Shortest path algorithms

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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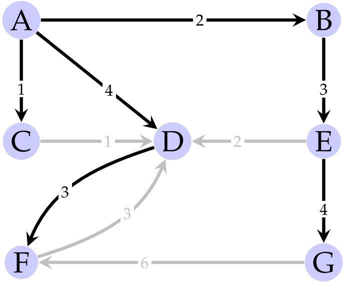
Shortest path

- Finding shortest paths on a weighted (directed) graph is one of the most common problems in many fields
- Applications include
  - Navigation
  - Routing in computer networks
  - Optimal construction of electronic circuits, VLSI chips
  - Robotics, transportation, finance, …

Shortest paths on unweighted graphs

BFS

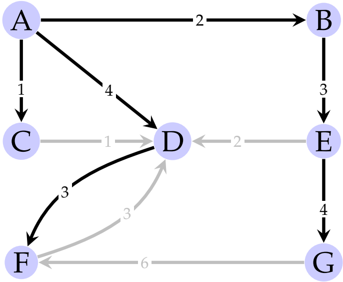
- A BFS search tree gives the shortest path from the source node to all other nodes



Shortest paths on unweighted graphs

BFS

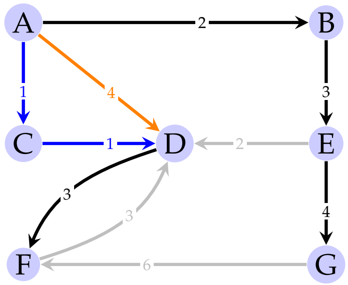
- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs



Shortest paths on unweighted graphs

BFS

- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs
- Shortest-cost path may be longer in terms of nodes visited



Shortest paths on weighted graphs

variation of the problem

- Different versions of the problem:
  - Single source shortest path: find shortest path from a source node to all others
  - Single target (sometimes called sink) shortest path: find shortest path from all nodes to a target node
  - Source to target: from a particular source node to a particular target node
  - All pairs: shortest paths between all pairs of nodes
- Restrictions on weights:
  - Euclidean weights
  - Non-negative weights
  - Arbitrary weights

Dijkstra's algorithm

intro

- Dijkstra's algorithm is a 'weighted' version of the BFS
- The algorithm finds shortest path from a single source node to all connected nodes
- Weights has to be non-negative
- It is a greedy algorithm that grows a 'cloud' of nodes for which we know the shortest paths from the source node
- The new nodes are included in the cloud in order of their shortest paths from the source node
- The algorithm is also similar to Prim-Jarník algorithm used for finding MST

Dijkstra's algorithm

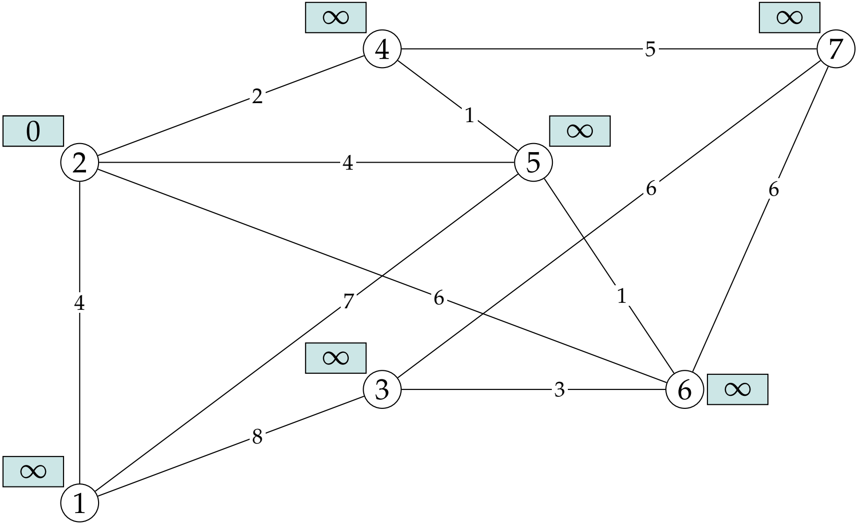
the algorithm

- We maintain a list D of minimum know distances to each node
- At each step
  - we take closest node out of Q
  - update the distances of all nodes
- Can be more efficient if Q is implemented using a (adaptable) priority queue

$\mathrm{D}[s] \leftarrow 0$
for each node $v \neq s$ **do**
    $\mathrm{D}[v] \leftarrow \infty$
Q $\leftarrow$ nodes
while Q is not empty **do**
    Find the node v with min $\mathrm{D}[v]$
    for each edge (v, w) **do**
        if $\mathrm{D}[v] + w[(v, w)] < \mathrm{D}[w]$ **then**
            $\mathrm{D}[w] \leftarrow \mathrm{D}[v] + w[(v, w)]$
D contains the shortest distances from s

Dijkstra's algorithm

demonstration



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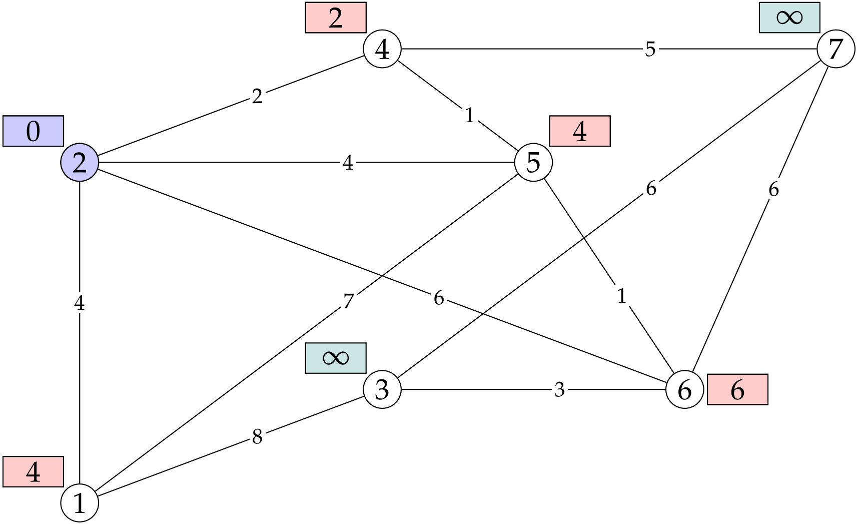

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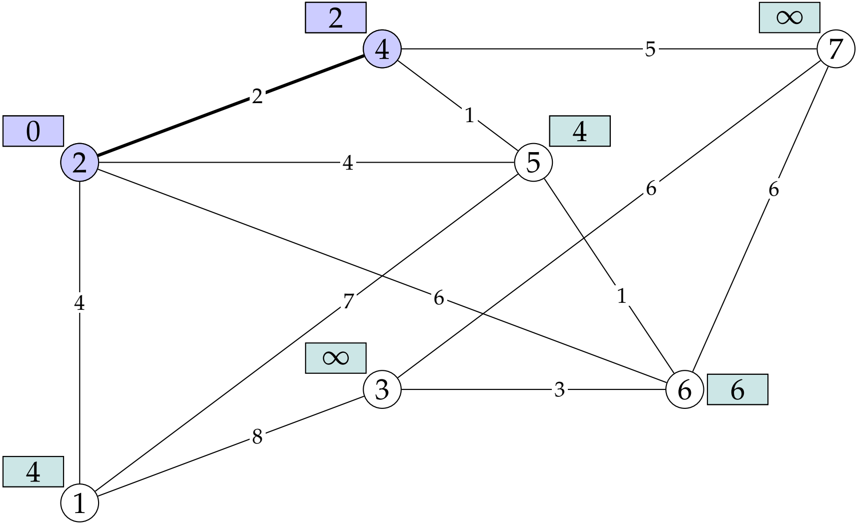

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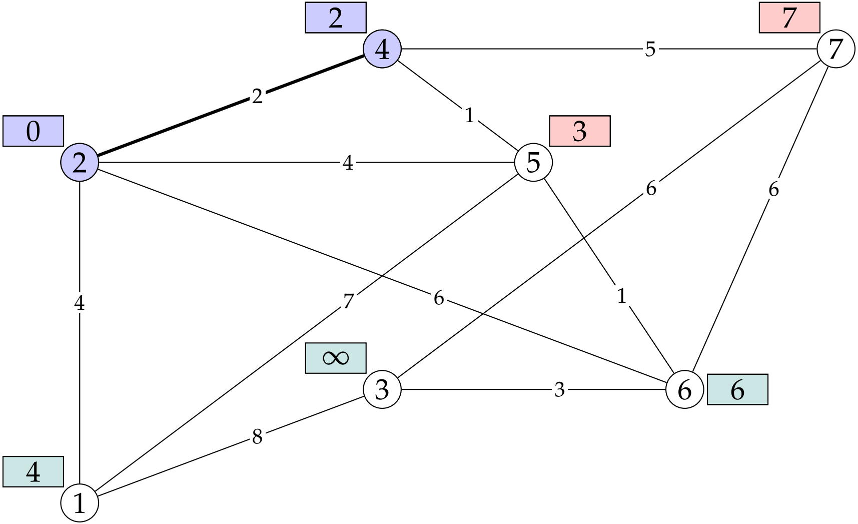

