AC Circuits
CSEE 102 S1
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Advantages of AC system over DC system

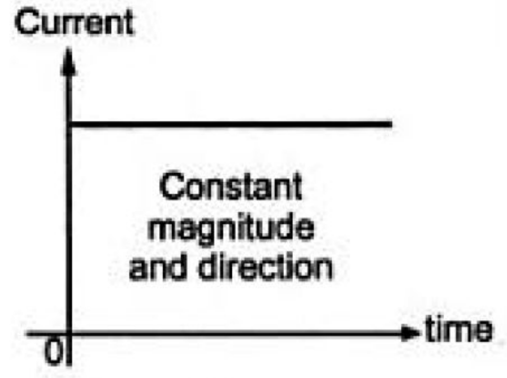
- An alternating voltage can be *stepped up and stepped down* efficiently by means of *transformer*. To *transmit huge power over a long distance*, the voltages are stepped up (upto 400 kV) for economical reasons at the generating stations whereas they are stepped down to a very low level (400/230 V) for the utilization of electrical energy from safety point of view.
- The *AC motors* (i.e., induction motors) are cheaper in cost, simple in construction, more efficient and robust as compared to DC motors.
- The *switchgear* (e.g., switches, circuit breakers, etc.) for AC system is *simpler* than DC system.

Thus, AC system is universally adopted for generation, transmission, distribution, and utilization of electrical energy

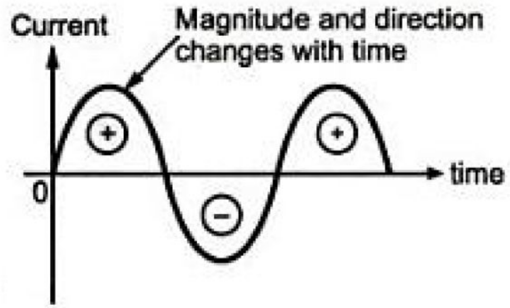
AC	DC
(1) An alternating current reverses periodically and its magnitude changes.	(1) Direct current flows only in one direction and remains unaltered.
(2) Amplitude and polarities are varying continuously.	(2) Amplitude and polarities are fixed.
(3) It has a particular frequency.	(3) It is independent of frequency.
(4) AC can be generated at higher voltages.	(4) DC cannot be generated at high voltages because of commutation difficulties.
(5) In case of AC, the cost of generation is less.	(5) In case of DC, the cost of generation is more.
(6) Alternating voltage can be increased (stepped up or decreased (stepped down) easily with the help of a transformer.	(6) Direction voltage cannot be increased or decreased easily.
(7) AC motors are of less cost, more robust, and durable.	(7) DC motors are costly and less durable.
(8) The maintenance cost of AC equipment and appliances is less.	(8) The maintenance cost of DC equipment and appliances is more.
(9) AC cannot be used directly for electroplating.	(9) Only DC can be used directly for electroplating.
(10) The speed of AC motors cannot be controlled easily.	(10) The speed control of DC motor is very easy and economical.

Introduction

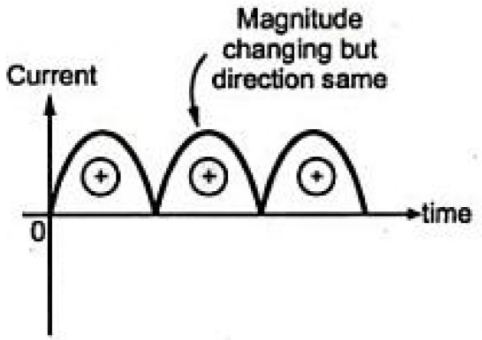
- The DC supply has constant magnitude with respect to time.
- Alternating Current changes periodically with respect to time.
- Changes in magnitude and direction is measured in terms of cycles.
- In Fig. b, current increases in one direction, attains maximum and starts decreasing, passing through zero it increases in opposite direction and behaves similarly.
- While plotting Fig. b, usually the instantaneous values of the alternating quantities are taken along Y-axis and time along X-axis.
- In practice, some waveforms have change in their magnitude but direction remains positive or negative, such waveforms are called as Pulsating DC (Fig. c).



(a) Direct current



(b) Alternating current



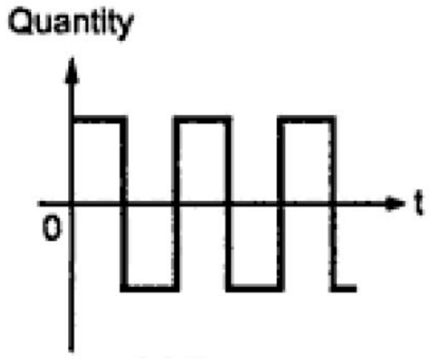
(c) Pulsating d.c.

Types of AC Waveforms

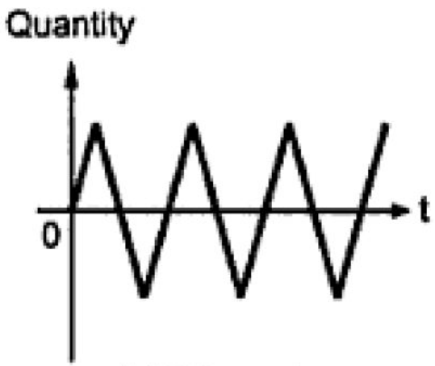
- Various types of alternating waveforms other than sinusoidal are shown.
- Out of all types alternating waveforms, sinusoidal w/f is preferred for AC systems.

<u>Advantages of Pure Sinusoidal w/f</u>

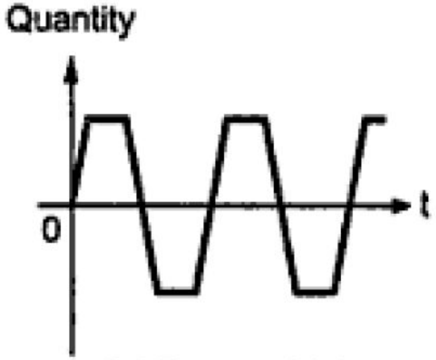
- Easy to write equation of pure sine w/f (Mathematically).
- Integration and differentiation of sine w/f is again sine function.
- Sine and cosine waves which can pass through linear circuit containing R, L, C without distortion. In case of other w/fs, there is possibility of distortion when it passes through linear circuit.
- Any other w/f can be resolved into a series of sine or cosine waves of fundamental and higher frequencies sum of all these waves gives original w/f. Hence, it is always better to have sine w/f as standard w/f.



(a) Square



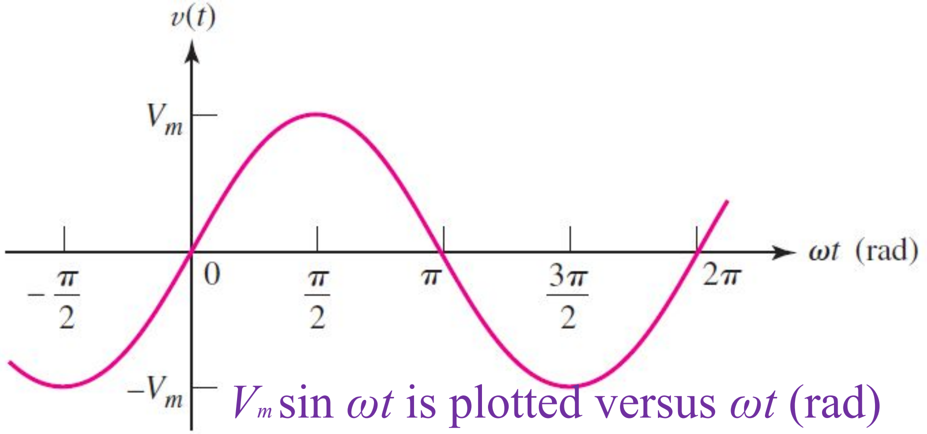
(b) Triangular



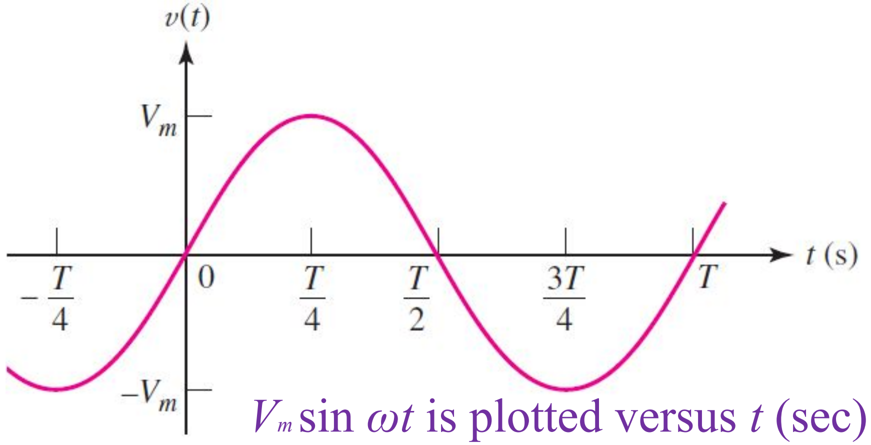
(c) Trapezoidal

Characteristics of Sinusoids

- Let $v(t)=V_m \sin \omega t$.
- Amplitude of sine wave is V_m.
- Radian/angular frequency is ω.

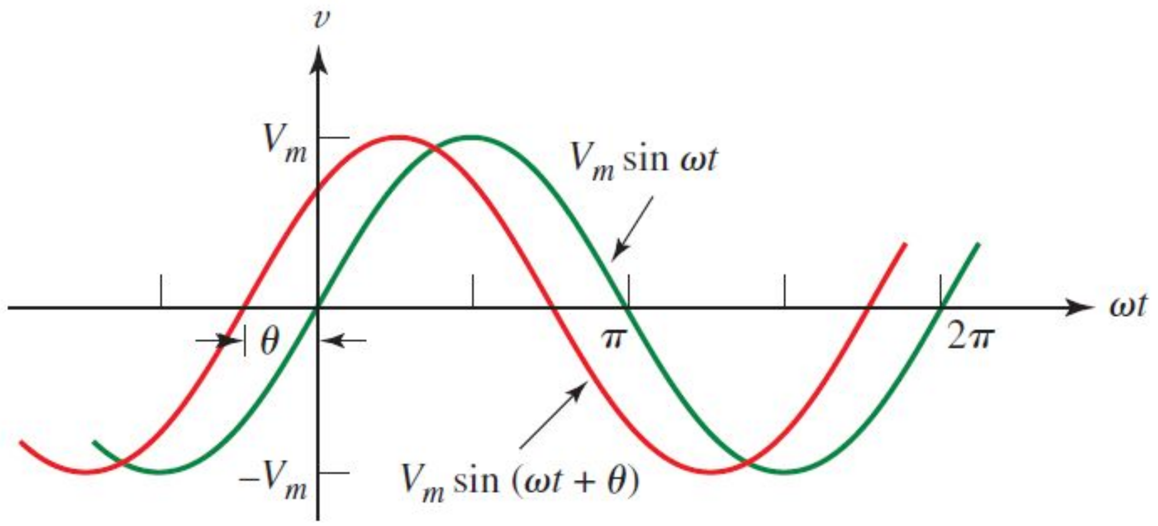


$V_m \sin \omega t$ is plotted versus ωt (rad)



$V_m \sin \omega t$ is plotted versus t (sec)

- The function repeats itself every 2π radians, and its ***period*** is therefore 2π radians.
- A sine wave having a period T must execute $1/T$ periods each second; its ***frequency*** f is $1/T$ hertz, abbreviated Hz. Thus
- *$f = 1/T$ and since, $\omega T=2\pi$.*
- Therefore, we obtain common relation between frequency (Hz) and angular frequency (rad/s) i.e, $\omega=2\pi f$.
- Lagging and Leading: $v(t)=V_m \sin(\omega t+\theta)$



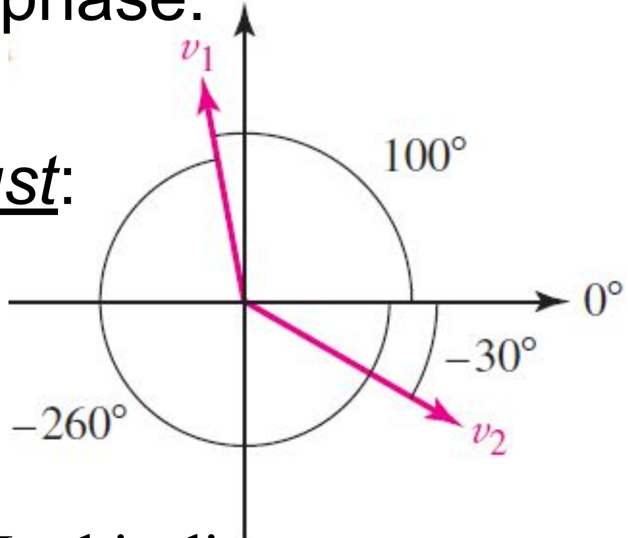
The sine wave **$V_m \sin(\omega t + \theta)$** leads **$V_m \sin \omega t$** by θ rad

Lagging and Leading

- sin ωt as lagging sin(ωt + θ) by θ rad, as leading sin(ωt + θ) by −θ rad, or as leading sin(ωt − θ) by θ rad.
- In either case, leading or lagging, we say that the sinusoids are out of phase.
- If the phase angles are equal, the sinusoids are said to be in phase.
- We customarily use, $v(t) = 100 \sin(2\pi 1000t - 30^\circ)$.

 __Two sinusoidal waves whose phases are to be compared must__:

1. Both be written as sine waves, or both as cosine waves.
2. Both be written with positive amplitudes.
3. Each have the same frequency.



Converting Sine to cosine or cosine to sine

$$\begin{aligned} v_1 &= V_{m_1}\cos(5t + 10^\circ) \\ &= V_{m_1}\sin(5t + 90^\circ + 10^\circ) \\ &= V_{m_1}\sin(5t + 100^\circ) \end{aligned}$$

$$v_2 = V_{m_2}\sin(5t - 30^\circ)$$

Note that:

$$-\sin\omega t = \sin(\omega t \pm 180^\circ)$$
$$-\cos\omega t = \cos(\omega t \pm 180^\circ)$$
$$\mp\sin\omega t = \cos(\omega t \pm 90^\circ)$$
$$\pm\cos\omega t = \sin(\omega t \pm 90^\circ)$$

In this diagram, v_1 leads v_2 by 100° + 30° = 130°, or it could also be argued that v_2 leads v_1 by 230°.

Important Terms

- An alternating voltage or current changes its magnitude and direction at regular intervals of time. *A sinusoidal voltage or current varies as a sine function of time t or angle ($\theta = \omega t$).*

- **Wave form:** The shape of the curve obtained by *plotting the instantaneous values of alternating quantity* (voltage or current) along Y-axis and time or angle ($\theta = \omega t$) along X-axis is called ‘wave form or wave shape’.

- **Instantaneous value:** The value of an alternating quantity, that is, *voltage or current at any instant* is called its instantaneous value and is represented by ‘*e*’ or ‘*i*’, respectively.

- **Cycle:** When an alternating quantity goes through a complete *set of positive and negative values* or goes through *360 electrical degrees,* it is said to have completed one cycle.

- **Alternation:** One *half-cycle* is called ‘alternation’. An alternation spans *180 electrical degrees.*

- **Time period:** The *time taken in seconds to complete one cycle* by an alternating quantity is called time period. It is generally denoted by ‘T’.

- **Frequency:** The *number of cycles made per second* by an alternating quantity is called ‘frequency’. It is measured in cycles per second (c/s) or hertz (Hz) and is denoted by ‘f ’.

- **Amplitude:** The *maximum value* (positive or negative) attained by an alternating quantity in one cycle is called its ‘amplitude or peak value or maximum value’. The maximum value of voltage and current is generally denoted by E_m (or V_m) and I_m, respectively

Relations

- **Relation between frequency and time period:** An alternating quantity has a frequency f c/s. Then, time taken to complete f cycles is 1 s.

  Time taken to complete 1 cycle is 1/f s.

  Hence, time period, $T=1/f$ s or $f=1/T$ c/s.

- **Relation between angular velocity and frequency:** An alternating quantity has a frequency f c/s.

  Angular distance covered in one cycle is 2π radian.

  Therefore, angular distance covered per second in f cycles is 2π radian. Hence $\omega = 2\pi f$ rad/s.

- Alternating voltage represented as $e=E_m \sin\theta = E_m \sin\omega t$

Values of alternating voltage and current

- The voltage and current in DC system are constant so that there is no problem of specifying their magnitudes, whereas in AC system, the alternating voltage and current vary from time to time.
- Hence, it is necessary to explain the ways to express the magnitude of alternating voltage and current.
- The following three ways are adopted to express the magnitude of these quantities:

 Peak Value

 Average Value or Mean Value

 Effective Value or RMS Value

Peak value

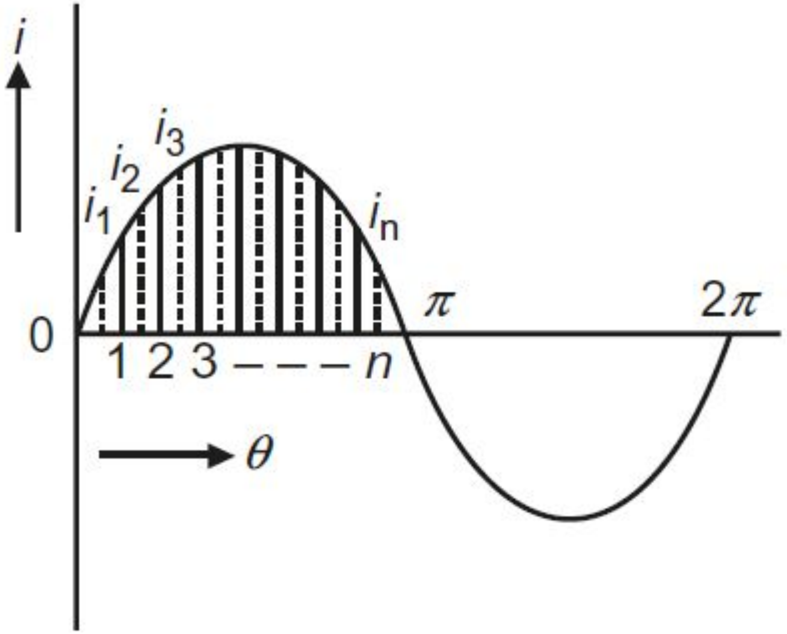
- The maximum value attained by an alternating quantity during one cycle is called 'peak value'/ 'Maximum Value'/ 'Crest Value'/ 'Amplitude'.
- A sinusoidal alternating quantity obtains its maximum value at 90°.
- The peak of an alternating voltage and current is represented by E_m and I_m.
- The knowledge of peak value is important in case of testing dielectric strength of insulating materials.

Average Value

- The arithmetic average of all the instantaneous values considered an alternating quantity (current or voltage) over one cycle is called average value.
- In case of symmetrical waves (such as sinusoidal current or voltage wave), the positive half is exactly equal to the negative half; therefore, the average value over a complete cycle is zero.
- Since work is being done by the current in both the positive and the negative half cycle, average value is determined regardless of signs.
- Hence, to determine average value of alternating quantities having symmetrical waves, only (*positive half*) cycle is considered.

Average Value

- Divide the positive half cycle into ‘n’ number of equal parts as shown in Figure. Let i_1, i_2, i_3, ..., i_n be the mid-ordinates.
- Average value of current, I_{av} = mean of mid ordinates.



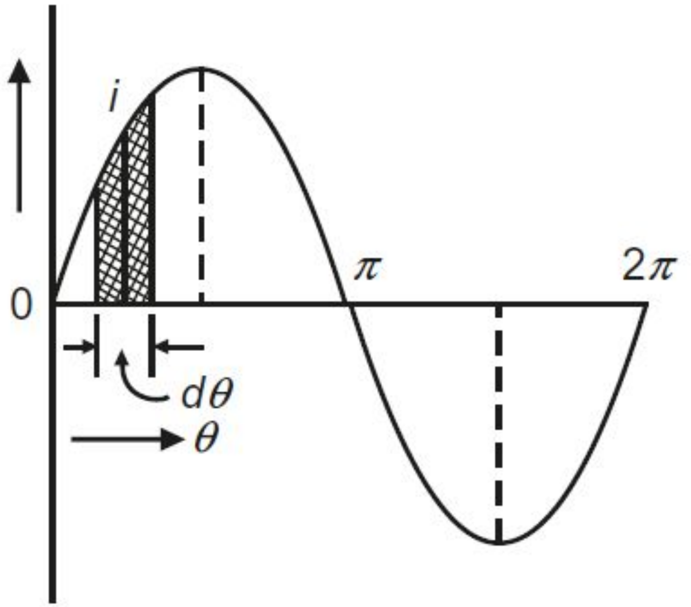
Positive half cycle divided into n equal parts

$$I_{av} = \frac{i_1 + i_2 + i_3 + \ldots + i_n}{n} = \frac{\text{Area of alternation}}{\text{Base}}$$

Average Value of Sinusoidal Current

$$i = I_m \sin\theta$$

Consider an elementary strip of thickness *dq* in the positive half cycle, *i* be its mid-ordinate. Then, Area of strip = *I dq*.



$$\text{Area of half cycle} \int_0 i\, d\theta = \int_0 I_m \sin\theta d\theta$$

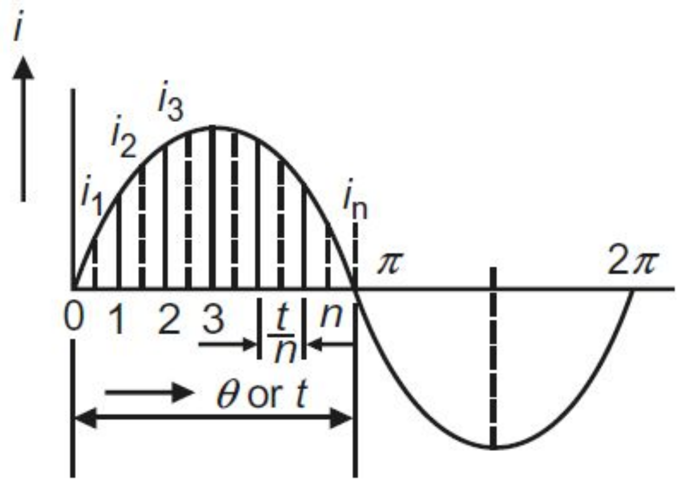
$$= I_m(-\cos\theta)_0^\pi = I_m(-(\cos\pi - \cos 0))$$

$$= I_m[-1(-1-1)] = 2I_m$$

$$\text{Base} = 0 \text{ to } \pi = \pi - 0 = \pi$$

$$I_{av} = \frac{\text{Area of alternation}}{\text{base}} = \frac{2I_m}{\pi} = \frac{I_m}{\pi/2} = 0.637\, I_m$$

Effective Value or RMS Value



- The steady current when flows through a resistor of known resistance for a given time produces the same amount of heat as produced by an alternating current when flows through the same resistor for the same time is called effective or rms value of an alternating current.
- Let *i* be an alternating current flowing through a resistor of resistance R for time t seconds which produces the same amount of heat as produced by I_{eff} (direct current).
- The base of one alternation is divided into n equal parts, so that interval is of (t/n) second. Let $i_1, i_2, i_3, ..., i_n$ be the mid-ordinates.
- Heat produced in the intervals 1, 2… n are given by
- $(i_1^2Rt/Jn), (i_2^2Rt/Jn), \ldots (i_n^2Rt/Jn)$ Calorie

$$\text{Total heat produced} = \frac{Rt}{J}\left(\frac{i_1^2 + i_2^2 + i_3^2 + \ldots\ldots i_n^2}{n}\right)$$

Effective Value or RMS Value

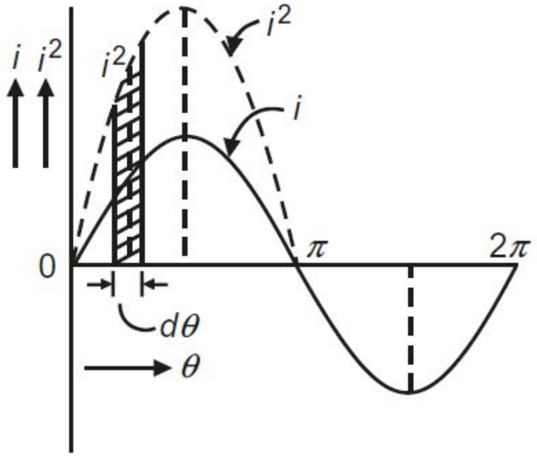
$$\frac{I_{eff}^2 Rt}{J} = \frac{Rt}{J}\left(\frac{i_1^2 + i_2^2 + i_3^2 + \ldots\ldots i_n^2}{n}\right)$$

$$I_{\text{eff}} = \sqrt{\frac{i_1^2 + i_2^2 + i_3^2 + \cdots + i_n^2}{n}} = \sqrt{\text{mean of squares of instantaneous values}}$$

I_{eff} = Square root of mean of squares of instantaneous values
= root-mean-square value

- It is the actual value of an alternating quantity which tells us the energy transfer capability of an AC source.
- For example, if we say that 5 A AC is flowing through a circuit, it means the rms value of an AC which flows through the circuit is 5 A. It transfers the same amount of energy as is transferred by 5 A DC.
- The ammeters and voltmeters record the rms values of alternating currents and voltages, respectively. The domestic single-phase AC supply is 230 V, 50 Hz. Where 230 V is the rms value of an alternating voltage.

RMS value of sinusoidal current



- Considering an elementary strip of thickness $d\theta$ in the first half-cycle of the squared wave, let i^2 be its mid-ordinate.
- Then, Area of strip = $i^2 d\theta$.
- Area of first half cycle of squared wave

$$= \int_0^{\pi} i^2 d\theta = \int_0^{\pi} (I_m \sin\theta)^2 d\theta$$

$$= I_m^2 \int_0^{\pi} \sin^2 \theta d\theta = I_m^2 \int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{I_m^2}{2} \int_0^{\pi} (1-\cos 2\theta)\, d\theta = \frac{I_m^2}{2} \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{I_m^2}{2} \left((\pi - 0) - \frac{\sin 2\pi - \sin 0}{2} \right)$$

$$= \frac{I_m^2}{2} [(\pi - 0) - (0 - 0) = \frac{\pi I_m^2}{2}$$

Effective or rms value

$$I_{rms} = \sqrt{\frac{\text{Area of first half of squared wave}}{\text{Base}}}$$

$$= \sqrt{\frac{\pi I_m^2}{2\pi}} = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707\, I_m$$

Form Factor and Peak Factor

- **Form Factor:** Ratio of RMS value to the average value of an alternating quantity. For a sinusoidal wave $FF=(I_{rms}/I_{avg})=(0.707I_m/0.636I_m)=1.11$.

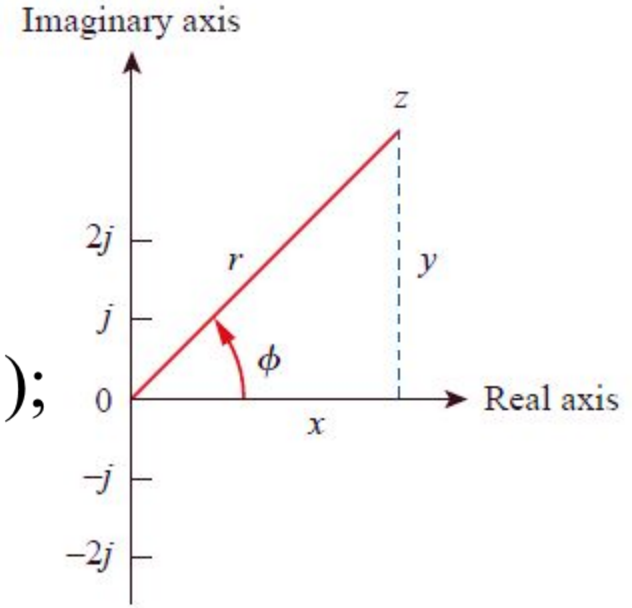
- **Peak Factor:** Ratio of Maximum value to RMS value to an alternating quantity. For a Sinusoidal wave $PF=(I_m/I_{rms})=(I_m/0.707I_m)=1.414$.

Phasors

- Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions.
- *A phasor is a complex number that represents the amplitude and phase of a sinusoid.*
- Phasors provide a simple means of analyzing linear circuits excited by sinusoidal sources.
- Before applying phasors for circuit analysis, we need to be familiar with complex numbers.
- Complex number z is given by $z = x + jy$; where $j = \sqrt{-1}$.
- x is the real part of z and y is the imaginary part of z.
- Complex number z can also be written in polar or exponential form as
- $z = r \angle \emptyset = re^{j\emptyset}$; where r is the magnitude of z and $\emptyset$ is phase angle.

Phasors

- $z = x + jy$; is rectangular form.
- $z = r \angle \emptyset$; is polar form.
- $z = re^{j\emptyset}$; is exponential form.
- The relation between rectangular form and polar form is as follows.
- If we know x and y, we can determine r and $\emptyset$.
- $r = \sqrt{x^2 + y^2}$ and $\emptyset = tan^{-1}(y/x)$.
- If we know r and $\emptyset$, we can determine x and y.
- $x = r\cos\emptyset, y = r\sin\emptyset$;
- Thus z, may be written as $z = x + jy = r \angle \emptyset = r(\cos\emptyset + j\sin\emptyset)$;



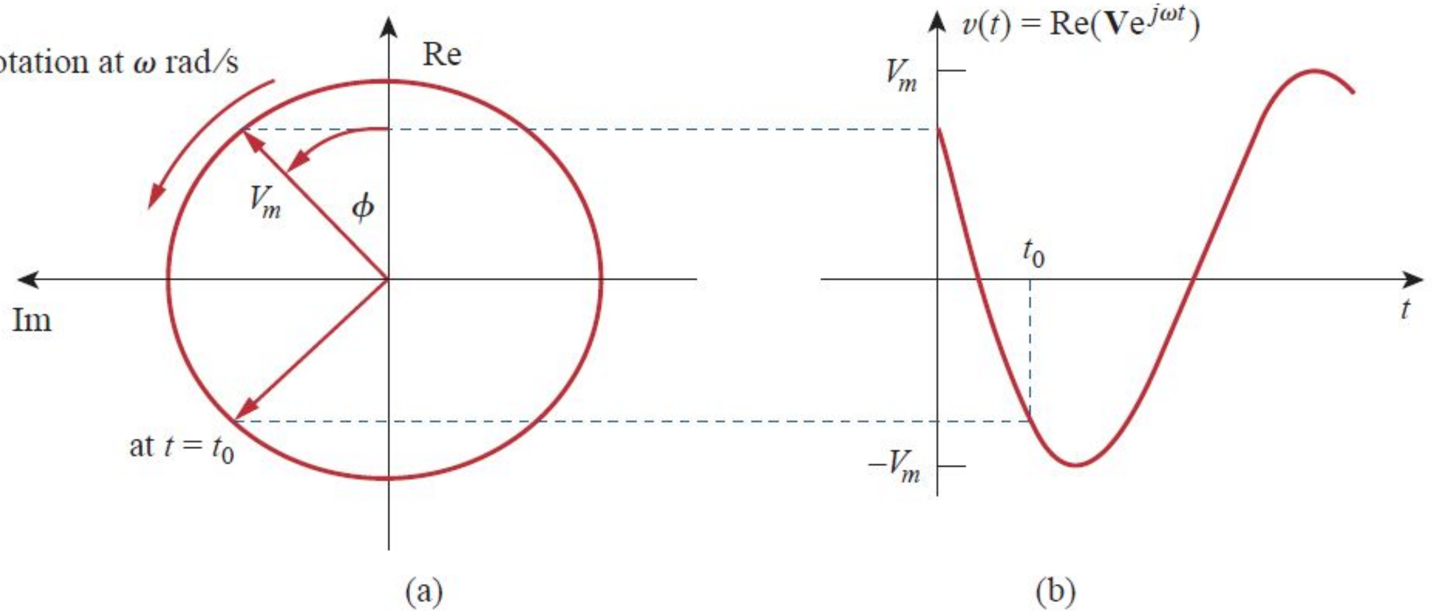
Addition and *subtraction* of complex numbers are better performed in *rectangular form*.

Multiplication and *Division* of complex numbers are better performed in *Polar form*.

- Let $z = x + jy = r \angle \emptyset$; $z_1 = x_1 + jy_1 = r_1 \angle \emptyset_1$; and $z_2 = x_2 + jy_2 = r_2 \angle \emptyset_2$
- ***Addition:*** $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
- ***Subtraction:*** $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
- ***Multiplication:*** $z_1 z_2 = r_1 r_2 \angle (\emptyset_1 + \emptyset_2)$
- ***Division:*** $z_1 / z_2 = r_1 / r_2 \angle (\emptyset_1 - \emptyset_2)$
- ***Reciprocal:*** $\frac{1}{z} = \frac{1}{r} \angle (-\emptyset)$
- ***Square root:*** $\sqrt{z} = \sqrt{r} \angle (\emptyset/2)$
- ***Complex Conjugate:*** $z^* = x + jy = r \angle (-\emptyset) = re^{-j\emptyset}$
- The idea of phasor representation is based on Euler's identity.
- $e^{\pm j\emptyset} = \cos\emptyset \pm j\sin\emptyset$
- $\cos\emptyset = Re\left(e^{j\emptyset}\right)$and $\sin\emptyset = Im(e^{j\emptyset})$
- $v(t) = V_m\cos(\omega t + \emptyset)$ =Re($V_m e^{j(\omega t+\emptyset)}$)=Re($V_m e^{j\omega t} e^{j\emptyset}$)
- Thus $v(t)$ =Re($\mathbf{V} e^{j\omega t}$); where $\mathbf{V} = V_m e^{j\emptyset} = V_m \angle (\emptyset)$.

- A phasor may be regarded as a mathematical equivalent of a sinusoid with the time dependence dropped.
- **V** is thus the phasor representation of the sinusoid $v(t)$.
- A phasor is a complex representation of the magnitude and phase of a sinusoid.
- If we use sine for the phasor instead of cosine, then $v(t) = V_m \sin(\omega t + \emptyset) = Im\left(V_m e^{j(\omega t+\emptyset)}\right)$.
- The plot of sinor can be obtained as follows: As time increases, the sinor rotates on a circle with radius V_m, at an angular velocity ω in counter clockwise direction.

Representation of $\mathbf{V}e^{j\omega t}$: (a) sinor rotating counterclockwise, (b) its projection on the real axis, as a function of time.

