Network Theorems

Network Theorems: Introduction

- A disadvantage of using Kirchoff’s Laws is that, for a large, complex circuit, tedious computation is involved.
- Network theorems are used to reduce a complex circuit to a simpler one, thereby making circuit analysis much simpler.
- The theorems that are most useful in analysing networks are the superposition, Thévenin, Norton, and maximum power transfer theorems.
- Since these theorems are applicable to linear circuits, we first discuss the concept of circuit linearity.

Linearity Property

- Linearity is the property of an element describing a linear relationship between cause and effect.
- The property is a combination of both the ***homogeneity*** (scaling) property and ***the additivity*** property.
- The homogeneity property requires that if the input (also called the excitation) is multiplied by a constant, then the output (also called the response) is multiplied by the same constant.

Linearity Property

- For a resistor, for example, Ohm's law relates the input i to the output v,

$$v = Ri$$

If the current is increased by a constant K, then the voltage increases correspondingly by K; that is,

$$KiR = Kv$$

The additivity property requires that ***the response to a sum of inputs*** is ***the sum of the responses to each input applied separately.***

Using the voltage-current relationship of a resistor, if

Linearity Property

- Applying $(i_1 + i_2)$gives,

$$v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2$$

Therefore, we say that a ***resistor is a linear element*** because the ***voltage-current relationship*** satisfies both the homogeneity and the additivity properties.

In general, a circuit is linear if ***it satisfies both additive and homogeneous properties.***

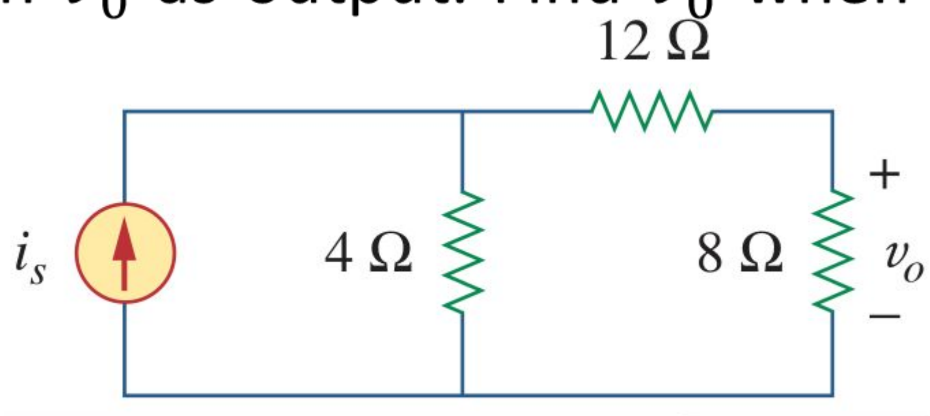
A linear circuit is one whose output is ***linearly related*** (or directly proportional) to ***its input***

A linear circuit consists of only linear elements, linear

Linear Property

- Note that since $p = \frac{v^2}{R} = i^2 R$ (making it a quadratic function rather than a linear one), ***the relationship between power and voltage (or current) is nonlinear.*** Therefore, the ***theorems covered*** in this chapter are ***not applicable to power***.

- For the circuit shown here., check if it satisfies linear property with v_0 as output. Find v_0 when is=15 and is=30 A.



For the circuit shown in Figure, find Io when Vs=12 V and Vs=24 V.

Applying KVL to the two loops, we obtain

$$12i_1 - 4i_2 + v_s = 0 \tag{4.1.1}$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \tag{4.1.2}$$

But $v_x = 2i_1$. Equation (4.1.2) becomes

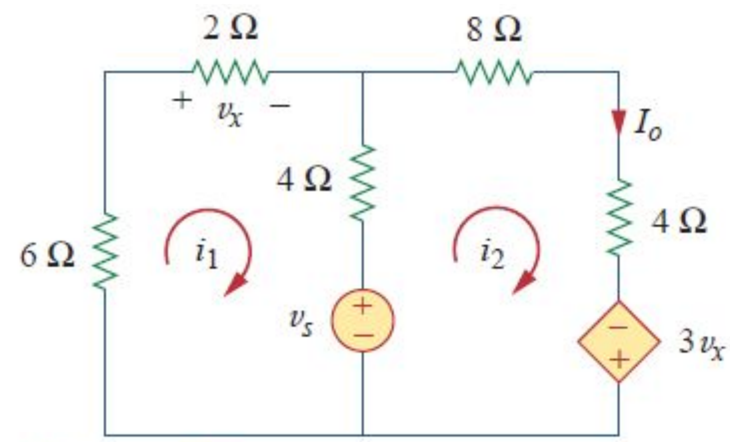
$$-10i_1 + 16i_2 - v_s = 0 \tag{4.1.3}$$

Adding Eqs. (4.1.1) and (4.1.3) yields

$$2i_1 + 12i_2 = 0 \quad \Rightarrow \quad i_1 = -6i_2$$

Substituting this in Eq. (4.1.1), we get

$$-76i_2 + v_s = 0 \quad \Rightarrow \quad i_2 = \frac{v_s}{76}$$



When $v_s = 12$ V,

$$I_o = i_2 = \frac{12}{76}\text{ A}$$

When $v_s = 24$ V,

$$I_o = i_2 = \frac{24}{76}\text{ A}$$

showing that when the source value is doubled, I_o doubles.

Superposition Theorem

If a circuit has two or more independent sources, ***one way*** to determine the value of a specific variable (voltage or current) is to use ***nodal or mesh analysis.***

Another way is to determine the contribution of each independent source to the variable and then add them up. This approach is known as the superposition.

The idea of ***superposition rests*** on ***the linearity property.***

Superposition Theorem

- The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

- The principle of superposition helps us to analyse a linear circuit ***with more than one independent source*** by calculating the contribution of each independent source separately.

Superposition Theorem

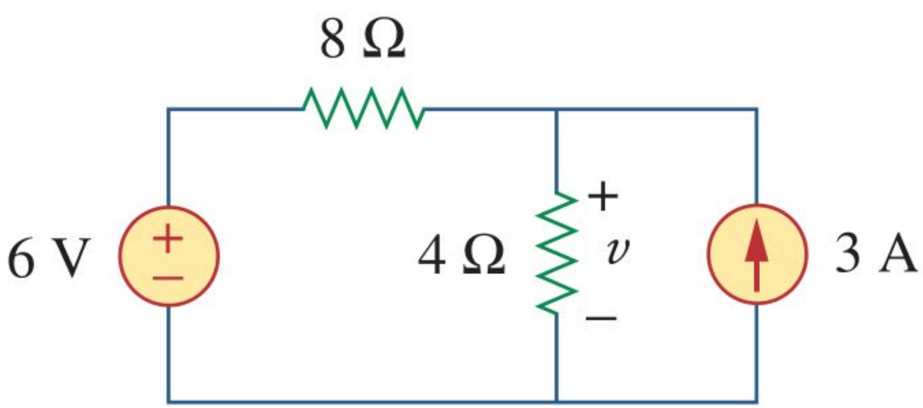
To apply the superposition principle:

Consider one independent source at a time while all ***other*** independent sources are ***set to zero.*** This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.

