x}\right)$$

$$= 4\frac{1}{D^3 - 3D^2 + 2D}e^{0x} + 60\frac{1}{D^3 - 3D^2 + 2D}e^{5x}$$

$$= 4x\frac{1}{3D^2 - 6D + 2}e^{0x} + 60\frac{1}{60}e^{5x}$$

$$= 2x + e^{5x}$$

$\therefore$ The general solution is

$$y = c_1 + c_2 e^{x} + c_3 e^{2x} + 2x + e^{5x}$$

11. Solve $(D^2 - 2D + 3)y = \cos x + x^2$.

Solution : The auxiliary equation is :

$$m^2 - 2m + 3 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm \sqrt{2}i$$

$$\therefore \quad y_c = e^x \left\{c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right\}$$

$$\text{Now } y_p = \frac{1}{D^2 - 2D + 3}\left(\cos x + x^2\right)$$

$$= \frac{1}{D^2 - 2D + 3}\cos x + \frac{1}{D^2 - 2D + 3}x^2$$

$$= \frac{1}{-2D+2}\cos x + \frac{1}{3}\left[1 + \left(-\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= -\frac{1}{2}\frac{D+1}{D^2-1}\cos x + \frac{1}{3}\left[1 - \left(-\frac{2D}{3} + \frac{D^2}{3}\right) + \left(-\frac{2D}{3} + \frac{D^2}{3}\right)^2\right] x^2$$

$$= -\frac{1}{4}(D+1)\cos x + \frac{1}{3}\left[1 + \frac{2}{3}D - \frac{D^2}{3} + \frac{4D^2}{9}\right] x^2$$

$$= \frac{1}{4}(-\sin x + \cos x) + \frac{1}{3}\left(x^2 + \frac{2}{3}(2x) + \frac{1}{9}\cdot 2\right)$$

$$= \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left(x^2 + \frac{4x}{3} + \frac{2}{9}\right)$$

$\therefore$ The general solution is

$$y = e^x\left\{c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right\} + \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left(x^2 + \frac{4x}{3} + \frac{2}{9}\right).$$

12. Solve $(D^2 + 16)y = x^4 + e^{3x} + \cos 3x$.

Solution : The auxiliary equation is :

$$m^2 + 16 = 0 \Rightarrow m = \pm 4i$$

$$\therefore \quad y_c = c_1 \cos 4x + c_2 \sin 4x.$$

$$\text{Now } y_p = \frac{1}{D^2 + 16}\left(x^4 + e^{3x} + \cos 3x\right)$$

$$= \frac{1}{D^2+16}x^4 + \frac{1}{D^2+16}e^{3x} + \frac{1}{D^2+16}\cos 3x$$

$$= \frac{1}{16}\left[1+\frac{D^2}{16}\right]^{-1} x^4 + \frac{1}{25} e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left[1-\frac{D^2}{16}+\frac{D^4}{256}\right] x^4 + \frac{1}{25} e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left[x^4 - \frac{1}{16}\,12x^2 + \frac{1}{256}\cdot 24\right] + \frac{1}{25} e^{3x} + \frac{1}{7}\cos 3x$$

$$= \frac{1}{16}\left[x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right] + \frac{1}{25} e^{3x} + \frac{1}{7}\cos 3x$$

$\therefore$ The general solution is

$$y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{16}\left(x^4 - \frac{3}{4}x^2 + \frac{3}{32}\right) + \frac{1}{25} e^{3x} + \frac{1}{7}\cos 3x$$

13. Solve $(D^3 + 2D^2 + D)\,y = e^{2x} + x^2 + x$.

Solution : The auxiliary equation is :

$$m^3 + 2m^2 + m = 0 \Rightarrow m\,(m^2 + 2m + 1) = 0 \Rightarrow m\,(m+1)^2 = 0 \Rightarrow m = 0, -1, -1.$$

$$\therefore\ y_c = c_1 + (c_2 + c_3 x)\, e^{-x}$$

Now $y_p = \dfrac{1}{D^3 + 2D^2 + D}\left(e^{2x} + x^2 + x\right)$

$$= \frac{1}{D^3+2D^2+D} e^{2x} + \frac{1}{D^3+2D^2+D}\left(x^2+x\right)$$

$$= \frac{1}{8+8+2} e^{2x} + \frac{1}{D}\left[1+(2D+D^2)\right]^{-1}(x^2+x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}\left[1-\left(2D+D^2\right)+\left(2D+D^2\right)^2\right]\left(x^2+x\right)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}\left[1-2D-D^2+4D^2\right]\left(x^2+x\right)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}\left[x^2+x-2\,(2x+1)+3(2)\right]$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D}\left[x^2-3x+4\right]$$

$$= \frac{1}{18} e^{2x} + \frac{x^3}{3} - 3\,\frac{x^2}{2} + 4x$$

$\therefore$ The general solution is

$$y = c_1 + (c_2 + c_3 x)\, e^{-x} + \frac{1}{18} e^{2x} + \frac{x^3}{3} - 3\,\frac{x^2}{2} + 4x$$

14. Solve $(D^4 + 2D^3 - 3D^2)\, y = x^2 + 3\, e^{2x} + 4\sin x.$

Solution : The auxiliary equation is :

$$m^4 + 2m^3 - 3m^2 = 0 \Rightarrow m^2\,(m^2 + 2m - 3) = 0$$

$$\Rightarrow m^2\,(m-1)\,(m+3) = 0 \Rightarrow m = 0, 0, 1, 3.$$

$$\therefore\ y_c = c_1 + c_2 x + c_3\, e^x + c_4\, e^{3x}.$$

Now $y_p = \dfrac{1}{D^4 + 2D^3 - 3D^2}\,(x^2 + 3e^{2x} + 4\sin x)$

$$= \frac{1}{D^4+2D^3-3D^2}\,x^2 + 3\,\frac{1}{D^4+2D^3-3D^2}\,e^{2x} + 4\,\frac{1}{D^4+2D^3-3D^2}\sin x$$

Now $\dfrac{1}{D^4 + 2D^3 - 3D^2}\,x^2 = -\dfrac{1}{3D^2}\left[1 - \left(\dfrac{2}{3}D + \dfrac{D^2}{3}\right)\right]^{-1} x^2$

$$= -\frac{1}{3D^2}\left[1 + \left(\frac{2}{3}D + \frac{D^2}{3}\right) + \left(\frac{2}{3}D + \frac{D^2}{3}\right)^2\right] x^2$$

$$= -\frac{1}{3D^2}\left[1 + \frac{2}{3}D + \frac{D^2}{3} + \frac{4}{9}D^2\right] x^2$$

$$= -\frac{1}{3D^2}\left[x^2 + \frac{2}{3}(2x) + \frac{7}{9}(2)\right]$$

$$= -\frac{1}{3}\left[\frac{x^4}{12} + \frac{4}{3}\,\frac{x^3}{6} + \frac{14}{9}\,\frac{x^2}{2}\right]$$

$$= -\frac{1}{3}\left[\frac{x^4}{12} + \frac{2}{9}x^3 + \frac{7}{9}x^2\right] \qquad \text{.......... (2)}$$

$$\frac{1}{D^4 + 2D^3 - 3D^2}\,e^{2x} = \frac{1}{20}\,e^{2x} \qquad \text{.......... (3)}$$

$$\frac{1}{D^4 + 2D^3 - 3D^2}\sin x = \frac{1}{4 - 2D}\sin x$$

$$= \frac{1}{2}\cdot\frac{2 + D}{4 - D^2}\sin x$$

$$= \frac{1}{10}(2 + D)\sin x$$

$$= \frac{1}{10}(2\sin x + \cos x) \qquad \text{.......... (4)}$$

∴ From (1), (2), (3) and (4),

$$y_p = -\frac{1}{3}\left[\frac{x^4}{12} + \frac{2}{9}x^3 + \frac{7}{9}x^2\right] + \frac{3}{20}e^{2x} + \frac{2}{5}(2\sin x + \cos x)$$

∴ The general solution is

$$y = c_1 + c_2 x + c_3 e^x + c_4 e^{3x} - \frac{1}{3}\left[\frac{x^4}{12} + \frac{2}{9}x^3 + \frac{7}{9}x^2\right] + \frac{3}{20}e^{2x} + \frac{2}{5}(2\sin x + \cos x)$$