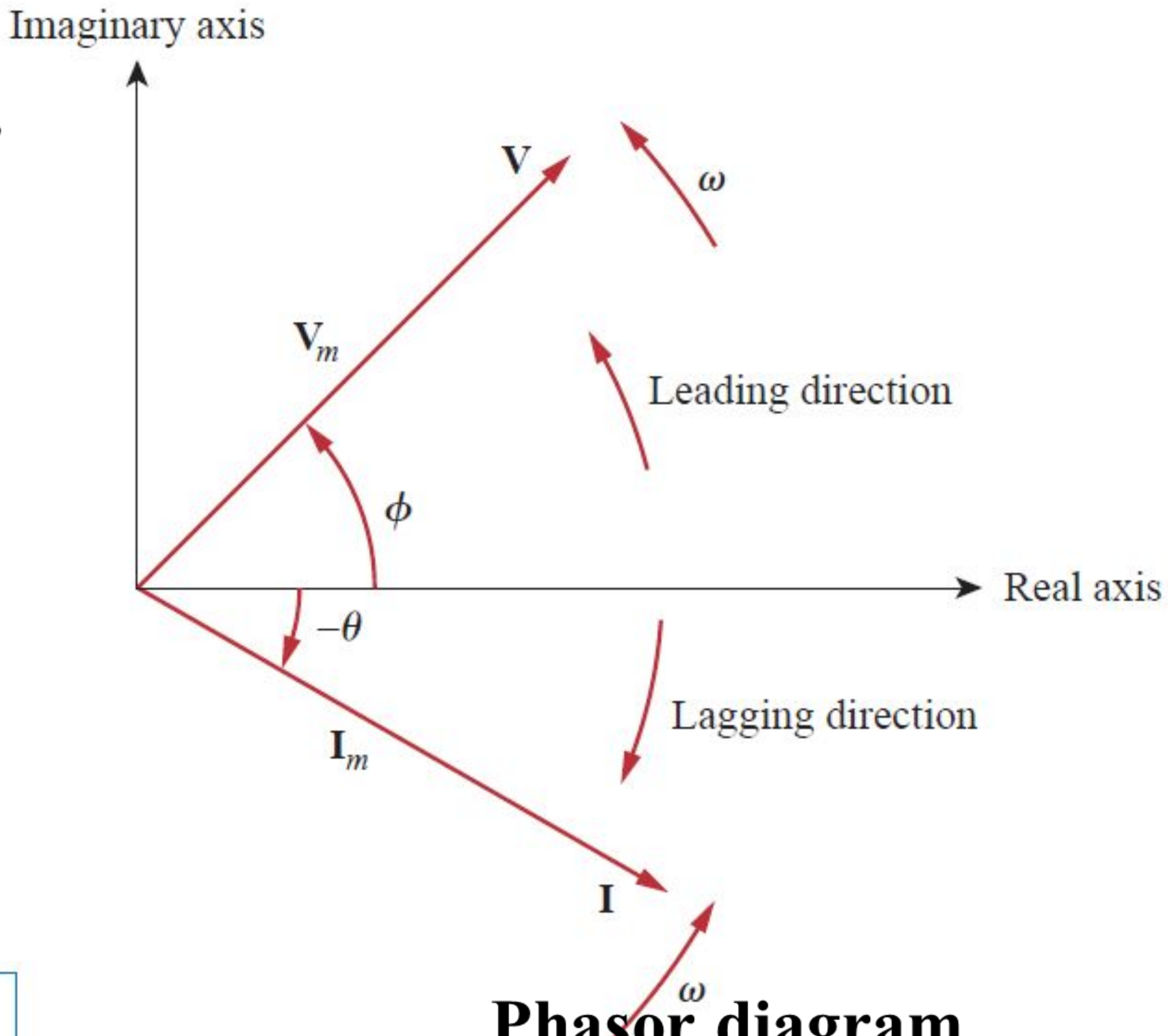


- Graphical representation of phasors is termed as Phasor Diagram.
- In the phasor diagram, Voltage leads Current by $(\emptyset + \theta)$ or Current lags voltage by $(\emptyset + \theta)$.

Time domain representation **Phasor domain representation**

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)



Phasor diagram

$\mathbf{V} = V_m : \emptyset$ and

$\mathbf{I} = I_m : (-\theta)$

Derivative and Integral of Sinusoid

- Let $v(t)$ $=\mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right) = V_m \cos(\omega t + \phi)$, so that its derivative is
- $\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90^0)$
- $=\mathrm{Re}\left(\omega V_m e^{j\omega t} e^{j\phi} e^{j90}\right)$ $=\mathrm{Re}(j\omega \mathbf{V} e^{j\omega t})$
- From this the derivative of *v(t)* is transformed to the phasor domain as $j\omega\mathbf{V}$.
- Similarly, the integral of *v(t)* is transformed to the phasor domain as $\mathbf{V}/j\omega$.

Time Domain	Phasor Domain

- The above table is useful in finding the steady-state solution, which does not require knowing the initial values of the variables involved. This is an important application of phasor.

- Besides time differentiation and integration, another important use of phasors is found in summing sinusoids of the same frequency. (Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors)
- The difference between $v(t)$ and **V** are as follows:

 $v(t)$ is the instantaneous or time domain representation, while **V** is the frequency or phasor domain representation.

 $v(t)$ is time dependent, while **V** is not.

 $v(t)$ is always real with no complex term, while **V** is generally complex.

- Finally, we should remember that phasor analysis applies only when frequency is constant; it applies in manipulating two or more sinusoidal signals only if they are of the same frequency.

Example

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

(b) Using polar-rectangular transformation, addition, multiplication, and division,

$$\frac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*} = \frac{8.66 - j5 + (3 - j4)}{(2 + j4)(3 + j5)}$$

$$= \frac{11.66 - j9}{-14 + j22} = \frac{14.73\angle -37.66^\circ}{26.08\angle 122.47^\circ}$$

$$= 0.565\angle -160.13^\circ$$

Example

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

(a) $i = 6\cos(50t - 40^\circ)$ has the phasor

$$\mathbf{I} = 6\angle{-40^\circ} \text{ A}$$

(b) Since $-\sin A = \cos(A + 90^\circ)$,

$$\begin{aligned} v = -4\sin(30t + 50^\circ) &= 4\cos(30t + 50^\circ + 90^\circ) \\ &= 4\cos(30t + 140^\circ) \text{ V} \end{aligned}$$

The phasor form of v is

$$\mathbf{V} = 4\angle{140^\circ} \text{ V}$$

Example

Express these sinusoids as phasors:

(a) $v = 7\cos(2t + 40^\circ)$ V
(b) $i = -4\sin(10t + 10^\circ)$ A

Answer: (a) $\mathbf{V} = 7\underline{/40^\circ}$ V, (b) $\mathbf{I} = 4\underline{/100^\circ}$ A.

Example

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

(a) $\mathbf{I} = -3 + j4 = 5\angle 126.87^\circ$. Transforming this to the time domain gives

$$i(t) = 5\cos(\omega t + 126.87^\circ)\text{ A}$$

(b) Since $j = 1\angle 90^\circ$,

$$\mathbf{V} = j8\angle -20^\circ = (1\angle 90^\circ)(8\angle -20^\circ)$$
$$= 8\angle 90^\circ - 20^\circ = 8\angle 70^\circ\text{ V}$$

Converting this to the time domain gives

$$v(t) = 8\cos(\omega t + 70^\circ)\text{ V}$$

Example

If $v_1 = -10 \sin(\omega t - 30°)$ V and $v_2 = 20 \cos(\omega t + 45°)$ V, find $v = v_1 + v_2$.

Answer: $v(t) = 12.158 \cos(\omega t + 55.95°)$ V.

Example

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

$$4\mathbf{I} + \frac{8\mathbf{I}}{j\omega} - 3j\omega\mathbf{I} = 50\underline{/75^\circ}$$

But $\omega = 2$, so

$$\mathbf{I}(4 - j4 - j6) = 50\underline{/75^\circ}$$

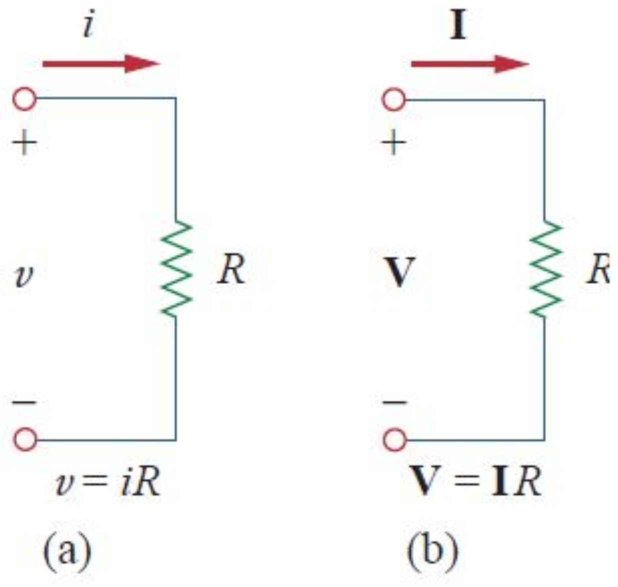
$$\mathbf{I} = \frac{50\underline{/75^\circ}}{4 - j10} = \frac{50\underline{/75^\circ}}{10.77\underline{/-68.2^\circ}} = 4.642\underline{/143.2^\circ}\text{ A}$$

Converting this to the time domain,

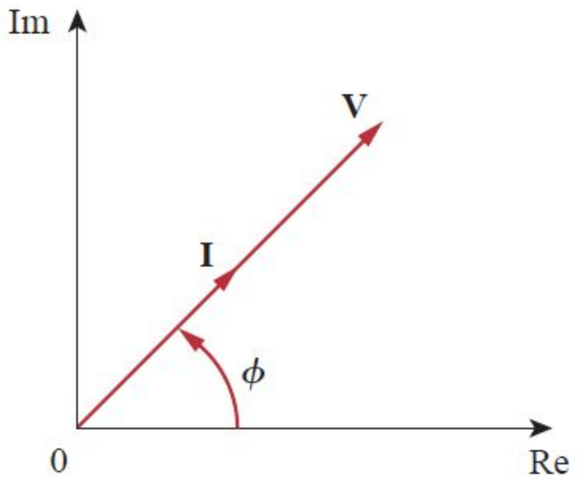
$$i(t) = 4.642\cos(2t + 143.2^\circ)\text{ A}$$

Phasor relationships in circuit elements

- In electrical circuits, we need to transform time domain to frequency domain for each element, to perform this passive sign convention is used.
- If the current through a resistor is $i(t) = I_m \cos(\omega t + \varphi)$; the voltage across it is given by $v = iR = RI_m \cos(\omega t + \varphi)$;
- The phasor form of voltage is $\mathbf{V} = RI_m \angle \varphi$
- But the phasor representation of the current is $\mathbf{I} = I_m \angle \varphi$
- Hence, $\mathbf{V} = \mathbf{I}R$;
- Showing that, the ohms law satisfies in phasor domain.
- In a Resistor, voltage and current are in In-phase.
- The *phase difference* between Voltage and Current in a *Resistor* is *Zero* Degree.



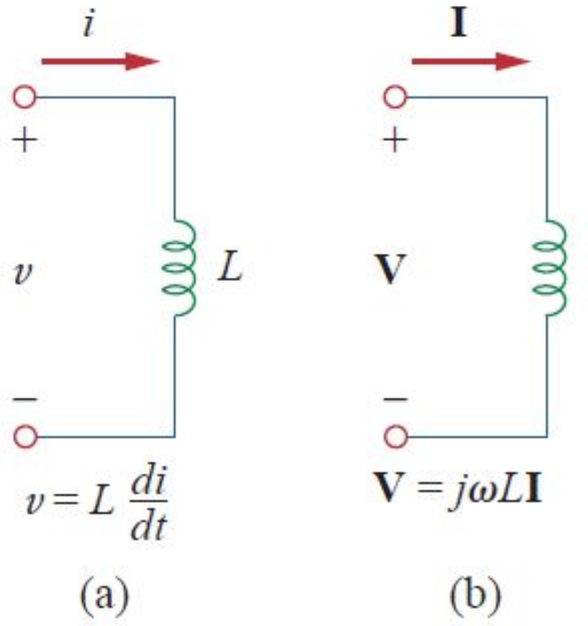
Voltage-current relations for a resistor in the: (a) time domain, (b) frequency domain

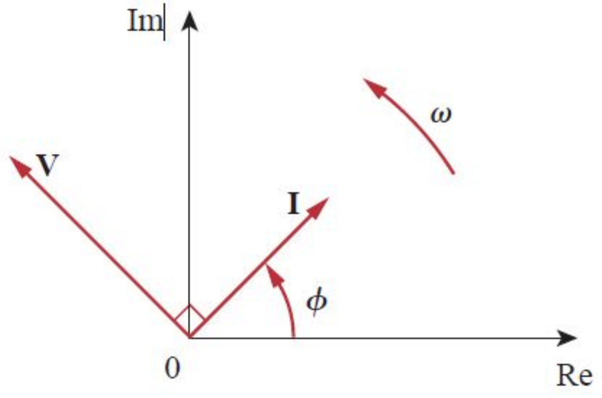


Phasor diagram for the resistor

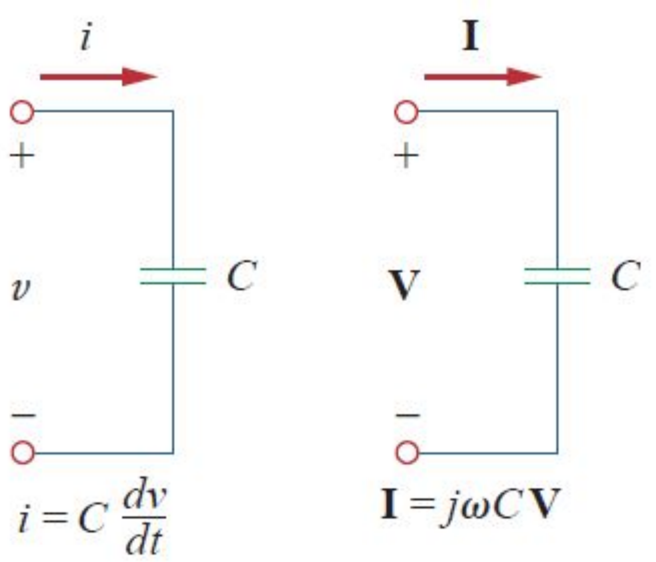
Phasor representation for Inductor

- For an inductor, assume current through it is $i(t) = I_m \cos(\omega t + \emptyset)$; The voltage across it is $v = L\,{}^{di}/_{dt}$
- $v = L\,{}^{di}/_{dt} = -\omega L I_m \sin(\omega t + \emptyset)$;
- $-\sin A$ can be written as $\cos(A + 90)$.
- Voltage can be written as $v = \omega L I_m \cos(\omega t + \emptyset + 90)$.
- In phasor domain $\mathbf{V} = \omega L I_m e^{j(\emptyset+90)} = \omega L I_m e^{j\emptyset} e^{j90}$
- $\mathbf{V} = \omega L I_m \angle (\emptyset + 90)$.
- But $I_m \angle \emptyset = \mathbf{I}$ and $e^{j90} = j$. Thus
- $\mathbf{V} = j\omega L\mathbf{I}$. Therefore voltage has magnitude $j\omega L I_m$ and a phase of $\emptyset + 90$.
- Phasor diagram of voltage-current for inductor is shown in figure.
- From the phasor diagram, current lags the voltage by 90^0.

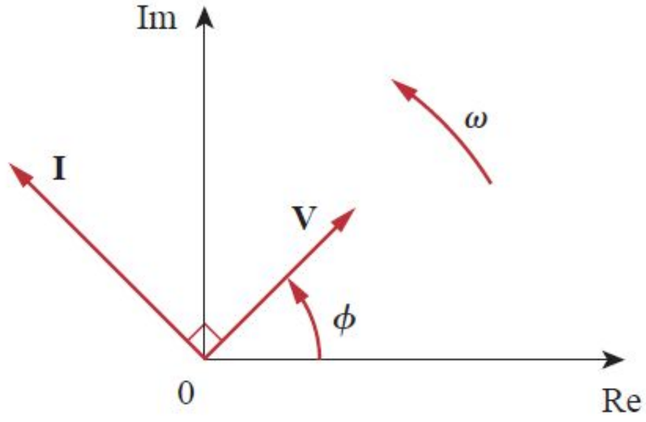




Phasor representation for Capacitor



- For an inductor, assume current through it is $v(t) = V_m \cos(\omega t + \emptyset)$; The voltage across it is $i = C\, {}^{dv}/_{dt}$
- $i = C\, {}^{dv}/_{dt} = -\omega C V_m \sin(\omega t + \emptyset)$;
- $-\sin A$ can be written as $\cos(A + 90)$.
- Voltage can be written as $i = \omega C V_m \cos(\omega t + \emptyset + 90)$.
- In phasor domain $\mathbf{I} = \omega C V_m e^{j(\emptyset+90)} = \omega C V_m e^{j\emptyset} e^{j90}$
- $\mathbf{I} = \omega C V_m \vdots (\emptyset + 90)$.
- But $V_m \vdots \emptyset = \mathbf{V}$ and $e^{j90} = j$. Thus
- $\mathbf{I} = j\omega C\mathbf{V}$. Therefore current has magnitude $j\omega C V_m$ and a phase of $\emptyset + 90$.
- Phasor diagram of voltage-current for capacitor is shown in figure.
- From the phasor diagram, voltage lags the current by 90^0.



Summary of voltage current relationships

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Example

- The voltage $v = 12\cos(60t + 45^0)$ is applied to a 0.1 H inductor. Find the steady-state current through the inductor.

For the inductor, $\mathbf{V} = j\omega L\mathbf{I}$, where $\omega = 60$ rad/s and $\mathbf{V} = 12\underline{/45^\circ}$ V. Hence,

$$\mathbf{I} = \frac{\mathbf{V}}{j\omega L} = \frac{12\underline{/45^\circ}}{j60 \times 0.1} = \frac{12\underline{/45^\circ}}{6\underline{/90^\circ}} = 2\underline{/-45^\circ} \text{ A}$$

Converting this to the time domain,

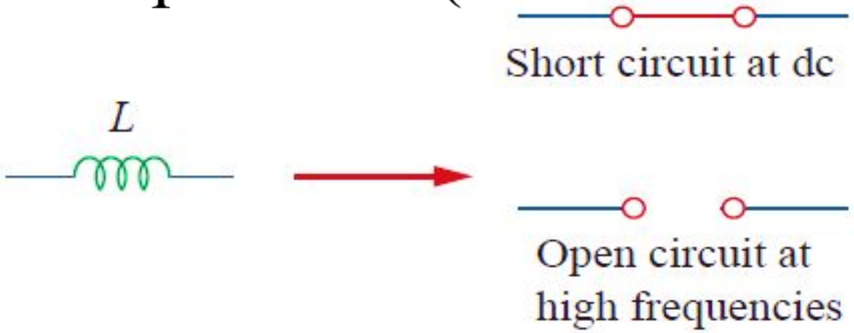
$$i(t) = 2\cos(60t - 45^\circ) \text{ A}$$

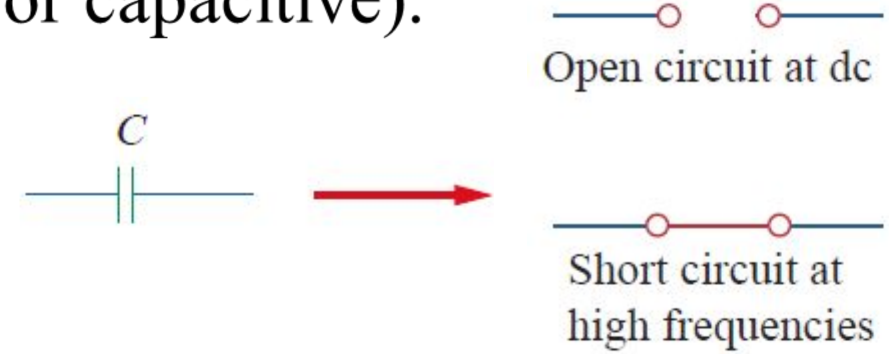
Impedance and Admittance

- The voltage-current relationships for passive elements as
- $\mathbf{V} = \mathbf{I}R$; $\mathbf{V} = j\omega L\mathbf{I}$; $\mathbf{V} = {}^{\mathbf{I}}/_{j\omega C}$;
- These equations may be written in terms of the ratio of the phasor voltage to phasor current as
- $\frac{\mathbf{V}}{\mathbf{I}} = R$; $\frac{\mathbf{V}}{\mathbf{I}} = j\omega L$; $\frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$;
- From the above, ohms law in any form for any element as
- $\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$; or $\mathbf{V} = \mathbf{Z}\mathbf{I}$; where Z is frequency dependent quantity known as impedance measured in ohms.
- The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms. Reciprocal of Impedance is called admittance (**Y**=1/**Z**).
- The impedance represents the opposition that the circuit exhibits to the flow of sinusoidal current.
- Impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity.

Impedance

- The impedances of resistors, inductors and capacitors can be written as $Z = R$; $Z_L = j\omega L$; $Z_C = {}^{1}\!/_{j\omega C}$; Let us consider two extreme cases of angular frequency, i.e, $\omega = 0$ and $\omega = \infty$;
- When $\omega = 0$ (i.e, DC Source), $Z_L = 0$; $Z_c \rightarrow \infty$; confirming that the inductor behaves like a short circuit and capacitor acts like open circuit.
- When $\omega \rightarrow \infty$ (i.e, High Frequency), $Z_L \rightarrow \infty$; $Z_c = 0$; confirming that the inductor behaves like a open circuit and capacitor acts like short circuit for high frequencies.
- As a complex quantity, the impedance can be expressed in rectangular form as $\mathbf{Z} = R + jX$; where R is real part of $\mathbf{Z}$ (resistance) and X is imaginary part of $\mathbf{Z}$ (reactance either inductive or capacitive).





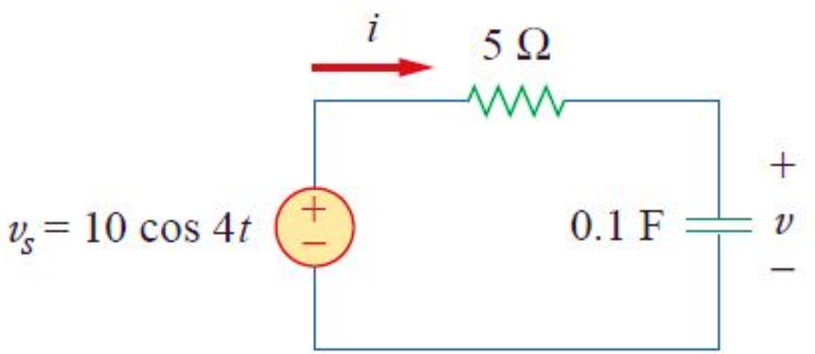
Impedance

- In the expression $\mathbf{Z} = R \pm jX$; the reactance may be positive or negative.
- Impedance is inductive when X is positive otherwise it is capacitive.
- Thus $\mathbf{Z} = R + jX$ is said to be inductive or lagging since current lags voltage.
- $\mathbf{Z} = R - jX$ is said to be capacitive or leading since current leads voltage.
- The impedance, resistance and reactance are measured in terms of ohms.
- In polar form the impedance can be expressed as $\mathbf{Z} = |\mathbf{Z}| \angle \theta$;
- $\mathbf{Z} = R + jX = |\mathbf{Z}| \angle \theta$ where $|\mathbf{Z}|=\sqrt{R^2 + X^2}$ and $\theta = \tan^{-1}(X/_R)$;
- $R = |\mathbf{Z}| \cos\theta$ and $X = |\mathbf{Z}| \sin\theta$.

Admittance

- It is the reciprocal of impedance and measured in siemens (S or mho).
- The admittance $\mathbf{Y}$ of an element is the ratio of phasor current through it to the phasor voltage across it. $\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$.
- As a complex quantity $\mathbf{Y} = \mathrm{G} + j\mathrm{B}$; where G is conductance and B is susceptance.
- Admittance, conductance and susceptance are expressed in siemens.
- $G + jB = \frac{1}{R+jX}$ on simplifying by using rationalization and separating real and imaginary parts
- $G = \frac{R}{R^2+X^2}$ and $B = -\frac{X}{R^2+X^2}$. Showing that $G \neq {}^1/_R$ as it is in resistive circuits. Of course if X=0, then $G = {}^1/_R$.
- Kirchoffs law also valid frequency domain (*phasor domain*).

Example: Find $i(t)$ and $v(t)$ for the circuit shown below.



From the voltage source $10\cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$
$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}$$

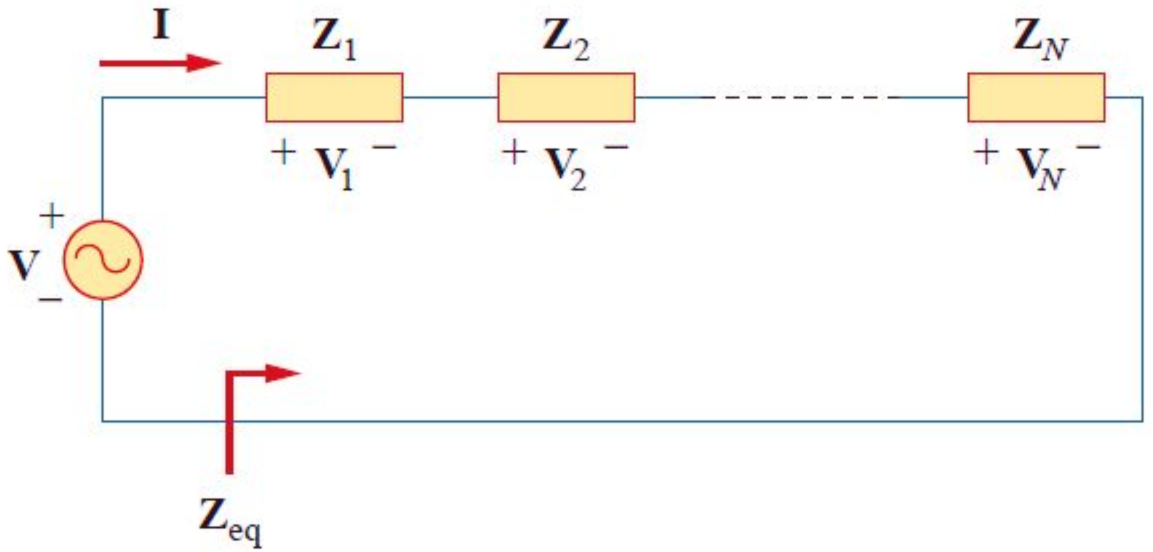
The voltage across the capacitor is

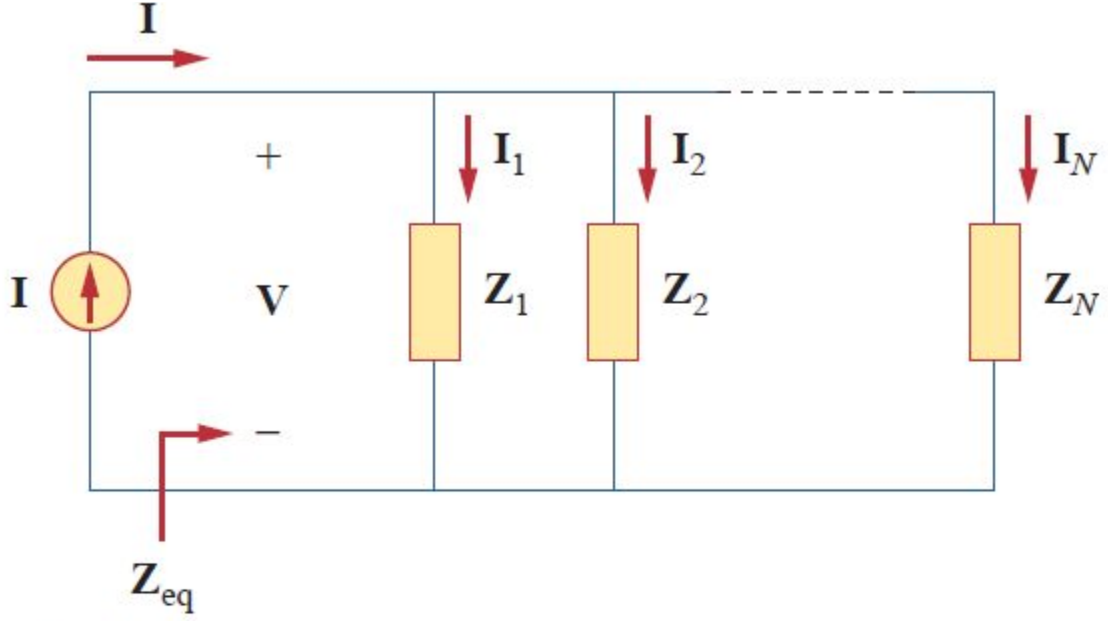
$$\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$
$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}$$

$$i(t) = 1.789\cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47\cos(4t - 63.43^\circ) \text{ V}$$

Thus, $i(t)$ leads $v(t)$ by 90 degree.

Impedance Combinations





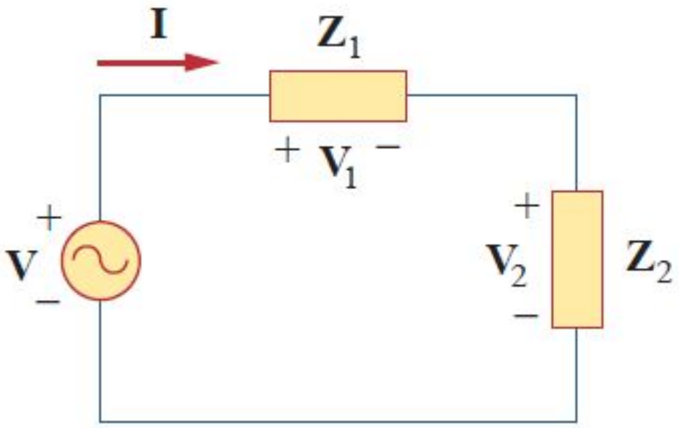
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

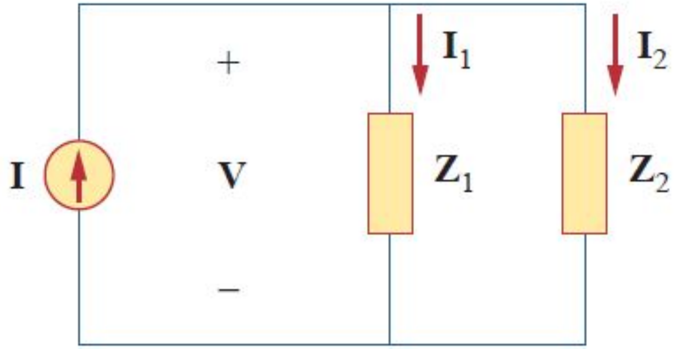
$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

$$\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N$$



$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

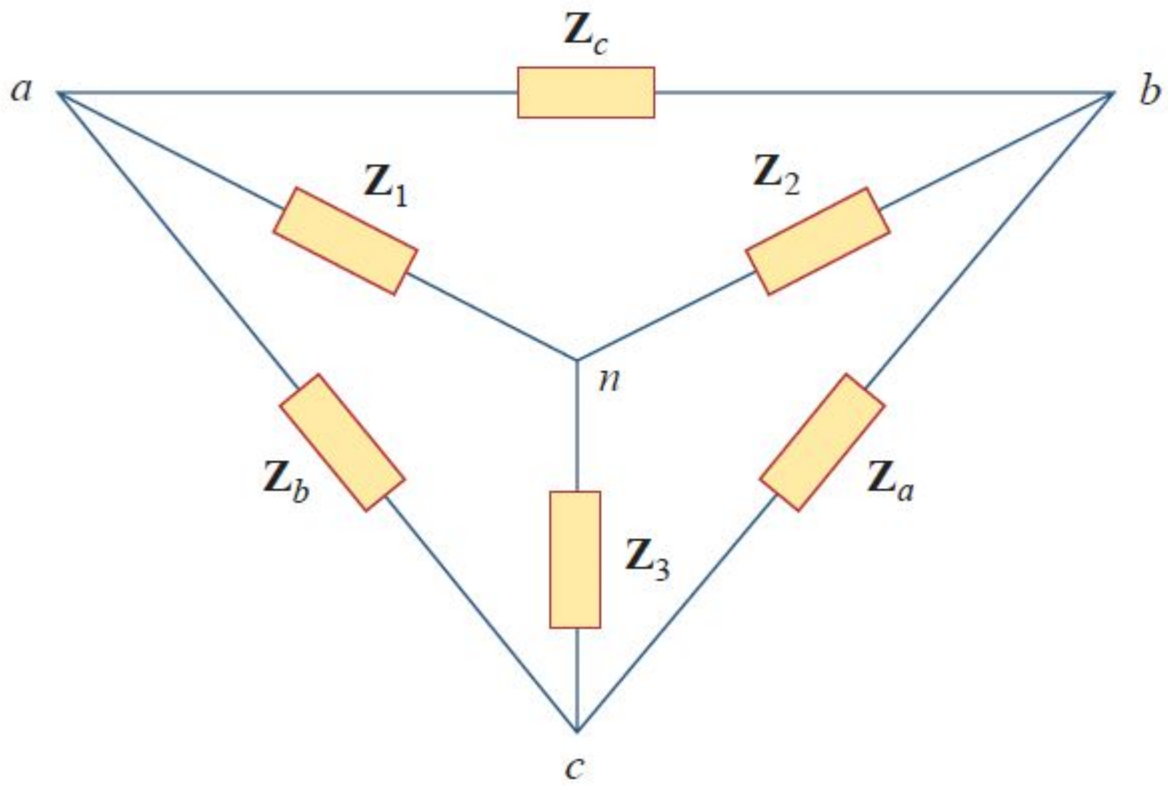
$$\mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$



$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

$$\mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

Star-Delta Networks



Y − Δ Conversion

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ − YConversion

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

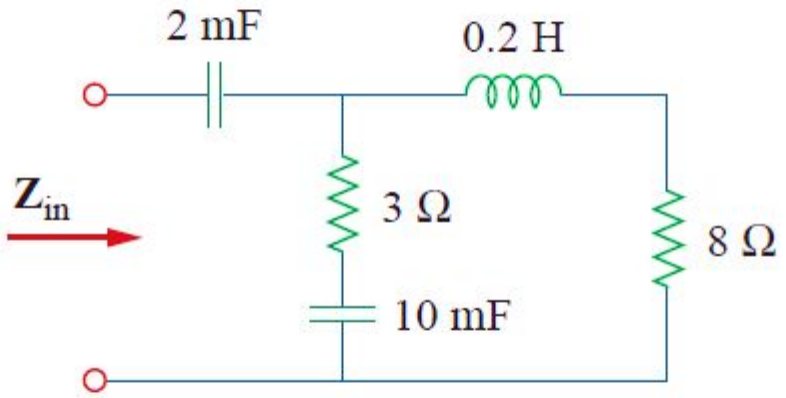
$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

Under Balanced Condition

A delta or wye circuit is said to be balanced if it has equal impedances in all three branches.

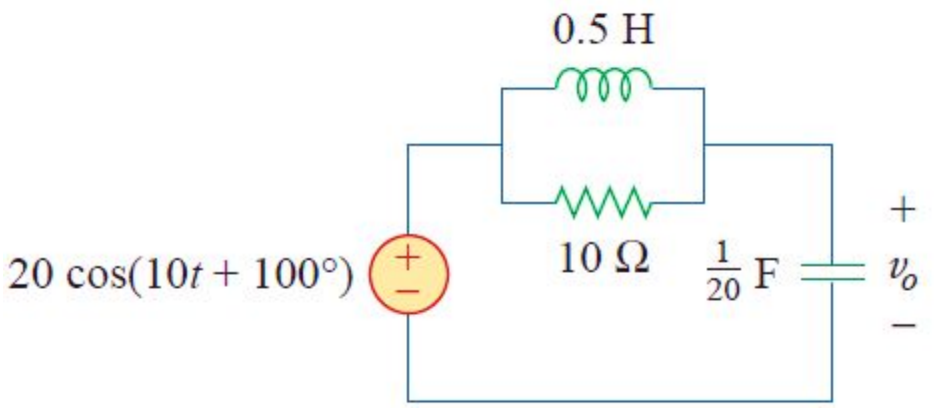
$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

Example: Find the input impedance of the circuit shown in figure below. Assume $\omega = 50\ rad/s$



$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$

Example: Calculate v_o in the circuit



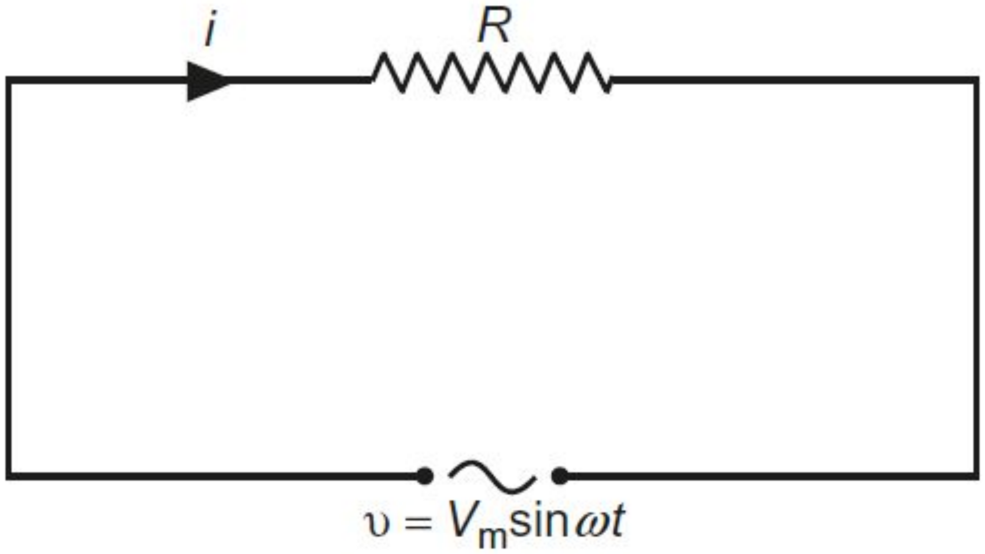
$$v_o(t) = 14.142 \cos(10t - 35°) \text{ V}$$

Single phase AC circuits

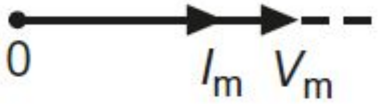
- In DC circuits, the opposition to the flow of current is only the resistance of the circuit. While in AC circuits, the opposition to the flow of current is due to Resistance (R), Inductive Reactance (X_L) and Capacitive Reactance (X_C) of the circuit.
- In AC circuits, frequency plays an important rule, the currents and voltages are represented with magnitude and direction (phasors).
- The voltage and current may or may not be in phase with each other depending upon the parameters (R, L and C) of the circuit.
- Moreover, in AC circuits, the currents as well as voltages are added and subtracted vectorially instead of arithmetically as in DC circuits.