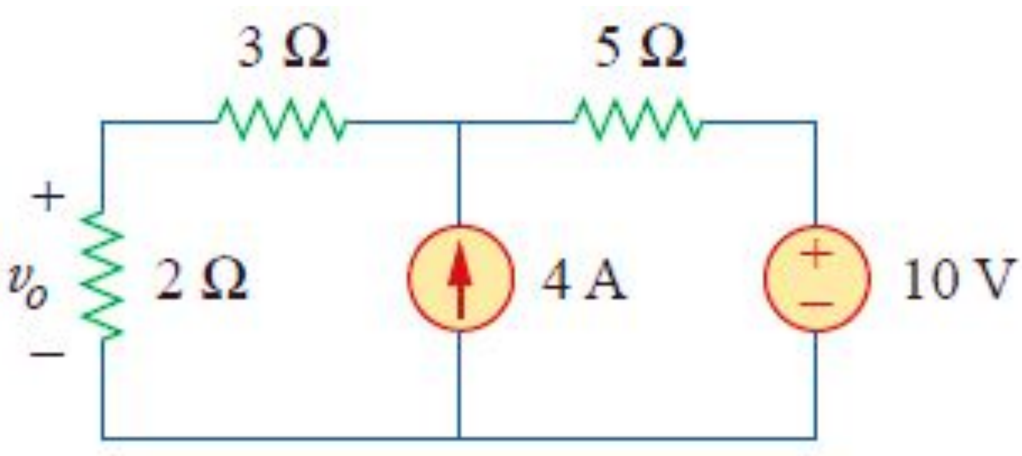
Dependent sources are left intact because they are controlled by circuit variables.

Example

- Use the superposition theorem to find v in the circuit shown.



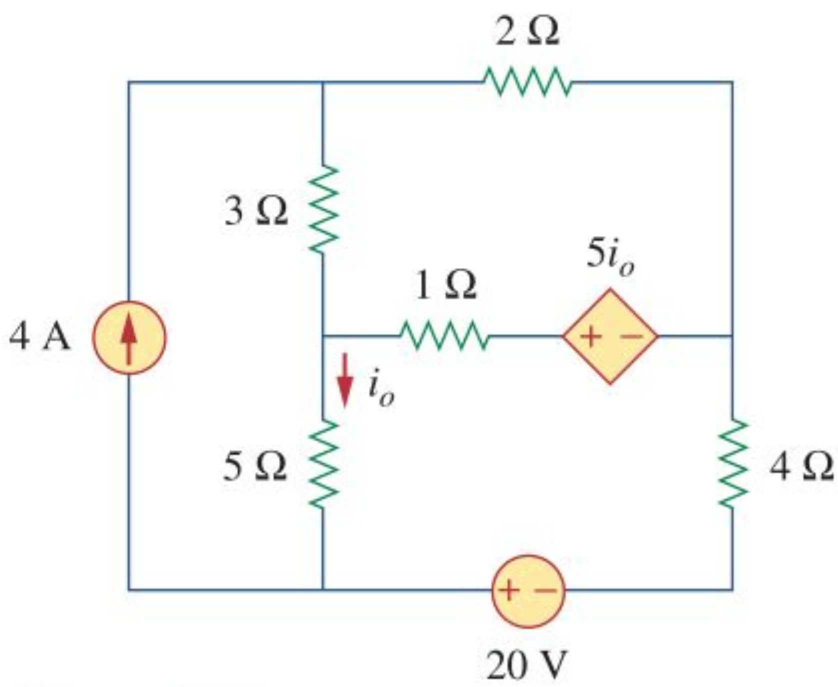
Using superposition theorem find v_o in the circuit shown below.



Answer: 6 V.

Example

Find i_0 in the circuit shown using superposition.



Source Transformation

- Source transformation is another tool for simplifying circuits.
- A common goal in source transformation is to ***end up*** with ***either all current sources or all voltage sources in the circuit.***
- An ideal independent voltage source maintains a constant voltage between its terminals regardless of the current that flows through it. But an ideal voltage source does not exist. Similarly, an ideal current source does not exist.

Source Transformation

A practical voltage source has an internal resistance which, to be accounted for, it is represented with an external resistance in series with the voltage source.

Likewise a practical current source has an internal conductance which is represented as a resistance (or conductance) in parallel with the current source.

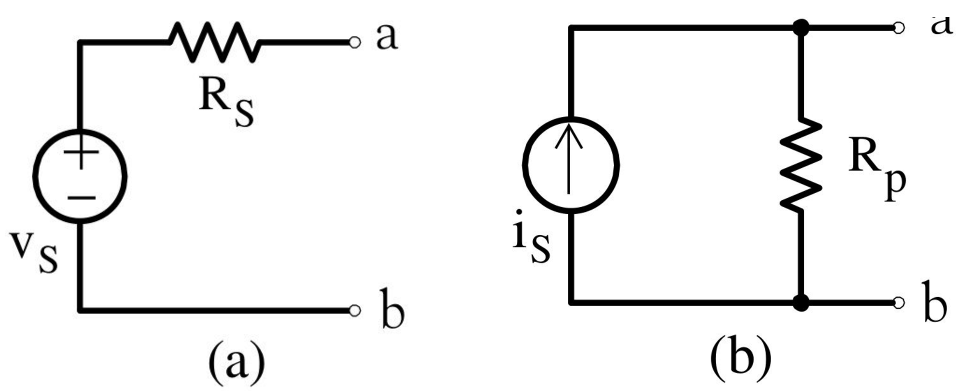


Figure: Practical voltage and current sources

Source Transformation

A ***source transformation*** is the process of ***replacing a voltage source*** v_s in series with a resistor R_s by a ***current source*** i_s in parallel with a resistor R_p, or vice versa.

The two circuits shown are equivalent—provided they have the same voltage-current relation at terminals a and b. That is, a load resistor R_L in either circuits will give same v_{ab} and i_{ab}. That is possible only when $v_s = i_s R_P$ and $R_s = R_p$