EXERCISE – 3.5

❑ **Solve the following differential equations :**

1. $(D^3 + 3D^2 + 2D)\,y = x.$ **Ans. :** $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{4}(x^2 - 3x)$

2. $(D^3 - 3D - 2)\,y = x^2.$ **Ans. :** $y = c_1 e^{2x} + (c_2 + c_3 x)e^{-x} - \frac{1}{2}\left(x^2 - 3x + \frac{9}{2}\right)$

3. $(D^4 + D^2 + 16)\,y = 16x^2 + 256.$

Ans. : $y = c_1 e^{-\frac{\sqrt{7}}{2}x}\left\{c_1\cos\frac{3}{2}x + c_2\sin\frac{3}{2}x\right\} + x^2 + \frac{127}{8}$

4. $(D^2 - 2D + 1)\,y = x - 1.$ **Ans. :** $y = (c_1 + c_2 x)\,e^x + x + 1$

5. $(D^3 - 8)\,y = x^3.$ **Ans. :** $y = c_1 e^{2x} + e^{-x}\left(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right) - \frac{1}{32}\left(3 + 4x^3\right)$

6. $(D^2 - 4D + 4)\;y = 8\,(x^2 + \sin 2x + e^{3x}).$

Ans. : $y = (c_1 + c_2 x)\,e^{2x} + 2x^2 + 4x + 3 + \cos 2x + 4x^2 e^{2x}$

7. $(D^2 + 2D + 1)\;y = \cos 2x + x^2.$

Ans. : $y = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(3\cos 2x - 2\sin 2x) + x^2 - 2x$

8. $(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x.$

Ans. : $y = (c_1 + c_2 x)\,e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2} + e^x - \frac{1}{8}\sin 2x\right)$

9. $(D^2 + 4)y = e^{-2x} + x.$ **Ans. :** $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}e^{-2x} + \frac{1}{4}x.$

10. $(D^2 - 4D + 3)\;y = x^3.$

Ans. : $y = c_1 e^x + c_2 e^{3x} + \frac{1}{27}(9x^3 + 36x^2 + 78x + 80)$

Case – IV : To find P. I. when R(x) is of the form e^{ax}V, where V is any function.

Result : Prove that

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$$

Proof : By successive differentiation, we have

$$D(e^{ax}u) = e^{ax} Du + ae^{ax}u = e^{ax} (D + a)u$$

where u is any function of x.

$$D^2 (e^{ax}u) = D \{e^{ax} (D + a)u\}$$
$$= e^{ax} D (D + a)u + ae^{ax} (D + a)u$$
$$= e^{ax} (D + a)^2u$$

Similarly, we get

$$D^n (e^{ax}u) = e^{ax} (D + a)^n u$$

$$\therefore \; f(D) (e^{ax}u) = \{D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0\} (e^{ax}u)$$
$$= e^{ax} [(D + a)^n + a_{n-1} (D + a)^{n-1} + ... + a_1 (D + a) + a_0]u$$
$$= e^{ax} f(D + a)u$$

Put $f(D + a)u = V \Rightarrow u = \frac{1}{f(D+a)} V$

$$\therefore \; f(D) e^{ax} \frac{1}{f(D+a)} V = e^{ax}V$$

operating $\frac{1}{f(D)}$ on both sides, we get

$$\frac{1}{f(D)} f(D) e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} e^{ax} V$$

$$\therefore \; \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$$

SOLVED EXAMPLES

1. Solve $(D^2 + 3D + 2)y = e^{2x} \sin x$.

Solution : The auxiliary equation is :

$$m^2 + 3m + 2 = 0 \Rightarrow (m + 2)(m + 1) = 0 \Rightarrow m = -1, -2.$$

$$\therefore \; y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Now $y_p = \frac{1}{D^2 + 3D + 2} e^{2x} \sin x$

$$= e^{2x} \frac{1}{(D+2)^2 + 3(D+2) + 2} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 7D + 12} \sin x$$

$$= e^{2x} \frac{1}{7D + 11} \sin x$$

$$= e^{2x} \frac{7D - 11}{49D^2 - 121} \sin x$$

$$= -\frac{e^{2x}}{170} (7D - 11) \sin x$$

$$= -\frac{e^{2x}}{170} (7 \cos x - 11 \sin x)$$

$\therefore$ The general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{e^{2x}}{170} (7 \cos x - 11 \sin x).$$

2. Solve $(D^2 - 2D + 1)\, y = x^2 e^{3x}$.

Solution : The auxiliary equation is :

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

$\therefore$ $y_c = (c_1 + c_2 x) e^x$.

Now $y_p = \frac{1}{D^2 - 2D + 1} x^2 e^{3x}$

$$= \frac{1}{(D-1)^2} x^2 e^{3x}$$

$$= e^{3x} \frac{1}{(D+2)^2} x^2$$

$$= \frac{e^{3x}}{4} \left[1 + \frac{D}{2}\right]^{-2} x^2$$

$$= \frac{e^{3x}}{4} \left[1 - 2\frac{D}{2} + 3\frac{D^2}{4}\right] x^2$$

$$= \frac{e^{3x}}{4} \left[x^2 - 2x + \frac{3}{2}\right]$$

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$\therefore$ The general solution is

$$y = (c_1 + c_2x)\, e^x + \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right)$$

3. Solve $(D + 1)^3 y = x^2 e^{-x}$.

Solution : The auxiliary equation is :

$$(m + 1)^3 = 0 \Rightarrow m = -1, -1, -1.$$

$$\therefore\ y_c = (c_1 + c_2x + c_3x^2)\, e^{-x}$$

Now $y_p = \dfrac{1}{(D+1)^3}\, x^2 e^{-x}$

$$= e^{-x}\, \frac{1}{D^3}\, x^2$$

$$= e^{-x}\, \frac{1}{D^2}\, \frac{x^3}{3}$$

$$= e^{-x}\, \frac{1}{D}\, \frac{x^4}{12}$$

$$= \frac{e^{-x} x^5}{60}$$

$\therefore$ The general solution is

$$y = (c_1 + c_2x + c_3x^2)\, e^{-x} + \frac{1}{60}\, e^{-x}\, x^5$$

4. Solve $(D^2 - 1)\, y = \cosh x \cos x$.

Solution : The auxiliary equation is :

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

$$\therefore\ y_c = c_1\, e^x + c_2\, e^{-x}.$$

Now $y_p = \dfrac{1}{D^2 - 1}\ \cosh x \cos x$

$$= \frac{1}{D^2 - 1}\left(\frac{e^x + e^{-x}}{2}\right)\cos x$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 1}\, e^x \cos x + \frac{1}{D^2 - 1}\, e^{-x} \cos x\right]$$

$$= \frac{1}{2}\left[e^x\, \frac{1}{(D+1)^2 - 1}\cos x + e^{-x}\, \frac{1}{(D-1)^2 - 1}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\,\frac{1}{D^2+2D}\cos x + e^{-x}\,\frac{1}{D^2-2D}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\,\frac{1}{2D-1}\cos x + e^{-x}\,\frac{1}{-2D-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\,\frac{2D+1}{4D^2-1}\cos x - e^{-x}\,\frac{2D-1}{4D^2-1}\cos x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{5}(2D+1)\cos x + \frac{e^{-x}}{5}(2D-1)\cos x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{5}(-2\sin x+\cos x) + \frac{e^{-x}}{5}(-2\sin x-\cos x)\right]$$

$$= \frac{1}{2}\left[\frac{2}{5}\sin x\left(e^{x}-e^{-x}\right) - \frac{1}{5}\cos x\left(e^{x}+e^{-x}\right)\right]$$

$$= \frac{2}{5}\sin x\sinh x - \frac{1}{5}\cos x\cosh x$$

$\therefore$ The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{2}{5}\sin x\sinh x - \frac{1}{5}\cos x\cosh x$$

5. Solve $(D^2-5D+6)y = xe^{4x}$.

Solution : The auxiliary equation is :

$$m^2-5m+6=0 \Rightarrow (m-2)(m-3)=0 \Rightarrow m = 2, 3.$$

$\therefore$ $y_c = c_1 e^{2x} + c_2 e^{3x}$.