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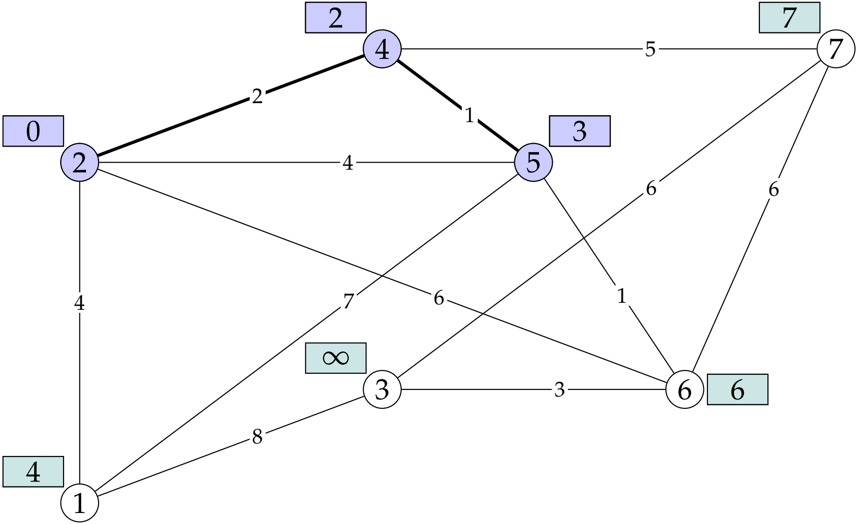

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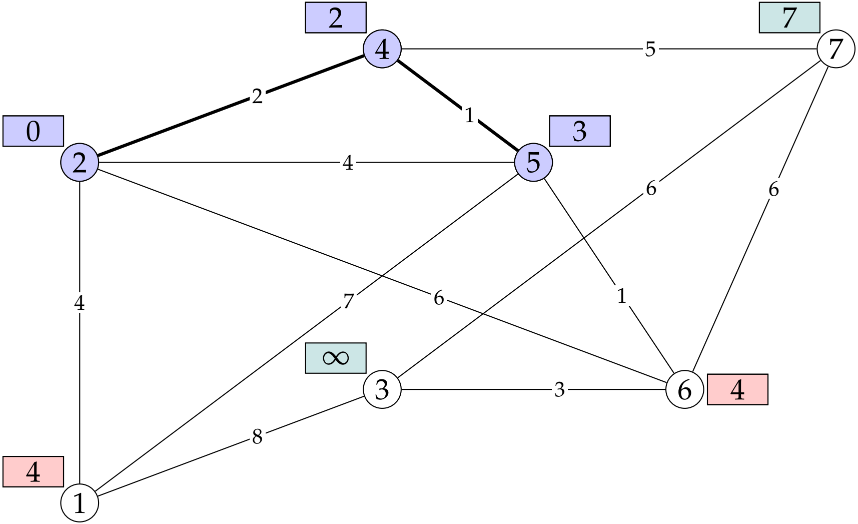

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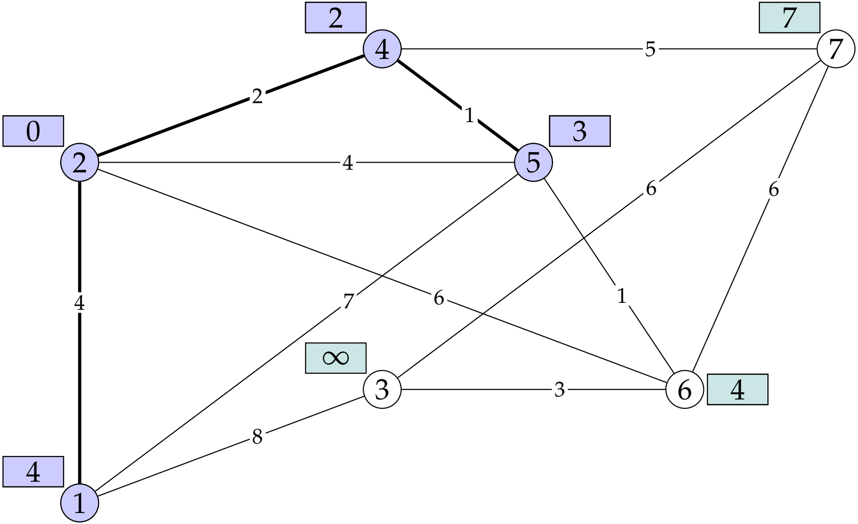

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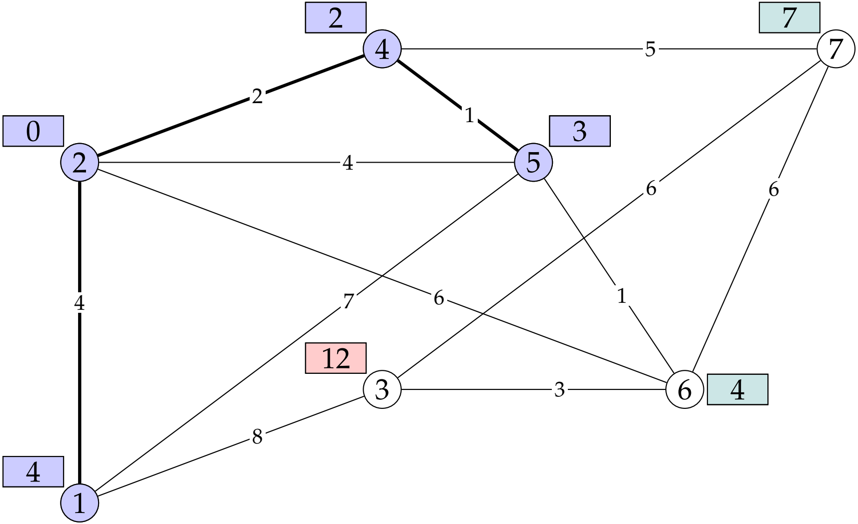

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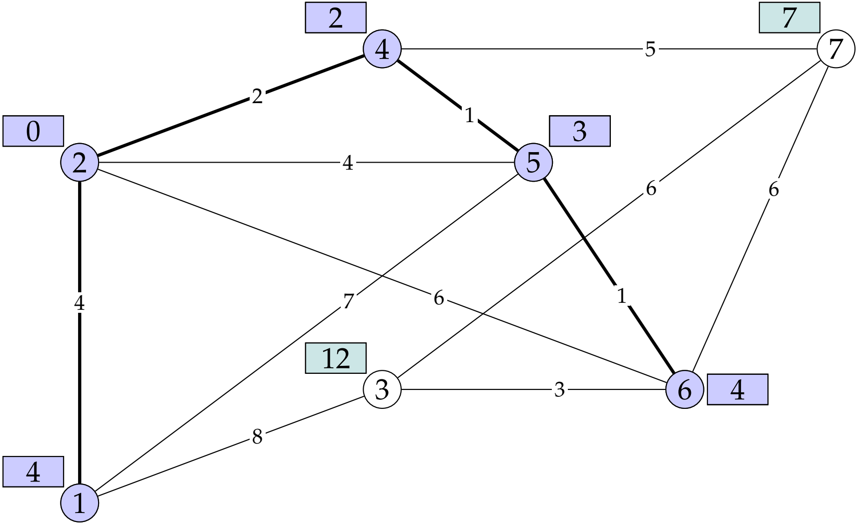

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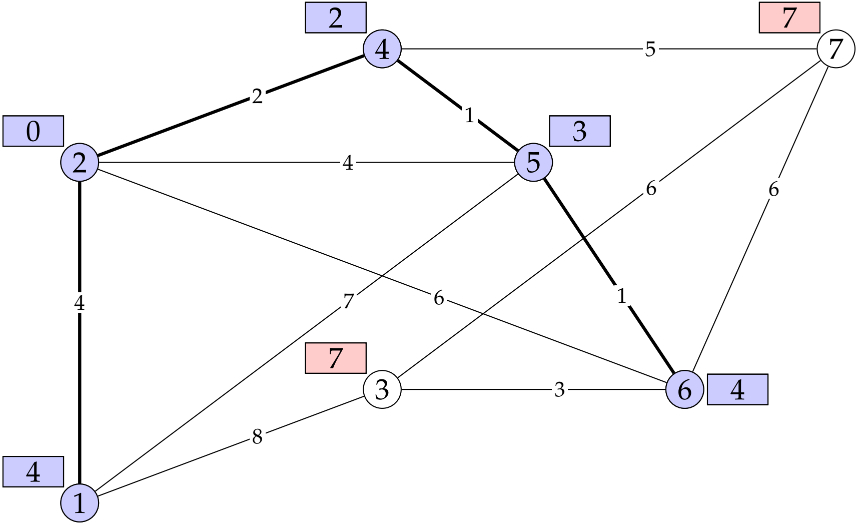