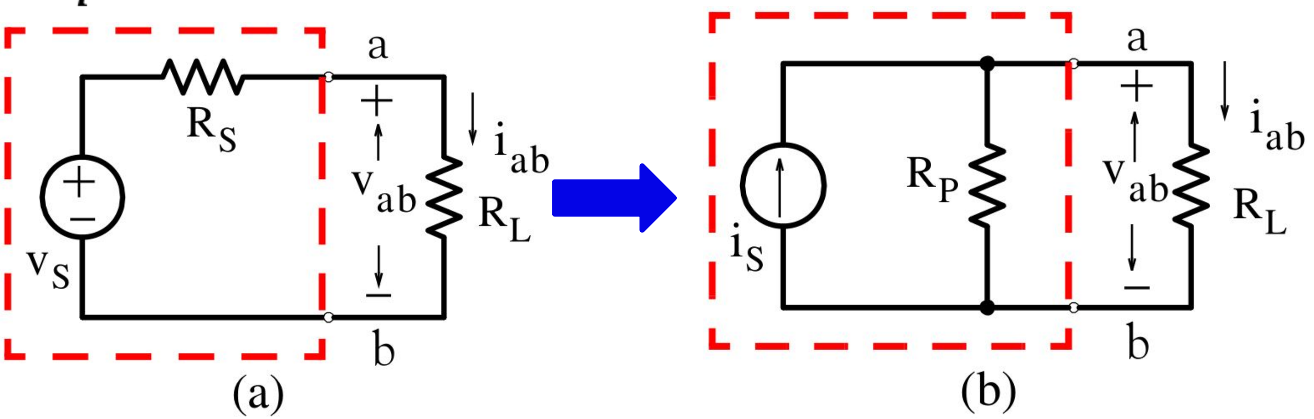
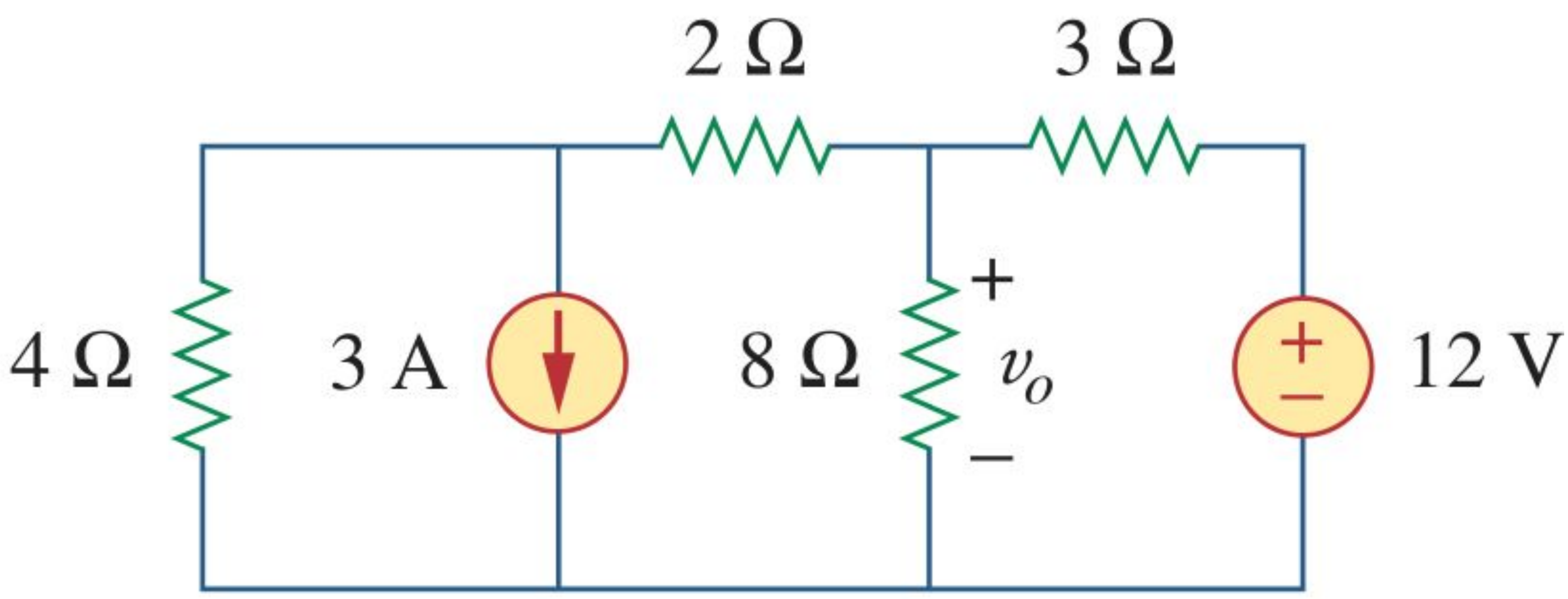


Figure: Equivalent sources

Source Transformation: Example

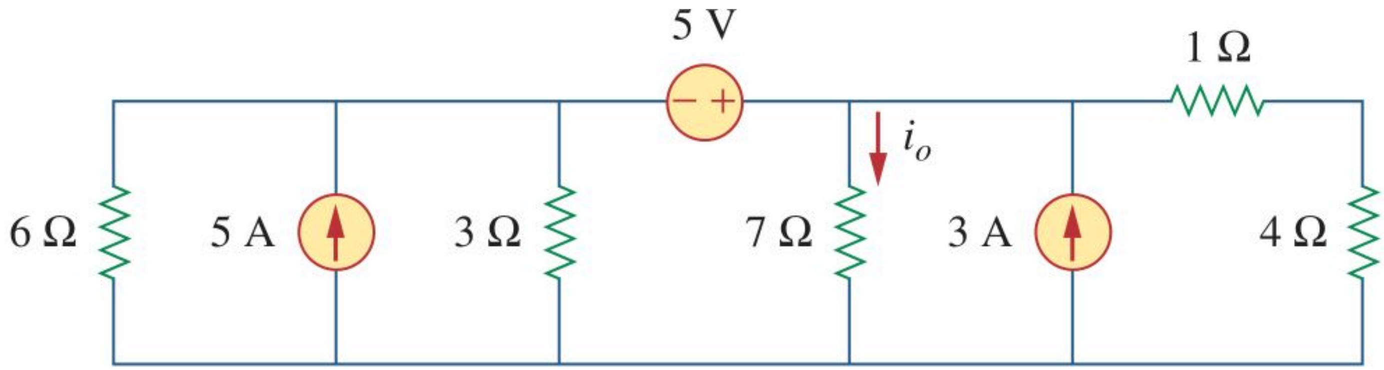
- Use source transformation to find v_0 in the circuit shown.



Note: ***In a source transformation, the head of the current source arrow corresponds to the "+" terminal of the voltage source.***

Practice Problem

- Find i_0 using source transformation



Thevenin's Theorem

- We often find ourselves where a particular element in a circuit is variable (usually called the load) while other elements are fixed. Each time the variable element is changed, the entire circuit has to be analysed all over again.
- To handle such problems, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.
- ***It states*** that a ***linear two-terminal circuit*** can be ***replaced by*** an equivalent circuit consisting of a

Thevenin's Theorem

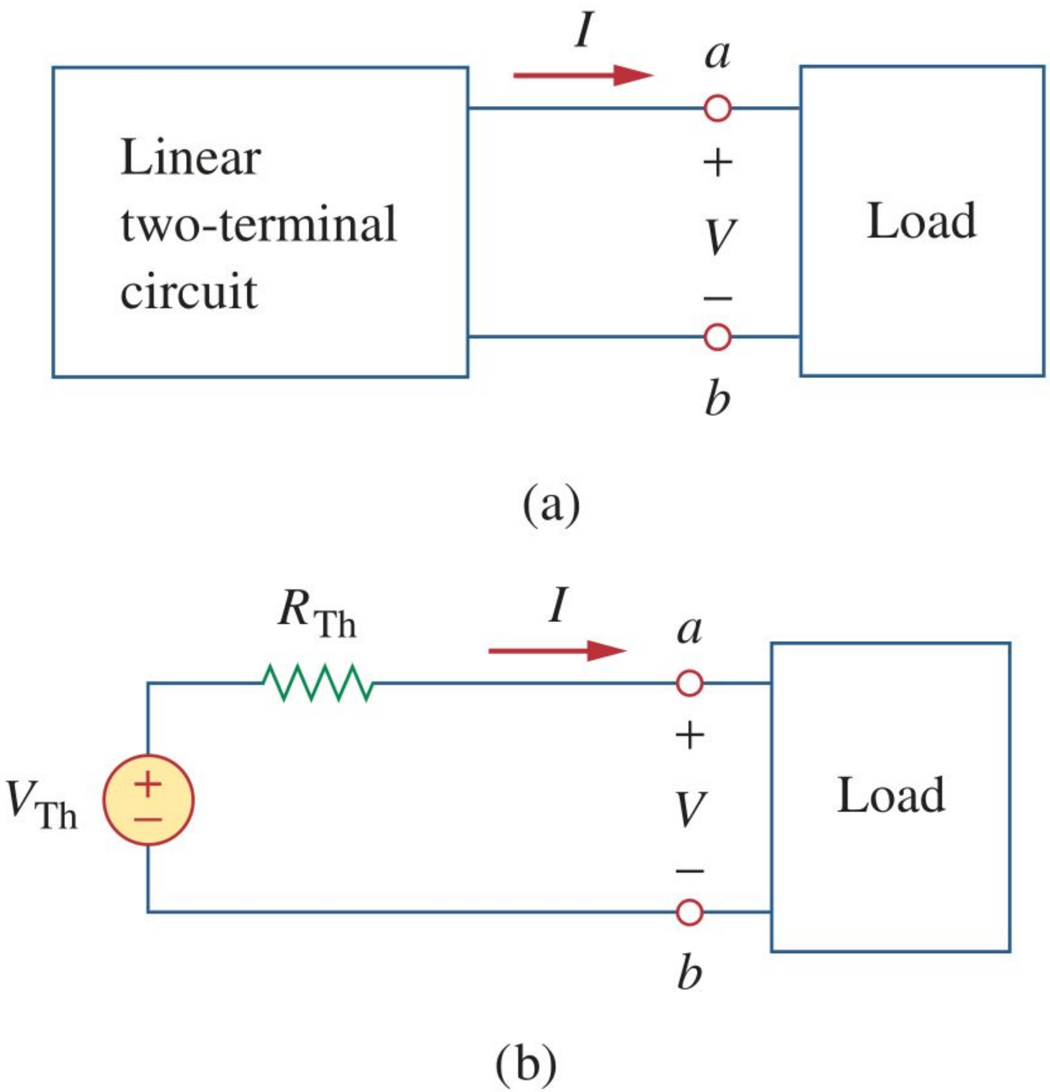


Figure: ***Replacing a linear two-terminal circuit by its Thevenin equivalent: (a) original circuit, (b) the Thevenin equivalent circuit.***