AC circuit containing Resistance only

- Alternating voltage applied to resistor as shown in figure, the current and voltage are given by: $v = V_m \sin \omega t$ and $i = \frac{v}{R} = \frac{V_m}{R} \sin \omega t$



- The value of current is maximum @ $\omega t = 90^o$.
- From the phasor diagram, the current and voltage are in same phase.

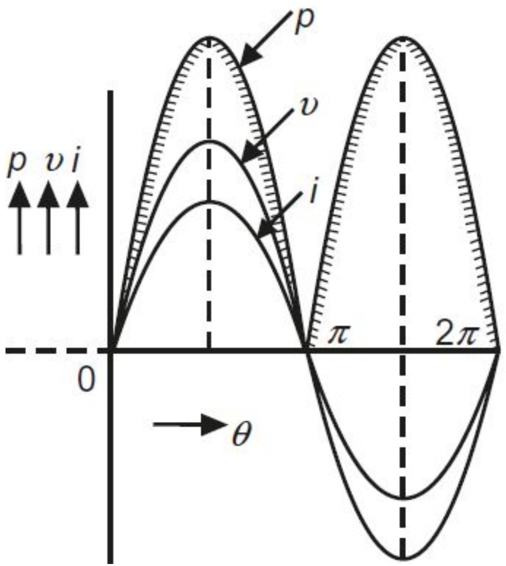


Phasor diagram

- Instantaneous power:
- $p = vi$ = $(V_m \sin \omega t)\,(I_m \sin \omega t) = (V_m I_m)(\sin^2 \omega t)$

$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} (1 - \cos 2\omega t) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} - \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos 2\omega t$$

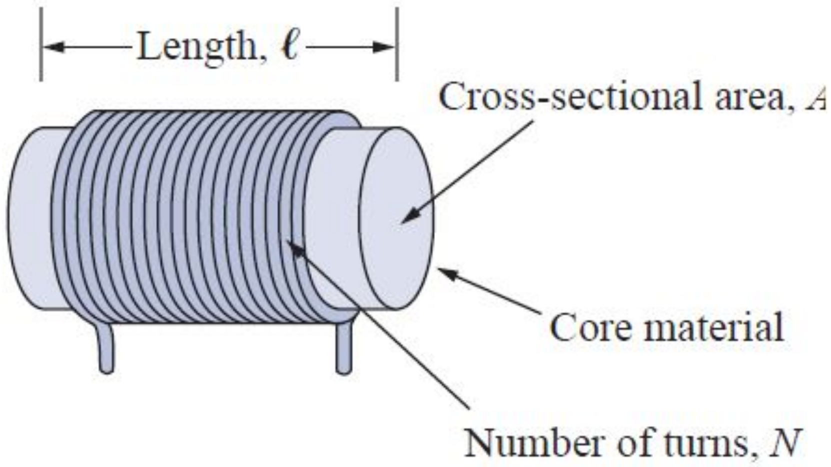
$$\mathrm{P} = V_{rms} I_{rms} - 0 \text{ or } P = VI$$



Inductor

- An inductor is a passive element designed to store energy in magnetic field.
- A practical inductance is called inductor. It is a coil wound on a magnetic core (may be air core for small values of inductance).
- A magnetic core inductor has constant inductance only in a limited range of current (at high current the core saturates and inductance reduces).
- Inductor is linear in a limited range of currents.
- In electronic circuits the use of inductor is avoided except in high power circuits. In fact, inductance cannot be fabricated as such in semiconductor integrated circuits.

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

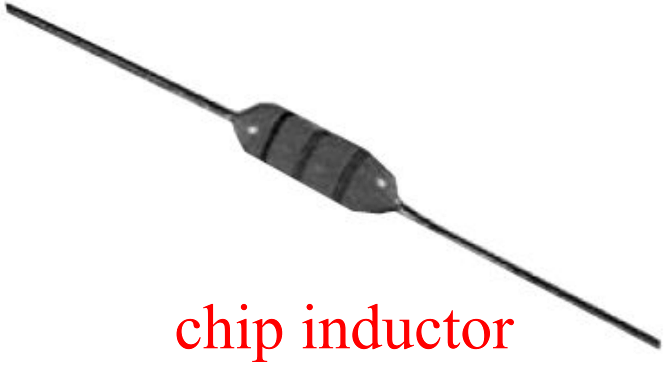


Inductor

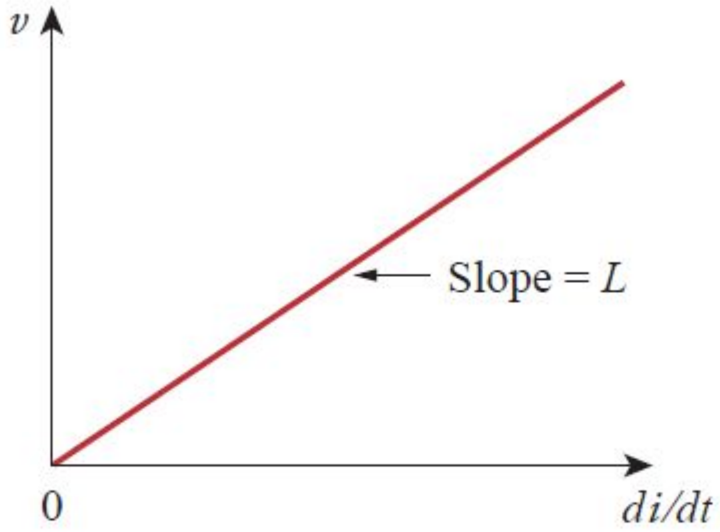
- Inductance of inductor depends on its physical dimension and construction.
- $L = \frac{N^2 \mu A}{l}$ Inductance of a solenoid inductor, where N is the number of turns, l is length, A is cross-sectional area and μ is the permeability of the core.
- The inductance can be increased by increasing number of turns od the coil, using material with high permeability as the core, increasing the cross-sectional area or reducing the length of the coil.
- The terms coil and choke are also used for inductors.

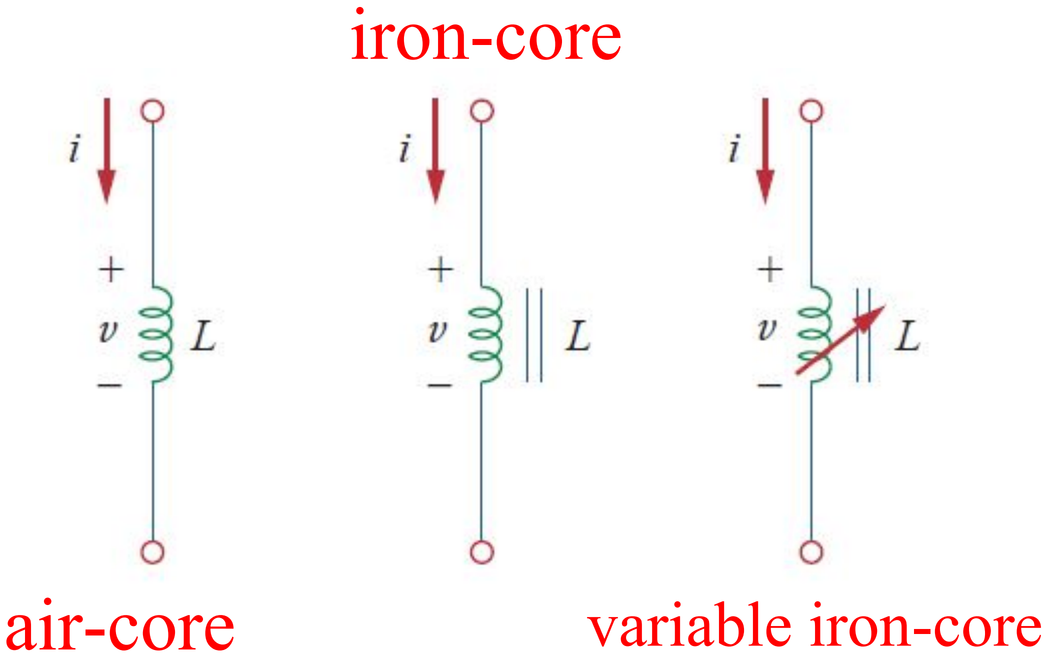
solenoidal wound inductor
(a)

toroidal inductor
(b)



chip inductor
(c)

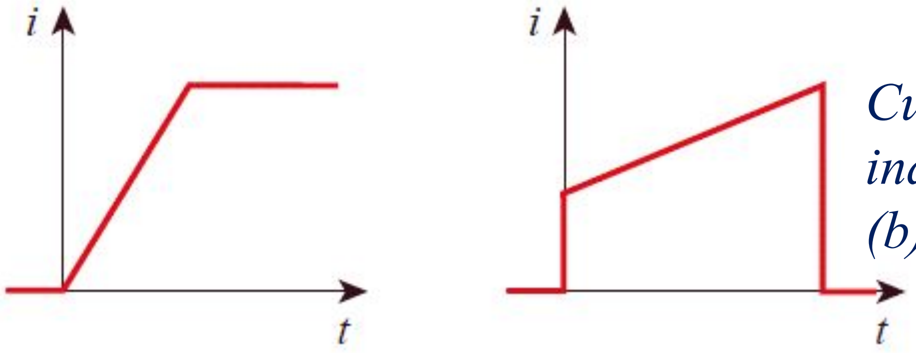




- vi relation of inductor is $v = L\dfrac{di}{dt}$.
- For an inductor whose inductance is independent of current. Such an inductor is known as linear inductor.
- For a non-linear inductor, the slope is not a constant because its inductance varies with current.

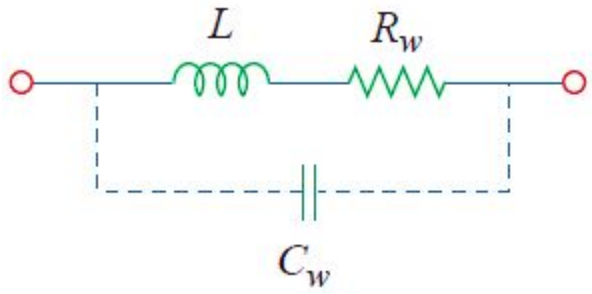
Important properties of inductor

- The voltage across an inductor is zero when current is constant. (An inductor acts like short circuit for DC).
- The current through inductor can’t change instantaneously. (A discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible)
- Ideal inductor does not dissipate the energy. The energy stored in it can be retrieved at a later time.
- A practical, nonideal inductor equivalent circuit is shown in figure below.

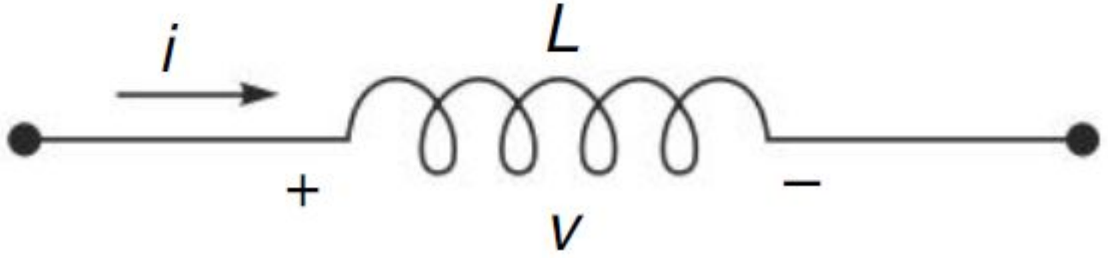


(a) (b)

Current through an inductor: (a) allowed, (b) not allowable



Inductance



- It is a two-terminal storage element in which energy is stored in the magnetic field.
- The changing magnetic field set up by the time varying current through the inductance reacts to induce voltage in it to oppose the change of current.
- $v = L\frac{di}{dt}$ where L is inductance (H), on Integrating $i = \frac{1}{L}\int_0^t v\,dt + i(0)$
- Where i(0) is inductance current at t=0. For convenience i(0)=0.
- Current through inductance cannot change instantly.
- Linearity of Inductance: $v = \alpha_1 v_1 + \alpha_2 v_2$

$$i = \frac{1}{L}\int_0^t [\alpha_1 v_1 + \alpha_2 v_2]\ \mathrm{d}t = \alpha_1 \frac{1}{L}\int_0^t v_1\,\mathrm{d}t + \alpha_2 \frac{1}{L}\int_0^t v_2\ \mathrm{d}t = \alpha_1 i_1 + \alpha_2 i_2$$

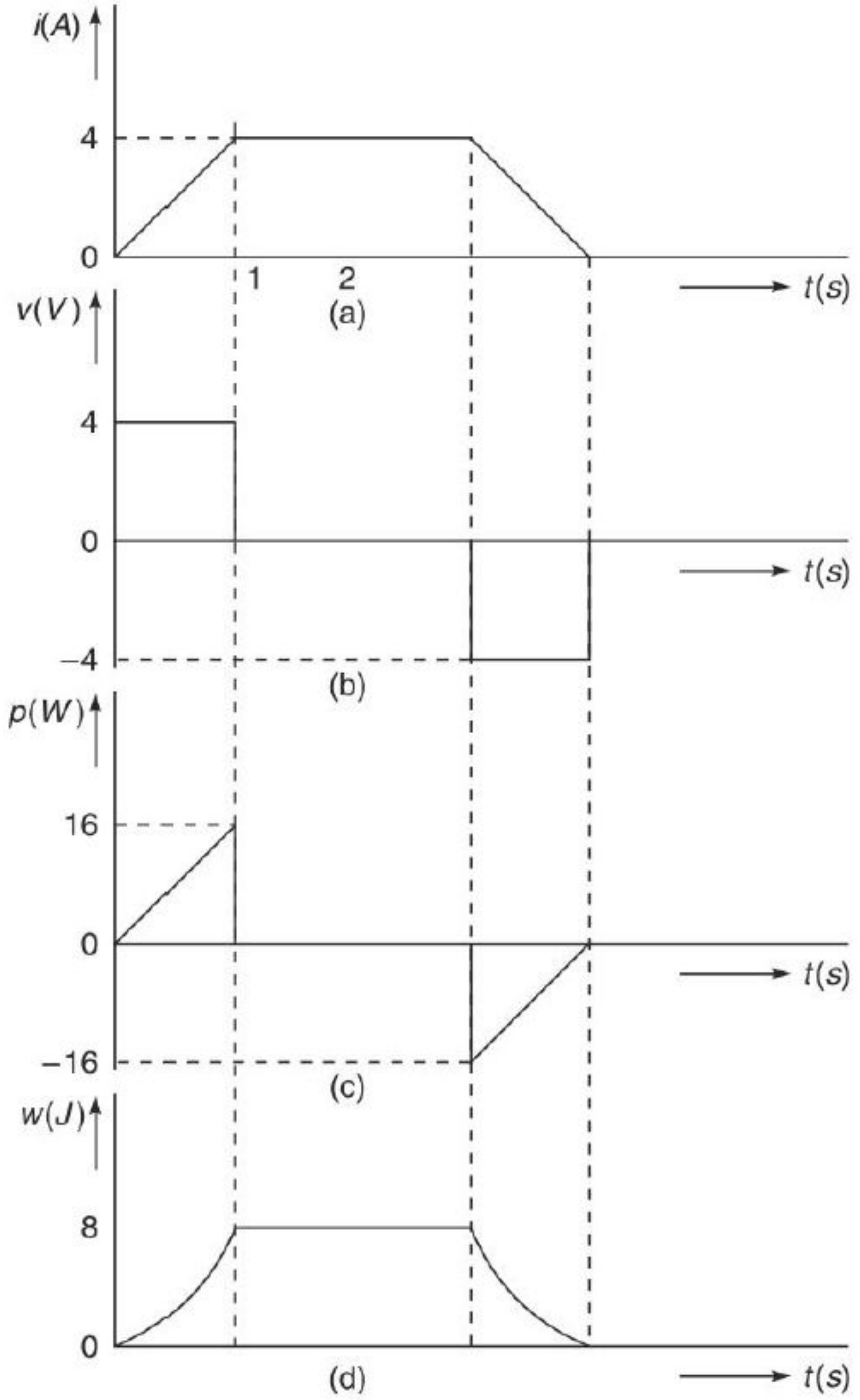
Energy stored in inductance

- Power fed to inductance is $p = vi = L\frac{di}{dt}$
- The stored energy is found by integrating (p) as $w_L = \int_0^t p dt = \int_0^t L\frac{di}{dt} dt$
- $w_L = L\int_0^t i dt = \frac{1}{2} L i^2$ (J)
- The energy stored in the inductance depends upon the instantaneous current and is independent of history of the current.

Example

➢Consider that an inductance of 1H is excited with current waveform shown in figure find voltage across it, draw the power waveform and stored energy waveform.

- Voltage w/f of upto 1s, the i(t) rises to 4 A at rate of 4A/s. so the voltage $v = L\frac{di}{dt} = 1 * 4 = 4V$.
- From 1 to 3s, the rate of change of current is zero, and so voltage is zero.
- From 3 to 4s, the current reduces to zero at rate of 4 A/s, so the voltage is constant -4V.
- Power is product of voltage and current$p = vi$ (P=4*4=16w)
- Energy stored = $W_L = \frac{1}{2}Li^2 = \frac{1}{2}1(4^2) = 8$J.



Example

- Find the current through a 5H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also, find the energy stored at t=5s, Assume $i(v)$>0.

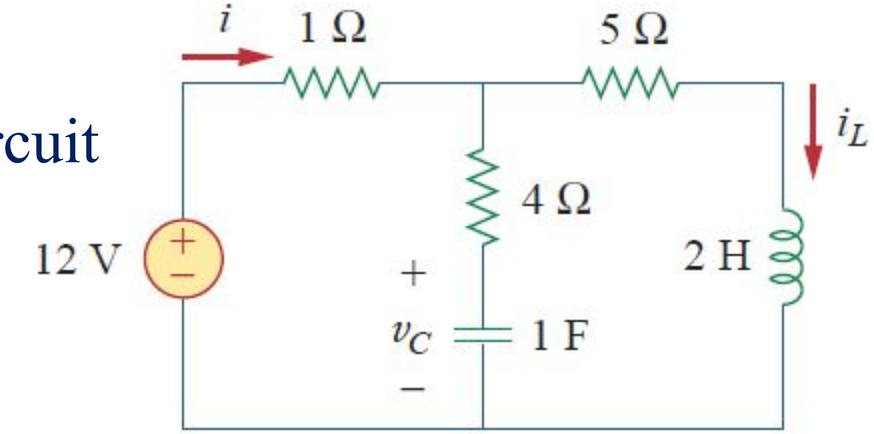
Since $i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_{0}^{t} 30t^2\,dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_{0}^{5} 60t^5\,dt = 60\frac{t^6}{6}\Big|_0^5 = 156.25 \text{ kJ}$$

Consider the circuit in Fig. Under dc conditions, find: (a) i, v_c and i_L (b) the energy stored in the capacitor and inductor.

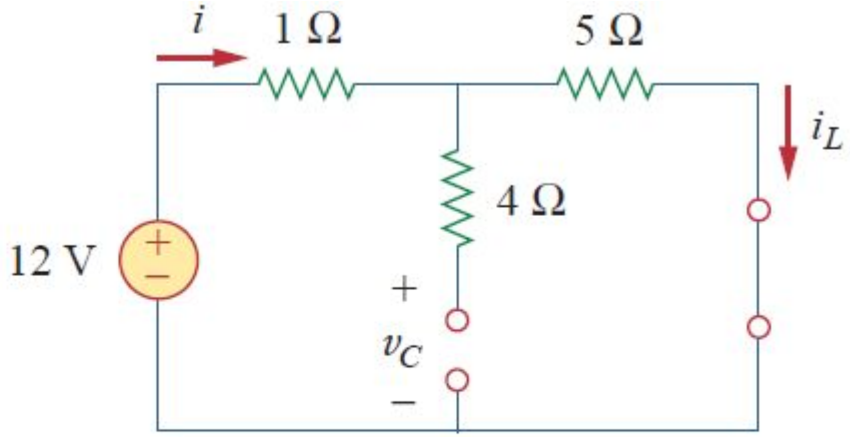


Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit

$$i = i_L = \frac{12}{1+5} = 2\text{ A}$$

The voltage v_C is the same as the voltage across the 5-Ω resistor. Hence,

$$v_C = 5i = 10\text{ V}$$



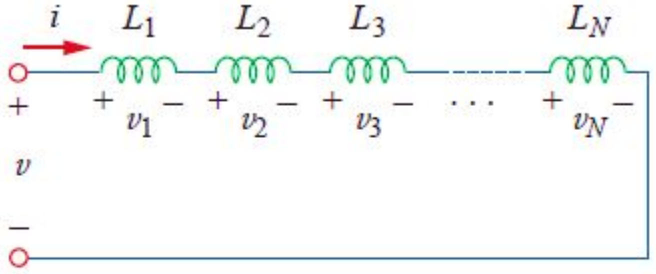
(b) The energy in the capacitor is

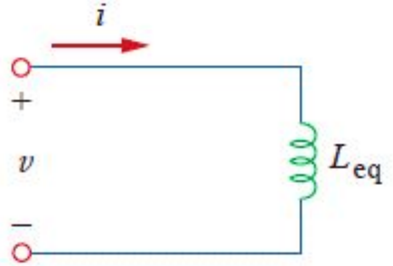
$$w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\text{ J}$$

and that in the inductor is

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4\text{ J}$$

Series connected Inductors



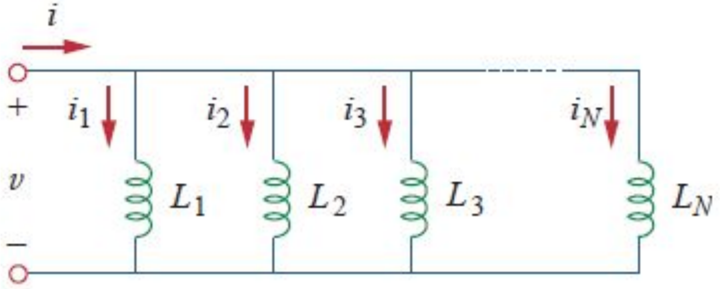


- Consider a series of N inductors as shown in Fig. a and its equivalent circuit is shown in Fig. b.
- The inductor have the same current through them. Apply KVL to loop. $v = v_1 + v_2 + v_3 + \cdots + v_N$.
- Substituting $v_k = L_k \, {}^{di}/_{dt}$results in
- $v = L_1 \, {}^{di}/_{dt} + L_2 \, {}^{di}/_{dt} + L_3 \, {}^{di}/_{dt} + \cdots + L_N \, {}^{di}/_{dt}$.
- $v = (L_1 + L_2 + L_3 + \cdots + L_N) \dfrac{di}{dt}$
- $v = \left(\sum_{k=1}^{N} L_k\right) \dfrac{di}{dt} = L_{eq} \dfrac{di}{dt}$; where $L_{eq} = L_1 + L_2 + \cdots + L_N$

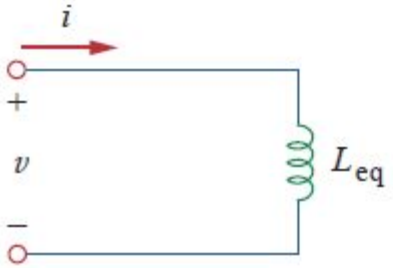
The equivalent inductance of series-connected inductors is the sum of the individual inductances

Parallel connected inductors

- Parallel connected inductors have same voltage across them. Using KCL: $i = i_1 + i_2 + i_3 + \cdots + i_N$.
- But we know that $i_k = \frac{1}{L_k}\int_{t_0}^{t} v\,dt + i_k(t_0)$.
- $i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0) + \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$

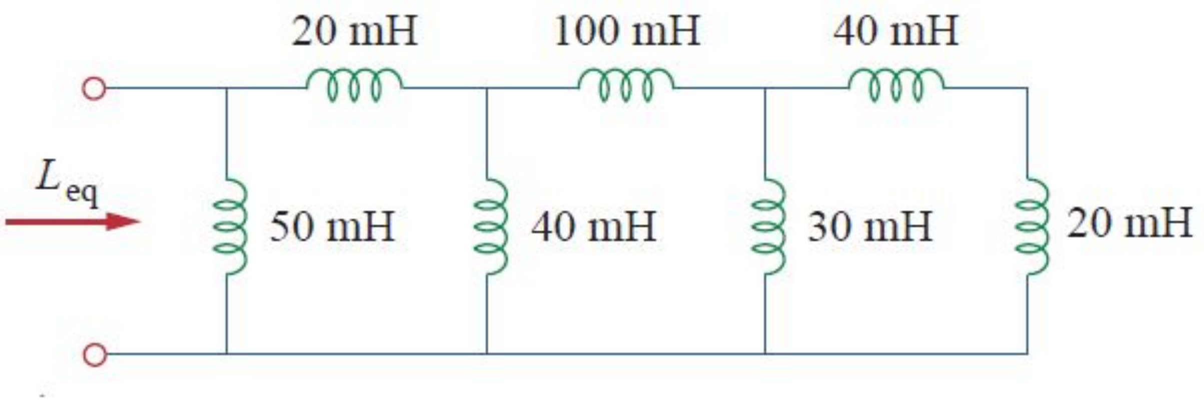


- $i = \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_o) + i_2(t_o) + \cdots + i_N(t_0)$.
- $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$



- The equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.

Calculate the equivalent inductance for the inductive ladder network in Fig.



Ans: 25 mH

AC circuit containing Pure Inductance only

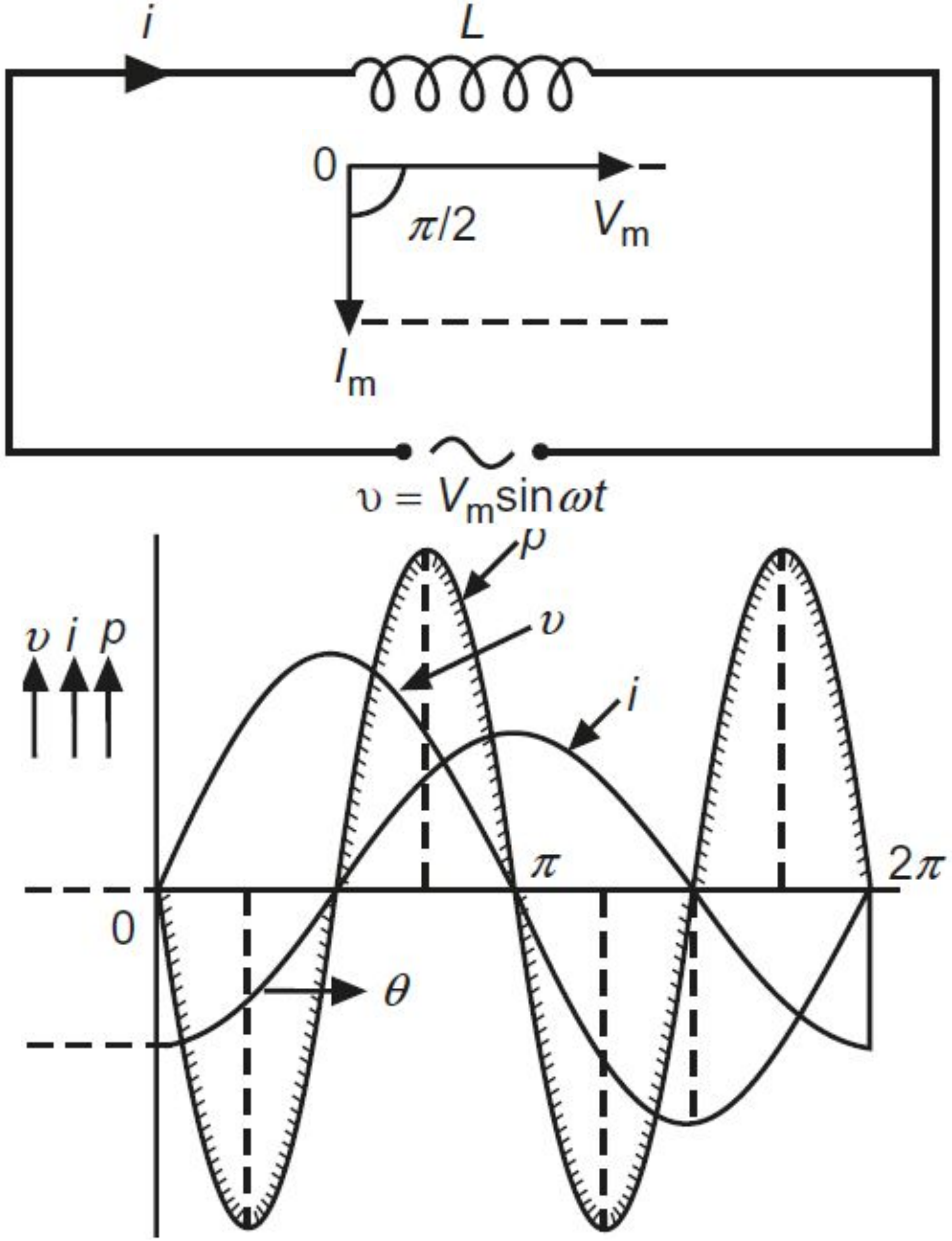
- Let an alternating voltage $v(t)=V_m \sin \omega t$ is applied to a pure inductor then an alternating current flows through inductance and induces an emf, i.e $e=-L(di/dt)$;
- This induced emf is equal and opposite to the applied voltage i.e, $v=-e=-L(di/dt)$;

$$V_m \sin \omega t = L\frac{di}{dt} \quad \text{or} \quad di = \frac{V_m}{L} \sin \omega t \, dt$$

$$\int di = \int \frac{V_m}{L} \sin \omega t \, dt \quad \text{or} \quad i = \frac{V_m}{\omega L}(-\cos \omega t)$$

$$i = \frac{V_m}{\omega L} \sin(\omega t - \pi/2) = \frac{V_m}{X_L} \sin(\omega t - \pi/2)$$

where $X_L = w L$ is the opposition offered to the flow of AC by a pure inductance and it is called inductive reactance



AC circuit containing Pure Inductance only

- The value of current will be maximum when sin (ω t - π/2) = 1; i.e., $I_m = V_m / X_L$. $i = I_m \sin(\omega t - \pi/2)$
- From the phasor diagram, it is clear that current flowing through pure inductive circuit lags behind the applied voltage *v* by 90^O.

Instantaneous power, $p = vi = V_m \sin \omega t \times I_m \sin(\omega t - \pi/2)$

$$= V_m I_m \sin \omega t \cos \omega t = \frac{V_m I_m}{2} 2 \sin \omega t \cos \omega t$$

$$= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \sin 2\omega t$$

It is very clear that average power in a half cycle (one alternation) is zero, as the negative and positive loop area under the power curve is the same.

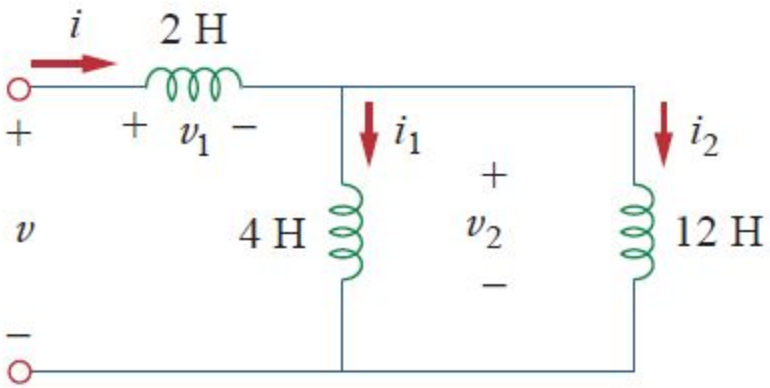
Average Power consumed in the circuit over a complete cycle,

$$P = \text{average} \frac{V_m}{\sqrt{2}} . \frac{I_m}{\sqrt{2}} \sin 2\omega t = 0$$

Average Power consumed in the circuit over a complete cycle,

It is interesting to note that during the first quarter cycle, whatever power (or energy) is supplied by the source to the inductance (or coil) is stored in the magnetic field set-up around it. However, in the next quarter cycle, the magnetic field collapses and the power (or energy) stored in the field is returned to the source. This process is repeated in each and every alternation. Hence, no power or energy is consumed in this circuit.

For the circuit shown in Figure below, $i(t) = 4(2 - e^{-10t})mA$. If $i_2(0) = -1\ mA$, find: (a) $i_1(0)$; (b) $v(t), v_1(t)$and $v_2(t)$; (c) $i_1(t)$and $i_2(t)$.



(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \| 12 = 2 + 3 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV} = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV} = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\, dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\, dt + 5 \text{ mA}$$
$$= -3e^{-10t}\Big|_0^t + 5 \text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA}$$

Similarly,

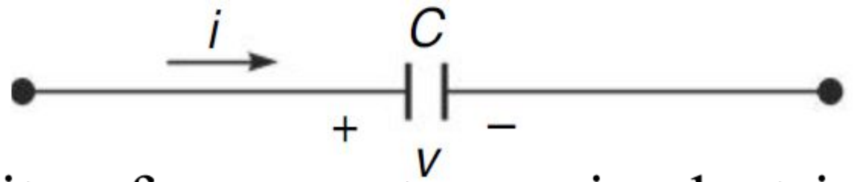
$$i_2(t) = \frac{1}{12}\int_0^t v_2\, dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\, dt - 1 \text{ mA}$$
$$= -e^{-10t}\Big|_0^t - 1 \text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA}$$

Note that $i_1(t) + i_2(t) = i(t)$.

Capacitor

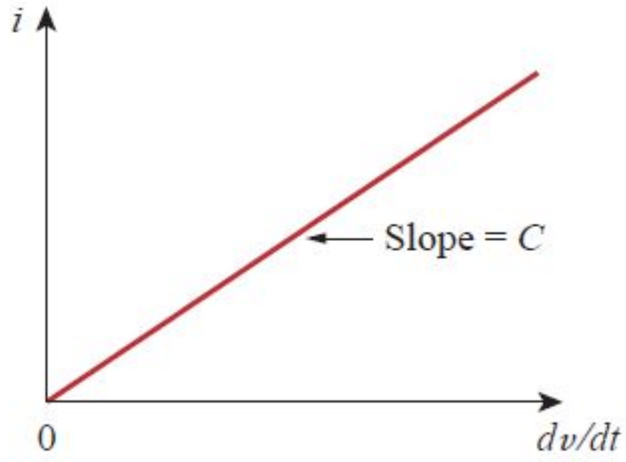
- A Capacitor is passive element designed to store energy in its electric field.
- The capacitor is said to store the electric charge.
- An element possessing the property of capacitance is known as a capacitor.
- A capacitor consists of two conducting plates separated by an insulator (dielectric).
- Capacitance of a parallel plate capacitor is given by $C = \varepsilon_o \varepsilon_r \frac{A}{d}$
- ε_o is permittivity of free space; $8.854\text{x}10^{-12}$; and ε_r is 1.
- Note 1: Larger the area, greater the capacitance.
- Note 2: Smaller the spacing, greater the capacitance.
- Note 3: Higher the permittivity, greater the capacitance.