Now $y_p = \dfrac{1}{D^2-5D+6}\,xe^{4x}$

$$= e^{4x}\,\frac{1}{(D+4)^2-5(D+4)+6}\,x$$

$$= e^{4x}\,\frac{1}{D^2+3D+2}\,x$$

$$= \frac{e^{4x}}{2}\left[1+\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]^{-1} x$$

$$= \frac{e^{4x}}{2}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)\right]x$$

$$= \frac{e^{4x}}{2}\left[x-\frac{3}{2}\right]$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^{4x}}{2}\left(x-\frac{3}{2}\right)$$

6. Solve $(D^2 - 4D + 1)y = e^{2x}\sin x$.

Solution : The auxiliary equation is :

$$m^2 - 4m + 1 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore \quad y_c = c_1 e^{2x} \cosh\left(\sqrt{3}x + c_2\right)$$

$$\text{Now } y_p = \frac{1}{D^2 - 4D + 1} e^{2x}\sin x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 1}\sin x$$

$$= e^{2x} \frac{1}{D^2 - 3}\sin x$$

$$= -\frac{e^{2x}}{4}\sin x$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} \cosh\left(\sqrt{3}x + c_2\right) - \frac{e^{2x}}{4}\sin x.$$

7. Solve $(D^2 - 5D + 6)y = e^{2x}\sin 2x$.

Solution : The auxiliary equation is :

$$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3.$$

$$\therefore \quad y_c = c_1 e^{2x} + c_2' e^{3x}$$

$$\text{Now } y_p = \frac{1}{D^2 - 5D + 6} e^{2x}\sin 2x$$

$$= e^{2x} \frac{1}{(D+2)^2 - 5(D+2) + 6}\sin 2x$$

$$= e^{2x} \frac{1}{D^2 - D} \sin 2x$$

$$= e^{2x} \frac{1}{-4 - D} \sin 2x$$

$$= - e^{2x} \frac{D + 4}{D^2 - 16} \sin 2x$$

$$= \frac{e^{2x}}{20} (D - 4) \sin 2x$$

$$= \frac{e^{2x}}{20} (2\cos 2x - 4\sin 2x)$$

$\therefore$ The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^{2x}}{20} (2\cos 2x - 4\sin 2x)$$

8. Solve $(D^2 - 4D + 4)\, y = e^{2x} \sin 3x$

Solution : The auxiliary equation is :

$$m^2 - 4m + 4 = 0 \Rightarrow (m - 2)^2 = 0 \Rightarrow m = 2, 2.$$

$$\therefore\ y_c = (c_1 + c_2 x)\, e^{2x}$$

Now $y_p = \dfrac{1}{D^2 - 4D + 4} e^{2x} \sin 3x$

$$= \frac{1}{(D - 2)^2} e^{2x} \sin 3x$$

$$= e^{2x} \frac{1}{D^2} \sin 3x$$

$$= - \frac{e^{2x}}{3} \frac{1}{D} \cos 3x$$

$$= - \frac{e^{2x}}{3} \sin 3x$$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)\, e^{2x} - \frac{e^{2x}}{3} \sin 3x.$$

9. **Solve $(D^4 - 1)y = e^x \cos x$**

Solution : The auxiliary equation is :

$$m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = \pm 1, \pm i$$

$\therefore \quad y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

$$\text{Now } y_p = \frac{1}{D^4 - 1} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^4 - 1} \cos x$$

$$= e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D} \cos x$$

$$= e^x \frac{1}{1 - 4D - 6 + 4D} \cos x$$

$$= -\frac{e^x}{5} \cos x$$

$\therefore$ The general solution is

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{e^x}{5} \cos x.$$

10. **Solve $(D^2 - 1)y = \cosh x \cos x - a^x$.**

Solution : The auxiliary equation is :

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}.$

$$\text{Now } y_p = \frac{1}{D^2 - 1} [\cosh x \cos x + a^x]$$

$$\text{Where } \frac{1}{D^2 - 1} \cosh x \cos x = \frac{1}{D^2 - 1} \left(\frac{e^x + e^{-x}}{2} \right) \cos x$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 1} e^x \cos x + \frac{1}{D^2 - 1} e^{-x} \cos x \right]$$

$$= \frac{1}{2} \left[e^x \frac{1}{(D+1)^2 - 1} \cos x + e^{-x} \frac{1}{(D-1)^2 - 1} \cos x \right]$$

$$= \frac{1}{2} \left[e^x \frac{1}{D^2 + 2D} \cos x + e^{-x} \frac{1}{D^2 - 2D} \cos x \right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{2D-1}\cos x + e^{-x}\frac{1}{-2D-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{2D+1}{4D^2-1}\cos x - e^{-x}\frac{2D-1}{4D^2-1}\cos x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{5}(2D+1)\cos x + \frac{e^{-x}}{5}(2D-1)\cos x\right]$$

$$= \frac{1}{2}\left[-\frac{e^{x}}{5}(-2\sin x+\cos x) + \frac{e^{-x}}{5}(-2\sin x-\cos x)\right]$$

$$= \frac{1}{2}\left[\frac{2\sin x}{5}\left(e^{x}-e^{-x}\right) - \frac{\cos x}{5}\left(e^{x}+e^{-x}\right)\right]$$

$$= \frac{2}{5}\sin x\cdot\sinh x - \frac{\cos x}{5}\cosh x$$

$$\frac{1}{D^2-1}a^x = \frac{1}{D^2-1}e^{x\log a}$$

$$= \frac{1}{(\log a)^2-1}e^{x\log a} = \frac{1}{(\log a)^2-1}a^x$$

$\therefore$ The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{2}{5}\sin x\sinh x - \frac{1}{5}\cos x\cosh x + \frac{1}{(\log a)^2-1}a^x$$

11. Solve $(D^3 - 1) = xe^x + \cos^2 x$.

Solution : The auxiliary equation is :

$$m^3 - 1 = 0 \Rightarrow (m-1)(m^2+m+1) = 0 \Rightarrow m = 1, \frac{-1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow m = 1, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

$$\therefore\ y_c = c_1 e^{x} + e^{-\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

Now $y_p = \dfrac{1}{D^3-1}[xe^x + \cos^2 x]$

Now $\dfrac{1}{D^3-1}\, xe^x = e^x \dfrac{1}{(D+1)^3-1}\, x$

$$= e^x \frac{1}{D^3+3D^2+3D}\, x$$

$$= \frac{e^x}{3D}\left[1+\left(D+\frac{D^2}{3}\right)\right]^{-1} x$$

$$= e^x \cdot \frac{1}{3D}\left[1-\left(D+\frac{D^2}{3}\right)\right] x$$

$$= e^x \frac{1}{3D}[x-1]$$

$$= \frac{e^x}{3}\left(\frac{x^2}{2}-x\right)$$

$$\frac{1}{D^3-1}\cos^2 x = \frac{1}{D^3-1}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^3-1}e^{0x} + \frac{1}{2}\frac{1}{D^3-1}\cos 2x$$

$$= -\frac{1}{2}e^{0x} + \frac{1}{2}\frac{1}{-4D-1}\cos 2x$$

$$= -\frac{1}{2} - \frac{1}{2}\frac{4D-1}{16D^2-1}\cos 2x$$

$$= -\frac{1}{2} + \frac{1}{130}(4D-1)\cos 2x$$

$$= -\frac{1}{2} + \frac{1}{130}(-8\sin 2x - \cos 2x)$$

$\therefore$ The general solution is

$$y = c_1 e^x + e^{-\frac{x}{2}}\left\{c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right\} + \frac{e^x}{3}\left(\frac{x^2}{2}-x\right) - \frac{1}{2} + \frac{1}{130}(-8\sin 2x - \cos$$

