FSA and regular languages

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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Recap: languages and automata

- Recognizing strings from a language defined by a grammar is a fundamental question in computer science
- The efficiency of computation, and required properties of computing device depending on the grammar (and the language)
- A well-known hierarchy of grammars both in computer science and linguistics is the *Chomsky hierarchy*
- Each grammar in the Chomsky hierarchy corresponds to an abstract computing device (an automaton)
- The class of *regular grammars* are the class that corresponds to *finite state automata*

Chomsky hierarchy and automata

Grammar class	*Rules*	*Automata*
Unrestricted grammars	$\alpha \rightarrow \beta$	Turing machines
Context-sensitive grammars	$\alpha A \beta \rightarrow \alpha \gamma \beta$	Linear-bounded automata
Context-free grammars	$A \rightarrow \alpha$	Pushdown automata
Regular grammars	$A \rightarrow a$, $A \rightarrow aB$ | $A \rightarrow a$, $A \rightarrow B\,a$	Finite state automata

Regular grammars: definition

A regular grammar is a tuple $G = (\Sigma, N, S, R)$ where

Σ is an alphabet of terminal symbols

N are a set of non-terminal symbols

S is a special 'start' symbol $\in N$

R is a set of rewrite rules following one of the following patterns ($A, B \in N$, $a \in \Sigma$, ϵ is the empty string)

Left regular

1. $A \rightarrow a$
2. $A \rightarrow Ba$
3. $A \rightarrow \epsilon$

Right regular

1. $A \rightarrow a$
2. $A \rightarrow aB$
3. $A \rightarrow \epsilon$

Regular languages: some properties/operations

$\mathcal{L}_1\mathcal{L}_2$ Concatenation of two languages $\mathcal{L}_1$ and $\mathcal{L}_2$: any sentence of $\mathcal{L}_1$ followed by any sentence of $\mathcal{L}_2$

$\mathcal{L}^*$ Kleene star of $\mathcal{L}$: $\mathcal{L}$ concatenated by itself 0 or more times

$\mathcal{L}^R$ Reverse of $\mathcal{L}$: reverse of any string in $\mathcal{L}$

$\overline{\mathcal{L}}$ Complement of $\mathcal{L}$: all strings in $\Sigma_{\mathcal{L}}^*$ except the ones in $\mathcal{L}$ ($\Sigma_{\mathcal{L}}^* - \mathcal{L}$)

$\mathcal{L}_1 \cup \mathcal{L}_2$ Union of languages $\mathcal{L}_1$ and $\mathcal{L}_2$: strings that are in any of the languages

$\mathcal{L}_1 \cap \mathcal{L}_2$ Intersection of languages $\mathcal{L}_1$ and $\mathcal{L}_2$: strings that are in both languages

Regular languages are closed under all of these operations.

Three ways to define a regular language

- A language is regular if there is regular grammar that generates/recognizes it
- A language is regular if there is an FSA that generates/recognizes it
- A language is regular regular if we can define a regular expressions for the language

Regular expressions

- Every regular language (RL) can be expressed by a regular expression (RE), and every RE defines a RL
- A RE `e` defines a RL $\mathcal{L}(\texttt{e})$
- Relations between RE and RL
  - $\mathcal{L}(\varnothing) = \varnothing$,
  - $\mathcal{L}(\epsilon) = \epsilon$,
  - $\mathcal{L}(\texttt{a}) = a$
  - $\mathcal{L}(\texttt{ab}) = \mathcal{L}(a)\mathcal{L}(b)$
  - $\mathcal{L}(\texttt{a*}) = \mathcal{L}(a)^*$
  - $\mathcal{L}(\texttt{a|b}) = \mathcal{L}(a) \cup \mathcal{L}(b)$ (some author use the notation `a+b`, we will use `a|b` as in many practical implementations)

  where, $a, b \in \Sigma$, ϵ is empty string, $\varnothing$ is the language that accepts nothing (e.g., $\Sigma^* - \Sigma^*$)
- Note: no standard complement and intersection in RE

Regular
some extensions

- Kleene star (`a*`), Concatenation (`ab`) and union (`a|b`) are the common operations
- Parentheses can be used to group the sub-expressions. Otherwise, the priority of the operators as specified above `a|bc*` = `a|(b(c*))`
- In practice some short-hand notations are common
  - `.` = $(a_1|\ldots|a_n)$, for $\Sigma = \{a_1, \ldots, a_n\}$
  - `a+` = `aa*`
  - `[a-c]` = `(a|b|c)`
  - `[^a-c]` = `. - (a|b|c)`
  - `\d` = `(0|1|...|8|9)`
  - ...
- And some non-regular extensions, like `(a*)b\1` (sometimes the term *regexp* is used for expressions with non-regular extensions)

