

Graphs

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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Introduction

- A graph is collection of **vertices (nodes)** connected pairwise by **edges (arcs)**.
- A graph is a useful abstraction with many applications
- Most problems on graphs are challenging



Example applications

City map

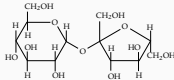
- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics



Example applications

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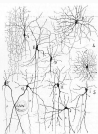
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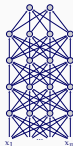
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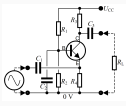
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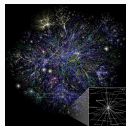
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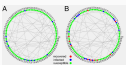
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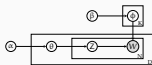
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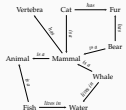
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Example applications

many more...

- Food web
- Course dependencies
- Social media
- Scheduling
- Games
- Academic networks
- Inheritance relations in object-oriented programming
- Flow charts
- Financial transactions
- World's languages
- PageRank algorithm
- ...

Definition

- A (simple) graph G is a pair (V, E) where
 - V is a set of nodes (or vertices),
 - $E \subseteq \{[x, y] \mid x, y \in V \text{ and } x \neq y\}$ is a set of ordered or unordered pairs
- A graph represent a set of objects (nodes) and the relations between them (edges)
- Edges in a graph can be either **directed**, or **undirected**
 - directed edges are 2-tuples, or *ordered pairs* (order is important)
 - undirected edges are unordered pairs, or pair sets (order is not important)



Types of graphs

- An **undirected graph** is a graph with only undirected edges
 - Transportation (e.g., railway) networks
- A **directed graph** (digraph) is a graph with only directed edges
 - course dependencies
- A **mixed graph** contains both directed and undirected edges
 - a city map



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More graphs types

- A graph is **simple** if there is only a single edge between two nodes (our earlier definition)
- If the edges of a graph has associated weights, it is called a **weighted graph**
- A **complete graph** contains edges from each node to every other node
- A **bipartite graph** has two disjoint sets of nodes, where edges are always across the sets
- A graph is called a **multi-graph** if there are multiple edges (with the same direction) between a pair of nodes
- A graph is called a **hyper-graph** if a single edge can link more than two nodes



More definitions

- Two nodes joined by an edge are called the **endpoints of the edge**
- An edge is called **incident** to a node if the node is one of its endpoints. Two nodes are **adjacent** (or they are neighbors) if they are incident to the same edge
- The **degree** (or valency) of a node is the number of its incident edges
- In a digraph **indegree** of a node is the number of incoming edges, and **outdegree** of a node is the number of outgoing edges



A and B are endpoints of edge 1

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edge 1 is incident to A and B

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 $\deg(A) = 4$

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 $\text{indeg}(A) = 1, \text{outdeg}(A) = 3$

More definitions

- Two edges are **parallel** if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A path is a sequence of alternating edges and nodes
- A cycle is a path that starts and ends at the same node
- A path or a cycle is a **simple** if every node on the path is visited only once



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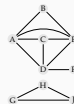
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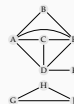
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- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components



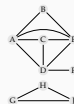
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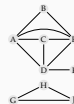
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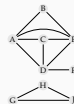
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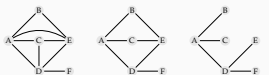
More definitions

- A *spanning subgraph* of a graph is a subgraph that includes all nodes of the graph
- A *tree* is a connected graph without cycles
- A *spanning tree* is a spanning subgraph which is a tree
- A *forest* is a disconnected acyclic graph



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Some properties

sum of degrees

- For an undirected graph with m edges and set of nodes V

$$\sum_{v \in V} \deg(v) = 2m$$

- All edges are counted twice for each node they are incident to
- The total contribution of each node is twice its degree
- For a directed graph with m edges and set of nodes V

$$\sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v) = m$$

Some properties

relation between the number of edges and nodes

- For a simple undirected graph with n nodes and m edges

$$m \leq \frac{n(n-1)}{2}$$

- If the graph is simple
 - there are no parallel edges
 - there are no self loops
 - the maximum degree of a node is $n-1$
- Putting this together with the previous property

$$2m \leq n(n-1) \Rightarrow m \leq \frac{n(n-1)}{2}$$

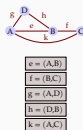
- For a directed graph with n nodes and m edges

$$m \leq n(n-1)$$

The graph ADT

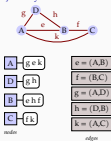
- A graph is a collection of nodes and edges
- Basic operations include
 - `add_node(v)` add a new node
 - `remove_node(v)` remove an existing node
 - `adjacent(u, v)` return true if the nodes are adjacent (for a digraph true only if there is a directed link from u to v)
 - `neighbors(v)` enumerate the neighbors of the node (for a digraph we list the nodes reachable through outgoing edges by default)
 - `remove_edge(u, v)` remove an existing edge
 - `add_edge(u, v)` add a new edge
 - `nodes()` enumerate the nodes in the graph
 - `edges()` enumerate the edges in the graph

Edge list



- We keep a simple a simple list of edges (and possibly nodes)
- Simple structure, but not very efficient (n nodes, m edges):
 - `add_edge(v)` $O(1)$
 - `remove_edge(v)` $O(m)$
 - `remove_node(v)` $O(m)$
 - `adjacent(u, v)` $O(m)$
 - `neighbors(v)` $O(m)$

Adjacency list



- We keep simple lists for nodes and edges
- Very simple structure, but not very efficient:
 - `add_node(v)` $O(1)$
 - `remove_node(v)` $O(\deg(v))$
 - `adjacent(u, v)` $O(\min(\deg(u), \deg(v)))$
 - `neighbors(v)` $O(\deg(v))$

Adjacency matrix



- We keep simple lists for nodes and edges
- Very simple structure, but not very efficient:
 - `add_node(v)` $O(n)$
 - `remove_node(v)` $O(n)$
 - `adjacent(u, v)` $O(1)$
 - `neighbors(v)` $O(n)$

Interesting problems on graphs

- Is there a (directed) path between two nodes?
- What is the shortest path between two nodes?
- Is there a cycle in the graph?
- Is there a cycle that uses each edge exactly once? (Eulerian path)
- Is there a cycle that uses each node exactly once? (Hamiltonian path)
- Are all nodes of the graph connected?
- Is there a node that breaks the connectivity if removed?
- Is the graph planar: can it be drawn without crossing edges?
- Are two graphs isomorphic (have the same structure)?
- What is the importance of a web page, based on the links pointing to it?

Summary

- Graphs are data structures with many applications
- Reading on graphs: Goodrich, Tamassia, and Goldwasser (2013, chapter 14), Next:
 - Graph traversals
- Reading: Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references

- The map on slide 2 is from OpenStreetMap, The other images are from Wikipedia, except the infectious disease graph which comes from Thurner et al. (2020).

 Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. [9781118476734](https://doi.org/10.1002/9781118476734).