Thevenin's Theorem

- V_{Th} ***is the open-circuit voltage*** at the terminals and R_{Th} is the input or ***equivalent resistance*** at the terminals when the ***independent sources are turned off.***

$$V_{Th} = v_{oc} \text{ and } R_{Th} = R_{in}$$

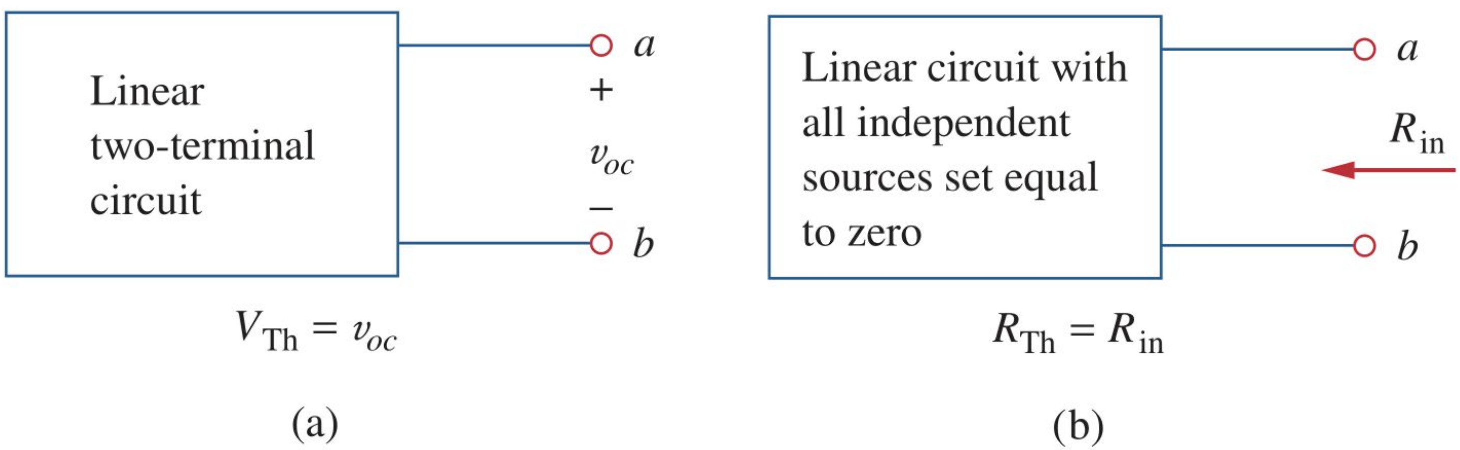


Figure: Finding V_{Th} and R_{Th}.

The following steps provide a technique that converts any circuit into its Thévenin equivalent:

1. Identify and remove the load resistance from the circuit and label the resulting two terminals as a and b.
2. Determine the open-circuit voltage between the terminals a and b. superposition theorem may use if the circuit has more than one source.
3. Set all sources in the circuit to zero: Voltage sources are set to zero by replacing them with short circuits (zero volts). Current sources are set to zero by replacing them with open circuits (zero amps).

Thevenin's Theorem

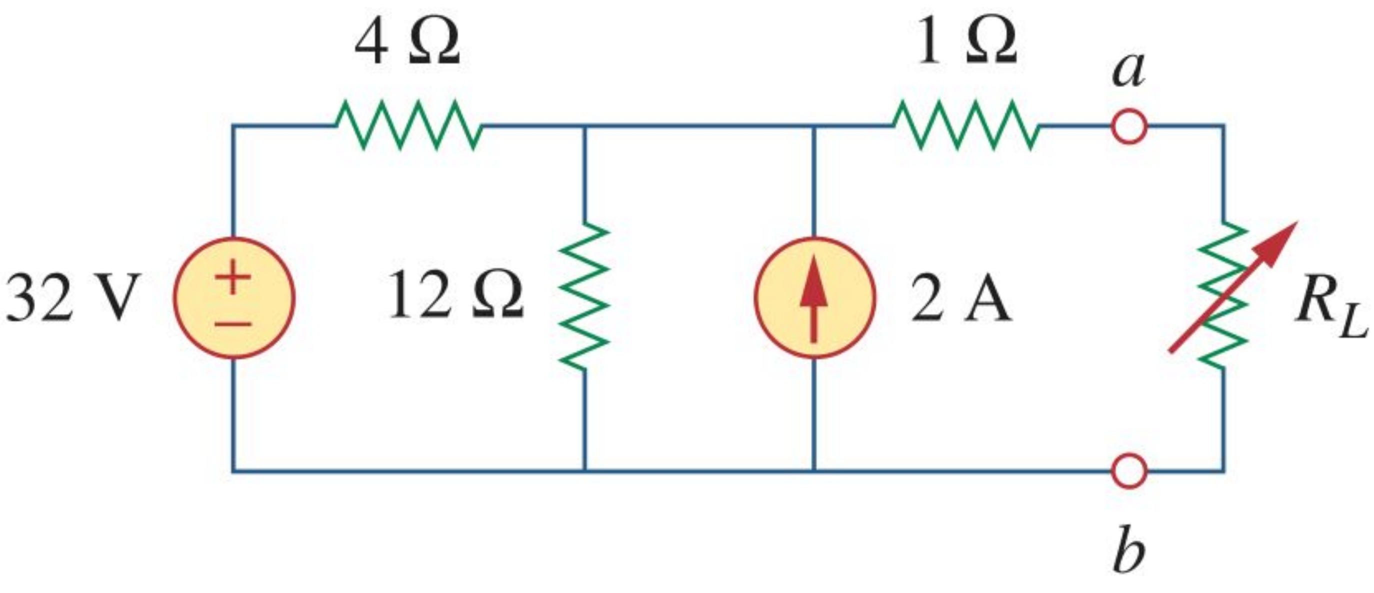
- In finding the R_{Th}, we need to consider two cases.

CASE 1:If the network has no dependent sources, we turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b.

CASE 2:If the network has dependent sources, we turn off all independent sources and dependent sources are not to be turned off because they are controlled by circuit variables. Then $R_{Th} = \frac{v_{ab}\, with\, ab\, open}{i_{ab}\, with\, ab\, shorted} = v_{oc}/i_{sc}$.