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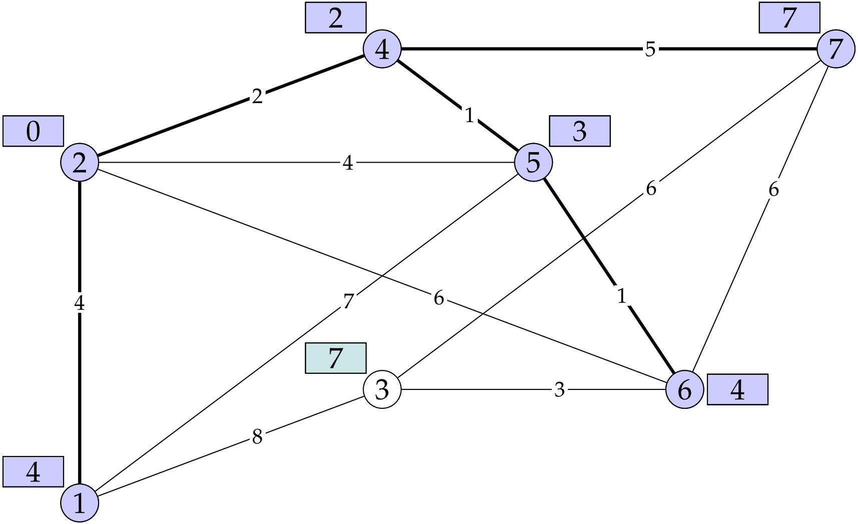

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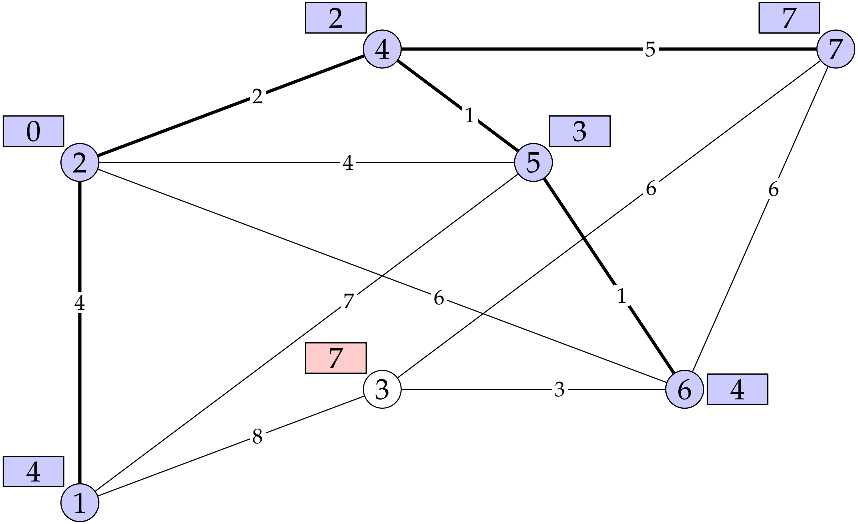

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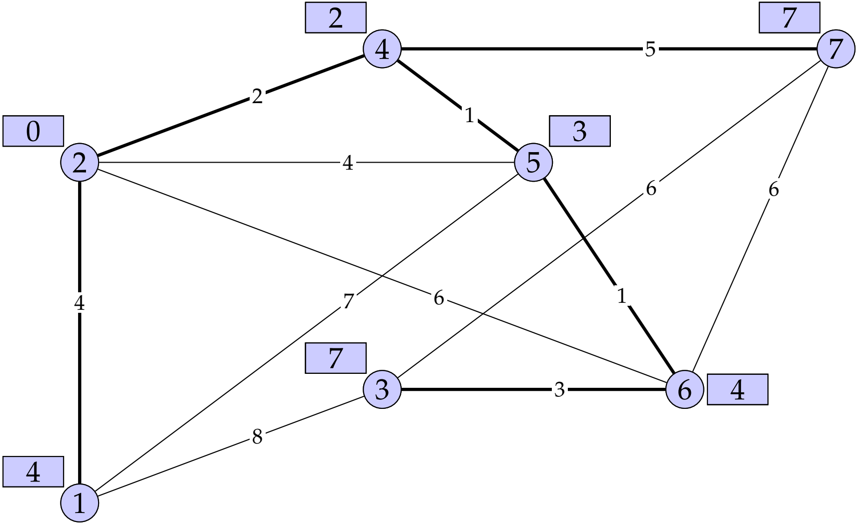
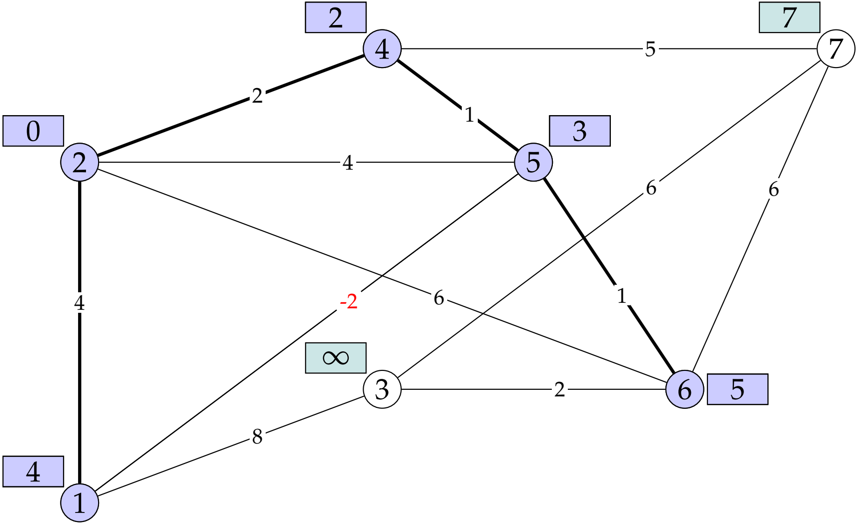


Dijkstra's algorithm and negative weights



Dijkstra's algorithm

the algorithm

- In general, complexity is $O(t_{\text{find\_min}} n + t_{\text{update\_key}} m)$
- With list-based implementation of Q: $O(m + n^2) = O(n^2)$
- With a priority queue: $O((m + n) \log n)$

```
D[s] ← 0
for each node v ≠ s do
    D[v] ← ∞
Q ← nodes
while Q is not empty do
    Find the node v with min D[v]
    for each edge (v, w) do
        if D[v] + w[(v, w)] < D[w] then
            D[w] ← D[v] + w[(v, w)]
D contains the shortest distances from s
```

Shortest-path tree

- The way we introduced, the Dijkstra's algorithm does not give the shortest-path tree
- Similar to traversal algorithms, we can extract it from distances D
- Running time is $O(n^2)$ (or $O(n + m)$)

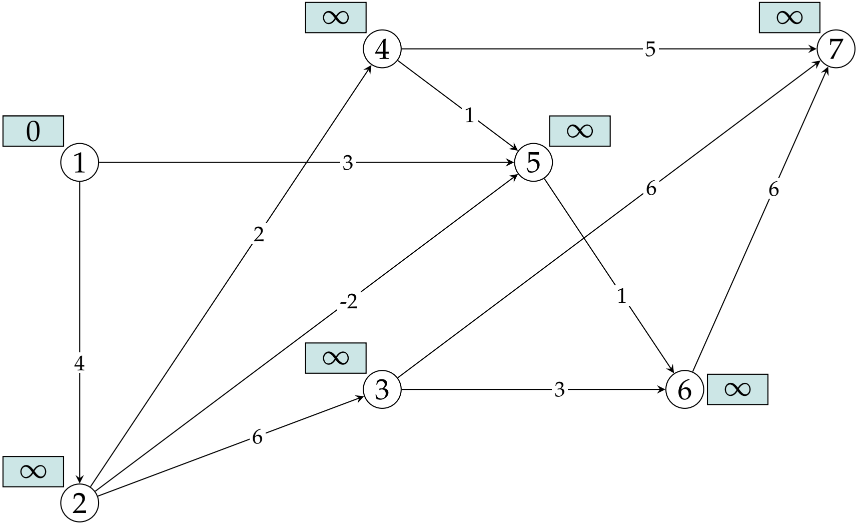
$T \leftarrow \varnothing$
for $v \in D - \{s\}$ **do**
    for each edge (w, v) **do**
        if $D[v] == D[u] + \text{weight}(w, v)$ **then**
            $T \leftarrow T \cup (w, v)$

Shortest-paths on DAGs

- The shortest path can be found more efficiently, if the graph is a DAG
- The algorithm is similar to Dijkstra's, but simpler and faster
- Only difference is we follow a topological order
- The algorithm will also work with negative edge weights

Shortest-paths on DAGs

demonstration



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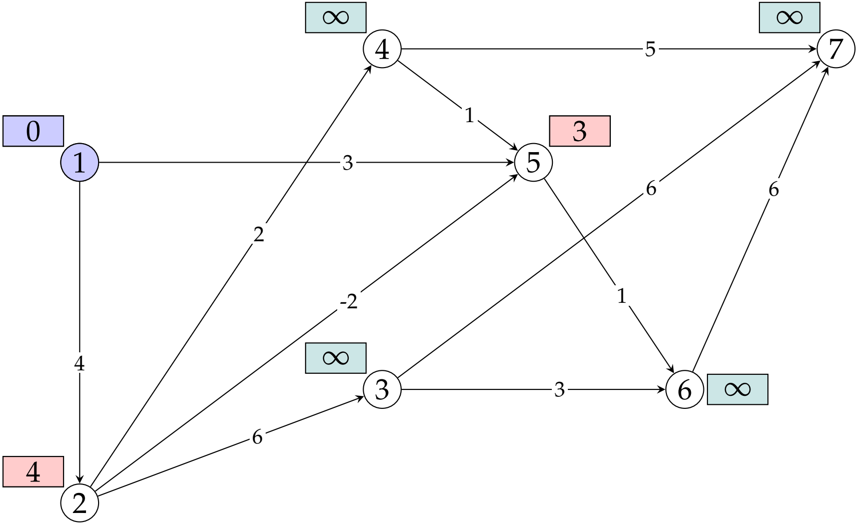