12. Solve $(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + 2^x$.

Solution : The auxiliary equation is :

$$m^2 - 6m + 13 = 0 \Rightarrow m = \frac{6 \pm \sqrt{36-52}}{2} = 3 \pm 2i$$

$\therefore\ y_c = e^{3x}\{c_1 \cos 2x + c_2 \sin 2x\}$

$$\text{Now } y_p = \frac{1}{D^2 - 6D + 13}[8e^{3x}\sin 4x + 2^x]$$

$$= 8\frac{1}{D^2 - 6D + 13}e^{3x}\sin 4x + \frac{1}{D^2 - 6D + 13}2^x$$

$$\text{Now } \frac{1}{D^2 - 6D + 13}e^{3x}\sin 4x = e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\sin 4x$$

$$= e^{3x}\frac{1}{D^2 + 4}\sin 4x$$

$$= -\frac{e^{3x}}{12}$$

$$\frac{1}{D^2 - 6D + 13}2^x = \frac{1}{D^2 - 6D + 13}e^{x\log 2}$$

$$= \frac{1}{(\log 2)^2 - 6(\log 2) + 13}e^{x\log 2}$$

$$= \frac{1}{(\log 2)^2 - 6(\log 2) + 13}2^x.$$

$\therefore$ The general solution is

$$y = e^{3x}\{c_1\cos 2x + c_2\sin 2x\} - \frac{e^{3x}}{2} + \frac{2^x}{(\log 2)^2 - 6(\log 2) + 13}$$

EXERCISE – 3.6

Solve the following differential equations :

1. $(D^2 - 2D + 1)y = x^2e^x$ **Ans. :** $y = (c_1 + c_2x)e_x + \frac{1}{12}x^4e^x$

2. $(D^2 - 3D + 2)y = xe^x.$ **Ans. :** $y = c_1e^x + c_2e^{2x} - e^x\left(\frac{x^2}{2} + x\right)$

3. $(D^3 - 3D - 2)y = 540\ x^3 e^{-x}$
Ans. : $y = c_1e^{2x} + (c_2 + c_3x)e^{-x} - e^{-x}(9x^5 + 15x^4 + 20x^3 + 20x^2)$

4. $(D - 2)^3 y = xe^{2x}.$ **Ans. :** $y = (c_1 + c_2x + c_3x^2)e^{2x} + \frac{1}{24}x^4e^{2x}$

5. $(D^2 - 2D + 5)y = e^{2x} \sin x$. **Ans. :** $y = e^x (c_1 \cos 2x + c_2 \sin 2x) - \frac{e^{2x}}{10} (\cos x - 2...$

6. $(D^3 + 3D^2 - 4) y = xe^{-2x}$. **Ans. :** $y = c_1 e^x + (c_2 + c_3 x) e^{-2x} - \frac{1}{18} (x^3 + x^2) ...$

7. $(D^2 + 4D - 12) y = (x - 1)e^{2x}$ **Ans. :** $y = c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{16} x^2 e^{2x} - \frac{9}{64} ...$

8. $(D^2 - 1) y = e^x \cos x$. **Ans. :** $y = c_1 e^x + c_2 e^{-x} - \frac{e^x}{5} (\cos x - 2 ...$

9. $(D^2 - 4D + 4) y = e^{2x} \sin 3x$. **Ans. :** $y = (c_1 + c_2 x)e^{2x} - \frac{e^{2x}}{9} ...$

10. $(D^2 - 3D + 2) y = 2x^2 e^{4x} + 5e^{3x}$.

Ans. : $y = c_1 e^x + c_2 e^{2x} + \frac{e^{4x}}{54} [18x^2 - 30x + 19] + ...$

Case – V : To find P. I. when R(x) is of the form x^mV, where V is some function of x and m is a positive integer.

Result : Prove that

$$\frac{1}{f(D)} x V = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V \qquad (m = 1)$$

Proof : By Leibnitz's theorem, we have

$$D^n (xV) = x D^n V + n D^{n-1} V = x D^n V + \frac{d}{dD} D^n V$$

Now $f(D) = (D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0)$

$\therefore \; f(D) (xV) = x f(D)V + f'(D)V$

Replace V by $\frac{1}{f(D)}$ V, we get

$$f(D) \left\{ x \frac{1}{f(D)} V \right\} = x f(D) \frac{1}{f(D)} V + f'(D) \frac{1}{f(D)} V$$

$$= xV + \frac{f'(D)}{f(D)} V$$

Operating $\frac{1}{f(D)}$ on both sides, we get

$$x \frac{1}{f(D)} V = \frac{1}{f(D)} x V + \frac{1}{f(D)} \frac{f'(D)}{f(D)} V$$

$$\therefore \quad \frac{1}{f(D)}\, x\, V = x\frac{1}{f(D)}\, V - \frac{f'(D)}{[f(D)]^2}\, V$$

Remark – 1. Proceeding by repeated application of above formula we can obtain

$$\frac{1}{f(D)}\, x^m\, V.$$

2. When $V = \cos ax$ or $\sin ax$ and as $e^{axi} = \cos ax + i\sin ax$, we can use :

$$\frac{1}{f(D)}\, x^m \cos ax = \text{Real Part of } \frac{1}{f(D)}\, x^m\, e^{axi}$$

$$= \text{R. P. of } e^{axi}\, \frac{1}{f(D+ai)}\, x^m$$

and $\dfrac{1}{f(D)}\, x^m \sin ax = \text{Imaginary Part of } \dfrac{1}{f(D)}\, x^m\, e^{axi}$

$$= \text{I. P. of } e^{axi}\, \frac{1}{f(D+ai)}\, x^m$$

SOLVED EXAMPLES

1. Solve $(D^2 + 9)\, y = x \sin x$.

Solution : The auxiliary equation is :

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$\therefore \quad y_c = c_1 \cos 3x + c_2 \sin 3x$$

Now $y_p = \dfrac{1}{D^2 + 9}\, x \sin x$

$$= x\, \frac{1}{D^2+9} \sin x - \frac{2D}{(D^2+9)^2} \sin x$$

$$= \frac{x}{8} \sin x - \frac{2}{64} D \sin x$$

$$= \frac{x}{8} \sin x - \frac{1}{32} \cos x$$

$\therefore$ The general solution is

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{8} \sin x - \frac{1}{32} \cos x$$

2. Solve $(D^2 - 2D + 1)\,y = x\sin x$.

Solution : The auxiliary equation is :

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

$$\therefore \quad y_c = (c_1 + c_2 x)\,e^x.$$

$$\text{Now } y_p = \frac{1}{D^2 - 2D + 1}\, x\sin x$$

$$= x\,\frac{1}{D^2 - 2D + 1}\sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2}\sin x$$

$$= x\,\frac{1}{-2D}\sin x - \frac{2D-2}{4D^2}\sin x$$

$$= -\frac{x}{2}\,\frac{1}{D}\sin x + \frac{1}{2}(D-1)\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)\,e^x + \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

3. Solve $(D^4 - 1)\,y = x\sin x$.

Solution : The auxiliary equation is :

$$m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = \pm 1, \pm i$$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{Now } y_p = \frac{1}{D^4 - 1}\, x\sin x$$

$$= x\,\frac{1}{D^4 - 1}\,x\sin x - \frac{4D^3}{(D^4 - 1)^2}\sin x$$

$$= x^2\,\frac{1}{4D^3}\sin x - 4D^3\left\{x\,\frac{1}{8D^7 - 8D^3}\sin x\right\} \quad (\because (D^4-1)^2 = D^8 - 2D^4$$

$$= -\frac{x^2}{4}\cdot\frac{1}{D}\sin x - \frac{4}{8}D^3\left\{x^2\,\frac{1}{7D^6 - 3D^2}\sin x\right\}$$

$$= \frac{x^2}{2}\cos x - \frac{1}{2}D^3\left\{-\frac{x^2}{4}\sin x\right\}$$

$$= \frac{x^2}{2}\cos x + \frac{1}{8}D^2\{2x\sin x + x^2\cos x\}$$

$$= \frac{x^2}{2}\cos x + \frac{1}{8}D\{2\sin x + 4x\cos x - x^2\sin x\}$$

$$= \frac{x^2}{2}\cos x + \frac{1}{8}\{2\cos x + 4\cos x - 4x\sin x - 2x\sin x - x^2\cos x\}$$

$$= \frac{x^2}{2}\cos x + \frac{1}{8}\{6\cos x - 6x\sin x - x^2\cos x\}$$

$$= \frac{3}{4}\cos x - \frac{3}{4}x\sin x + \frac{3}{8}x^2\cos x$$

$\therefore$ The general solution is

$$y = c_1 e^x + c_2 e^{-x} + c_3\cos x + c_4\sin x + \frac{3}{4}\cos x - \frac{3}{4}x\sin x + \frac{3}{8}x^2\cos x$$

$$= c_1 e^x + c_2 e^{-x} + c_3'\cos x + c_4\sin x - \frac{3}{4}x\sin x + \frac{3}{8}x^2\cos x$$

where $c_3' = c_3 + \frac{3}{4}$.