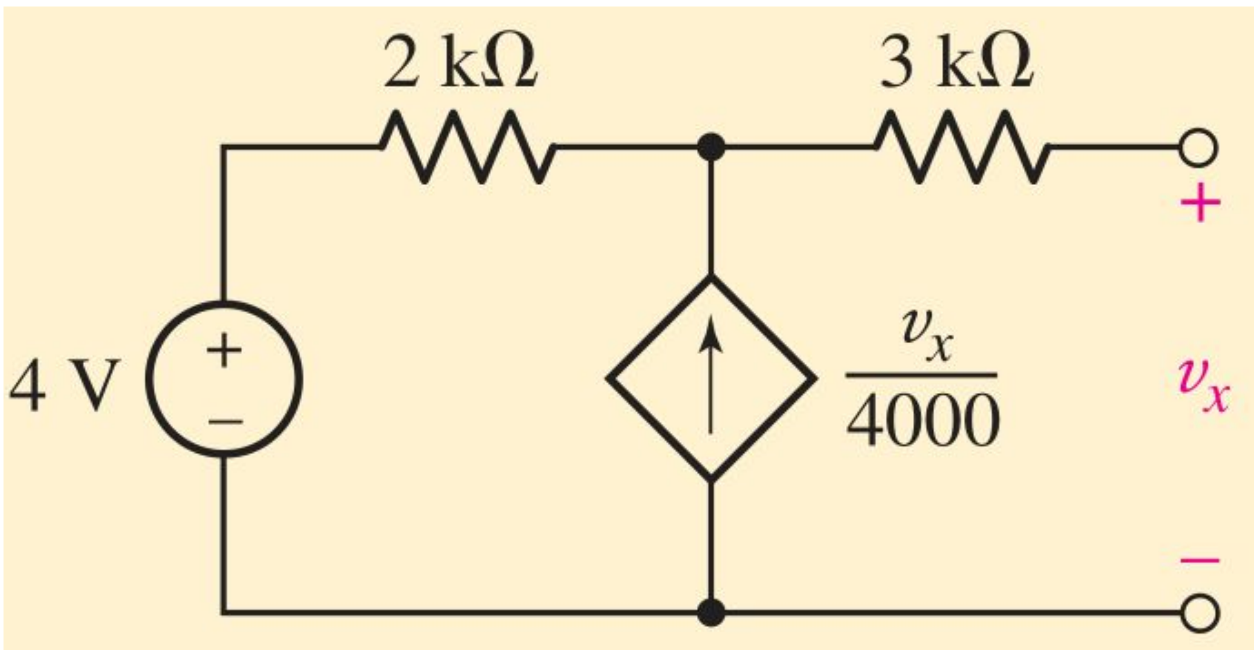
Example

- Find the Thevenin equivalent circuit of the circuit shown, to the left of the terminals a-b. Then find the current through $R_L = 6, 16, 32\Omega$.



Example

Determine the Thévenin equivalent of the circuit shown.



Norton’s Theorem

- Norton’s theorem is a circuit analysis technique that is similar to Thévenin’s theorem. By using this theorem we reduce the circuit to a single current source and one parallel resistor.
- ***Norton’s theorem*** states that a ***linear two-terminal circuit*** can be ***replaced by*** an equivalent circuit consisting of a ***current source*** I_N ***in parallel*** with a ***resistor*** $\boldsymbol{R_N}$.
- $\boldsymbol{I_N}$ ***is the short-circuit current*** through the terminals and $\boldsymbol{R_N}$ ***is the input or equivalent resistance*** at the

Norton's Theorem

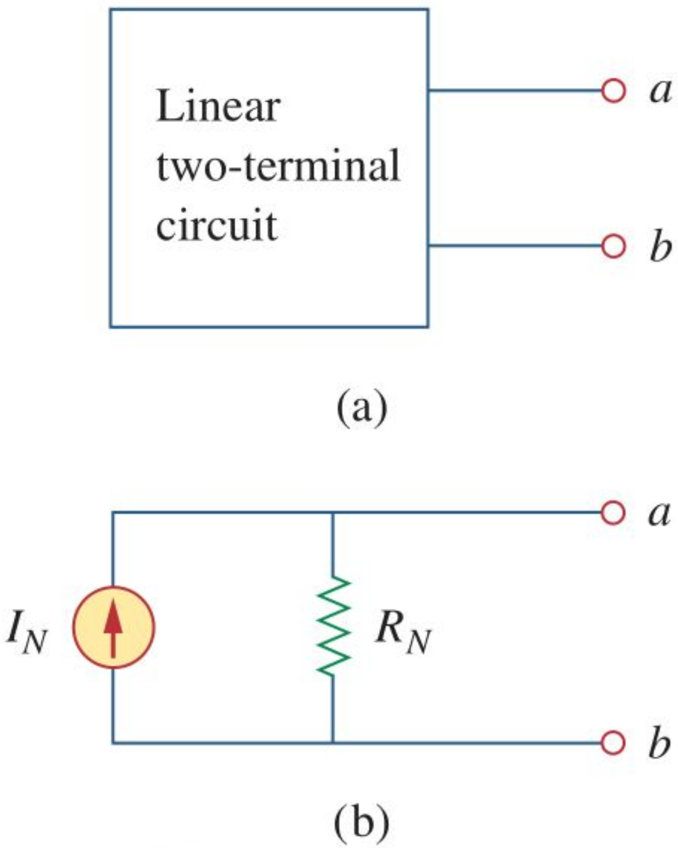


Figure 1: (a) Original circuit, (b) Norton equivalent circuit.

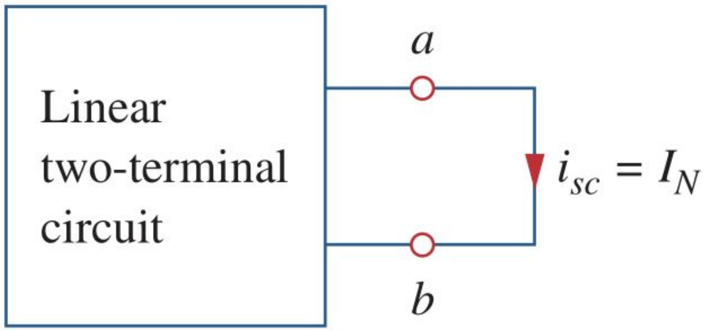


Figure 2: Finding Norton current I_N.

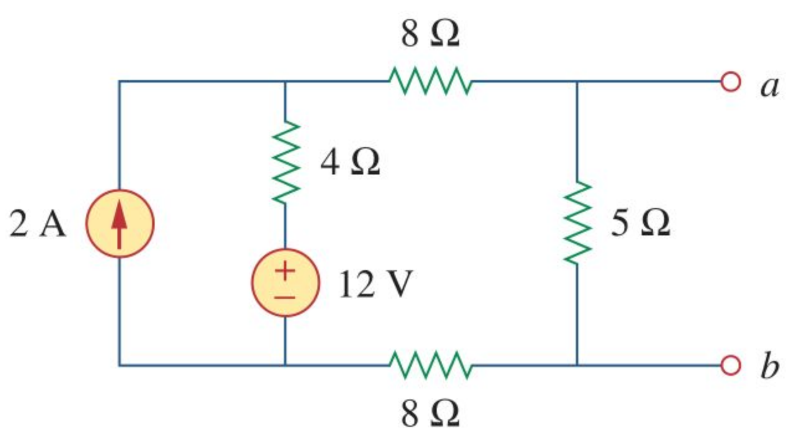
We find R_N in the same way we find R_{Th}

Therefore, $R_N = R_{Th}$.

Norton's Theorem

- The Norton equivalent circuit may also be determined directly from the Thévenin equivalent circuit by using the source conversion technique. $I_N = \mathrm{V_{Th}}/\mathrm{R_{Th}}$ and $R_N = R_{Th}$.

❖ Find the Norton equivalent circuit of the circuit in show here at terminals a-b.



Practice Problem

- Using Norton's theorem, find I_N and R_N of the circuits shown at terminals a-b.