Some properties of regular expressions

Kleene algebra

These identities are useful for simplifying regular expressions:

- $\epsilon\texttt{u} = \texttt{u}$
- $\varnothing\texttt{u} = \varnothing$
- $\texttt{u(vw)} = \texttt{(uv)w}$
- $\varnothing\texttt{*} = \epsilon$
- $\epsilon\texttt{*} = \epsilon$
- $\texttt{(u*)*} = \texttt{u*}$
- $\texttt{u|v} = \texttt{v|u}$
- $\texttt{u|u} = \texttt{u}$
- $\texttt{u|}\varnothing = \texttt{u}$
- $\texttt{u|}\epsilon = \texttt{u}$
- $\texttt{u|(v|w)} = \texttt{(u|v)|w}$
- $\texttt{u(v|w)} = \texttt{uv|uw}$
- $\texttt{(u|v)*} = \texttt{(u*|v*)*}$

Note: most of these follow from set theory, and some can be derived from others.

Some properties of regular expressions

Kleene algebra

These identities are useful for simplifying regular expressions:

- ϵu $=$ u
- $\varnothing$u $= \varnothing$
- u(vw) $=$ (uv)w
- $\varnothing$* $= \epsilon$
- ϵ* $= \epsilon$
- (u*)* $=$ u*
- u|v $=$ v|u
- u|u $=$ u
- u|$\varnothing$ $=$ u
- u|ϵ $=$ u
- u|(v|w) $=$ (u|v)|w
- u(v|w) $=$ uv|uw
- (u|v)* $=$ (u*|v*)*

An exercise

Simplify a|ab*

Note: most of these follow from set theory, and some can be derived from others.

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- $\varnothing* = \epsilon$
- $\epsilon* = \epsilon$
- $\mathtt{(u*)*} = \mathtt{u*}$
- $\mathtt{u|v} = \mathtt{v|u}$
- $\mathtt{u|u} = \mathtt{u}$
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- $\mathtt{u(v|w)} = \mathtt{uv|uw}$
- $\mathtt{(u|v)*} = \mathtt{(u*|v*)*}$

An exercise

Simplify $\mathtt{a|ab*}$

$$\mathtt{a|ab*} \quad = \quad \mathtt{a}\epsilon\mathtt{|ab*}$$

Note: most of these follow from set theory, and some can be derived from others.

Some properties of regular expressions

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An exercise

Simplify $\mathtt{a|ab*}$

$$\begin{aligned}\mathtt{a|ab*} &= \mathtt{a}\epsilon\mathtt{|ab*}\\ &= \mathtt{a(}\epsilon\mathtt{|b*)}\end{aligned}$$

Note: most of these follow from set theory, and some can be derived from others.

Some properties of regular expressions

Kleene algebra

These identities are useful for simplifying regular expressions:

- $\epsilon\mathtt{u} = \mathtt{u}$
- $\emptyset\mathtt{u} = \emptyset$
- $\mathtt{u(vw)} = \mathtt{(uv)w}$
- $\emptyset* = \epsilon$
- $\epsilon* = \epsilon$
- $\mathtt{(u*)*} = \mathtt{u*}$
- $\mathtt{u|v} = \mathtt{v|u}$
- $\mathtt{u|u} = \mathtt{u}$
- $\mathtt{u}|\emptyset = \mathtt{u}$
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- $\mathtt{u(v|w)} = \mathtt{uv|uw}$
- $\mathtt{(u|v)*} = \mathtt{(u*|v*)*}$

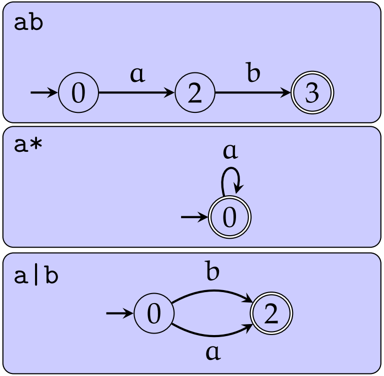
An exercise

Simplify $\mathtt{a|ab*}$

$$\begin{aligned} \mathtt{a|ab*} &= \mathtt{a}\epsilon\mathtt{|ab*} \\ &= \mathtt{a}(\epsilon\mathtt{|b*}) \\ &= \mathtt{ab*} \end{aligned}$$

Note: most of these follow from set theory, and some can be derived from others.