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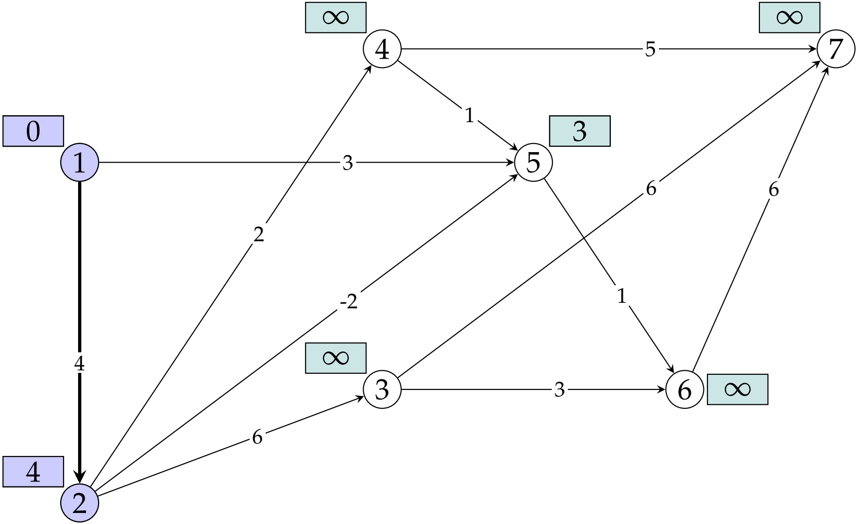

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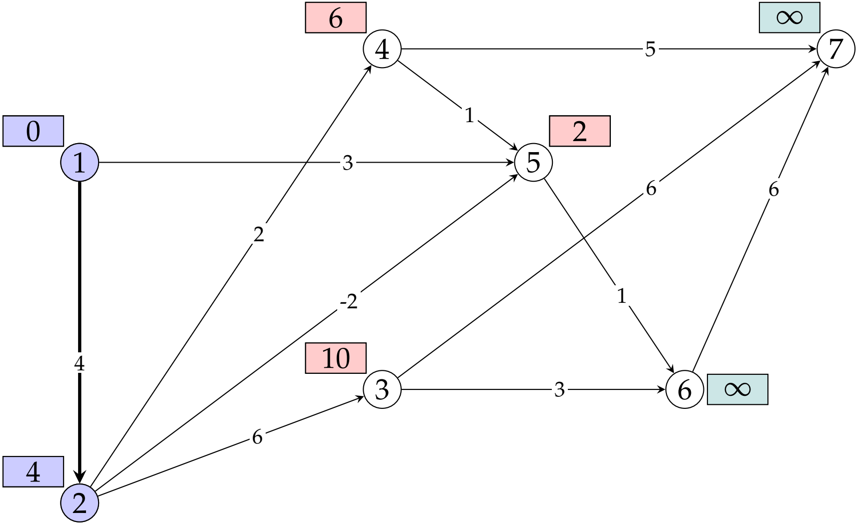

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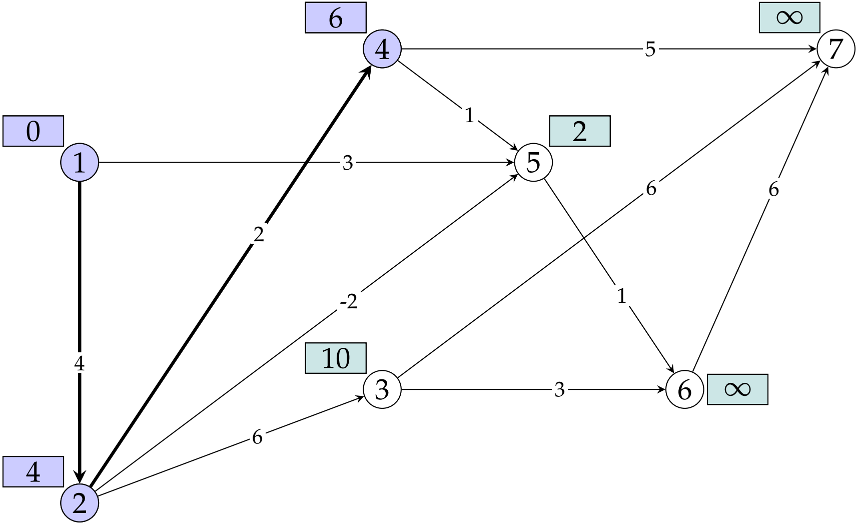

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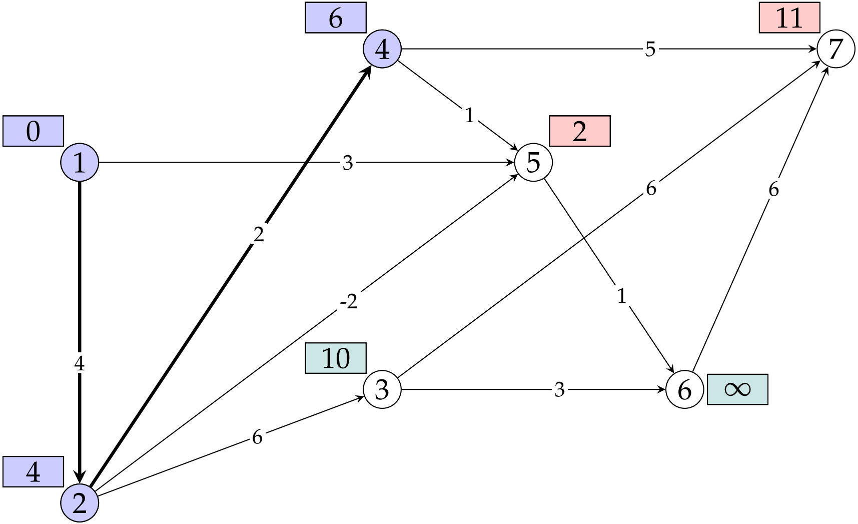

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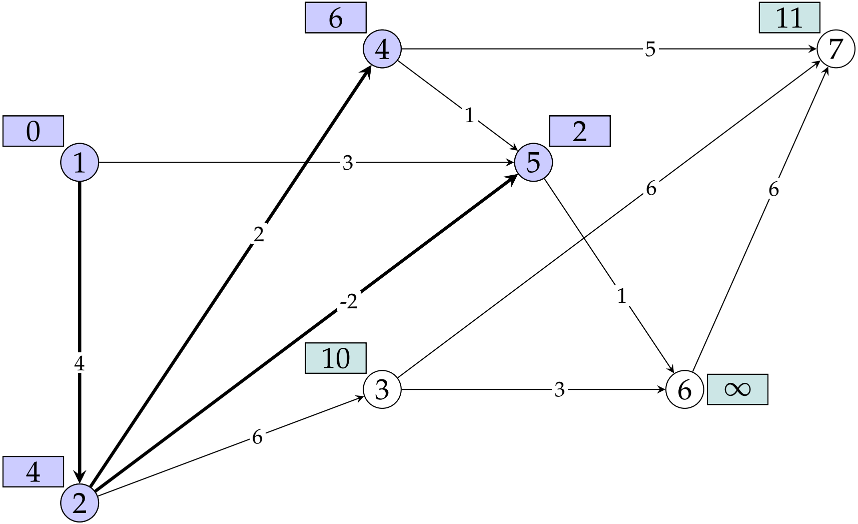

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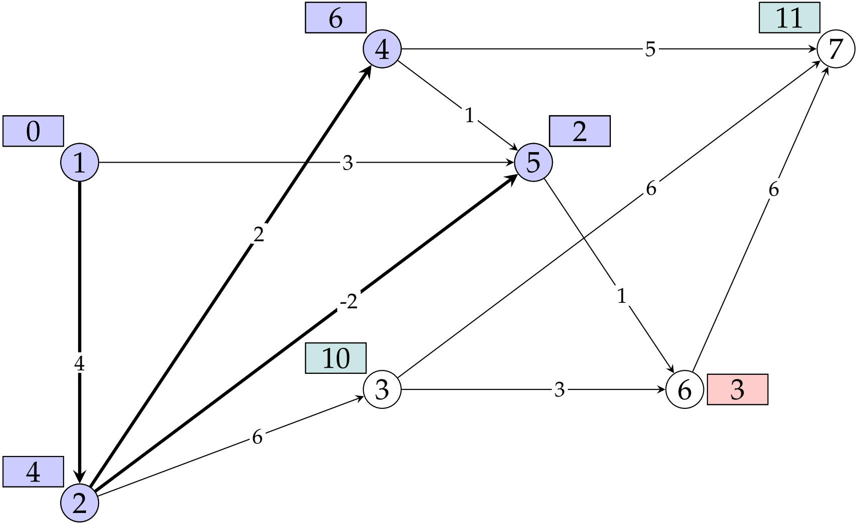