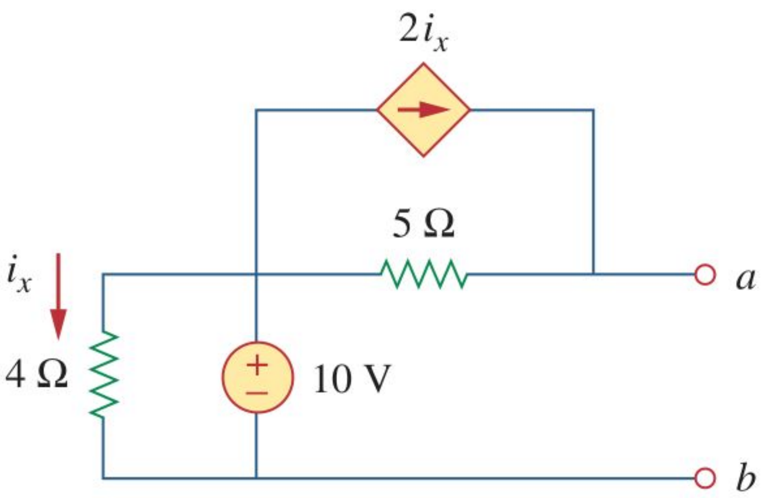


Figure: Example one

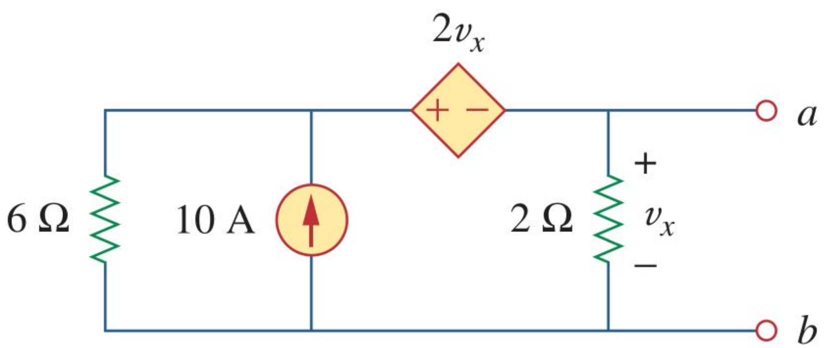


Figure: Example two

Maximum Power Transfer Theorem

The ***maximum power transfer theorem*** states

A load resistance will ***receive maximum power*** from a circuit when ***the resistance of the load*** is exactly the same as ***the Thévenin (Norton) resistance*** looking back at the circuit.

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load.

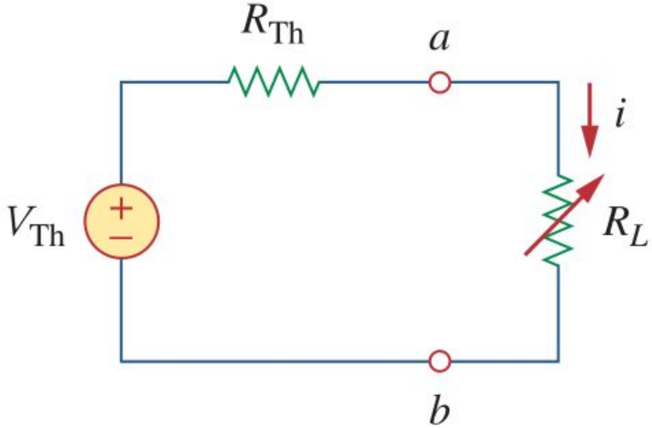


Figure: The circuit used for maximum power transfer.

Maximum Power Transfer Theorem

- If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown in previous Fig., the power delivered to the load is

$$p = \left(\frac{V_{Th}}{R_{Th}+R_L}\right)^2 \times R_L$$

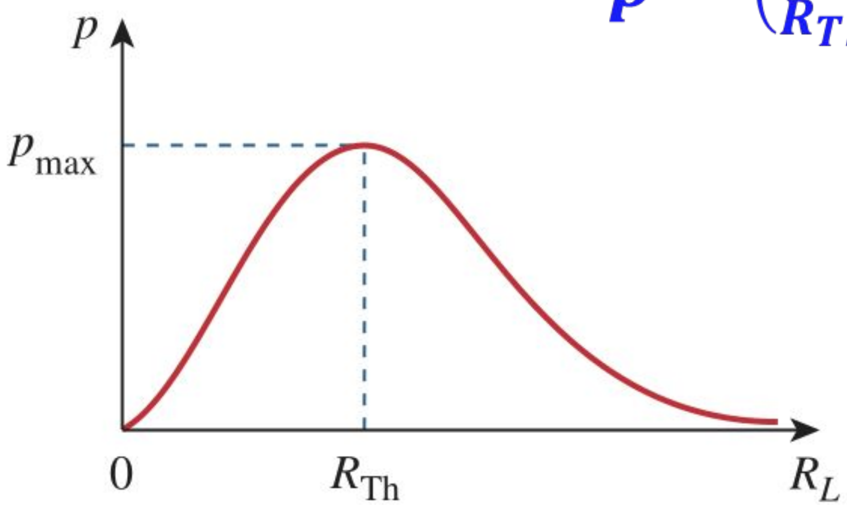


Figure: Power delivered to the load as a function of R_L.

- For a given circuit, V_{Th} and R_{Th} are fixed. By varying the load resistance R_L, the power delivered to the load varies as illustrated in Fig.

Maximum Power Transfer Theorem

To prove the maximum power transfer theorem, we differentiate the power delivered to load p with respect to R_L and set the result equal to zero.

$$\frac{dp}{dR_L} = V_{Th}^2\left[\frac{(R_{Th}+R_L)^2 - 2R_L(R_{Th}+R_L)}{(R_{Th}+R_L)^4}\right] = 0$$

$$\Longrightarrow \quad R_{Th} + R_L - 2R_L = 0$$

$$\boldsymbol{R_L = R_{Th}}$$

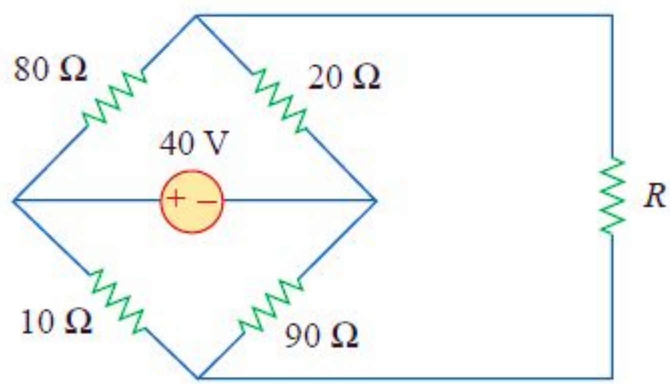
Therefore, the maximum power transferred to the load resistance is

$$\boldsymbol{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$

The variable resistor *R* in Fig. below is adjusted until it absorbs the maximum power from the circuit.

(a) Calculate the value of *R* for maximum power.