Converting regular expressions to FSA



- For more complex expressions, one can replace the paths for individual symbols with corresponding automata
- Using ϵ transitions may ease the task
- The reverse conversion (from automata to regular expressions) is also easy:
  - identify the patterns on the left, collapse paths to single transitions with regular expressions

Exercise

convert `b((ab)*|a)` to an NFA

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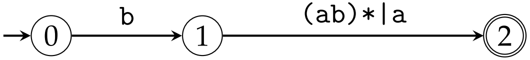
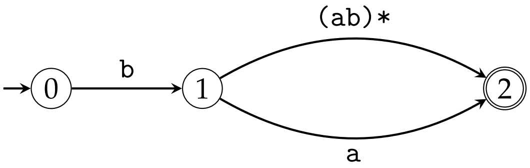


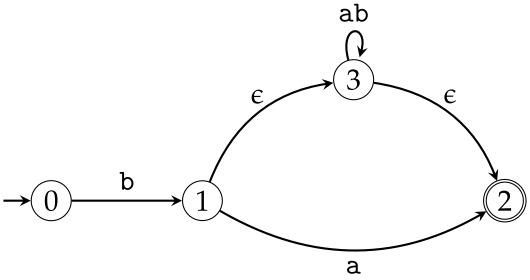
Exercise

convert `b((ab)*|a)` to an NFA



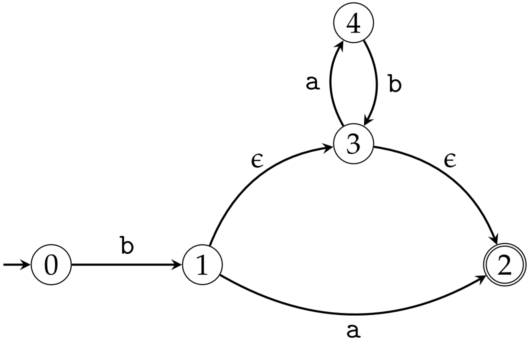
Exercise

convert `b((ab)*|a)` to an NFA

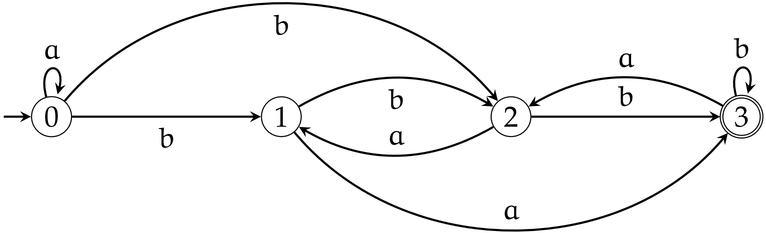


Exercise

convert `b((ab)*|a)` to an NFA

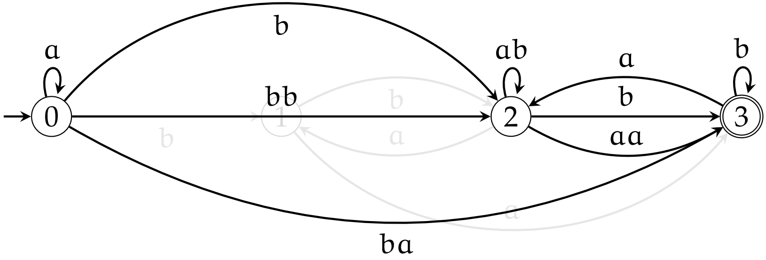


Converting FSA to regular expressions



- The general idea: remove (intermediate) states, replacing edge labels with regular expressions

Converting FSA to regular expressions



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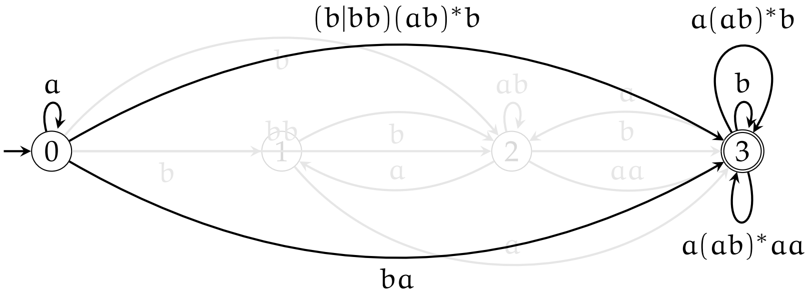
Converting FSA to regular expressions