Capacitance



- It is a two-terminal element that has the capability of energy storage in electric field. The stored energy can be fully retrieved.
- There is a voltage drop in the direction of current with the terminal where the current flows in acquiring positive polarity with respect to the terminal at which the current leaves the element.
- When a voltage source connected to capacitor, the source deposits +ve charge on one plate and –ve charge on other plate.
- The current through capacitor is given by $i = c\frac{dv}{dt}$(A) where C is in Farad.
- On integrating $v = \frac{1}{c}\int_0^t i\,dt + v_c(0)$ where $v_c(0)$ is capacitance voltage at t=0; For an initially uncharged capacitor, $v_c(0)=0$; so that $c = \frac{q}{v}$ as $q = \int_0^t i\,dt$
- The voltage (or charge) of a capacitance cannot change instantly.
- Capacitance is the ratio of charge on one plate of capacitor to voltage difference between two plates.
- Linearity of capacitor is demonstrated by $v = \alpha_1 v_1 + \alpha_2 v_2$

vi relation in a capacitor

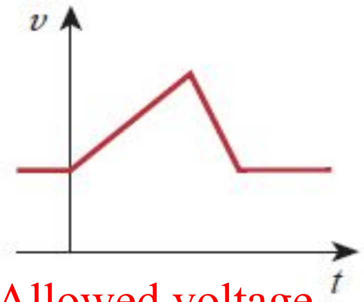


Energy stored in capacitance

- Power fed into capacitance is $p = vi = Cv\frac{dv}{dt}$, integrating and assuming initially uncharged capacitance (*v*=0 at t=0), the stored energy is found to be $W_c = \int_0^t p\, dt = C\int_0^t v\frac{dv}{dt}\, dt = C\int_0^t v dv$
- $W_c = \frac{1}{2}Cv^2$(J)
- Energy stored in capacitance is a function of its (instantaneous) voltage magnitude and is independent of the history of how this voltage is reached.
- As the voltage reduced to zero, all the energy stored in the capacitor is returned to the circuit in which the capacitor is connected.
- As per passive sign convention, if $v>0$ and $i>0$, $v<0$ and $i<0$, the capacitor is being charged. If $vi<0$, the capacitor is discharging.

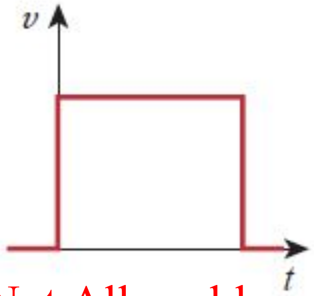
Important Properties of capacitor

- Voltage across capacitor is not changing with time (i.e, dc voltage), the current through capacitance is zero. Thus a capacitor is an open circuit to DC. However, if a battery (DC supply) is connected across a capacitor, the battery charges.



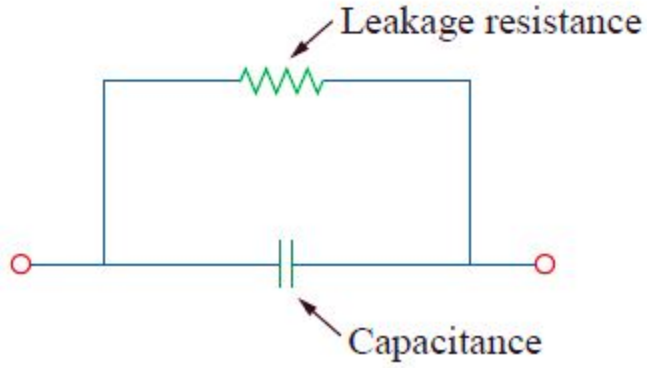
Allowed voltage across a capacitor

- The voltage on the capacitor must be continuous. The voltage on capacitor cannot change abruptly. The capacitor resists abrupt change in the voltage across it. A continuous change in voltage requires infinite current, which is impossible.



Not Allowable voltage across a capacitor

- The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
- Non-ideal capacitor has a parallel-model leakage resistance. The parallel resistance is as high as 100 MΩ and can be neglected for most practical applications.



Example

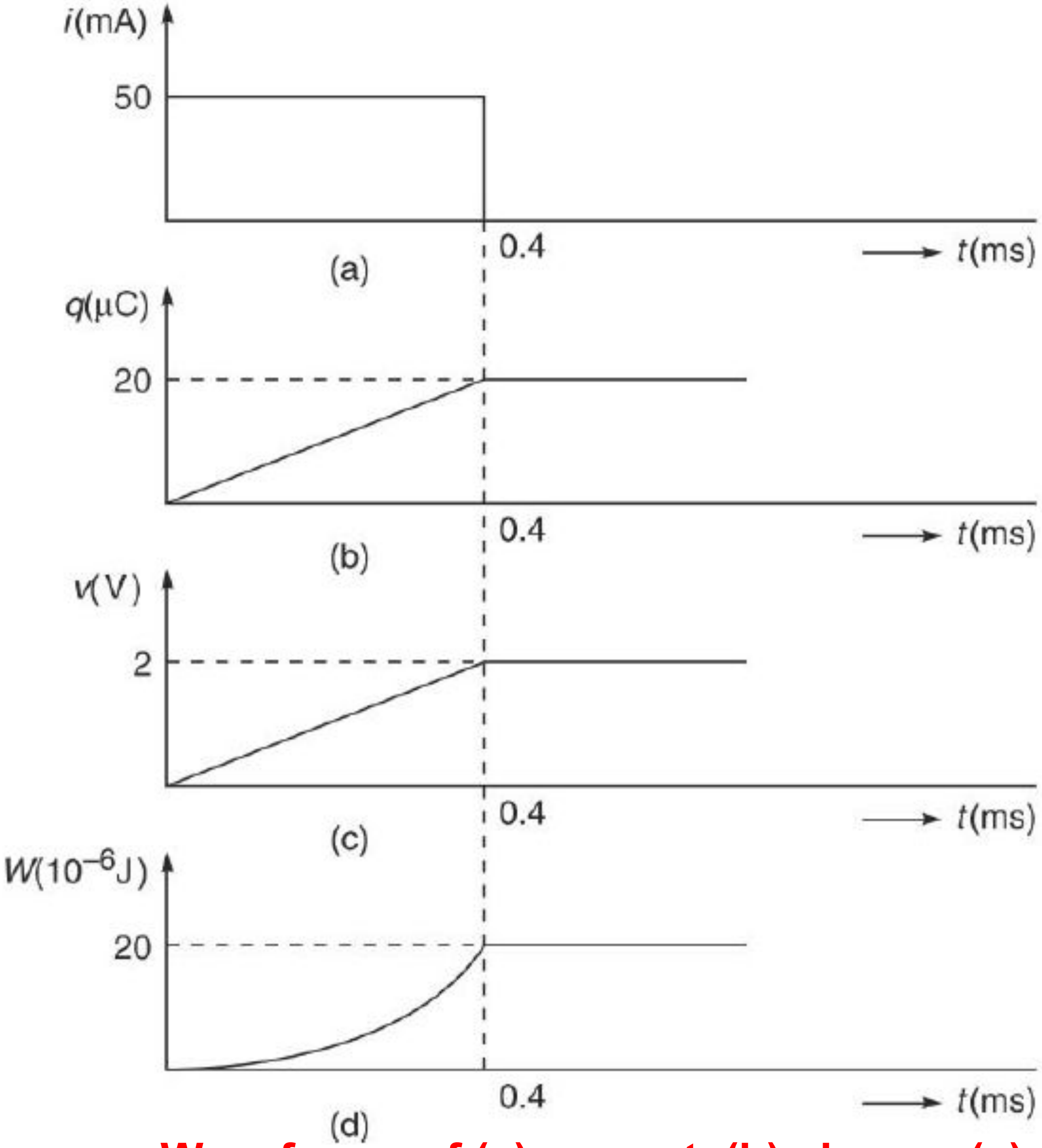
- Consider a capacitor of 10 μF to which is applied 50 mA rectangular pulse of duration 0.4 ms as shown in figure. Then find charge, capacitor voltage and energy stored in the capacitor.

$$q = \int_0^t i \; dt = 50 \int_0^{0} dt = 50\, dt$$

$$Q = 50 \text{ mA} \times 0.4 \text{ ms} = 20 \text{ μC}$$

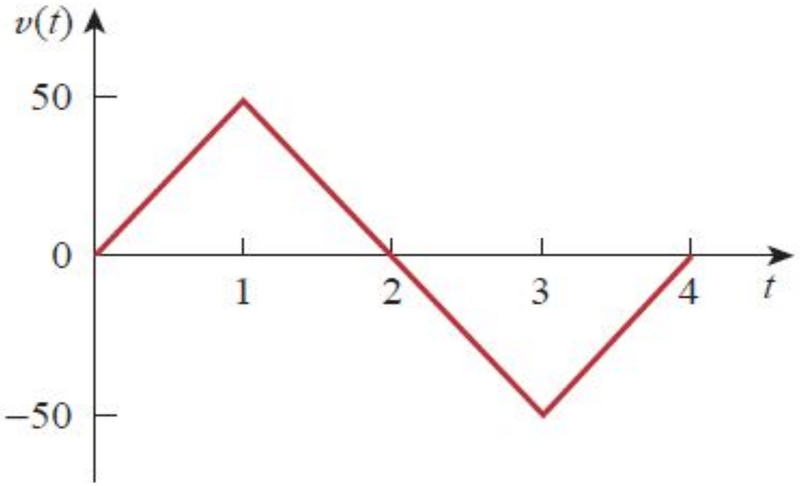
$$v = \frac{20 \text{ μC}}{10 \text{ μF}} = 2 \text{ V}$$

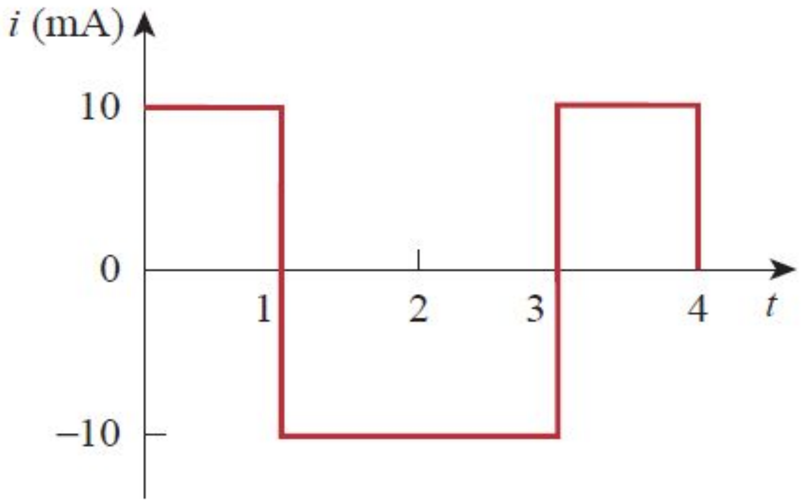
$$w = \frac{1}{2} C v^2 \qquad w = \frac{1}{2} \times (10 \text{ μF}) \times (2)^2 = 20 \text{ μJ}$$



Waveforms of (a) current, (b) charge, (c) voltage and (d) energy in a capacitor

- Determine the current through a 200 micro farad capacitor whose voltage is as shown in Figure.





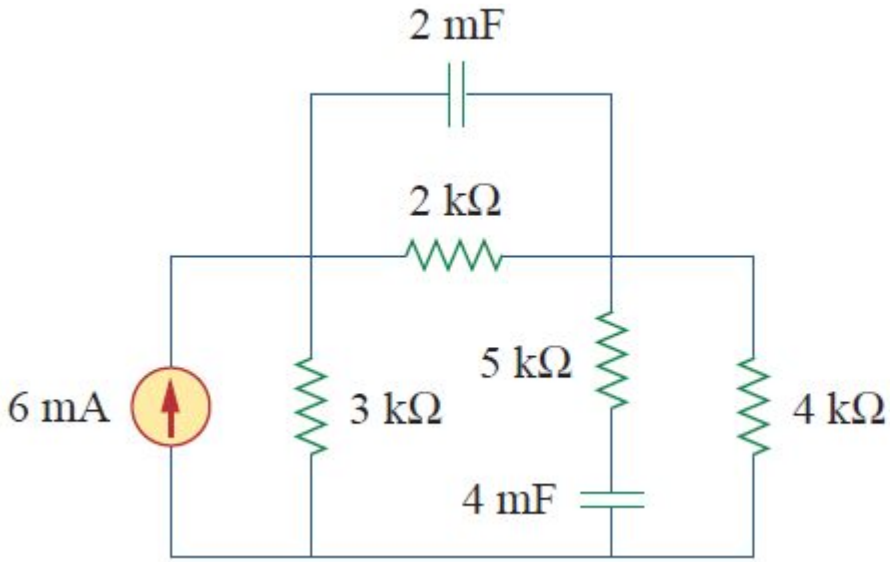
The voltage waveform can be described mathematically as

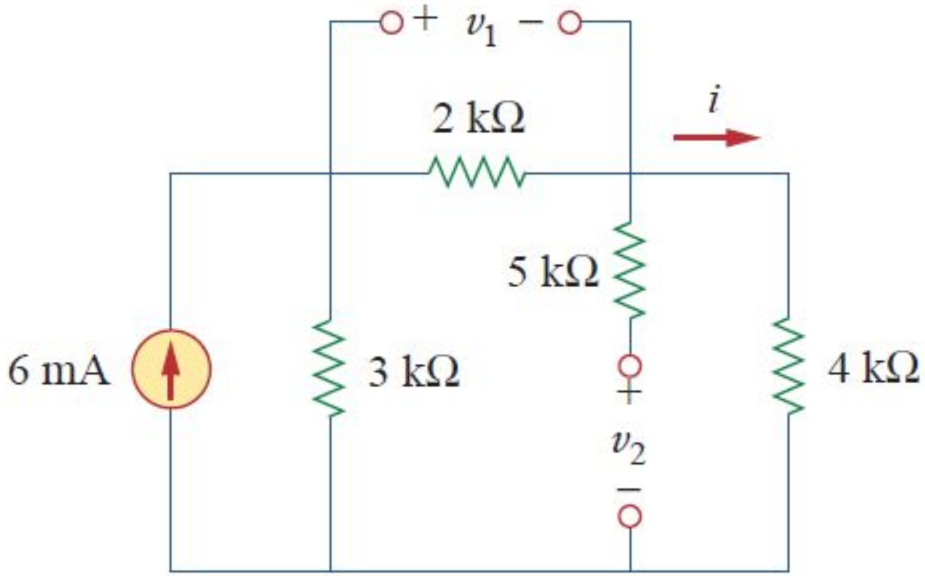
$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

i is the derivative of v, hence $i = C\dfrac{dv}{dt}$

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 10 \text{ mA} & 0 < t < 1 \\ -10 \text{ mA} & 1 < t < 3 \\ 10 \text{ mA} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Example: Obtain the energy stored in each capacitor in Figure under dc conditions.





Under dc conditions, we can replace each capacitor with an open circuit.

$$i = \frac{3}{3+4+2} * 6mA = 2\ mA$$

Hence, the voltage *v*1 and *v*2 across capacitor are

$$v_1 = 2000i = 4V \ and \ v_2 = 4000i = 8V$$

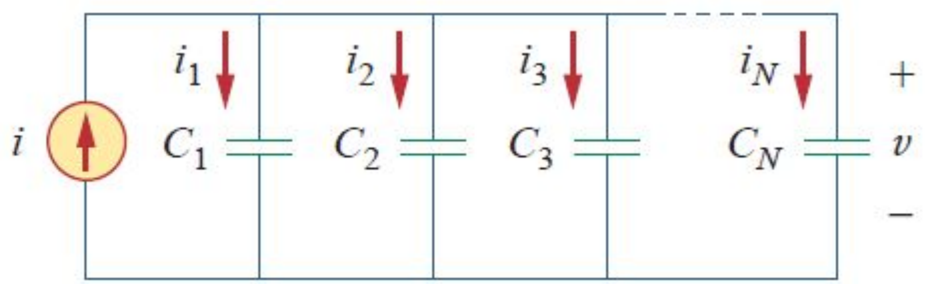
$$w_1 = \frac{1}{2}Cv_1^2 = 16\ mJ \ and \ w_2 = \frac{1}{2}Cv_2^2 = 128\ mJ$$

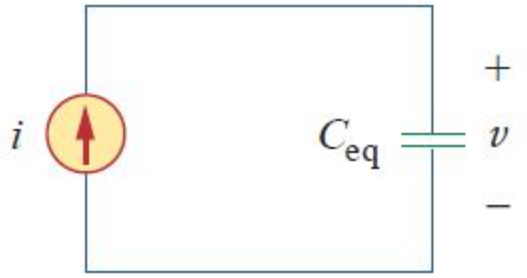
Observations

- The differential-integral equations governing the elemental behavior of inductance and capacitance
- Inductance $v = L\frac{di}{dt}; i = \frac{1}{L}\int V\,dt$
- Capacitance $i = C\frac{dv}{dt}; v = \frac{1}{c}\int i\,dt$
- From the above relations $v \leftrightarrow i$; $L \leftrightarrow C$; This interchangeability is Duality.
- Duality is extended to resistance by interchanging $R \leftrightarrow G$;

Parallel connection of capacitors

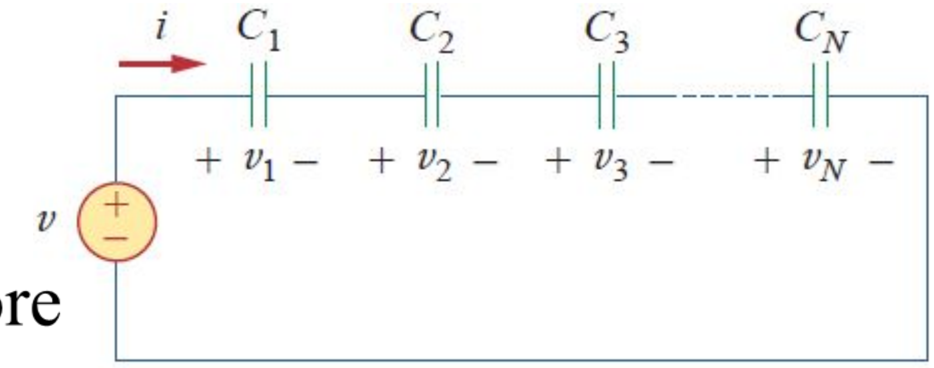
- From circuit, $i = i_1 + i_2 + i_3 + \cdots + i_N$
- We know $i = C\frac{dv}{dt}$, Hence
- $i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + C_3\frac{dv}{dt} + \cdots + C_N\frac{dv}{dt}$
- $i = \sum_{k=1}^{N} C_k\left(\frac{dv}{dt}\right) = C_{eq}\frac{dv}{dt}$.
- $C_{eq} = C_1 + C_2 + C_3 + \ldots + C_N$



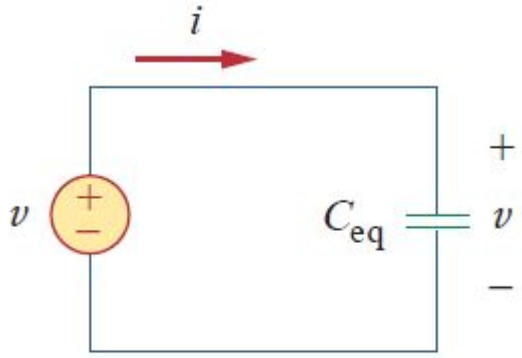


The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances

Series connection of capacitors



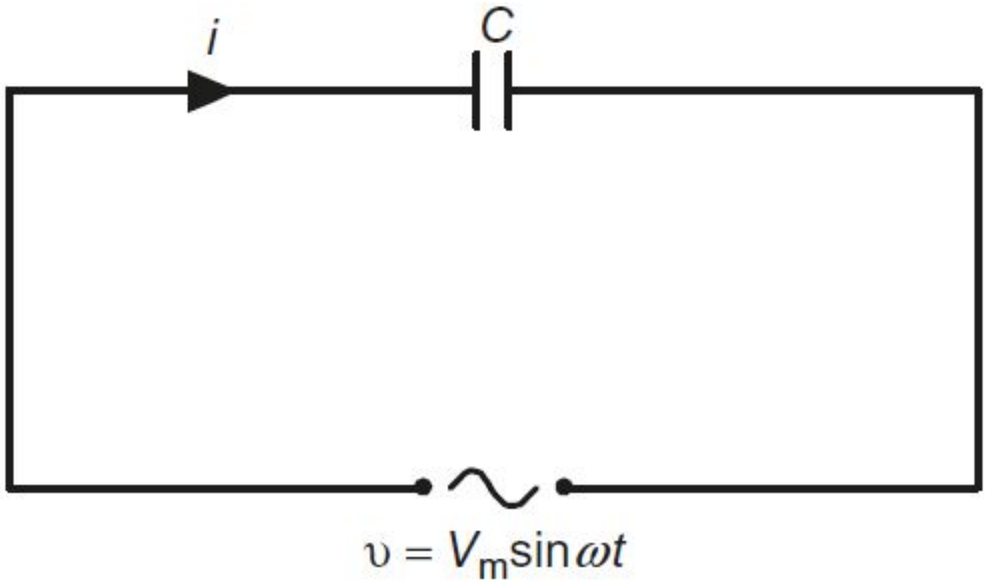
- From circuit, $v = v_1 + v_2 + v_3 + \cdots + v_N$
- We know $v_k = \frac{1}{C}\int_{t0}^{t} i(t)dt + v_k(t_o)$, Therefore
- $v = \frac{1}{C1}\int_{t0}^{t} i(t)\ dt + v_1(t_o) + \frac{1}{C2}\int_{t0}^{t} i(t)\ dt + v_2(t_o)$
- $= (\frac{1}{C1} + \frac{1}{C2})\int_{t0}^{t} i(t)\ dt + v_1(t_o) + v_2(t_o)$
- $= \frac{1}{C_{eq}}\int_{to}^{t} i(t)dt + v(to)$
- $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$



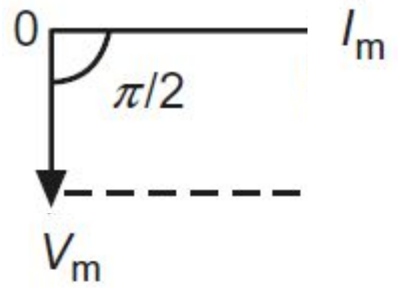
The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

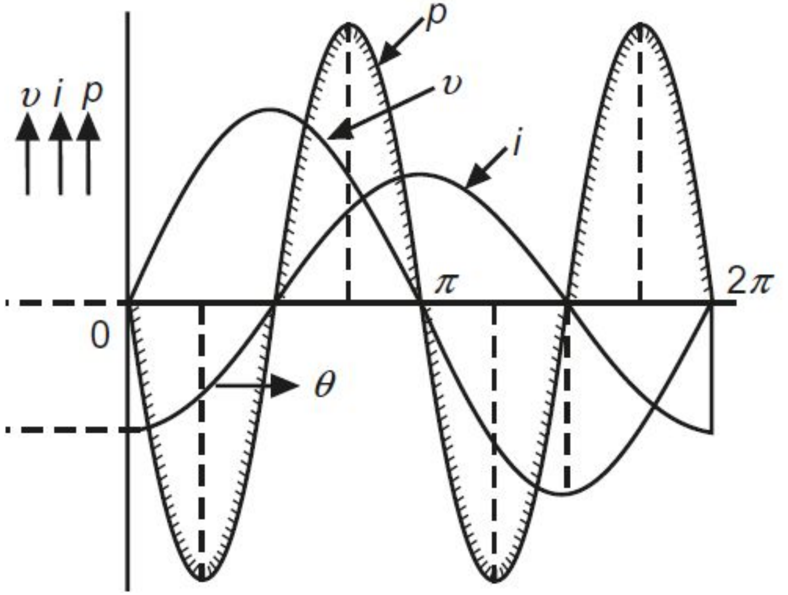
AC Circuit containing Pure Capacitor Only

- Let $v = V_m \sin \omega t$ applied to a capacitor, as a result charge on the capacitor at any instant,
- $q = Cv$;
- Current flowing through circuit
- $i = \frac{dq}{dt} = \frac{d}{dt} Cv = \frac{d}{dt} CV_m \sin \omega t = CV_m \cos \omega t$
- $i = \frac{V_m}{1/\omega C} \sin(\omega t + \pi/2) = \frac{V_m}{X_c} \sin(\omega t + \pi/2)$
- Where $X_c = \frac{1}{\omega C}$ is the opposition offered to flow of AC by capacitor is called capacitive reactance.
- The value of current will be maximum when $\sin\left(\omega t + \frac{\pi}{2}\right) = 1$ i.e $I_m = \frac{V_m}{X_c}$.
- $i = I_m \sin(\omega t + \pi/2)$



- The current flowing through pure capacitive circuit leads the applied voltage by 90^0.
- In a pure capacitance current leads voltage by 90^0.
- Instantaneous power $p = vi = V_m \sin(\omega t)\, I_m \sin(\omega t + \pi/2)$
- $p = V_m I_m \sin(\omega t)\cos(\omega t) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\sin(2\omega t)$
- Average power over a complete cycle, P=0;
- Hence average power consumed in a pure capacitive circuit is zero.
- From the power curve the average power in a half cycle is zero since the positive and negative loop area under the curve is the same.
- During the first quarter cycle, power supplied by source to capacitor is stored in electric field set-up between capacitor plates.
- In the next quarter cycle, electric field collapses and the power stored in the field is returned to zero. Hence no power stored in the circuit.





Example

- An AC circuit consists of a pure resistance of 10 Ω and is connected across an AC supply of 230 V, 50 Hz. Calculate (i) current, and (ii) power consumed; further, (iii) write down the equation for voltage and current.

- Current in the circuit $i = \frac{V}{R} = \frac{230}{10} = 23\ A.$
- Power Consumed $P = VI = 230*23 = 5290$ W.
- Maximum value of applied voltage $V_m = \sqrt{2}V = \sqrt{2}\,(230) = 325.27V$
- Maximum value of current $I_m = \sqrt{2}I = \sqrt{2}\,(23) = 32.53A$
- Angular Velocity $\omega = 2\pi f = 2\pi 50 = 314.16$ rad/s.
- Equation for applied voltage: $v = V_m \sin(\omega t) = 325.27 \sin(314.16t)$
- As in a pure resistive circuit, voltage and current are in phase therefore current is given by $i = 32.53 \sin 314.16t$.

An inductive coil having negligible resistance and 0.1 H inductance is connected across 200 V, 50 Hz supply. Find (i) the inductive reactance, (ii) rms value of current, (iii) power, and (iv) equations for voltage and current.

Inductive reactance, $X_L = 2\pi L = 2\pi \times 50 \times 0.1 = 31.416\ \Omega$

Current, $I = V/X_L = 200/31.416 = 6.366$ A

Power, $P = 0$

Now, $V_m = \sqrt{2}\,V = \sqrt{2} \times 200 = 282.84$ V;

$I_m = \sqrt{2}\,I = \sqrt{2} \times 6.366 = 9$A

and $\omega = 2\pi f = 314$ rad/s

$\therefore$ $v = V_m \sin \omega t = 282.84 \sin 314\,t$

In pure inductive circuit, current lags behind voltage by $\pi/2$ radian.

$\therefore$ $i = I_m \sin(\omega t - \pi/2) = 9 \sin(314\,t - \pi/2)$

A capacitor has a capacitance of 30 μF. Find its capacitive reactance for frequencies of 25 and 50 Hz. Find in each case the current if the supply voltage is 440 V.

Capacitance of the capacitor, $C = 30 \times 10^{-6}$ F

Supply voltage, $V = 440$ V

When supply frequency, $f_1 = 25$ Hz

$$\text{Capacitive reactance, } X_{C1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi f_1 C} = \frac{1}{2\pi \times 25 \times 30 \times 10^{-6}} = 212.2\ \Omega$$

$$\text{Current in the circuit, } I_1 = \frac{V}{X_{C1}} = \frac{440}{212.2} = 2.073\ \text{A}$$

When supply frequency $f_2 = 50$ Hz,

$$\text{capacitive reactance, } X_{C2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi f_2 C} = \frac{1}{2\pi \times 50 \times 30 \times 10^{-6}} = 106.1\ \Omega$$

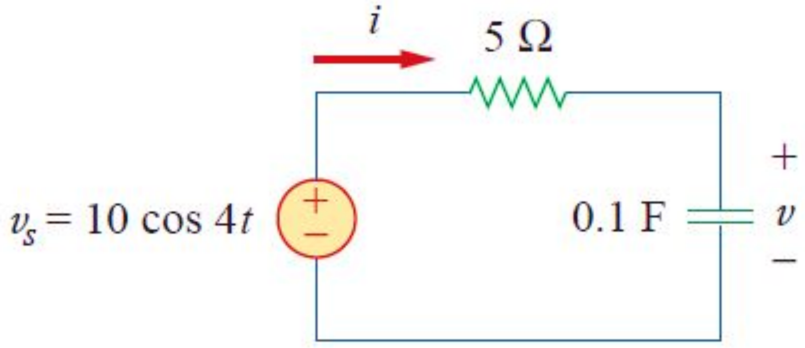
$$\text{Current in the circuit, } I_2 = \frac{V}{X_{C2}} = \frac{440}{106.1} = 4.146\ \text{A}$$

Important characteristics of Basic Elements

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Example

- Find $i(t)$ and $v(t)$ in the circuit shown below.

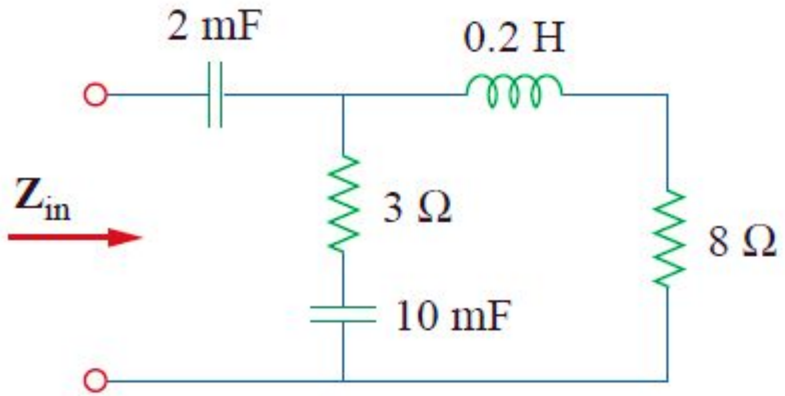


$$i(t) = 1.789 \cos(4t + 26.57°) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43°) \text{ V}$$

- Note: KCL ($\mathbf{I_1} + \mathbf{I_2} + \cdots + \mathbf{I_N} = 0$) and KVL ($\mathbf{V_1} + \mathbf{V_2} + \cdots + \mathbf{V_N} = 0$) holds for Phasors.

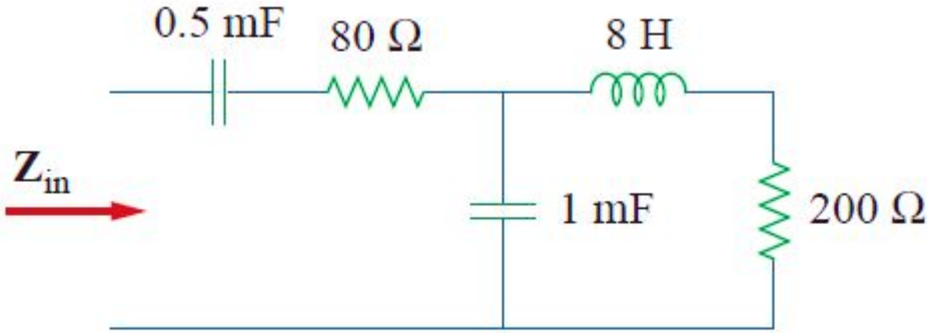
Example

- Find the input impedance of the circuit shown below. Assume that the circuit operates $\omega = 50\ rad/s$.



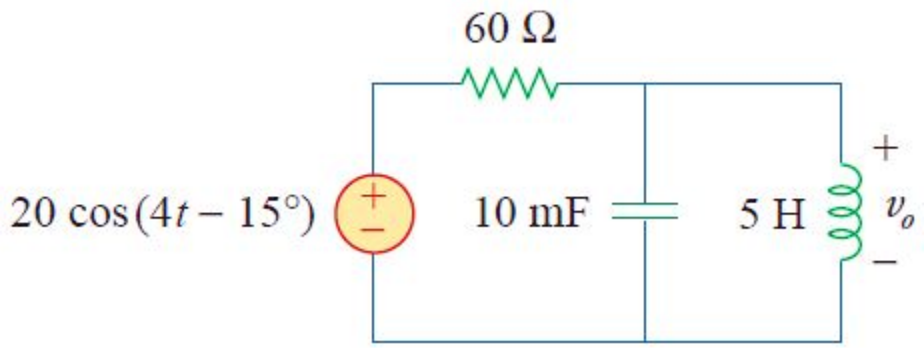
$\mathbf{Z}_{\text{in}} = 3.22 - j11.07\ \Omega$

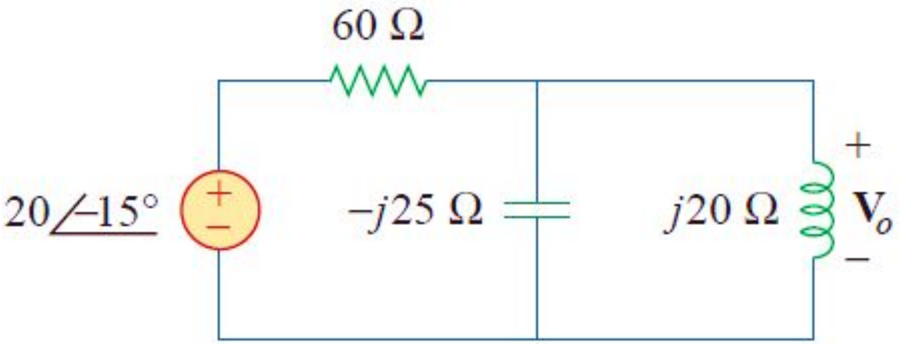
- Find the input impedance of the circuit shown below. Assume that the circuit operates $\omega = 10\ rad/s$.