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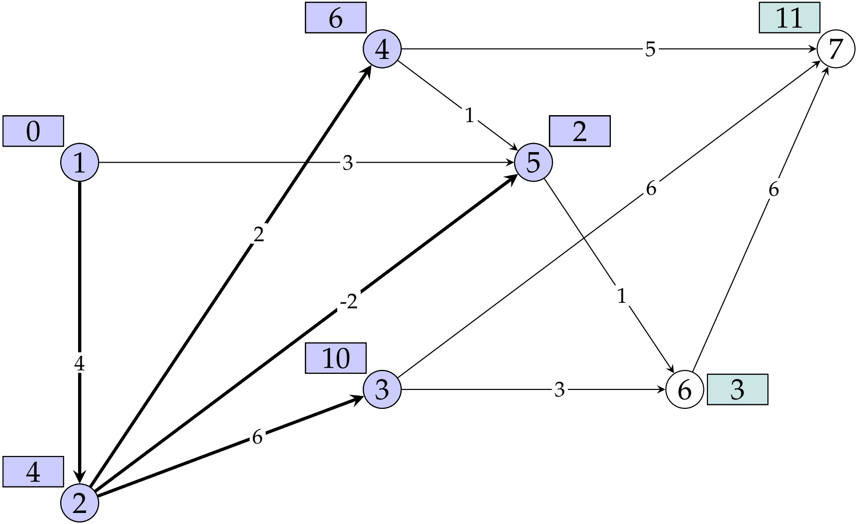

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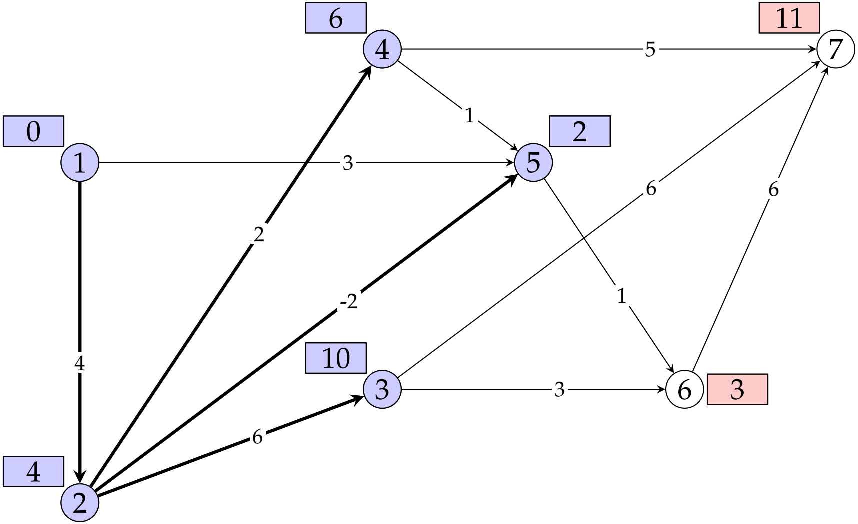

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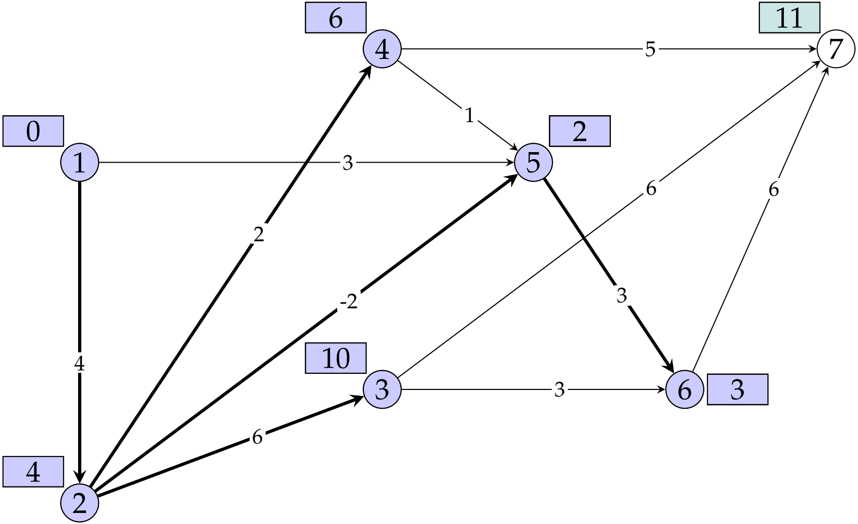

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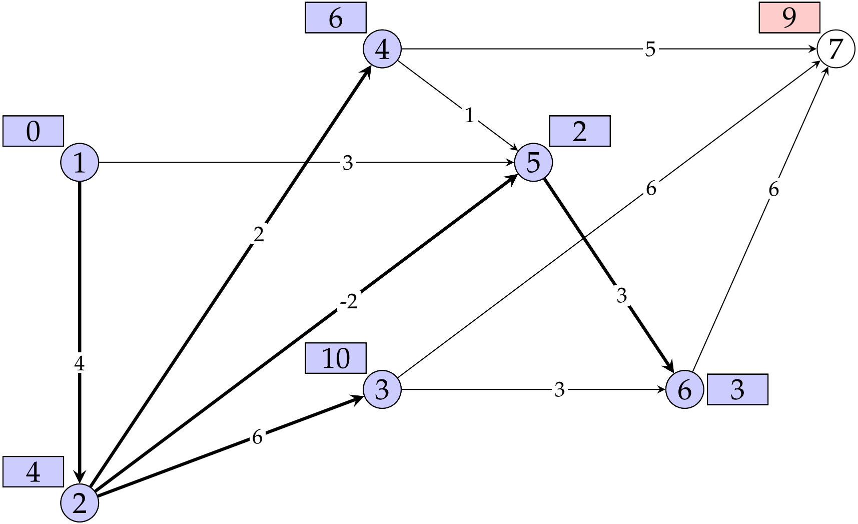

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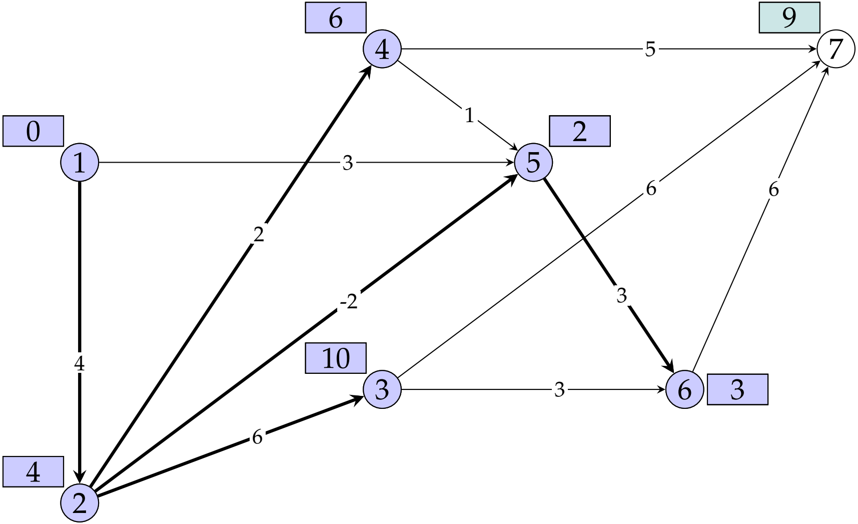

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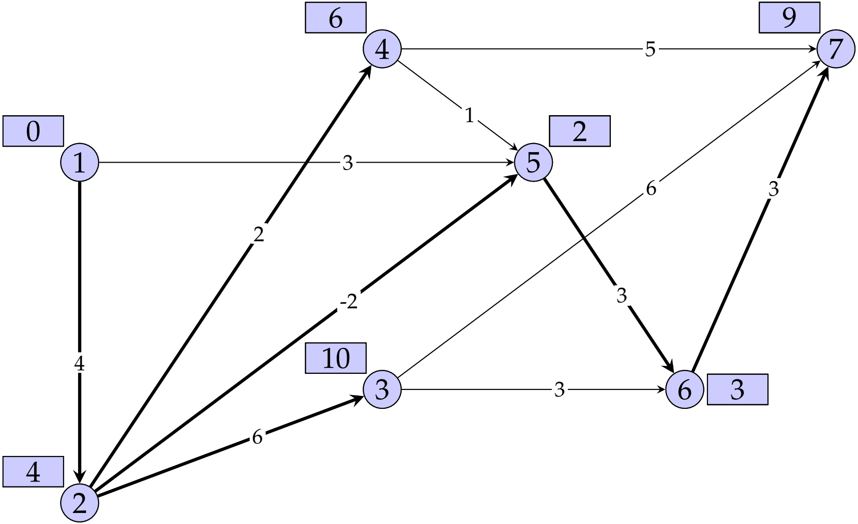


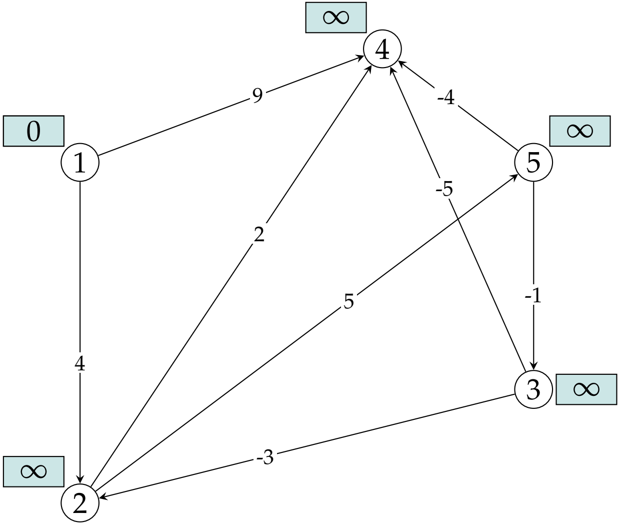
Shortest-paths on directed graphs

with negative wights – without negative cycles

- Single-source shortest path problem can also be solved efficiently for any directed graph
  - including cycles (no DAG requirement)
  - including negative weights
  - *excluding* negative cycles
- The algorithms is known as Bellman-Ford algorithm
  - Similar to earlier, initialize $D[s] = 0$, $D[v] = \infty$
  - Make n passes over the edges
    - Update distances for each edge (relax edges)
    - Stop if there were no changes at the end of a pass