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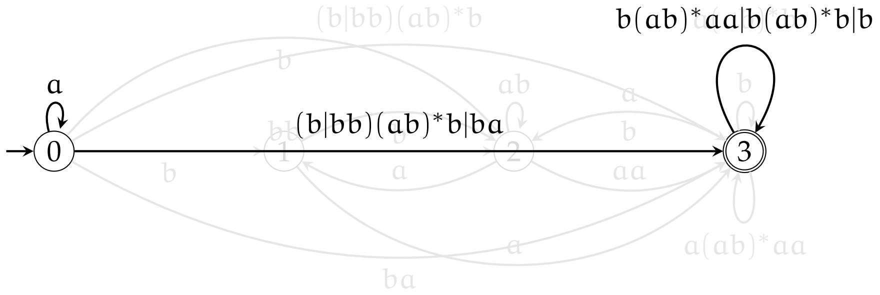

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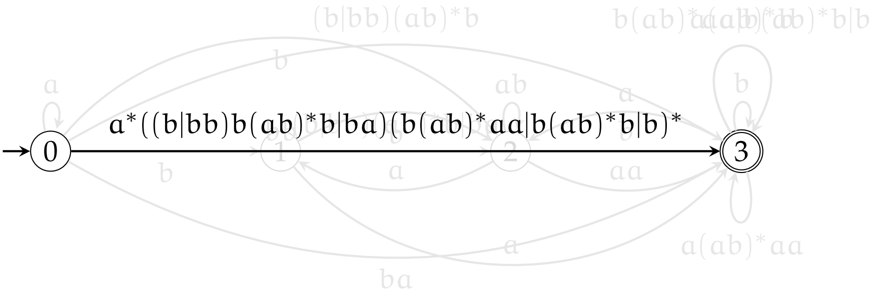
Converting FSA to regular expressions



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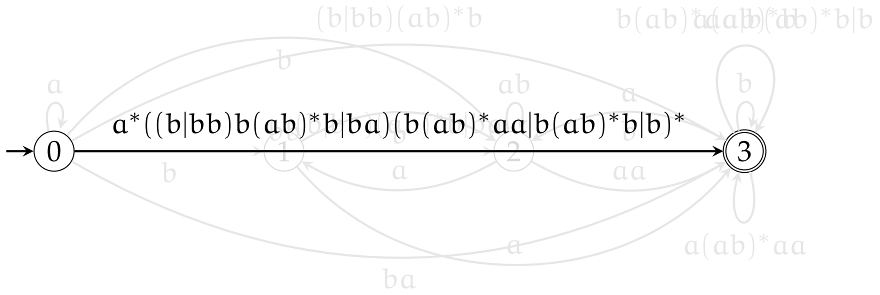
Converting FSA to regular expressions



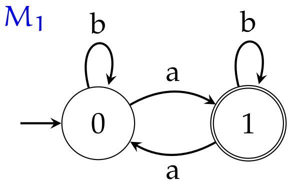
- The general idea: remove (intermediate) states, replacing edge labels with regular expressions

An exercise: simplify the resulting regular expressions

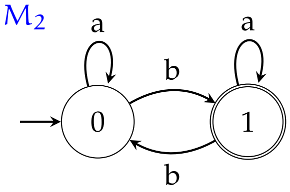
Two example FSA

what languages do they accept?

$L_1 = \mathcal{L}(M_1)$



$L_2 = \mathcal{L}(M_2)$



Two example FSA

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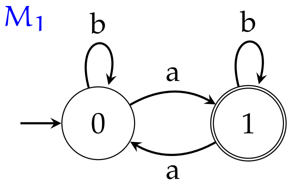

Odd number of a's over $\{a, b\}$.

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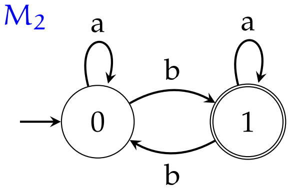
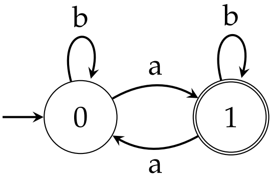


Odd number of b's over $\{a, b\}$.