4. Solve $(D^2 + m^2)y = x\cos mx$.

Solution : The auxiliary equation is :

$$M^2 + m^2 = 0 \Rightarrow M = \pm mi$$

$\therefore\ y_c = c_1\cos mx + c_2\sin mx$

Now $y_p = \dfrac{1}{D^2 + m^2}x\cos mx$

$$= x\frac{1}{D^2 + m^2}\cos mx - \frac{2D}{(D^2 + m^2)^2}\cos mx$$

$$= x^2\frac{1}{2D}\cos mx - 2D\left\{x\frac{1}{2(D^2 + m^2)2D}\cos mx\right\}$$

$$= \frac{x^2}{2m}\sin mx - \frac{D}{2}\left\{x\frac{1}{D^2 + m^2}\frac{\sin mx}{m}\right\}$$

$$= \frac{x^2}{2m}\sin mx - \frac{1}{2m}D\left\{x^2\frac{1}{2D}\sin mx\right\}$$

$$= \frac{x^2}{2m} \sin mx + \frac{1}{4m^2} D\left\{x^2 \cos mx\right\}$$

$$= \frac{x^2}{2m} \sin mx + \frac{1}{4m^2} \left\{2x \cos mx - x^2 m \sin mx\right\}$$

$$= \frac{x^2}{2m} \sin mx + \frac{x}{2m^2} \cos mx - \frac{x^2}{4m} \sin mx$$

$$= \frac{x^2}{4m} \sin mx + \frac{x}{2m^2} \cos mx$$

$\therefore$ The general solution is

$$y = c_1 \cos mx + c_2 \sin mx + \frac{x^2}{4m} \sin mx + \frac{x}{2m^2} \cos mx.$$

5. Solve $(D^2 + 4)y = x \sin 2x$.

Solution : The auxiliary equation is :

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$$

Now $y_p = \dfrac{1}{D^2 + 4} x \sin 2x$

$$= x \frac{1}{D^2 + 4} \sin 2x - \frac{2D}{(D^2 + 4)^2} \sin 2x$$

$$= x^2 \frac{1}{2D} \sin 2x - 2D \left\{x \frac{1}{2(D^2 + 4)2D} \sin 2x\right\}$$

$$= -\frac{x^2}{4} \cos 2x + \frac{D}{2} \left\{x \frac{1}{D^2 + 4} \left(\frac{\cos 2x}{2}\right)\right\}$$

$$= -\frac{x^2}{4} \cos 2x + \frac{D}{4} \left\{x^2 \frac{1}{2D} \cos 2x\right\}$$

$$= -\frac{x^2}{4} \cos 2x + \frac{1}{8} D \left\{x^2 \frac{\sin 2x}{2}\right\}$$

$$= -\frac{x^2}{4} \cos 2x + \frac{1}{16} \left\{2x \sin 2x + 2x^2 \cos 2x\right\}$$

$$= -\frac{x^2}{4} \cos 2x + \frac{x}{8} \sin 2x + \frac{x^2}{8} \cos 2x$$

$$= \frac{x}{8} \sin 2x - \frac{x^2}{8} \cos 2x$$

$\therefore$ The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8}\sin 2x - \frac{x^2}{8}\cos 2x$$

6. **Solve $(D^4 + 2D^2 + 1)\,y = x^2 \cos x$.**

Solution : The auxiliary equation is :

$$m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0 \Rightarrow m = \pm\, i \text{ twice.}$$

$$\therefore\ y_c = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$$

Now $y_p = \dfrac{1}{D^4 + 2D^2 + 1}\, x^2 \cos x$

$$= \text{Real Part of } \frac{1}{\left(D^2+1\right)^2}\, x^2 e^{ix}$$

$$= \text{R. P. of } e^{ix} \frac{1}{\left[(D+i)^2+1\right]^2}\, x^2$$

$$= \text{R. P. of } e^{ix} \frac{1}{\left[D^2 + 2iD\right]^2}\, x^2$$

$$= \text{R. P. of } -\frac{e^{ix}}{4D^2}\left[1 + \frac{D}{2i}\right]^{-2} x^2$$

$$= \text{R. P. of } -\frac{e^{ix}}{4}\cdot\frac{1}{D^2}\left[1 - 2\,\frac{D}{2i} + 3\,\frac{D^2}{-4}\right] x^2$$

$$= \text{R. P. of } -\frac{e^{ix}}{4}\cdot\frac{1}{D^2}\left[x^2 + i2x - \frac{3}{2}\right]$$

$$= \text{R. P. of } -\frac{e^{ix}}{4}\cdot\left[\frac{x^4}{12} + i\,\frac{x^3}{3} - \frac{3}{4}x^2\right]$$

$$= \text{R. P. of } -\frac{1}{4}\,(\cos x + i\sin x)\left(\frac{x^4}{12} + i\,\frac{x^3}{3} - \frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x - \frac{x^3}{3}\sin x\right]$$

$\therefore$ The general solution

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - \frac{1}{4}\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x + \frac{x^3}{12}\sin x$$

7. Solve $(D^2 + 1)\, y = x^2 \sin 2x$.

Solution : The auxiliary equation is :

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad y_c = c_1 \cos x + c_2 \sin x$$

$$\text{Now } y_p = \frac{1}{D^2+1} x^2 \sin 2x$$

$$= \text{Imaginary Part of } \frac{1}{D^2+1} x^2 e^{2xi}$$

$$= \text{I. P. of } e^{2xi} \frac{1}{(D+2i)^2+1} x^2$$

$$= \text{I. P. of } e^{2xi} \frac{1}{D^2+4Di-3} x^2$$

$$= \text{I. P. of } -\frac{e^{2xi}}{3}\left[1-\left(\frac{4iD}{3}+\frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I. P. of } -\frac{e^{2xi}}{3}\left[1+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)^2\right] x^2$$

$$= \text{I. P. of } -\frac{e^{2xi}}{3}\left[1+\frac{4i}{3}D+\frac{D^2}{3}-\frac{16}{9}D^2\right] x^2$$

$$= \text{I. P. of } -\frac{e^{2xi}}{3}\left[x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)\right]$$

$$= \text{I. P. of } -\frac{1}{3}(\cos 2x + i\sin 2x)\left(x^2+\frac{8xi}{3}-\frac{26}{9}\right)$$

$$= -\frac{1}{3}\left[\frac{8}{3}x\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$

$\therefore$ The general solution is,

$$y = c_1 \cos x + c_2 \sin x - \frac{8}{3} x \cos 2x - \frac{1}{3}\left(x^2 - \frac{26}{9}\right)\sin 2x.$$

8. **Solve $(D^2 - 2D + 1)\, y = x\, e^x \sin x$.**

Solution : The auxiliary equation is :

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

$$\therefore\ y_c = (c_1 + c_2 x)\, e^x$$

$$\text{Now } y_p = \frac{1}{D^2 - 2D + 1} x e^x \sin x$$

$$= \frac{1}{(D-1)^2} x e^x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x \frac{1}{D^2} \sin x - \frac{2D}{D^4} \sin x \right]$$

$$= e^x \left[-x \sin x - \frac{2}{D^3} \sin x \right]$$

$$= e^x \left[-x \sin x + 2 \frac{1}{D} \sin x \right]$$

$$= e^x \left[-x \sin x - 2 \cos x \right]$$

$\therefore$ The general solution is

$$y = (c_1 + c_2 x)\, e^x + e^x (-x \sin x - 2 \cos x).$$

9. **Solve $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$.**

Solution : The auxiliary equation is :