$(129.52 - j295)$

Determine $v_o(t)$ in the circuit shown below





$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 20\angle{-15^\circ}\text{ V}, \qquad \omega = 4$$

$$10\text{ mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j25\ \Omega$$

$$5\text{ H} \quad \Rightarrow \quad j\omega L = j4 \times 5 = j20\ \Omega$$

Then $\mathbf{Z}_1 = 60\ \Omega$ and

$$\mathbf{Z}_2 = -j25 \parallel j20 = \frac{-j25 \times j20}{-j25 + j20} = j100\ \Omega$$

By the voltage-division principle,

$$\mathbf{V}_o = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}_s = \frac{j100}{60 + j100}(20\angle{-15^\circ})$$
$$= (0.8575\angle{30.96^\circ})(20\angle{-15^\circ}) = 17.15\angle{15.96^\circ}\text{ V}$$

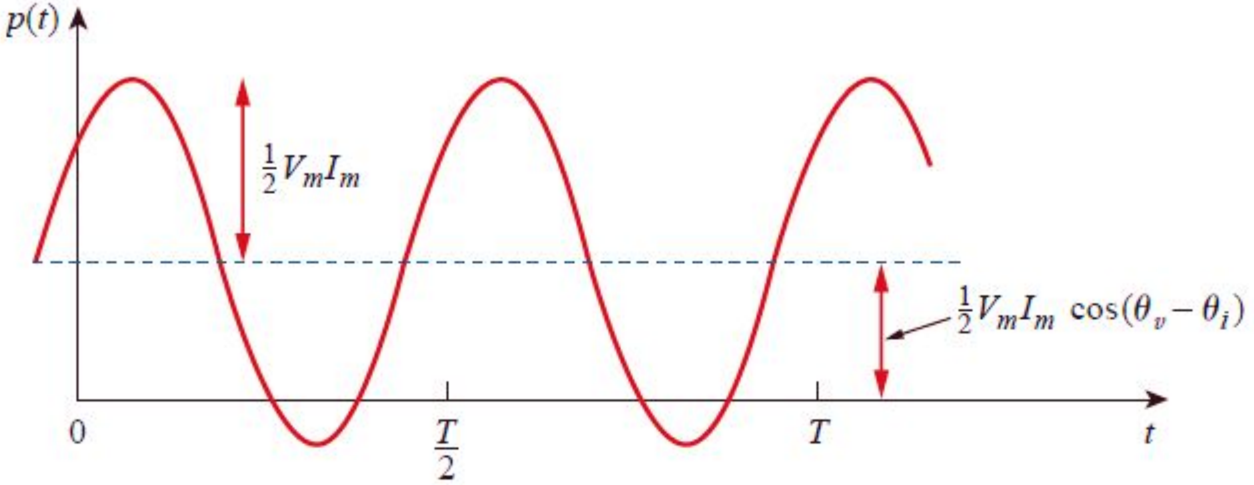
We convert this to the time domain and obtain

$$v_o(t) = 17.15\cos(4t + 15.96^\circ)\text{ V}$$

Instantaneous Power $p(t)$

- It is the power at any instant of time. $p(t) = v(t)i(t)$.
- The instantaneous power $p(t)$, absorbed by any element is the product of instantaneous voltage $v(t)$ and $i(t)$ through it.
- It is the rate at which an element absorbs energy.
- Let the voltage and current at the terminals of the circuit be
- $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$.
- $p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$.
- We apply trigonometry identity, $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$.
- $p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$.
- In above equation, 1^{st} part is constant or time independent. Its value depends on the phase difference between voltage and current.
- 2^{nd} part, is a sine function whose frequency is 2ω, which is twice the angular frequency of the voltage or current.

- A sketch of $p(t)$ is shown in figure below, where $T = {}^{2\pi}/_{\omega}$ is the period of voltage or current.
- $p(t)$ is periodic, i.e, $p(t)= p(t+T_0)$ and has a period of $T_0=T/2$, since its frequency is twice that of voltage or current.
- We can observe that $p(t)$ is positive for some part of each cycle and negative for the rest of the cycle.
- When $p(t)$ is positive, power is absorbed by the circuit. When $p(t)$ is negative, power is absorbed by the source. i.e, power is transferred from the circuit to source.
- This is possible because of storage elements (inductor and capacitor) in the circuit.

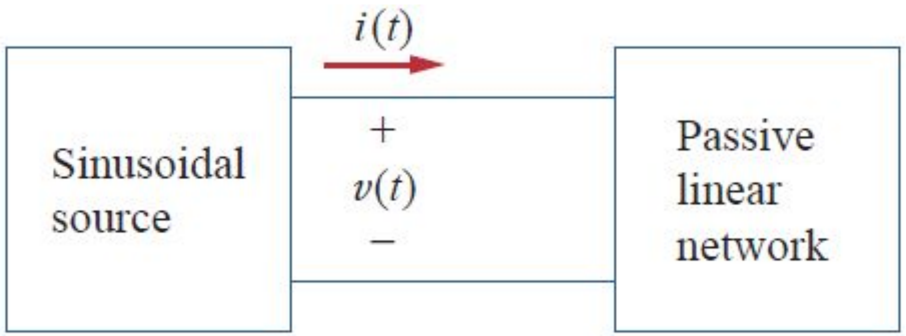


- The instantaneous power changes with time and is therefore difficult to measure. The *average power* is more convenient to measure. Wattmeter is used to measure average power.
- The average power in watts, is the average of the instantaneous power over one period.
- The average power is given by: $\mathrm{P} = \frac{1}{T}\int_0^t p(t)dt.$
- Above equation shows that, the averaging done over *T*, we will get the same result if we perform the integration over actual period of *p*(*t*) is $T_o = {}^T\!/_2.$
- $P = \frac{1}{T}\int_0^T \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)\, dt + \frac{1}{T}\int_0^T \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)\, dt.$
- $P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T}\int_0^T dt + \frac{1}{2} V_m I_m \frac{1}{T}\int_0^T \cos(2\omega t + \theta_v + \theta_i)\, dt.$
- The 1st integrand is constant and the average of a constant is the same as constant. The 2nd integrand is a sinusoid.

- The average of a sinusoid over its period is zero, because area under positive sinusoid during +ve half cycle is cancelled by the area under it during –ve half cycle. Thus, the second term vanishes and the average power becomes.
- $P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$.
- Note: $p(t)$ is time varying while P does not depend on time.
- In phasor form P is calculated using phasors **V** and **I**, i.e,
- $\frac{1}{2}\mathbf{VI}^* = \frac{1}{2} V_m I_m \vdots (\theta_v - \theta_i) = \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$.
- In the above expression, the real part is average power P.
- $P = \frac{1}{2} \mathrm{Re}[\mathbf{VI}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$.
- When $\theta_v = \theta_i$, the voltage and current are in in-phase. This implies a purely resistive load R. $P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R$. where, $|\mathbf{I}|^2 = \mathbf{I} \text{ X } \mathbf{I}^*$.
- For a purely reactive circuit $\theta_v - \theta_i = \pm 90^0$, $P = \frac{1}{2} V_m I_m \cos 90^0 = 0$.

A resistive load (R) absorbs power at all times, while a reactive load (L or C) absorbs zero average power

Practice Problem: Calculate the instantaneous power and average power absorbed by the passive linear network, if $v(t) = 165\cos(10t + 20^0)$, and $i(t) = 20\sin(10t + 60^0)$.



Answer: $1.0606 + 1.65\cos(20t - 10^\circ)$ kW, 1.0606 kW.

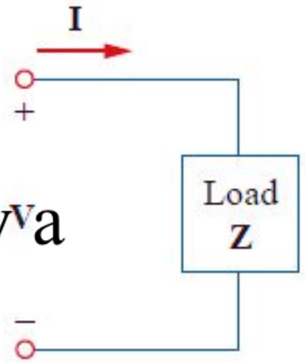
Apparent power and Power factor

- If the voltage and current at the terminals of a circuit are: $v(t) = V_m \cos(\omega t + \theta_v)$ and $i(t) = I_m \cos(\omega t + \theta_i)$.
- In phasor form $\mathbf{V} = V_m \angle \theta_v$ and $\mathbf{I} = I_m \angle \theta_i$, the average power is
- $P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$.
- $P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = S \cos(\theta_v - \theta_i)$.
- A new term called apparent power $S = V_{rms} I_{rms}$.
- The average power is the product of terms called apparent power (S) and $\cos(\theta_v - \theta_i)$ called Power Factor (PF).
- The apparent power (VA) is the product of RMS values of voltage and current.
- S is measured in VA to distinguish it from average or real power.
- The Power factor is dimensionless, since it is the ratio of S to P.

Power factor

- PF=$\frac{P}{S} = \cos(\theta_v - \theta_i)$. The angle $(\theta_v - \theta_i)$ is called power factor angle.
- The PF angle is equal to angle of load impedance if **V** is the voltage across the load and **I** is the current through it.
- $Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle (\theta_v - \theta_i) = \frac{V_{rms}}{I_{rms}} \angle (\theta_v - \theta_i)$.
- *The power factor is the cosine of the phase difference between voltage and current. It is also cosine of the angle of the load impedance.*
- For a purely resistive load, $(\theta_v - \theta_i)$=0; and PF=1. (S=P)
- For a purely reactive load, $(\theta_v - \theta_i)$=$\pm$90; and PF=0. (P=0)
- Leading PF means the current leads the voltage, which implies capacitive load. Lagging PF implies inductive load.

Complex Power



- Complex power contains all information about power absorbed by a given load.
- Consider the ac load, Given the phasor, $\mathbf{V} = V_m \angle \theta_v$, and $\mathbf{I} = I_m \angle \theta_i$ of voltage $v(t)$ and $i(t)$.
- The complex power S absorbed by ac load is the product of the voltage and complex conjugate of the current, i.e, $S = \frac{1}{2}\mathbf{VI}^* = V_{rms} I_{rms}^*$
- where $V_{rms} = \frac{V}{\sqrt{2}} = V_m \angle \theta_v$ and $I_{rms} = \frac{I}{\sqrt{2}} = I_m \angle \theta_i$
- Thus we may write $S = V_{rms} I_{rms} \angle (\theta_v - \theta_i)$
- $S = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + jV_{rms} I_{rms} \sin(\theta_v - \theta_i)$
- The magnitude of complex power is the apparent power, hence the complex power is measured in VA. The angle of complex power is the power factor angle.

- The complex power may be expressed in terms of the load impedance Z.
- $Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{V}_{rms}}{\mathbf{I}_{rms}} = \frac{V_{rms}}{I_{rms}} : (\theta_v - \theta_i)$.
- Thus $\mathbf{V}_{rms} = \mathbf{Z}\mathbf{I}_{rms}$. $\mathbf{S} = I_{\mathrm{rms}}^2 \mathbf{Z} = \frac{V_{\mathrm{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\mathrm{rms}} \mathbf{I}_{\mathrm{rms}}^*$
- since, $\mathbf{Z} = R + jX$; becomes $S = I_{rms}^2(R + jX) = P + jQ$.
- where P and Q are the real and imaginary parts of complex power, i.e,
- $P = \mathrm{Re}(\mathbf{S}) = I_{rms}^2 R$. and $Q = \mathrm{Im}(\mathbf{S}) = I_{rms}^2 X$.
- P is the average real power and it depends on load's resistance R.
- Q depends on load's reactance X called reactive power.
- $P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$ (W) and $Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$ (VAR).
- *The energy storage elements neither dissipate nor supply power, but exchange power back and forth with the rest of the network. In the same way, reactive power is being transferred back and forth between load and source.*

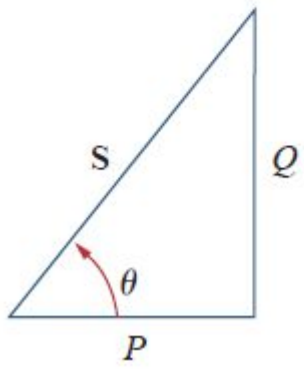
Complex Power

- If Q=0 for resistive loads
- Q < 0 for capacitive loads (Leading PF)
- Q > 0 for inductive Loads (Lagging PF)
- *Complex power (in VA) is the product of the rms voltage phasor and the complex conjugate of the rms current phasor.*
- As a complex quantity (S), its real part is real power P and its imaginary part is reactive power Q.

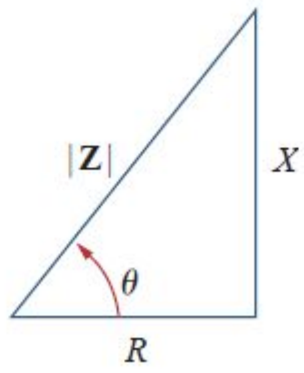
$$\begin{aligned}
\text{Complex Power} &= \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^* \\
&= \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}} \angle \theta_v - \theta_i \\
\text{Apparent Power} &= S = |\mathbf{S}| = \mathbf{V}_{\text{rms}}\mathbf{I}_{\text{rms}} = \sqrt{P^2 + Q^2} \\
\text{Real Power} &= P = \operatorname{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i) \\
\text{Reactive Power} &= Q = \operatorname{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i) \\
\text{Power Factor} &= \frac{P}{S} = \cos(\theta_v - \theta_i)
\end{aligned}$$

Power Triangle

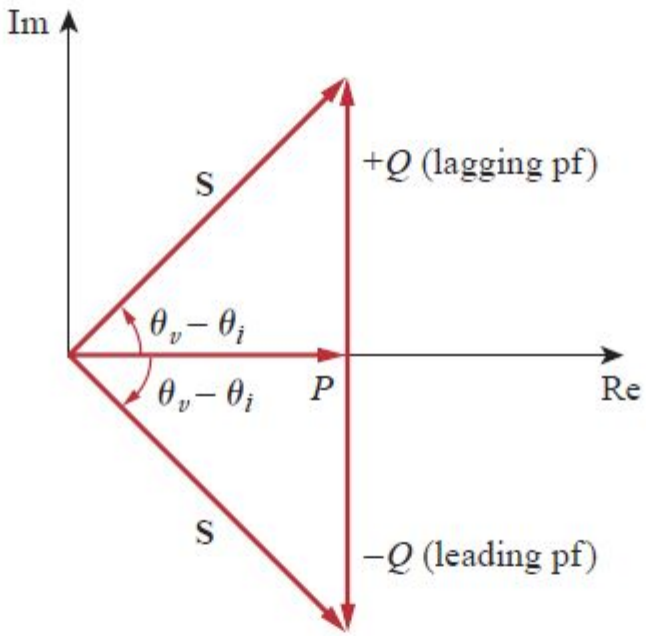
- This is similar to impedance triangle.
- Power Triangle has four items: Apparent Power, Real Power, Reactive Power and Power Factor angle.



Power Triangle



Impedance Triangle

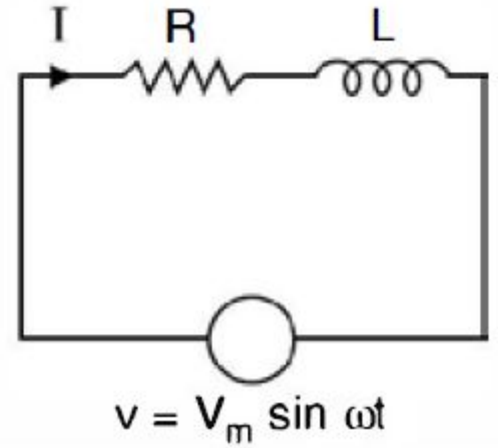


Power Triangle indicating Lagging and Leading PF

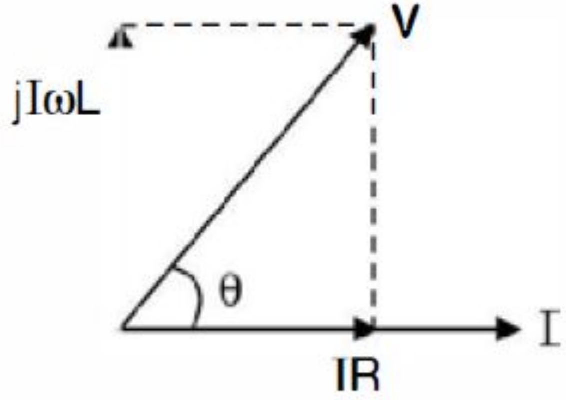
Significance of reactive power

- Reactive power is the component of power which is supplied to reactive components of the load from source during positive half cycle while it is returned back to supply from the components to the source during negative half cycle.
- Reactive Power never gets consumed by a circuit but flows alternatively back and forth from the source to reactive components and vice-versa.

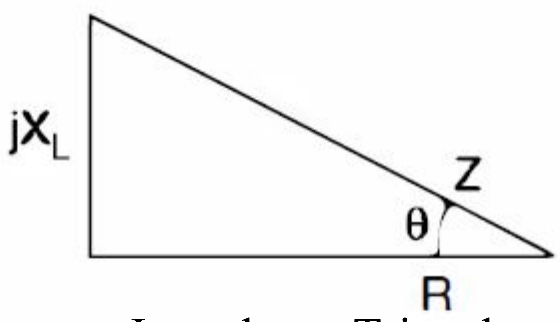
AC through series RL circuit



- As R and L are connected in series, current flowing through R and L is same. Current is taken as reference phasor.
- The voltage across resistance is in phase with current *I*, hence *I*R drop has drawn on same phase of *I*.
- Voltage across inductor *L* is leading the current hence a phasor has drawn in quadrature with current. The voltage across inductor leads current by 90 degree.
- The sum of the voltage across resistor R and inductor L should equal the supply voltage **V**.
- From the phasor diagram $V = IR + jI\omega L$. or $\frac{V}{I} = R + jX_L$.
- It also gives that phase angle is neither 0 nor 90, therefore it represents impedance rather than resistance or reactance.



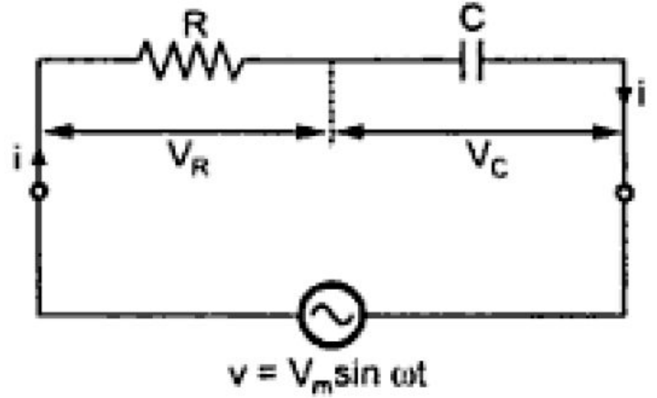
Phasor Diagram



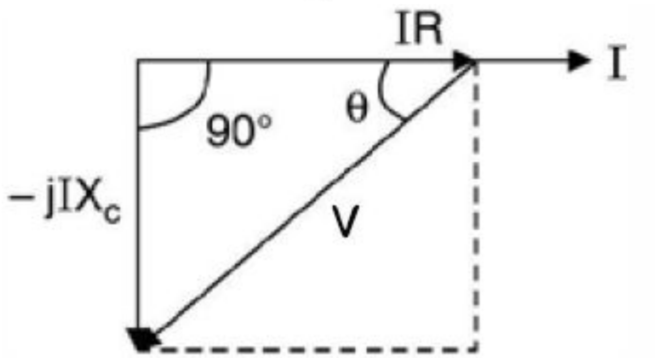
Impedance Triangle

- Impedance is the ratio of two phasors, but it is not a phasor by itself as R and X_L are not function of time and hence while representing impedance triangle arrows should be placed.
- Angle θ between R and Z is same as the angle between I and V.
- From impedance triangle for RL circuit $Z = R + jX_L$;
- $R = Z\cos\theta$; $X_L = Z\sin\theta$ and $Z = \sqrt{R^2 + {X_L}^2}$; $\theta = \tan^{-1} {X_L}/_{R}$.
- The power in the circuit is given by $P = VI\cos\theta$.
- PF of the circuit is $\cos\theta = \frac{P}{VI} = \frac{Real\ Power}{Apparent\ Power}$.
- The component of current along the voltage phasor ($I\cos\theta$) gives real power whereas the component of current at right angle to voltage phasor gives reactive power (which does not do any useful work).

AC through series RC circuit



- The voltage across R is in phase with I and voltage across C lags the current by 90º as shown in phasor diagram.
- The resultant phasor is sum of IR and $-jIX_c$ is the supply voltage **V**.
- $V = IR - jIX_c$. or $\frac{V}{I} = R - jX_c = Z$.
- Hence, $Z = R - jX_c = \sqrt{R^2 + {X_c}^2}$. $R = Z\cos\theta$, $X_c = Z\sin\theta$ and $\tan\theta = -\frac{X_c}{R}$.



Example: The impedance of a circuit placed across a 120 V, 50 Hz source is (10+j20). Find the current and power.

Solution: Taking voltage as the reference we have $V = (120 + j0)$

Hence current $$I = \frac{V}{Z} = \frac{120 + j0}{(10 + j20)}$$

$$I = \frac{(120 + j0)(10 - j20)}{(10 + j20)(10 - j20)} = \frac{12 - j24}{5}$$

$$= 2.4 - j4.8 \text{ Amp.}$$

This shows that the current lags the voltage by an angle

$$\theta = \tan^{-1} \frac{-4.8}{2.4} = -63.4°$$

$$I = \sqrt{2.4^2 + 4.8^2} \angle -63.4$$

$$= 5.36 \angle -63.4$$

Power $$P = VI \cos\theta = 120 \times 5.36 \cos 63.4$$

$$= 288 \text{ watt}$$

The same problem can be solved using polar coordinates as follows :

Taking voltage as reference

$$V = 120 \angle 0$$

Impedance $$Z = \sqrt{10^2 + 20^2} \tan^{-1} \frac{20}{10}$$

$$= 22.36 \angle 63.4$$

$$I = \frac{V}{Z} = \frac{120 \angle 0}{22.36 \angle 63.4}$$

$$= 5.37 \angle -63.4$$

Power $$P = 120 \times 5.37 \cos 63.4 = 288 \text{ watts}$$

Power can also be calculated using the relation

$$P = \text{Real } [VI^*]$$

In Cartesian co-ordinates

$$V = 120 + j0$$

$$I = 2.4 - j4.8$$

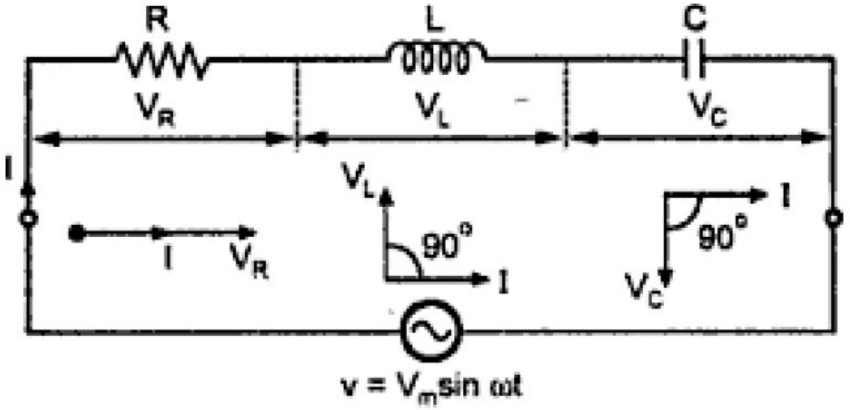
Hence $$I^* = 2.4 + j4.8$$

$$P = Re\ [(120 + j0)(2.4 + j4.8)]$$

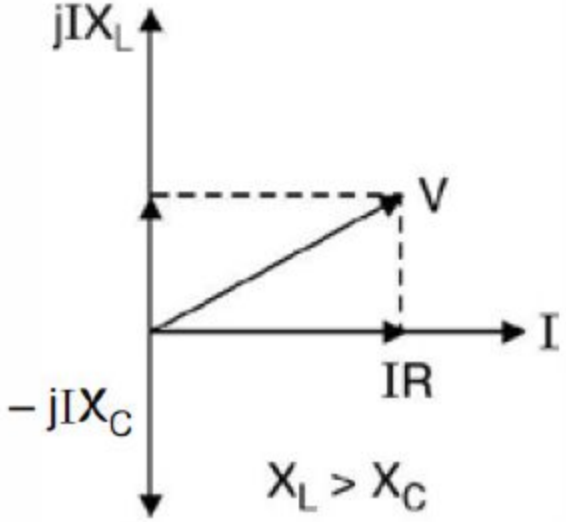
$$= 120 \times 2.4$$

$$= 288 \text{ watts}$$

AC through series RLC circuit



- By connecting ac voltage source, the circuit draws a current I. Due to current, there are different voltage drops across R, L and C which are given by
- Drop across R is V_R=IR.
- Drop across L is V_L=IX_L.
- Drop across C is V_C=IX_C.
- The values of I, V_R, V_L and V_C are RMS values.
- The characteristics of these drops are, V_R is in phase with I. V_L leads I by 90º. V_C lags I by 90º.
- As per KVL: The phasor sum $V = V_R + V_L + V_C$.
- $V = I(R + j(X_L - X_C))$.



Depending upon which reactance is larger, the current will lead or lag the supply voltage

If X_C>X_L, current will lead the supply voltage (capacitive circuit), otherwise it will lag (inductive circuit).

- Impedance $Z = R + j(X_L - X_C)$
- If $X_L > X_C$, circuit is inductive,
- If $X_L < X_C$, circuit is capacitive,
- If $X_L = X_C$, circuit is resistive. and $\tan\theta = \frac{X_L - X_C}{R}$,
- $Z = \sqrt{R^2 + (X_L - X_L)^2}$; $\cos\theta = {}^{R}/_{Z}$.
- $P_{avg} = P_R + P_L + P_C$

Example: A series circuit having pure resistance of 40 ohms, pure inductance of 50.07 mH, and a capacitor is connected across a 400 V, 50 Hz, AC supply. This R, L, C combination draws a current of 10A. Calculate (i) PF of the circuit, and (ii) capacitor value

$$X_L = 2\pi f L$$
$$= 2\pi \times 50 \times 50.07 \times 10^{-3}$$
$$= 15.73\ \Omega, \quad X_C = \frac{1}{2\pi f C}$$

$$Z = R + j(X_L - X_C)$$
$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (15.73 - X_C)^2}$$
$$|Z| = \frac{|V|}{|I|} = \frac{400}{10} = 40\ \Omega$$
$$40 = \sqrt{R^2 + (15.73 - X_C)^2} \quad \text{i.e.} \quad 1600 = R^2 + (15.73 - X_C)^2$$

Substitute $R = 40\ \Omega$ in (3), $1600 = 1600 + (15.73 - X_C)^2$

$$\therefore \quad (15.73 - X_C)^2 = 0$$
$$\therefore \quad X_C = 15.73\ \Omega \quad \text{i.e.} \quad 15.73 = \frac{1}{2\pi f C}$$
$$\therefore \quad C = 2.023 \times 10^{-4}\ \text{F}$$
$$\therefore \quad Z = 40 + j(15.73 - 15.73) = 40 + j\,0\ \Omega$$
$$= 40 \angle 0°\ \Omega$$
$$\therefore \quad \text{p.f.} = \cos(0°) = 1$$

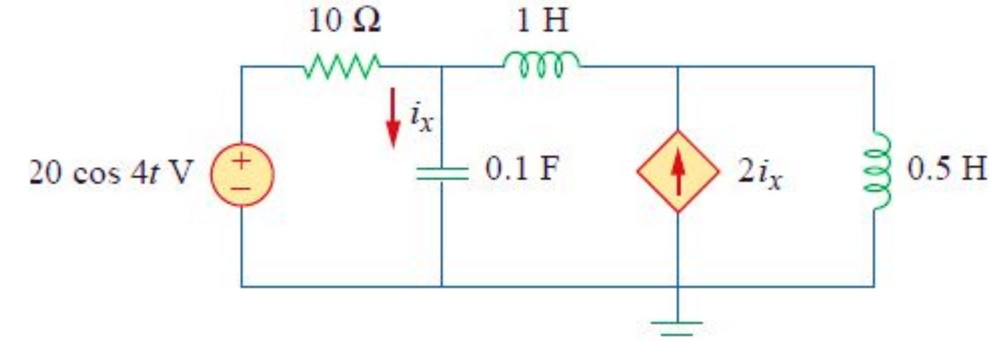
Steps to Analyze AC Circuits

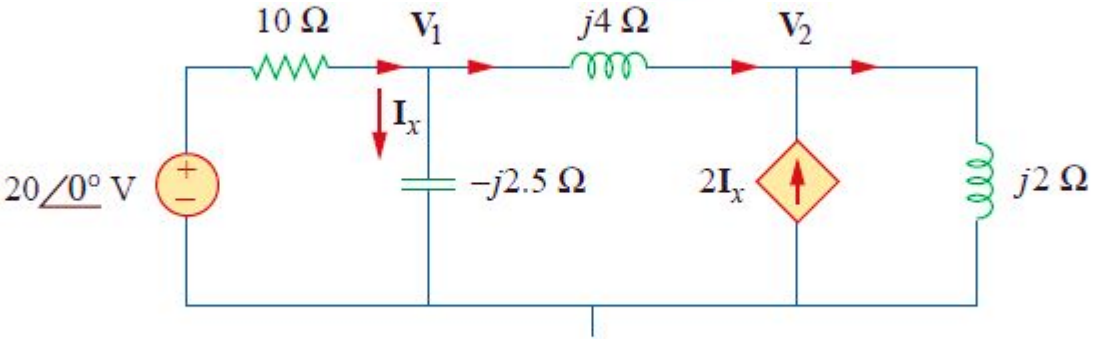
- Transform the circuit to the phasor or frequency domain.
- Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
- Transform the resulting phasor to the time domain.

Nodal Analysis

- The basis of nodal analysis is Kirchoff's current law, since KCL is also valid for phasors.
- We can analyze AC circuits by using nodal analysis.

Numerical: Find i_x in the circuit using nodal analysis





$$20\cos 4t \quad\Rightarrow\quad 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s}$$

$$1 \text{ H} \quad\Rightarrow\quad j\omega L = j4$$

$$0.5 \text{ H} \quad\Rightarrow\quad j\omega L = j2$$

$$0.1 \text{ F} \quad\Rightarrow\quad \frac{1}{j\omega C} = -j2.5$$

Applying KCL at node 1,

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

or

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20$$

At node 2,

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $\mathbf{I}_x = \mathbf{V}_1/-j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

By simplifying, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0$$

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

We obtain the determinants as

$$\Delta = \begin{vmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1 + j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\underline{/18.43^\circ}\text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\underline{/198.3^\circ}\text{ V}$$

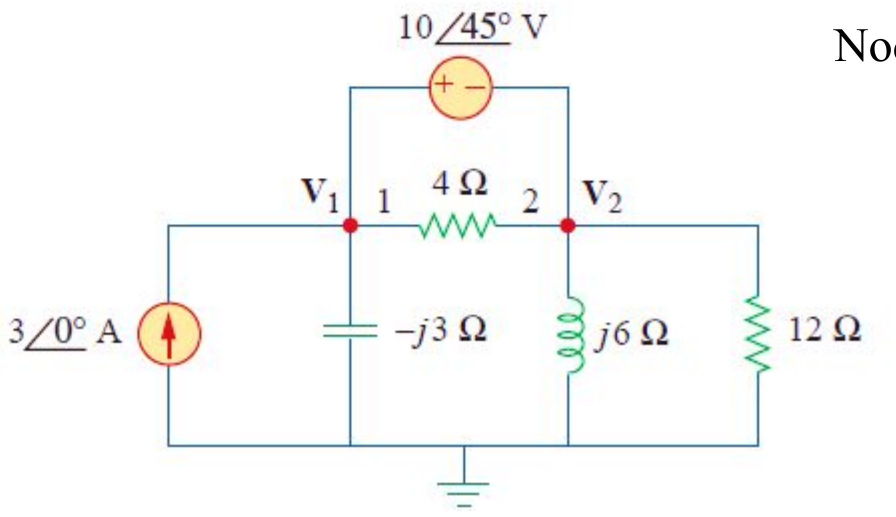
The current $\mathbf{I}_x$ is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\underline{/18.43^\circ}}{2.5\underline{/-90^\circ}} = 7.59\underline{/108.4^\circ}\text{ A}$$

Transforming this to the time domain,

$$i_x = 7.59\cos(4t + 108.4^\circ)\text{ A}$$

Numerical: Compute V_1 and V_2 for the circuit shown below



Nodes 1 and 2 form a supernode, Applying KCL at supernode gives

$$3 = \frac{\mathbf{V}_1}{-j3} + \frac{\mathbf{V}_2}{j6} + \frac{\mathbf{V}_2}{12} \qquad 36 = j4\mathbf{V}_1 + (1 - j2)\mathbf{V}_2$$

But a voltage source is connected between nodes 1 and 2, so that

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ$$

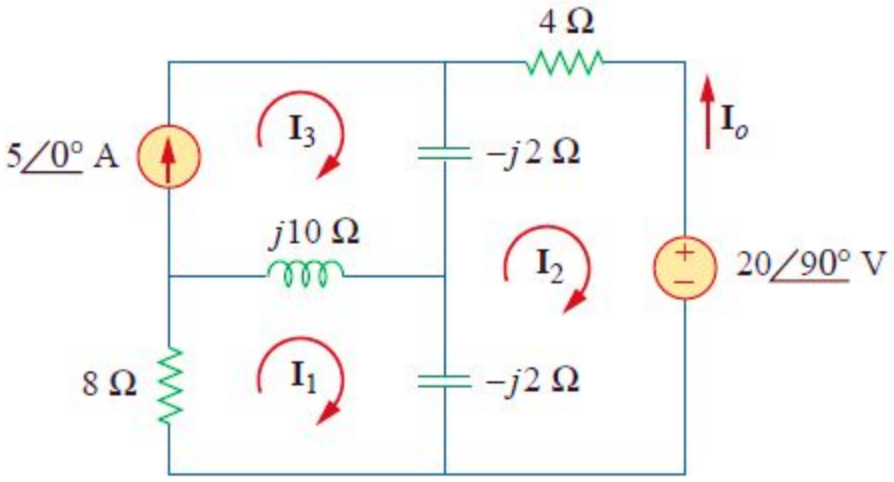
On simplification $36 - 40\angle 135^\circ = (1 + j2)\mathbf{V}_2 \quad \Rightarrow \quad \mathbf{V}_2 = 31.41\angle -87.18^\circ \text{ V}$

$$\mathbf{V}_1 = \mathbf{V}_2 + 10\angle 45^\circ = 25.78\angle -70.48^\circ \text{ V}$$

Mesh Analysis

- Kirchhoff’s voltage law (KVL) forms the basis of mesh analysis.