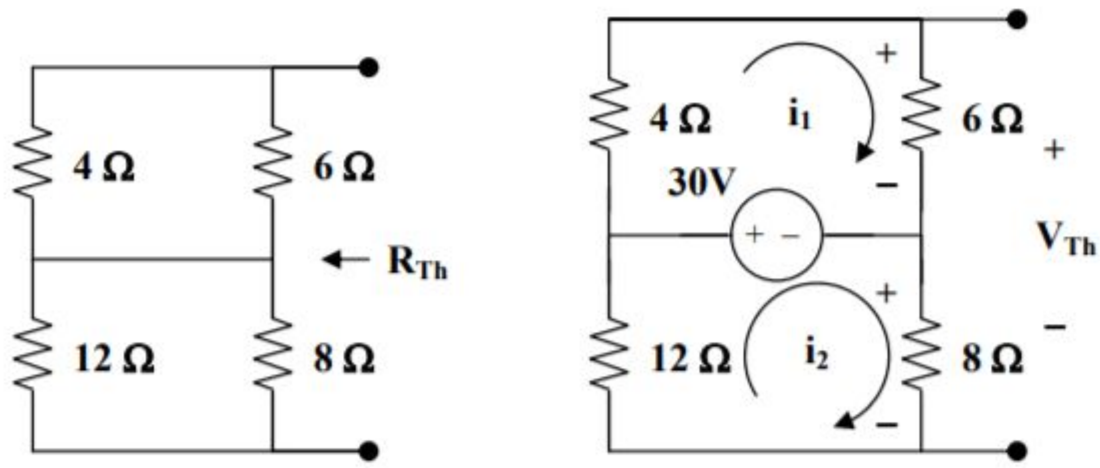
(b) Determine the maximum power absorbed by *R*.



$$R_{Th} = 4||6 + 8||12 = 2.4 + 4.8 = \underline{\textbf{7.2 ohms}}$$

From Fig. (b),

$$10i_1 - 30 = 0, \text{ or } i_1 = 3$$



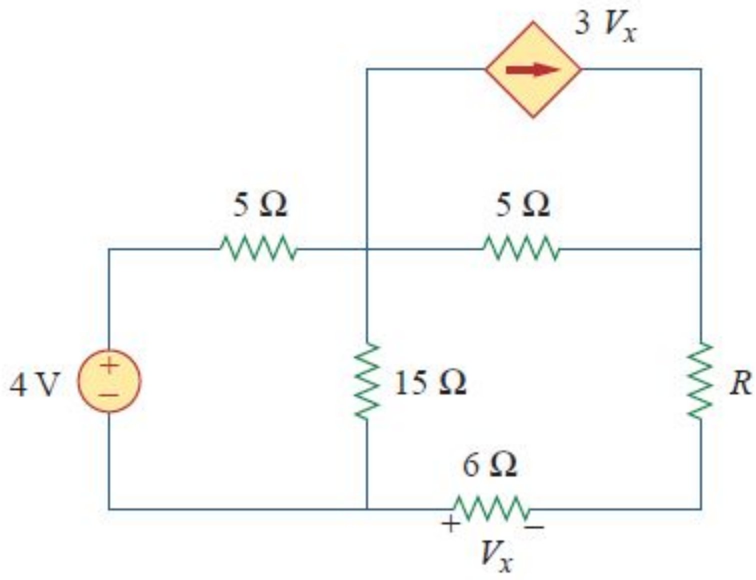
(a) (b)

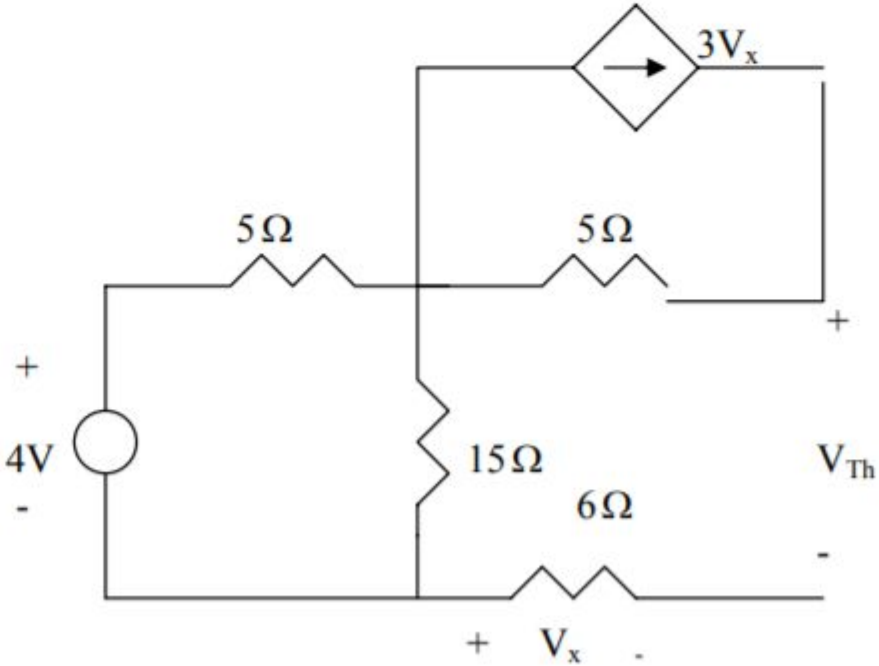
$$20i_2 + 30 = 0, \text{ or } i_2 = 1.5, \; V_{Th} = 6i_1 + 8i_2 = 6x3 - 8x1.5 = \underline{\textbf{6 V}}$$

For maximum power transfer,

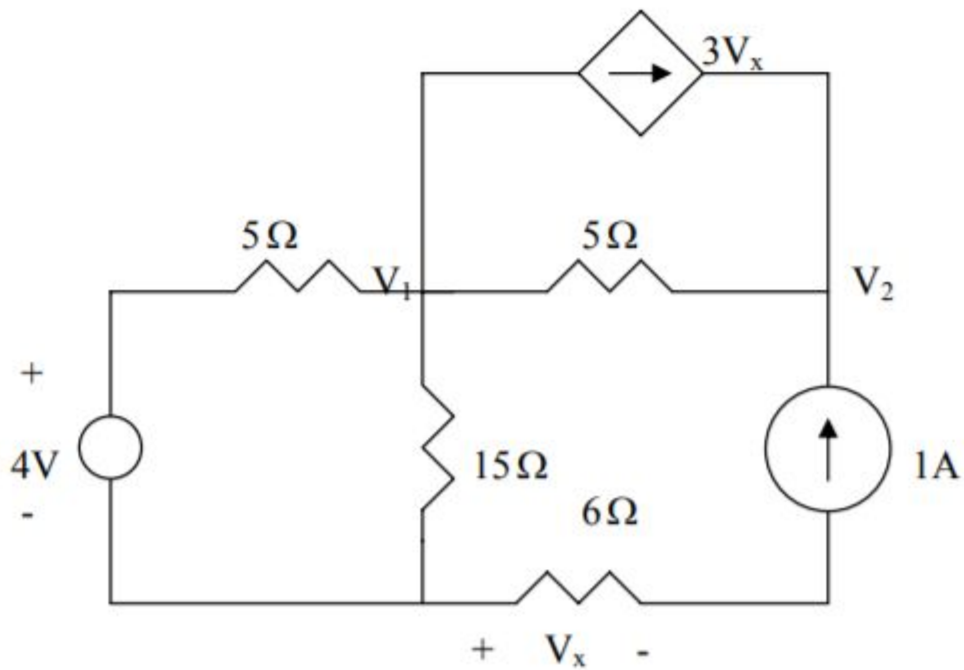
$$p = V_{Th}^2/(4R_{Th}) = (6)^2/[4(7.2)] = \underline{\textbf{1.25 watts}}$$

Determine the maximum power delivered to the variable resistor R shown in the circuit of Fig.





$$V_x = 0, \qquad V_{Th} = \frac{15}{15+5}(4) = 3\text{V}$$



At node 1,

$$\frac{4-V_1}{5} = 3V_x + \frac{V_1}{15} + \frac{V_1 - V_2}{5}, \qquad V_x = 6x1 = 6 \quad \longrightarrow \quad 258 = 3V_2 - 7V_1$$

At node 2,

$$1 + 3V_x + \frac{V_1 - V_2}{5} = 0 \quad \longrightarrow \quad V_1 = V_2 - 95$$

Solving (1) and (2) leads to $V_2 = 101.75$ V

$$R_{Th} = \frac{V_2}{1} = 101.75\Omega, \qquad p_{\max} = \frac{V_{Th}^{\ 2}}{4R_{Th}} = \frac{9}{4x101.75} = \underline{22.11\,\text{mW}}$$

Reciprocity Theorem

- The reciprocity theorem is a theorem that can ***only be used with single-source circuits***. However, this theorem may be applied to either voltage sources or current sources.