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$$

$$\therefore\ y_c = (c_1 + c_2 x)\, e^{2x}$$

$$\text{Now } y_p = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x$$

$$= 8 \frac{1}{(D-2)^2} x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= \text{I. P. of } 8e^{2x} \frac{1}{D^2} x^2 e^{2xi}$$

$$= \text{I. P. of } 8e^{2x} e^{2xi} \frac{1}{(D+2i)^2} x^2$$

$$= \text{I. P. of } 8e^{2x}\, e^{2xi}\, \frac{1}{-4}\left[1+\frac{D}{2i}\right]^{-2} x^2$$

$$= \text{I. P. of } -2e^{2x}\, e^{2xi}\left[1-2\,\frac{D}{2i}+3\,\frac{D^2}{-4}\right]x^2$$

$$= \text{I. P. of } -2e^{2x}\, e^{2xi}\left[x^2+i2x-\frac{3}{4}\cdot 2\right]$$

$$= \text{I. P. of } -2e^{2x}\,(\cos 2x + i\sin 2x)\left(x^2+2xi-\frac{3}{2}\right)$$

$$= -2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

$\therefore$ The general solution is

$$y = (c_1 + c_2x)\, e^{2x} - 2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

10. Solve $(D^2 - 1)\, y = x \sin x + (1 + x^2)\, e^x$.

Solution : The auxiliary equation is :

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$$\therefore\ y_c = c_1e^x + c_2e^{-x}$$

Now $y_p = \dfrac{1}{D^2-1}\,[x \sin x + (1 + x^2)\, e^x]$

$$= \frac{1}{D^2-1}\, x\sin x + \frac{1}{D^2-1}\,(1+x^2)\, e^x$$

Now $\dfrac{1}{D^2-1}\, x \sin x = x\,\dfrac{1}{D^2-1}\sin x - \dfrac{2D}{(D^2-1)^2}\sin x$

$$= -\frac{x}{2}\sin x - \frac{2D}{4}\sin x$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x$$

Again $\dfrac{1}{D^2-1}\,(1+x^2)\, e^x = e^x\,\dfrac{1}{(D+1)^2-1}\left(1+x^2\right)$

$$= e^x\,\frac{1}{D^2+2D}\left(1+x^2\right)$$

$$= e^x \cdot \frac{1}{2D}\left[1+\frac{D}{2}\right]^{-1}\left(1+x^2\right)$$

$$= e^x \cdot \frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{2}\right]\left(1+x^2\right)$$

$$= e^x \cdot \frac{1}{2D}\left[1+x^2-\frac{1}{2}(2x)+\frac{1}{4}(2)\right]$$

$$= e^x \cdot \frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{2}-\frac{x^2}{2}+\frac{3}{2}x\right]$$

$$\therefore\ y_p = -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3}{2}x\right)$$

$\therefore$ The general solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(\cos x + x\sin x) + \frac{1}{12}e^x\left(2x^3 - 3x^2 + 9x\right).$$

11. Solve $(D^2 - 2D + 2)\, y = e^x \tan x$.

Solution : The auxiliary equation is :

$$m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$\therefore\ y_c = e^x\left(c_1 \cos x + c_2 \sin x\right)$$

Now $y_p = \dfrac{1}{D^2 - 2D + 2} e^x \tan x$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}\tan x$$

$$= e^x \frac{1}{D^2+1}\tan x$$

$$= e^x \frac{1}{(D+i)(D-i)}\tan x \qquad \text{.......... (1)}$$

$$= \frac{e^x}{2i}\left[\frac{1}{(D-i)} - \frac{1}{(D+i)}\right]\tan x$$

Now $\frac{1}{D-i}\tan x = e^{ix}\int e^{-ix}\tan x\,dx$

$$= e^{ix}\int(\cos x - i\sin x)\tan x\,dx$$

$$= e^{ix}\int\left(\sin x - i\,\frac{\sin^2 x}{\cos x}\right)dx$$

$$= e^{ix}\int(\sin x - i(\sec x - \cos x))\,dx$$

$$= e^{ix}\left[-\cos x - i\log(\sec x + \tan x) + i\sin x\right] \quad \text{.......... (2)}$$

Replacing i by $-i$, we get

$$\frac{1}{D+i}\tan x = e^{-ix}\left[-\cos x + i\log(\sec x + \tan x) - i\sin x\right] \quad \text{.......... (3)}$$

$\therefore$ From (1), (2) and (3), we have,

$$y_p = \frac{e^x}{2i}\left[-\cos x\{e^{ix} - e^{-ix}\} - i\log(\sec x + \tan x)\{e^{ix} + e^{-ix}\} + i\sin x\{e^{ix}, \right.$$

$$= e^x\left[-\cos x\sin x - \cos x\log(\sec x + \tan x) + \sin x\cos x\right]$$

$$= -e^x\cos x\log(\sec x + \tan x)$$

$\therefore$ The general solution is

$$y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\cdot\log(\sec x + \tan x)$$

12. Solve $(D^2 + 5D + 6)y = e^{-2x}\sec^2 x(1 + 2\tan x)$

Solution : The auxiliary equation is :

$$m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3$$

$$\therefore\ y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

Now $y_p = \frac{1}{D^2 + 5D + 6}e^{-2x}\sec^2 x(1 + 2\tan x)$

$$= \frac{1}{(D+3)(D+2)}e^{-2x}\sec^2 x(1 + 2\tan x)$$

$$= \frac{1}{D+3}e^{-2x}\int e^{2x}\cdot e^{-2x}\sec^2 x(1 + 2\tan x)\,dx$$

$$= \frac{1}{D+3}e^{-2x}\int\sec^2 x(1 + 2\tan x)\,dx$$

$$= \frac{1}{D+3}e^{-2x}(\tan x + \tan^2 x)$$

$$= e^{-3x} \int e^{3x} \cdot e^{-2x} (\tan x + \tan^2 x)\, dx$$

$$= e^{-3x} \int e^{x} (\tan x + \tan^2 x)\, dx$$

$$= e^{-3x} \left\{ \int e^{x} (\tan x + \sec^2 x)\, dx - \int e^{x}\, dx \right\}$$

$$= e^{-3x} [e^{x} \tan x - e^{x}] \qquad (\because \int e^{x} [f(x) + f'(x)]\, dx = e^{x} f(x))$$

$$= e^{-2x} (\tan x - 1)$$

$\therefore$ The general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x} (\tan x - 1)$$

EXERCISE – 3.7

❑ **Solve the following differential equations :**

1. $(D^2 + 4)\, y = x \sin x$. **Ans. :** $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cos x$.

2. $(D^2 + 2D + 1)\, y = x \cos x$. **Ans. :** $y = (c_1 + c_2 x)e^{-x} + \frac{x}{2} \sin x + \frac{1}{2} (\sin x + \cos x)$

3. $(D^2 + D)\, y = x \cos x$. **Ans. :** $y = c_1 + c_2 e^{-x} - \frac{x}{2} (\cos x - \sin x) + \cos x + \frac{1}{2} \sin x$

4. $(D^2 - 1)\, y = x^2 \cos x$. **Ans. :** $y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2} (x^2 - 1) \cos x + x \sin x$

5. $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$.

   Ans. : $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - \frac{1}{48} (x^4 - 9x^2) \cos x + \frac{1}{12} x^3 \sin x$

6. $(D^2 + 1)^2\, y = 24x \cos x$.

   Ans. : $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - x^3 \cos x + 3x^2 \sin x$

7. $(D^2 + 4)\, y = x \sin^2 x$. **Ans. :** $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \left[\frac{1}{2} + \sin 2x\right]$

8. $(D^2 - 1)\, y = x \sin 3x + \cos x$.

   Ans. : $y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{10} \left[\frac{3}{5} \cos 3x + x \sin 3x + 5 \cos x\right]$

3.10.3 METHOD OF VARIATION OF PARAMETERS :

The method of variation of parameters is due to Lagrange is more general method of constructing a particular integral and is useful when other methods fail to give the particular integral. This method can also be used when the differential equation has variable coefficients.