Bellman-Ford algorithm

demonstration



edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
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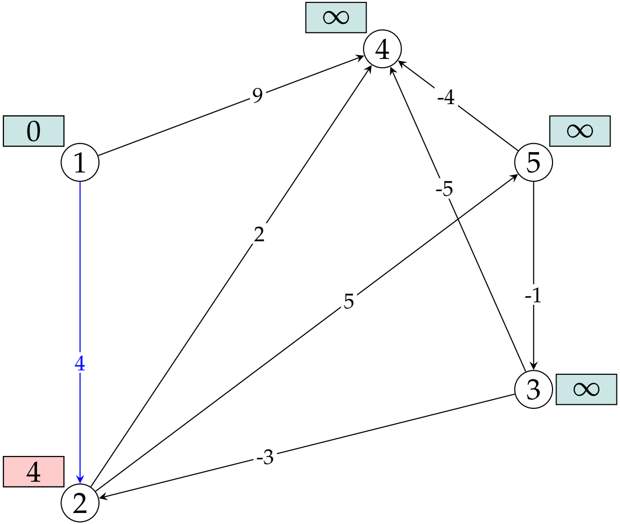
Bellman-Ford algorithm

demonstration



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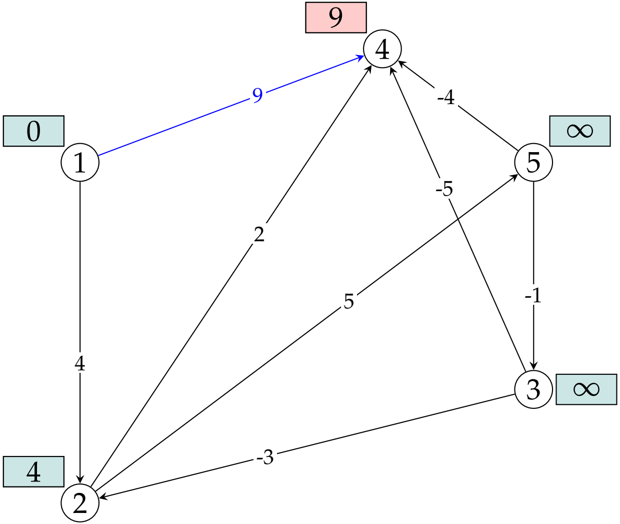

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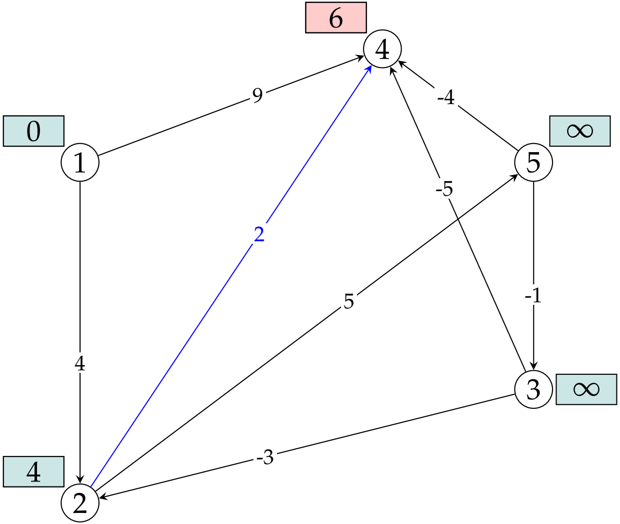

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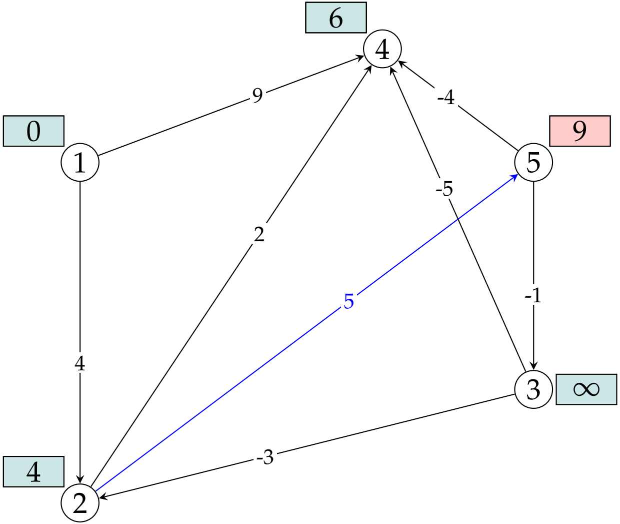

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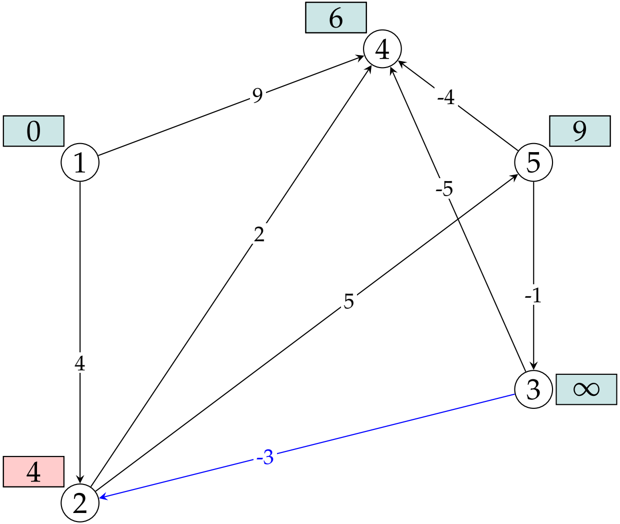

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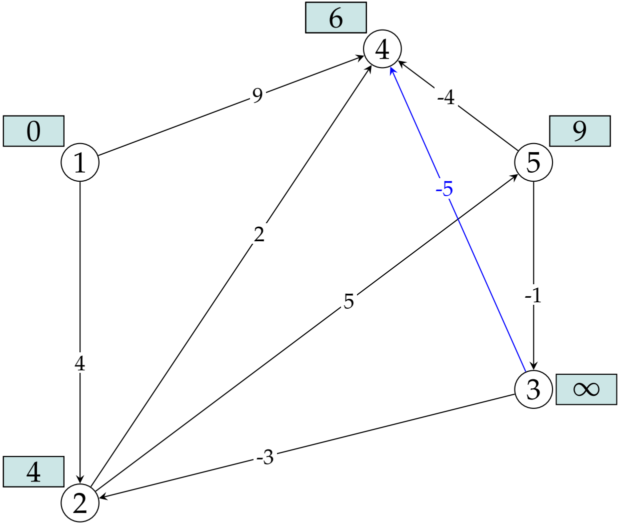

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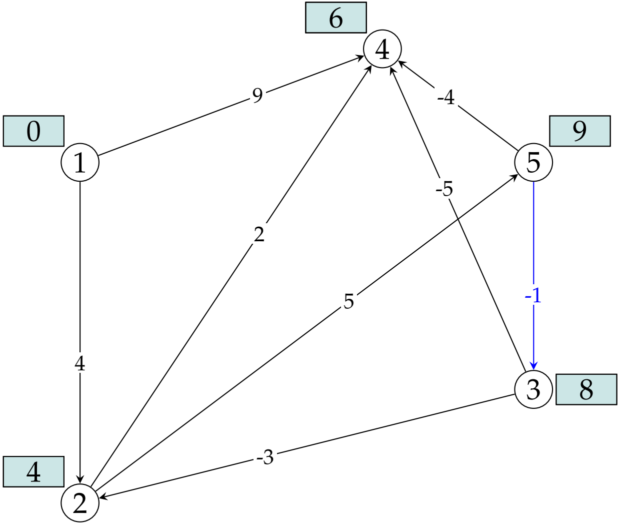

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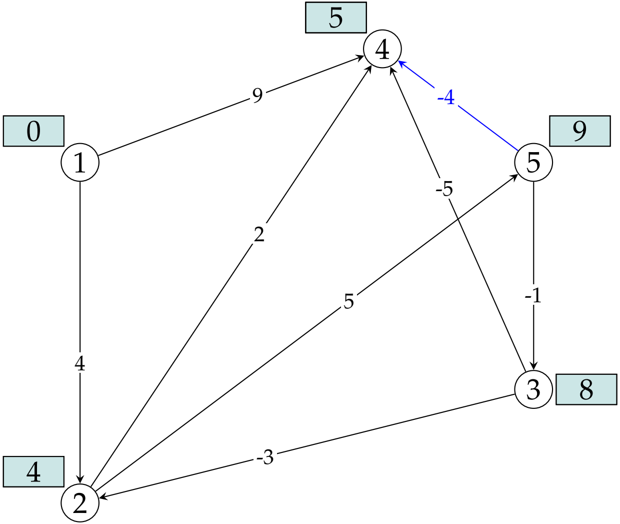

