Graphs
Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

Çağrı Çöltekin
ccoltekin@sfs.uni-tuebingen.de

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Introduction

- A graph is collection of **vertices (nodes)** connected pairwise by **edges (arcs)**.
- A graph is a useful abstraction with many applications
- Most problems on graphs are challenging

Example applications
City map

- **City maps**
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics

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Example applications
many more...

- Food web
- Course dependencies
- Social media
- Scheduling
- Games
- Academic networks
- Inheritance relations in object-oriented programming
- Flow charts
- Financial transactions
- World's languages
- PageRank algorithm
- ...

Definition

- A (simple) *graph* G is a pair (V, E) where
  - V is a set of *nodes* (or *vertices*),
  - E ⊆ {{x, y} | x, y ∈ V and x ≠ y} is a set of ordered or unordered pairs
- A graph represent a set of objects (nodes) and the relations between them (edges)
- Edges in a graph can be either *directed*, or *undirected*
  - directed edges are 2-tuples, or *ordered pairs* (order is important)
  - undirected edges are unordered pairs, or pair sets (order is not important)

Types of graphs

- An *undirected graph* is a graph with only undirected edges
  - Transportation (e.g., railway) networks
- A *directed graph* (digraph) is a graph with only directed edges
  - course dependencies
- A *mixed graph* contains both directed and undirected edges
  - a city map

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More graphs types

- A graph is *simple* if there is only a single edge between two nodes (our earlier definition)
- If the edges of a graph has associated weights, it is called a *weighted graph*
- A *complete graph* contains edges from each node to every other node
- A *bipartite graph* has two disjoint sets of nodes, where edges are always across the sets
- A graph is called a *multi-graph* if there are multiple edges (with the same direction) between a pair of nodes
- A graph is called a *hyper-graph* if a single edge can link more than two nodes

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More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are *neighbors*) if they are incident to the same edge
- The *degree* (or *valency*) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* of a node is the number of outgoing edges

A and B are endpoints of edge 1

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edge 1 is incident to A and B

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deg(A) = 4

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indeg(A) = 1, outdeg(A) = 3

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A *self-loop* is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

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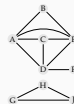- A path or a cycle is a *simple* if every node on the path is visited only once

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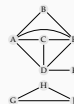- A path or a cycle is a *simple* if every node on the path is visited only once

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More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components
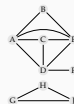

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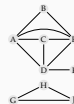

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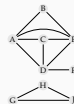

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More definitions

- A *spanning subgraph* of a graph is a subgraph that includes all nodes of the graph
- A *tree* is a connected graph without cycles
- A *spanning tree* is a spanning subgraph which is a tree
- A *forest* is a disconnected acyclic graph

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Some properties
sum of degrees

- For an undirected graph with m edges and set of nodes V

$$\sum_{v \in V} \deg(v) = 2m$$

- All edges are counted twice for each node they are incident to
- The total contribution of each node is twice its degree
- For a directed graph with m edges and set of nodes V

$$\sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v) = m$$


Some properties
relation between the number of edges and nodes

- For a simple undirected graph with n nodes and m edges

$$m \leq \frac{n(n-1)}{2}$$

- If the graph is simple
  - there are no parallel edges
  - there are no self loops
  - the maximum degree of a node is $n - 1$
- Putting this together with the previous property

$$2m \leq n(n-1) \Rightarrow m \leq \frac{n(n-1)}{2}$$

- For a directed graph with n nodes and m edges

$$m \leq n(n-1)$$

The graph ADT

- A graph is a collection of nodes and edges
- Basic operations include

  `add_node(v)` add a new node

  `remove_node(v)` remove an existing node

  `adjacent(u,v)` return true if the nodes are adjacent (for a digraph true only if there is a directed link from u to v)

  `neighbors(v)` enumerate the neighbors of the node (for a digraph we list the nodes reachable through outgoing edges by default)

  `remove_edge(u,v)` remove an existing edge

  `add_edge(u,v)` add a new edge

  `nodes()` enumerate the nodes in the graph

  `edges()` enumerate the edges in the graph

Edge list
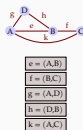
- We keep a simple list of edges (and possibly nodes)
- Simple structure, but not very efficient (n nodes, m edges):

  `add_edge(v)` O(1)

  `remove_edge(v)` O(m)

  `remove_node(v)` O(m)

  `neighbors(v)` O(m)

```
e = (A,B)
f = (B,C)
g = (A,D)
h = (D,B)
k = (A,C)
```
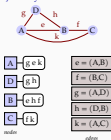
Adjacency list

- We keep simple lists for nodes and edges
- Very simple structure, but not very efficient:

  `add_node(v)` O(1)

  `remove_node(v)` O(deg(v))

  `adjacent(u,v)` O(min(deg(u), deg(v)))

  `neighbors(v)` O(deg(v))

```
A → g e k        e = (A,B)
D → g h          f = (B,C)
B → e h f        g = (A,D)
C → i k          h = (D,B)
nodes            k = (A,C)
                 edges
```

Adjacency matrix

- We keep simple lists of nodes and edges
- Very simple structure, but not very efficient:

  `add_node(v)` O(n)

  `remove_node(v)` O(n)

  `adjacent(u,v)` O(1)

  `neighbors(v)` O(n)

	A	B	C	D
A		e	k	g
B			f	h
C				
D				

Interesting problems on graphs

- Is there a (directed) path between two nodes?
- What is the shortest path between two nodes?
- Is there a cycle in the graph?
- Is there a cycle that uses each edge exactly once? (Eulerian path)
- Is there a cycle that uses each node exactly once? (Hamiltonian path)
- Are all nodes of the graph connected?
- Is there a node that breaks the connectivity if removed?
- Is the graph planar: can it be drawn without crossing edges?
- Are two graphs isomorphic (have the same structure)?
- What is the importance of a web page, based on the links pointing to it?

Summary

- Graphs are data structures with many applications
- Reading on graphs: Goodrich, Tamassia, and Goldwasser (2013, chapter 14),

Next:

- Graph traversals
- Reading: Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references

- The map on slide 2 is from OpenStreetMap, The other images are from Wikipedia, except the infectious disease graph which comes from Thurner et al. (2020).

📖 Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python.* John Wiley & Sons, Incorporated. ISBN: 9781118476734.