Consider the second order linear differential equation

$$y'' + a_1(x)\, y' + a_0(x) y = R(x). \qquad \text{.......... (1)}$$

Let the solution of the homogeneous part of equation (1) be,

$$y_c = c_1\, y_1(x) + c_2\, y_2(x) \qquad \text{.......... (2)}$$

where $y_1(x)$ and $y_2(x)$ satisfies the equation

$$y'' + a_1(x)\, y' + a_0(x)\, y = 0 \qquad \text{.......... (3)}$$

Now assume that the particular integral is

$$y_p = u(x)\, y_1(x) + v(x)\, y_2(x) \qquad \text{.......... (4)}$$

Where $u(x)$ and $v(x)$ are functions of x to be determined in such a way that equation (satisfies equation (1).

Differentiating (4), we get

$$y_p' = u(x) y_1'(x) + v(x)\, y_2'(x) + u'(x) y_1(x) + v'(x)\, y_2(x) \qquad \text{.......... (5)}$$

To simplify (5) we impose the condition

$$u'(x) y_1(x) + v'(x)\, y_2(x) = 0 \qquad \text{.......... (6)}$$

Thus equation (5) becomes

$$y_p' = u(x)\, y_1'(x) + v(x)\, y_2'(x) \qquad \text{.......... (7)}$$

Again differentiating, we get

$$y_p'' = u(x) y_1''(x) + v(x)\, y_2''(x) + u'(x) y_1'(x) + v'(x)\, y_2'(x) \qquad \text{.......... (8)}$$

Substituting y_p, y_p', y_p'' from (4), (7), (8) in (1), we get

$$y_p'' + a_1(x)\, y_p' + a_0(x)\, y_p = R(x)$$

$$\Rightarrow u(x) y_1''(x) + v(x) y_2''(x) + u'(x) y_1'(x) + v'(x) y_2'(x) + a_1(x) [u(x) y_1'(x) + v(x) y_2'($$

$$+ a_0(x)\, [u(x)\, y_1(x) + v(x)\, y_2(x)] = R(x).$$

$$\Rightarrow u(x)[y_1''(x) + a_1(x) y_1'(x) + a_0(x)\, y_1(x)] + v(x)[y_2''(x) + a_1(x) y_2'(x) + a_0(x)\, y_2($$

$$+ u'(x)\, y_1'(x) + v'(x) y_2'(x) = R(x)$$

Since $y_1(x)$ and $y_2(x)$ satisfies equation (3), we have

$$u'(x)\, y_1'(x) + v'(x) y_2'(x) = R(x) \qquad \text{.......... (9)}$$

Thus from (8) and (9), we have the following system of equations :

$$u'(x)\, y_1(x) + v'(x)\, y_2(x) = 0$$

$$u'(x)\, y_1'(x) + v'(x)\, y_2'(x) = R(x)$$

The nontrivial solution for the above system of equations for $u'(x)$ and $v'(x)$ exist, if the coefficient determinant does not vanish.

That is

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

$\Rightarrow W(y_1, y_2)(x) \neq 0.$

$\therefore$ The required solution of the system of equations is

$$u'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ R(x) & y_2'(x) \end{vmatrix}}{W(y_1, y_2)(x)} = -\frac{y_2(x)\, R(x)}{W(y_1, y_2)(x)}$$

$$\Rightarrow u(x) = -\int \frac{y_2(x)\, R(x)}{W(y_1, y_2)(x)}\, dx \qquad \text{.......... (10)}$$

and $$v'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & R(x) \end{vmatrix}}{W(y_1, y_2)(x)} = \frac{y_1(x)\, R(x)}{W(y_1, y_2)(x)}$$

$$\Rightarrow v(x) = \int \frac{y_1(x)\, R(x)}{W(y_1, y_2)(x)}\, dx \qquad \text{.......... (11)}$$

$\therefore$ The particular integral of (1) is given by

$$y_p = -y_1(x) \int \frac{y_2(x)\, R(x)}{W(y_1, y_2)(x)}\, dx + y_2(x) \int \frac{y_1(x)\, R(x)}{W(y_1, y_2)(x)}\, dx \qquad \text{.......... (12)}$$

Hence the general solution of (1) is given by

$$y = y_c + y_p.$$

SOLVED EXAMPLES

1. Solve $y'' + 4y = \sec 2x$.

Solution : The auxiliary equation is :

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$

$\therefore \quad y_1(x) = \cos 2x,\ y_2(x) = \sin 2x$ and $R(x) = \sec 2x$.

$$\therefore \quad W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Let $y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$.

$$\therefore \quad u(x) = -\int \frac{y_2(x)\, R(x)}{W}\, dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\, dx$$

$$= -\frac{1}{2}\int \tan 2x\, dx = -\frac{1}{4}\log \sec 2x$$

$$v(x) = \int \frac{y_1(x)\, R(x)}{W}\, dx$$

$$= \int \frac{\cos 2x \sec 2x}{2}\, dx = \frac{1}{2}\int dx = \frac{x}{2}$$

$$\therefore \quad y_p = -\frac{1}{4}\cos 2x \log \sec 2x + \frac{x}{2}\sin 2x.$$

$\therefore$ The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log \sec 2x + \frac{x}{2}\sin 2x.$$

2. Solve $y'' + a^2 y = \tan ax$.

Solution : The auxiliary equation is :

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore \quad y_c = c_1 \cos ax + c_2 \sin ax$

$\therefore \quad y_1(x) = \cos ax,\ y_2(x) = \sin ax,\ R(x) = \tan ax$

Let $y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$

Now $W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = a$ (as above example)

$$\therefore \; u(x) = -\int \frac{y_2(x)\,R(x)}{W}\,dx$$

$$= -\int \frac{\sin ax \cdot \tan ax}{a}\,dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)\,dx$$

$$= -\frac{1}{a}\left[\frac{1}{a}\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$$= \frac{1}{a^2}\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax)$$

$$\text{and } v(x) = \int \frac{y_1(x)\,R(x)}{W}\,dx$$

$$= \int \frac{\cos ax \cdot \tan ax}{a}\,dx = \frac{1}{a}\int \sin ax\,dx = -\frac{\cos ax}{a^2}$$

$$\therefore \; y_p = \frac{1}{a^2}\sin ax\cos ax - \frac{1}{a^2}\cos ax\log(\sec ax + \tan ax) - \frac{1}{a^2}\cos ax\sin ax$$

$$= -\frac{1}{a^2}\cos ax\log(\sec ax + \tan ax)$$

$\therefore$ The general solution is

$$y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax \cdot \log(\sec ax + \tan ax)$$

3. **Solve $(D^2 + a^2)y = \operatorname{cosec} ax$**

Solution : The auxiliary equation is :

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore \; y_c = c_1\cos ax + c_2\sin ax$

$\therefore \; y_1(x) = \cos ax,\; y_2 = \sin ax,\; R(x) = \operatorname{cosec} ax$

$\therefore \quad W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = a$

Let $y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$

$$\therefore \quad u(x) = -\int \frac{y_2(x) \cdot R(x)}{W}\, dx$$

$$= -\int \frac{\sin ax \cdot \operatorname{cosec} ax}{a}\, dx$$

$$= -\frac{1}{a}\int dx = -\frac{x}{a}$$

$$v(x) = \int \frac{y_1(x) \cdot R(x)}{W}\, dx$$

$$= \int \frac{\cos ax \cdot \operatorname{cosec} ax}{a}\, dx$$

$$= \frac{1}{a}\int \cot ax\, dx = \frac{1}{a^2} \log \sin ax$$

$$\therefore \quad y_p = -\frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax \log \sin ax$$

$\therefore$ The general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x}{a} \cos ax + \frac{1}{a^2} \sin ax \cdot \log \sin ax$$