Numerical: Determine current I_o in the circuit shown below



Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0$$

For mesh 2, $(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\angle 90^\circ = 0$

For mesh 3, I_3=5A

On simplification

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10$$

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix} = 32(1 + j)(1 - j) + 4 = 68$$

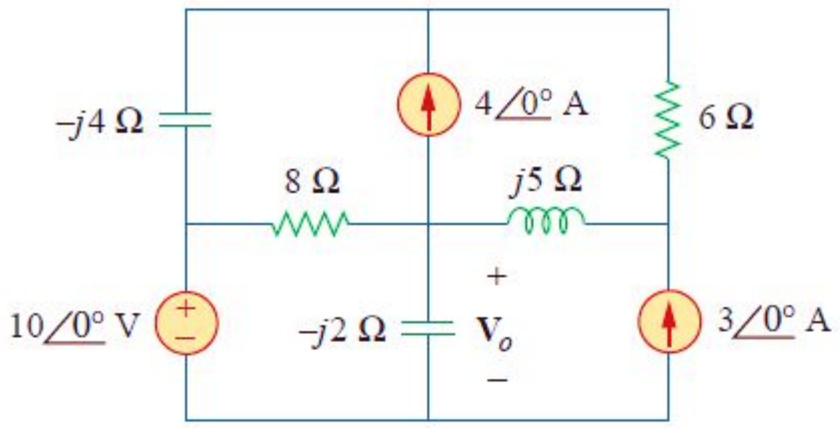
$$\Delta_2 = \begin{vmatrix} 8 + j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\angle -35.22^\circ$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\angle -35.22^\circ}{68} = 6.12\angle -35.22^\circ \text{ A}$$

The desired current is

$$\mathbf{I}_o = -\mathbf{I}_2 = 6.12\angle 144.78^\circ \text{ A}$$

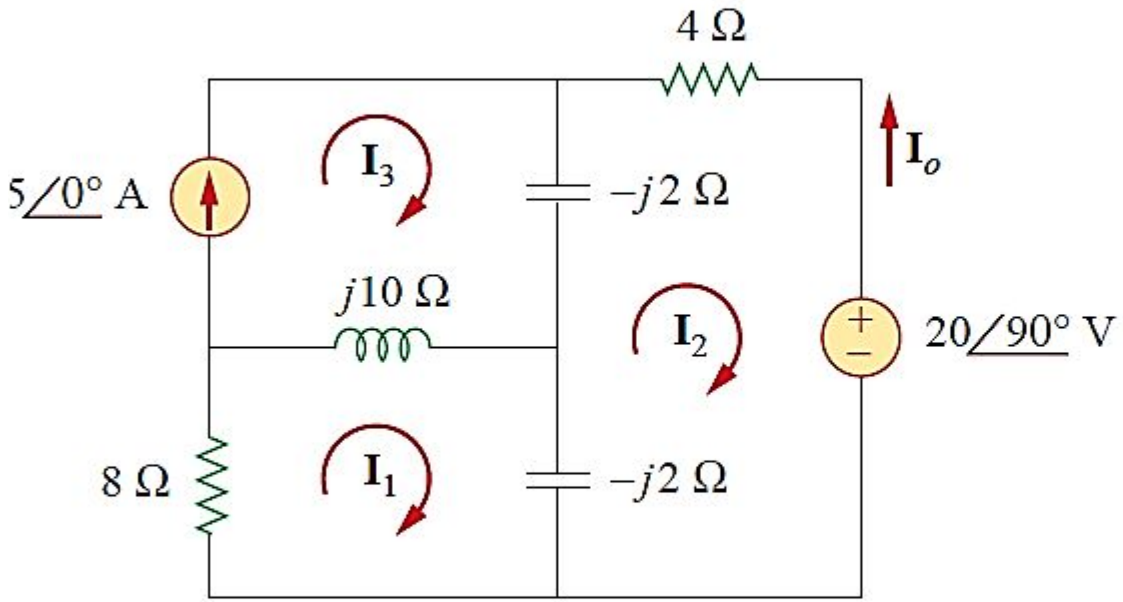
Solve for Vo in the circuit shown below using mesh analysis



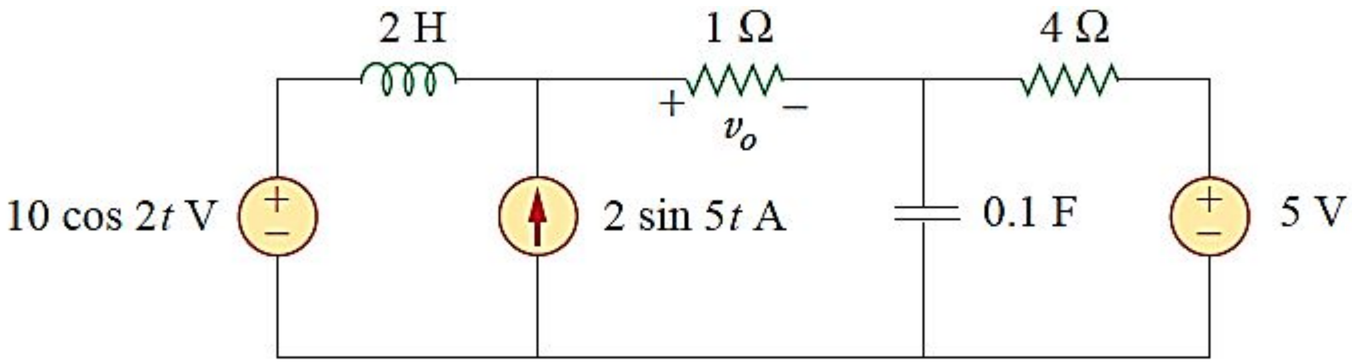
Superposition Theorem

- The superposition theorem applies to ac circuits in the same way it applies to dc circuits.
- This theorem is important if the circuit has operating at different frequencies. In this case, since the impedance depend on the frequency, we must have a different frequency domain circuit for each frequency.
- The superposition theorem makes the problem into single frequency problem by breaking the circuit.
- So the circuit can be simplified by operating individual source with different frequency at a time.

Numerical: Use the superposition theorem to find I_o in the circuit shown below.



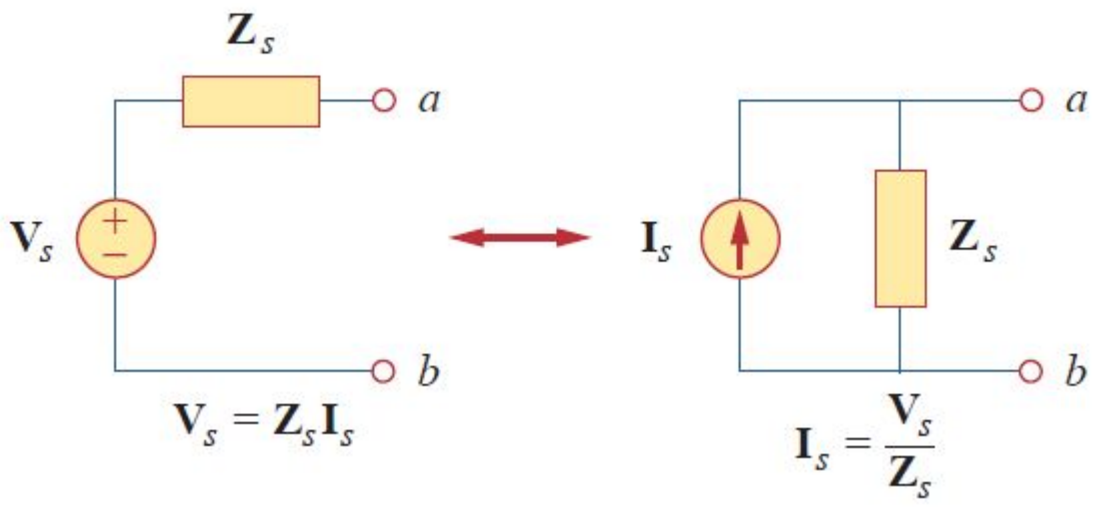
Numerical: Find Vo of the circuit using superposition theorem



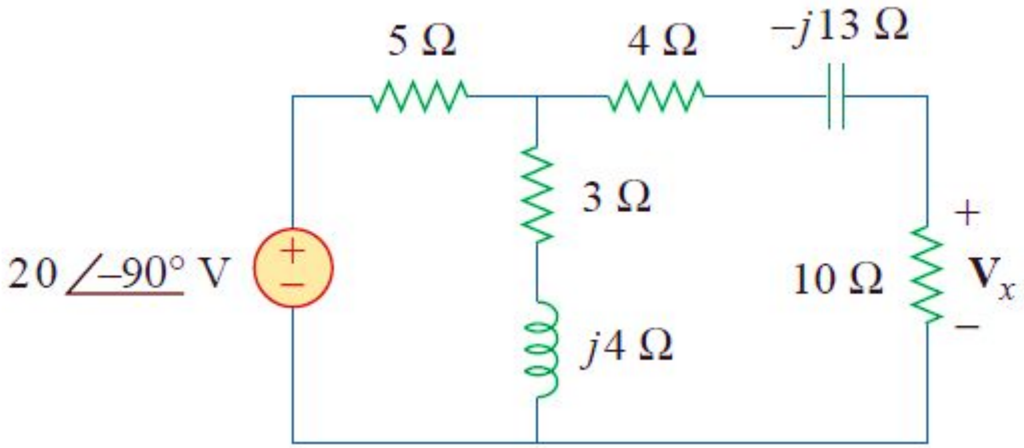
Source Transformation

- Source transformation in the frequency domain involves transforming voltage source in series with an impedance into a current source in parallel with an impedance or vice versa.

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Leftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$

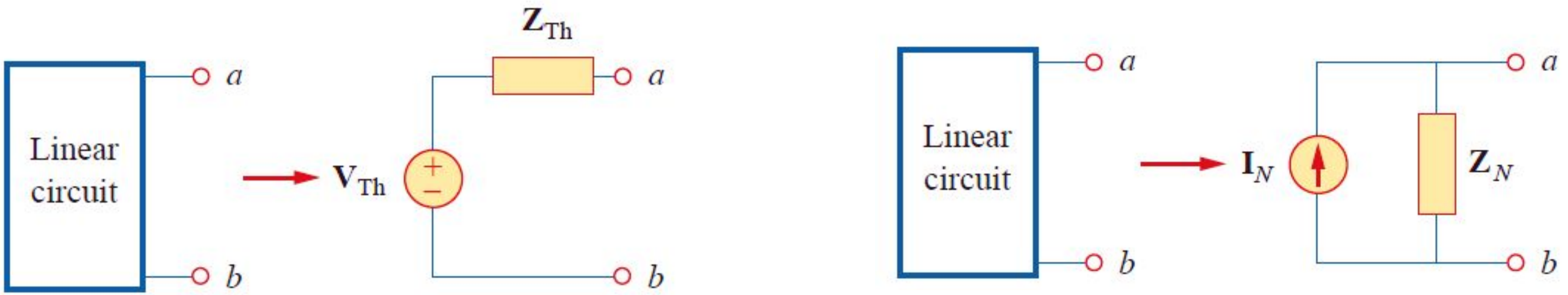


Numerical: Calculate Vx in the circuit using superposition theorem.

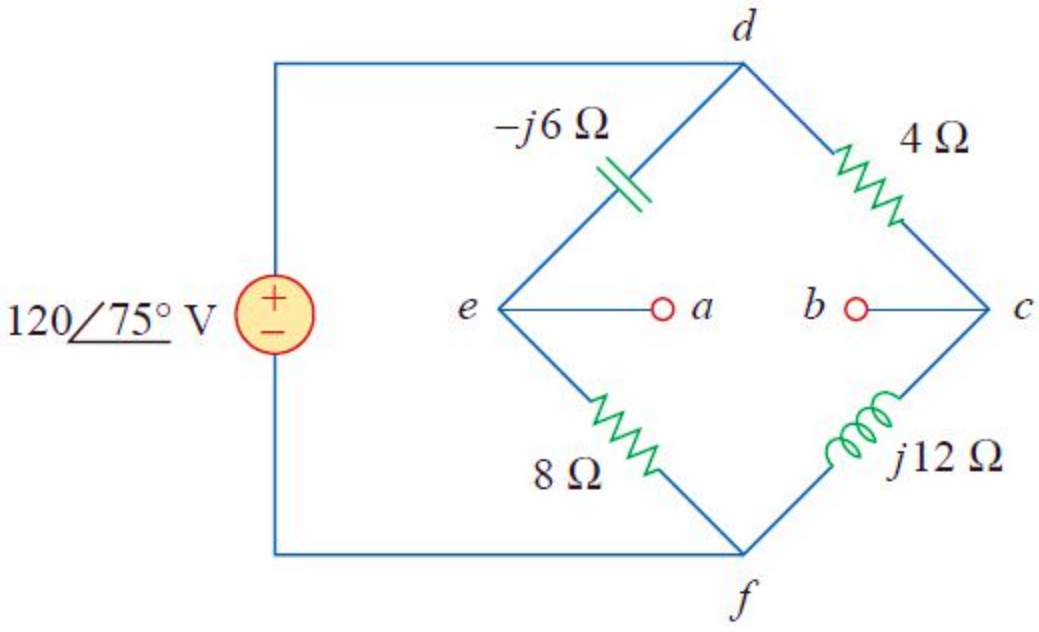


Thevenin and Norton equivalent circuit

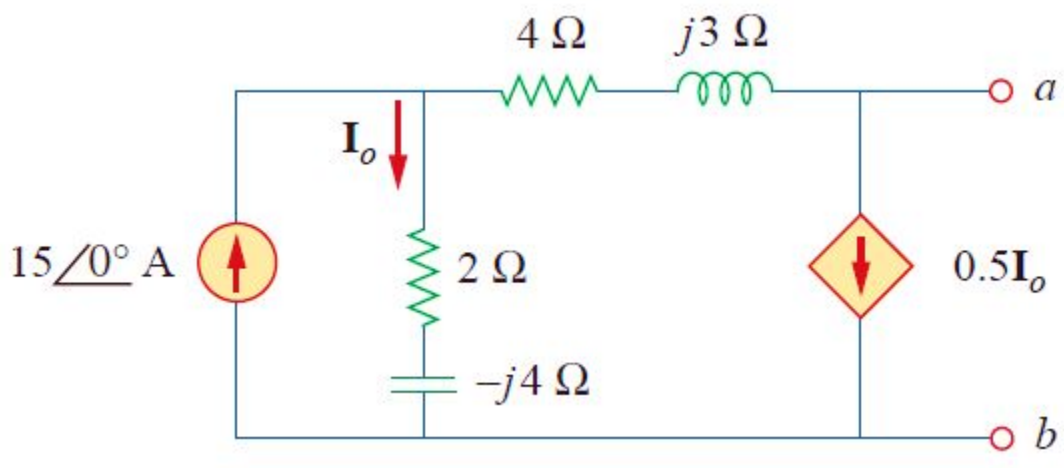
- Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits.
- A linear circuit is replaced by a voltage source in series with an impedance.
- A linear circuit is replaced by a current source in parallel with an impedance.



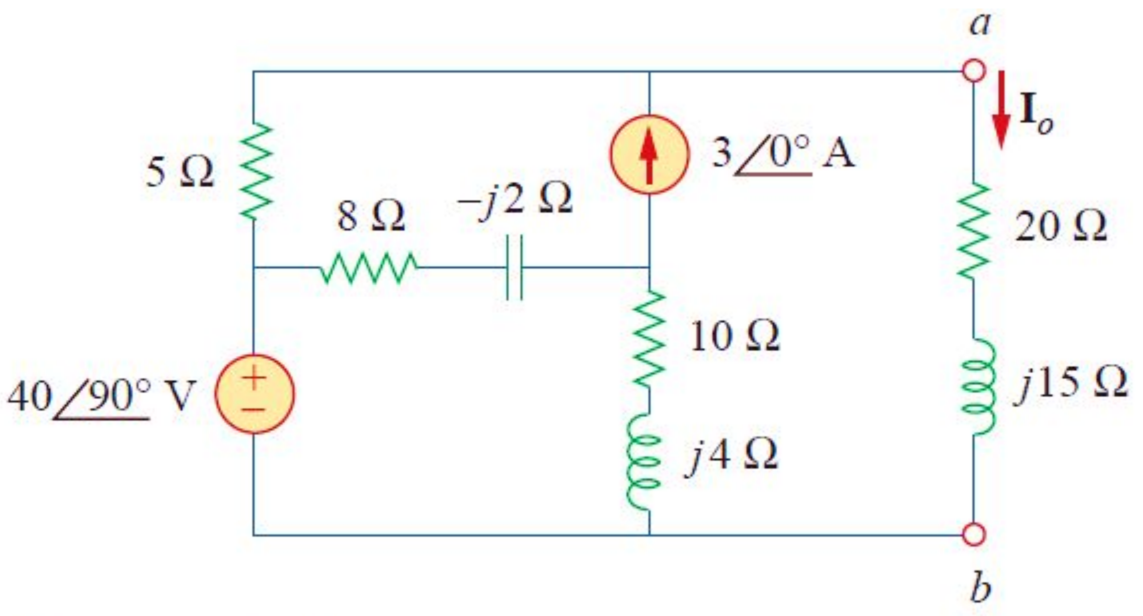
Numerical: Obtain the Thevenin equivalent at terminals a-b of the circuit



Numerical: Find the Thevenin's equivalent of the circuit as seen in figure below across terminals a-b.

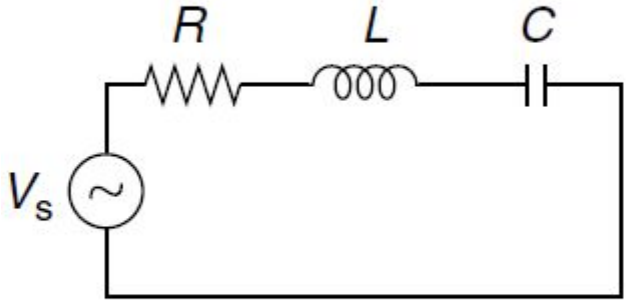


Numerical: Obtain current I_o for the circuit shown below using Nortons theorem.



Maximum Power Transfer Theorem

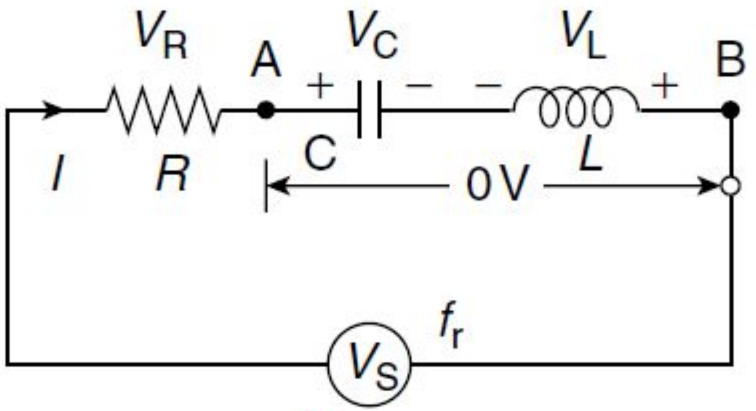
RLC series circuit with variable frequency voltage source

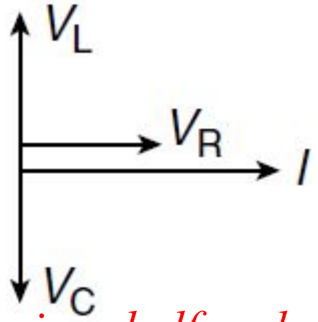


- In the circuit shown, the voltage source is has variable frequency.
- The inductive reactance (X_L) causes the total current lag the source voltage, similarly capacitive reactance (X_C) has opposite effect.
- It also causes total current to lag, lead or in phase with voltage source.
- When they are in in-phase X_L and X_C cancel each other and the circuit becomes purely resistive.
- With the supply voltage of variable frequency, at a particular frequency, X_L will be equal to X_C, such a condition is called resonance.

Series Resonance

- When R-L-C series circuit connected to variable frequency voltage source, if the frequency is increased, X_L will increase, whereas X_C will decrease because $X_L = 2\pi f L$ and $X_C = {}^{1}/_{2\pi f C}$.
- Series resonance will occur when $X_L = X_C$. The frequency at which resonance occur is called Resonant frequency f_r.
- At resonant frequency f_r, the inductive and capacitive reactance's are equal in magnitude and they cancel each other, thus total impedance of the circuit is resistive.
- The current flowing through the series circuit is same, voltage drop across the capacitor will be equal to voltage drop across the inductor.





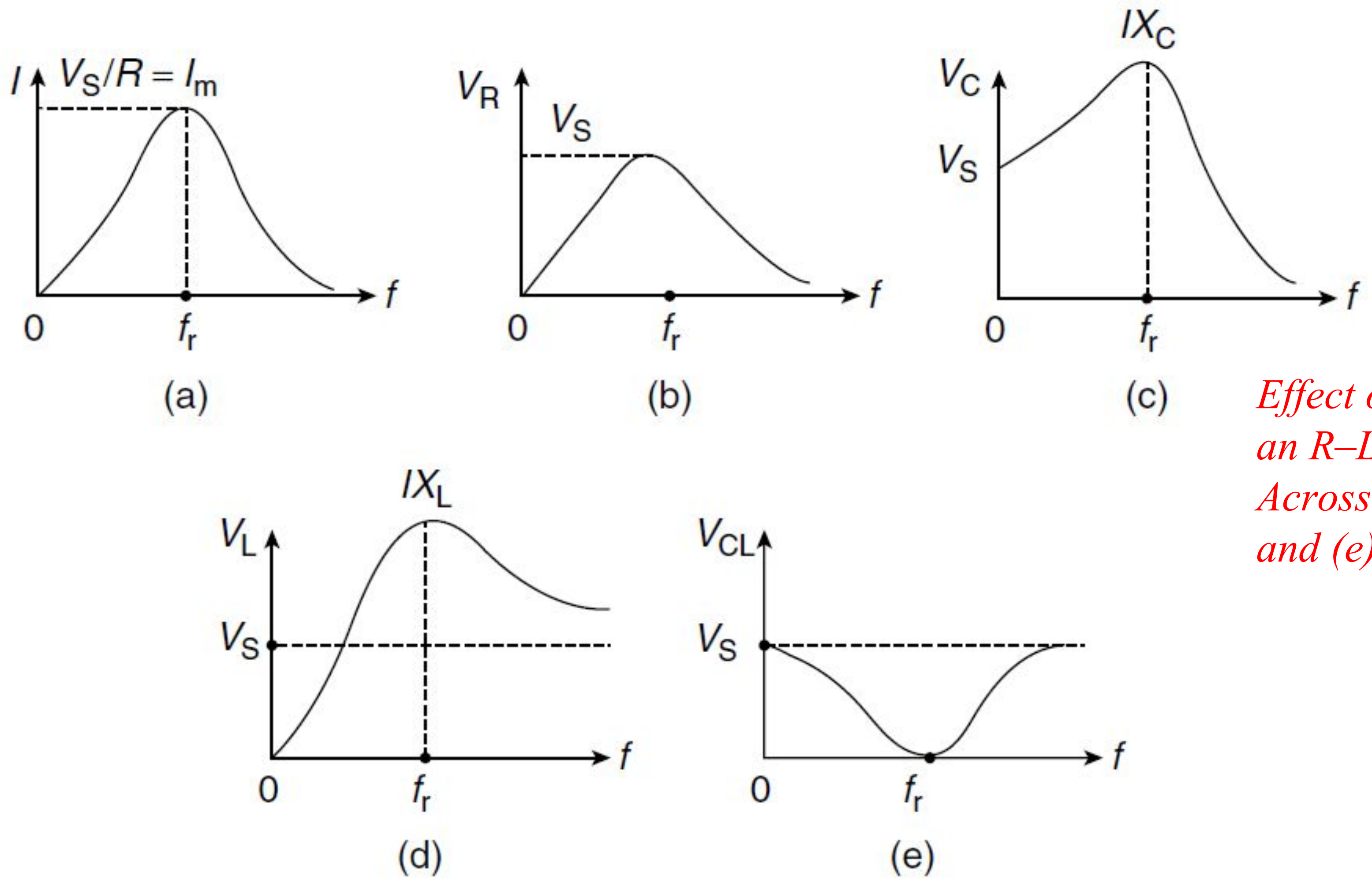
During any given half cycle of the input voltage, polarities of the voltage across inductor and capacitor are of opposite sign.

The condition of resonance can also be written as follows:

$2\pi f_r L = {}^{1}/_{2\pi f_r C}$.

$f_r = {}^{1}/_{2\pi\sqrt{LC}}$.

Effect of variation of frequency on current and voltage drops



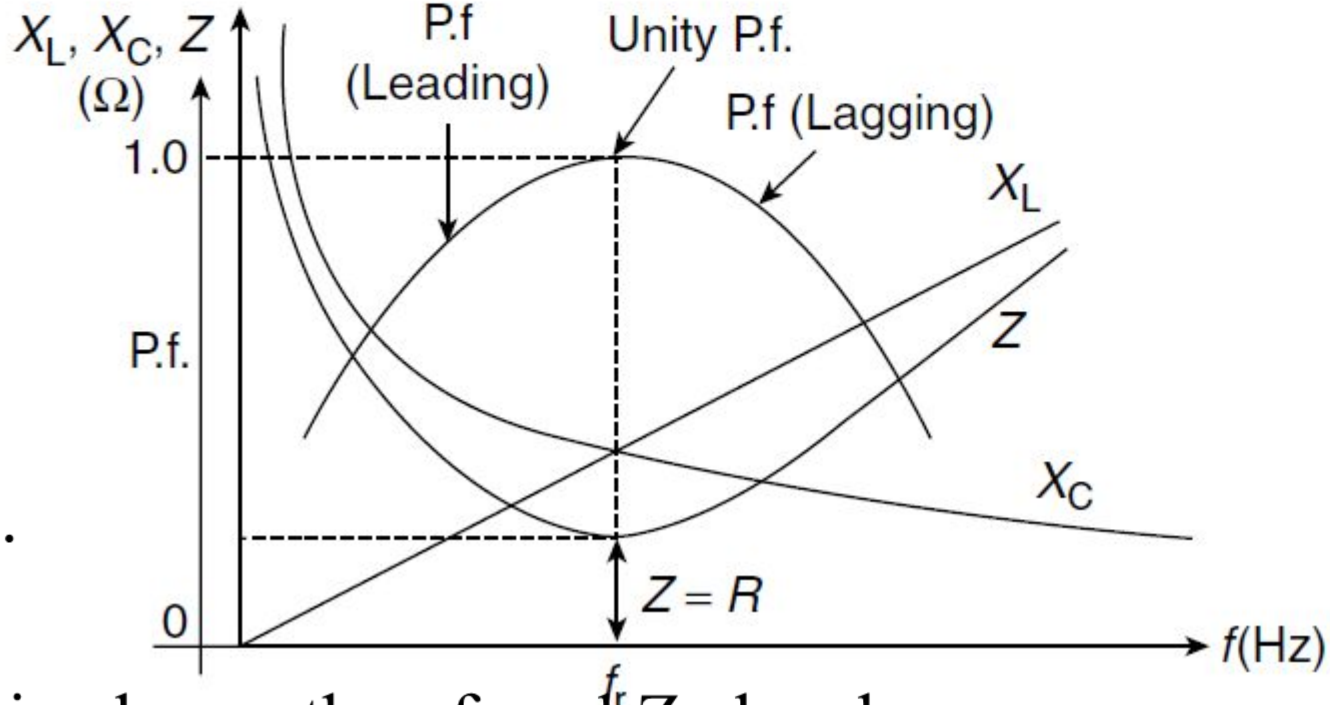
Effect of Variation of Frequency of Supply Voltage in an R–L–C Series Circuit: (a) Current; (b) Voltage Across R; (c) Voltage Across C; (d) Voltage Across L and (e) Voltage Across C and L

At resonant frequency, Z=R, and hence the current is maximum, as Z is minimum. The voltage across capacitor and inductor Can be much larger than the supply voltage but these two voltages oppose each other making their combined voltage as zero.

- Under resonance condition, the impedance of the circuit is minimum and is equal to the resistance of circuit, the current is maximum in the circuit and the voltage across capacitor is equal to inductor which is more than the source voltage (several times).
- So a series resonant circuit is called voltage resonant circuit.
- As frequency is increased above resonant frequency, X_L continues to increase and X_C continues to decrease.
- The total reactance X_L-X_C will increase, and hence the current will decrease.
- As current decreases, the voltage across resistance decreases. Further V_L and V_C will decrease but the difference of V_L and V_C will be increasing.
- When frequency increased to a high value, the circuit current will approach zero,
- Accordingly V_R and V_C will approach zero and V_L approach V_S.

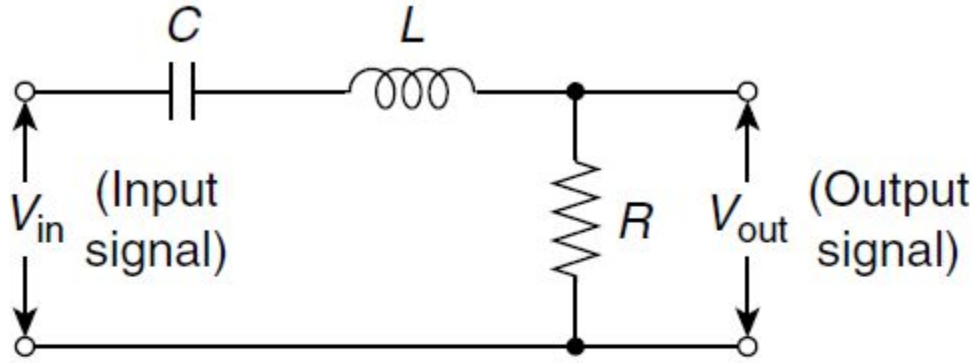
Effect of variation of frequency on Z and PF

- The variation of X_L, X_C and Z with frequency as shown in figure.
- It can be observed that at resonant frequency, impedance of the circuit is minimum (Z=R) and PF=1.
- @f=0, both X_C and Z are ∞ and X_L=0.
- As f↑, X_L ↑ and X_C↓.

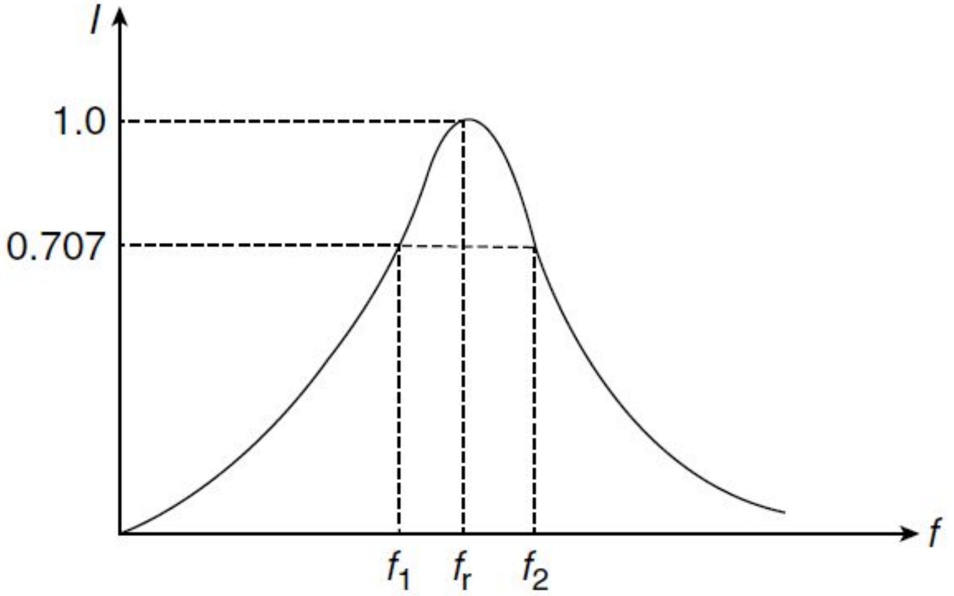


- Since, X_C is larger than X_L at frequencies lower than f_r and Z also decreases along with X_C.
- At frequencies higher than f_r, X_L increases and is higher than X_C, and Z ↑.
- @fq, X_C>X_L and hence I will lead the source voltage V_S.
- @fq > f_r, X_L>X_C, and hence I will lag the source voltage V_S.

RLC series circuit as Band Pass Filter



- A series RLC circuit with LC part is placed in between input and output as shown. The output is taken as resistor.
- At fr, $X_C = X_L$, and cancel each other, the circuit works as band pass filter.
- Signals at fr are allowed to pass from input to output without any reduction in amplitude because LC part is not offering any opposition.



- In fact for a range of frequencies extending below and above f_r, a significant strength of input signal will pass to the output circuit. This band of frequencies is called pass band.
- The signals at frequencies lower or higher than the pass band appearing at input are rejected by the circuit.

Band width of pass band

- The band width of the pass band filter is the range of frequencies at which the current in the circuit is equal to or greater than 70.7 % of its value at resonant frequency.
- The frequencies at which the output of a filter is 70.7 % of its maximum value are called cut-off frequencies f_1 and f_2.
- f_1 is called lower cut off frequency and f_2 is called upper cut-off frequency.
- These two frequencies are also called band frequencies, -3 db frequencies or half-power frequencies.
- Thus, band width (BW)= f_2 - f_1.
- *Half power frequency*: the power from the source at these frequencies is one-half the power delivered at resonant frequency. @resonance: $P_{max} = I_{max}^2 R$.
- The power @f_1 and f_2 are: $(0.707 I_{max})^2 R = 0.5 I_{max}{}^2 R$=0.5 P_{max}.

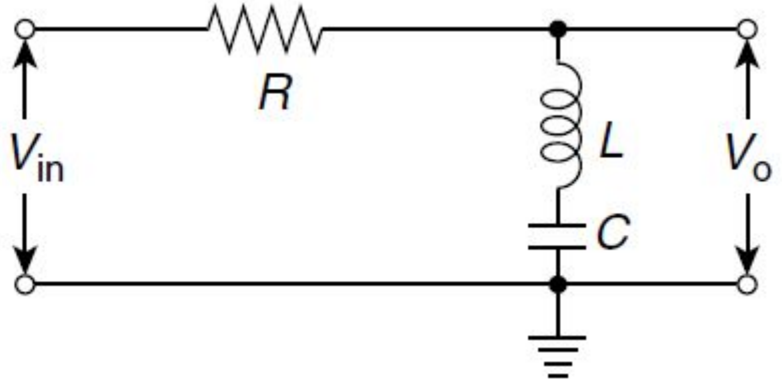
Q-factor of a coil

- Reciprocal of power factor of a coil is known as Q-factor (Figure of merit of coil). Mathematically $Q = \frac{1}{PF} = \frac{1}{\cos\varphi} = \frac{Z}{R}$.
- If the value of R is very small, in comparison to X_L or X_C, then.
- $Q = \frac{X_L}{R} = \frac{\omega L}{R}$.
- Q is also defined as $Q = \frac{Energy\ stored}{Energy\ dissipated}$.

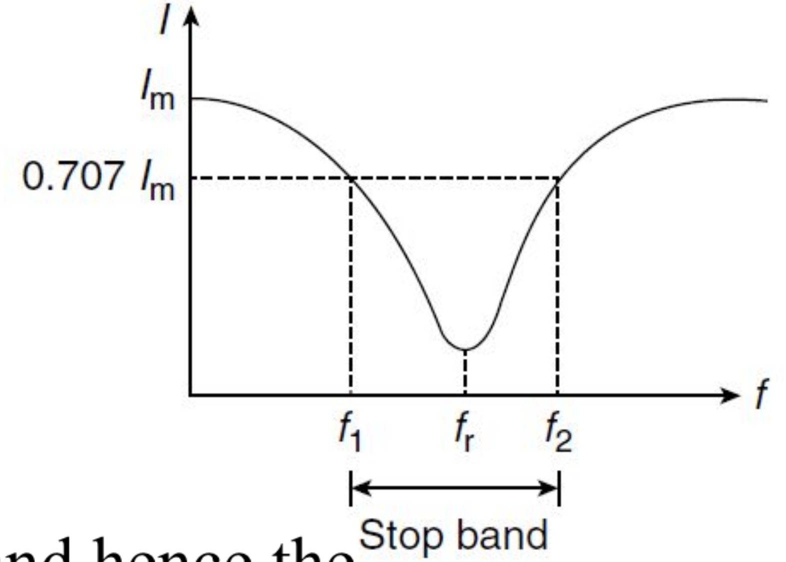
Q-factor of a series RLC resonant circuit

- Q-factor is the ratio of energy stored in the inductor (reactive power) to the true power in the resistance of the circuit.
- It is the ratio of power in L to the power in R.
- $Q = \frac{Energy\ Stored}{Energy\ Dissipated} = \frac{reactive\ power}{real/active\ power} = \frac{I^2 X_L}{I^2 R} = \frac{X_L}{R}$.
- where X_L is the reactance at resonant frequency f_r.
- Selectivity: it indicates how the resonant circuit discriminates certain frequencies against all other frequencies of the signal.
- If the BW is narrow, selectivity is high.
- Relation B/N BW and Q: $BW = \frac{f_r}{Q}$.

Series RLC circuit as band stop filter



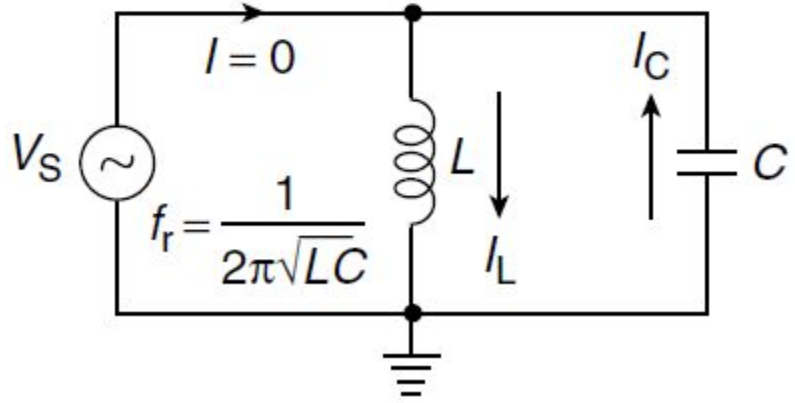
- When the output is taken from LC portion of an RLC circuit, series resonant circuit acts as band stop filter.
- It rejects the signals with the frequencies between lower and upper cutoff frequencies and will pass those signals with frequencies lower than lower cut-off frequency and higher than higher cut-off frequency.
- This filter is called band stop/reject filter (BSF).
- For BSF, output current and voltage are minimum at f_r.