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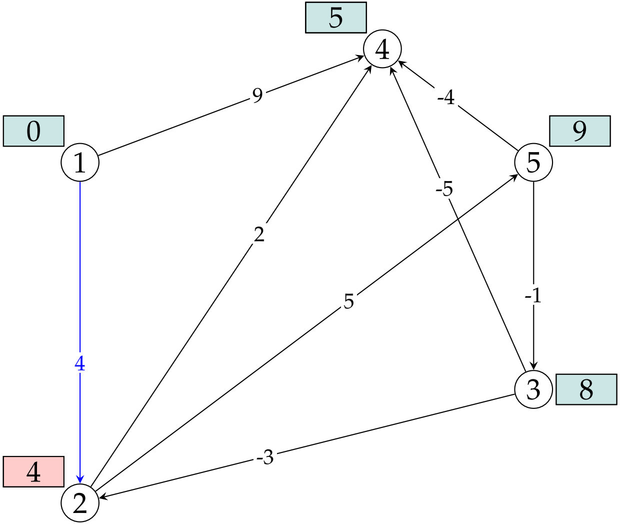

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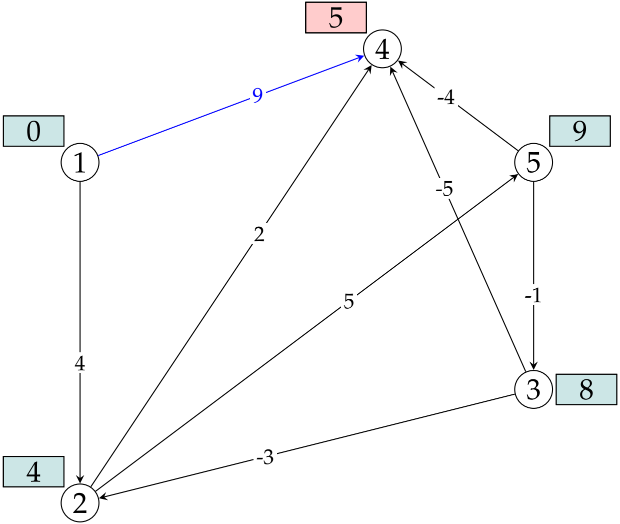

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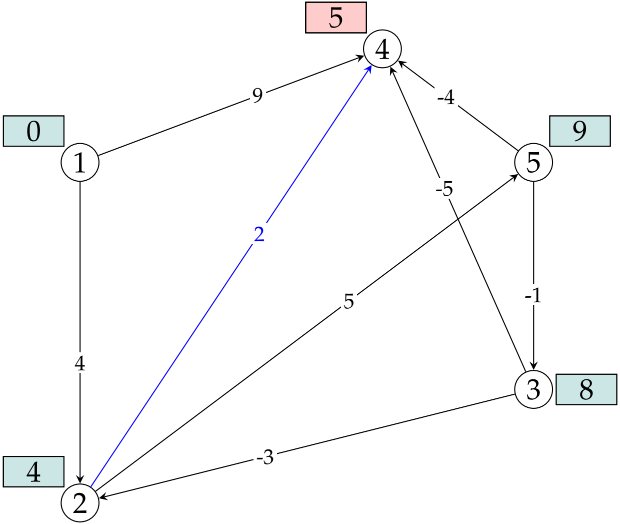

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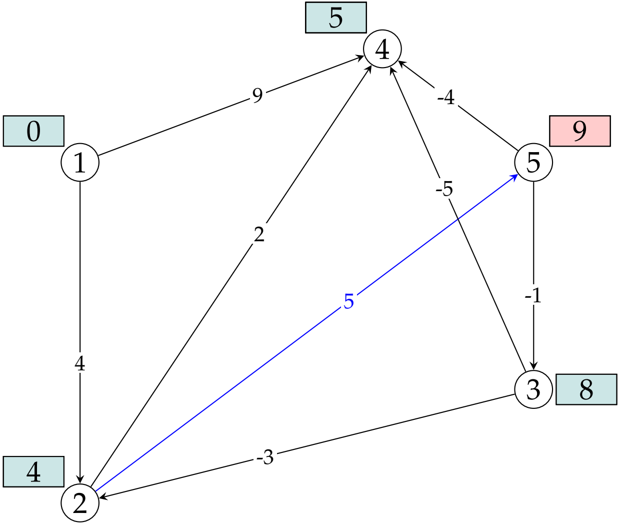

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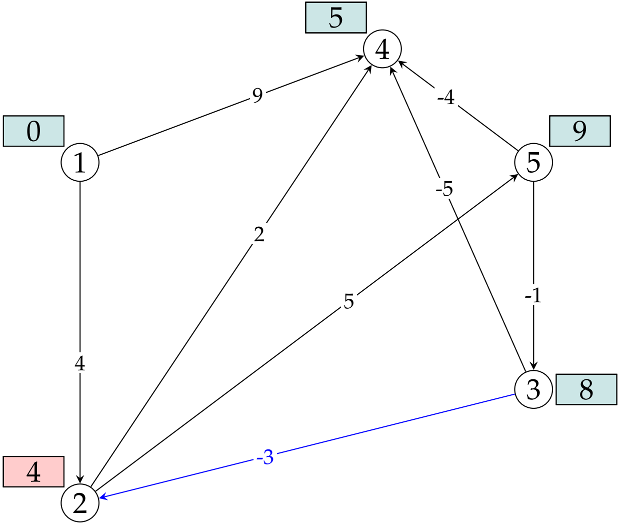

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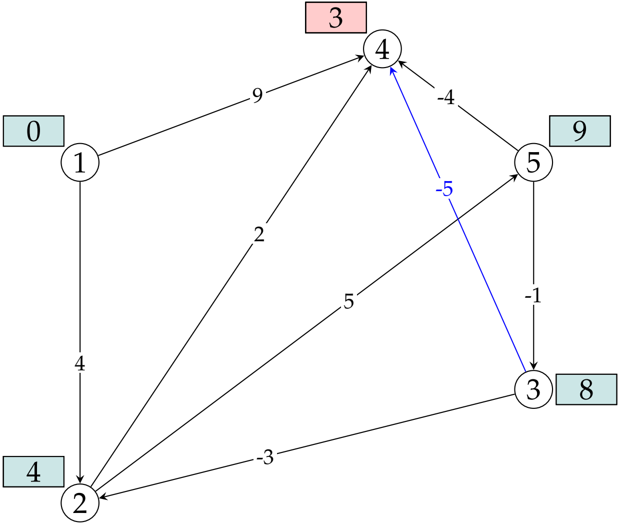

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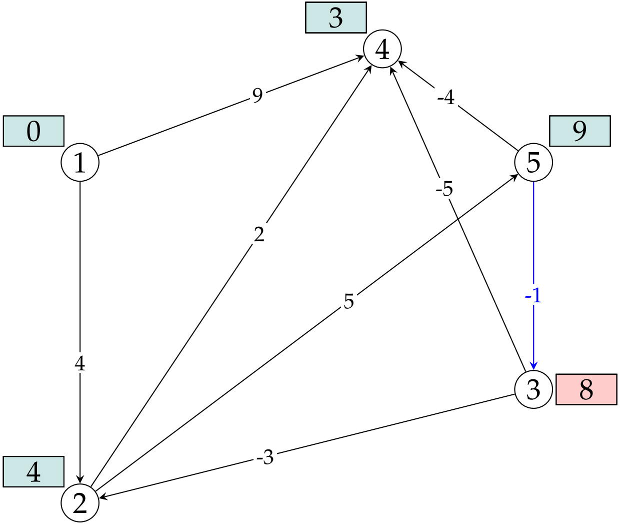

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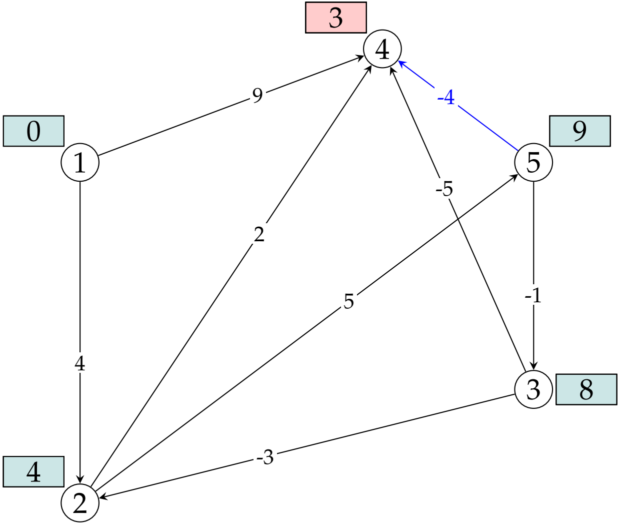