Reciprocity theorem for voltage sources: A voltage source causing a current I in any branch of a circuit may be removed from the original location and placed into that branch having the current I. The voltage source in the new location will produce a current in the original source location that is exactly equal to the originally calculated current, I.

Reciprocity Theorem

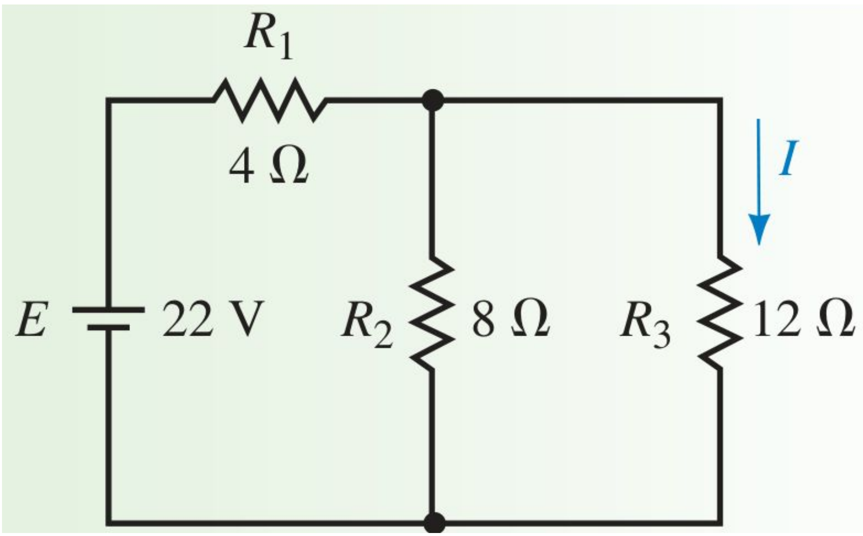
When applying the reciprocity theorem for a voltage source, the following steps must be followed:

1. The voltage source is replaced by a short circuit in the original location.
2. The polarity of the source in the new location is such that the current direction in that branch remains unchanged.

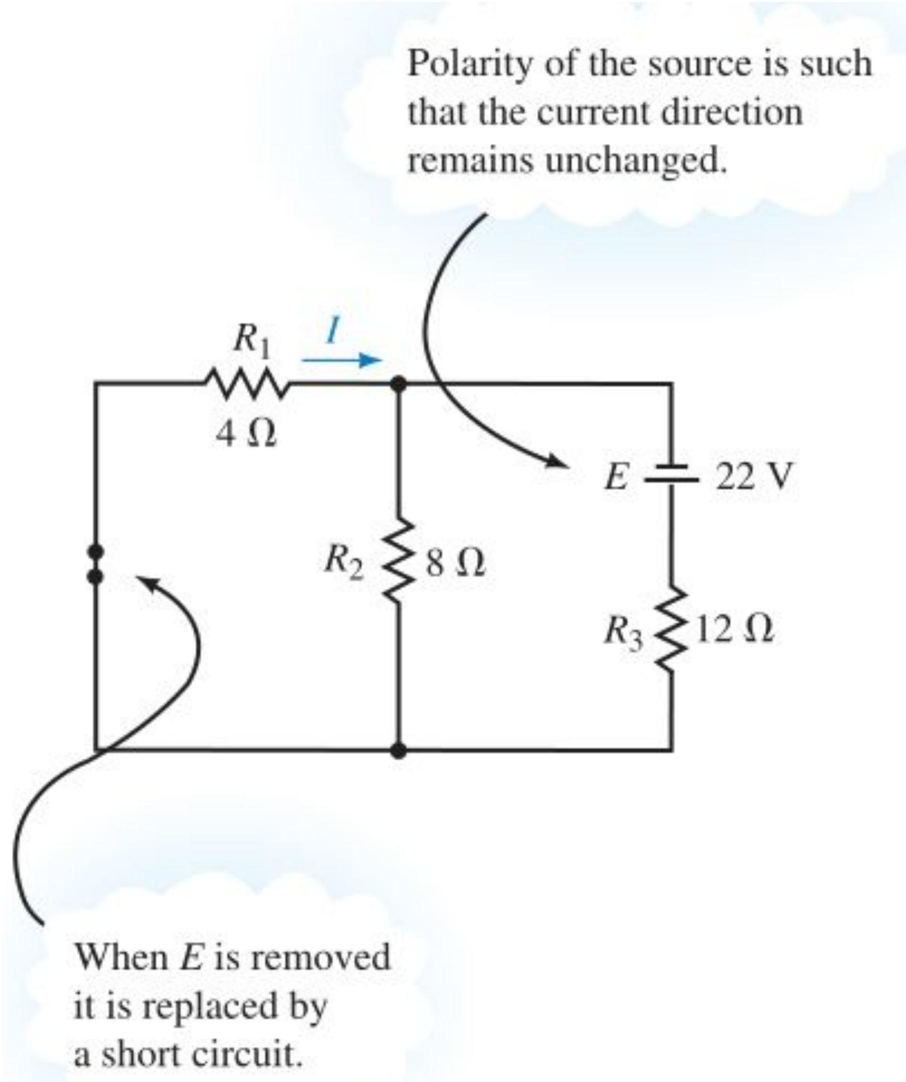
Practice Problem 1

Consider the circuit of Figure shown here:

1. Calculate the current I.
2. Remove voltage source E and place it into the branch with R_3 . Show that the current through the branch which formerly had E is now the same as the current I.



Practice Problem 1 (Continuation)



After removing the voltage source from its original location and moving it into the branch containing the current I, we obtain the circuit shown here.

Reciprocity Theorem: Current Source

For Current Sources: A current source causing a voltage V at any node of a circuit may be removed from the original location and connected to that node. The current source in the new location will produce a voltage in the original source location that is exactly equal to the originally calculated voltage, V.

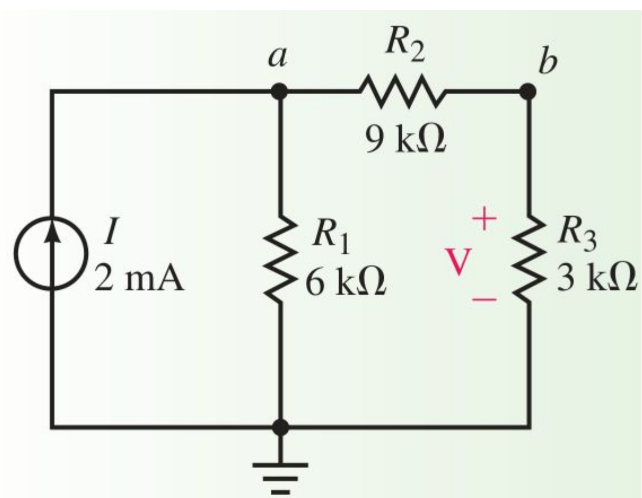
The following conditions must be met:

1. The current source is replaced by an open circuit in the original location.
2. The direction of the source in the new location is such that the polarity of the voltage at the node to which the current source is now connected remains unchanged.

Practice Problem2

- Consider the circuit shown in Figure below:

1. Determine the voltage V across resistor R_3.
2. Remove the current source I and place it between node b and the reference node. Show that the voltage across the former location of the current source (node a) is now the same as the voltage V.



Practice Problem 2 (Continuation)

After relocating the current source from the original location, and connecting it between node b and ground, we obtain the circuit shown here.