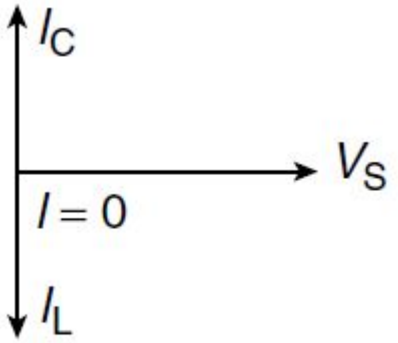


- At very low frequency, XC is very high and XL is very low and hence the combination of LC network appears to be nearly open.
- Hence, almost whole input voltage will appear at output. @fr, impedance offered by LC network is minimum, whole input is grounded and output is negligible.
- As frequency increases, impedance of LC network increases and O/P voltage ↑.

Parallel Resonance

- Let us consider resonance in parallel LC circuit, where the resistance of the inductance coil is neglected.
- The parallel resonance occurs when $X_L = X_C$.
- The frequency at which resonance occurs is resonant frequency. In the circuit shown at f_r, $X_L = X_C$ and
- $I_L = \frac{V_S}{X_L}$ and $I_C = \frac{V_s}{X_C}$. Therefore, $I_L = I_C$.
- As I_L and I_C are in opposite directions, total current I=0.
- At resonant frequency, X_L and X_C are equal, I_L and I_C cancel each other as these are equal in magnitude and opposite in phase. Total current is zero.
- The impedance $Z = \frac{V_S}{I} = \frac{V_S}{0} = \infty$.

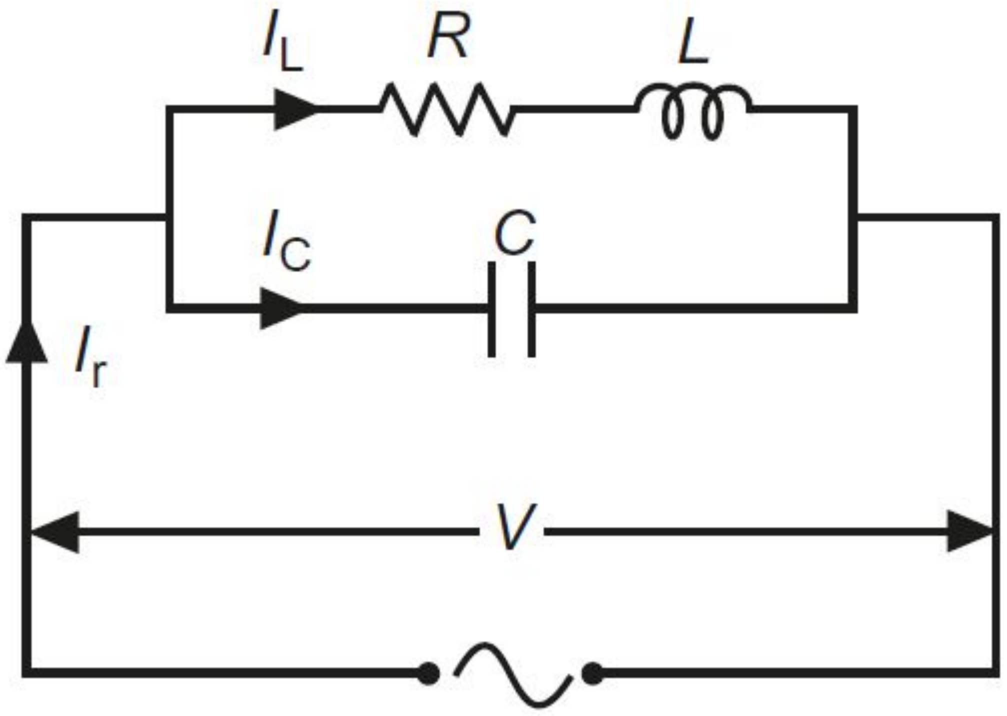


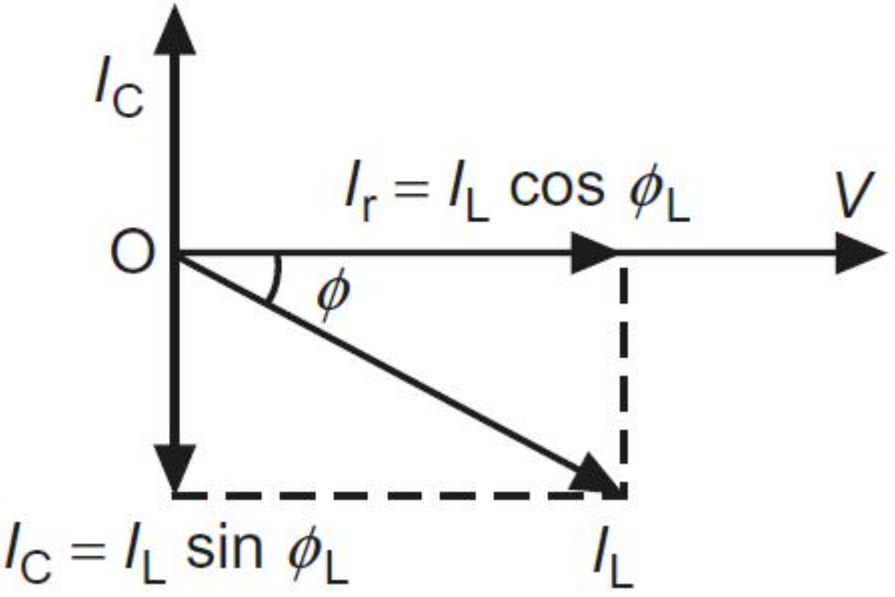


Ideal tank circuit

- In the tank circuit, the current is zero, there exist current in the inductor and capacitor.
- Such a parallel resonant circuit is often called as tank circuit.
- Normally, a tank stores water or some liquid. Here, the circuit stores energy.
- At resonance, energy is stored in the magnetic field of the current carrying inductive coil and in the electric field of the capacitor.
- This stored energy is transferred back and forth between the inductor and capacitor on alternate half cycles.
- On alternate half cycles, the inductor gets energized while the capacitor is de-energized and vice-versa and this process continues definitely.

Parallel Resonance (Non-Ideal Tank circuit)

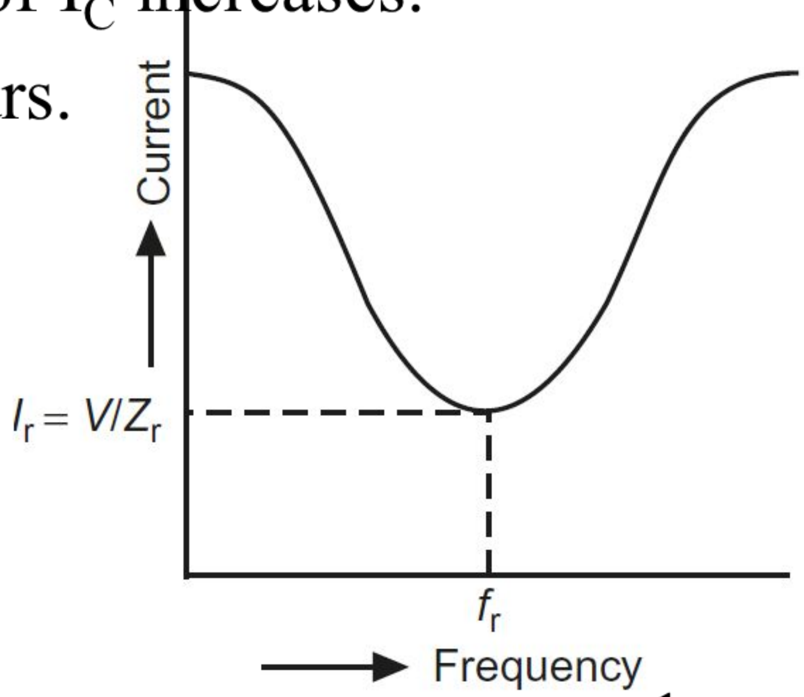




- The circuit current I_r will be in phase with supply voltage when $I_C = I_L \sin \varphi_L$.
- At resonance, the reactive component of current is suppressed, the circuit draws minimum current under this condition.

Resonant Frequency

- The value of $X_L = 2\pi f L$ and $X_C = {}^1/_{2\pi f C}$ can be changed by changing supply frequency.
- When frequency increases, the value of X_L and Z_L ↑. This decreases the magnitude of current I_L that also lags behind the V by a greater angle.
- On the other hand, the value of X_C decreases, the value of I_C increases.
- At some frequency f_r, $I_C = I_L \sin \varphi_L$ and resonance occurs.
- where $I_L = \frac{V}{Z_L}$, $\sin \varphi_L = \frac{X_L}{Z_L}$, and $I_C = \frac{V}{X_c}$.
- $\frac{V}{X_C} = \frac{V}{Z_L}\frac{X_L}{Z_L}$ or $X_L X_C = Z_L^2$ or $\frac{\omega L}{\omega C} = Z_L^2 = (R^2 + X_L^2)$
- $\frac{L}{C} = (R^2 + (2\pi f_r L)^2)$ or $2\pi f_r L = \sqrt{\frac{L}{C} - R^2}$
- $f_r = \frac{1}{2\pi L}\sqrt{\frac{L}{C} - R^2} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$. If R is small as compared to L, then $f_r = \frac{1}{2\pi\sqrt{LC}}$.



Effect of parallel resonance

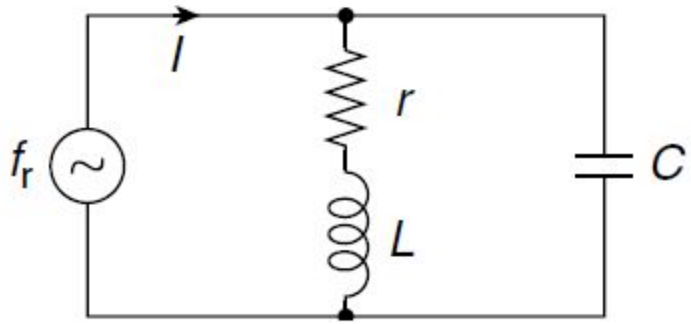
- At parallel resonance, line current $I_r = I_L \cos\varphi$.
- $\frac{V}{Z_r} = \frac{R}{L/C} = \frac{RC}{L}$ (since, $Z_L^2 = {}^L/_C$), ∴circuit impedance, $Z_r = {}^L/_{CR}$.

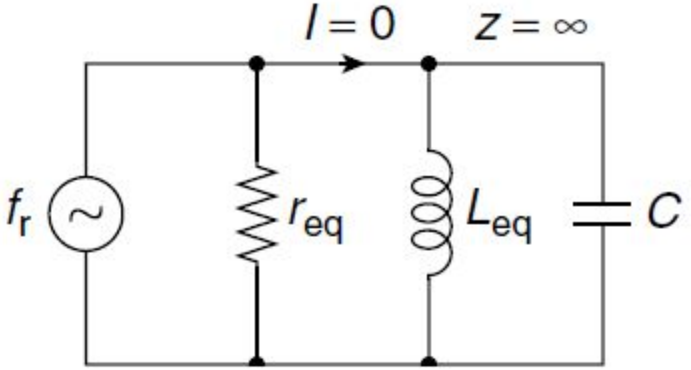
1) Circuit impedance ($Z_r = {}^L/_{CR}$) is a pure resistive because there is no frequency terms.
2) The value of Zr is very high, because the ratio L/C is very large at parallel resonance.
3) The value of circuit current $I_r = {}^V/_{Z_r}$ is very small because the value of Z_r is very high.
4) The current flowing through the capacitor and coil is much greater than the line current because of impedance of each parallel branch is quite low than circuit impedance Z_r.

Q-factor of parallel resonant circuit

- In parallel resonance case, the current circulating between two branches is many times greater than the line current drawn from the mains.
- This current simplification produced by resonance called Q-factor of parallel resonant circuit.
- $Q - factor = \frac{current\ circulating\ between\ L\ and\ C}{Line\ Current} = \frac{I_C}{I_r}$.
- Now, $I_C = \frac{V}{X_C} = 2\pi f_r CV$ and $I_r = {}^{V}/_{L/CR}$.
- Q-factor$=\frac{2\pi f_r CV}{V}\frac{L}{CR} = \frac{2\pi f_r L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$ ….(same as series RL circuit)

Non-ideal Tank circuit





- Now, let us consider the resonance in tank circuit by considering the resistance of inductor coil.
- The equivalent inductance L_{eq} and resistant equivalent r_{eq} are given as follows: $L_{eq} = L(\frac{Q^2+1}{Q^2})$ and $r_{eq} = r(Q^2+1)$. $Q = {}^{X_L}/_{R}$.
- At parallel resonance; $X_{L(eq)} = X_C$.
- The LC branches acts as ideal tank. Since, current I=0, the tank circuit will have infinite impedance at resonance.
- The total impedance of the circuit at resonance is equal to equivalent resistance r_{eq} can be written as $Z_r = r_{eq}(Q^2+1)$

Resonant frequency

- The resonant frequency of the circuit is derived as: $X_{L(eq)} = X_C$
- $2\pi f_r L\left(\frac{Q^2+1}{Q^2}\right) = \frac{1}{2\pi f_r C}$ or $f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{Q^2}{Q^2+1}}$.
- For high values of Q, f_r can be written as follows:
- $f_r = \frac{1}{2\pi\sqrt{LC}}$, therefore, the resonant frequency for parallel resonant circuit is approximately the same as series resonant frequency.

Comparison of Series and Parallel Resonance

Particulars	Series Circuit	Parallel Circuit
Impedance	Minimum, i.e., $Z_r = R$	Maximum, i.e., $Z_r = L/CR$
Current	Maximum, i.e., $I_r = V/R$	Minimum, i.e., $I_r = V/Z_r$
Resonant frequency	$f_r = \dfrac{1}{2\pi\sqrt{LC}}$	$f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$
Power factor	Unity	Unity
Q-factor	X_L/R	X_L/R
Amplification	It amplifies voltage	It amplifies current

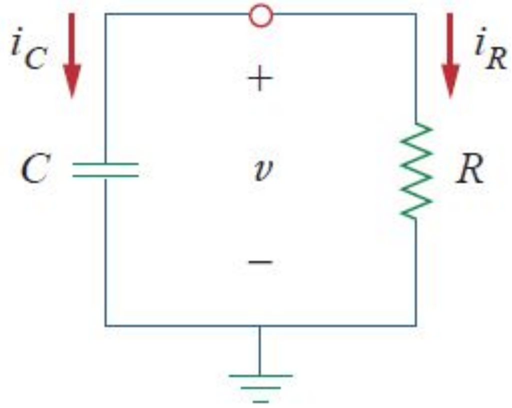
Transient Condition in networks

- Electrical networks contain resistors, inductors, and capacitors. Inductors and capacitors are energy storing devices.
- The energy stored in L & C changes from their initial level to the final level called settling time (generally fraction of a second).
- The time taken by a circuit to change from one steady state condition to another steady state condition is called transient time.
- A transient condition in networks occurs due to switching operations.
- During the transient period, the current and voltages change from their initial values to the new values.

- When a circuit is switched on, there exist two sources of energy in the circuit.
- One source is the initial stored energy in inductances and capacitances at the time of switching.
- The second energy source is the energy, that is, applied externally in the form of voltage or current sources.
- The complete response of a circuit representing a system can be represented in two parts, namely, its forced response or steady state response and transient response.
- The transient response or solution shows the way the circuit responds when a forcing function (a voltage) as input changes in energy state.
- Transient response depends upon the circuit parameters and their initial charge condition.
- Transient response of electrical circuits can be determined either by using differential equations or by using Laplace transform.

- Usually, the analysis of RL or RC circuits can be performed by using kirchoffs laws.
- The application of kirchoffs laws to resistive circuits results in algebraic equations, while applying the to RL and RC circuits leads to differential equations, which are difficult to solve than algebraic equations.
- The differential equations resulting from analyzing RL and RC circuits are of the first order circuits.
- A first order circuit is characterized by a first order differential equations.
- There are two ways to excite the first order circutis: The first is by initial conditions of the storage elements in the circuits so called source-free circuits (energy is initially stored in the capacitive and inductive element).
- Source free circuits are by definition free of independent sources, they may have dependent sources.
- Second way of exciting the first order circuits is by independent sources.

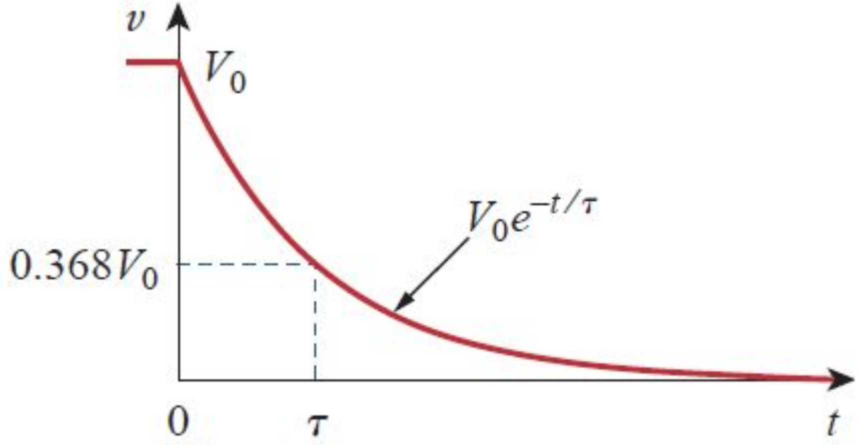
Source free RC circuit



- Source free RC circuit obtained by when its dc source is suddenly disconnected.
- The energy already stored in the capacitor is released to the resistors.
- Let us consider a series combination of resistor and an initially charged capacitor as shown in figure.
- We assume the voltage across the capacitor is $v(t)$, since the capacitor is initially charged, we can assume that at time t=0, the initial voltage is $v(0)=V_0$. The corresponding stored energy is $w(0) = \frac{1}{2}CV_o^2$.
- Applying KCL at the top node: $i_c + i_R = 0$; we know that,
- $i_c = C \, {}^{dv}/_{dt}$ and $i_R = {}^{v}/_{R}$. Thus $C\frac{dv}{dt} + \frac{v}{R} = 0$ or $\frac{dv}{dt} + \frac{v}{RC} = 0$.
- This is the first order differential equation.

- The above equation can be rearranged as $\frac{dv}{v} = -\frac{1}{RC}dt$.
- Integrating on both sides, we get $\ln v = -\frac{t}{RC} + \ln A$.
- where $\ln A$ is the integration constant. Thus $\ln\frac{v}{A} = -\frac{t}{RC}$.
- Taking powers of e produces, $v(t) = Ae^{-t/RC}$.
- But from the initial conditions, $v(0) = A = V_0$.Hence, $v(t) = V_o e^{-t/RC}$.
- This shows that voltage response of RC circuit is exponential decay of the initial voltage. Since the response is due to initial energy stored and physical characteristics of the circuit and not due to some external voltage or current source (it is called natural response).
- *The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.*

- The natural response depends on the nature of the circuit alone, with no external sources. In fact, the circuit has a response only because of the energy initially stored in the capacitor.
- The natural response is illustrated graphically, at t=0, the initial condition can be identified. As t increases, the voltage decreases towards zero.
- The rapidity with which voltage decreases is expressed in terms of time constant, denoted by τ.
- The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

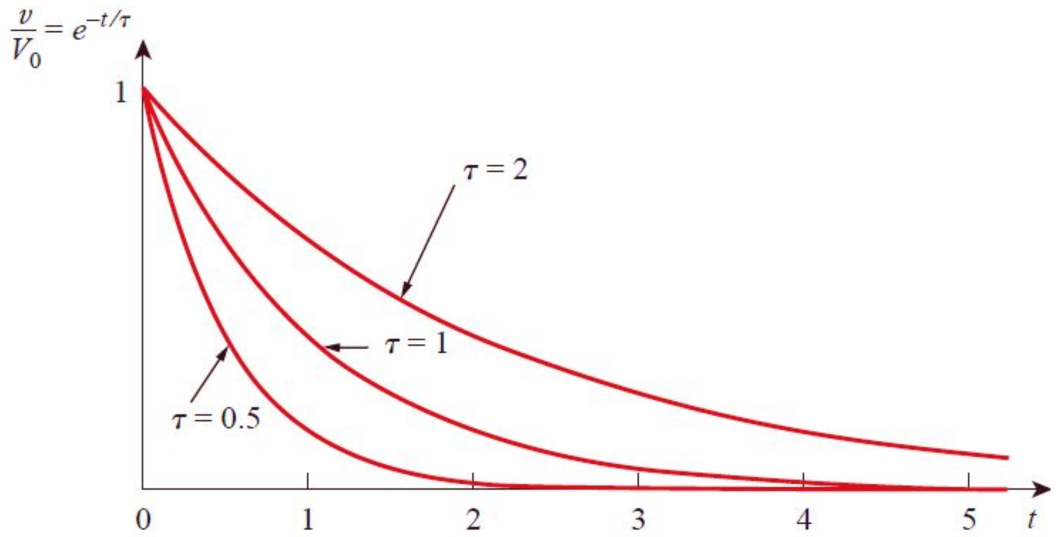


- This implies that at t=τ, $V_o e^{-\tau/RC} = V_o e^{-1} = 0.368V_o$ or $\tau = RC$.
- In terms of time constant, $v(t) = V_o e^{-t/\tau}$.
- From the table, at t=5τ, $v(t)$ is less tan 1% of V_o.

Values of $v(t)/V_0 = e^{-t/\tau}$.

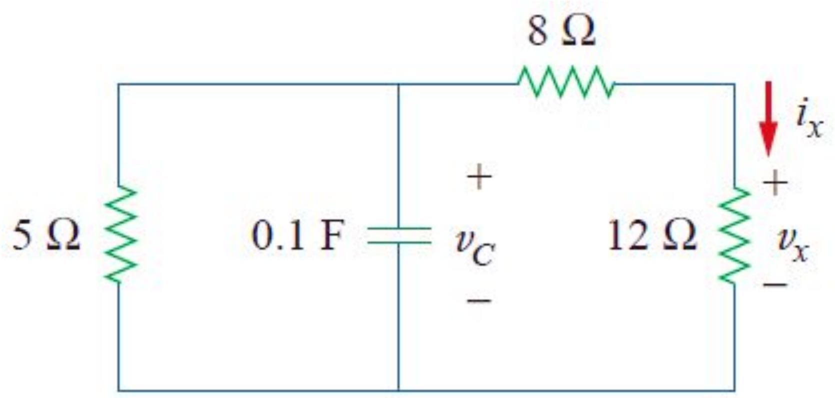
t	$v(t)/V_0$
τ	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674

- It can be assumed that the capacitor is fully discharged after five time constants.
- In other words, it takes 5 time constants for the circuit to reach to its final state or steady state when no changes takes place with time.
- For every time interval of τ, the voltage is reduced to 36.8% of its previous value, $v(t+\tau) = {}^{v(t)}/_{e} = 0.368v(t)$, regardless the value of t.
- If the time constant is smaller, voltage decreases rapidly.
- A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state.

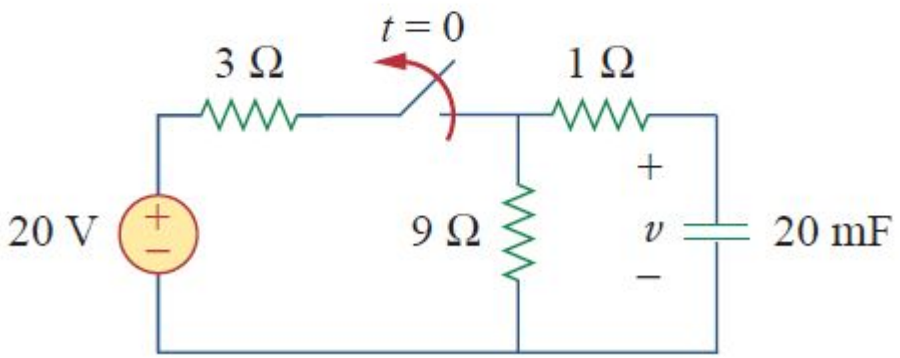


- Even if the time constant is larger or smaller, the circuit reaches the steady state only after five time constants.
- With the voltage $v(t)$, we can find $i_R(t)$, $i_R(t) = \frac{v(t)}{R} = \frac{V_o}{R} e^{-t/\tau}$.
- The power dissipated in the resistor is $p(t) = v i_R = \frac{V_o^2}{R} e^{-2t/\tau}$.
- The energy absorbed by the resistor up to time *t* is:
- $w_r(t) = \int_0^t p\, dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt = -\frac{\tau V_o^2}{2R} e^{-\frac{2t}{\tau}} |_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau})$
- Note that, as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_c(0)$, the energy initially stored in the capapcitor.
- The energy that was initially stored in the capacitor is eventually dissipated in the resistor.

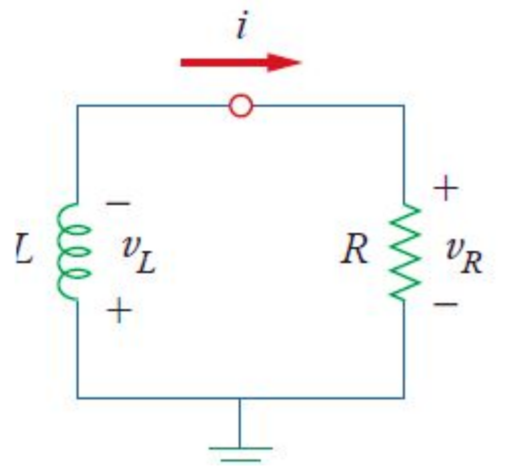
Example: For the circuit shown below, let $v_c(0) = 15V$.
Find v_c, v_x and i_x for t>0.



Example: The switch in the circuit shown below, has been closed for long time, and it opened at t=0. find $v(t)$ for $t \geq 0$. calculate the initial energy stored in the capacitor.

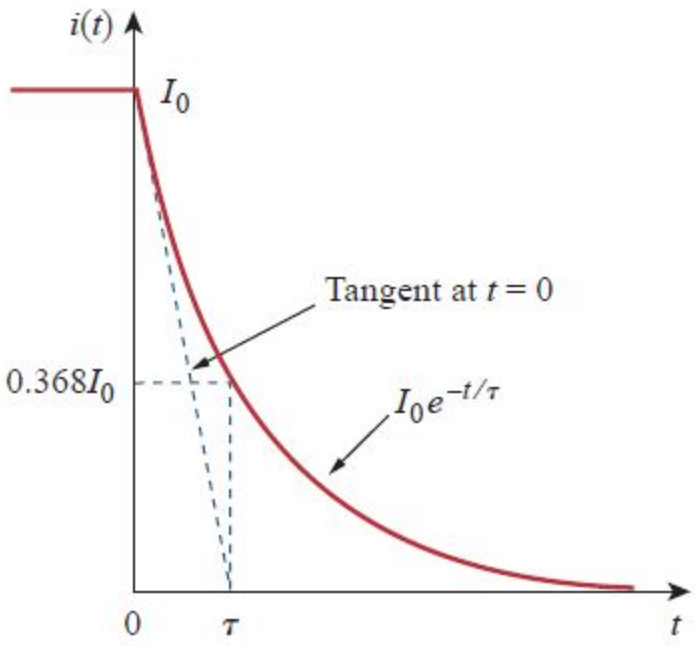


Source free RL circuit

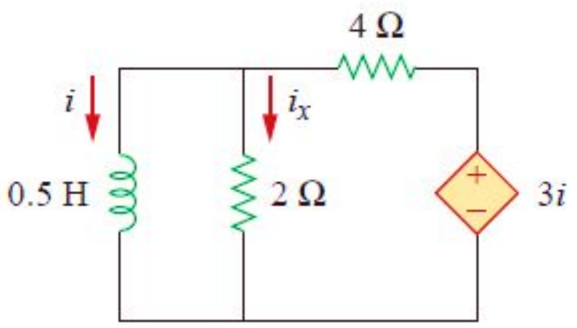


- Source free RL circuit obtained by when its dc source is suddenly disconnected.
- The energy already stored in the inductor is released to the resistor.
- Let us consider a series combination of resistor and an initially charged inductor as shown in figure.
- We assume the current through the inductor is $i(t)$, the inductor current as the response in order to take advantage of the idea that the inductor current cannot change instantaneously. At time t=0, we assume that the inductor has an initial current is I_0 or $i(0) = I_o$. The corresponding stored energy is $w(0) = \frac{1}{2} L I_o^2$.
- Applying KVL around the top node: $v_L + v_R = 0$; we know that,
- $v_L = L \, {}^{di}/_{dt}$ and $v_R = iR$. Thus $L\frac{di}{dt} + iR = 0$ or $\frac{di}{dt} + \frac{iR}{L} = 0$.
- This is the first order differential equation.

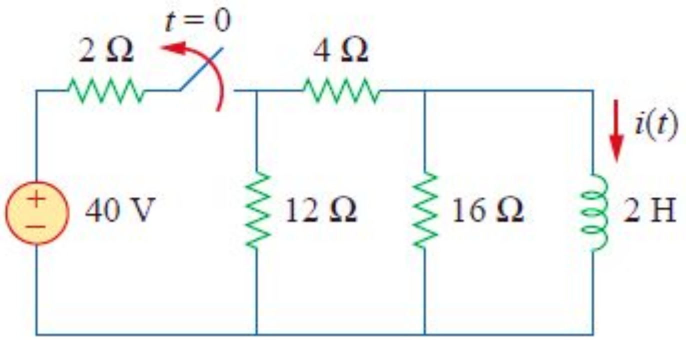
- The above equation can be rearranged as $\int_{o}^{i(t)} \frac{di}{i} = -\int_{0}^{t} \frac{R}{L} dt$.
- Integrating on both sides, we get $\ln(\frac{i(t)}{I_o}) = -\frac{Rt}{L}$.
- Taking powers of *e* produces, $i(t) = I_o e^{-Rt/L}$.
- This shows that the natural response of the RL circuit is an exponential decay of the initial current. The time constant of RL circuit is $\tau = \frac{L}{R}$.
- $i(t) = I_o e^{-t/\tau}$
- We can find the voltage across resistor as $v_R(t) = iR = I_o R e^{-t/\tau}$.
- Power dissipated=$p = v_R i = I_o^2 R e^{-2t/\tau}$.
- The energy absorbed by resistor is
- $w_R(t) = \int_0^t P.dt = \int_0^t I_o^2 R e^{-2t/\tau} dt = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$
- As $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L I_0^2$, which is same as $w_L(0)$.



Example: Assuming that, $i(0)$=10A, calculate $i(t)$ and $i_x(t)$ in the circuit shown below.



Example: The switch in the circuit has been closed for a long time. At time t=0, the switch is opened. Calculate $i(t)$ for t>0.



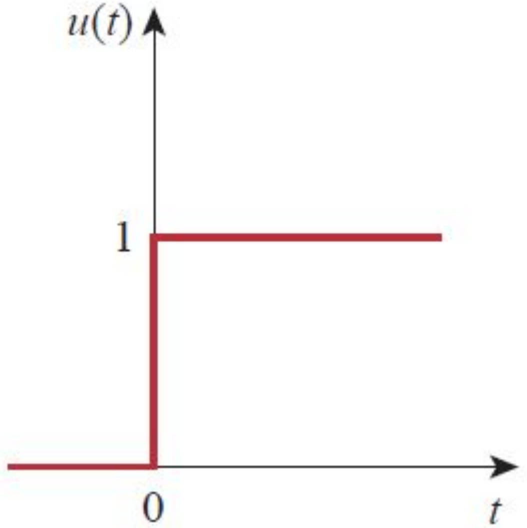
Singularity functions

- Singularity functions also called as switching functions. They serve as good approximations to the switching signals that arise in circuits with switching operations.
- Singularity functions are the functions that either are discontinuous or have discontinuous derivatives.
- The three most widely used singularity functions are : Unit Step, Unit Impulse and Unit Ramp.

Unit Step function

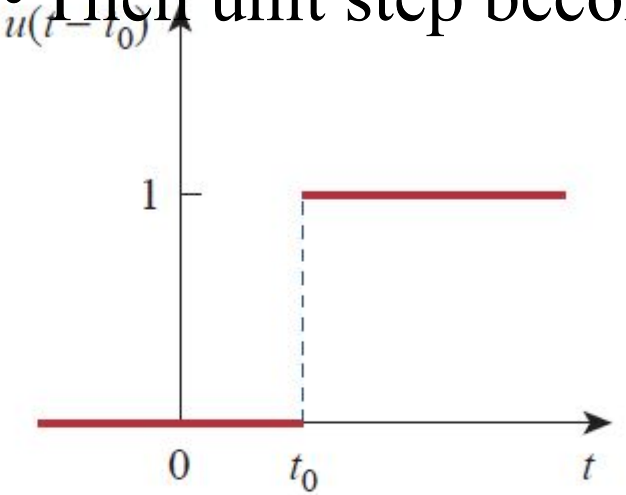
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

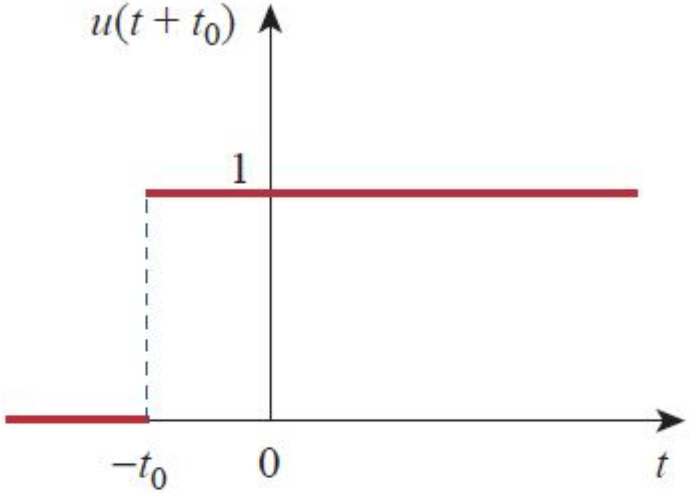
- The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.
- It is undefined at t=0.
- It changes abruptly from 0 to 1.
- It is dimensionless, like other functions sine and cosine.
- If the abrupt change occurs at $t=t_o$, instead of t=0,
- Then unit step becomes



$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

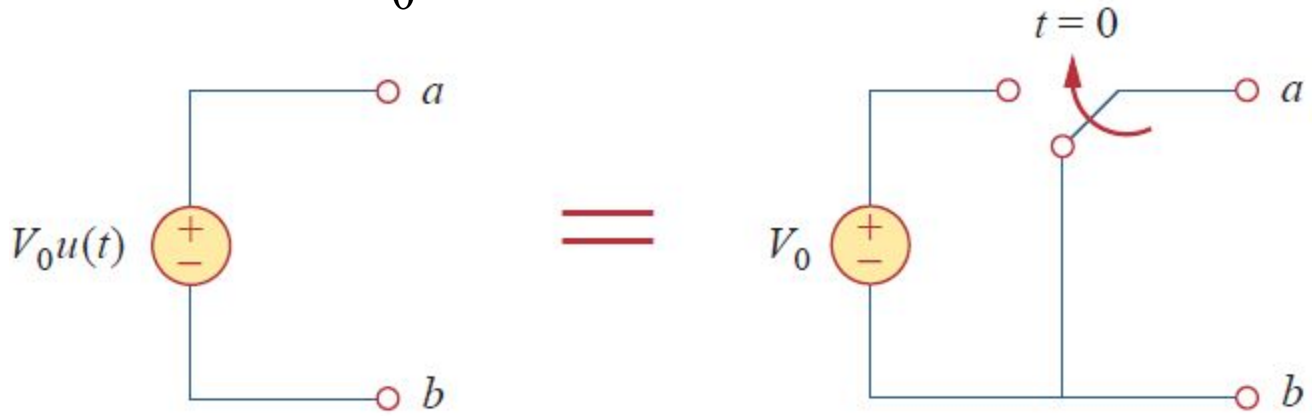




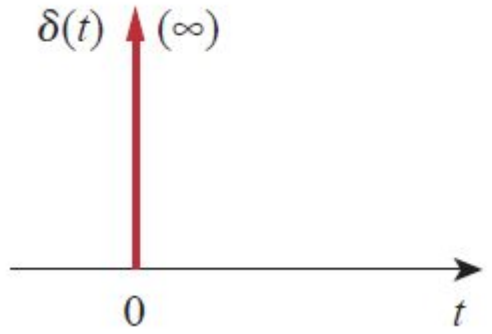
- The step function is used to represent an abrupt change in voltage, current, like other changes that occur in circuits of control systems and digital computers. For example, the voltage may be expressed in terms of step function as $v(t) = V_0 u(t - t_0)$

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

- If we let $t_0=0$, then $v(t)$ is simply the step voltage $V_0 u(t)$.
- A voltage of source of $V_0 u(t)$ is as shown, its equivalent circuit is shown in figure below. From the figure, the terminals ab short circuited for $t<0$ and $v=V_0$, appear at terminals for $t>0$



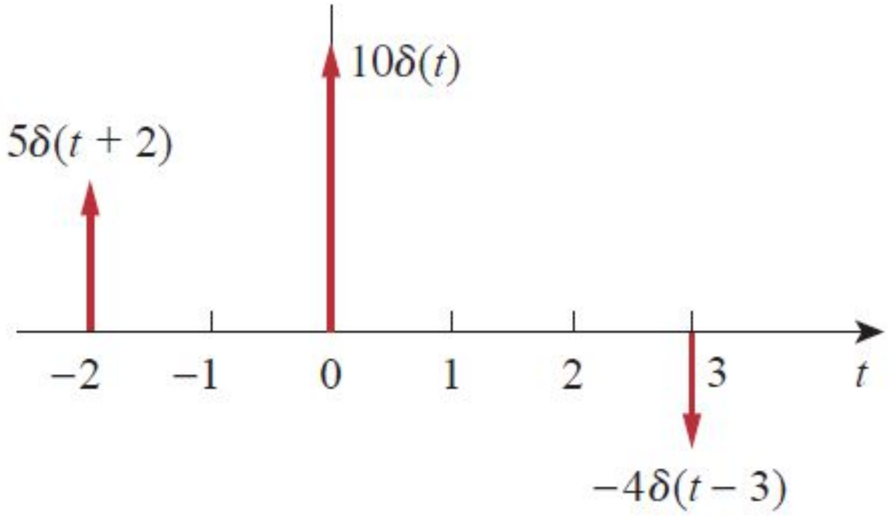
- The derivative of the unit step function $u(t)$ is the unit impulse function $\delta(t)$, which can be written as



$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

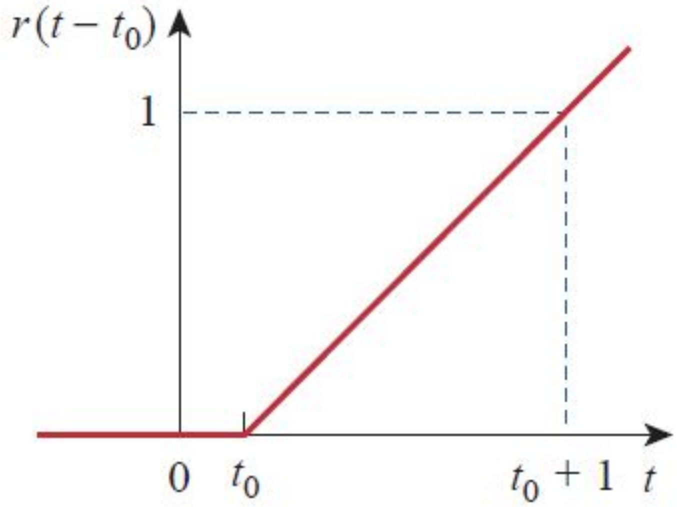
- The unit impulse function also called as delta function.
- The unit impulse function $\delta(t) = 0$ everywhere except at t=0, where it is undefined.
- Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources.
- The unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematical tool.
- The unit impulse may be regarded as an applied or resulting shock.
- It may be visualized as a very short duration pulse of unit area.