4. Solve $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$.

Solution : The auxiliary equation is :

$$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2.$$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{2x}$$

$$\therefore \quad y_1(x) = e^x,\ y_2(x) = e^{2x} \text{ and } R(x) = \frac{1}{1 + e^{-x}}$$

$$\therefore \quad W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

Let $y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$

$$\therefore \; u(x) = -\int \frac{y_2(x)\,R(x)}{W}\,dx$$

$$= -\int e^{2x}\,\frac{1}{1+e^{-x}}\cdot\frac{1}{e^{3x}}\,dx$$

$$= -\int \frac{e^{-x}}{1+e^{-x}}\,dx = \log\left(1+e^{-x}\right)$$

$$v(x) = \int \frac{y_1(x)\,R(x)}{W}\,dx$$

$$= \int e^{x}\,\frac{1}{1+e^{-x}}\cdot\frac{1}{e^{3x}}\,dx$$

$$= \int \frac{e^{-2x}}{1+e^{-x}}\,dx = \int\left\{e^{-x} - \frac{e^{-x}}{1+e^{-x}}\right\}dx$$

$$= -\,e^{-x} + \log(1+e^{-x})$$

$$\therefore \; y_p = e^{x}\log(1+e^{-x}) + e^{2x}\left[-e^{-x} + \log(1+e^{-x})\right]$$

$$= (e^{x}+e^{2x})\log(1+e^{-x}) - e^{x}$$

$\therefore$ The general solution is

$$y = c_1e^{x} + c_2e^{2x} + (e^{x}+e^{2x})\log(1+e^{-x}) - e^{x}.$$

5. Solve $y'' + y = x\sin x$.

Solution : The auxiliary equation is :

$$m^2 + 1 = 0 \Rightarrow m = \pm\, i$$

$$\therefore \; y_c = c_1\cos x + c_2\sin x$$

$\therefore$ $y_1(x) = \cos x$, $y_2(x) = \sin x$ and $R(x) = x\sin x$.

Now $W(y_1, y_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore$ Let $y_p = u(x)\,y_1(x) + v(x)\,y_2(x)$

$$\therefore \; u(x) = -\int \frac{y_2(x)\,R(x)}{W}\,dx$$

$$= -\int \frac{\sin x\cdot x\sin x}{1}\,dx$$

$$= -\int x \sin^2 x \, dx$$

$$= -\frac{1}{2}\int x\,(1-\cos 2x)\,dx$$

$$= -\frac{1}{2}\left[(x)\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x\right]$$

and $v(x) = \int \frac{y_1(x)\cdot R(x)}{W}\,dx$

$$= \int \frac{\cos x \cdot x \sin x}{1}\,dx$$

$$= \frac{1}{2}\int x \sin 2x \, dx$$

$$= \frac{1}{2}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]$$

$$\therefore\ y_p = -\frac{1}{2}\cos x\left[\frac{x^2}{2} - \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right] + \frac{1}{2}\sin x\left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right]$$

$\therefore$ The general solution is

$y = y_c + y_p$.

Remark : When the coefficients of second order differential equation are variables then the two solutions $y_1(x)$ and $y_2(x)$ of the homogeneous part of the equation (and hence the complementary function) can be obtained as follows :

Consider the equation $y'' + P(x)\,y' + Q(x)\,y = 0$ .... (1)

Case – I : $y = e^{mx}$ is the solution of (1) if $m^2 + Pm + Q = 0$.

Deductions : 1. $y = e^{x}$ is a solution of (1) if $1 + P + Q = 0$

2. $y = e^{-x}$ is a solution of (1) if $1 - P + Q = 0$

Case – II : If $y = x^m$ is the solution of (1) then

$$m\,(m-1)x^{m-2} + Pm\,x^{m-1}\ Q\,x^m = 0$$

$$\Rightarrow m\,(m-1) + Pmx + Q\,x^2 = 0$$

Deductions : 1. $y = x$ is the solution of (1) if $P + Qx = 0$

2. $y = 1$ is the solution of (1) if $Q = 0$

3. $y = x^2$ is the solution of (1) if $2 + 2Px + Qx^2 = 0$

6. **Solve** $(1 - x)y'' + xy' - y = (1 - x)^2, x \neq 1$

Solution : Rewriting the equation into normal form

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 1 - x$$

$$\therefore \quad P = \frac{x}{1-x}, Q = -\frac{1}{1-x}, R(x) = 1 - x$$

$$\text{Now } 1 + P + Q = 1 + \frac{x}{1-x} - \frac{1}{1-x} = 0$$

$$\text{Also, } P + Qx = \frac{x}{1-x} - \frac{x}{1-x} = 0$$

$\therefore$ e^x and x are the solutions of homogeneous part of the equation.

Let $y_1(x) = e^x, y_2(x) = x \Rightarrow y_c = c_1e^x + c_2x$

Let $y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$.

$$\text{Now } W(y_1, y_2)(x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1 - x)$$

$$\therefore \quad u(x) = -\int \frac{y_2(x)\,R(x)}{W}\,dx$$

$$= -\int \frac{x(1-x)}{e^x(1-x)}\,dx$$

$$= -\int x e^{-x}\,dx = -\left[(x)\left(-e^{-x}\right) - (1)\left(e^{-x}\right)\right] dx$$

$$= e^{-x}(x + 1)$$

$$v(x) = \int \frac{y_1(x)\,R(x)}{W}\,dx$$

$$= \int \frac{e^x(1-x)}{e^x(1-x)}\,dx = \int dx = x$$

$\therefore$ $y_p = e^{-x}(x + 1)e^x + x^2 = x^2 + x + 1.$

$\therefore$ The general solution is

$$y = c_1e^x + c_2x + x^2 + x + 1.$$

7. Solve $x^2y'' - 6xy' + 10y = 3x^4 + 6x^3$, $x \neq 0$.

Solution : The normal form of the equation is

$$y'' - \frac{6}{x}y' + \frac{10}{x^2}y = 3x^2 + 6x$$

$$\therefore \quad P = -\frac{6}{x}, Q = \frac{10}{x^2}, R(x) = 3x^2 + 6x$$

Now $m(m-1) + xPm + Qx^2 = 0$

$$\Rightarrow m(m-1) - \frac{6}{x}x\,m + \frac{10}{x^2}x^2 = 0$$

$$\Rightarrow m(m-1) - 6m + 10 = 0 \Rightarrow m^2 - 7m + 10 = 0 \Rightarrow m = 2, 5.$$

$\therefore$ x^2 and x^5 are the solutions of the homogeneous part of the equation.

Let $y_1(x) = x^2$, $y_2(x) = x^5$

$\therefore$ $y_c = c_1x^2 + c_2x^5$

Now $W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^5 \\ 2x & 5x^4 \end{vmatrix} = 3x^6$

Let $y_p = u(x)\,y_1(x) + v(x)\,y_2(x)$

$$\therefore \quad u(x) = -\int \frac{y_2(x)\,R(x)}{W}dx$$

$$= -\int \frac{x^5(3x^2 + 6x)}{3x^6}dx = -\int (x+2)\,dx = -\left(\frac{x^2}{2} + 2x\right)$$

$$v(x) = \int \frac{y_1(x)\,R(x)}{W}dx$$

$$= \int \frac{x^2(3x^2 + 6x)}{3x^6}dx = \int \left(x^{-2} + 2x^{-3}\right)dx = -\frac{1}{x} - \frac{1}{x^2}$$

$$\therefore \quad y_p = -\left(\frac{x^2}{2} + 3x\right)x^2 - \left(\frac{1}{x} + \frac{1}{x^2}\right)x^5$$

$$= -\frac{x^4}{2} - 3x^3 - x^4 - x^3 = -\left(\frac{3}{2}x^4 + 4x^3\right)$$

$\therefore$ The general solution is

$$y = c_1x^2 + c_2x^5 - \left(\frac{3}{2}x^4 + 4x^3\right)$$

EXERCISE – 3.8

Q. Solve the following differential equations by M. V. P. :

1. $(D^2 + 4)y = \tan 2x.$

Ans. : $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$

2. $(D^2 - 2D + 2)y = e^x \tan x.$

Ans. : $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$

3. $(D^2 + 1)y = \sec x.$ **Ans. :** $y = c_1 \cos x + c_2 \sin x + \cos x \log(\cos x) + x \sin x$

4. $(D^2 + 4)y = 4\sec^2 2x.$

Ans. : $y = c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$

5. $(D^2 - 1)y = e^{-x} \sin e^{-x} + \cos e^{-x}.$

Ans. : $y = c_1 e^x + c_2 e^{-x} - e^{-x} \sin e^{-x}.$