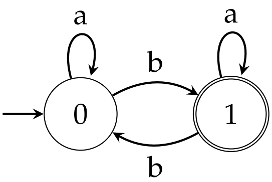
We will use these languages and automata for demonstration.

Concatenation

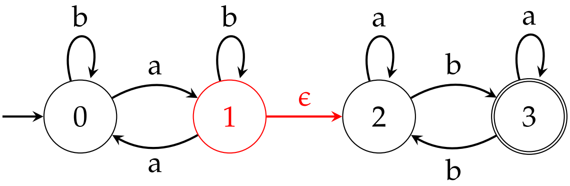
L_1



L_2

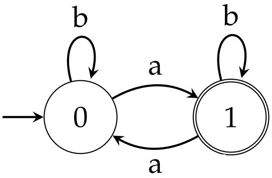


$L_1 L_2$

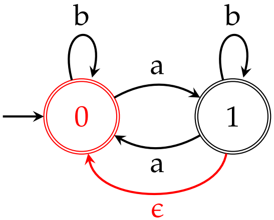


Kleene star

L_1

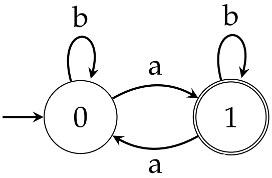


L_1^*

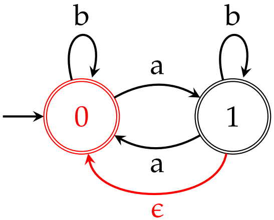


Kleene star

L_1



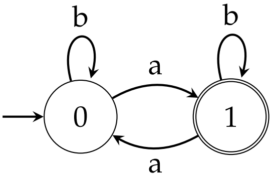
L_1^*



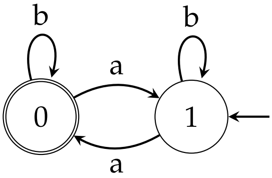
- What if there were more than one accepting states?