- This may be expressed as $\int_{0^-}^{0^+} \delta(t)\, dt = 1$
- where $t = 0^-$ denotes the time just before t=0 and $t = 0^+$ is the time just after t=0.
- For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function.
- The unit area is known as the strength of the impulse function.
- When an impulse function has a strength other than unity, the area of the impulse is equal to its strength.



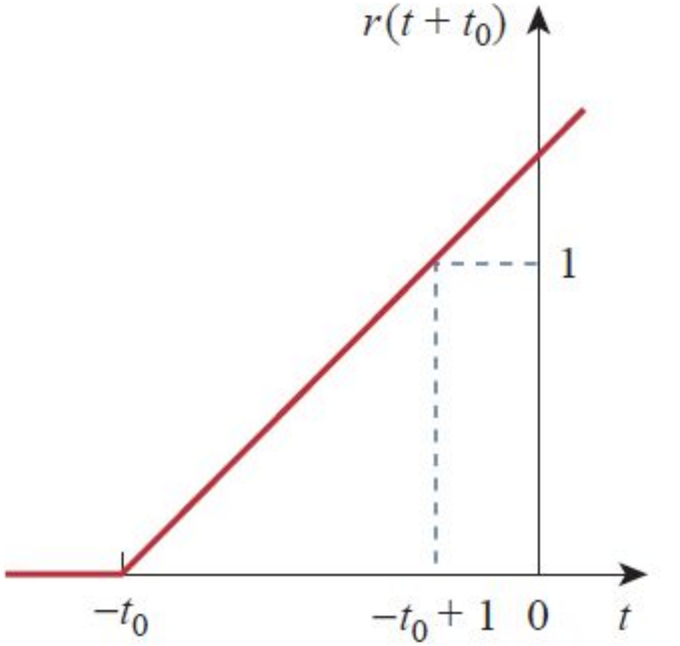
- To understand how the impulse function affects other functions, let us evaluate the integral $\int_a^b f(t)\delta(t-t_o)dt$, where a< t_o<b. since $\delta(t-t_o)$=0 except at t=t_o, the integrand is zero except at t_o. Thus,
- $\int_a^b f(t)\delta(t-t_o)dt = \int_a^b f(t_o)\delta(t-t_o)dt = f(t_o)\int_a^b \delta(t-t_o)dt = f(t_o)$.
- When a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs.
- This is a highly useful property of the impulse function known as sampling or shifting property.
- The special case for $t_0 = 0$. then above equation becomes,
  $\int_{0^-}^{0^+} f(t)\delta(t)dt = f(0)$
- Integrating the unit step function $u(t)$ results in unit ramp function $r(t)$, thus
- $$r(t) = \int_{-\infty}^{t} u(t)dt = tu(t) \qquad r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

- The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.
- In general, a ramp is a function that changes at a constant rate.
- The unit ramp function may be delayed or advanced.



$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t-t_0, & t \ge t_0 \end{cases}$$

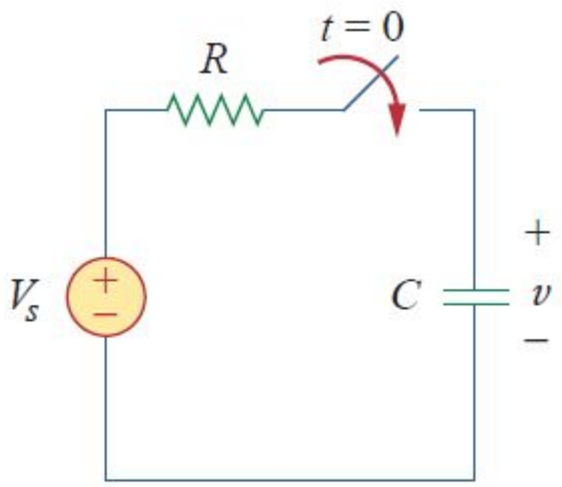
$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t+t_0, & t \ge -t_0 \end{cases}$$

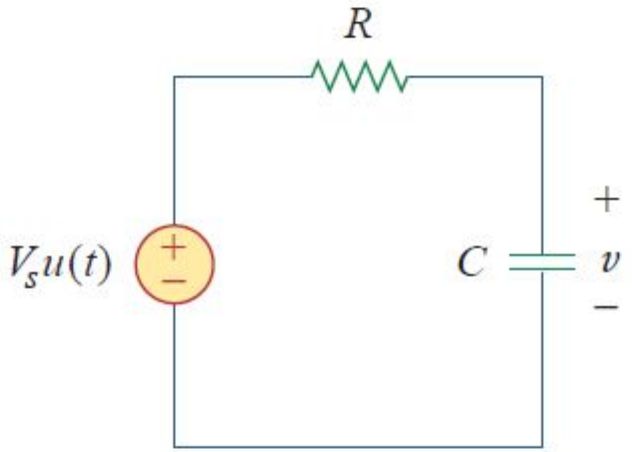


- Singularity functions related by differentiation as
- $r(t), u(t) = \dfrac{dr(t)}{dt}, \delta(t) = \dfrac{du(t)}{dt}$. Similarly by integration
- $\delta(t), u(t) = \int_{-\infty}^{t} \delta(t)dt, r(t) = \int_{-\infty}^{t} u(t)dt$

Step response of an RC circuit

- When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a step response.
- ***The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.***
- The figure shown (top) can be replaced by figure (bottom).
- Let us assume an initial voltage across capacitor is V_o, although this is not necessary for the step response. Since the voltage across capacitor can not change instantaneously, $v(0^-) = v(0^+) = V_o$.





- On applying KVL to the above circuit, $C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$ $\quad \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$
- where v, is the voltage across capacitor. For t > 0, becomes $\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$
- On rearranging the terms, $\frac{dv}{dt} = -\frac{v - V_s}{RC}$ $\quad \frac{dv}{v - V_s} = -\frac{dt}{RC}$
- Integrating on both sides and introducing initial conditions,

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t \qquad \ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0 \qquad \ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

- Taking exponential on both sides, $\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$

$$v - V_s = (V_0 - V_s)e^{-t/\tau} \qquad v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$$

- Thus

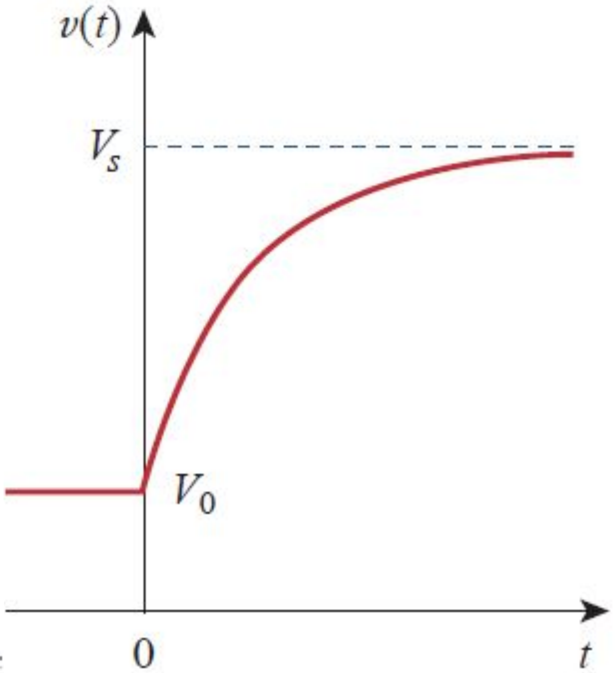
$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

- This is known as complete response of the circuit to a sudden application of a dc voltage source, assuming the capacitor voltage is initially charged.

- If we assume that the capacitor is uncharged initially, we set $V_0 = 0$. Then

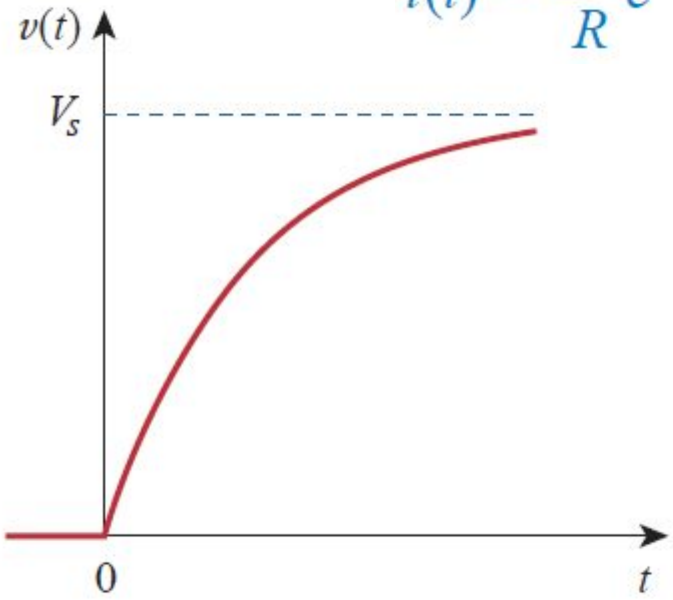
$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

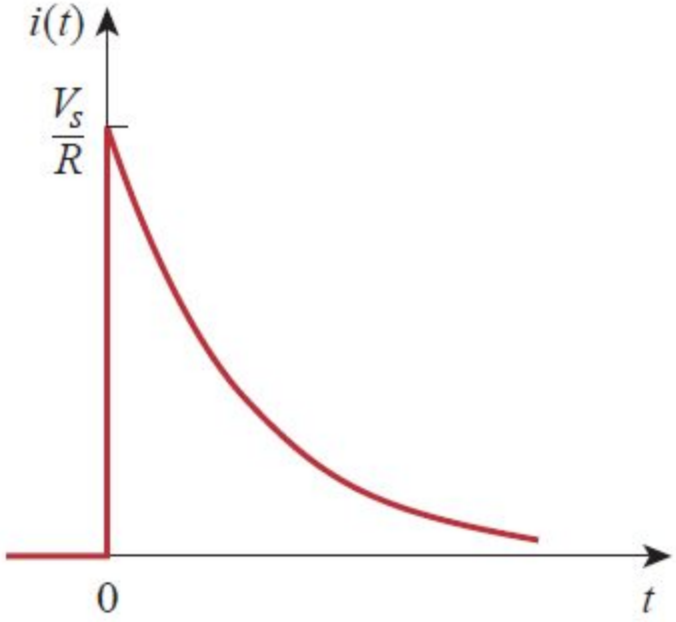
- Or can be written as $v(t) = V_s(1 - e^{-t/\tau})u(t)$
- This is the complete step response of the RC circuit when the capacitor is initially uncharged. Current through the capacitor is given by using $i(t) = C\frac{dv}{dt}$. $i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}$



Response of an *RC* circuit with initially charged capacitor

$$i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$





Step response of an *RC* circuit with initially uncharged capacitor: (a) voltage response, (b) current response.

- $v(t)$ has two components, there are two ways of decomposing this into two components.
- The first is to break it into a natural response and a forced response. The second is to break into transient and a steady state response.
- Starting with the natural response and forced response, we write the total response as $\underset{\text{stored energy}}{\text{Complete response} = \text{natural response}} + \underset{\text{independent source}}{\text{forced response}}$ $\quad v = v_n + v_f$
- where $v_n = V_o e^{-t/\tau}$ $\qquad v_f = V_s(1 - e^{-t/\tau})$
- The natural response of the circuit is v_n and v_f is known as forced response because it is produced by circuit when the external force is applied.
- The natural response eventually dies out along with the transient component of the force response, leaving only the steady state component of the forced response.

- Another way of looking at the complete response is to break into two components—one temporary and the other permanent, i.e.,

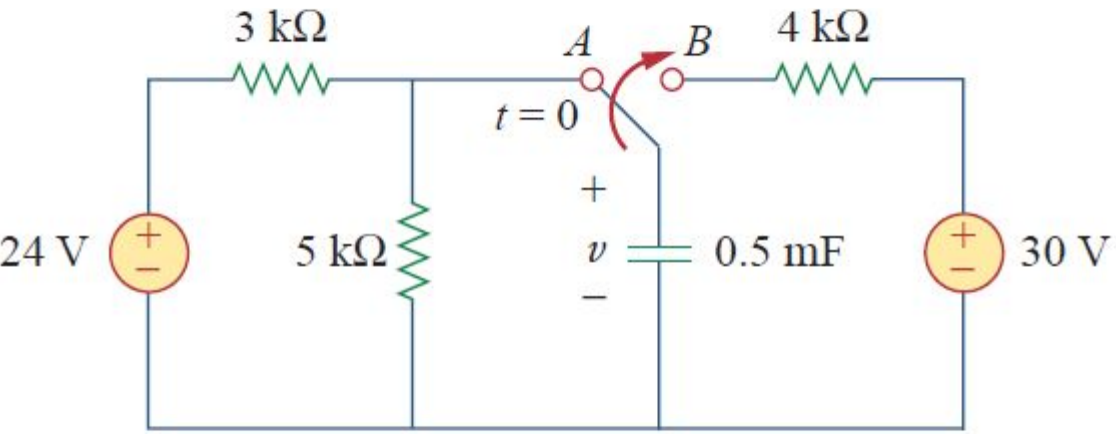
$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

$$v = v_t + v_{ss} \qquad v_t = (V_o - V_s)e^{-t/\tau} \qquad v_{ss} = V_s$$

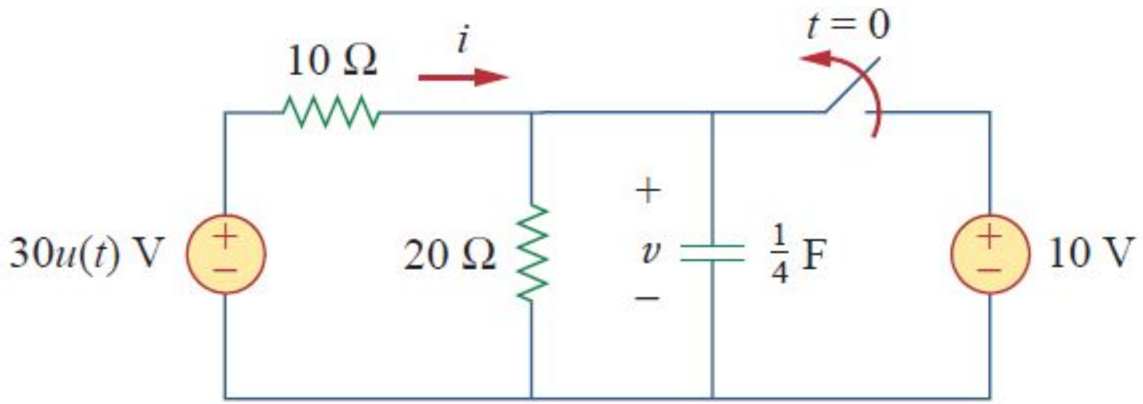
- The transient response is the circuit's temporary response that will die out with time.
- The steady-state response is the behavior of the circuit a long time after an external excitation is applied.
- The first decomposition of the complete response is in terms of the source of the responses, while the second decomposition is in terms of the permanency of the responses.
- Under certain conditions, the natural response and transient response are the same. The same can be said about the forced response and steady-state response.

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \qquad v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau}$$

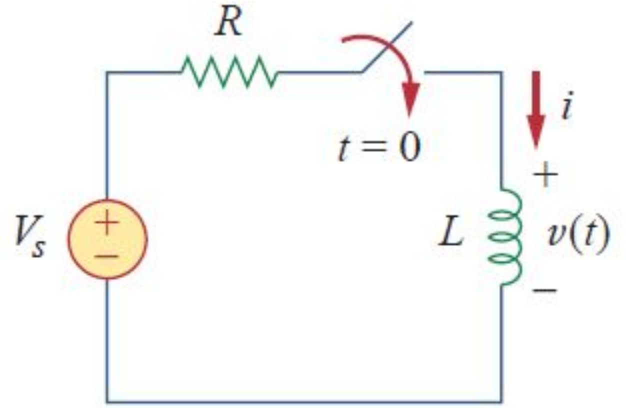
- Example: The switch in Figure has been in position A for a long time. At t=0, the switch moves to B. Determine $v(t)$ for t>0 and calculate its value at t =1 s and 4 s.

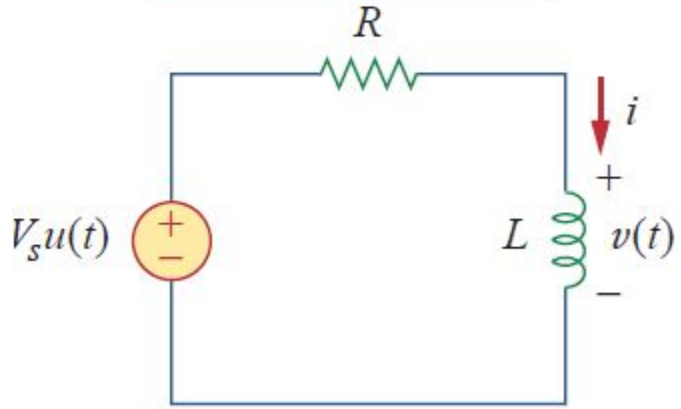


- Example: In figure shown below, the switch has been closed for a long time and is opened at t=0. Find *i* and *v* for all time.



Step response of RL circuit





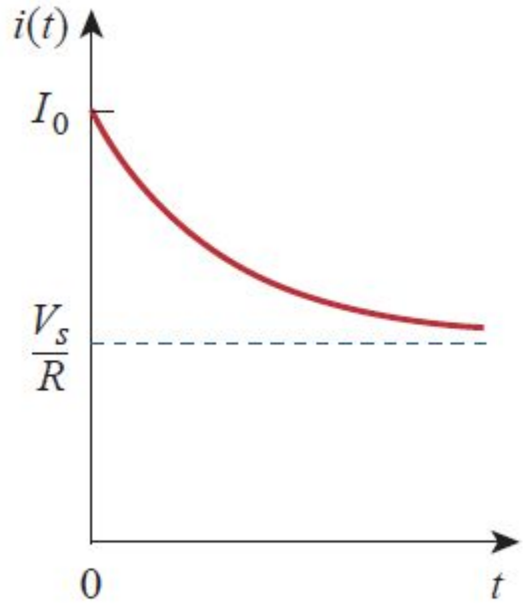
- Consider RL circuit shown in figure (top) can be replaced as figure (bottom).
- Our goal is to find the inductor current *i* as the circuit response. Rather than apply kirchoffs laws, we will use simple techniques.
- Let the response be the sum of the transient response and the steady-state response. $i = i_t + i_{ss}$.
- The transient response is a decaying exponential i.e,

$i_t = Ae^{-t/\tau}$ where $\tau = L/R$ and A is constant to be found.

The steady-state response is the value of the current a long time after the switch in Figure (top) is closed. The transient response essentially dies out after five time constants. *At that time, the inductor becomes a short circuit, and the voltage across it is zero. The entire source voltage V_s appears across R.*

- Thus, the steady state response is $i_{ss} = {}^{V_S}/_{R}$, $\quad i = Ae^{-t/\tau} + \frac{V_s}{R}$
- Now we will determine the constant A from the initial value of i.
- Let I_o be the initial current through the inductor, which may come from a source other than V_s.
- Since, the current through the inductor cannot change instantaneously,
- $i(0^+) = i(0^-) = I_o$. Thus at t=0, $I_o = A + \frac{V_s}{R}$. From this we obtain A as
- $A = I_o - \frac{V_s}{R}$, thus $i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$
- This is the complete response of the RL circuit. It is shown as.

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$



- where $i(0)$ and $i(\infty)$ are the initial and final values of i.
- Thus, to find the step response of an RL circuit 3 things:

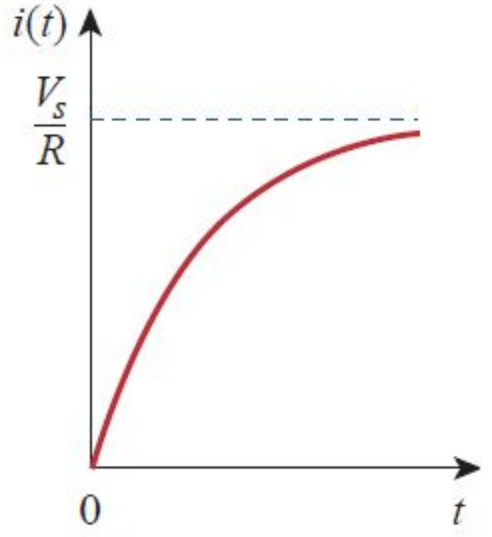
1. The initial inductor current $i(0)$ at $t = 0$
2. The final inductor current $i(\infty)$.
3. The time constant τ.

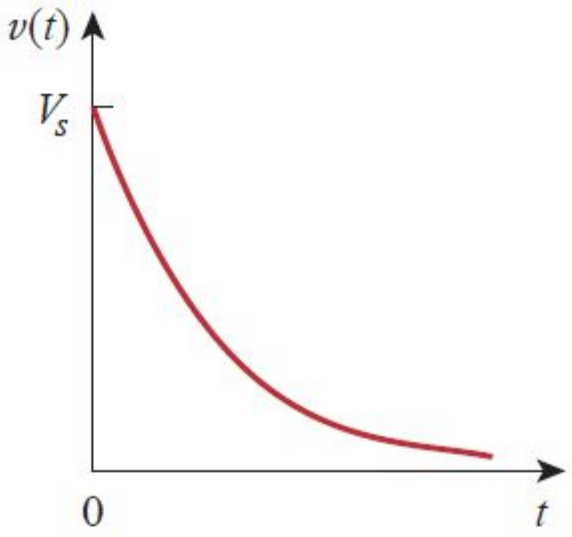
- We can obtain item1 from the given circuit for t<0 and items 2 and 3 from the circuit for t>0.
- Switching takes place at time t=t_o, instead of t=0, becomes,
- $$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$
- If I_o=0, then

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \qquad v(t) = V_s e^{-t/\tau} u(t)$$

$$i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

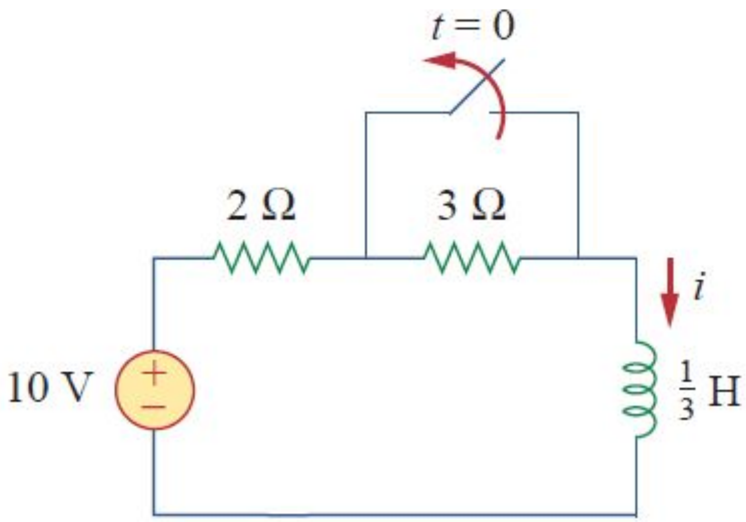
- This is the step response of the *RL* circuit with no initial inductor current.
- The voltage across inductor can be obtained as $v = L\,{}^{di}/_{dt}$.
- $$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \qquad t > 0 \qquad v(t) = V_s e^{-t/\tau}u(t)$$



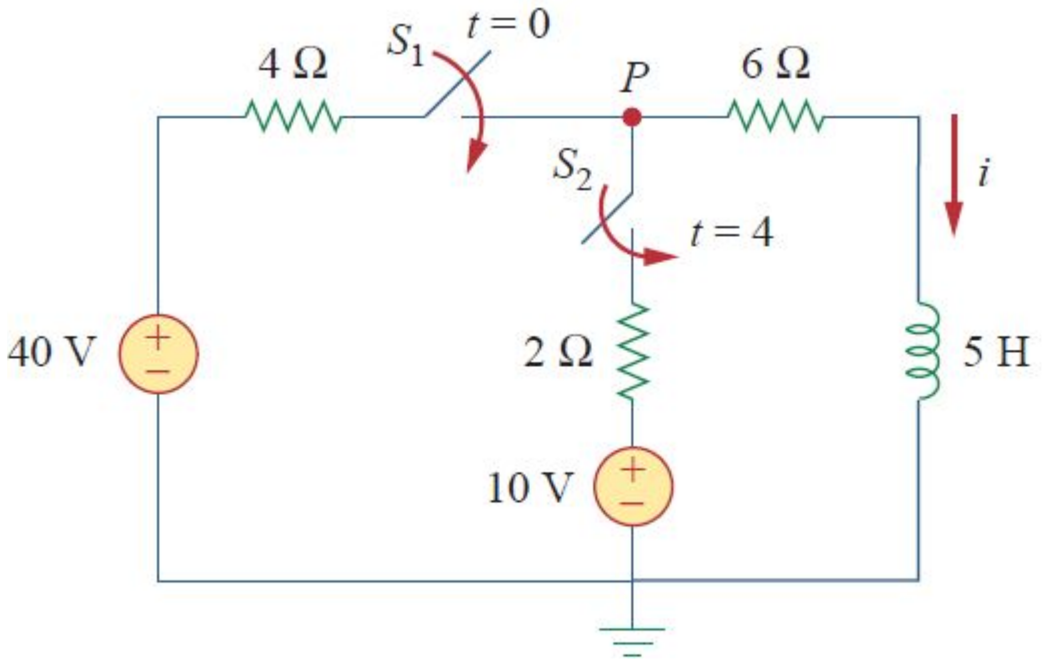


Step responses of an *RL* circuit with no initial inductor current: (a) current response, (b) voltage response

- Example: Find $i(t)$ in the circuit shown below for $t>0$, Assume that the switch has been closed for long time.



- Example: At t=0, switch in figure is closed, and switch 2 is closed 4s later. Find $i(t)$ for t>0. Calculate i for t=2s, and t=5s.



Laplace transform

- Usually differential equations are used to describe the complete behavior of the circuit.
- Laplace transform is a powerful tool for turning differential equations into algebraic equations, thus greatly facilitates the solution process.
- The Laplace transform method follows the same process (as in phasor approach): we use the Laplace transformation to transform the circuit from the time domain to the frequency domain, obtain the solution, and apply the inverse Laplace transform to the result to transform it back to the time domain.
- The Laplace transform is significant for a number of reasons.
- First, it can be applied to a wider variety of inputs than phasor analysis.
- Second, it provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.
- Third, the Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses.

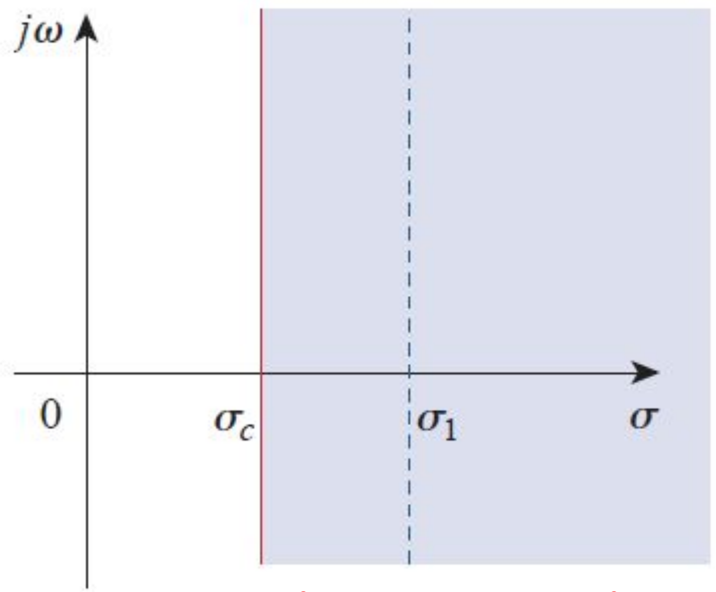
- Definition of the Laplace transform:
- Given a function $f(t)$, its laplace transform denoted by F(s) or L[f(t)], is defined by

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

where s is a complex variable given by $s = \sigma + j\omega$

- Since, the argument st of the exponent e, must be dimensionless, it follows that s has the dimensions of frequency and units of inverse second (s^{-1}) or frequency.
- The lower limit is specified as 0^- to indicate a time just before t=0.
- We use $t=0^-$, as the lower limit to include the origin and capture any discontinuity of $f(t)$ at t=0, this will accommodate functions, such as singularity functions, that may be discontinuous at t=0.
- The *Laplace Transform* is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.
- When the laplace transform is applied to circuit analysis, the differential equations represent in the circuit in time domain. The terms in the differential equations take the place of f (t). Their Laplace transform, which corresponds to F(s),

- In order f(t) to have laplace transform the integral must converge to a finite value. Since $\left|e^{j\omega t}\right| = 1$, for any value of t, the integral converges, when $\int_{0^-}^{\infty} e^{-\sigma t}|f(t)|dt < \infty$. For some real value $\sigma = \sigma_c$.
- Thus, the region of convergence for the laplace transform is Re(s)= $\sigma > \sigma_c$.
- In this region, |F(s)|< ∞ and F(s) exist. F(s) is undefined outside the ROC.



Region of convergence for the Laplace transform.

Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$, and (c) $\delta(t)$.

Determine the Laplace transform of $f(t) = \sin \omega t\, u(t)$.

Properties of the Laplace Transform

- Linearity: $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$
- Scaling: $\mathcal{L}[f(at)] = \int_{0^-}^{\infty} f(at) e^{-st}\, dt$ $\quad \mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$
- Time Shift: $\mathcal{L}[f(t-a)u(t-a)] = \int_{0^-}^{\infty} f(t-a)u(t-a)e^{-st}\, dt$

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

- Frequency Shift: $\mathcal{L}[e^{-at} f(t) u(t)] = F(s+a)$
- Time Differentiation: $\mathcal{L}[f'(t)] = sF(s) - f(0^-)$

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - s^0 f^{(n-1)}(0^-)$$

- Time Integration:

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \frac{1}{s} F(s)$$

- Frequency Differentiation: $\mathcal{L}[tf(t)] = -\dfrac{dF(s)}{ds}$
- Time Periodicity: $F(s) = \dfrac{F_1(s)}{1 - e^{-Ts}}$
- Initial and Final Values: The initial-value and final-value properties allow us to find the initial value $f(0)$ and the final value $f(\infty)$ of $f(t)$ directly from its Laplace transform F(s).
- To obtain these properties, we begin with the differentiation property
- $sF(s) - f(0) = \mathcal{L}\left[\dfrac{df}{dt}\right] = \displaystyle\int_{0^-}^{\infty} \dfrac{df}{dt} e^{-st}\, dt$
- If we let $s \to \infty$, the integrand vanishes due to the damping exponential factor, and becomes $\lim_{s\to\infty}[sF(s) - f(0)] = 0$.
- Since $f(0)$ is independent of s, we can write, $f(0) = \lim\limits_{s\to\infty} sF(s)$
- This is known as initial value theorem.

- If we let $s \to 0$, then $\lim_{s\to 0}[sF(s) - f(0^-)] = \int_{0^-}^{\infty} \frac{df}{dt} e^{0t}\, dt = \int_{0^-}^{\infty} df = f(\infty) - f(0^-)$
- or $f(\infty) = \lim_{s\to 0} sF(s)$ this is referred to as final value theorem.
- In order for the final value theorem to hold, all poles of F(s) must be located in the left half of the s plane; that is, the poles must have negative real parts.
- The only exception to this requirement is the case in which F(s) has a simple pole at $s=0$ because the effect $1/s$ of will be nullified by sF(s).
- The initial-value and final-value theorems depict the relationship between the origin and infinity in the time domain and the s-domain.

Inverse Laplace Transform

- Given F(s), how do we transform it back to the time domain and obtain the corresponding f (t)?
- Steps to Find the Inverse Laplace Transform:
- 1. Decompose F(s) into simple terms using partial fraction expansion.
- 2. Find the inverse of each term by matching entries.
- **Simple Poles**: $F(s) = \dfrac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t})u(t)$$

- **Repeated Poles:** $F(s) = \dfrac{k_n}{(s+p)^n} + \dfrac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \dfrac{k_2}{(s+p)^2} + \dfrac{k_1}{s+p} + F_1(s)$

$$f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)$$

Complex Poles

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s)$$

$$f(t) = (A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t) u(t) + f_1(t)$$

TRANSIENT RESPONSE OF R–L SERIES CIRCUITS HAVING DC EXCITATION

- For the circuit shown, at t=0, switch is closed and DC voltage applied to series RL network.
- Let $i(t)$ current flowing through the circuit after closing the switch.