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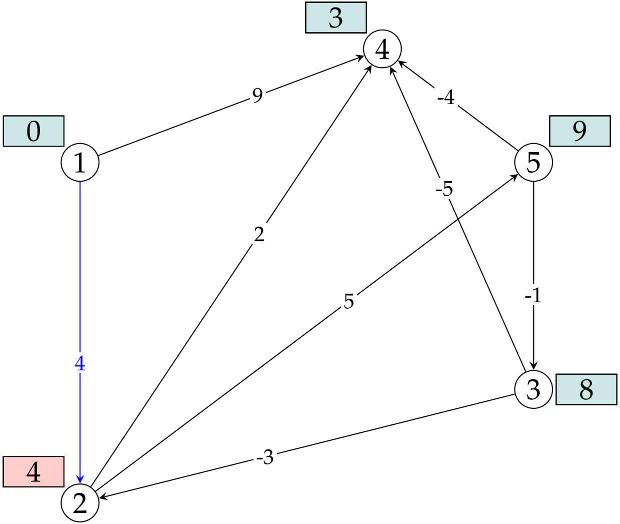

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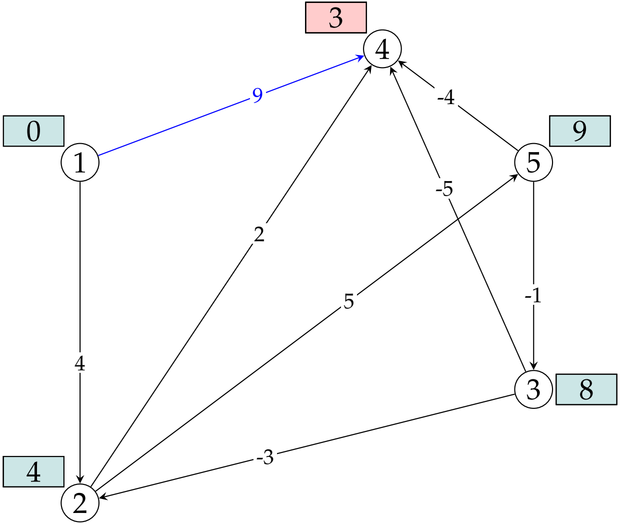
$5 \rightarrow 4$

Bellman-Ford algorithm

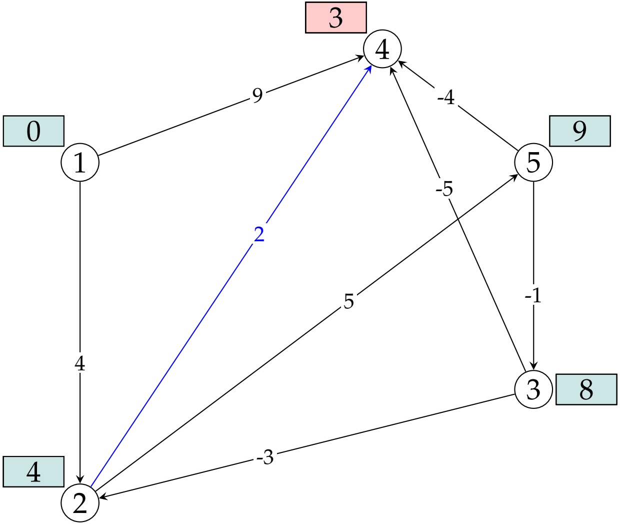
demonstration



edges
$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

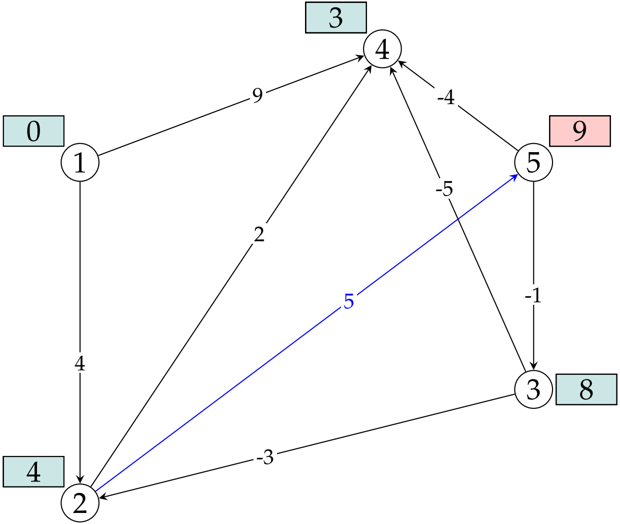


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

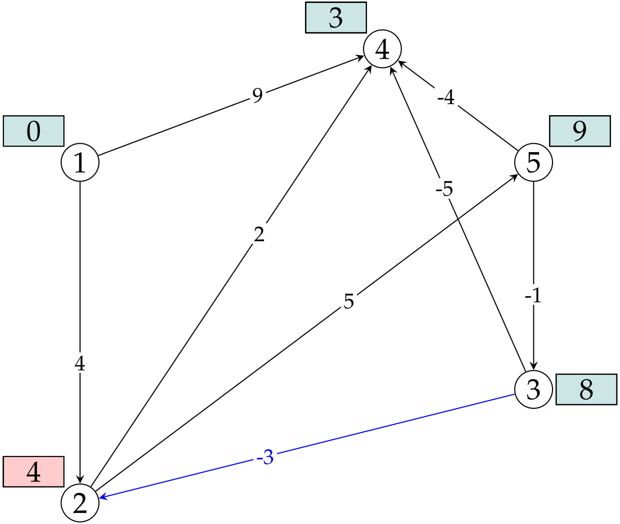


edges

$1 \to 2$
$1 \to 4$
$2 \to 4$
$2 \to 5$
$3 \to 2$
$3 \to 4$
$5 \to 3$
$5 \to 4$

Bellman-Ford algorithm

demonstration

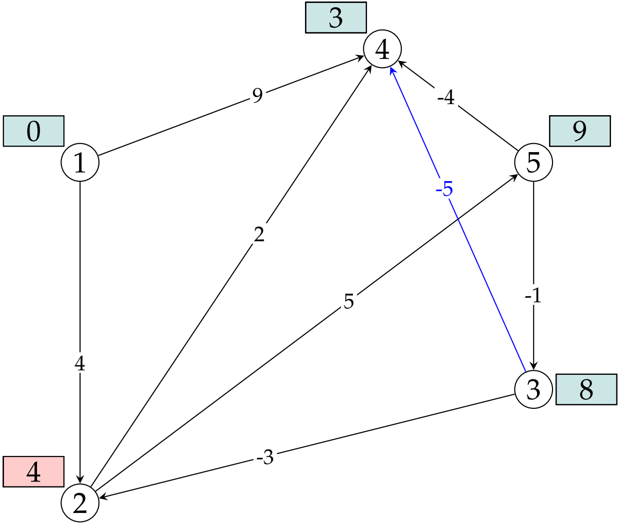


edges

$1 \to 2$
$1 \to 4$
$2 \to 4$
$2 \to 5$
$3 \to 2$
$3 \to 4$
$5 \to 3$
$5 \to 4$

Bellman-Ford algorithm

demonstration

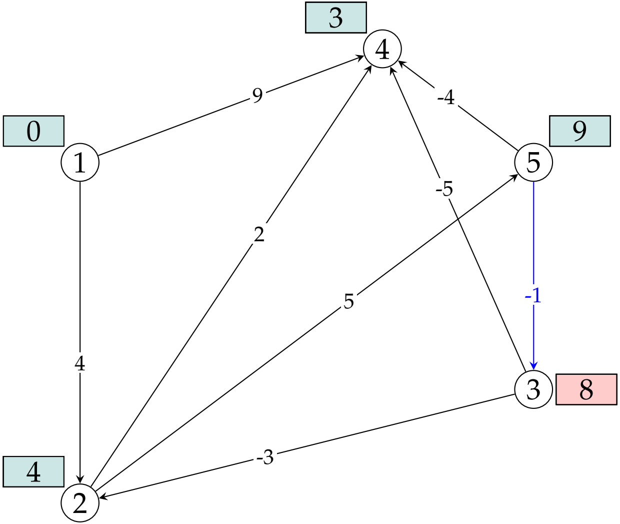


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

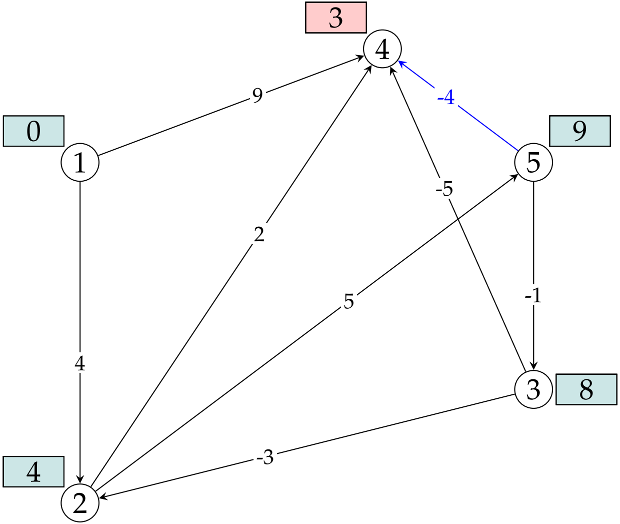


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration



edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Summary

- Shortest path algorithms are one of the most applied graph algorithms
- We revised three algorithms
  - Dijkstra's: non-negative weights, general algorithm
  - For DAGs: unrestricted weights, following topological order
  - Bellman-Ford: no negative cycles, digraphs
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Next:

- Maps and hashing
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 10)

Acknowledgments, credits, references

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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