Reversal

L_1

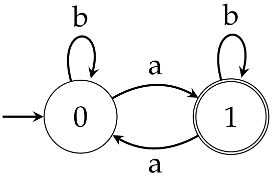


L_1^R

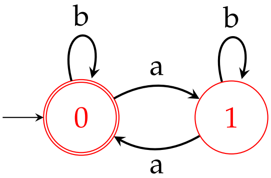


Complement

L_1

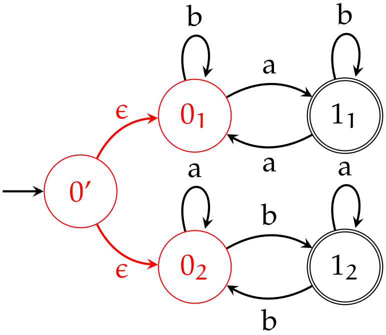


$\overline{L_1}$

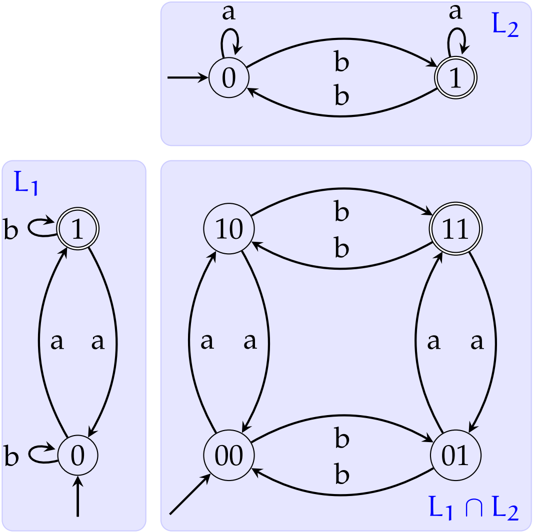


Union

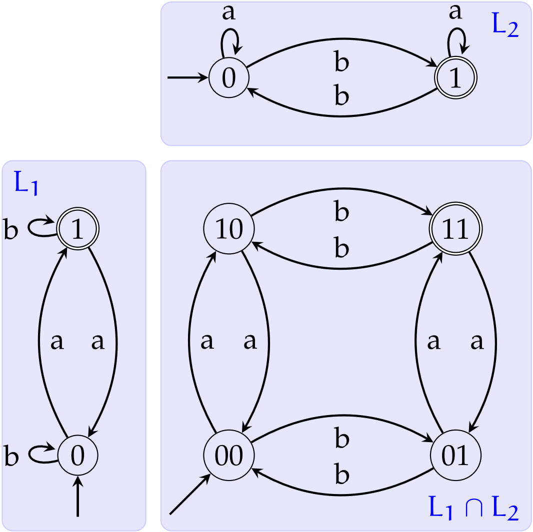
$L_1 \cup L_2$



Intersection



Intersection



...or

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

Closure properties of regular languages

- Since results of all the operations we studied are FSA: Regular languages are closed under
  - Concatenation
  - Kleene star
  - Reversal
  - Complement
  - Union
  - Intersection

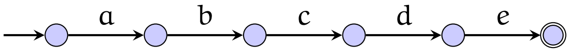
Is a language regular?

— or not

- To show that a language is regular, it is sufficient to find an FSA that recognizes it.
- Showing that a language is *not* regular is more involved
- We will study a method based on *pumping lemma*

Pumping lemma

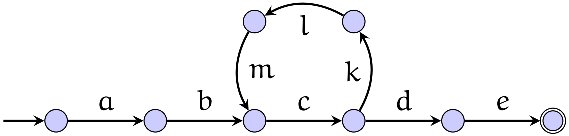
intuition



- What is the length of longest string generated by this FSA?

Pumping lemma

intuition



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Pumping lemma

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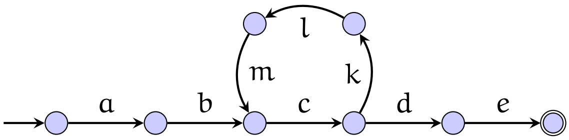

- What is the length of longest string generated by this FSA?
- Any FSA generating an infinite language has to have a loop (application of recursive rule(s) in the grammar)
- Part of every string longer than some number will include repetition of the same substring ('cklm' above)

Pumping lemma

definition

For every regular language L, there exist an integer p such that a string $x \in L$ can be factored as $x = uvw$,

- $uv^iw \in L, \forall i \geqslant 0$
- $v \neq \epsilon$
- $|uv| \leqslant p$

Pumping lemma

definition

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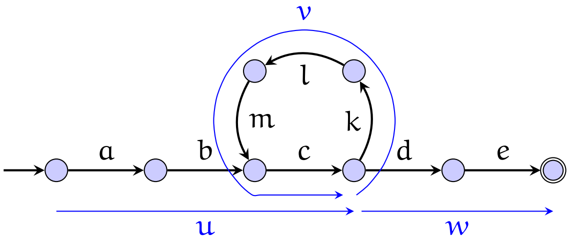

How to use pumping lemma

- We use pumping lemma to prove that a language is not regular
- Proof is by contradiction:
  - Assume the language is regular
  - Find a string x in the language, for all splits of $x = uvw$, at least one of the pumping lemma conditions does not hold
    - $uv^iw \in L \ (\forall i \geqslant 0)$
    - $v \neq \epsilon$
    - $|uv| \leqslant p$

Pumping lemma example

prove $L = a^n b^n$ is not regular

- Assume L is regular: there must be a p such that, if uvw is in the language
  1. $uv^iw \in L$ $(\forall i \geqslant 0)$
  2. $v \neq \epsilon$
  3. $|uv| \leqslant p$
- Pick the string $a^p b^p$
- For the sake of example, assume $p = 5$, $x = aaaaabbbbb$
- Three different ways to split

$\underbrace{a}_{u}\underbrace{aaa}_{v}\underbrace{abbbbb}_{w}$	violates 1
$\underbrace{aaaa}_{u}\underbrace{ab}_{v}\underbrace{bbbb}_{w}$	violates 1 & 3
$\underbrace{aaaaab}_{u}\underbrace{bbb}_{v}\underbrace{b}_{w}$	violates 1 & 3

Wrapping up

- FSA and regular expressions express regular languages
- Regular languages and FSA are closed under
  - Concatenation
  - Kleene star
  - Complement
  - Reversal
  - Union
  - Intersection
- To prove a language is regular, it is sufficient to find a regular expression or FSA for it
- To prove a language is not regular, we can use pumping lemma

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Next:

- Finite state transducers (FSTs)
- Applications of FSA and FSTs
- Summary exam preparation/discussion

Acknowledgments, credits, references

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